Effective classes and Lagrangian tori in symplectic four-manifolds

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January 3, 2007

Abstract:
An effective class in a closed symplectic four-manifold (X, ω) is a two-dimensional homology class which is realized by a J-holomorphic cycle for every tamed almost complex structure J. We prove that effective classes are orthogonal to Lagrangian tori in $H_2(X; \mathbb{Z})$.

1.1 Results

Let (X, ω) be a closed symplectic four-manifold. A two-dimensional homology class $d \in H_2(X; \mathbb{Z})$ is called an effective class if it is realized by a J-holomorphic two-cycle for every almost-complex structure J tamed by the symplectic form ω.

Theorem 1.1 *Let (X, ω) be a closed symplectic four-manifold. Let A_ω be the subspace of $H_2(X; \mathbb{Z})$ generated by Lagrangian tori and B_ω the subspace generated by effective classes. Then, A_ω and B_ω are orthogonal to each other.*

Corollary 1.2 *Let L be a Lagrangian torus and S be an embedded symplectic (-1)-sphere in a closed symplectic four-manifold (X, ω). Then, L and S have vanishing intersection index.* □

We indeed know from Lemma 3.1 of [4] that such embedded symplectic (-1)-spheres define effective classes. Do there exist a Lagrangian torus and symplectic (-1)-sphere such that, though they have vanishing intersection index, they have to intersect? Otherwise, it means that the space A_ω comes from an underlying minimal symplectic four-manifold.

Let L be a torus equipped with a flat metric, S^*L be its unit cotangent bundle and $\pi : S^*L \to L$ the canonical projection. The manifold S^*L is

equipped with a canonical contact form λ, namely the restriction of the Liouville one-form of its cotangent bundle. We denote by R_λ the subgroup of $H_1(S^*L;\mathbb{Z})$ generated by its closed Reeb orbits.

Lemma 1.3 *The restriction of $\pi_* : H_1(S^*L;\mathbb{Z}) \to H_1(L;\mathbb{Z})$ to R_λ is an isomorphism.*

Proof:

The Reeb flow on S^*L coincides with the geodesic flow. Closed Reeb orbits are thus the lifts of closed geodesics on L. Now S^*L is diffeomorphic to a product of L with the sphere S of directions in L, and π is the projection onto the first factor. Since geodesics of L have a constant direction, the projection onto the second factor maps every Reeb orbit to a point of S. From Künneth formula, we get the isomorphism $H_1(S^*L;\mathbb{Z}) \cong H_1(L;\mathbb{Z}) \times H_1(S;\mathbb{Z})$ and, from what we have just noticed, that this isomorphism maps R_λ into $H_1(L;\mathbb{Z}) \times \{0\}$. Since generators of $H_1(L;\mathbb{Z})$ can be realized by closed geodesics, the latter map is onto. Since π_* is the projection onto the first factor, it is an isomorphism once restricted to R_λ. $\square$

Proof of Theorem 1.1:

Let L be a Lagrangian torus and d be an effective class. Following the principle of symplectic field theory [1], we stretch the neck of the symplectic manifold in the neighbourhood of L until the manifold splits in two parts, one part being the cotangent bundle of the torus and the other one being $X \setminus L$. We produce this splitting in such a way that both parts have the contact manifold (S^*L, λ) at infinity. Let J_∞ be a CR-structure on this contact three-manifold which we extend to an almost-complex structure J with cylindrical end on both parts T^*L and $X \setminus L$. The latter is the limit of a sequence J_n of almost-complex structures of (X, ω). Since d is effective, we may associate a sequence C_n of J_n-holomorphic two-cycles homologous to d. From the compactness Theorem in SFT [2], we extract a subsequence converging to a broken J-holomorphic curve C, which we assume for convenience to have only two levels -the general case follows easily from this one-. Denote by C^L the part of C in T^*L and by C^X the part in $X \setminus L$. Both curves C^L and C^X have cylindrical ends asymptotic to the same set of closed Reeb orbits with same multiplicities. Let $C^L_1, \ldots, C^L_k$ denote the irreducible components of C^L, and $R_1, \ldots, R_k$ be the corresponding set of closed Reeb orbits. These sets $R_1, \ldots, R_k$ define integral one-cycles in S^*L and we denote by $[R_1], \ldots, [R_k]$ their homology classes. These one-cycles are boundaries of the two-chains $C^L_1, \ldots, C^L_k$ in T^*L , so that with the notation of Lemma

1.3, $\pi_*([R_i])$ vanishes for every $i \in \{1, \ldots, k\}$. Since $[R_1], \ldots, [R_k]$ belong to the subgroup R_λ, we deduce from Lemma 1.3 that $[R_1], \ldots, [R_k]$ actually vanish. Let $S_1, \ldots, S_k$ be integral two-chains of S^*L having $R_1, \ldots, R_k$ as boundaries, and S be the sum of these k chains. Then, $C^L - S$ is an integral two-cycle contained in T^*L, $C^X + S$ is an integral two-cycle contained in $X \setminus L$, and the sum of these cycles is homologous to d. Now, L and $C^X + S$ are disjoint from each other and the second homology group of T^*L is generated by $[L]$ itself. Since the latter has vanishing self-intersection, we deduce that the intersection index of L and $C^L - S$ vanishes. As a consequence, the intersection index of d and $[L]$ vanishes. Since this holds for any Lagrangian torus or effective class, Theorem 1.1 is proved. □

1.2 Remarks

1. We have actually proved more, namely for a class $d \in H_2(X; \mathbb{Z})$ to be orthogonal to a Lagrangian torus $[L]$, it suffices that d be realized by a sequence of J_n-holomorphic two-cycles, for a sequence J_n having a flat neck stretching to infinity.

2. From the results of Taubes [6], any Seiberg-Witten basic class is effective and thus, from Theorem 1.1, we deduce that SW-basic classes are orthogonal to Lagrangian tori. This fact was already known, it indeed follows from the adjunction inequality [3], [5]. Our space B_ω might however be bigger than the one generated by SW-basic classes? Also, our proof remains in the symplectic category and offers possibilities to have counterparts in higher dimensions.

3. If (X, ω) is Kähler, then the Poincaré dual of B_ω is contained in the intersection of $H^{1,1}(X; \mathbb{Z})$ on every complex structure of X tamed by ω. How smaller can it be?

4. From Hodge' signature Theorem follows that the intersection $A_\omega \cap B_\omega$ vanishes for every Kähler surface. This intersection indeed lies in the isotropic cone of the Lorentzian $H^{1,1}(X; \mathbb{R})$ and is orthogonal to the symplectic form which lies in the positive cone (compare Example 1 of §1.3). I don't see at the moment whether or not this holds for every closed symplectic four-manifold (and am grateful to Stéphane Lamy for raising the question to me). More generally, one may wonder whether the intersection form restricted to B_ω has to be non-degenerated. This is the case at least for rational surfaces from Example 2 of §1.3 and for

Kähler surfaces with $b_2^+ \geq 2$ and $K_X^2 > 0$ since from Taubes' results [6], B_ω contains the canonical class K_X.

5. We made a crucial use of a property of the contact manifold (S^*L, λ), namely that the subgroup R_λ generated by its closed Reeb orbits has a rather big index in $H_1(S^*L; \mathbb{Z})$. We may then more generally wonder, given a contact manifold, how small can this subgroup R_λ of "effective" homology classes be?

1.3 Examples

1. If (X, ω) has a Lorentzian intersection form, then A_ω vanishes. Indeed, each Lagrangian torus should be in the isotropic cone of the intersection form and should be orthogonal to the class of the symplectic form which lies in the positive cone.

2. If (X, ω) is a blow up of the projective plane, then $B_\omega = H_2(X; \mathbb{Z})$. Indeed, exceptional divisors are effective classes, and the strict transform of a line has non-trivial GW-invariants.

3. If X is a product of two curves (C_1, ω_1) and (C_2, ω_2) with symplectic form $\omega_1 \ominus \omega_2$, then A_ω contains the index two subgroup $H_1(C_1; \mathbb{Z}) \otimes H_1(C_2; \mathbb{Z})$ given by Künneth formula. When in addition, (C_1, ω_1) and (C_2, ω_2) are symplectomorphic tori, A_ω contains the graph of the symplectomorphism. Note that A_ω cannot have codimension less than one since it lies in the orthogonal of the symplectic form.

4. If X is a product of two genus one curves, then B_ω vanishes, since for a generic complex structure, $H^{1,1}(X; \mathbb{Z})$ vanishes. If instead one of the curve is not elliptic, then we know from Taubes' results [6] that B_ω contains the canonical class of X.

Acknowledgements: I am grateful to the French Agence Nationale de la Recherche for its support.

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