# On a randomized PNG model with a columnar defect

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**Abstract** We study a variant of poly-nuclear growth where the level boundaries perform continuous-time, discrete-space random walks, and study how its asymptotic behavior is affected by the presence of a columnar defect on the line. We prove that there is a non-trivial phase transition in the strength of the perturbation, above which the law of large numbers for the height function is modified.

**Keywords** Polynuclear growth · interacting random walks · zero-temperature Glauber dynamics · polymer pinning

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## 1 Introduction

The study of growth processes in $1+1$ dimensions has led to a very rich theory. One particular model, known as poly-nuclear growth (or PNG for short) can be described as follows: On a flat substrate, layers appear on top of each other according to a Poisson process, and then grow laterally with constant speed, merging whenever they collide. This model is interesting in the sense that it lends itself to the explicit computation of various quantities, such as growth speed and interface fluctuations (see [2,6]). However, the proofs depend on exact algebraic computations, and do not apply as soon as the dynamics are modified in a non-translation-invariant way. In particular, the response of the system to a localized perturbation (such as adding an independent Poisson source at the origin) is largely unknown.

Similar questions are often raised in the setting of first-passage percolation (see [5] for a review). More specifically, consider an FPP process in which the waiting times are shortened along one of the axes, is it true that the asymptotic shape of the cluster is always modified, or is there a phase transition for the amount of perturbation ? Very related is the problem of polymer pinning in a random environment (*i.e.* in the presence of bulk disorder), or that of exclusion processes with a localized obstruction to the flow (see [4]). In all of these, the central object of interest is the speed obtained from a law of large numbers.

The process we study in this paper can be informally described as follows. On each edge of the discrete line $\mathbb{Z}$, with some positive rate $\lambda$, pedestals of width and height 1 appear. Every pedestal will evolve into a plateau, its endpoints performing independent continuous-time, discrete-space random walks with jump rate 1, until one of the following happens:

- If two plateaux *collide*, *i.e.* if the right edge of one occupies the same site as the left edge of the next one, then they merge and only the two extremal endpoints continue to perform random walks;
- If one of the two endpoints of a plateau jumps to the location of the other one, so that the plateau shrinks to width 0, then it vanishes.

If a pedestal appears on an edge which is already spanned by one of the plateaux, it appears on an additional layer on top of it. Only two plateaux evolving within the same layer can merge, and besides the growth of the plateaux at different layers has to obey the following constraint: Every plateau which is not on the ground level must always lie entirely on top of a plateau at the layer below it. Each jump which would break this constraint is simply suppressed. For any fixed value of $\lambda$ (see Prop. 2 below) the process has linear growth, *i.e.* if by $h_t(e)$ we denote the height of the aggregate at time $t$ above the edge $e$, then the following limit exists:

$$v(\lambda) := \lim_{t\to+\infty} h_t(e)/t > 0. \tag{1}$$

Notice that in the case of a strictly growing process (such as first- or last-passage percolation), the law of large numbers is usually a direct consequence of a statement of sub-additivity, while if the dynamics is allowed to both grow and shrink, like in the present case, even the existence of the asymptotic speed is non-trivial.

The dynamics we are considering corresponds exactly to Glauber dynamics for the two-dimensional Ising model on a solid surface at temperature 0, with

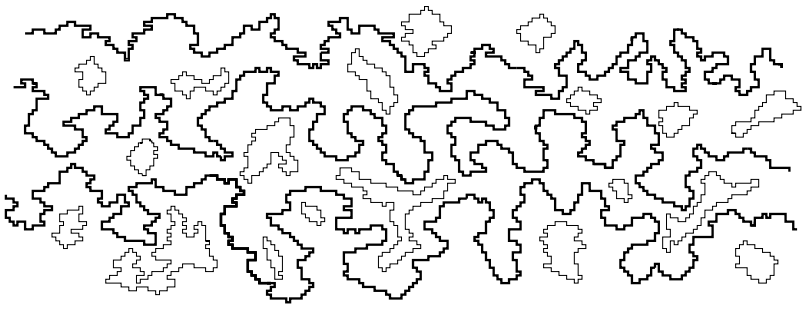

**Fig. 1** Illustration of an RPNG process. (Level lines are bold.)

additional nucleations. More precisely, consider an Ising model on $\mathbb{Z}\times\mathbb{Z}_+$, initially filled with $(-)$ spins, and with a $(+)$ boundary condition. Every site with two neighbors of each spin flips with rate 1; every site with three neighbors of a given spin acquire it with rate 1; and besides, each site with spin $(-)$ whose bottom neighbor has spin $(+)$ switches to $(+)$ with rate $\lambda$. It is very easy to check that vertical phase boundaries perform constrained, continuous-time random walks and that both the annihilation and interaction rules are the same as for RPNG.

To the best of our knowledge, a law of large numbers in such a setup has not been given before, and while this one-sided nucleation might seem a little artificial, it may give information about mixing times for low-temperature Ising model in a finite box. It might also be a decent way of describing one of the limiting regimes of low-temperature Ising model in a constant external field, where both the temperature and field intensity go to 0.

Next we will introduce a *"columnar defect"* to the system by changing the nucleation rate on the edge $e_0 :=< 0,1 >$ from $\lambda$ to $\lambda+\lambda_0$, with $\lambda_0 > 0$, and the main question which we address is how the asymptotic growth rate of the aggregate above the origin is affected. More precisely, let $h_t^{\lambda_0}(e_0)$ be the height of the aggregate above $e_0$ at time $t$ under the modified dynamics (we make the dependency on $\lambda$ implicit in this notation, since the parameter of interest is $\lambda_0$): We are interested in whether the inequality

$$v(\lambda,\lambda_0) := \lim_{t\to+\infty} h_t^{\lambda_0}(e_0)/t > v(\lambda) \tag{2}$$

holds for all $\lambda_0 > 0$.

Define

$$\lambda_c(\lambda) = \inf\{\lambda_0 : v(\lambda,\lambda_0) > v(\lambda)\}. \tag{3}$$

Our main result, Theorem 1 below, is that $0 < \lambda_c(\lambda) < +\infty$. Moreover, the phase transition is non-trivial even without nucleation in the bulk (*i.e.* $\lambda = 0$). Even in this simplified situation, for any value $\lambda_0 > 0$ the origin is eventually covered by plateaux for all times, however $0 < \lambda_c' := \lambda_c(0) < +\infty$ so that if $\lambda_0 < \lambda_c'$ then $v(0,\lambda_0) = 0$, *i.e.* there is sub-linear growth, while for $\lambda_0 > \lambda_c'$ we have $v(0,\lambda_0) > 0$.

### 1.1 Structure of the paper

The dynamics we are considering is coming from two sources, one in the bulk (the nucleations on every edge with rate $\lambda$), and one on a particular edge (the additional nucleation on $e_0$). In section 3 we first give a formal construction of the process, in terms of a Harris-type representation. Then in section 4 we study the homogeneous process for its own sake, without the defect at $e_0$. More specifically, we prove a law of large numbers for the height of the interface above edge $e_0$; a key step towards this result is the proof of eventual covering of any finite-size interval of the line.

In section 5 we study a related model in the half-line, which, while simpler, exhibits the same kind of behavior as the original one. It allows us to introduce a description of the system in terms of interacting random walks.

Finally, in section 6 we extend the methods of the previous section to the full-space case, first for a system with nucleation at only one edge (corresponding to the case $\lambda = 0$ in the previous description), then for the full model, thus completing the proof of the main result of the paper.

## 2 Preliminary considerations and statement of the results

### 2.1 Description of the interaction

One can describe this process in terms of an interacting particle system. Each step up (or equivalently each left-most endpoint of a plateau) is represented by a particle of type $+$ at the corresponding site of $\mathbb{Z}$, and each step down as a particle of type $-$ (which can be interpreted as an anti-particle). If several plateaux have there endpoints at the same site, this site will contain several particles, but notice that a site will never contain particles of both types.

The dynamics then looks like a zero-range process with annihilation and nucleation, *i.e.*:

- For each non-empty site, with rate 1, one of the particles jumps with equal probability to a neighboring site; and if the target site contains a particle of the other type, then the jumping particle and one of the target particles annihilate instantaneously;
- On every edge $(x, x+1)$ of $\mathbb{Z}$, with rate $\lambda$, there is nucleation, meaning that a particle of type $+$ appears at $x$ and a particle of type $-$ appears at $x+1$; and similarly, each one is instantaneously annihilated if a particle of the opposite type is already present at the corresponding site.

With this description, notice that the behavior of a tagged particle is essentially that of a continuous-time random walk, jumping at rate $1/n$ whenever it is at a site containing $n$ particles, and which is killed whenever it meets a particle of the opposite type.

### 2.2 Stick percolation

We sidetrack for a moment and introduce a toy model sharing some of the spirit of the way the defect will affect the system (in the $\lambda = 0$ case). Let $(t_i)_{i\in\mathbb{N}}$ be a

Poisson point process of intensity $\gamma > 0$ on the positive real line. We assume that they are ordered, $t_0$ being the point closest to the origin. To each $t_i$ we associate a positive random variable $S_i$ (a "length") and assume that the $(S_i)_{i\in\mathbb{N}}$ are i.i.d. with common distribution function $F$.

The system we consider is then the following: For every $i \in \mathbb{N}$, construct the segment $\widehat{S}_i = [t_i, t_i + S_i]$ (which we will call the $i$-th "stick"). We say that the sticks $\widehat{S}_i$ and $\widehat{S}_j$, $i < j$ are *connected* if $t_j < t_i + S_i$, *i.e.* if they have non-empty intersection; the system *percolates* if and only if there is an infinite chain of distinct, pairwise connected sticks, which (with probability 1) is equivalent to saying that the union of the sticks contains a half-line.

It is easy to see that the system percolates with probability 0 or 1 (it is a tail event for the obvious filtration); and in fact there is a complete characterization of both cases. The proof of the following proposition is not difficult, so we omit it at this point.

**Proposition 1** *Let $R(t) = P(S_1 > t)$ and $\varphi(t) = \int_0^t R(u)du$. Then the system percolates with probability* 1 *if and only if*

$$\int_0^{+\infty} e^{-\gamma\varphi(s)}\mathrm{d}s < +\infty. \tag{4}$$

A similar process was studied by Shepp [7], the only difference being that in his case the support of the birth process is the whole line instead of positive real line. In particular, it is easy to see that in the full-line case any non-integrable distribution for $S$ covers the whole space, whereas here there are non-integrable distributions which do not percolate.

The natural distribution for the length of the sticks, in view of the previous discussion of our original problem, is the following: At time $t_i$, start two continuous-time symmetric simple random walks, from 0 and $+1$ respectively, and let $t_i + S_i$ be the first time when they occupy the same site. It is well known that, up to multiplicative constants, $P(S_i > t)$ behaves as $t^{-1/2}$ for large $t$. It is easy to check that the system percolates, for any $\gamma > 0$. On the other hand, it is also easy to see that there is not linear growth, in the sense that if we let $N_t$ be the number of sticks that contain the point $t$, then

$$E(N_t) = \gamma \int_0^t R(s)\mathrm{d}s = \Theta(\sqrt{t}).$$

In the RPNG case, the walks will interact, thus leading to a more interesting behavior and to the existence of a non-trivial phase transition.

### 2.3 Statement of the results

We are now ready to precisely state our main result. Recall that $h_t^{\lambda_0}(e_0)$ is the height of the interface above $e_0$, for a system with bulk nucleation rate $\lambda \geq 0$ and defect nucleation rate $\lambda_0$.

**Theorem 1** *1. There is an asymptotic speed for $h_t^{\lambda_0}$*, i.e.

$$v(\lambda,\lambda_0) := \lim_{t\to+\infty} \frac{h_t^{\lambda_0}(e_0)}{t}$$

*exists, and it is a constant almost surely; $v(\lambda,\lambda_0) \geq 0 \vee (\lambda_0 - \lambda)$; moreover, $v(\lambda,\lambda_0) > 0$ if $\lambda > 0$.*

*2. Let $\lambda_c(\lambda) := \inf\{\lambda_0 : v(\lambda,\lambda_0) > v(\lambda,0)\}$ be the critical value of $\lambda_0$: then,*

$$\lambda_c(\lambda) \in (0,+\infty).$$

In proving this theorem, we actually obtain a more precise description of the behavior of the system in the supercritical phase $\lambda_0 > \lambda_c(\lambda)$: particles actually accumulate linearly fast on both vertices 0 and 1, but the behavior of the system outside of those two vertices is perturbed only sub-linearly. In other words, the defect creates a kind of "antenna" at the origin. This contrasts with the expected behavior of the usual PNG with a supercritical localized defect, for which the region of influence of the additional nucleation process is believed to extend linearly in space.

## 3 Construction of the process

### 3.1 Graphical construction

Let $\mathbb{E} := \{< x, x+1 >, x \in \mathbb{Z}\}$ be the set of edges of the integer line $\mathbb{Z}$. We first construct the process using a Harris representation, as follows. Define the following three independent Poisson systems:

- $\mathscr{L} = \{\mathscr{L}_e, e \in \mathbb{E}\}$ is a collection of independent Poisson processes of left-arrows, each with intensity $\mathrm{d}t/2$; the individual arrows (or *marks*) of the process at edge $e$ will be denoted by $\mathscr{L}_e^n$, $n \geqslant 1$, with similar notations for the other two systems;
- $\mathscr{R} = \{\mathscr{R}_e, e \in \mathbb{E}\}$ is a collection of independent Poisson processes of right-arrows, each with intensity $\mathrm{d}t/2$;
- $\mathscr{N} = \{\mathscr{N}_e, e \in \mathbb{E}\}$ is a collection of independent Poisson processes of undirected edges, each with intensity $\lambda\,\mathrm{d}t$.

$\mathscr{L}$ will correspond to jumps of particles to the left, $\mathscr{R}$ to jumps to the right and $\mathscr{N}$ to nucleation times.

Given $T > 0$, we first construct the process up to time $T$, as follows. There is positive density of edges carrying no mark of any of the Poisson before time $T$; between any two such edges, there are almost surely only finitely many marks appearing by time $T$, so that they can be ordered them chronologically, and we need only describe the action of each of the marks.

- If there is a directed arrow at edge $e$ and time $t$, and if at this time the site at the source of the arrow is not empty, then one particle at this site jumps in the direction of the arrow; moreover, if after this jump it meets a particle of the opposite type, they annihilate instantaneously.

– If there is a mark of $\mathcal{N}$ at edge $e =< x, x+1 >$ and time $t$, then at this time a particle of type $+$ appears at site $x$ and a particle of type $-$ appears at site $x+1$; and if the corresponding sites contain particles of the opposite type, a pair is annihilated instantaneously.

It is then easy to see that the obtained processes for different values of $T$ are consistent, and letting $T$ go to infinity gives the process for all time. We will refer to this version of the construction as the process in *quenched environment*, *i.e.* seeing the Poisson processes as a random scenery in which a deterministic process evolves.

**Remark:** If one wanted to construct the process of interacting random walks in higher dimension (for which it certainly does exist), a different method should be used.

### 3.2 Construction in terms of interacting random walks

To follow the path of a tagged particle in the previous construction, one needs a rule to determine which of the particles at a given site will jump when an arrow appears, and which will be annihilated when a particle of the opposite type arises. Obviously the process itself will not be affected by the choice, so we have some freedom here. Some of the more natural choices are to declare that the oldest particle (*i.e.* the one that appeared earliest in the system), or the youngest particle, or the one that has been at the given site for the longest time, or the shortest time, is "active". The rule can even be different for jumps and annihilations. We will refer to such a rule as a *priority rule*.

For a similar process with finitely many particles at each time (which for instance will be the case if nucleation occurs only at finitely many sites), the previous graphical construction can be replaced with the following interpretation of the dynamics. Whenever a particle $i$ is born into the system, it comes together with a "project", *i.e.* a family $(T_i^n)$ of independent exponential times of rate 1 and a family $(\varepsilon_i^n)$ of independent signs, and a clock. Those will represent waiting times and directions of jumps for the path of this particle, if it were alone in an otherwise empty system.

Choose any priority rule. When a particle is alone, it is always "active", and its clock runs deterministically with speed 1. When there are several particles at a given site, only one of them is active, according to the chosen rule, and the others are "sleeping". The clock of the active particle runs deterministically at rate 1, the clocks of the other particles do not run at all. Whenever the clock of any particle reaches the next waiting time for the project of this particle, the particle jumps and the clock is reset.

Another natural choice for a jumping rule could be that at a site with $n$ particles, the clock of each of them runs with speed $1/n$; again the distribution of the numbers of particles at different sites would be the same as in the graphical construction.

Notice that while this construction still makes sense for infinitely many particles as long as they don't accumulate, the technical details of the construction would become significantly more technical.

The previous construction is *Abelian*, in the following sense: one particular priority rule is to order all the particles *a priori* with some indices, and to declare that the active particle at a site is the one with the lowest, or highest, index. Then, the distribution of the numbers of particles does not depend on the ordering of the particles. This allows one to add finitely many new nucleation points to an otherwise infinite system (where the process is constructed *e.g.* using the previous graphical construction), by declaring that the added particles have a lower priority than all the others, and do not see the marks. The Abelian property ensures that the obtained process is well-defined (again in terms of the distribution of the number of particles per site if not in terms of trajectories) and coincides with the process constructed graphically from the full family of nucleation points.

### 3.3 Back to the PNG-like description

We saw how to go from a PNG-like description to a family of interacting particles, but our final goal is to describe the long term behavior of the height (*i.e.* the number of layers) at a given edge, so we need a way to go back to the layer structure. As we saw already the particles of type $+$ correspond to left boundaries of plateaux (*i.e.* to steps upward in the profile), and the particles of type $-$ correspond to right boundaries (*i.e.* to steps downward in the profile). All that remains to do is to compute the height at edge $e_0 =< 0,1 >$.

Let $(\phi_t^+)$ be the flow of particles of type $+$ through the edge $e_0$: $\phi_0^+ = 0$ and $\phi^+$ increases (resp. decreases) by 1 whenever a particle of type $+$ crosses $e_0$ to the right (resp. to the left). Define $(\phi_t^-)$ similarly as the flow of particles of type $-$. Let $N_t$ be the number of nucleation marks occurring at edge $e_0$ by time $t$ and $h_t(e_0)$ be the height of the profile at edge $e_0$ and time $t+$ (*i.e.*, the càdlàg version in time of the height process). Then,

$$h_t(e_0) = N_t - \phi_t^+ + \phi_t^-. \tag{5}$$

From $h_t(e_0)$ and the previous discussion, it is then easy to obtain the whole profile of plateaux.

From here on, we will refer to this particular aspect of the process as the *Randomized Polynuclear Growth process*, or RPNG for short.

### 3.4 Adding a columnar defect

We want to study how the long term behavior of the system (say, in terms of the law of large numbers for $h_t(e_0)$) is sensitive to a columnar modification, which can be described as adding an auxiliary nucleation process $\mathscr{N}_0$ with rate $\lambda_0 > 0$ localized on the edge $e_0$. In other words, the nucleation rate will be $\lambda$ everywhere except at $e_0$ where it is $\lambda + \lambda_0$.

The construction of the new process can be done exactly the same way as for the unperturbed system, using a graphical construction and letting the marks of $\mathscr{N}_0$ act the same way as those of $\mathscr{N}$.

The rest of the paper is organized as follows: In Section 4, we study the homogeneous system and obtain the deterministic asymptotic speed for the height of

the profile at $e_0$. In Section 5, we use a similar process in the half-space as a simplified model of the propagation of influence. In Section 6 we extend the analysis to the full-space process, and finish the proof of Theorem 1.

## 4 Behavior of the homogeneous process ($\lambda > 0,\ \lambda_0 = 0$)

### 4.1 Eventual covering of the first level line

In deriving the asymptotic speed for the height of the RPNG at $e_0$ for each $\lambda > 0$, we first need to prove that it indeed goes to infinity. The first step is to show that for every interval of the integer line, there is a (random) time after which it remains covered by one of the plateaux of the first RPNG layer.

Let $X_t$ (resp. $Y_t$) be the position of the left (resp. right) end of the first-layer plateau containing $e_0$ at time $t$, if there is one *i.e.* if $h_t(e_0) \geqslant 1$ — for completeness, one can let $X_t = +\infty$ and $Y_t = -\infty$ if $h_t(e_0) = 0$.

Let $\tau$ be the first time at which $Y_t = X_t + 2$. In the absence of nucleation and of other layers, the behavior of both $X_t$ and $Y_t$ until the first time when $Y_t = X_t + 1$ would exactly be that of two independent, centered continuous-time random walks. The effect of the presence of the second and above layers, by the previous description, can only be the suppression of left jumps of $Y_t$ and right jumps of $X_t$ — notice that if there are several particles at $Y_t$ at time $t$ and one has to jump to the right, then so does $Y_t$.

Moreover, with rate $\lambda$ there is nucleation on the edge $< Y_t, Y_t + 1 >$, and when this occurs, $Y_t$ jumps to the right by at least one step. The last thing that can affect $Y_t$ is if the plateau $[X_t, Y_t]$ merges to the next plateau to its right, which again can only increase the value of $Y_t$. The same remarks apply *mutatis mutandis* to $X_t$.

Taking all those effects into account, one obtains that the process $(Y_t)$ stochastically dominates the biased random walk $(\tilde{Y}_t)$ that has rates $1/2$ to jump to the left and $1/2 + \lambda$ to jump to the right. Similarly, $(X_t)$ is stochastically dominated by $(\tilde{X}_t)$ which is biased to the left. Since the probability

$$p = P(\forall t > 0,\ \tilde{Y}_t - \tilde{X}_t > 1) \tag{6}$$

is positive, one gets that the probability that the plateau $[X_t, Y_t]$ eventually covers the whole space also is positive. One also gets that there is some deterministic and finite $T_1$ such that:

$$P\left(\forall t \geqslant T_1,\ Y_t > \frac{\lambda t}{2} \text{ and } X_t < -\frac{\lambda t}{2}\right) > \frac{p}{2} \tag{7}$$

If the first plateau appearing at $e_0$ vanishes at a finite time, one waits for a new nucleation at $e_0$ after this time, with the new plateau having a positive probability to survive forever. The argument can be repeated each time the plateau vanishes, always with the same positive lower bound for the probability of survival, from which the result easily follows.

A related fact that we will need in the sequel is the following: Assuming that the plateau of $e_0$ is already large at some time $t$, then its probability of survival is

very close to 1. Again the estimate uses standard facts about random walk with drift: If $\tau$ is the vanishing time of the plateau, then

$$P(\tau = +\infty | Y_t - X_t > l) > 1 - c_1 e^{-c_2 l},$$
$$P(\tau = +\infty | \tau > t) > 1 - c_3 e^{-c_4 t}.$$

**Lemma 1** *With the above notations, almost surely $Y_t/t \to +\infty$. More quantitatively, there exist finite, positive constants $C$, $c$ and $T_2$ such that*

$$P\left(\forall t > T_2,\, Y_t > c_5 e^{c_6 t}\right) \geqslant 1 - C e^{-cT_2}.$$

Before giving a formal proof of this Lemma, here is a non-rigorous explanation of the phenomenon. The key idea is the following: The plateau $[X_t, Y_t]$ which we are following grows whenever it merges with another plateau of layer 0, and such a plateau, having the same "age" as $[X_t, Y_t]$, will have comparable length. In other words, $(Y_t - X_t)$ essentially double with positive rate, which give it exponential growth.

*Proof* Let $T > 0$ be some fixed time, and $K_T$ be the size of a box around site 0, to be fixed later. For every $x$ in $[-K_T, K_T]$ and each $t > 0$, let $X_t^x$ (resp. $Y_t^x$) be the left (resp. right) boundary of the layer 0 plateau containing $< x, x+1 >$ — or $+\infty$ (resp. $-\infty$) if there is no such plateau. As we saw previously, for every $t_0$ large enough, we have

$$P\left[\forall t > t_0, \quad X_t^x < x - \frac{\lambda}{2} t < x + \frac{\lambda}{2} t < Y_t^x\right] \geqslant 1 - C e^{-ct_0}.$$

Applying this estimate to all the edges of $[-K_T, K_T]$ simultaneously, we obtain the following estimate:

$$P\left[\forall x \in [-K_t, K_T],\, \forall t > t_0, \quad X_t^x < x - \frac{\lambda}{2} t < x + \frac{\lambda}{2} t < Y_t^x\right] \geqslant 1 - 2K_T C e^{-ct_0}.$$

This last event clearly implies that the involved plateaux overlap, which in turn leads to

$$P[\forall t \geqslant t_0,\, X_t < -K_T < K_T < Y_t] \geqslant 1 - 2K_T C e^{-ct_0}.$$

Now taking $T > t_0$ and say $K_T = e^{ct_0/2}$, and applying Borel-Cantelli, we obtain the required estimate.

This is already sufficient to ensure that the height at $e_0$ grows at least linearly in time. Indeed, one can apply the same reasoning to the second layer: The rightmost edge of the second layer droplet containing $e_0$, say $Y_t^{(2)}$, will drift to the right with speed at least $\lambda$ as long as it does not touch $Y_t$, and if this happens the behavior of $Y_t - Y_t^{(2)}$ will be dominated by that of a random walk reflected at 0. In other words, one gets that after some finite time, $Y_t^{(2)} > c_1 e^{c_2 t} - K t^{1/2+\varepsilon}$.

This can be repeated for every new layer, and proves that each new nucleation does survive with positive probability. In other words,

$$\text{(a.s.)} \quad \liminf \frac{h_t(e_0)}{t} > 0. \tag{8}$$

A trivial counterpart of that is that the speed of growth at $e_0$ is bounded above by the sum of the nucleation and immigration rates, which implies that

$$(\text{a.s.}) \quad \limsup \frac{h_t(e_0)}{t} \leqslant \lambda + 1 < +\infty. \tag{9}$$

### 4.2 Level lines

In this subsection, we give an additional description of the time evolution of the system, in terms of a family of space-time curves similar to a Hammersley process (cf. [1]). The main difference is that the paths of the particles between meeting points are random walks instead of straight lines.

More precisely, one can see the path of any given particle, from its birth to its eventual annihilation, is a sequence of vertical and horizontal line segments. Let $\mathscr{C}_1$ be the union of the space-time paths of all the particles corresponding to endpoints of plateaux of the first layer.

From the study in the previous subsection, we know that with probability one, every plateau either vanishes in finite time, or grows and merges with others to eventually cover the whole line. This means that $\mathscr{C}_1$ is the disjoint union of a countable family of finite loops (corresponding to plateaux with a finite life span) and one unbounded curve, which we will denote by $C_1$ and call the *first level line* of the process.

Similarly, let $\mathscr{C}_k$ be the union of all the space-time paths of the endpoints of plateaux of layer $k$, then $\mathscr{C}_k$ is the disjoint union of a countable family of finite loops, and one unbounded curve $C_k$. Notice that in general a component of $\mathscr{C}_k$ and a component of $\mathscr{C}_l$ for $l \neq k$ need not be disjoint.

Using this description, it is possible to compute the height of the RPNG process at any edge $e$ and (almost) any time $t$: Indeed, it is equal to the sum of the number of level lines that separate $(e,t)$ from the line $t = 0$ (*i.e.* that pass below it an odd number of times), plus the number of loops containing $(e,t)$. This means that the law a of large numbers for $h_t$ can be translated into a statement about the asymptotic density of the level lines.

### 4.3 The asymptotic speed

The first important remark to make in order to get a law of large numbers is that the height of the RPNG at a given edge and a given time only depends on the marks present in a particular finite region, which we now construct. Let $e$ be an edge and let $t > 0$, we say that $(x,s)$ is in the *past cone* $V_{e,t}$ of $(e,t)$ if there is a sequence of marks $(e_n,t_n)_{1\leqslant n\leqslant N}$ (where $N \geqslant 0$) in the union of the three Poisson systems $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{N}$ previously defined, such that $x$ is one of the endpoints of $e_1$, $e_n$ is adjacent to $e_{n+1}$ for every $n \in [\![1,N]\!]$, $e_{N+1} = e$, and $s \leqslant t_1 < \cdots < t_{N+1} \leqslant t$.

Then, $h_t(e)$ only depends on marks with both endpoints in the cone $V_{e,t}$, which in particular means that it depends only on finitely many marks (notice that $V_{e,t}$ itself can be constructed by looking at finitely many marks).

**Proposition 2** *Recall that $e_0$ is the edge at he origin $< 0,1 >$. For each $\lambda > 0$, there exists $v(\lambda) > 0$ such that, almost surely,*

$$\frac{h_t(e_0)}{t} \to v(\lambda).$$

*Proof* The proof is based on a sub-additivity argument. Let $t_1$, $t_2$ be two large times (*i.e.*, larger than a constant to be determined later). As we saw, the height $h_{t_1+t_2}(e_0)$ depends only on what happens within the cone $V_{e_0,t_1+t_2}$.

First look at what happens inside the cone $V_{e_0,t_1-\sqrt{t_1}}$. At time $t_1-\sqrt{t_1}$, there are $h_{t_1-\sqrt{t_1}}(e_0)$ layers in the RPNG process at $e_0$. This height is equal to the number of pairs of particle paths exiting the cone from opposite sides (the $+$-particle exiting at position 1 or to the right of 1, and the $-$-particle at position 0 to the left of 0). We know in advance that this number is at most linear in $t_1$, but we will not use this fact.

By a previous estimate, any one of those lines has a positive probability, uniformly bounded below by $p > 0$ (as in equation 6), not to annihilate before time $t_1+\sqrt{t_2}$. Besides, survival of one of them ensures the survival of all the older (*i.e.* lower) ones, so that the number of annihilating lines is stochastically dominated by a geometric distribution of parameter $p > 0$.

Lemma 1 ensures that a surviving plateau is very likely to have its width increase at least exponentially fast, so similarly the number $Z$ of lines created in $V_{e_0,t_1-\sqrt{t_1}}$ and which either vanish before time $t_1+\sqrt{t_2}$, or survive up to this time but do not reach width $c_5 e^{c_6(\sqrt{t_1}+\sqrt{t_2})}$ is stochastically dominated by a geometric variable of parameter $p'$ for some $p' > 0$ independent of $t_1$ and $t_2$.

Let $N_1$ be the number of pairs of lines created in $V_{e_0,t_1-\sqrt{t_1}}$, $N_1'$ be the number of those pairs that exit $V_{e_0,t_1+t_2}$ from opposite sides before time $t_1+\sqrt{t_2}$, and let $N_2$ be the number of pairs of lines created in the cone of depth $t_2-\sqrt{t_2}$ with tip at $(e_0,t_1+t_2)$ and exiting that cone from opposite sides. By the monotonicity of the dynamics in terms of the height function, together with the definition of $V_{t_1+t_2}$, we almost surely have

$$h_{t_1+t_2}(e_0) \geqslant N_1' + N_2.$$

Let $(H_t)$ and $(H_t')$ be two independent copies of the process $(h_t(e_0))$: the previous construction leads to the following stochastic domination statement:

$$H_{t_1+t_2} \succcurlyeq H_{t_1-\sqrt{t_1}} + H'_{t_2-\sqrt{t_2}} - \mathscr{G}(p') \tag{10}$$

where the notation $\mathscr{G}(p')$ stands for an independent geometric variable with parameter $p'$. Use the fact that $H_t$ grows at most linearly in $t$ to obtain

$$H_{t_1+t_2} \succcurlyeq H_{t_1} + H'_{t_2} - \mathscr{G}(p') - C\left[\sqrt{t_1}+\sqrt{t_2}\right].$$

This super-convolutive relation readily implies a strong law of large numbers (see [3]).

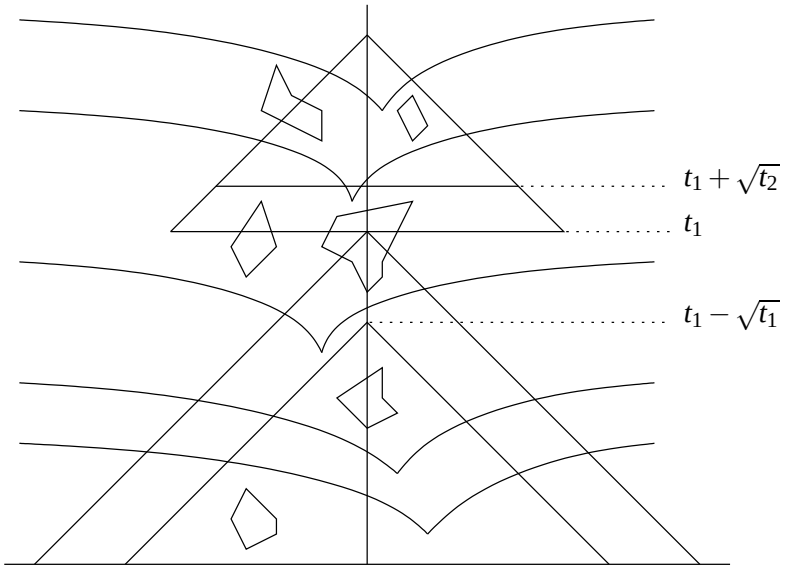


**Fig. 2** The interaction between the two cones in the proof of the proposition

## 5 The half-space system

### 5.1 Description of the simplified model

We first consider a simplified model, where we can study the effect of the slow-down present in the real model when several particles occupy the same site. The initial configuration is empty; at each occurrence of a Poisson process in $\mathbb{R}_+$ of intensity $\lambda$, a particle is created at site 1 — and this is the only source of particles in the system. Particles move according to the same mechanism as described in the basic homogeneous model, and are annihilated upon hitting the site 0. Note that this implies that the negative half-space remains empty for all time.

Analogous to the question treated in Section 4 we consider the validity of the law of large numbers for the total number $N_t$ of particles present at time $t$.

**Theorem 2** *There exists $\lambda_c^{(1)} \in (0, +\infty)$ such that:*

1. *If $\lambda < \lambda_c^{(1)}$, then $N_t/t$ tends to zero almost surely, as $t \to +\infty$, and the law of the particle system converges to a measure under which the expected number of particles at any site is finite;*
2. *If $\lambda > \lambda_c^{(1)}$, then there exists $v^{(1)}(\lambda) \in (0, +\infty)$ so that $N_t/t \to v^{(1)}(\lambda)$ almost surely, as $t \to +\infty$, and the number of particles at site 1 at time $t$ tends to $+\infty$ linearly in $t$ almost surely;*
3. *If $\lambda = \lambda_c^{(1)}$, then $N_t/t$ tends to zero almost surely, as $t \to +\infty$, but the expected number of particles at any site goes to infinity with time.*

We will give two independent proofs of this theorem, corresponding to two distinct interpretations of the behavior of the system. The first one uses a descrip-

tion of the dynamics in terms of a zero-range process; it is likely that it already appeared in some form in the literature, but we still give it for sake of completeness. The second proof, which uses the interpretation of the paths of the individual particles as interacting random walks, provides a better understanding of the nature of the phase transition.

### 5.2 Interpretation in terms of a zero-range process

We start with the study of the system for a low rate of particle creation, namely with $\lambda < 1/2$. For any $\rho \in (0,1)$, let $\mu_\rho$ be the measure on particle configurations on $\mathbb{Z}_+$, defined as follows: $\mu_\rho$-almost surely there is no particle at site 0, and the numbers of particles at the sites $i > 0$ are i.i.d. geometric random variables of parameter $\rho$. In particular, the probability that site $i$ is non-empty is equal to $\rho$.

Notice that the particle system in this case is a zero-range process on $\mathbb{Z}_+$ with a particular boundary condition: Indeed, from each non-empty site $i > 0$, with rate 1 one particle jumps, with equal probability to either neighboring site. The boundary condition at the origin is the combination of a source and a sink: A particle jumping from site 1 to the left disappears, and besides there is a rate $\lambda$ of particle creation at site 1.

It is easy to see that for the particular choice $\rho = 2\lambda$, the measure $\mu_\rho$ is stationary: Indeed, products of geometric distributions are always stationary for the zero-range process, all that needs to be checked is the behavior at the boundary. Under $\mu_\rho$, site 1 loses particles with rate $\rho$ (*i.e.* with rate 1 when it is non-empty), and gains particles with rate $\lambda + \rho/2$ (from the source and from site 2). So, if $\rho = 2\lambda$, $\mu_\rho$ is stationary, and if $\rho < 2\lambda$, the occupation probability of site 1 increases with time.

Since the dynamics is attractive, we have convergence starting from the empty configuration, and by what precedes, the limit must be $\mu_{2\lambda}$. Moreover, if $p_t(i)$ denotes the probability that site $i$ is occupied at time $t$, then $p_t(i)$ increases to $2\lambda$ as $t \to +\infty$, and besides the expected number $N_T$ of particles entering the system between times 0 and $T$ (counted algebraically, *i.e.* counting the particles that leave the system negatively) satisfies:

$$E(N_T) = \int_0^T \left(\lambda - \frac{p_t(1)}{2}\right) \mathrm{d}t = o(T).$$

Now, assume that $\lambda > 1/2$: From what precedes, $p_t(i)$ is still non-decreasing as a function of $t$, and it is bounded by 1, so that it converges. But again by monotonicity, the limit has to be at least equal to $2\lambda'$ for any $\lambda' < 1/2$. Hence, we get that

$$\forall i > 0, \quad p_t(i) \to 1.$$

Then, by the same reasoning as previously, we see that particles enter the system at rate $\lambda$ and leave it at a rate that converges to $1/2$, so that with probability 1, as $t \to +\infty$,

$$\frac{N_t}{t} \to \lambda - \frac{1}{2} > 0.$$

The last thing we need to prove is that, for $\lambda > \lambda_c^{(1)}$, particles accumulate at site 1. Once again, one sees that at time $t$, particles arrive at site 1 with rate $\lambda + p_t(2)/2$, and leave with rate $p_t(1)$. Since both $p_t(1)$ and $p_t(2)$ converge to 1, one gets that the number of particles at site 1 grows linearly at the same rate as $N_t$.

Comparison with the cases $\lambda < \lambda_c^{(1)}$ and $\lambda > \lambda_c^{(1)}$ directly tells us that, at criticality, $N_t/t \to 0$ almost surely, while the expected number of particles at site $i$ and time $t$ is bounded below by that of the system for any $\lambda < \lambda_c^{(1)}$, which converges to $1/(1-2\lambda)$. In other words, the expected number of particles at site $i$ goes to infinity with time.

The construction also tells us that, still for $\lambda > \lambda_c^{(1)}$, almost surely there is a (random) time $\tau$ after which site 1 will always be occupied. This means exactly that, seen from the sites $i \geqslant 2$, it will act like a source of rate $1/2$, so the restriction of the system to $\{2,3,\ldots\}$ behaves asymptotically the same way as the whole system taken at $\lambda = \lambda_c^{(1)}$. In particular, the number of particles that are present in $\{2,3,\ldots\}$ grows sub-linearly (which we knew already by comparison of $N_t$ and the number of particles at site 1).

### 5.3 Interpretation in terms of random walks

Here we will present an alternative proof of Theorem 2 in terms of the interacting random walks described in the beginning of this Section. As a warm-up, let us prove that there is indeed linear accumulation of particles at site 1 if $\lambda > 1$. At any given time $t$, there is a certain number of particles, and each of them has to wait for some time, as given by its project (but only one of them is waiting, the others are sleeping). Let $L_t$ be the sum of these waiting times. Then by our rules, up to time $t + L_t$ at least one of these particles will still be at site 1.

Between times $t$ and $t+h$, a few things can happen. If there is no new particle born or arriving, then $L_{t+h} = L_t - h$; if on the other hand a new particle is born (which happens with probability of order $\lambda h$ if $h$ is small, and if $\tau$ is the first waiting time of its project, then the new value of $L$ is at least equal to $L_t - h + \tau$. If a particle is arriving at site 1 (from site 2), again $L$ is increased by an $\mathrm{Exp}(1)$ random variable, but we omit this for now.

In other words, $L_t$ stochastically dominates a process in $\mathbb{R}_+$ with constant drift $-1$ and rate $\lambda$ to make jumps to the right that are distributed as i.i.d. $\mathrm{Exp}(1)$ random variables. If $\lambda$ is larger than 1, then this process has a positive probability to never return to 0 after its first jump. This means that there will always be particles at site 1 after some (random) finite time.

Now, we can obtain linear growth as follows: Split the incoming particles into two Poisson processes, one with intensity $1+\varepsilon$ where $\varepsilon$ is chosen small enough that $1+\varepsilon < \lambda$, and an independent one with intensity $\lambda - 1 - \varepsilon > 0$. Chose a jumping rule such that particles of the second process have lower priority than those of the first one. Then, by the previous reasoning there is a random finite time after which there will always be at least one particle of the first process at site 1; and after this time, every particle of the second process born at 1 will never move.

In particular, letting $\varepsilon$ go to 0, we see that the number of particles at 1 will grow linearly with time, with speed at least $\lambda - 1$. Of course this is far from optimal, as was seen in the previous subsection.

We now address the case when $\lambda$ is smaller than $\lambda_c^{(1)}$. The general idea is the following: It is well known that if $X_t$ is a continuous time random walk started at site 1 and killed at the first time it hits 0, then for each $k \geqslant 1$ the expected time $X_t$ spends at site $k$ is equal to 2. Let $\eta_t(k)$ be 1 if there is at least one particle at site $k$ at time $t$, and 0 in the opposite case. Let $C_t$ be the total number of particles created by time $t$. By the previous construction, we have for every $k$:

$$\int_0^t \eta_s(k)\mathrm{d}s \leqslant \sum_{i=1}^{C_t} \tau_i(k), \tag{11}$$

where $\tau_i(k)$ is the total time spent at site $k$ by the project of the $i$-th walk. Since $E(\tau_i(k)) = 2$, we immediately get that

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t \eta_s(k)\mathrm{d}s \leqslant 2\lambda.$$

What we want to quantify is the following remark: If $\lambda < 1/2$, the *pedestals* created by the particles when they interact do not cover the whole time line at any level (because their density is smaller than 1), meaning that any added particle, even with very low priority, will have a chance to jump in finite time, and its actual path will look like a time-stretched random walk. Then by comparison with stick-percolation, we automatically obtain a sub-linear growth.

To make these remarks precise, we need some quantitative estimates on the distribution of the lengths of pedestals at different levels. First of all, consider what happens at level 1. Particles are created with rate $\lambda$, immigrate from site 2 at some of the times of a Poisson process of intensity $1/2$ (*i.e.* whenever there is at least one particle at site 2, one of these jumps to 1 with rate $1/2$); and when site 1 is occupied, it loses particles with rate 1, either to the boundary or to site 2. Let $T$ be any (possibly random) stopping time at which site 1 is occupied. As long as it remains so, the number of particles at 1 behaves like a jump process, jumping by $-1$ with rate 1 and by $+1$ with rate at most $\lambda + 1/2 < 1$. Hence, the drift is negative, and if we denote by $T + \tau$ the first time after $T$ when site 1 becomes empty, we have an exponential estimate of the form

$$\forall t \in \mathbb{R}_+, \quad P(\tau > t) \leqslant M_1 e^{-m_1 t}, \tag{12}$$

where $M_1$ and $m_1$ are positive constants depending on the value of $\lambda$. Let $\lambda' = \frac{1}{2}(\lambda + 1/2)$: then by Cramér's theorem, for every $\lambda_1 \in (\lambda', 1/2)$, we get the following estimate, for positive $M_1'$ and $m_1'$ depending on $\lambda$ and $\lambda_1$:

$$\forall T_1 < T_2, \quad P\left(\int_{T_1}^{T_2} \eta_s(1)\mathrm{d}s > 2\lambda_1(T_2 - T_1)\right) \leqslant M_1' e^{-m_1' t}. \tag{13}$$

(Notice that if we restricted ourselves to $T_1 = 0$, then this kind of estimate would hold all the way down to $\lambda_1 > \lambda$, because the integral on the left is then bounded above by the sum of the times particles spend at site 1.)

Now consider the next level. Particles immigrate to site 2 from site 1 with intensity at most $\lambda_1 < 1/2$, and from site 3 with intensity at most $1/2$. Using the previous estimates, but letting $\tau$ be the survival time of a pedestal at site 2, we obtain

$$\forall t \in \mathbb{R}_+, \quad P(\tau > t) \leqslant M_2 e^{-m_2 t}, \tag{14}$$

where $M_2$ and $m_2$ depend on $\lambda$ and $\lambda_1$. Then one can chose $\lambda_2 \in (\lambda_1', 1/2)$, and continue similarly for each successive level.

Let $K$ be a positive integer. We split the particles into two classes, those that remain within $\{1,\dots,K\}$ at all times (which we call *shallow* particles), and those that visit site $K+1$ at least once (which we call *deep* particles). Fix a priority rule such that deep particles have lower priority than shallow particles. So, we first consider the shallow particles; let $N_t^K$ be the number of shallow particles that are present at time $t$. Assume that it grows linearly, *i.e.* that there exist two (random) real numbers $a > 0$ and $b$ such that for all $t > 0$, $N_t^K > at + b$. For ease of notation, let us assume that $b = 0$ — up to a time shift, it does not reduce the generality of the argument.

Then, for each $t > 0$, there is an integer $k_t \in \{1,\dots,K\}$ such that at time $t$, site $k_t$ contains at least $at/K$ particles. The survival time of the corresponding pedestal is then bounded below by the sum of $\lfloor at/K \rfloor$ exponential variables of parameter 1 — indeed, before the site becomes empty again, all these particles need to jump out, which they do with rate 1. So we get a lower bound for the survival time which is of order $t$. By the previous computations and the Borel-Cantelli lemma, this can only happen finitely many times, hence the number of shallow particles cannot grow linearly in time.

The last remark is the following: The number of particles that are present in the system at time $t$ is equal to the sum of the number of shallow particles, of which there are sub-linearly many, and the number of deep particles. But a given particle is shallow with probability exactly equal to $1 - 1/(K+1)$, and deep with probability $1/(K+1)$, in other words the number of deep particles cannot grow faster than $\lambda t/(K+1)$, and we obtain the following upper bound: Almost surely,

$$\limsup_{t\to+\infty} \frac{N_t}{t} \leqslant \frac{\lambda}{K+1}.$$

Since this holds for every $K > 0$, we can conclude that the total number of particles indeed grows sub-linearly.

## 6 The full-space system

### 6.1 Without bulk nucleation

We begin the study of the full-space system with the case where the bulk nucleation rate is equal to 0. The dynamics are the following: At time 0, the system is empty; at the occurrences of a Poisson process of intensity $\lambda$, a particle is created at site 1 and an anti-particle is created at site 0. The jump mechanism is the same as previously: Whenever there are $k$ particles at a given site, each of them jumps away symmetrically with rate $1/k$; or equivalently, one of them, chosen according

to some priority rule, jumps away symmetrically with rate 1. Last, whenever a particle and an anti-particle occupy the same site (be it because one was just created or through a jump), they annihilate instantly, so that cohabitation of particles of different types at any site is impossible.

As previously, let $N_t$ be the number of particles alive at time $t$ — note that this is also the number of anti-particles present in the system, since they are created and annihilated in pairs.

**Theorem 3** *There exists $\lambda_c(0) \in (0,+\infty)$ such that:*

1. *If $\lambda_0 < \lambda_c(0)$, then $N_t/t$ tends to zero almost surely as $t \to +\infty$;*
2. *If $\lambda_0 > \lambda_c(0)$, then there exists $v(0,\lambda_0) \in (0,+\infty)$ so that $N_t/t \to v(0,\lambda_0)$ almost surely as $t \to +\infty$; moreover, the number of particles at site 1 at time $t$ also tends to $+\infty$ linearly with the same speed.*

We first give a general stochastic domination lemma. For every $x \in \mathbb{Z}$, let $n_x^+(t)$ be the number of particles present at site $x$ at time $t$, $n_x^-(t)$ be the number of anti-particles present at site $x$ at time $t$, and let $n_x(t) := n_x^+(t) - n_x^-(t)$.

**Lemma 2** *There exists a countable family $(X_\cdot^x)_{x\in\mathbb{Z}}$ of continuous-time random walks on $\mathbb{Z}_+$, with jump rate 1, and a coupling of the $(X^x)$ with the previously described particle system, satisfying the following condition with probability 1:*

$$\forall x \in \mathbb{Z}\setminus\{0,1\}, \quad \forall t > 0, \quad |n_x(t)| \leqslant X_t^x.$$

*Proof* Let $x \in \mathbb{Z}$, and in the graphical construction of the process (cf. Section 3), let $(T_{x,k})_{k>0}$ (resp. $(T'_{x,k})$) be the $k$-th time at which there is an arrow pointing toward (resp. away from) site $x$. Define $X^x$ as follows: $X_0^x = 0$, $X^x$ jumps up by 1 at each of the $T_{T_{x,k}}$, and jumps down by 1 at each $t \in \{T'_{x,k}\}$ such that $X_{t-}^x > 0$. Clearly, $X^x$ behaves like a continuous-time random walk whenever it is positive; and it can only jump up when it is equal to 0.

Now assume $x \notin \{0,1\}$. This means that $n_x$ can only change when a particle at $x$ jumps away (in which case $|n_x|$ decreases by 1), or when a particle at $x \pm 1$ jumps onto $x$ (in which case, depending on the types of particles involved, $|n_x|$ either increases on decreases by 1). $|n_x|$ remains constant otherwise. In other words, whenever $|n_x|$ jumps up, so does $X^x$, and whenever $X^x$ jumps down, so does $|n_x|$ — provided it is positive. It is then easy to verify that for each $t > 0$, $|n_x(t)| \leqslant X_t^x$.

Notice that the $X^x$ are not independent from each other, since $X^x$ and $X^{x+1}$ (say) share some of their jump times (however, $X^x$ and $X^y$ are independent as soon as $|y-x| \geqslant 2$). Nevertheless, the existence of the $(X^x)$ is sufficient to obtain the following fact: With probability 1,

$$\forall x \in \mathbb{Z}\setminus\{0,1\}, \quad \lim_{t\to+\infty} \frac{n_x(t)}{t} = 0. \tag{15}$$

In other words, the number of particles at a site which is not a source can never grow linearly. A slight generalization of this fact is the following:

**Lemma 3** *For each $x > 1$, the expected total number of particles to the right of $x$ grows sub-linearly; equivalently, as a consequence of the previous Lemma, with probability* 1

$$\frac{1}{t} E\left[\sum_{k=2}^{+\infty} |n_k(t)|\right] \to 0.$$

*Proof* Let $p_x(t)$ be the probability that site $x$ is non-empty at time $t$. Imagine for a moment that there were not anti-particles in the system; then, clearly we would have

$$\partial_t E\left[\sum_{k=x+1}^{+\infty} n_k(t)\right] = \frac{p_x(t) - p_{x+1}(t)}{2}.$$

Indeed, particles move into the half-line to the right of $x$ with rate $1/2$ whenever site $x$ is non-empty, and they exit with the same rate whenever site $x+1$ is non-empty. Taking the existence of anti-particles into account (which can only decrease the total number of particles to the right of $x$ through annihilation) and integrating over $t$, one gets:

$$E\left[\sum_{k=x+1}^{+\infty} |n_k(t)|\right] \leqslant \int_0^t \frac{p_x(s) - p_{x+1}(s)}{2}\,\mathrm{d}t. \tag{16}$$

Now, let

$$\varphi := \limsup_{t\to+\infty} \frac{1}{t} E\left[\sum_{k=2}^{+\infty} |n_x(t)|\right].$$

As a consequence of Lemma 2, the same holds when summing over any half-line $\{x, x+1, \ldots\}$. Besides, for every $K \in \mathbb{Z}_+$,

$$\begin{aligned}\int_0^t \frac{p_2(s) - p_{2+K}(s)}{2}\,\mathrm{d}t &= \sum_{k=0}^{K-1} \int_0^t \frac{p_{2+k}(s) - p_{2+k+1}(s)}{2}\,\mathrm{d}t \\ &\geqslant \sum_{k=0}^{K-1} E\left[\sum_{x=2+k+1}^{+\infty} |n_x(t)|\right].\end{aligned}$$

Letting $t$ go to infinity, one gets

$$\limsup_{t\to+\infty} \frac{1}{t} \int_0^t \frac{p_2(s) - p_{2+K}(s)}{2}\,\mathrm{d}t \geqslant K\varphi.$$

On the other hand, the $p_k(t)$ are bounded by 1, so we obtain $2K\varphi \leqslant 1$. Since this holds for all $K$, it implies that $\varphi = 0$, which concludes the proof.

We are now ready to prove Theorem 3. We already know from previous considerations that if $\lambda$ is large enough (compared to the jump rate), then particles accumulate at site 1 and anti-particles at site 0 — if (say) $\lambda > 2$ it works very easily. Conversely, assume now that $\lambda$ is chosen such that there is linear growth of the total number of particles in the system, *i.e.* that there exists $v > 0$ such that,

for $t$ large enough, $E[N(t)] \geqslant vt$. From what precedes, this implies that particles accumulate linearly at site 1, and anti-particles accumulate linearly at site 0.

In particular, after some random time sites 0 and 1 remain constantly occupied. The number of particles at site 1 can then change for several reasons:

- Particles can jump from 1 to one of its neighbors. This happens with rate 1, and decreases $n_1$ by 1;
- Anti-particles can jump from 0 to 1. This happens with rate $1/2$ (remember that we are in a case when site 0 is never empty), and decreases $n_1$ by 1;
- Particles or anti-particles can jump from site 2 to site 1. This happens with rate at most $1/2$, and can increase or decrease $n_1$ by 1;
- A nucleation can happen on $<0,1>$. This happens with rate $\lambda$, and increases $n_1$ by 1.

Taking all these influences into account, we find that the rate of change of $n_1$ is bounded above by $-1-1/2+1/2+\lambda = \lambda - 1$. In particular, if $\lambda < 1$, comparison with a random walk with negative drift ensures that site 1 will empty eventually, which contradicts our assumption. In other words, we obtain $\lambda_c(0) \geqslant 1 > 0$, as claimed.

The process can then be compared to the half-space version of the previous section: Once we know that site 1 never empties, then it behaves like a source of particles with critical intensity, and in particular the occupation probability of site 2 increases to 1. Re-injecting this into the above accounting of the variations of $n_1$, the third source (particles jumping from 2 to 1) has rate approaching $1/2$, meaning that the upper bound $\lambda - 1$ is sharp as $t \to +\infty$. So, if $\lambda > 1$, we do have accumulation of particles at site 1, which allows us to conclude that indeed $\lambda_c(0) = 1$.

### 6.2 With bulk nucleation

We are now ready to put all the pieces of the construction together. Start with a full-space RPNG as described in Section3; let $h_t(x)$ be its height function, *i.e.* the height at time $t$ of the highest layer covering the edge $<x,x+1>$. Notice that the whole dynamics can be described in terms of this function; in particular, with the previous notation we have $n_x(t) = h_t(x-1) - h_t(x)$.

Now add an extra rate $\lambda_0$ of nucleation on the edge $e_0 =< 0,1 >$, and let $h'_t$ be the height function of the modified process. By monotonicity of the dynamics, it is obvious that $h'_t \geqslant h_t$; let $\tilde{h}_t(x) = h'_t(x) - h_t(x) \geqslant 0$. We are interested in differences in the law of large numbers between the two versions of the process, and this can be interpreted as asymptotic properties of $\tilde{h}_t(0)/t$.

We introduce an auxiliary particle system, composed of what we call *virtual particles*, defined as follows: Let $\tilde{n}_x(t)\tilde{h}_t(x-1) - \tilde{h}_t(x)$; if $\tilde{n}_x(t)$ is positive (resp. negative), we say that at time $t$ there are $|\tilde{n}_x(t)|$ virtual particles (resp. virtual anti-particles) at site $x$. A good way of thinking of this process is as the symmetric difference between the two original ones.

The dynamics of the virtual particles is easy to describe at this point: They use the same arrows as in the graphical construction of the original process, except that the presence of a true particle inhibits the jumps of virtual particles at the

same site (equivalently, a virtual particle only sees arrows that are "uncovered" by the original process); and they annihilate with virtual anti-particles, as expected.

We want to apply the same strategy to this process as we did in the previous subsection, in particular we will need equivalents to lemmas 2 and 3.

**Lemma 4** *With probability* 1, *for every* $x \in \mathbb{Z} \setminus \{0,1\}$, $\tilde{n}_x(t) = o(t)$.

*Proof* From the previous subsection, we know that for every $x \in \mathbb{Z} \setminus \{0,1\}$, $n_x(t)$ is at most of order $t^{1/2}$. Assume that for some $x_0 \in \mathbb{Z} \setminus \{0,1\}$,

$$\psi := \limsup_{t \to +\infty} \frac{\tilde{n}_{x_0}(t)}{t} > 0.$$

Because $\tilde{n}$ is defined as a difference of height functions, we know that

$$\begin{aligned}\tilde{n}_x(t) - \tilde{n}_{x+1}(t) &= (\tilde{h}_t(x-1) - \tilde{h}_t(x)) - (\tilde{h}_t(x) - \tilde{h}_t(x+1)) \\ &= -(2h'_t(x) - h'_t(x+1) - h'_t(x-1)) \\ &\quad + (2h_t(x) - h_t(x+1) - h_t(x-1)) \\ &= (n'_x(t) - n'_{x+1}(t)) - (n_x(t) - n_{x+1}(t)),\end{aligned}$$

where as above $n'_x(t) := h'_t(x-1) - h'_t(x)$. This last expression is at most of order $t^{1/2}$, by Lemma 2.

Now let $(t_k)$ be a sequence of times increasing to $+\infty$, such that $\tilde{n}_{x_0}(t_k) \sim \psi t_k$. By the previous estimate, we obtain that for every $x \in \mathbb{Z} \setminus \{0,1\}$ of the same sign as $x_0$, we also have $\tilde{n}_x(t_k) \sim \psi t_k$. On the other hand, the total number of virtual particles is at most equivalent to $\lambda_0 t$, because they are created by the extra nucleations at the edge $< 0,1 >$. Hence $\psi = 0$.

**Lemma 5** *For each* $x > 1$, *the expected total number of virtual particles to the right of x grows sub-linearly; as a consequence of the previous lemma, this is equivalent to saying that with probability* 1,

$$\frac{1}{t} E\left[\sum_{k=2}^{+\infty} |\tilde{n}_k(t)|\right] \to 0.$$

*Proof* The proof is the same as that of Lemma 3, replacing $p_x(t)$ with

$$\tilde{p}_x(t) := P[\tilde{n}_x(t) > 0,\ n_x(t) = 0],$$

and of course making use of Lemma 4 instead of Lemma 2.

The end of the proof of Theorem 1 is the same as in the previous subsection: If there is a change in the law of large numbers when adding nucleation on $< 0,1 >$, then this implies that there is linear accumulation of virtual particles at 1 and virtual anti-particles at 0; and because the number of particles at these sites for the original process grows less than linearly (by Lemma 2), this implies accumulation of real particles in the modified process — because as we saw above, $\tilde{n}_x = n'_x - n_x$. In particular, after some random time sites 0 and 1 remain constantly occupied.

All that remains to be done is to account for all the sources of change of $n_1'$. In addition to the ones mentioned in the previous subsection, we need to take care of bulk nucleation at edges $<0,1>$ and $<1,2>$, but this adds a martingale term to $n_1'$, hence the end-result is the same: The drift of $n_1'$ is bounded above by $\lambda_0 - 1$, and in particular $\lambda_c(\lambda) \geqslant 1$.

One thing that is missing though in this case is a precise description of the half-space process, which is not the same as above because of bulk nucleation. In particular we cannot prove that the occupation probability of site 2 increases to 1, and thus are unable to determine the exact value of $\lambda_c(\lambda)$; the best we obtain here is $\lambda_c(\lambda) \in [1,2]$.

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