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Restricted version of the infinitesimal Hilbert 16th problem*

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1 Introduction

1.1 Restricted Infinitesimal Hilbert 16th Problem

The original Infinitesimal Hilbert 16th Problem is stated as follows. Consider a real polynomial H in two variables of degree $n+1$. The space of all such polynomials is denoted by $\mathcal{H}_n$.

Connected components of closed level curves of H are called *ovals* of H. Ovals form continuous families, see Fig. 1. Fix one family of ovals, say Γ, and denote by $\gamma(t)$ an oval of this family that belongs to the level curve $\{H = t\}$.

Consider a polynomial one-form

$$\omega = Adx + Bdy$$

with polynomial coefficients of degree at most n. The set of all such forms is denoted by Ω_n. The main object to study below is the integral

$$I(t) = \int_{\gamma(t)} \omega. \tag{1.1}$$

Infinitesimal Hilbert 16th Problem . *Let H and ω be as above. Find an upper bound of the number of isolated real zeros of integral (1.1) for a polynomial $H \in \mathcal{H}_n$ and any family Γ of real ovals of H. The estimate should be uniform in ω and H, thus depending on n only.*

This problem stated more than 30 years ago is not yet solved. The existence of such a bound was proved by A.N.Varchenko [23] and A.G.Khovanskii [12]. A weaker version of the problem is called *restricted.* In order to formulate it we need the following

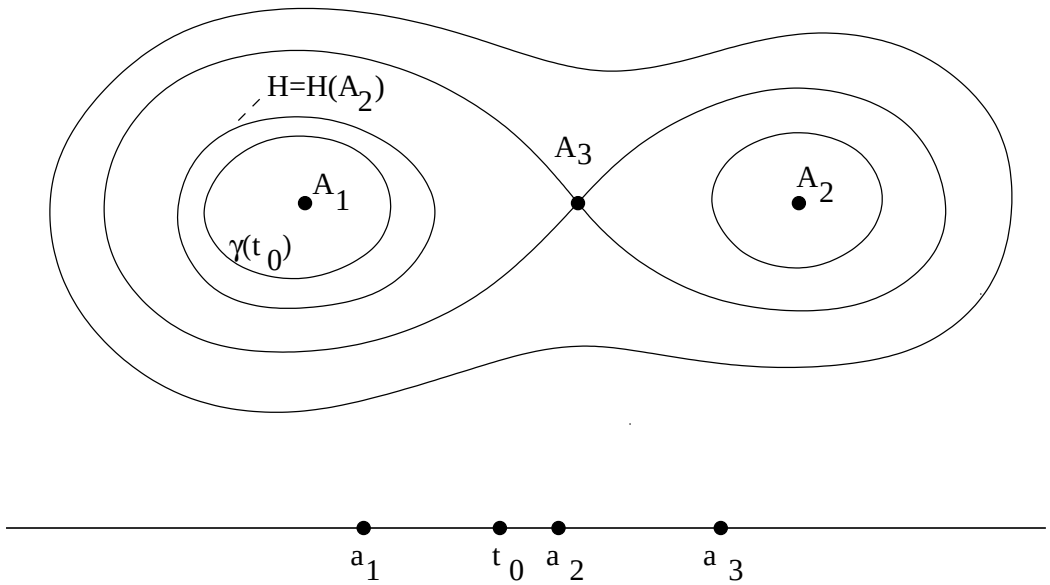

Figure 1: Families of ovals; an oval around A_1 that belongs to the level curve $H = H(A_2)$ is distinguished.

Definition 1.1 A polynomial $H \in \mathcal{H}_n$ is *ultra-Morse* provided that it has n^2 complex Morse critical points with pairwise distinct critical values, and the sum h of its higher order terms has no multiple linear factors.

Denote by $\mathcal{U}_n$ the set of all ultra-Morse polynomials in $\mathcal{H}_n$. The complement to this set is denoted by Σ_n and called *the discriminant set.* The integral (1.1) may be identically zero. The following theorem shows that for ultra-Morse polynomials this may happen by a trivial reason only.

Theorem 1.2 (Exactness theorem [7, 8, 20]) *Let H be a real ultra-Morse polynomial of degree higher than 2. Let the integral (1.1) be identically zero for some family of real ovals of the polynomial H. Then the form ω is exact: $\omega = df$.*

Denote by Ω_n^* the set of all non-exact polynomial one-forms from Ω_n.

Restricted version of the Infinitesimal Hilbert 16th Problem (RIHP) . *For any compact subset $\mathcal{K}$ of the set of ultra-Morse polynomials find an upper bound of the number of all real zeros of the integral (1.1) over the ovals of a polynomial $H \in \mathcal{K}$. The bound should be uniform with respect to $H \in \mathcal{K}$ and $\omega \in \Omega_n^*$. It may depend on n and $\mathcal{K}$ only.*

This problem is solved in the present paper. The solution is based on the results of [2], [3] and [10]. All the four papers ([2], [3], [10] and the present one) were written as the parts of one and the same project. The preliminary draft of the joint text may be found at [4] (preprint, 2001). After 2001, the results were strongly improved and the text had grown up. So, we decided to split the large text in a series of four papers. Each one of the papers [2], [3], [10] is independent on the others. The present paper is the main one in the series. It contains the survey of results of all the four papers, as well as the solution of the RIHP.

Numerous results obtained during more than 30 years of the study of the infinitesimal Hilbert problem are presented in section 7 of a survey paper [9]. Partial solution of the RIHP was claimed in that survey paper. The present paper contains a complete solution to RIHP (modulo [2], [3], [10]). The results of the paper with a brief proof were announced in [5].

1.2 Main results

To measure a gap between a compact set $\mathcal{K} \subset \mathcal{U}_n$ and the discriminant set Σ_n, let us first normalize ultra-Morse polynomials by an affine transformation in the target space. This transformation does not change the ovals of H, thus the number of zeros of the integral (1.1) remains unchanged.

Say that two polynomials G and H are equivalent iff

$$G = aH + b, \quad a > 0, \ b \in \mathbb{C}.$$

Definition 1.3 A polynomial is *balanced* if all its complex critical values belong to the closed disk of radius 2 centered at zero, and there is no smaller disk that contains all the critical values.

Remark 1.4 Any polynomial with at least two distinct critical values is equivalent to one and unique balanced polynomial. If the initial polynomial has real coefficients, then so does the corresponding balanced polynomial.

Define two positive functions on $\mathcal{U}_n$ such that at least one of them tends to zero as H tends to Σ_n. For any compact set $\mathcal{K} \subset \mathcal{U}_n$ the minimal values of these functions on $\mathcal{K}$ form a vector in $\mathbb{R}^+ \times \mathbb{R}^+$ that is taken as a size of the gap between $\mathcal{K}$ and Σ_n.

Definition 1.5 For any $H \in \mathcal{U}_n$ let $c_1(H)$ be n multiplied by the smallest distance between two lines in the locus of h, the higher order form of H. The distance between two lines is taken in sense of Fubini-Study metric on the projective line $\mathbb{C}P^1$. Let $c'(H) = \min(c_1(H), 1)$.

Denote by $\mathcal{V}_n$ the set of all polynomials with more than one critical value and more than one line in the locus of the higher order homogeneous form. By Definition 1.1, $\mathcal{U}_n \subset \mathcal{V}_n$.

Definition 1.6 For any $H \in \mathcal{V}_n$, let G be the balanced polynomial equivalent to H. Let $c_2(H)$ be the minimal distance between two critical values of G multiplied by n^2. Let $c''(H) = \min(c_2(H), 1)$.

Note that inequality $c'(H)c''(H) > 0$ is equivalent to the statement that H is ultra-Morse.

In what follows, we deal with balanced ultra-Morse polynomials only. This may be done without loss of generality: any ultra-Morse polynomial is equivalent to a balanced one; equivalent polynomials have the same number of zeros of the integral (1.1) over the corresponding families of ovals.

Theorem A. *Let H be a real ultra-Morse polynomial of degree $n+1$. Let $\Gamma = \{\gamma(t)\}$ be an arbitrary continuous family of real ovals of H. There exists a universal positive c such that the integral (1.1) has at most $(1 - \log c'(H)) e^{\frac{c}{c''(H)} n^4}$ isolated zeros.*

Appendix. The statement of Theorem A holds with $c = 5.000$.

An approach to the Infinitesimal Hilbert 16th Problem itself presented below motivates the following complex counterpart of Theorem A, namely, Theorem B that gives an estimate of the number of zeros of the integral (1.1) in the complex domain. Consider an ultra-Morse polynomial H and let

$$\nu = \nu(H) := \frac{c''(H)}{4n^2}. \tag{1.2}$$

Fix any real noncritical value t_0 of H,

$$|t_0| < 3,$$

whose distance to the complex critical values of H is no less than ν. Consider a real oval $\gamma_0 \subset \{H = t_0\}$. We suppose that such an oval exists. Let $a = a(t_0) < t_0 < b(t_0) = b$ (or $a(H, t_0)$, $b(H, t_0)$ for variable H) be the nearest real critical values of H to the left and to the right from t_0 respectively; or $-\infty, +\infty$ if there are none. Denote by $\sigma(t_0)$ the interval $(a(t_0), b(t_0))$ and let $\Gamma(\gamma_0)$ be the continuous family of ovals that contains γ_0 :

$$\Gamma(\gamma_0) = \{\gamma(t) \,|t \in \sigma(t_0),\ \gamma(t_0) = \gamma_0\}. \tag{1.3}$$

The following cases for $(a, b) = \sigma(t_0)$ are possible:

$$(a, b),\ b > a; -\infty < a < b < +\infty,\ (a, +\infty);\ (-\infty, b).$$

If a is finite, and $\lim\ \mathrm{top}_{\ t \to a} \gamma(t)$ contains a saddle critical point of H, then a is a logarithmic branch point of I. If $\lim\ \mathrm{top}_{\ t \to a} \gamma(t)$ is a singleton, or contains no critical point of H, then a is called an apparent singularity. The same for b.

Denote by $B = B_H$ the set of all noncritical values of H :

$$B = \mathbb{C} \setminus \{a_1 \ldots, a_\mu\},\ \mu = n^2,\ a_j \text{ are the complex critical values.}$$

Let W be the universal cover over B with the base point t_0 and projection

$$\pi : W \to B \subset \mathbb{C}.$$

Definition 1.7 Any point $\hat{t} \in W$ is represented by a class $[\lambda]$ of curves in B starting at t_0 and terminating at $t = \pi\hat{t}$; all the curves of the class are homotopic on B. Any cycle γ from $H_1(S_{t_0}, \mathbb{Z})$ may be continuously extended over λ as an element of the homology groups of level curves of H; the resulting cycle $\gamma(\hat{t})$ from $H_1(S_t, \mathbb{Z})$ is called an *extension* of γ corresponding to $\hat{t}$.

This construction allows us to extend the integral (1.1) to W : for any $\hat{t} \in W$,

$$I(\hat{t}) = \int_{\gamma(\hat{t})} \omega. \tag{1.4}$$

Let $D(t, r)$ be the disk centered at t of radius r. Denote by $a + re^{i\varphi} \in W$ a point represented by a curve $\Gamma_1\Gamma_2 \subset B$, where Γ_1 is an oriented segment from t_0 to $t_1 = a + r \in \sigma(t_0)$, $\Gamma_2 = \{a + re^{i\theta} \mid \theta \in [0, \varphi]\}$; Γ_2 is oriented from t_1 to t. In the same way $b - re^{i\varphi} \in W$ is defined. Let

$$\Pi(a) = \{a + re^{i\varphi} \in W \mid 0 < r \leq \nu, |\varphi| \leq 2\pi\},\ \text{for } a \neq -\infty \tag{1.5}$$

$$\Pi(b) = \{b - re^{i\varphi} \in W \mid 0 < r \leq \nu, |\varphi| \leq 2\pi\}, \text{ for } b \neq +\infty$$

Let

$$D(l,a) = \{a + re^{i\varphi} \in W \mid a + re^{\frac{i\varphi}{l}} \in \Pi(a)\}$$

$$D(l,b) = \{b - re^{i\varphi} \in W \mid b - re^{\frac{i\varphi}{l}} \in \Pi(b)\},$$

$$D(l,a) = \emptyset, \text{ if } a = -\infty; \; D(l,b) = \emptyset, \text{ if } b = +\infty.$$

Let $DP_R = DP_R(H, t_0)$ be the disk of radius R in the Poincaré metric of W centered at t_0. Denote

$$S_t = \{H = t\} \subset \mathbb{C}^2.$$

For any real polynomial H, the choice of a cycle γ_0 determines a family of ovals (1.3) over which the integral (1.1) is taken. When we want to specify this choice we write I_{H,γ_0} or I_H instead of I. The integral I_{H,γ_0} may be analytically extended not only as a function of $\hat{t} \in W$, but also as a function of H.

An analytic extension of the integral I to W is denoted by the same symbol I. For any positive R and natural l denote by $G = G(l, R, H, t_0)$ the domain

$$G = DP_R(H, t_0) \;\cup\; D(l, a(H, t_0)) \;\cup\; D(l, b(H, t_0)).$$

Theorem B. *For any real ultra-Morse polynomial H, any real oval γ_0 of H, any natural l and any positive $R > \frac{288n^4}{c''(H)}$, the number of zeros of the integral I_{H,γ_0} in $G = G(l, R, H, t_0)$, where $t_0 = H \mid \gamma_0$, is estimated as follows:*

$$\#\{\hat{t} \in G(l, R, H, t_0) | I_{H,\gamma_0}(\hat{t}) = 0\} \leq (1 - \log c'(H)) \cdot \left(e^{7R} + A^{4800} e^{\frac{481l}{c''(H)}}\right), \; A = e^{\frac{n^4}{c''(H)}}. \quad (1.6)$$

The lower bound on R in the statement of the theorem is motivated by the remark in Subsection 2.4 below.

1.3 An approach to a solution of the Infinitesimal Hilbert 16th Problem

Conjecture. *For any n there exist $\delta(n), l(n), R(n)$ with the following property. Let H_0 be an arbitrary real polynomial from $\mathcal{H}_n$, t_0 be its real noncritical value and γ_0 be a real oval of H_0 that belongs to $\{H_0 = t_0\}$ (we suppose that such an oval exists). Let I_H be the integral (1.1). The integral I_H depends on H as a parameter. Let $t_1 \in \sigma(t_0), I_{H_0}(t_1) = 0$ and $t(H)$ be a germ of an analytic function defined by the equation $I_H(t(H)) \equiv 0$, $t(H_0) = t_1$. The required property is the following. There exists a path $\lambda \subset \mathcal{H}_n$ depending on H_0 only starting at H_0 and ending at some $H_1 \in \mathcal{H}_n$ such that:*

$$c'(H_1) \geq \delta(n), \; c''(H_1) \geq \delta(n);$$

the analytic extension $t(H_1)$ of the function $t(H)$ along λ starting at the value t_1 belongs to the domain $G(l(n), R(n), H_1, t_0)$.

The conjecture above implies the solution of the Infinitesimal 16th Problem. Indeed, suppose that the conjecture is true. Let $N(n)$ be the right hand side of the inequality (1.6)

with $c'(H)$ and $c''(H)$ replaced by $\delta(n)$; R and l replaced by $R(n)$ and $l(n)$ respectively. Then the number of real zeros of integral I_{H_0} can not exceed $N(n)$. If not, any of real zeros of I_{H_0} would be extended along λ up to a zero of a polynomial H_1 located in $G = G(l(n), R(n), H_1, t_0)$. Thus the number of zeros of the integral I_{H_1} in G will exceed $N(n)$. But Theorem B implies that the number of zeros of H_1 in G is no greater than $N(n)$, a contradiction.

The paper is structured as follows. In Section 2 we present the main ideas of the proof of Theorems A and B. Section 2 contains also a survey of the previous investigations and describes some results of [3]; these results may be called "quantitative algebraic geometry". Moreover, we prove in this section a part of Theorem A, namely, Theorem $A1$, modulo the Main Lemma. In Section 3 we prove the Main Lemma. The proof relies upon two statements: formula for the determinant of periods, and upper estimates of Abelian integrals provided by quantitative algebraic geometry. These two statements are proved in two separate papers, [2] and [3] respectively. Theorem A is finally proved in Section 5. Theorem B is proved in Sections 4 and 5. The Main Lemma is an important tool for both sections.

2 Main ideas of the proof and survey of the related results

2.1 Historical remarks

A survey of the history of the Infinitesimal Hilbert 16th Problem may be found in [9], and we will not repeat it here. In particular, a much weaker version of Theorem A is claimed there as Theorem 7.7. The first solution to restricted Hilbert problem was suggested in [18]. An explicit upper bound for the same numbers of zeros as in Theorem A was suggested there as a tower of four exponents with coefficients "that may be explicitly written following the proposed constructive solution." It is unclear how much efforts is needed to write these constants down. Moreover, exponential of a polynomial presented in Theorem A is much simpler (though still very excessive) than the tower of four exponentials.

The result of [18] is a crown of a series of papers [15] - [17]. Solution to the restricted version of the Infinitesimal Hilbert 16th Problem presented there is only one application of a vast theory. This theory presents an upper bound of the number of zeros of solutions to linear systems of differential equations. Similar results for components of vector solutions to linear systems are obtained. Abelian integrals are considered as solutions to Picard-Fuchs equations.

On the contrary, our presentation is focused on the study of Abelian integrals given by formula (1.1) "as they are" and not as solutions of differential equations.

2.2 Quantitative algebraic geometry

Our main tool is Growth-and-Zeros theorem for holomorphic functions stated in the next subsection. It requires, in particular, an upper bound of the integral under consideration. We fix an integrand, say $w = x^k y^{n-k} dx$. Depending on a scale in $\mathbb{C}^2$, a cycle γ in the integral $\int_\gamma \omega$ may be located in a small or in a large ball. According to this, the integrand will be small or large. We want to estimate the integral at a certain point of the universal cover W represented by an arc that connects a base point t_0 with some point, say t, with $|t| \leq 3$. To make this restriction meaningful, the scale in the range of the polynomial should be chosen;

in other words, the polynomial should be balanced. The argument above shows that it should be also *rescaled* in sense of the following definitions.

Definition 2.1 The *norm* of a homogeneous polynomial h is the maximal value of its module on the unit sphere; this norm is denoted by $\|h\|_{\max}$.

Definition 2.2 A balanced polynomial $H \in \mathbb{C}[x, y]$ is *rescaled* provided that the norm of its higher order form h equals one: $||h||_{\max} = 1$, and the origin is a critical point for H. Briefly, a balanced rescaled polynomial will be called *normalized.*

Remark 2.3 Any ultra-Morse polynomial may be transformed to a normalized one by homotheties and shifts in the source and target spaces (not in the unique way). The functions c' and c'' remain unchanged under such transformations.

Definition 2.4 We say that the topology of a complex level curve $S_t = H^{-1}(t)$ of a polynomial $H \in \mathcal{H}_n$ is *located in a bidisk*

$$D_{X,Y} = \{(x, y) \in \mathbb{C}^2 \,|\, |x| \leq X,\ |y| \leq Y\}$$

provided that the difference $S_t \setminus D_{X,Y}$ consists of $n + 1 = \deg H$ punctured topological disks, and the restriction of the projection $(x, y) \mapsto x$ to any of these disks is a biholomorphic map onto $\{x \in \mathbb{C} | X < |x| < \infty\}$.

Theorem C [3]. *For a normalized polynomial, the Hermitian basis in $\mathbb{C}^2$ may be so chosen that the topology of all level curves S_t for $|t| \leq 5$ will be located in a bidisk $D_{X,Y}$ with*

$$X \leq Y \leq (c'(H))^{-14n^3} n^{65n^3} = R_0.$$

This theorem is of independent interest, providing one of the first results in *quantitative algebraic geometry.* On the other hand, it implies upper estimates of Abelian integrals used in the proof of Theorem A and required by the Growth-and-Zeros theorem below.

In the rest of this section, we describe the main ideas of the proof of a simplified version of Theorem A, namely Theorem A1 stated below. It provides an upper bound for the number of zeros of the integral (1.1) on a real segment that is ν-distant from critical values of H and belongs to the disk $\overline{D}_3 = \{t \mid |t| \leq 3\}$, thus being distant from infinity; recall that $\nu = \nu(H)$ is given by (1.2).

By the use of Theorem A1, we get in Section 5 an estimate of the number of zeros of the integral I_{H,γ_0} near the endpoints of $\sigma(t_0)$, as well as near infinity (Theorem A2 stated in 2.5). Together with Theorem A1, this completes the proof of Theorem A. Theorem B is split into two parts. The first one (Theorem B1 stated in 4.1) is proved by extending an upper bound given with the help of Theorem C from a disk $|t| \leq 5$ into a larger domain. The second one, Theorem B2 stated in 5.6, is proved (in the same place) by the same tools as Theorem A2. These tools include Petrov method and a so called KRY theorem. The latter one is a recent result in one-dimensional complex analysis [13, 21]. Its improved version is proved by the second author in a separate paper [10] and stated in Section 5. In this form it provides a powerful tool to estimate the number of zeros of analytic functions near logarithmic singularities.

2.3 Growth-and-Zeros Theorem for Riemann surfaces

The idea of the proof of Theorem $A1$ is to consider an analytic extension of the integral (1.1) to the complex domain and to make use of the following Growth-and-Zeros theorem. The symbol diam_{int} used in the statement of the theorem denotes the intrinsic diameter, see Definition 2.6 below. We need a notion of a π-gap between a set and its subset on a Riemann surface.

Definition 2.5 Let W be a Riemann surface, $\pi : W \to \mathbb{C}$ be a holomorphic function (called projection) with non-zero derivative. Let ρ be the metric on W lifted from $\mathbb{C}$ by projection π. Let $U \subset W$ be a connected domain, and $K \subset U$ be a compact set. For any $p \in U$ let $\varepsilon(p, \partial U)$ be the supremum of radii of disks centered at p, located in U and such that π is bijective on these disks. The π-gap between K and ∂U, is defined as

$$\pi\text{-gap } (K, \partial U) = \min_{p \in K} \varepsilon(p, \partial U).$$

Growth-and-zeros theorem. *Let W, π, ρ be the same as in Definition 2.5. Let $U \subset W$ be a domain conformally equivalent to a disk. Let $K \subset U$ be a path connected compact subset of U (different from a single point). Suppose that the following two assumptions hold:*

Diameter condition:

$$diam\ _{int} K \leq D;$$

Gap condition:

$$\pi\text{-}gap(K, \partial U) \leq \varepsilon.$$

Let I be a bounded holomorphic function on $\bar{U}$. Then

$$\#\{z \in K | I(z) = 0\} \leq e^{\frac{2D}{\varepsilon}} \log \frac{\max_{\overline{U}} |I|}{\max_K |I|} \tag{2.1}$$

The definition of the intrinsic diameter is well known; yet we recall it for the sake of completeness.

Definition 2.6 The *intrinsic distance* between two points of a path connected set in a metric space is the infinum of the length of paths in K that connect these points (if exists). The *intrinsic diameter* of K is the supremum of intrinsic distances between two points taken over all the pairs of points in K.

Definition 2.7 The second factor in the right hand side of (2.1) is called the Bernstein index of I with respect to U and K and denoted $B_{K,U}(I)$:

$$B_{K,U}(I) = \log \frac{M}{m}, \ M = \sup_U |I|, \ m = \max_K |I|. \tag{2.2}$$

Proof of the Growth-and-Zeros theorem. The above theorem is proved in [11] for the case when $W = \mathbb{C}, \pi = Id$. In fact, in [11] another version of (2.1) is proved with (2.1) replaced by

$$\#\{z \in K | I(z) = 0\} \leq B_{K,U}(I) e^{\rho}, \tag{2.3}$$

where ρ is the diameter of K in the Poincaré metric of U. In this case it does not matter whether U belongs to $\mathbb{C}$ or to a Riemann surface.

Proposition 2.8 *Let K, U be two sets in the Riemann surface W from Definition 2.5, and let the Diameter and Gap conditions from the Growth-and-Zeros theorem hold. Then the diameter of K in the Poincaré metric of U admits the following upper estimate:*

$$\rho \leq 2D/\varepsilon. \tag{2.4}$$

Proof Denote by $|v|_{PU}$ the length of a vector v in sense of the Poincaré metric of U. By the monotonicity property of the Poincaré metric, the length $|v|_{PU}$ of any vector v attached at any point $p \in K$ is no greater than two times the Euclidean length of v divided by the π-gap between K and ∂U. This implies (2.4) □

Together with (2.3), this proves (2.1). □

2.4 Theorem A1 and Main lemma

In what follows, H will be an ultra-Morse polynomial unless the converse is stated. Consider a normalized polynomial H. Let a_j be its complex critical values, $j = 1, \ldots, n^2$; ν, t_0, W and π be the same as in 1.2. Let I be the integral (1.1) as in Theorem A (well defined for $t = t_0$). It admits an analytic extension to W, which will be denoted by the same symbol I.

Let $a = a(t_0), b = b(t_0)$ be the same as in 1.2, and ν be from (1.2). Let

$$l(t_0) = \begin{cases} a + \nu \text{ for } a \neq -\infty \\ -3 \text{ for } a = -\infty, \end{cases}$$

$$r(t_0) = \begin{cases} b - \nu \text{ for } b \neq +\infty \\ 3 \text{ for } b = +\infty. \end{cases}$$

Let

$$\sigma(t_0, \nu) = [l(t_0), r(t_0)].$$

We identify $\sigma(t_0, \nu) \subset \mathbb{C}$ with its lift to W that contains t_0.

Theorem A1. *In the assumptions at the beginning of the subsection, for any complex form $\omega \in \Omega_n^*$,*

$$\#\{t \in \sigma(t_0, \nu) \mid I(t) = 0\} < (1 - \log c')A^{578}, \ A = e^{\frac{n^4}{c''}(H)}. \tag{2.5}$$

This theorem is an immediate corollary of the Growth-and-Zeros theorem and the Main Lemma stated below. Let

$$L^{\pm}(t_0) = \begin{cases} \{a + \nu e^{\pm i\varphi} \in W \mid \varphi \in [0, 2\pi]\} \text{ for } a \neq -\infty \\ \{-3e^{\pm i\varphi} \in W \mid \varphi \in [0, 2(n+1)\pi]\}, \text{ for } a = -\infty, \end{cases} \tag{2.6}$$

$$R^{\pm}(t_0) = \begin{cases} \{b - \nu e^{\pm i\varphi} \in W \mid \varphi \in [0, 2\pi]\} \text{ for } b \neq +\infty \\ \{+3e^{\pm i\varphi} \in W \mid \varphi \in [0, 2(n+1)\pi]\}, \text{ for } b = +\infty, \end{cases} \tag{2.7}$$

$$\Gamma_a = L^+(t_0) \cup L^-(t_0),\ \Gamma_b = R^+(t_0) \cup R^-(t_0),\ \Sigma = \Gamma_a \cup \Gamma_b \cup \sigma(t_0, \nu).$$

Main Lemma. *Let H be a normalized polynomial of degree $n+1 \geq 3$ with critical values $a_j :\ j = 1, ..., n^2$, ω be a complex polynomial 1-form of degree no greater than n. Let W, ν, Σ be the same as at the beginning of this subsection. Then there exists a path connected compact set $K \subset W$, $K \supset \Sigma$, $\pi K \subset \overline{D_3}$, with the following properties:*

$$diam_{int} K < 36n^2; \tag{2.8}$$

$$\rho(\pi K, a_j) \geq \nu \text{ for any } j = 1, ..., n^2. \tag{2.9}$$

Moreover, let U be the minimal simply connected domain in W that contains the $\nu/2$ neighborhood of K. Then the Bernstein index of the integral (1.1) *admits the following upper bound:*

$$B_{K,U}(I) < (1 - \log c')A^2. \tag{2.10}$$

The Main Lemma is proved in Section 3. It is used also in the estimate of the number of zeros of the integral in the intervals $(a, l(t_0))$, $(r(t_0), b)$. In fact, a much better estimate for the Bernstein index holds:

$$B_{K,U}(I) < \frac{2700n^{18}}{c''(H)} - 30n^6 \log c'(H) := B(n, c', c''). \tag{2.11}$$

Inequality (2.11) is proved in 3.7. Together with the elementary inequality

$$B(n, c', c'') < (1 - \log c')A^2, \tag{2.12}$$

it implies (2.10).

Proof of Theorem A1. Let us apply Growth-and-Zeros theorem to the function I in the domain U in order to estimate the number of zeros of I in K; note that $K \supset \sigma(t_0)$. The intrinsic diameter of K is estimated from above by (2.8). The gap condition for U and K has the form

$$\varepsilon(K,\ \partial U) = \frac{\nu}{2} = \frac{c''}{8n^2}$$

by the definition of U. Hence,

$$e^{\frac{2D}{\varepsilon}} < e^{\frac{72n^2}{c''}8n^2} = A^{576}.$$

The Bernstein index $B_{K,U}(I)$ is estimated from above in (2.10). By Growth-and-Zeros theorem

$$\#\{t \in \sigma(t_0) \mid I(t) = 0\} < B_{K,U}(I)A^{576} < (1 - \log c')A^{578}.$$

This proves (2.5). □

The following remark motivates the restriction on R in Theorem B.

Remark 2.9 Let K be the set from the Main Lemma, $\rho_W K$ be its diameter in the Poincaré metric of W. Then

$$\rho_W K < (c'')^{-1} 288 n^4. \tag{2.13}$$

Indeed, $\rho_W K$ is no greater than the ratio of the double intrinsic diameter of K divided by its minimal distance to the critical values of H. Together with (2.8) and (2.9) this implies (2.13). On the other hand, in the proof of Theorem B, we apply Growth-and-Zeros theorem in the case, when the Poincaré disc $DP_R(H, t_0)$ is large enough, namely, contains the set K. Hence, the maximum of $|I|$ over the disk is no less than $\max |I|$ over K. The latter maximum is estimated from below in the proof of the Main Lemma, see inequality (3.17) below.

2.5 Theorem A2 and proof of Theorem A

Theorem A2. *Let H, $t_0, a = a(t_0), b = b(t_0)$ be the same as in the previous Subsection. Let ω be a real 1- form in Ω_n^*. Then, in assumptions of Theorem A1,*

$$\#\{t \in (a, l(t_0)) \cup (r(t_0), b) \mid I(t) = 0\} < (1 - \log c') A^{4800} \tag{2.14}$$

Proof of Theorem A. By Theorems A1 and A2

$$\#\{t \in (a, b),\ I(t) = 0\} < (1 - \log c') A^{578} + (1 - \log c') A^{4800} < 2(1 - \log c') A^{4800}. \tag{2.15}$$

This implies the estimate of the number of zeros given by Theorem A on the interval (a, b).

Let $\sigma' \subset \mathbb{R}$ be the maximal interval of continuity of the family Γ of real ovals that contains γ_0. Then σ' is bounded by a pair of critical values, at most one of them may be infinite. In general, the interval σ' may contain critical values (see Fig.1, which presents a possible arrangement of level curves of H in this case: A_1, A_2, A_3 are critical points of H, $a_j = H(A_j)$, $a_2 \in \sigma' = (a_1, a_3)$, $t_0 \in (a_1, a_2)$). In this case $\sigma' \neq (a, b) = (a_1, a_2)$. Let us estimate the number of zeros on σ'. The interval σ' is split into at most n^2 subintervals bounded by critical values. On each subinterval the number of zeros of I is estimated by (2.15), as before. Therefore, the number of zeros of I on σ' is less than $2n^2(1 - \log c') A^{4800} < (1 - \log c') A^{4801}$. This proves Theorem A. □

3 An upper bound for the number of zeros on a real segment distant from critical values

In this section we prove the Main Lemma and hence Theorem A1. We also prove the Modified Main Lemma, see Subsection 3.8 below, and prepare necessary tools for the proof of other results: Theorems A2, B1 and B2.

3.1 Sketch of the proof of the Main Lemma

The proof of the Main lemma is based on the following idea. The integral (1.1) is extended onto the universal cover W of the set of noncritical values of the real ultra-Morse polynomial H; the base point of this cover belongs to $(-3,3)$. The upper estimate of the Bernstein index of this integral in the pair of domains U, K requires an upper bound of the maximal module of the integral in $\overline{U}$, and a lower bound in K. When we consider these maximums instead of their ratio, we have to normalize the form ω, multiplying it by a complex factor.

Definition 3.1 A polynomial 1-form is normalized if the maximal magnitude of its coefficients equals 1, and some coefficients equal 1.

The upper bound of the integral is provided by the quantitative algebraic geometry. The main difficulty is to obtain the lower bound. For this we consider μ^2 integrals instead of a single one; recall that $\mu = n^2$. Namely, we introduce a special set of μ forms $\omega_i, i = 1, \ldots, \mu$ and a special set of vanishing cycles on the level curves $S_t = \{H = t\}$: $\delta_1(t), \ldots, \delta_\mu(t)$. The matrix $\mathbb{I}(t)$ with the entries $I_{ij}(t) = \int_{\delta_j(t)} \omega_i$ is called *a matrix of periods.* The determinant $\Delta(t) = \det \mathbb{I}(t)$ is univalent. The first step is to evaluate its determinant and to provide a lower bound for $\Delta(t)$ when t is distant from the critical values of H. This is done in [2] and [3]. The second step is to give an upper estimate for the entries of $\mathbb{I}$. This estimate based on the results of [3] is obtained below. The main step is to construct the set $K \subset W$. This set is constructed in such a way that the assumption "$m := \max_K |I|, I(t) = \int_{\gamma(t)} \omega$, is small" implies that *all the integrals* $\int_{\delta_j(t_0)} \omega$, $j = 1, ..., \mu$ are small. This implication makes use of the Picard-Lefschets theorem, and the connectedness of the intersection graph of the special system of vanishing cycles.

The implication above is used in the following way. For a normalized form ω, one may replace some row of the matrix $\mathbb{I}$ by the row $\int_{\delta_1(t)} \omega, \ldots, \int_{\delta_\mu(t)} \omega$ without changing the main determinant. All the entries of $\mathbb{I}$ are estimated from above; the determinant of $\mathbb{I}$ is estimated from below. This implies that none of the rows of $\mathbb{I}$ may be too small, and thus provides a lower bound for m. The domain U is chosen as a slightly modified ε-neighborhood of K for appropriate ε. The upper estimate of $M = \max_U |I|$ is obtained as explained above. Upper estimate of M and lower bound for m imply an upper estimate of the Bernstein index $B_{U,K}(I)$ and thus prove the Main Lemma.

Let us now pass to the detailed proof.

3.2 Special set of vanishing cycles and modified Main Lemma

All along this section H is a real normalized ultra-Morse polynomial of degree $n+1 \geq 3$, $\mu = n^2$; $a_1, \ldots, a_\mu$ are critical values of H, ν is the same as in (1.2), $\varepsilon = \nu/2$. For t close to a_j, $\delta_j(t)$ is a local vanishing cycle corresponding to a_j on a level curve

$$S_t = \{H = t\};$$

Recall the definition of this cycle.

Consider an ultra-Morse polynomial in $\mathbb{C}^2$ having a (Morse) critical point with a critical value a. An intersection of a level curve of this function corresponding to a value close to a with an appropriate neighborhood of the critical point is diffeomorphic to an annulus.

This follows from the Morse lemma. The annulus above may be called a local level curve corresponding to the a critical value a.

Definition 3.2 A generator of the first homology group of the local level curve corresponding to a is called a *local vanishing cycle* corresponding to a.

A local vanishing cycle is well defined up to change of orientation.

A path $\alpha_j : [0,1] \to \mathbb{C}$ is called *regular* provided that

$$\alpha_j(0) = t_0,\ \alpha_j(1) = a_j,\ \alpha_j[0,1) \subset B \tag{3.1}$$

Definition 3.3 Let α_j be a regular path, $s \in [0,1]$ be close to 1, $\delta_j(t)$, $t = \alpha_j(s)$, be a local vanishing cycle on S_t corresponding to a_j. Consider the extension of δ_j along the path α up to a continuous family depending on s of cycles $\delta_j(\alpha_j(s))$ in complex level curves $H = \alpha_j(s)$. The homology class $\delta_j = \delta_j(t_0) \in H_1(S_{t_0}, \mathbb{Z})$ (corresponding to $s = 0$) is called a *cycle vanishing along* α_j.

Definition 3.4 Consider a set of regular paths $\alpha_1, \ldots, \alpha_\mu$, see (3.1). Suppose that these paths are not pairwise and self intersected. Then the set of cycles $\delta_j \in H_1(S_{t_0}, \mathbb{Z})$ vanishing along α_j, $j = 1 \ldots, \mu$, is called a *marked set of vanishing cycles* on the level curve $H = t_0$.

Recall that $W = W(t_0, H)$ is the universal cover over the set of noncritical values of H with the base point t_0 and the projection $\pi : W \to \mathbb{C}$.

Let $\delta_1, ..., \delta_\mu$ be a marked set of vanishing cycles. For any cycle δ_l from this set, consider an integral

$$I_l(t) = \int_{\delta_l(t)} \omega,$$

over local vanishing cycles, for t close to a_l. This integral is holomorphic at a_l, and takes zero value at a_l. Denote by W_l the Riemann surface of the analytic extension of this integral. Note that the Riemann surface W_l contains the disc $D_\nu(a)$.

Lemma 3.5 (Modified Main Lemma). *The Main Lemma from Subsection 2.4 holds true provided that the real oval $\gamma(t)$ of integration (1.1) is replaced by a local vanishing cycle $\delta_l(t)$ close to the corresponding critical value a_l, W is replaced by W_l and Σ is replaced by the disk $\overline{D}_\nu(a_l)$.*

This lemma is proved in 3.8.

3.3 Matrix of periods

Consider and fix an arbitrary marked set of vanishing cycles δ_j, $j = 1, \ldots, \mu$. For any $\hat{t} \in W$, let $\delta_j(\hat{t})$ be the extension of δ_j corresponding to $\hat{t}$.

Definition 3.6 Consider a set Ω of μ forms ω_j of the type

$$\omega_i = yx^k y^l dx,\ k, l \geq 0,\ k + l \leq 2n - 2 \tag{3.2}$$

(k, l) depends on i, such that all the forms with $k + l \leq n - 1$ are included in the set, and the number of forms with monomials of degree $2n - k$ equals k for $1 \leq k \leq n$. In what follows, such a set is called *standard*.

A *matrix of periods* $\mathbb{I} = (I_{ij}),\ 1 \le i \le \mu,\ 1 \le j \le \mu$, is the matrix function defined on W by the formula:

$$I_{ij}(\hat{t}) = \int_{\delta_j(\hat{t})} \omega_i,\ \mathbb{I}(\hat{t}) = (I_{ij}(\hat{t})) \tag{3.3}$$

where $\delta_j,\ j = 1, \ldots, \mu$, form a marked set of vanishing cycles; $\{\omega_i | i = 1, \ldots, \mu\}$ is a standard set of forms (3.2).

When we want to specify dependence on H, we write $\mathbb{I}(\hat{t}, H)$ instead of $\mathbb{I}(\hat{t})$.

3.4 Upper estimates of integrals

Denote by $|\lambda|$ the length of a curve λ, and by $U^\varepsilon(A)$ the ε-neighborhood of a set A.

The main result of the quantitative algebraic geometry that we need is the following

Theorem 3.7 *Let δ_j be a vanishing cycle from a marked set, see Definition 3.4, corresponding to a curve $\alpha_j, |\alpha_j| \le 9$ (recall that $|t_0| \le 3$). Let $\lambda \subset B$ be a curve starting at t_0 (denote by t its endpoint) such that*

$$|\lambda| \le 36n^2 + 1,\ |t| \le 5. \tag{3.4}$$

Let the curve $\alpha_j \cap U^\varepsilon(a_j)$ be a connected arc of α_j, and the curves $\alpha_j \setminus U^\varepsilon(a_j)$ and λ have an empty intersection with ε-neighborhoods of the critical values a_k, where $\varepsilon = \nu/2$, ν is from (1.2). Let ω be a form (3.2), $\hat{t} \in W$ corresponds to $[\lambda]$, and $\delta_j(\hat{t})$ be the extension of δ_j to $\hat{t}$. Then

$$|\int_{\delta_j(\hat{t})} \omega| < 2^{\frac{2600n^{16}}{c''(H)}} (c'(H))^{-28n^4} := M_0 \tag{3.5}$$

This result is based on Theorem C from 2.2. Both results are proved in the forthcoming paper [3].

We have to give an upper bound of the integral not over a vanishing cycle, but over a real oval. The following Lemma shows that the real oval is always a linear combination of some (at most μ) vanishing cycles with coefficients ± 1.

Lemma 3.8 (Geometric lemma). *Let H be a real ultra-Morse polynomial and γ be a real oval of H. Let $H|_\gamma = t_0$. Denote by s the number of critical points of H located inside γ in the real plane. Let $a_1, \ldots, a_s$ be the corresponding critical values. Let $\alpha_j,\ j = 1, \ldots, s$, be nonintersecting and nonself-intersecting paths that connect t_0 with these critical values and satisfy assumption (3.1). Moreover, suppose that all these paths belong to the upper halfplane and for any a_j (which is real), an open domain bounded by a path α_j and a real segment (connecting the endpoints of α_j) contains no critical value of H (see Figures 2 and 3). Let δ_j be the vanishing cycles that correspond to the paths α_j. Then*

$$[\gamma] = \Sigma_1^s \varepsilon_j \delta_j,\ \textit{where } \varepsilon_j = \pm 1. \tag{3.6}$$

The authors believe that Lemma 3.8 is well known to specialists, but they did not find it in literature. Its proof is given in 3.10.

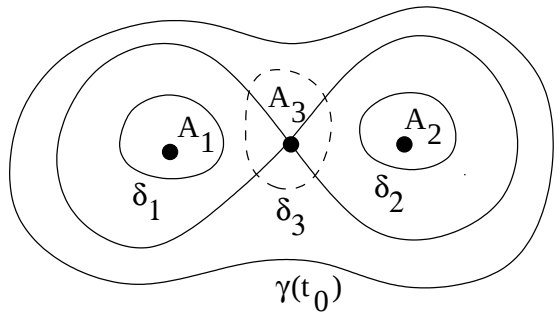

Figure 2: The cycle $\gamma = \gamma(t_0)$ and local vanishing cycles $\delta_j = \delta_j(t_j)$; the points t_j close to a_j are marked at Fig.3.

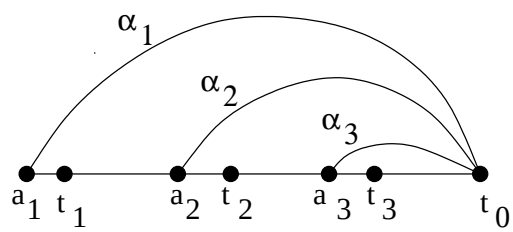

Figure 3: The paths for the extension of the local vanishing cycles $\delta_j(t)$.

Upper estimates of the integrals of monomial forms over vanishing cycles are provided by Theorem 3.7. When we replace a monomial form by a polynomial one, the following changes are needed. Let $\omega \in \Omega_n^*$ be the form in the integral I. There exists another form of type

$$\omega' = \sum_{k+l \leq n-1} a_{kl} x^k y^{l+1} dx, \tag{3.7}$$

such that the difference $\omega - \omega'$ is exact. We may replace the form ω by ω' in (1.1); the integral I will be preserved. Moreover, we can replace the form ω' by a normalized form $\alpha\omega', \alpha \in \mathbb{C}$, see Definition 3.1. Hence, we may assume that the form ω in the integral I has the type (3.7) and is normalized from the very beginning. When we replace a monomial form by a normalized one, the previous upper bound of the integral should be multiplied by the number of monomials, namely, by $\frac{n(n+1)}{2}$. When the vanishing cycle is replaced by a real one, the integral is replaced by a sum of $s \leq n^2$ integrals over vanishing cycles, by the Geometric Lemma. This results in another multiplication by n^2.

Corollary 3.9 *In the condition of the previous Theorem let H be a real polynomial, $\gamma(\hat{t})$ be the extension to $\hat{t}$ of a real oval, ω be a normalized form* (3.7)*. Then*

$$|I_{\gamma(\hat{t})}\omega| \leq \frac{n^3(n+1)}{2} M_0. \tag{3.8}$$

3.5 Determinant of periods

The determinant of the matrix of periods (3.3) is called the *determinant of periods.* It appears that this determinant is single-valued on B, thus depending not on a point of the universal cover W, but rather on the projection of this point to B. Let

$$\Delta(t) = \det \mathbb{I}(\hat{t}),\ t = \pi\hat{t}.$$

The main determinant is single-valued; this follows from the Picard-Lefschetz theorem. Indeed, a circuit around one critical value adds the multiple of the correspondent column to some other columns of the matrix of periods. Thus the determinant remains unchanged.

When we want to specify the dependence of the main determinant on H, we write $\Delta_H(t)$. This function is a polynomial in t, and an algebraic function in the coefficients of H. The formula for the main determinant (with ω_i of appropriate degrees) with a sketch of the proof was claimed by A.Varchenko [24]; this formula is given up to a constant factor not precisely determined. The complete answer (under the same assumption on the degrees of ω_i) is obtained by the first author (A.Glutsyuk, [2]). Moreover, the following lower estimate holds:

Theorem 3.10 *For any normalized ultra-Morse polynomial H, the tuple Ω of standard forms (3.2) may be so chosen that for any $t \in \mathbb{C}$ lying outside the $\nu = \frac{c''}{4n^2}$- neighborhoods of the critical values of H the following lower estimate holds:*

$$|\Delta(t,H)| \geq (c'(H))^{6n^3}(c''(H))^{n^2} n^{-62n^3} := \Delta_0 \tag{3.9}$$

This result is proved in [3] with the use of the explicit formula for the Main Determinant mentioned before, and results of the quantitative algebraic geometry.

3.6 Construction of the set K

We can now pass to the construction of the set K mentioned in the Main Lemma. We first construct a smaller set K'.

Lemma 3.11 (Construction lemma). *Let $\gamma \subset S_{t_0}$ be a real oval of an ultra-Morse polynomial. Then there exist:*

a set of regular paths α_j, $j = 1, \ldots, \mu$, see Definition 3.3, such that $|\alpha_j| \leq 9$, and the paths α_j are not pairwise and self intersected;

a path connected set $K' \subset W$, $t_0 \in K'$, $\pi K' \subset D_3$, such that for any cycle $\delta_j \in H_1(S_{t_0}, \mathbb{Z})$ vanishing along α_j there exist two points $\tau_1, \tau_2 \in K' \cap \pi^{-1}(t_0)$ with the property

$$[\gamma(\tau_1)] - [\gamma(\tau_2)] = l_j[\delta_j],\ l_j \in \mathbb{Z} \setminus 0. \tag{3.10}$$

Moreover,

$$diam_{int} K' < 19n^2, \tag{3.11}$$

and $\pi K'$ is disjoint from ν-neighborhoods of the critical values a_j, $j = 1, \ldots, \mu$.

The next modification of this Lemma will be used in the proof of the Modified Main Lemma.

Lemma 3.12 (Construction lemma for vanishing cycles). *Construction lemma holds true if $\gamma \subset S_{t_0}$ is replaced by any vanishing cycle $\delta_l = \delta_l(t_0)$ from an arbitrary marked set of vanishing cycles, and W is replaced by W_l (see 3.2). In the conclusion, (3.10) should be replaced by*

$$[\delta_l(\tau_1)] - [\delta_l(\tau_2)] = l_j[\delta_j(t_0)], \text{ for } j \neq l, \ l_j \in \mathbb{Z} \setminus 0.$$

Both lemmas are proved in 3.9. In what follows we deduce the Main Lemma from Lemma 3.11 and Theorems 3.7, 3.10.

Corollary 3.13 (of Lemma 3.11). *For any form ω (not necessarily of type (3.2)) and any marked set of vanishing cycles consider the vector function*

$$\mathbb{I}_\omega : W \to \mathbb{C}^\mu, \ \hat{t} \mapsto \left(\int_{\delta_1(\hat{t})} \omega, \ldots, \int_{\delta_\mu(\hat{t})} \omega \right). \tag{3.12}$$

Let $||\cdot||$ denote the Euclidean length in $\mathbb{C}^\mu$. Then

$$m_0 := \max_{\hat{t} \in K' \cap \pi^{-1}(t_0)} |I(\hat{t})| \geq \frac{1}{2n} ||\mathbb{I}_\omega(t_0)||. \tag{3.13}$$

Proof Consider a component of the vector $\mathbb{I}_\omega(t_0)$ with the largest magnitude. Let its number be j. Then

$$\left| \int_{\delta_j(t_0)} \omega \right| \geq \frac{1}{n} ||\mathbb{I}_\omega(t_0)||. \tag{3.14}$$

By Lemma 3.11, there exist $\tau_1, \tau_2 \in K' \cap \pi^{-1}(t_0)$ such that

$$I(\tau_1) - I(\tau_2) = l_j \int_{\delta_j(t_0)} \omega, \ l_j \in \mathbb{Z} \setminus 0.$$

Hence, at least one of the integrals $I(\tau_l)$ in the left hand side, say $I(\tau_l)$, $l \in \{1, 2\}$, admits a lower estimate:

$$|I(\tau_l)| \geq \frac{1}{2} \left| \int_{\delta_j(t_0)} \omega \right|. \tag{3.15}$$

Together with (3.14) this proves the Corollary. □

Let us now take

$$K = K' \cup \Sigma, \Sigma = \sigma(t_0) \cup L^\pm(t_0) \cup R^\pm(t_0), \tag{3.16}$$

see (2.6), (2.7).

In the following Section we will check that this K satisfies the requirements of the Main Lemma.

3.7 Proof of the Main Lemma

Let us take K as in (3.16). Let ν be the same as in (1.2). Let U be the smallest simply connected set that contains the ε-neighborhood of K, $\varepsilon = \nu/2$. Then (2.8) follows from (3.11), (3.16). The last statement of Lemma 3.11 implies (2.9).

Let us now check (2.10), that is, estimate from above the Bernstein index $B_{K,U}(I)$ for the integral (1.1).

Let the form ω in the integral (1.1) be normalized, and let, as before, $M = \max_{\bar{U}} |I|$, $m = \max_K |I|$. By Corollary 3.9,

$$M \leq \frac{n^3(n+1)}{2} M_0 := M_0'$$

where M_0 is from (3.5). Let us now estimate m from below, following the ideas presented at the beginning of the Section.

Let in (3.7) $|a_{k_0 l_0}| = 1$, $\omega_i = y x^{k_0} y^{l_0} dx$. Without loss of generality we may assume that $a_{k_0 l_0} = 1$. Let us now replace the ith row of the matrix $\mathbb{I}$ by the vector $\mathbb{I}_\omega$. This transformation is equivalent to adding a linear combination of rows of $\mathbb{I}$ to the ith row, so the determinant $\Delta(t_0)$ remains unchanged.

By Theorem 3.7 and (2.8), all the entries in other rows are estimated from above by M_0, see (3.5). (The corresponding paths α_j used in the construction of K are chosen as in Lemma 3.11, so, the inequality $|\alpha_j| \leq 9$ of Theorem 3.7 holds true.) Hence, all the vector-rows except for the ith one have the length at most nM_0. By (3.13), the ith row has the length at most $2nm_0$. We can now obtain a lower bound for m. Indeed, $m \geq m_0$. On the other hand,

$$\Delta_0 \leq |\Delta(t_0)| \leq 2m_0 M_0^{\mu-1} n^\mu, \mu = n^2,$$

where Δ_0 is the same as in (3.9). Therefore,

$$m \geq m_0 \geq \frac{1}{2} \Delta_0 M_0^{1-\mu} n^{-\mu}. \tag{3.17}$$

We can now estimate $B_{K,U}(I)$ from above. Indeed,

$$B_{K,U}(I) = \log M - \log m \geq \log M_0' - \log m_0.$$

Elementary estimates (together with (3.17)) imply that

$$\log M_0' - \log m_0 \geq (1 - \log c')A^2. \tag{3.18}$$

This proves the Main Lemma.

3.8 Modified Main Lemma and zeros of integrals over (complex) vanishing cycles

Proof of the Modified Main Lemma. The arguments of the previous Subsection work almost verbatim. The previous Corollary for the integral $I = I_l$ taken over δ_l instead of γ, is stated and proved in the same way.

Let K' be the same as in Lemma 3.12. Instead of (3.16), let

$$K = K' \cup \alpha_l \cup \overline{D}_\nu(a_l).$$

Let U be the smallest simply connected set that contains the ε-neighborhood of K.

By Theorem 3.7,

$$\max_{\overline{V}} |I_l| \leq M_0, \text{ where } V = U \setminus D_\nu(a_l).$$

But I_l is holomorphic in $D_\nu(a_l)$. Hence, by the maximum modulus principle, the previous inequality holds in U instead of V. After that, the rest of the arguments of the previous Subsection work. This proves the Modified Main Lemma. □

Theorem 3.14 *The number of zeros of the integral I_l in the disk $D_\nu(a_l)$ satisfies the inequality:*

$$\#\{\hat{t} \in D_\nu(a_l) | I_l(\hat{t}) = 0\} \leq (1 - \log c'(H))A^{578}. \tag{3.19}$$

The proof is the same as for Theorem A1, section 2.4.

3.9 Proof of the Construction Lemmas

Proof of Lemma 3.11. We prove the lemma in four steps. The set K' is constructed in the first three steps. In the fourth step we check that the resulting set has the required properties.

Step 1: special path set. Let α'_j be a segment $[t_0, a_j]$ oriented from t_0. Note that the disks $D_i = D_\nu(a_i)$ are pairwise disjoint, by definition of ν, see (1.2). If the intersection $\beta_j = \alpha'_j \cap D_i$ is nonempty, it is replaced by the smallest arc of the circle ∂D_i with the chord β_j. If both arcs are equal, and $\alpha'_j \subset \mathbb{R}$, the detour arc is chosen in the upper halfplane. In the opposite case, the semicircles are chosen lying on one side of α'_j, no matter, what one.

The pathes thus modified are slightly perturbed in order to become nonintersecting outside t_0. These pathes are denoted α_j and form a special path set.

Recall that $t_0 \in D_3$, $a_j \in \overline{D}_2$. Therefore, the length of any segment α'_j is less than 5. Hence,

$$|\alpha_j| \leq \frac{5\pi}{2} < 9. \tag{3.20}$$

Step 2: special loop set. For any α_j take a loop λ_j *associated to* α_j and defined as follows:

$$\lambda_j = \alpha''_j \partial D_\nu(a_j)(\alpha''_j)^{-1}, \ \alpha''_j = \alpha_j \setminus D_\nu(a_j),$$

$\partial D_\nu(a_j)$ is positively oriented, and the path α''_j is oriented from t_0. The set K' we are looking for will be the union of appropriate n^2 liftings of the loops λ_j (one lifting for each λ_j) associated with α_j, to the Riemann surface W.

Step 3: construction of K'. Denote by G the intersection graph of $\gamma(t_0)$ and all the vanishing cycles δ_j (along the previously constructed paths α_j). (Recall the definition of the intersection graph: its vertices are identified with the cycles; two of them are connected by an edge, if and only if the corresponding intersection index is nonzero.) *This graph is connected.* This follows from the two lemmas below.

Lemma 3.15 *The intersection graph of the marked set of vanishing cycles of an ultra-Morse polynomial is connected. The set itself forms a basis in the group $H_1(S_{t_0}, \mathbb{Z})$.*

Lemma 3.15 is implied by the following statements from [1]: theorem 1 in 2.1 and theorem 3 in 3.2.

Lemma 3.16 *Consider a maximal family of real ovals that contains $\gamma(t_0)$. The union of the ovals of the family forms an open domain. The boundary of this domain consists of one or two connected components. Any of these components belongs to a critical level of H and contains a unique critical point. Fix any of these critical points and denote by δ the corresponding local vanishing cycle. Then the cycle δ may be extended to a cycle $\delta(t_0)$ that belongs to a marked set of vanishing cycles constructed above. Moreover,*

$$(\delta(t_0), \gamma(t_0)) \neq 0, \text{ more precisely, it is equal to } \pm 1, \ \pm 2.$$

The proof of this lemma is written between the lines of [6], pp 12,13. It is illustrated by Fig.4.

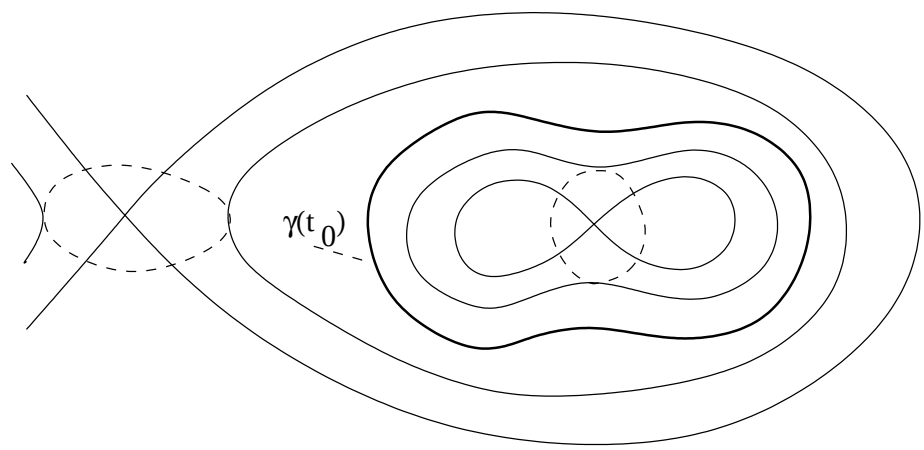

Figure 4: Local vanishing cycles having nonzero intersection index with the ovals of the family that contains $\gamma(t_0)$.

Let us define a metric on the set of the vertices of the graph G. Suppose that each edge of G has length 1. Then the distance D_G between any two vertices of G is well defined as the length of the shortest path in G that connects the vertices. For any j let

$$\rho(\delta_j) = D_G(\gamma(t_0), \delta_j).$$

Let T be a maximal tree in G with the root $[\gamma(t_0)]$ such that the distance in T (defined as in D_G but with paths in T) of any vertex to the root $[\gamma(t_0)]$ coincides with D_G (see Fig.5: the tree T is marked bold).

The set $K' \subset W$ we are looking for is the image of the tree T in W under a continuous map $\Phi : T \to W$ so that the vertices are mapped to $\pi^{-1}(t_0)$. This map is defined by induction in $r = \rho(\delta_j)$ as follows.

Base of induction: $r = 0$. The cycle $\gamma(t_0)$ is mapped to t_0.

Induction step. Let δ_j be an arbitrary vanishing cycle with $\rho(\delta_j) = r$, and δ' be the ancestor of δ_j. Then $\rho(\delta') = r - 1$, and $\Phi(\delta') := \tau_1 \in \pi^{-1}(t_0)$ is defined by the induction

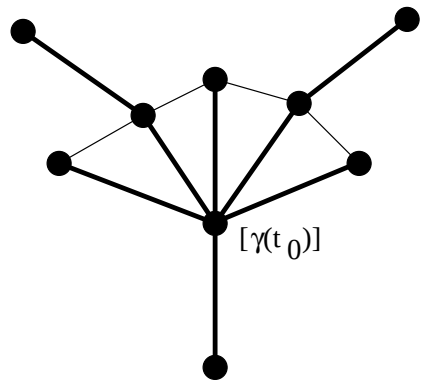

Figure 5: The intersection graph and the maximal tree.

hypothesis. Let $[\delta', \delta_j]$ be the edge of G with the vertices δ', δ_j. Let us lift the special loop λ_j to a covering loop $\tilde{\lambda}_j \subset W$ with the initial point τ_1. Let $\tilde{\lambda}_j = \Phi([\delta', \delta_j])$; the continuation of Φ to the edge $[\delta', \delta_j]$ is an arbitrary homeomorphism with $\Phi(\delta') := \tau_1$. Let $\Phi(\delta_j) := \tau_2 \in \pi^{-1}(t_0)$. The induction step is over. By definition, $K' = \Phi(T)$.

Step 4: properties of the set K'. The set K' is a curvilinear tree and thus, path connected. Its intrinsic diameter admits the upper estimate

$$\mathrm{diam}_{int} K' \leq n^2 \max_j |\lambda_j| < 19n^2, \text{ since } |\lambda_j| = 2|\alpha_j''| + 2\pi\nu < 19.$$

The set K' is projected to the loops λ_j, which lie in D_3 and are disjoint from the ν- neighborhoods of the critical values by definition. Hence, the same is true for $\pi(K')$.

Let us prove (3.10). For any vanishing cycle $\delta_j(t_0)$ let L_j be the branch of the tree T from $[\gamma(t_0)]$ to $\delta_j(t_0)$. Let $[\gamma(t_0)]$, $\delta_{j_1}(t_0)$, ..., $\delta_{j_r}(t_0) = \delta_j(t_0)$ be its vertices ordered from the beginning to the end of the branch. By definition, the intersection index of any cycle in this sequence with its two neighbors is nonzero, and that of any two nonneighbor cycles is zero. Let us call this *the regularity property* of L_j.

We will prove (3.10) by induction in r. Let $\tau_1, \tau_2 \in \pi^{-1}(t_0)$ be the same, as in the previous induction step: $\tau_2 = \Phi(\delta_j)$, $\tau_1 = \Phi(\delta') = \Phi(\delta_{j_{r-1}})$. Moreover, we will prove by induction that

$$\gamma(\tau_2) = \gamma(t_0) + \sum_{m=1}^{r} l_m \delta_{j_m}(t_0), \; l_m \in \mathbb{Z} \setminus 0, \tag{3.21}$$

The induction assumption is

$$\gamma(\tau_1) = \gamma(t_0) + \sum_{m=1}^{r-1} l_m \delta_{j_m}(t_0), \; l_m \in \mathbb{Z} \setminus 0. \tag{3.22}$$

Now, (3.21) follows from (3.22) by the regularity property of L_j, and the Picard–Lefschetz theorem. On the other hand, (3.21) and (3.22) imply (3.10). The Construction Lemma is proved. □

Lemma 3.12 is proved in the same way with the following minor changes: G is now the intersection graph of the marked set of vanishing cycles concidered, and in the lifting process, W should be replaced by W_l.

3.10 Proof of the Geometric Lemma

We prove Lemma 3.8 by induction in s. For $s = 1$, it is a direct consequence of the definition of the vanishing cycle.

Let $N > 1$ and suppose that the Lemma is proved for all $s < N$. Let us prove it for $s = N$. Consider the family of ovals of H that contains γ. The ovals of the family located inside γ fill a domain bounded by γ and by a connected component of a critical level curve of H. As $N > 1$, this component is not a singleton. Hence, it contains a saddle. Therefore, it is a union of two separatrix loops, because it is compact and contains only one critical point of H; recall that H is ultra-Morse. Denote these loops by Γ_1 and Γ_2, the corresponding critical point of H may be considered to be zero, as well as the corresponding critical value. This may be achieved by a shift in the source and target of H. Denote by δ the local vanishing cycle at 0. Let $t_0 := H|_\gamma > 0$; if not, we reverse the sign of H.

The separatix loops Γ_1 and Γ_2 may form an eight shaped figure or be located one inside the other, see Figures 6 and 7. We will study the first case in full detail, then the second one briefly.

Suppose that $\Gamma_1 \cup \Gamma_2$ is an eight shaped figure. Let t_1 be a negative number of a small magnitude. For such t_1, the part of the level curve $H = t_1$ inside γ is a pair of ovals γ_1 and γ_2, located inside Γ_1 and Γ_2 respectively. The arrangements of the real curves γ, Γ_1, Γ_2, γ_1, γ_2 and a loop δ on the complex curve S_{t_1} are shown in Figure 6. On all the curves γ, Γ_1, Γ_2, γ_1, γ_2 we choose the clockwise orientation.

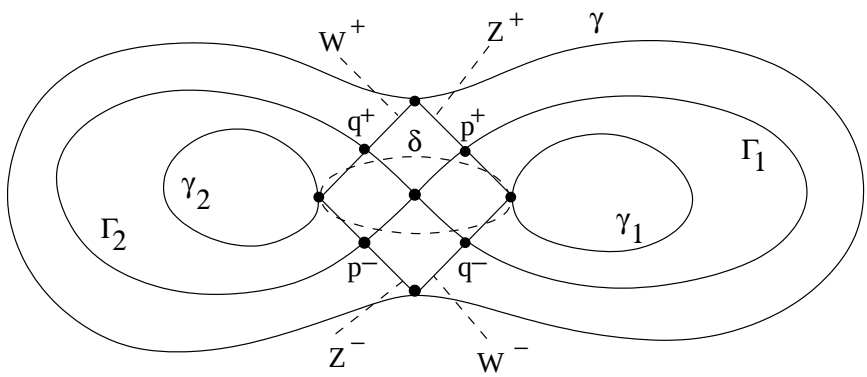

Figure 6: Eight shaped figure.

Suppose first that t_0 is small. Let $t_0 = -t_1 = \tau^2$. Consider a semicircle path in the set of noncritical values of H, named λ_0 :

$$t(\theta) = t_0 e^{i\theta},\ \lambda_0 = \{t(\theta) \mid \theta \in [0, \pi]\}.$$

Let $[\gamma(\theta)] \in H_1(S_{t(\theta)}, Z)$ be the homology class, depending continuously on θ and such that $[\gamma(0)] = [\gamma]$. We will prove that

$$[\gamma(\pi)] = [\gamma_1] + [\gamma_2] + \varepsilon[\delta],\ \varepsilon \in \{-1; 1\} \tag{3.23}$$

Any of the loops γ_1, γ_2 contains less than N critical points of H inside. By the induction assumption, $[\gamma_1]$ and $[\gamma_2]$ may be represented as sums of vanishing cycles like (3.6). After a

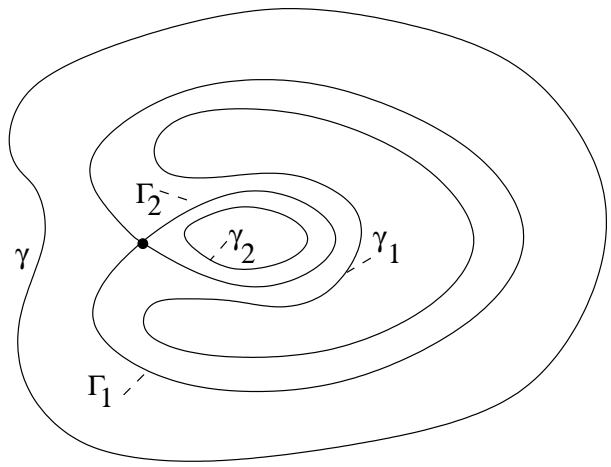

Figure 7: One separatrix loop inside the other.

proper numeration, this representation has the form

$$[\gamma_1(t_1)] = \Sigma_1^{N_1}\varepsilon_j\delta_j(t_1),\ [\gamma_2(t_1)] = \Sigma_{N_1+1}^{N-1}\varepsilon_j\delta_j(t_1).$$

Then

$$[\gamma(\pi)] = \Sigma_1^N\varepsilon_j\delta_j(t_1),$$

where $\varepsilon_N = \varepsilon, \delta_N = \delta$. Let the cycles $\delta_j(t_1)$ correspond to the pathes $\alpha_j(t_1)$. Let $\alpha_j(t_0) = \lambda_0\alpha_j(t_1)$, $\delta_j(t_0)$ correspond to $\alpha_j(t_0)$. Then (3.6) holds with $\delta_j = \delta_j(t_0)$.

Let us now prove (3.23). Let z, w be Morse coordinates for H in a neighborhood of 0, so that locally $S_t = \{zw = t\}$ for small t. Let $Z^\pm$ and $W^\pm$ be the following cross-sections of the foliation $zw = const$ in the previous neighborhood of 0: $Z^\pm = \{z = \pm\tau\}, W^\pm = \{w = \pm\tau\}$. Let $p^+ = Z^+ \cap \Gamma_1$, $p^- = Z^- \cap \Gamma_2$, $q^+ = W^+ \cap \Gamma_2$, $q^- = W^- \cap \Gamma_1$ (see Fig.6). Let $\Gamma_1(0)$ be the arc of Γ_1 from p^+ to q^-; $\Gamma_2(0)$ be the arc of Γ_2 from p^- to q^+. Consider a continuous family of curves $\Gamma_1(t)$ defined for small t by the following assumptions: $\Gamma_1(t) \subset S_t$, the initial point of $\Gamma_1(t)$ belongs to Z^+, the endpoint belongs to W^-; $\Gamma_1(0)$ is the same as above; $\Gamma_1(t_1) = \gamma_1$, $\Gamma_1(t_0)$ is an arc of γ from $(z, w) = (\tau, \tau)$ to $(z, w) = (-\tau, -\tau)$.

In a similar way, the family $\{\Gamma_2(t)\}$ is defined for t small: $\Gamma_2(0)$ is the same as above; $\gamma = \Gamma_1(t_0)\Gamma_2(t_0)$, see Fig.8.

Let $\alpha^\pm \subset Z^\pm$, $\beta^\pm \subset W^\pm$ be the following semicircles:

$$\alpha^\pm = \{\alpha^\pm(\theta) \mid \theta \in [0, \pi]\},\ \alpha^\pm(\theta) = \pm(\tau, \tau e^{i\theta}),$$

$$\beta^\pm = \{\beta^\pm(\theta) \mid \theta \in [0, \pi]\},\ \beta^\pm(\theta) = \pm(\tau e^{i\theta}, \tau).$$

Note that $\alpha^+(\theta)$ and $\beta^-(\theta)$ are the endpoints of $\Gamma_1(t(\theta))$, $\alpha^-(\theta)$ and $\beta^+(\theta)$ are the endpoints of $\Gamma_2(t(\theta))$. Consider the following curves $\Gamma^\pm(\theta) \subset S_{t(\theta)}$:

$$\Gamma^\pm(\theta) = \{\pm(\tau e^{i(\theta-\psi)}, \tau e^{i\psi}) \mid 0 \le \psi \le \theta\}$$

Note that $\Gamma^\pm(\theta)$ connects $\beta^\pm(\theta)$ and $\alpha^\pm(\theta)$, being oriented from $\beta^\pm$ to $\alpha^\pm$. Now, let

$$\gamma(\theta) = \Gamma_1(t(\theta))\Gamma^-(\theta)\Gamma_2(t(\theta))\Gamma^+(\theta).$$

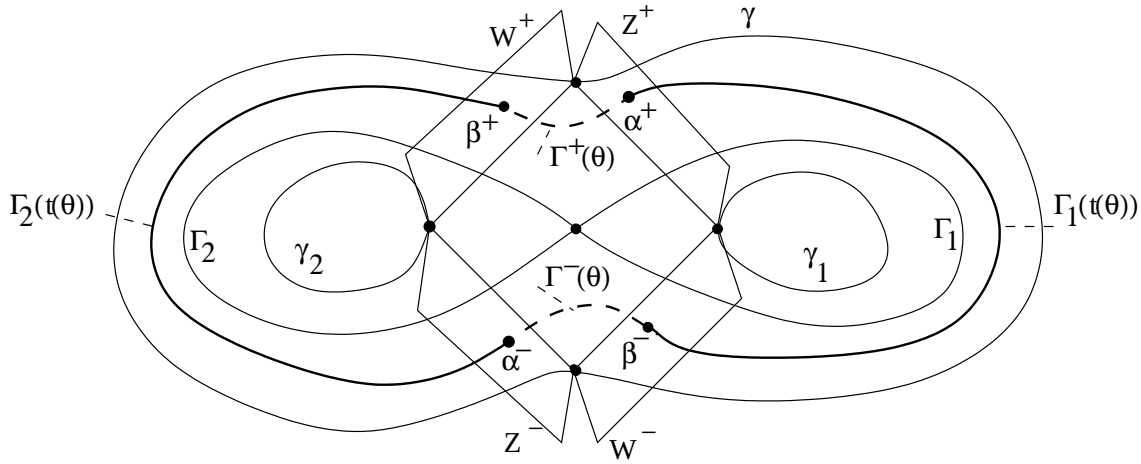

Figure 8: Extension of oval γ.

We have: $[\gamma(\pi)]$ satisfies (3.23), because $\Gamma_1(t(\pi)) = \gamma_1,\ \Gamma_2(t(\pi)) = \gamma_2,\ \Gamma^+(\pi)\Gamma^-(\pi) = \delta$.

This completes the induction step for $H|_\gamma$ small. Let now $t_0 = H|_\gamma$be large. Take small $t_2 > 0$ such that the oval $\gamma(t_2)$ belongs to the same family of ovals as $\gamma = \gamma(t_0)$ and formula (3.6) for $\gamma(t_2)$ and $\delta_j = \delta_j(t_2)$ holds. Suppose that vanishing cycles δ_j in this formula correspond to paths α_j. Let $\alpha_0 \subset B$ be a path in the upper halfplane that connects t_2 and t_0 and is close to the segment $[t_2, t_0]$. If this segments contains no critical points of H, then $\alpha_0 = [t_0, t_2]$. Clearly, γ is a continuation of $\gamma(t_2)$ along α_0. On the other hand, let the vanishing cycles $\delta_j(t_0)$ correspond to the paths $\alpha_0\alpha_j$. Then (3.6) holds for γ and $\delta_j = \delta_j(t_0)$. This proves the Geometric Lemma in case when $\Gamma_1 \cup \Gamma_2$ is an eight shaped figure.

In case when Γ_2 lies inside Γ_1, there is another component of $H = t_0$, except for γ, that lies inside Γ_2, see Fig.7. Denote this component by γ_2. For t_0 small, the part of the curve S_{-t_0} close to $\Gamma_1 \cup \Gamma_2$ is one oval; denote it by γ_1. Let δ be the local vanishing cycle corresponding to zero. Then $[\gamma_1] = [\gamma] + [\gamma_2] + \varepsilon[\delta]$. This is proved in the same way as above. After that the proof follows the same lines as for the eight shaped figure.

4 Number of zeros of abelian integrals in complex domains distant from critical values

In this section we prove the first part of Theorem B, namely, Theorem B1 mentioned in 2.2 and stated below.

4.1 Upper estimates in Euclidean and Poincaré disks

All through this Section the notations of Subsection 1.2 hold. Moreover, $\delta_1, \ldots, \delta_\mu$ is a marked set of vanishing cycles on S_{t_0}, $K \subset W$ is a compact set from the Main Lemma, see 2.4. Denote by $|\alpha|$ the Euclidean length of the curve $\alpha \subset W$.

Theorem 4.1 *Fix a normalized complex ultra-Morse polynomial H. Let $\hat{t} \in W$ be a point represented by a curve $\lambda \subset B$. Let δ_j be a vanishing cycle from a marked set corresponding to a curve α_j. Let $\alpha = \lambda^{-1}\alpha_j$. Let $0 < \beta < 1$. Suppose that $\alpha \cap D_\beta(a_j)$ is a connected arc of the path α, and α avoids the β- neighborhoods of the critical values distinct from a_j of the polynomial H. Then for any 1- form ω of type (3.2)*

$$\left| \int_{\delta_j(\hat{t})} \omega \right| < 2^{-2n} M_{1,j}, \ M_{1,j} = 2^{10n^{12}\frac{|\alpha|+5}{\beta}} (c'(H))^{-28n^4} \tag{4.1}$$

Theorem 4.1 is proved in [3]. It is used in the estimate of the number of zeros in Euclidean disc. The following upper bound (Theorem 4.2 also proved in [3]) of integrals is used to prove an upper bound of the number of zeros in Poincaré disc that is exponential in the radius of the disc.

Denote by $\mathrm{V}_\gamma f$ the variation of the argument of the function f along an oriented curve γ.

Theorem 4.2 *[3]. Let H be a normalized complex ultra-Morse polynomial of degree $n+1 \geq 3$. Let $\hat{t} \subset W$ be a point represented by a curve $\lambda \subset B$. Let δ be a vanishing cycle from the marked set; δ corresponds to a curve α_j, $\alpha = \lambda^{-1}\alpha_j : [0,1] \to B$. Let $0 < \beta \leq \nu = \frac{c''(H)}{4n^2}$,*

$$t' = \alpha(0) = \pi(\hat{t}), \ a = \alpha(1) = \alpha_j(1), \ \tau' = \min\{\tau \in [0,1] \mid \alpha(\tau,1] \subset D_\beta(a)\}, \ \hat{\alpha} = \alpha \setminus \alpha(\tau',1],$$

$$\widetilde{\alpha} = \alpha \cap (\overline{D}_3 \setminus \cup_i D_\beta(a_i)), \tag{4.2}$$

$$V = V_{\alpha,\beta} = \beta \sum_i \mathrm{V}_{\hat{\alpha} \cap D_\beta(a_i)} f_i + 3\, \mathrm{V}_{\hat{\alpha} \setminus \overline{D}_3} f_0, \ f_i(t) = t - a_i, \ f_0(t) = t.$$

Let $\delta \in H_1(S_{t'}, \mathbb{Z})$ be the cycle vanishing along α. Let ω be a monomial 1- form of degree at most $2n-1$ with unit coefficient. Then

$$\left| \int_{\delta(\hat{t})} \omega \right| < 2^{-2n} M_{2,j}, \ M_{2,j} = 2^{20n^{12}\frac{|\widetilde{\alpha}|+V+5}{\beta}} (c'(H))^{-28n^4} \max\{1, (\frac{|t'|}{5})^2\}. \tag{4.3}$$

Remark 4.3 One can estimate the number of zeros in Poincaré disc by using Theorem 4.1 instead of Theorem 4.2 (see the proof for Euclidean disc below). But the upper bound of the number of zeros obtained in this way is double exponential in the radius.

Let $DE_{R,\beta}$ be an Euclidean disk in W with β- neighborhoods of critical values deleted. More precisely, $DE_{R,\beta}$ is the set of all those $\hat{t} \in W$ that may be represented by a curve λ, whose length is no greater than R, provided that λ avoids β-neighborhoods of critical values.

Theorem 4.4 *Let H be a normalized complex ultra-Morse polynomial of degree $n+1 \geq 3$, ω be arbitrary 1- form of degree at most n. Then the number of zeros of integral (1.1), which is an analytic extension of an integral over real ovals or over marked vanishing cycles of a normalized polynomial H, is estimated from above as follows:*

$$\#\{\hat{t} \in DE_{R,\beta} \mid I(t) = 0\} < (1 - \log c'(H)) e^{\frac{9R}{\beta}} \tag{4.4}$$

provided that

$$R \geq 36n^2, \ \beta \leq \nu/2 \tag{4.5}$$

The following statement is an analogue of Theorem 4.4 for Euclidean metric replaced by the Poincaré one.

Theorem 4.5 (Theorem B1). *In the assumptions of Theorem 4.4 the number of zeros of integral (1.1) over real ovals or over marked vanishing cycles of a normalized polynomial H is estimated as follows:*

$$\#\{\hat{t} \in DP_R | I(\hat{t}) = 0\} < (1 - \log c'(H))e^{7R}, \tag{4.6}$$

provided that

$$R \geq \frac{288n^4}{c''(H)}. \tag{4.7}$$

Recall that DP_R is the disk in the Poincaré metric of W of radius R centered at the base point t_0.

Theorem B1 forms the first part of Theorem B. The second part of Theorem B, Theorem B2, is presented in Section 5. Theorems B1, B2 imply Theorem B.

4.2 Idea of the proof

Theorems 4.4 and B1 are proved as Theorem A1, making use of Growth-and-Zeros Theorem. The set K, both from the Main Lemma and from the Modified Main Lemma, belongs to DE_R by (4.5) and to DP_R by (4.7), see Remark 2.9.

Thus we have the main ingredient in the estimate of the Bernstein index, namely, the lower bound for m, see (3.17).

The upper bound is provided by the same arguments that prove Corollary 3.9.

Corollary 4.6 *Let in assumptions of Theorem 4.4, ω be a normalized form, and $\gamma(\hat{t})$ be an extension of a real cycle $\gamma(t_0)$ of a normalized polynomial H. Then*

$$|\int_{\gamma(\hat{t})} \omega| < M_1, \ M_1 = \max_j M_{1,j}, \tag{4.8}$$

where $M_{1,j}$ are the same as in (4.1).

Indeed, $n^3(n+1)/2 < 2^{2n}$ for any positive integer n.

Corollary 4.7 *Let in assumptions of Theorem 4.2, ω be a normalized form, and $\gamma(\hat{t})$ be an extension of a real cycle $\gamma(t_0)$ of a normalized polynomial H. Then*

$$|\int_{\gamma(\hat{t})} \omega| < M_2 = \max_j M_{2,j}, \tag{4.9}$$

where $M_{2,j}$ are the same as in (4.3).

These results allow us to get an upper estimate of the Bernstein index of I in a pair of domains K'' and U, provided that K'' is large enough, namely, contains the set K from the Main Lemma. To complete the proof of Theorems 4.4 and B1, we need to adjust the geometric assumptions of Theorems 4.1, 4.2 to the geometry of the Euclidiean and Poincaré disks in W.

4.3 Number of zeros in Euclidean disks

Theorem 4.4 is proved by a straightforward application of the Growth-and-Zeros theorem in the form (2.1).

Proof of Theorem 4.4. Denote the closure of the domain $DE_{R,\beta}$ by K. Let $\varepsilon' = \beta/2$, U be the smallest simply connected domain in W that contains the ε'-neighborhood of K. Then

$$D := \mathrm{diam}_{int} K \leq 2R, \ \pi\text{-gap } (K, \partial U) = \varepsilon'.$$

Hence,

$$e^{\frac{2D}{\varepsilon'}} \leq e^{\frac{8R}{\beta}}.$$

This is the main factor in the estimate (4.4).

Let us now estimate from above the Bernstein index $B = B_{K,U}(I)$. Let K' be the set from Lemma 3.11 (case of real oval) or Lemma 3.12 (case of vanishing cycle).

As in the proof of the Main Lemma, we assume (without loss of generality) that ω is normalized. Then by Corollary 3.9 (case of real cycle) or Theorem 4.1 (case of vanishing cycle), one has

$$\max_{\overline{U}} |I| \leq M_1,$$

M_1 is from (4.8). On the other hand, $K' \subset K$ by (3.11) and (4.5). Therefore, $\log \max_K |I| \geq \log m_0$, where m_0 is from (3.13). Then

$$B_{K,U}(I) \leq \log M_1 - \log m_0.$$

Relations (4.1), (3.17) together with elementary estimates imply that

$$\log M_1 - \log m_0 < (1 - \log c'(H)) e^{\frac{R}{\beta}}.$$

In the definition of $M_1 = \max_j M_{1,j}$ we take λ to be the path of length at most $R + \varepsilon'$ (recall that $\hat{t} \in DE_{R,\beta}$) and α_j with length $|\alpha_j| \leq 9$. Together with Growth-and-Zeros theorem, this completes the proof of Theorem 4.4. □

4.4 Number of zeros in Poincaré disks

The proof of Theorem B1 is carried on by application of version (2.3) of the Growth-and-Zeros Theorem to the sets

$$K_R = \overline{DP_R}, \quad U_R = DP_{R+1}:$$

$$\#\{\hat{t} \in \overline{DP_R} \mid I(\hat{t}) = 0\} \leq B_{K_R,U_R}(I) e^{\rho_R}, \ \rho_R = diam_{PU_R} K_R. \tag{4.10}$$

The right-hand side of the latter inequality is estimated below. Let us first estimate the Bernstein index in (4.10).

Recall that the form ω in the integral I is normalized.

The set K from the Main Lemma is contained in K_R (this follows from (2.13)), and as before, this yields immediately a lower bound of m. Indeed, $m \geq m_0$, and m_0 is estimated

from below in (3.17). The principal part of the proof of Theorem B1 is the justification of the following upper bound for the integral on the set U_R:

$$max_{\overline{U}_R}|I(\hat{t})| < M(R),\ \log M(R) = (1 - \log c'(H))e^{0.1R}. \tag{4.11}$$

The proof of (4.11) is based on Corollary 4.7. Namely, given R, take any $\hat{t} \in U_R$ and choose a path λ that represents $\hat{t}$. Our goal is to estimate from above the entries in (4.3): $|\tilde{\alpha}|, V, \beta$ and $|t'|$, $t' = \pi\hat{t}$ (we choose $\beta = \nu$). This is done in the technical Propositions below. In all these Propositions, R satisfies (4.7).

Proposition 4.8 *For U_R defined above and any $\hat{t} \in U_R$,*

$$|\pi\hat{t}| < M_R, \quad \textit{where}\ \log M_R = 6e^R \log R, \tag{4.12}$$

Proposition 4.9 *Let λ be a geodesic that connects $\hat{t}$ and t_0. Let $\beta = \nu = \frac{c''}{4n^2}$. Then, for $\tilde{\alpha}$ defined in (4.2),*

$$|\tilde{\alpha}| < 12R\log R \tag{4.13}$$

Proposition 4.10 *For $V_{\alpha,\nu}$ defined in (4.2),*

$$V_{\alpha,\nu} < 37e^R R\log R. \tag{4.14}$$

These Propositions will be proved at the end of the Section. Substituting (4.12), (4.13), (4.14) in the expression (4.3), we get:

$$\log M_2 < 20n^{12}\frac{12R\log R + 37e^R R\log R + 5}{\nu} - 28n^4\log c' + 2\log\frac{M_R}{5}.$$

By elementary inequalities and (4.7), the latter right-hand side is less than

$$e^R R^5 \log R - R\log c' + 12e^R\log R < (1 - \log c')R^6 e^R.$$

Once more, elementary inequalities and (4.7) show that the latter right-hand side is less than $\log M(R)$ from (4.11). This proves (4.11).

We can now get an upper estimate of the Bernstein index in (4.10). The lower bound $m = \max_{K_R}|I|$ is estimated from below in (3.17). The upper bound $M = \max_{\overline{U}_R}|I|$ is estimated in (4.11). Elementary estimates now imply

$$B_{K_R,U_R} < (1 - \log c'(H))e^{1.3R}. \tag{4.15}$$

To end up the proof of Theorem B1, modulo three propositions above, we need to get an upper estimate of the second factor in the right hand side of (4.10), namely, e^{ρ_R}.

Let, as above, K_R and U_R be two disks centered at t_0 of radius R and $R+1$ respectively in the Poincaré metric of W. Let ρ_R be the diameter of K_R in the Poincaré metric of U_R. Then

$$\rho_R < 5R. \tag{4.16}$$

This follows from the fact that the diameter of $K_R = DP_R$ in the Poincaré metric of W is equal to $2R$ (by definition), and the inequality

$$\frac{PU_R}{PW}|_{K_R} \leq \frac{e+1}{e-1} < \frac{5}{2}.$$

The latter inequality is a particular case of the following more general statement.

Proposition 4.11 *Let W be a hyperbolic Riemann surface, $U \subset W$ be a domain, $K \Subset U$ be a compact set. Let $dist_{PW}(K, \partial U) \geq \sigma > 0$. Then*

$$\frac{PU}{PW}|_K \leq \frac{e^\sigma + 1}{e^\sigma - 1}.$$

Proof By monotonicity of the Poincaré metric as a function of domain, it suffices to prove the Proposition in the case, when $W = D_1$, $K = \{0\}$, U is the Poincaré disc of radius σ centered at 0: in this case we prove the equality. Indeed, let r be the Euclidean radius of the latter disc. By definition and conformal invariance of the Poincaré metric, $\frac{PU}{PD_1}(0) = r^{-1}$. We have: $r^{-1} = \frac{e^\sigma+1}{e^\sigma-1}$, since, by definition,

$$\sigma = \int_0^r 2\frac{ds}{1-s^2} = \log\frac{1+r}{1-r}.$$

This proves the Proposition. □

Estimates (4.16), (4.15) and (4.10) prove Theorem B1.

4.5 Proofs of technical propositions

The proofs of the three Propositions above are based on the following lower bounds of the Poincaré metric. Given a domain $G \subset \mathbb{C}$, $\#(\mathbb{C} \setminus G) > 1$, denote by $P(G)$ the ratio of the Poincaré metric of G to the Euclidean one; $P(G)$ is a function in $t \in G$.

Inequality (follows from theorem 2.17 in [22]). For any distinct $a, b \in \mathbb{C}$ one has

$$P(\mathbb{C} \setminus \{a,b\})(t) > [\min_{c=a,b} |t-c|(\min_{c=a,b} |\log|\frac{t-c}{a-b}|| + 5)]^{-1} \tag{4.17}$$

Corollary 4.12 *Let H be a balanced polynomial, B be the complement of $\mathbb{C}$ to its critical values. Then*

$$P(B)(t) > [|t-a|(|\log|t-a|| + C)]^{-1}, C = 2\log n - \log c''(H) + 5, \text{ for any critical value } a. \tag{4.18}$$

The Corollary follows from the previous Inequality and monotonicity of the Poincaré metric.
Proof of Proposition 4.8 By (4.7),

$$C < \log R, \ -\log\nu < \log R. \tag{4.19}$$

Let $R, U_R, \hat{t}$ be the same as in the Proposition, $t = \pi\hat{t}$. Our goal is to prove that $|t| < M_R$. Let a be a critical value of H, $t \in \overline{DP_{R+1}}$. From (4.18) and (4.19) we have:

$$R + 1 \geq \int_{|t_0-a|}^{|t-a|} \frac{|ds|}{s(|\log s| + C)}, \text{ where } C < \log R. \tag{4.20}$$

By definition, $|a| \le 2$, $|t_0| \le 3$, so, $|t_0 - a| \le 5$. Suppose $|t| > 7$ (if not, then the inequality $|t| < M_R$ follows immediately, since $M_R > 7$, by (4.7)). Hence, $|t - a| > 5$. Put $u = \log s$. Then the latter integral is no less than

$$\int_5^{|t-a|} \frac{ds}{s(\log s + C)} = \log(u + C)|_{\log 5}^{\log|t-a|} > \log\log|t - a| - \log(C + 2).$$

Together with (4.20), this implies that

$$\log|t - a| < e^{R+1}(C + 2).$$

Together with inequalities $|a| \le 2$ and (4.19), this proves (4.12). □

Proof of Proposition 4.9 By definition, the curve $\widetilde{\alpha}$ consists of the arcs of paths λ and α_i lying in $\overline{D}_3 \setminus \cup_i D_\nu(a_i)$. Those contained in α_i have total length less than 9, since $|\alpha_i| \le 9$. Those contained in λ have total length no greater than

$$|\lambda|_P M_3, \;\; M_3 = (\min_{\widetilde{\alpha}} P(B))^{-1},$$

where $|\,.\,|_P$ is the Poincaré length. As λ is a geodesic, $|\lambda|_P \le R + 1$. Hence,

$$|\tilde{\alpha}| < 9 + (R + 1)M_3.$$

Let us estimate M_3 from above. Recall that the curve $\widetilde{\alpha}$, where the minimum in M_3 is taken, lies in $\overline{D}_3$ and its gap from the critical values is no less than ν. Hence, by (4.18) and the inequality $|a| \le 2$, we have that

$$M_3 \le \max_{|t|\le 3} |t - a|(-\log\nu + C) \le 5(-\log\nu + C).$$

By (4.19), $M_3 < 10\log R$. Together with (4.7), this implies that

$$|\widetilde{\alpha}| < 9 + 10(R + 1)\log R < 11(R + 1)\log R < 12R\log R.$$

This proves Proposition 4.9. □

Proof of Proposition 4.10 We have to prove first that the projection of the Poincaré disc considered is not too close to the critical values a_j. Namely,

$$dist(\overline{\pi(DP_{R+1})}, a_i) > \beta_R, \;\; \text{where } \beta_R = M_R^{-1}. \tag{4.21}$$

It suffices to show that for any critical value a

$$|t - a| > \beta_R \text{ for any } t \in \overline{\pi(DP_{R+1})}. \tag{4.22}$$

It follows from formula for β_R in (4.21), inequality (4.7), choice of t_0 and elementary inequalities that

$$\beta_R < \nu = \frac{c''(H)}{4n^2} \le |t_0 - a|. \tag{4.23}$$

Thus, if $|t-a| \geq \nu$, then inequality (4.22) holds. Let us prove (4.22) assuming that $|t-a| < \nu$. To do this, we use once more Corollary 4.12. Like in (4.20), we have,

$$R+1 \geq |\int_{\nu}^{|t-a|} \frac{ds}{s(|\log s|+C)}| = \log(u+C)|_{-\log\nu}^{-\log|t-a|}$$

$$= \log(C - \log|t-a|) - \log(C - \log\nu).$$

Together with (4.19), this implies (4.22), hence, (4.21).

Let us now prove (4.14). The expression $V = V_{\alpha,\nu}$ is a linear combination of variations of arguments along the pieces of the path α that lie either inside $\beta = \nu$- neighborhoods of the critical values of H, or outside $\overline{D}_3$. To estimate it from above, we use the following a priori upper bounds of variations.

Let a be a critical value. By definition, for any curve $l \subset B$

$$\mathrm{V}_l(t-a) = \int_l \frac{|dt|}{|t-a|} = \int_l |dt|_P \frac{(P(B))^{-1}}{|t-a|} \leq |l|_P \max_l \frac{(P(B))^{-1}}{|t-a|}$$

(here by $|l|_P$ we denote the Poincaré length). The latter ratio is estimated by (4.18):

$$\frac{(P(B))^{-1}}{|t-a|} < |\log|t-a|| + C < 7e^R \log R, \text{ whenever } t \in \overline{\pi(DP_{R+1})} \tag{4.24}$$

(the last inequality follows from (4.19) and (4.21)). Then by (4.24),

$$\mathrm{V}_l(t-a) < 7|l|_P e^R \log R, \text{ whenever } l \subset \overline{\pi(DP_{R+1})}. \tag{4.25}$$

Analogously, for any critical value a

$$\mathrm{V}_l\, t \leq |l|_P \max_l \frac{(P(B))^{-1}}{|t|} \leq |l|_P \max_l \frac{|t-a|}{|t|} \max_l \frac{(P(B))^{-1}}{|t-a|}. \tag{4.26}$$

Now let $l \subset \overline{\pi(DP_{R+1})} \setminus \overline{D}_3$. Then

$$\max_l \frac{|t-a|}{|t|} \leq \frac{5}{3},$$

since $|t| > 3$ on l and $|a| \leq 2$. Substituting this inequality and (4.24) to the right-hand side of (4.26), yields

$$\mathrm{V}_l\, t < \frac{5}{3} 7|l|_P e^R \log R < 12|l|_P e^R \log R, \text{ whenever } l \subset \overline{\pi(DP_{R+1})} \setminus \overline{D_3}. \tag{4.27}$$

Let us estimate the expression $V = V_{\alpha,\nu}$. By definition, the variations in this expression are taken along the arcs of the path $\alpha = \lambda^{-1}\alpha_i$ that lie either inside $D_\nu(a_j)$, or outside $\overline{D}_3$ (except for its final arc $\alpha(\tau', 1] \subset D_\nu(a_i)$, $\alpha(\tau') \in \partial D_\nu(a_i)$). By definition, the latter arc coincides with an arc of the path α_i, and its complement in α_i is a curve lying in $\overline{D}_3$ outside the ν- neighborhoods of the critical values (see 3.9). Therefore, the previous arcs, where the variations are taken, are disjoint from the path α_i and thus, are those of the path λ. The first sum in the expression of $V_{\alpha,\nu}$, which is ν times the sum of the variations along pieces of

α near the critical values, is less than $7\nu(R+1)e^R \log R$. This follows from inequality (4.25) applied to each piece and the inequality $|\lambda|_P \leq R+1$. Analogously, by the latter inequality and (4.27), the remaining term in the expression of $V_{\alpha,\nu}$ is less than $36(R+1)e^R \log R$. The two previous upper bounds of the terms in $V_{\alpha,\nu}$ together with (4.7) and inequality $\nu \leq \frac{1}{16}$ imply that

$$V_{\alpha,\nu} < (36+7\nu)(R+1)e^R \log R < 37Re^R \log R.$$

This proves (4.14). □

5 Estimates of the number of zeros of Abelian integrals near critical values

In this section we prove Theorem A2, see 2.5, and Theorem B2, stated below. Together with Theorem A1 (whose proof is completed in Section 2) Theorem A2 implies Theorem A. Together with Theorem B1 (whose proof is completed in Section 4), Theorem B2 implies Theorem B.

We have three statements to discuss:

1. Theorem A2 in the case when the endpoints of the interval considered are all finite;
2. Theorem A2 in the case when one of these endpoints is infinite;
3. Theorem B2.

These statements will be referred to as cases 1, 2, 3 below.

It appears that cases 1 and 3 are very close to each other.

5.1 Argument principle, KRY theorem and Petrov's method

All the three cases are treated in a similar way. We want to apply the argument principle.

The estimates near infinity are based on the argument principle only. The estimates near finite critical points use the Petrov's method that may be considered as a generalization of the argument principle for multivalued functions. The increment of the argument is estimated through the Bernstein index of the integral, bounded from above in the previous sections. The relation between these two quantities is the subject of the Khovanskii-Roitman-Yakovenko (KRY) theorem and Theorem 5.3 stated below. It seems surprising that these theorems were not discovered in the classical period of the development of complex analysis. The latter theorem is proved in [10]; the proof is based on the KRY theorem and methods of [21], [13].

At this spot we begin the proof of Theorem A2 in case 1. Recall the statement of the Theorem in case 1.

Theorem A2 (Case 1). *Let $a \neq \infty, b \neq \infty,$. Then*

$$\#\{t \in (a, l(t_0)) \cup (r(t_0), b) \mid I(t) = 0\} < (1 - \log c')e^{\frac{4700}{c''}n^4},$$

where $l(t_0)$ and $r(t_0)$ are the same as at the beginning of 2.4.

We will prove that

$$\#\{t \in (a, l(t_0)) \mid I(t) = 0\} < \frac{1}{2}(1 - \log c')e^{\frac{4700}{c''}n^4}. \tag{5.1}$$

Similar estimate for $(r(t_0), b)$ is proved in the same way. These two estimates imply Theorem A2.

Let $\Pi = \Pi(a)$ be the same as in (1.5), namely

$$\Pi = \{t \in W \mid 0 < |t-a| \leq \nu, |\arg(t-a)| \leq 2\pi\}.$$

Lemma 5.1 *Inequality (5.1) holds provided that in (5.1) the interval* $(a, l(t_0))$ *is replaced by* Π.

Lemma 5.1 implies (5.1) because $(a, l(t_0)) \subset \Pi$. Let

$$\Pi_\psi = \{t \in \Pi \mid \psi \leq |t-a| \leq \nu\}$$

Lemma 5.2 *Lemma 5.1 holds provided that in (5.1) the domain* Π *is replaced by* Π_ψ.

Lemma 5.2 implies Lemma 5.1, because

$$\Pi = \cup_{\psi>0} \Pi_\psi.$$

Proof of Lemma 5.2. The proof of this Lemma occupies this and the next four Subsections. We have

$$\partial\Pi_\psi = \Gamma_1\Gamma_2\Gamma_3\Gamma_4.$$

As sets, the curves Γ_j are defined by the formulas below; the orientation is defined separately:

$$\Gamma_1 = \{t \mid |t-a| = \nu, |arg(t-a)| \leq 2\pi\} = \Gamma_a$$

$$\Gamma_3 = \{t \mid |t-a| = \psi, |arg(t-a)| \leq 2\pi\}$$

$$\Gamma_{2,4} = \{t \mid \psi \leq |t-a| \leq \nu, arg(t-a) = \pm 2\pi\}.$$

The curve Γ_1 is oriented counterclockwise, Γ_2 is oriented from the right to the left, Γ_3 is oriented clockwise, Γ_4 is oriented from the left to the right.

Let $\#\{t \in (a+\psi, l(t_0)) \mid I(t) = 0\} = N_\psi$. Denote by $R_\Gamma(f)$ the increment of the argument of a holomorphic function f along a curve Γ (R of Rouchet). Recall that $V_\Gamma(f)$ denotes the variation of the argument of f along Γ. Obviously, $\mid R_\Gamma(f) \mid \leq V_\Gamma(f)$. □

In assumption that $I \neq 0$ on $\partial\Pi_\psi$, the argument principle implies that

$$N_\psi = \frac{1}{2\pi} R_{\partial\Pi_\psi}(I) = \frac{1}{2\pi} \sum_1^4 R_{\Gamma_j}(I). \tag{5.2}$$

The first term in this sum is estimated by the modified KRY theorem, the second and the forth one by the Petrov method, the third one by the Mardesic theorem. The case when the above assumption fails is treated in 5.3.

5.2 Bernstein index and variation of argument

The first step in establishing a relation between variation of argument and the Bernstein index was done by the following KRY theorem.

Let U be a connected and simply connected domain in $\mathbb{C}$, $\Gamma \subset U$ be a (nonoriented compact) curve, f be a bounded holomorphic function on U.

KRY theorem, [13] *For any tuple $U, \Gamma \subset U$ as above and a compact set $K \subset U$ there exists a geometric constant $\alpha = \alpha(U, K, \Gamma)$, such that*

$$V_\Gamma(f) \leq \alpha B_{K,U}(f).$$

In [13] an upper estimate of the Bernstein index through the variation of the argument along $\Gamma = \partial U$ is given; we do not use this estimate. On the contrary, we need an improved version of the previous theorem with α explicitly written and U being a domain on a Riemann surface. These two goals are achieved in the following Theorem.

Let $|\Gamma|$ be the length, and $\kappa(\Gamma)$ be the total curvature of a curve on a surface endowed with a Riemann metric.

Theorem 5.3 *Let $\Gamma \subset U'' \subset U' \subset U \subset W$ be respectively a curve, and three open sets in a Riemann surface W. Let $f : U \to \mathbb{C}$ be a bounded holomorphic function, $f|_\Gamma \neq 0$. Let $\pi : W \to \mathbb{C}$ be a projection which is locally biholomorphic, and the metric on W be a pullback of the Euclidean metric in $\mathbb{C}$. Let $\varepsilon < \frac{1}{2}$ and the following gap conditions hold:*

$$\pi\text{-gap } (\Gamma, U'') \geq \varepsilon, \ \pi\text{-gap } (U'', U') \geq \varepsilon, \ \pi\text{-gap } (U', U) \geq \varepsilon. \tag{5.3}$$

Let $D > 1$ and the following diameter conditions hold:

$$diam\ _{int} U'' \leq D, \ diam\ _{int} U' \leq D \tag{5.4}$$

Then

$$V_\Gamma(f) \leq B_{U'',U}(f)(\frac{|\Gamma|}{\varepsilon} + \kappa(\Gamma) + 1)e^{\frac{5D}{\varepsilon}}. \tag{5.5}$$

This theorem is proved in [10].

Recall that intrinsic diameter and π-gap are defined in 2.3.

We can now estimate from above the first term in the sum (5.2). The estimate works in both cases when a is finite or infinite.

Lemma 5.4 *Let H be a normalized polynomial of degree $n+1 \geq 3$. Let I be the same integral as in (1.1). Let K be a compact set mentioned in the Main Lemma, and $\Gamma_1 = \Gamma_a$ be the same as in this Lemma (a may be infinite). Then*

$$V_{\Gamma_1}(I) < (1 - \log c'(H))A^{4600}, \ A = e^{\frac{n^4}{c'H}}. \tag{5.6}$$

In what follows, we write c', c'' instead of $c'(H), c''(H)$.

Proof The Lemma follows immediately from Theorem 5.3 and the Main Lemma. To apply Theorem 5.3, let us take I for f, the universal cover over B for W with the natural projection $\pi : W \to \mathbb{C}$ and the metric induced from $\mathbb{C}$ by this projection. This metric on W is called Euclidean. Let K and U be the same as in the Main Lemma. Take this U for the domain U to apply Theorem 5.3. Recall that U is the minimal simply connected domain that contains the $\frac{\nu}{2}$-neighborhood of K in U in the Euclidean metric on W, ν is the same as in (1.2). Let $\varepsilon = \frac{\nu}{6}$, that is

$$\varepsilon = \frac{\nu}{6} = \frac{c''}{24n^2}. \tag{5.7}$$

Let U'' and U' be the minimal simply connected domains in W that contain ε-neighborhood of K and U'' respectively. Take Γ_1 for Γ from the statement of Theorem 5.3. Note that $\Gamma_1 \subset K$. Then gap condition (5.3) with ε from (5.7) holds. Moreover, $diam_{int}U' \leq diam_{int}U'' + 2\varepsilon \leq diam_{int}K + 4\varepsilon$. Hence, diameter condition (5.4) holds with $D < 38n^2$ by (2.8). Thus

$$e^{\frac{5D}{\varepsilon}} < A^c, \text{ where } c_0 = 5 \times 38 \times 24.$$

The factor A^{c_0} is the largest one in the estimate for $V_{\Gamma_1}(I)$.

By inequality (2.10) from the Main Lemma, $B_{K,U} < (1 - \log c')A^2$. By the monotonicity of Bernstein index (that follows directly from its definition), $B_{U'',U} < B_{K,U}$. At last,

$$\frac{|\Gamma_1|}{\varepsilon} + \kappa(\Gamma_1) + 1 = 24\pi + 4\pi + 1 << A.$$

Note, that $c_0 + 3 < 4600$. Now, inequality (5.5) proves the Lemma. □

Remark 5.5 Lemma 5.4 remains valid if in its hypothesis the integral I is replaced by an integral J over the cycle vanishing at the critical value a of H. The proof of this modified version of Lemma 5.4 repeats that of the original one with the following change: we use the Modified Main Lemma instead of the Main Lemma.

Corollary 5.6 *Suppose that the integral J with a real integrand ω is taken over a local vanishing cycle δ_t corresponding to the real critical value a. Then the number of zeros of J in the disk centered at a of radius $\nu = \frac{c''}{4n^2}$ admits the following upper estimate:*

$$N_J := \#\{t \in \mathbb{C} \mid |t - a| < \nu, J(t) = 0\} < \frac{1}{2\pi}(1 - \log c')A^{4600} \tag{5.8}$$

This follows from the modified Lemma 5.4 and the argument principle.

5.3 Application of the Petrov's method

The Petrov's method applied below is based on the remark that the magnitude of the increment of the argument of a nonzero function along an oriented curve is no greater than the number of zeros of the imaginary part of this function increased by 1 and multiplied by π. Indeed, at any half circuit around zero, a planar curve crosses an imaginary axis at least

once. The method works when the imaginary part of a function appears to be more simple than the function itself.

Let $\delta(t) \in H_1(t)$ be the local vanishing cycle at the point a. Let ω be the same real form as in integral (1.1). Let J be the germ of integral $J(t) = \int_{\delta(t)} \omega$ along the cycle $\delta(t)$, which is a local vanishing cycle at $t = a$. Note that J is single-valued in any simply connected neighborhood of a that contains no other critical values of H. Let $l_0 = (\gamma(t), \delta(t)) \neq 0$ be the intersection index of the cycles $\gamma(t)$ and $\delta(t)$. As the cycle $\gamma(t)$ is real and H is ultra-Morse, l_0 may take values $\pm 1, \pm 2$ only (Lemma 3.16). Let

$$\Gamma_0 = \{t \in \mathbb{R} \mid te^{2\pi i} \in \Gamma_2\}.$$

Then by the Picard-Lefschetz theorem

$$I\,|_{\Gamma_2} = (I + l_0 J)\,|_{\Gamma_0}, \;\; I\,|_{\Gamma_4} = (I - l_0 J)\,|_{\Gamma_0}\,.$$

Proposition 5.7 *The integral J is purely imaginary on the real interval (a, b).*

Proof Recall that the form ω and the polynomial H are real. Then

$$J(t) = -\overline{J(\overline{t})}.$$

Indeed, $\omega = Q(x, y)dx$. The involution $\mathbf{i} : (x, y) \mapsto (\overline{x}, \overline{y})$ brings the integral $J(t) = \int\limits_{\delta(t)} Q\,dx$ to $\int\limits_{\mathbf{i}\delta(t)} \overline{Q}\,d\overline{x} = \int\limits_{-\delta(\overline{t})} \overline{Q}\,d\overline{x} = -\overline{\int_{\delta(\overline{t})} Q\,dx} = -\overline{J(\overline{t})}$. On the other hand, for real t we have $t = \overline{t}$ and $\delta(\overline{t}) = \delta(t)$. Hence, $J(t) = -\overline{J(t)}$ for $t \in (a, b)$. This implies Proposition 5.7. □

Corollary 5.8 *Let, as above, $l_0 \neq 0$ be the intersection index of the cycles $\gamma(t)$ and $\delta(t)$. Then*

$$ImI\,|_{\Gamma_{2,4}} = \pm l_0 J\,|_{\Gamma_0}\,.$$

Proof This follows from Proposition 5.7, Picard-Lefschetz theorem and the reality of I on Γ_0. □

Suppose first that I has no zeros on Γ_2 and Γ_4. Then

$$\left|R_{\Gamma_{2,4}}(I)\right| \leq \pi(1 + N), \text{ where } N = \#\{t \in \Gamma_0 \mid J(t) = 0\}. \tag{5.9}$$

Obviously, $N \leq N_J$, see (5.8). The right hand side of this inequality is already estimated from above in Corollary 5.6. Hence,

$$\left|R_{\Gamma_{2,4}}(I)\right| \leq \pi + \frac{1}{2}(1 - \log c')A^{4600}.$$

Suppose now that I has zeros on Γ_2 (hence on Γ_4, by Proposition 5.7). Indeed, its real part is the same at the corresponding points of $\Gamma_2, \Gamma_0, \Gamma_4$, and the imaginary parts of $I|_{\Gamma_2}$ and $I|_{\Gamma_4}$ are opposite at these points. In this case we replace the domain Π_ψ by Π'_ψ defined as follows.

The curves $\Gamma_{2,4}$ should be modified. A small segment of Γ_2 centered at zero point of I that contains no other zeros of J, should be replaced by a lower half-circle having this segment as

a diameter and containing no zeros of J. A similar modification should be done for Γ_4 making use of upper half-circles. Denote the modified curves by $\Gamma'_{2,4}$. Let Π'_ψ be the domain bounded by the curve

$$\partial\Pi'_\psi = \Gamma_1\Gamma'_2\Gamma_3\Gamma'_4. \tag{5.10}$$

It contains Π_ψ, and we will estimate from above the number of zeros of I in Π'_ψ still using the argument principle. The increment of $\arg I$ along Γ_1 is already estimated in 5.2. Here we give an upper bound for the increment of $\arg I$ along $\Gamma'_{2,4}$. The increment along Γ_3 is estimated in the next Subsection.

Proposition 5.9 *Let N be the same as in (5.9). Then*

$$\mid R_{\Gamma'_{2,4}}(I) \mid \leq \pi(2N+1). \tag{5.11}$$

Proof We will prove the Proposition for Γ'_2; the proof for Γ'_4 is the same. Let I have zeros $b_j \in \Gamma_2,\ j = 1, ..., k$, the number of occurrences of b_j in this list equals its multiplicity. Note that

$$\text{Im } I|_{\Gamma_2} = l_0 J \tag{5.12}$$

Hence, at the points b_j, J has zeros of no less multiplicity than I. Hence, the total multiplicity k' of zeros of J at the points $b_j \in \Gamma_2,\ j = 1, ..., k$, is no less than k. Let J have s zeros on Γ'_2. We have: $k' \geq k,\ s \leq N - k' \leq N - k$. Let $\sigma_1, ..., \sigma_q,\ q \leq k+1$, be the open intervals, the connected components of the difference of Γ'_2 and the half-circles constructed above. Let s_j be the number of zeros of J on σ_j, $\sum_1^q s_j = s$. Let

$$R_j = R_{\sigma_j}(I).$$

Then

$$R_j \leq \pi(s_j + 1).$$

Hence,

$$\mid R_{\Gamma'_2}(I) \mid \leq \pi(k + \sum_1^q (s_j+1)) \leq \pi(2k+1+s) \leq \pi(2k'+1+s) \leq \pi(2N+1), \tag{5.13}$$

where $N \leq N_J < \frac{1}{2\pi}(1 - \log c')A^{4600}$, see (5.8). □

5.4 Application of the Mardesic theorem

Proposition 5.10 *Let I be the integral (1.1), and Γ_3 be the same as in Subsection 5.1. Then for ψ small enough,*

$$|R_{\Gamma_3}(I)| \leq \pi(4n^4 + 1). \tag{5.14}$$

Proof Let J and l_0 be the same as in the previous subsection. Let $a = 0$, and $I(e^{2\pi i}t)$ means the result of the analytic extension of I from a value $I(t)$ along a curve $e^{2\pi\varphi}t$, $\varphi \in [0,1]$. By the Picard-Lefshetz theorem, for small t

$$I(e^{2\pi i}t) = I(t) + l_0 J(t).$$

Consider the function

$$Y(t) = I(t) - l_0 \frac{\log t}{2\pi i} J(t).$$

This function is single-valued because the increments of both terms I and Y under the analytic extension over a circle centered at 0 cancel. The function I is bounded along any segment ending at zero, and J is holomorphic at zero, with $J(0) = 0$. Hence, Y is holomorphic and grows no faster than $\log|t|$ in a punctured neighborhood of zero. (In fact, it is bounded in the latter neighborhood: $|J(t)\log t| \le c|t||\log t| \to 0$, as $t \to 0$.) By the removable singularity theorem, it is holomorphic at zero. Hence,

$$I(t) = Y(t) + l_0 \frac{\log t}{2\pi i} J(t) \tag{5.15}$$

with Y and J holomorphic. We claim that *the increment of the argument of I along Γ_3 for ψ small is bounded from above through $\mathrm{ord}_0 J$, the multiplicity of zero of J at zero.* The latter order is estimated from above by the following theorem by Mardesic:

Theorem 5.11 *[14]. The multiplicity of any zero of the integral I (or J) taken at a point where the integral is holomorphic does not exceed n^4.*

The function (5.15) is multivalued. The proof of Proposition 5.10 is based on the following simple remark. Let f_1, f_2 be two continuous functions on a segment $\sigma \subset \mathbb{R}$, and $|f_1| \ge 2|f_2|$. Then $|R_\sigma(f_1 + f_2)| \le |R_\sigma(f_1)| + \frac{\pi}{3}$. Indeed, the value $R_\sigma(f_1 + \varepsilon f_2)$ cannot change more than by $\frac{\pi}{3}$, as ε ranges over the segment $[0,1]$.

To complete the proof of Proposition 5.10, we need to consider three cases. Let $\nu = \mathrm{ord}_0 Y$, $\mu = \mathrm{ord}_0 J$, $f(\varphi) = Y(\psi e^{2\pi i\varphi})$, $g(\varphi) = l_0 \left(J \frac{\log}{2\pi i}\right)(\psi e^{2\pi i\varphi})$. Note that $\mu \le n^4$.

Case (i): $\nu < \mu$. Then, for ψ small, $2|g| \le |f|$. By the previous remark, applied to $f_1 = f$, $f_2 = g$, we get

$$|R_{\Gamma_3}(I)| \le \pi(4\nu + 1) \le \pi(4\mu + 1) \le \pi(4n^4 + 1).$$

Case (ii): $\nu = \mu$. Then, for ψ small, $2|f| \le |g|$, because of the logarithmic factor in g. In the same way as before, we get

$$|R_{\Gamma_3}(I)| \le \pi(4\mu + 1) \le \pi(4n^4 + 1).$$

Case (iii): $\nu > \mu$. In the same way, as in Case (ii), we get (5.14). □

5.5 Proof of Theorem A2 in case 1 (endpoints of the interval considered are finite)

Proof It is sufficient to prove Lemma 5.2. We prove a stronger statement

$$N(I,\Pi'_\psi) := \#\{t \in \Pi'_\psi \mid I(t) = 0\} < \frac{1}{2}(1 - \log c')A^{4600} \tag{5.16}$$

By the argument principle

$$2\pi N(I,\Pi'_\psi) \le V(\Gamma_1) + \mid R_{\Gamma'_2}(I) \mid + \mid R_{\Gamma_3}(I) \mid + \mid R_{\Gamma'_4}(I) \mid \tag{5.17}$$

The first term in the r.h.s is estimated in (5.6). The second and the fourth terms are estimated from above in (5.11) (the N in the r.h.s. of (5.11) is estimated from above by N_J, see (5.8)). The third term is estimated in (5.14). Altogether this proves (5.16), hence, Lemma 5.2 and implies a stronger version of (5.1):

$$N(I,\Pi'_\psi) < \frac{1}{2}(1 - \log c')A^{4600}.$$

This proves Theorem A2 in case 1. □

5.6 Proof of Theorem B

Theorem B follows from Theorem B1 and the following statement.

Theorem B2. *For any real normalized ultra-Morse polynomial H, any family Γ of real ovals of H, and any l, let $\Pi(a)$ and $\Pi(b)$, $D(l,a)$ and $D(l,b)$ be the same domains, as in Subsection 1.2. Let I be the analytic extension to W of the integral (1.1) over the ovals of the family Γ :*

$$\int_{\gamma(t)} \omega = I(t),\ \gamma(t) \in \Gamma.$$

Then the number of zeros of I in $D(l,a) \cup D(l,b)$ (denoted by $N(l,H)$), is no greater than

$$N(l,H) \le (1 - \log c')e^{4600\frac{n^4}{c''} + \frac{481l}{c''}}.$$

Proof We will prove the Theorem for the case when $a = a(t_0)$ is a logarithmic branch point of the integral I at the left end of the segment $\sigma(t_0)$. The case of the right end is treated in the same way. The case when $a(t_0)$ is a critical value of H which is not a singular point of the integral I, is even more elementary. In this case the integral is univalent in a small neighborhood of a, the number of zeros to be estimated does not depend on $l \ge 1$, and the estimate follows from Theorem A2.

Let for simplicity $D(l) = D(l,a)$. For any $\psi \in (0,\nu)$ consider the set $\Pi'_\psi \subset W$, see (5.10). Let

$$\Pi'_{\psi,l} = \{a + re^{i\varphi} \mid a + re^{\frac{i\varphi}{l}} \in \Pi'_\psi\}.$$

Let $\Gamma_{1,l}$, $\Gamma'_{2,l}$, $\Gamma_{3,l}$, $\Gamma'_{4,l}$ be the curves defined by the relations:

$$\partial\Pi'_{\psi,l} = \Gamma_{1,l}\Gamma'_{2,l}\Gamma_{3,l}\Gamma'_{4,l};\ \pi\Gamma_{j,l} = \pi\Gamma_j,\ j = 1;3;\ \pi\Gamma'_{j,l} = \pi\Gamma'_j,\ j = 2;4.$$

Let $R_\Gamma(f)$ and $V_\Gamma(f)$ be the same as in Subsection 5.1. Then, by the argument principle

$$2\pi N(l,H) \leq V_{\Gamma_{1,l}}(I) + \mid R_{\Gamma'_{2,l}}(I) \mid + \mid R_{\Gamma_{3,l}}(I) \mid + \mid R_{\Gamma'_{4,l}} \mid . \tag{5.18}$$

The four terms in the right hand side are estimated in a similar way as the corresponding terms in (5.17).

Proposition 5.12 *For I and $\Gamma'_{j,l}$ above:*

$$\mid R_{\Gamma'_{j,l}}(I) \mid \leq \pi(2N+1),\ j = 2,4, \tag{5.19}$$

where N is the same as in (5.9).

Proof The proposition is proved in the very same way as Proposition 5.9 with the only difference: (5.12) should be replaced by

$$\text{Im } I \mid_{\Gamma_{j,l}} = \pm l l_0 J \circ \pi \mid_{\Gamma_0} .$$

The factor l in the r.h.s. does not change the number of zeros. □

Moreover,

$$\mid R_{\Gamma'_{3,l}}(I) \mid \leq \pi(4n^4 l + 1). \tag{5.20}$$

This is proved in the same way as (5.14) with the only difference that the increment of the argument of t along $\Gamma_{3,l}$ is now $4\pi l$.

Proposition 5.13 *For I and $\Gamma_{1,l}$ above,*

$$V_{\Gamma_{1,l}}(I) \leq (1 - \log c')A^{4600}e^{481l},\ A = e^{\frac{n^4}{c''}}. \tag{5.21}$$

Proof of Proposition 5.13. The proof follows the same lines as that of Lemma 5.4. We will estimate the variation of argument under consideration making use of Theorem 5.3. For this we need first to choose the curve Γ and domains U'', U', U. Let

$$\Gamma = \Gamma_{1,l} = \{a + \nu e^{i\varphi} \mid \varphi \in [-2\pi l, 2\pi l]\}.$$

Take the same $\varepsilon = \frac{\nu}{6}$ as in (5.7). For any set $G \subset W$ take G^ε to be the ε-neighborhood of G in the Euclidean metric of W, and $\widetilde{G^\varepsilon} \subset W$ be the minimal simply connected domain that contains G^ε. Let K be the same as in the Main Lemma. Take

$$U'' = (\widetilde{K \cup \Gamma})^\varepsilon,\ U' = (\widetilde{K \cup \Gamma})^{2\varepsilon},\ U = (\widetilde{K \cup \Gamma})^{3\varepsilon}.$$

Note that for any point $p \in K \cup \Gamma$, the 6ε-neighborhood of p in W is bijectively projected to a 6ε-disk in $\mathbb{C}$. Hence, the gap condition (5.3) holds for $\Gamma, U'', U', U, \varepsilon$ so chosen.

Note that $K \cap \Gamma = \Gamma_1 \neq \emptyset$. Hence, the set $K \cup \Gamma$, as well as U'', U', U is path connected. Then we have:

$$\text{diam } K < 36n^2$$

by (2.8),

$$\text{diam }_{int}(K \cup \Gamma) \leq 36n^2 + 4\pi l\nu := D_1,$$

$$\text{diam } {}_{int}U'' \leq D_1 + 2\varepsilon,$$

$$\text{diam } {}_{int}U' \leq D_1 + 4\varepsilon.$$

Hence, diameter condition (5.4) holds with

$$D_2 = 36n^2 + 16l\nu = 36n^2 + 96l\varepsilon. \tag{5.22}$$

Let us now estimate from above the Bernstein index $B_1 = B_{U'',U}(I)$. Let U_0 be the domain denoted by U in the Main Lemma. Then $K \subset U''$, $U_0 \subset U$. Let $B_0 = B_{K,U_0}(I)$ be the Bernstein index estimated in the Main Lemma. By (2.10),

$$B_0 < (1 - \log c')A^2.$$

□

Proposition 5.14

$$B_1 \leq B_0 + \log(4l + 1).$$

Proof By definition,

$$B_1 = \log \frac{M_1}{m_1}, \ B_0 = \log \frac{M_0}{m},$$

where $M_1 = \max_{\overline{U}} |I|$, $m_1 = \max_{\overline{U''}} |I|$, $M_0 = \max_{\overline{U_0}} |I|, m = \max_K |I|$. Note that $K \subset U''$, hence, $m \leq m_1$.

On the other hand, let

$$M_J = \max |J| \text{ on the closure of } D_{\nu+3\varepsilon}(a).$$

By definition, $\Gamma^{3\varepsilon} \subset U_0$, $\pi\Gamma^{3\varepsilon} \subset D_{\nu+3\varepsilon}(a)$, U is the minimal simply connected domain containing $U_0 \cup \Gamma^{3\varepsilon}$. By the Picard-Lefschetz theorem and Lemma 3.16

$$M_1 \leq M_0 + |l_0| l M_J, \ |l_0| \leq 2.$$

Let us estimate the integral J from above. Over each point of $\partial D_{\nu+3\varepsilon}(a)$ (except for $a+\nu+3\varepsilon$) there are two points of $\partial(\Gamma_1^{3\varepsilon}) \subset \overline{U}_0$, where $\Gamma_1 = \Gamma_a$ is the same, as in 5.1. The difference of the values of I at the two latter points is equal to $\pm l_0 J$, $0 < |l_0| \leq 2$. Therefore,

$$M_J \leq 2M_0. \text{ Hence,}$$

$$M_1 \leq M_0(4l + 1),$$

$$B_1 = \log \frac{M_1}{m_1} \leq \log \frac{M_0(4l+1)}{m} = B_0 + \log(4l + 1).$$

□

Let us now estimate from above other geometric characteristics used in Theorem 5.3. We have:

$$|\Gamma| = 4\pi l\nu = 24\pi l\varepsilon, \ \kappa(\Gamma) = 4\pi l.$$

Hence,

$$\frac{|\Gamma|}{\varepsilon} + \kappa(\Gamma) + 1 = 28\pi l + 1.$$

Moreover, by (5.22),

$$\frac{5D_2}{\varepsilon} = 5 \cdot 36 \cdot 24 \frac{n^4}{c''} + 480l.$$

Then, by Proposition 5.14 and Theorem 5.3,

$$V_{\Gamma_{1,l}}(I) \leq c_{n,l} A^{4320} e^{480l},$$

where

$$c_{n,l} = (B_0 + \log(4l+1))(28\pi nl + 1), A = e^{\frac{n^4}{c''}}.$$

By the Main Lemma,

$$B_0 < (1 - \log c')A^2.$$

Elementary estimates imply:

$$c_{n,l} \leq ((1 - \log c')A^3 e^l.$$

This implies (5.21).

Together, inequalities (5.19) - (5.21) imply Theorem B2 (by the argument principle). Theorems B1 and B2 imply Theorem B. □

5.7 Proof of Theorem A2 in Case 2 (near an infinite endpoint)

Here we prove Theorem A2 for a segment with one endpoint, say, b, infinite.

Proposition 5.15 *The integral I has an algebraic branching point at infinity of order $n+1$.*

Proof of Proposition 5.15. Let S_R be the circle $|t| = R$, $R \geq 3$, Γ_R be the $(n+1)$ sheet cover of S_R with the base point $-R$. Consider the real ovals $\gamma(t)$ extended for $t \in W$. For any arc $\Gamma' \subset \Gamma_R$ going from $-R$ to $t_\varphi = -Re^{i\varphi}$ let $[\Delta_{\Gamma'}]$ be the class of all the covering homotopy maps $\{H = -R\} \to \{H = t_\varphi\}$. Let h be the highest homogeneous part of H. If $H = h$, then for any R the class $[\Delta_{\Gamma'}]$ contains the simple rotation:

$$R_0 : (x, y) \mapsto (e^{\frac{i\varphi}{n+1}} x, e^{\frac{i\varphi}{n+1}} y)$$

In the general case, for R large enough the class $[\Delta_{\Gamma'}]$ contains a map $\Delta_{\Gamma'}$ close to the rotation. Let us prove this statement. To do this, consider the extension of the foliation $H = const$ by complex level curves of H to the projective plane $\mathbb{P}^2$ obtained by pasting the infinity line to the coordinate plane $\mathbb{C}^2$. The foliations $H = const$ and $h = const$ are topologically equivalent near infinity. More precisely, for any $r > 0$ large enough there exists a homeomorphism Φ of the complement $\mathbb{P}^2 \setminus D_r$ (D_r is the ball of radius r centered at 0) onto a domain in $\mathbb{P}^2$ that preserves the infinity line such that $h \circ \Phi = H$. This follows from the statements that the singularities of these foliations at infinity are the same and of the same topological type (nodes), and the holonomy mappings corresponding to clockwise

circuits around these singularities in the infinity line are rotations $t \mapsto e^{\frac{2\pi i}{n+1}} t$ in the transversal coordinate $t = H^{\frac{1}{n+1}}$. The last statement follows from the fact that for a generic $C \in \mathbb{C}$

$$H(x,y)|_{x=Cy} = (\widetilde{C}x)^{n+1}(1+o(1)), \text{ as } x \to \infty, \ \widetilde{C} \neq 0.$$

This already implies that the previous holonomies are rotations. The previous statement on the angle of these rotations holds true, since this is the case for the homogeneous polynomial h, and the foliation $H = const$ can be made arbitrarily close (near infinity) to $h = const$ (by appropriate rescaling).

The homeomorphism Φ is close to identity near infinity. For any $r > 0$ there exists a $T(r) > 0$ such that for any t, $|t| > T(r)$, $S_t \cap D_r = \emptyset$. The map $\Delta_{\Gamma'}$ we are looking for is obtained from the map R_0 corresponding to h by conjugation by the homeomorphism Φ. By construction, its $n+1$- iterate is identity. □

Proof of Theorem A2 near infinity. As before, consider the case $b = +\infty$; the case $a = -\infty$ is treated in the same way. Let W_I be the Riemann surface of the integral I. Let $\Gamma \subset W_I$ be the degree $n+1$ cover of the circle $|t| = 3$ with the base point $t_1 = +3$. This is a closed curve on W_I. This curve is a boundary of a domain on W_I that covers $n+1$ times a neighborhood of infinity on the Riemann sphere. Let us denote this domain by W_I^∞. We will estimate from above

$$N_\infty = \{t \in W_I^\infty \mid I(t) = 0\}.$$

This will give an upper estimate to the number of zeros of I on $\sigma^+ = (3, +\infty)$ because $\sigma^+ \subset W_I^\infty$. We will use the argument principle in the form

$$N_\infty \leq \frac{1}{2\pi} V_\Gamma(I) + n + 1. \tag{5.23}$$

This follows from the argument principle and the fact that the infinity is the only pole of $I|_{W_I^\infty}$, and its order is at most $n+1$. The latter bound on the order follows from the condition that the 1- form under the integral (1.1) has degree at most n, and the fact that the integration oval $\gamma(t)$ has size (and length) of the order $O(|t|^{\frac{1}{n+1}})$, as $t \to \infty$, $t \in \mathbb{R}$.

The variation in the right hand side will be estimated by Theorem 5.3. To apply this Theorem we need to define all the entries like in the previous Subsection.

We have: $\Gamma = \partial W_I^\infty$. Without loss of generality we consider that $I|_\Gamma \neq 0$ (one can achieve this by slight contraction of the circle $|t| = 3$). Let K be the same as in the Main Lemma. Denote by U_0 the set U from that Lemma: both K and U_0 are taken projected to the Riemann surface of the integral I. Let ε be the same as in (5.7). By (2.7), $K \supset \Gamma$.

Let

$$U'' = \widetilde{K^\varepsilon}, \ U' = \widetilde{K^{2\varepsilon}}, \ U = \widetilde{K^{3\varepsilon}}.$$

Then U coincides with the projection of U_0 to W_l (up to filling holes, if there are any). Therefore, $\max_{\overline{U_0}} |I| = \max_{\overline{U}} |I|$ (the maximum principle). Hence,

$$B_{U'',U}(I) \leq B_{K,U_0}(I) < (1 - \log c')A^2.$$

The latter inequality is (2.10). This provides the estimate of the Bernstein index required in Theorem 5.3. Other ingredients are the following.

By (2.8), the diameter condition (5.4) holds with

$$D = 36n^2 + 1.$$

The gap condition (5.3) for Γ, U'', U', U holds as well with ε from (5.7). Hence,

$$e^{\frac{5D}{\varepsilon}} \leq A^{4600}$$

Moreover,

$$\mid \Gamma \mid = 6\pi(n+1), \ \mid \kappa(\Gamma) \mid = 2\pi(n+1).$$

Altogether, by Theorem 5.3, this implies:

$$V_\Gamma(I) \leq (1 - \log c')C(n, c'')A^{4602},$$

with $C(n, c'') = \frac{6\pi(n+1)}{\varepsilon} + 2\pi(n+1) + 1 < A^{90}$. Together with (5.23) this proves Theorem A2, Case 2. □

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