The Common Origin of Linear and Nonlinear Chiral Multiplets in N=4 Mechanics

François Delduc, E. Ivanov

The Common Origin of Linear and Nonlinear Chiral Multiplets in $\mathcal{N}{=}4$ Mechanics

F. Delduc [a]**, E. Ivanov** [b]**,**

a)*Laboratoire de Physique de l'ENS Lyon, CNRS UMR 5672,*
46, allée d'Italie, 69364 Lyon Cedex 07, France
`francois.delduc@ens-lyon.fr`

b)*Bogoliubov Laboratory of Theoretical Physics, JINR,*
141980 Dubna, Moscow region, Russia
`eivanov@theor.jinr.ru`

Abstract

Elaborating on previous work (`hep-th/0605211, 0611247`), we show how the linear and nonlinear chiral multiplets of $\mathcal{N}{=}4$ supersymmetric mechanics with the off-shell content **(2,4,2)** can be obtained by gauging three distinct two-parameter isometries of the "root" **(4,4,0)** multiplet actions. In particular, two different gauge groups, one abelian and one non-abelian, lead, albeit in a disguised form in the second case, to the same (unique) nonlinear chiral multiplet. This provides an evidence that no other nonlinear chiral $\mathcal{N}{=}4$ multiplets exist. General sigma model type actions are discussed, together with the restricted potential terms coming from the Fayet-Iliopoulos terms associated with abelian gauge superfields. As in our previous work, we use the manifestly supersymmetric language of $\mathcal{N}{=}4, d{=}1$ harmonic superspace. A novel point is the necessity to use in parallel the λ and τ gauge frames, with the "bridges" between these two frames playing a crucial role. It is the $\mathcal{N}{=}4$ harmonic analyticity which, though being non-manifest in the τ frame, gives rise to both linear and nonlinear chirality constraints.

1   Introduction

The one-dimensional supersymmetry and related models of supersymmetric (quantum) mechanics reveal many specific surprising features and, at the same time, have a lot of links with higher-dimensional theories of current interest (see e.g. [1, 2] and refs. therein). This motivates many research groups towards thorough study and further advancing of this subject (see e.g. [2, 3]).

Recently, we argued [4, 5] that the plethora of relationships between various d=1 supermultiplets with the same number of fermionic fields, but different divisions of the bosonic fields into physical and auxiliary ones (so called "d=1 automorphic dualities" [6]), can be adequately understood in the approach based on the gauging of isometries of the invariant actions of some basic ("root") multiplets by non-propagating ("topological") gauge multiplets (these isometries should commute with supersymmetry, i.e. be triholomorphic). The key merit of our approach is the possibility to study these relationships in a manifestly supersymmetric superfield manner, including the choice of supersymmetry-preserving gauges. Previous analysis [6, 7] was basically limited to the component level and used some "ad-hoc" substitutions of the auxiliary fields. In the framework of the gauging procedure, doing this way corresponds to making use of the Wess-Zumino-type gauges.

In [4, 5] we focused on the case of $\mathcal{N}$=4 supersymmetric mechanics and showed that the actions of the $\mathcal{N}$=4 multiplets with the off-shell contents $(\mathbf{3}, \mathbf{4}, \mathbf{1})$, $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ and $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ can be obtained by gauging certain isometries of the general actions of the "root" multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ in the $\mathcal{N}$=4, d=1 harmonic superspace. Based on this, we argued that the latter is the underlying superspace for all $\mathcal{N}$=4 mechanics models. Here we confirm this by studying, along the same line, the remaining $\mathcal{N}$=4 multiplets, the chiral and nonlinear chiral off-shell multiplets with the field content $(\mathbf{2}, \mathbf{4}, \mathbf{2})$. They prove to naturally arise as a result of gauging some two-parameter isometry groups admitting a realization on the harmonic analytic superfield q^{+a} which describes the multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$. The origin of the difference between these two versions of the $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ multiplet is attributed to the fact that they emerge from gauging two essentially different isometries: the linear multiplet is associated with gauging of some purely shift isometries, while the nonlinear one corresponds to gauging the product of two "rotational" isometries, viz. the target space rescalings and a $U(1)$ subgroup of the $SU(2)$ Pauli-Gürsey group. We also recover the same nonlinear version of this multiplet, though in disguise, by gauging a mixture of the rescaling and shift isometries. It is the last possible two-parameter symmetry implementable on q^{+a}. Thus we show that two known off-shell forms of the $\mathcal{N}$=4, d=1 multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$, the linear and nonlinear ones, in fact exhaust all possibilities.

It should be emphasized that the existence of non-linear cousins of the basic d=1 supermultiplets is one of the most amazing features of extended d=1 supersymmetry. In superspace they are described by superfields satisfying some nonlinear versions of the standard constraints (e.g. of chirality constraints). As a result, the realization of the corresponding off-shell d=1 supersymmetry on the component fields of such supermultiplets is intrinsically nonlinear. The list of such multiplets known to date includes the nonlinear analogs of the $\mathcal{N}$=4 multiplets $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ [8, 9, 10, 4, 11], $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ [12, 13], $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ [13], as well as of the $\mathcal{N}$=8 multiplets $(\mathbf{4}, \mathbf{8}, \mathbf{4})$ [14] and $(\mathbf{2}, \mathbf{8}, \mathbf{6})$ [15]. As shown in [4], within the approach based on the gauging procedure the difference between linear and nonlinear $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplets originates from the fact that the first multiplet is related to the gauging of the shift or rotational $U(1)$ symmetries, and the second one to the gauging of the target space rescalings. In the case of the $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ multiplets we

find an analogous intimate relation between the type of multiplet and the q^{+a} two-parameter symmetry group which has to be gauged to generate it.

2 Brief preliminaries

Throughout the paper we use the same $\mathcal{N}{=}4, d{=}1$ harmonic superspace (HSS) techniques, conventions and notation as in [12, 4, 5]. The $\mathcal{N}{=}4$ "root" multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ is described by the harmonic analytic superfield $q^{+a}(\zeta, u)$ which is subjected to the Grassmann harmonic and bosonic harmonic constraints

$$\text{(a)} \ D^+ q^{+a} = \bar{D}^+ q^{+a} = 0, \quad \text{(b)} \ D^{++} q^{+a} = 0. \tag{2.1}$$

Here (ζ, u) are coordinates of the harmonic analytic $\mathcal{N}{=}4$ superspace [12], $(\zeta, u) = (t_A, \theta^+, \bar{\theta}^+, u_i^\pm)$, $u^{+i} u_i^- = 1$, they are related to the standard $\mathcal{N}{=}4$ superspace (central basis) coordinates $z = (t, \theta_i, \bar{\theta}^i)$ as

$$t_A = t - i(\theta^+ \bar{\theta}^- + \theta^- \bar{\theta}^+), \quad \theta^\pm = \theta^i u_i^\pm, \quad \bar{\theta}^\pm = \bar{\theta}^i u_i^\pm. \tag{2.2}$$

Respectively, the $\mathcal{N}{=}4$ covariant spinor derivatives and their harmonic projections are defined by

$$D^i = \frac{\partial}{\partial \theta_i} + i \bar{\theta}^i \partial_t, \quad \bar{D}_i = \frac{\partial}{\partial \bar{\theta}^i} + i \theta_i \partial_t, \quad \overline{(D^i)} = -\bar{D}_i, \quad \{D^i, \bar{D}_k\} = 2i \, \delta_k^i \partial_t, \tag{2.3}$$

$$D^\pm = u_i^\pm D^i, \quad \bar{D}^\pm = u_i^\pm \bar{D}^i, \quad \{D^+, \bar{D}^-\} = -\{D^-, \bar{D}^+\} = 2i \, \partial_{t_A}. \tag{2.4}$$

In the analytic basis, the derivatives D^+ and $\bar{D}^+$ are short,

$$D^+ = \frac{\partial}{\partial \theta^-}, \quad \bar{D}^+ = -\frac{\partial}{\partial \bar{\theta}^-}, \tag{2.5}$$

so the conditions (2.1a) become the harmonic Grassmann Cauchy-Riemann conditions stating that q^{+a} does not depend on the coordinates $\theta^-, \bar{\theta}^-$ in this basis. The analyticity-preserving harmonic derivative D^{++} and its conjugate D^{--} in the analytic basis are given by

$$D^{++} = \partial^{++} - 2i\theta^+ \bar{\theta}^+ \partial_{t_A} + \theta^+ \frac{\partial}{\partial \theta^-} + \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^-},$$

$$D^{--} = \partial^{--} - 2i\theta^- \bar{\theta}^- \partial_{t_A} + \theta^- \frac{\partial}{\partial \theta^+} + \bar{\theta}^- \frac{\partial}{\partial \bar{\theta}^+}, \quad \partial^{\pm\pm} = u_i^\pm \frac{\partial}{\partial u_i^\mp}, \tag{2.6}$$

and are reduced to the pure harmonic partial derivatives $\partial^{\pm\pm}$ in the central basis. They satisfy the commutation relations

$$[D^{++}, D^{--}] = D^0, \quad [D^0, D^{\pm\pm}] = \pm 2D^{\pm\pm}, \tag{2.7}$$

where D^0 is the operator counting external harmonic $U(1)$ charges. In the analytic basis it is given by

$$D^0 = u_i^+ \frac{\partial}{\partial u_i^+} - u_i^- \frac{\partial}{\partial u_i^-} + \theta^+ \frac{\partial}{\partial \theta^+} + \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^+} - \theta^- \frac{\partial}{\partial \theta^-} - \bar{\theta}^- \frac{\partial}{\partial \bar{\theta}^-}, \tag{2.8}$$

while in the central basis it coincides with its pure harmonic part. On the extra doublet index a of the superfield q^{+a} the so-called Pauli-Gürsey group $SU(2)_{PG}$ is realized. It commutes with

the $\mathcal{N}=4$ supersymmetry generators, as distinct from the R-symmetry $SU(2)_R$ group which acts on the doublet indices i, k of the Grassmann and harmonic coordinates, spinor derivatives and $\mathcal{N}=4$ supercharges.

The free action of q^{+a} can be written either in the analytic, or the central superspace

$$S_q^{\text{free}} = -\frac{1}{4} \int \mu_H \left(q^{+a} q_a^- \right) = \frac{i}{2} \int \mu_A^{(-2)} \left(q^{+a} \partial_t q_a^+ \right), \tag{2.9}$$

where $q^{-a} \equiv D^{--}q^{+a}$ and the integration measures are defined as

$$\mu_H = dudt d^4\theta = dudt_A (D^- \bar{D}^-)(D^+ \bar{D}^+) = \mu_A^{(-2)}(D^+ \bar{D}^+),$$
$$\mu_A^{(-2)} = dud\zeta^{(-2)} = dudt_A d\theta^+ d\bar{\theta}^+ = dudt_A (D^- \bar{D}^-). \tag{2.10}$$

The general sigma model-type action of q^{+a} (with a non-trivial bosonic target space metric) is given by

$$S_q = \int \mu_H \, \mathcal{L}(q^{+a}, q^{-b}, u^\pm). \tag{2.11}$$

The constraint (2.1) possesses a seven-parameter group of rigid symmetries commuting with supersymmetry (it includes $SU(2)_{PG}$ as a subgroup) [5]. One can single out the appropriate subclasses of the general action (2.11) (including the free action (2.9)) which are invariant with respect to one or another symmetry of this sort. For further use we give here the full list of non-equivalent two-parameter symmetries.

Abelian symmetries

$$\text{(a) } \delta q^{+a} = \lambda_1 \, u^{+a} + \lambda_2 \, c^{(ab)} u_b^+; \quad \text{(b) } \delta q^{+a} = \lambda_1 \, q^{+a} - \lambda_2 \, c^{(ab)} q_b^+. \tag{2.12}$$

Nonabelian symmetry

$$\delta q^{+a} = \lambda_1 \, q^{+a} + \lambda_2 \, u^{+a}. \tag{2.13}$$

Here the constant triplet c^{ab} is normalized as [1]

$$c^2 = c^{ab} c_{ab} = 2. \tag{2.14}$$

The algebra of the transformations (2.13) provides an example of two-generator solvable algebra. All other possible two-parameter symmetry groups listed in [5] can be reduced to (2.12), (2.13) by a redefinition of q^{+a}.

In what follows we shall gauge these symmetries and show that this gauging gives rise to three versions of the $\mathcal{N}=4$ multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$, with the corresponding general actions arising from the appropriate invariant subclasses of the general q^{+a} action (2.11). It turns out that the standard linear chiral $\mathcal{N}=4$ multiplet emerges as the result of gauging purely shift isometry (2.12a) while the two alternative gaugings give rise to two nonlinear versions of this multiplet. The nonlinear multiplet obtained from gauging the group (2.12b) is identical to the one discovered in [13]. The multiplet obtained from gauging (2.13), although looking different, can be identified with the previous one after suitable redefinitions.

---

[1] We use the same notation for the unrelated constant triplets in (2.12a) and (2.12b), hoping that this will not give rise to any confusion.

3   Chiral multiplet

Our gauging prescriptions are basically the same as in other cases [4, 5].

We start with the gauged version of the transformations (2.12a)

$$\delta q^{+a} = \Lambda_1\, u^{+a} + \Lambda_2 c^{ab}\, u_b^+ \,, \quad c^{ab} = c^{ba} \,, \tag{3.1}$$

where Λ_1 and Λ_2 are now charge-zero analytic superfields. The gauge covariantization of the q^{+a} harmonic constraint (2.1b) is given by

$$D^{++}q^{+a} - V_1^{++}u^{+a} - V_2^{++}c^{ab}\, u_b^+ = 0 \,, \tag{3.2}$$

where V_1^{++} and V_2^{++} are analytic gauge superfield transforming as

$$\delta V_1^{++} = D^{++}\Lambda_1 \,, \quad \delta V_2^{++} = D^{++}\Lambda_2 \,. \tag{3.3}$$

In the cases considered here (as distinct from the cases treated in [4, 5]), it is convenient to make use of the "bridge" representation of the gauge superfields [16, 17].

The analytic superfields V_1^{++} and V_2^{++} may be represented as

$$V_1^{++} = D^{++}v_1 \,, \quad V_2^{++} = D^{++}v_2 \,, \tag{3.4}$$

where $v_1(t,\theta,\bar\theta,u)$ and $v_2(t,\theta,\bar\theta,u)$ are non-analytic harmonic superfields which may be interpreted as "bridges" between the analytic and central superspace gauge groups (called the λ and τ gauge groups, see [17]). They transform under local shifts as

$$\delta v_1 = \Lambda_1(\zeta,u) - \tau_1(t,\theta,\bar\theta) \,, \quad \delta v_2 = \Lambda_2(\zeta,u) - \tau_2(t,\theta,\bar\theta) \,, \tag{3.5}$$

where $\tau_1(t,\theta,\bar\theta)$ and $\tau_2(t,\theta,\bar\theta)$ are non-analytic gauge superparameters bearing no dependence on the harmonic variables

$$D^{++}\tau_1(t,\theta,\bar\theta) = 0 \,, \quad D^{++}\tau_2(t,\theta,\bar\theta) = 0 \,. \tag{3.6}$$

We now define the non-analytic doublet superfield Q^{+a} by

$$Q^{+a} = q^{+a} - v_1 u^{+a} - v_2 c^{ab}\, u_b^+ \,. \tag{3.7}$$

As a consequence of (3.2), (3.4), it satisfies the simple harmonic constraint

$$D^{++}Q^{+a} = 0 \,, \tag{3.8}$$

which implies

$$Q^{+a}(t,\theta,\bar\theta,u) = Q^{ba}(t,\theta,\bar\theta)u_b^+ \,, \quad \overline{Q^{ba}} = Q_{ba} \,, \tag{3.9}$$

where the superfields $Q^{ba}(t,\theta,\bar\theta)$ are independent of the harmonic variables and form a real quartet. We also have

$$Q^{-a} \equiv D^{--}Q^{+a} = Q^{ba}(t,\theta,\bar\theta)u_b^- = q^{-a} - v_1 u^{-a} - v_2 c^{ab}u_b^- \,, \tag{3.10}$$

where

$$q^{-a} \equiv D^{--}q^{+a} - V_1^{--}u^{-a} - V_2^{--}c^{ab}u_b^- \tag{3.11}$$

and

$$V_1^{--} = D^{--}v_1, \quad V_2^{--} = D^{--}v_2, \quad \delta V_1^{--} = D^{--}\Lambda_1, \quad \delta V_2^{--} = D^{--}\Lambda_2. \tag{3.12}$$

From (3.10) and (3.12) one can find the gauge transformation law of the non-analytic harmonic superfield q^{-a}:

$$\delta q^{-a} = \Lambda_1 u^{-a} + \Lambda_2 c^{ab} u_b^-. \tag{3.13}$$

Now it is easy to determine how the superfields $Q^{\pm a}$ introduced in (3.7), (3.10) transform under the local shift symmetries. They are inert under the λ gauge transformations and have the following τ gauge transformation law

$$\delta Q^{\pm a} = \tau_1 u^{\pm a} + \tau_2 c^{ab} u_b^\pm \quad \Leftrightarrow \quad \delta Q^{ba} = \tau_2 c^{ab} - \tau_1 \epsilon^{ba}. \tag{3.14}$$

As a consequence of the Grassmann analyticity constraints (2.1), the superfield Q^{+a} satisfies the following fermionic constraints

$$\begin{aligned} D^+ q^{+a} = 0 &\Leftrightarrow D^+ Q^{+a} + (D^+ v_1)u^{+a} + (D^+ v_2)c^{ab} u_b^+ = 0, \\ \bar{D}^+ q^{+a} = 0 &\Leftrightarrow \bar{D}^+ Q^{+a} + (\bar{D}^+ v_1)u^{+a} + (\bar{D}^+ v_2)c^{ab} u_b^+ = 0. \end{aligned} \tag{3.15}$$

It is important that, due to the analyticity of the gauge superfields V_I^{++}, $I = 1, 2$, the fermionic connections $D^+ v_I$, $\bar{D}^+ v_I$ depend linearly on the harmonic variables. We shall use the notations

$$D^+ v_I(t, \theta, u) = A_I^a(t, \theta, \bar{\theta})u_a^+, \quad \bar{D}^+ v_I(t, \theta, u) = -\bar{A}_I^a(t, \theta, \bar{\theta})u_a^+. \tag{3.16}$$

Using (3.9) and (3.16), one can rewrite (3.15) in the following equivalent form with no harmonic dependence at all:

$$D^{(a}Q^{b)c} - A_1^{(a}\epsilon^{b)c} + A_2^{(a}c^{b)c} = 0, \quad \bar{D}^{(a}Q^{b)c} + \bar{A}_1^{(a}\epsilon^{b)c} - \bar{A}_2^{(a}c^{b)c} = 0. \tag{3.17}$$

Now let us more closely inspect the transformation laws (3.14). We start by choosing a frame in which the matrix c^{ab} has only one non-vanishing component:

$$c^{12} = c^{21} = i, \quad c^{11} = c^{22} = 0. \tag{3.18}$$

In this frame the transformations (3.14) look as

$$\delta Q^{\pm 1} = (\tau_1 - i\tau_2) u^{\pm 1}, \quad \delta Q^{\pm 2} = (\tau_1 + i\tau_2) u^{\pm 2}, \tag{3.19}$$

or, in terms of the $\mathcal{N}=4$ superfields $Q^{ab}(t, \theta, \bar{\theta})$ defined in (3.9),

$$\delta Q^{12} = (\tau_1 + i\tau_2), \quad \delta Q^{21} = -(\tau_1 - i\tau_2), \quad \delta Q^{11} = \delta Q^{22} = 0. \tag{3.20}$$

It is then convenient to choose the unitary-type gauge

$$Q^{12} = Q^{21} = 0. \tag{3.21}$$

Now the constraints (3.17) determine the spinor connection superfields A_I^a and their conjugate $\bar{A}_I^a$ in terms of the remaining superfield $Q^{11} \equiv \Phi$ and its complex conjugate $Q^{22} = \bar{\Phi}$

$$A_1^1 = \frac{1}{2} D^2\Phi, \ A_1^2 = -\frac{1}{2} D^1\bar{\Phi}, \quad A_2^1 = \frac{i}{2} D^2\Phi, \ A_2^2 = \frac{i}{2} D^1\bar{\Phi}, \quad \text{(and c.c.)}, \tag{3.22}$$

simultaneously with imposing the constraints on these superfields

$$D^1\Phi = \bar{D}^1\Phi = 0 , \quad D^2\bar{\Phi} = \bar{D}^2\bar{\Phi} = 0 . \tag{3.23}$$

These constraints may be interpreted as twisted chirality conditions. They can be given the standard form of the chirality conditions by relabelling the spinor derivative D^i, $\bar{D}_i$ in such a way that the R-symmetry $SU(2)$ acting on the indices i gets hidden, while another $SU(2)$ (which rotates D^i through $\bar{D}^i$), gets manifest [2]. Thus we have succeeded in deriving the linear chiral $\mathcal{N}=4, d=1$ multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ from the analytic multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ by gauging two independent shift isometries realized on the latter.

Let us now examine this correspondence on the level of the invariant actions. We start with the gauge covariantization of the free action (2.9) of the analytic superfield q^{+a}. In the full superspace the covariantized action reads

$$S_{cov}^{free} = \int \mu_H \left[q^{+a} D^{--} q_a^+ - 2V_1^{--} q^{+a} u_a^+ - 2V_2^{--} q^{+a} c_a^b u_b^+ + 2(V_1^{++} V_2^{--} - V_2^{++} V_1^{--}) c^{+-} \right] . \tag{3.24}$$

It is gauge invariant up to a total harmonic derivative in the integrand, i.e. it is of the Chern-Simons type. It may be equivalently rewritten in terms of the superfield Q^{+a} and the bridges v_1, v_2

$$S_{cov}^{free} = \int \mu_H \left[Q^{+a} D^{--} Q_a^+ + 2v_1 Q^{+a} u_a^- + 2v_2 Q^{+a} c_a^b u_b^- + (v_1 V_2^{++} - v_2 V_1^{++}) c^{--} + v_1^2 + v_2^2 \right] . \tag{3.25}$$

In this form it is invariant under both the λ and τ gauge transformations. In fact, the action can be written as a sum of two terms, the first of which transforms only under the τ gauge group, and the second only under the λ group

$$\begin{aligned}
S_{cov}^{free} &= S_\tau + S_\lambda , \\
S_\tau &= \int \mu_H \left[Q^{+a} D^{--} Q_a^+ - \frac{1}{4}(Q^{+a} u_a^- - Q^{-a} u_a^+)^2 - \frac{1}{4}(Q^{+a} c_a^b u_b^- - Q^{-a} c_a^b u_b^+)^2 \right] , \\
S_\lambda &= \int \mu_H \left[\frac{1}{4}(q^{+a} u_a^- - q^{-a} u_a^+)^2 + \frac{1}{4}(q^{+a} c_a^b u_b^- - q^{-a} c_a^b u_b^+)^2 \right. \\
&\qquad \left. - V_2^{++} q^{-a} c_a^b u_b^- - V_1^{++} q^{-a} u_a^- \right) \Big] .
\end{aligned} \tag{3.26, 3.27}$$

To check the invariance of S_τ we use the following transformation laws :

$$\begin{aligned}
\delta(Q^{+a} u_a^- - Q^{-a} u_a^+) &= 2\tau_1 , \quad \delta(Q^{+a} c_a^b u_b^- - Q^{-a} c_a^b u_b^+) = 2\tau_2 , \\
\delta(Q^{+a} Q_a^-) &= \tau_1(Q^{+a} u_a^- - Q^{-a} u_a^+) + \tau_2(Q^{+a} c_a^b u_b^- - Q^{-a} c_a^b u_b^+) .
\end{aligned} \tag{3.28}$$

It is worthwhile to note that all three terms in the action S_τ are $SU(2)$ singlets (independent of harmonic variables), so that the harmonic integral $\int du$ is in fact not necessary. After some simple algebra, making use of the integration by parts with respect to the harmonic derivatives, the constraint (3.2) and the definitions (3.4), (3.11) and (3.12), one can show that, up to a total harmonic derivative,

$$S_\lambda = \frac{1}{2} S_{cov}^{free} , \tag{3.29}$$

---

[2] Both these R-symmetry $SU(2)$ are manifest in the quartet notation $D^{i\underline{i}} = (D^i, \bar{D}^i) = (\hat{\bar{D}}^{\underline{i}}, \hat{D}^{\underline{i}})$, $\overline{D^{i\underline{i}}} = -D_{i\underline{i}} = (-\bar{D}_i, D_i) = (-\hat{D}_{\underline{i}}, \bar{D}_{\underline{i}})$.

whence one obtains the representation of the gauge-covariantized q^+ action (3.24) solely in terms of the superfields $Q^{\pm a}$:

$$S_{cov}^{free} = 2S_\tau = 2\int \mu_H \left[Q^{+a}Q_a^- - \frac{1}{4}(Q^{+a}u_a^- - Q^{-a}u_a^+)^2 - \frac{1}{4}(Q^{+a}c_a{}^b u_b^- - Q^{-a}c_a{}^b u_b^+)^2 \right]. \quad (3.30)$$

Now, the gauge condition (3.21) simply amounts to

$$Q^{+a}u_a^- - Q^{-a}u_a^+ = 0 \,, \quad Q^{+a}c_a{}^b u_b^- - Q^{-a}c_a{}^b u_b^+ = 0 \,, \quad (3.31)$$

and in this gauge the action (3.30) takes the standard form of the free action of the (twisted) chiral $(\mathbf{2},\mathbf{4},\mathbf{2})$ multiplet

$$S_{cov}^{free} = 2\int \mu_H\, Q^{+a}Q_a^- = 2\int dt d^4\theta\, \Phi\bar{\Phi} \,. \quad (3.32)$$

Thus the free action of the chiral $\mathcal{N}=4$ multiplet arises as a particular gauge of the properly gauge-covariantized free action of the analytic multiplet q^{+a}. Note that this equivalence, like in other cases [4, 5], was shown here in a manifestly $\mathcal{N}=4$ supersymmetric superfield approach, without any need to pass to the components. It is interesting to note that there exist two more equivalent useful forms of the action (3.24) in terms of the original superfields $q^{\pm a}$:

$$S_{cov}^{free} = \int \mu_H \left[q^{+a}q_a^- - V_1^{++}(q^{-a}u_a^-) - V_2^{++}(q^{-a}c_a{}^b u_b^-) \right] = \int \mu_H\, (D^{++}q^{-a})q_a^- \,. \quad (3.33)$$

Checking the gauge invariance of the action in the second form is especially simple: one uses the transformation law (3.13) and the fact that $D^{++}q^{-a}$ is analytic in virtue of the relation

$$D^{++}q^{-a} = q^{+a} + V_1^{++}u^{-a} + V_2^{++}c^{ab}u_b^- \,. \quad (3.34)$$

Let us now comment on the general sigma-model type action. The only invariant of the λ gauge transformations (3.1), (3.13) which one can construct from q^{+a} and q^{-a} is the quantity X defined as follows

$$X = (q^{+a}u_a^+)c^{--} - (q^{-a}u_a^-)c^{++} \,, \quad D^{\pm\pm}X = \pm 2(q^{\pm a}u_a^\pm)c^{+-} \mp (q^{+a}u_a^- + q^{-a}u_a^+)c^{\pm\pm} \,,$$
$$D^{++}D^{--}X = 2X \,, \quad (D^{++})^2 X = (D^{--})^2 X = 0 \,, \quad (3.35)$$

where

$$c^{\pm\pm} = c^{ab}u_a^\pm u_b^\pm \,, \quad \text{etc} \,.$$

It admits the equivalent representation, in which its τ gauge invariance becomes manifest

$$X = (Q^{+a}u_a^+)c^{--} - (Q^{-a}u_a^-)c^{++} = 2Q^{(ad)}c_d{}^b u_{(a}^+ u_{b)}^- \quad (3.36)$$

(the remaining relations in (3.35) preserve their form modulo the replacements $q^{\pm a} \Rightarrow Q^{\pm a}$). The subclass of the general sigma-model type q^+ action (2.11) invariant under the gauge transformations (3.1), (3.13) is then defined as follows

$$S_{cov} = \int \mu_H\, \mathcal{L}(X, D^{++}X, D^{--}X, u^\pm) \,. \quad (3.37)$$

In the gauge (3.21) and in the frame (3.18) we have

$$X = 2i \left(\Phi u_1^+ u_1^- - \bar{\Phi} u_2^+ u_2^- \right) \tag{3.38}$$

and, after integration over harmonics, the action (3.37) is reduced to the most general action of the twisted chiral superfields $\Phi, \bar{\Phi}$. Note the relation

$$\Phi = i \left[X \, u_2^- u_2^+ - \frac{1}{2} D^{++} X \, u_2^- u_2^- - \frac{1}{2} D^{--} X \, u_2^+ u_2^+ \right] . \tag{3.39}$$

The free action (3.24) can also be expressed through the universal invariant X:

$$S_{cov}^{free} = -\frac{3}{2} \int \mu_H \, X^2 \, . \tag{3.40}$$

This relation can be proved, starting from the τ form of the free action (3.30) and integrating by parts with respect to the harmonic derivatives. Another way to see this is to compare both sides in the original λ frame representation (i.e. in terms of $q^{\pm a}$) by choosing the Wess-Zumino gauge for the superfields $V_1^{\pm\pm}$ and $V_2^{\pm\pm}$

$$V_I^{\pm\pm} = \theta^\pm \bar{\theta}^\pm B_I \, .$$

Finally, we address the issue of the Fayet-Iliopoulos (FI) terms. In the present case one can define two independent gauge invariant FI terms

$$S_1^{FI} = i\xi_1 \int du d\zeta^{(-2)} \, V_1^{++} \, , \quad S_2^{FI} = i\xi_2 \int du d\zeta^{(-2)} \, V_2^{++} \, . \tag{3.41}$$

Let us consider the first one. Rewriting it in the full harmonic superspace

$$S_1^{FI} = i\xi_1 \int \mu_H \, \theta^- \bar{\theta}^- D^{++} v_1 = -i\xi_1 \int \mu_H \, (\theta^+ \bar{\theta}^- + \theta^- \bar{\theta}^+) v_1 \, , \tag{3.42}$$

expressing v_1 from the relation (3.7), integrating by parts with using the analyticity property of q^{+a} and V_1^{++}, V_2^{++} and performing in the end the integration over harmonics, this term in the gauge (3.21) and frame (3.18) can be transformed to the expression

$$S_1^{FI} = -\frac{i}{2}\xi_1 \int dt d^4\theta \left(\theta_1 \bar{\theta}^2 \Phi - \theta_2 \bar{\theta}^1 \bar{\Phi} \right) . \tag{3.43}$$

It is a particular case of twisted chiral superpotential term. Analogously,

$$S_2^{FI} = \frac{1}{2}\xi_2 \int dt d^4\theta \left(\theta_1 \bar{\theta}^2 \Phi + \theta_2 \bar{\theta}^1 \bar{\Phi} \right) . \tag{3.44}$$

It is unclear whether a general chiral superpotential can be generated from some gauge invariant q^{+a} action.

4 Nonlinear chiral multiplet

Let us now consider the gauging of the two-parameter abelian symmetry (2.12b)

$$\delta q^{+a} = \Lambda_1 q^{+a} + \Lambda_2\, c^a{}_b q^{+b}\,, \quad \Lambda_I = \Lambda_I(\zeta, u)\,,\ I = 1, 2\,. \tag{4.1}$$

The harmonic constraint (2.1b) is now covariantized as

$$D^{++} q^{+a} - V_1^{++} q^{+a} - V_2^{++} c^a{}_b q^{+b} = 0\,. \tag{4.2}$$

The analytic potentials V_I^{++}, $I = 1, 2$, possess the same gauge transformation laws (3.3) and are expressed through the bridges v_I with the mixed transformation rules (3.5) by the same relations (3.4). However, since now q^{+a} transforms homogeneously under the gauge transformations, the relation (3.7) between the λ and τ world objects has to be modified:

$$q^{+a} = e^{v_1}\left(\cos v_2\, Q^{+a} + \sin v_2 c^a{}_b\, Q^{+b}\right), \quad Q^{+a} = e^{-v_1}\left(\cos v_2\, q^{+a} - \sin v_2 c^a{}_b\, q^{+b}\right), \tag{4.3}$$

or, in another form,

$$q^{+a} + i c^a{}_b\, q^{+b} = e^{v_1 - i v_2}\left(Q^{+a} + i c^a{}_b\, Q^{+b}\right), \quad q^{+a} - i c^a{}_b\, q^{+b} = e^{v_1 + i v_2}\left(Q^{+a} - i c^a{}_b\, Q^{+b}\right). \tag{4.4}$$

A direct calculation shows that the constraint (4.2) entails, for Q^{+a},

$$D^{++} Q^{+a} = 0\,, \tag{4.5}$$

whence, in the central basis,

$$Q^{+a} = Q^{ia}(t, \theta, \bar\theta) u_i^+ \tag{4.6}$$

(cf. (3.9)). Also, using the transformation laws (3.5) and (4.1), it is easy to find

$$\delta Q^{\pm a} = \tau_1 Q^{\pm a} + \tau_2 c^a{}_b Q^{\pm b}\,. \tag{4.7}$$

In what follows it will be convenient to choose the $SU(2)$ frame (3.18) in which

$$q^{+1} = e^{(v_1 + i v_2)}\, Q^{+1}\,, \quad q^{+2} = e^{(v_1 - i v_2)}\, Q^{+2}\,, \tag{4.8}$$

$$\delta Q^{11} = \left(\tau_1 + i\tau_2\right) Q^{11}, \delta Q^{22} = \left(\tau_1 - i\tau_2\right) Q^{22}, \delta Q^{21} = \left(\tau_1 + i\tau_2\right) Q^{21}, \delta Q^{12} = \left(\tau_1 - i\tau_2\right) Q^{12}. \tag{4.9}$$

Like in the previous Section, the analyticity of q^{+a} implies the "covariant analyticity" for Q^{+a}:

$$D^+ q^{+1} = \bar D^+ q^{+1} = 0 \Leftrightarrow \left[D^+ + D^+(v_1 + i v_2)\right] Q^{+1} = \left[\bar D^+ + \bar D^+(v_1 + i v_2)\right] Q^{+1} = 0\,, \tag{4.10}$$

$$D^+ q^{+2} = \bar D^+ q^{+2} = 0 \Leftrightarrow \left[D^+ + D^+(v_1 - i v_2)\right] Q^{+2} = \left[\bar D^+ + \bar D^+(v_1 - i v_2)\right] Q^{+2} = 0\,. \tag{4.11}$$

From the analyticity of $V_{1,2}^{++}$ it follows that the gauge connections in (4.10), (4.11) are linear in harmonics

$$D^+(v_1 + i v_2) = A^a_{(+)}(t, \theta, \bar\theta) u_a^+\,, \quad D^+(v_1 - i v_2) = A^a_{(-)}(t, \theta, \bar\theta) u_a^+\,,$$
$$\bar D^+(v_1 - i v_2) = -\bar A^a_{(+)}(t, \theta, \bar\theta) u_a^+\,, \quad \bar D^+(v_1 + i v_2) = -\bar A^a_{(-)}(t, \theta, \bar\theta) u_a^+\,. \tag{4.12}$$

Now it is time to properly fix the τ gauge freedom (4.9). Assuming that $Q^{12} = -\overline{(Q^{21})}$ possesses a non-zero constant background, a convenient gauge is

$$Q^{12} = -Q^{21} = 1 \,. \tag{4.13}$$

Substituting this gauge into the covariant analyticity conditions for Q^{+a} in (4.11), taking into account the relations (4.6), (4.12), and equating to zero the coefficients of three independent products of harmonics $((u_1^+)^2, (u_2^+)^2$ and $u_1^+ u_2^+)$, we obtain

$$A^2_{(+)} = A^1_{(-)} = 0 \,, \quad A^1_{(+)} = D^2\Phi \,, \; A^2_{(-)} = -D^1\bar\Phi \,, \quad \text{(and c.c.)} \,,$$
$$D^1\Phi + \Phi D^2\Phi = \bar D^1\Phi + \Phi \bar D^2\Phi = 0 \,, \quad D^2\bar\Phi - \bar\Phi D^1\bar\Phi = \bar D^2\bar\Phi - \bar\Phi \bar D^1\bar\Phi = 0 \,, \tag{4.14}$$

where $Q^{11} \equiv \Phi$, $Q^{22} = \bar\Phi$.

Thus the only independent object that remains in the gauge (4.13) is a complex $\mathcal{N}{=}4$ superfield Φ subjected to the constraints (4.14). These constraints are a nonlinear version of the twisted chirality constraints (3.23) and are easily recognized as a twisted version of the *nonlinear $\mathcal{N}{=}4$ chirality constraints* [13]. It can be given the form of the ordinary nonlinear chirality constraints by relabelling the covariant derivatives just in the same way as in the case of the linear constraints (3.23).

It is worth mentioning a specific feature of the nonlinear chiral multiplet case as compared with the linear multiplet case. The original λ world constraints (2.1a), (4.2) preserve the whole automorphism $SU(2)_R$ group acting on the doublet indices of harmonics and Grassmann coordinates and break the Pauli-Gürsey $SU(2)_{PG}$ symmetry realized on the doublet index a of q^{+a} down to a $U(1)$ subgroup (due to the presence of the constant triplet c^{ab} in (4.2)). The same symmetry structure is exhibited by the τ world constraints (4.5), (4.11) which are equivalent to (4.2) and the analyticity condition (2.1a). Before fixing the τ frame gauge as in (4.13), the superfields Q^{ia} are transformed by $SU(2)_R$ linearly, in the same way as the spinor derivatives $D^i, \bar D^i$, i.e. as

$$\delta_R Q^{ia} \simeq Q^{ia\prime}(t,\theta\prime,\bar\theta\prime) - Q^{ia}(t,\theta,\bar\theta) = \lambda^i{}_k Q^{ka} \,, \quad \overline{(\lambda^{ik})} = \lambda_{ik} \,. \; \lambda^i{}_i = 0 \,, \tag{4.15}$$

Here λ^{ik} are constant $SU(2)_R$ parameters. After imposing the gauge (4.13), this transformation law becomes nonlinear, as it must be accompanied by the compensating τ gauge transformation needed for preserving (4.13)

$$(\tau_1 - i\tau_2)_{comp} = -\lambda^{12} + \lambda^{11}\bar\Phi \,, \quad (\tau_1 + i\tau_2)_{comp} = \lambda^{12} + \lambda^{22}\Phi \,, \tag{4.16}$$

whence

$$\delta_R\Phi = \lambda^{11} + 2\lambda^{12}\Phi + \lambda^{22}(\Phi)^2 \quad \text{and c.c.} \,, \tag{4.17}$$

i.e. Φ and $\bar\Phi$ are transformed as projective CP^1 coordinates of the 2-sphere $S^2 \sim SU(2)_R/U(1)_R$. The constraints (4.14), where D^i and $\bar D^i$ are still transformed linearly with respect to their doublet indices and Φ, $\bar\Phi$ are transformed according to the nonlinear transformation rule (4.17), are directly checked to be $SU(2)_R$ covariant. This interpretation of the nonlinear chiral $\mathcal{N}{=}4, d{=}1$ superfields as parameters of S^2 was the starting point of the derivation of the constraints (4.14) in [13] (in an $\mathcal{N}{=}4$ superspace parametrization twisted as compared to ours). Note that both Φ and $\bar\Phi$ are inert under the $U(1)$ remnant of the broken $SU(2)_{PG}$ symmetry.

Let us now discuss how the actions of the nonlinear chiral multiplet are reproduced from the gauged q^+ actions.

The standard free action of q^{+a} is obviously not invariant even under the rigid version of (4.1) due to the presence of the rescaling isometry in (4.1). This situation is quite similar to what we faced in [4] when deriving the action of the nonlinear $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet from the q^+ action with a gauged rescaling invariance. It was shown there that the simplest invariant action is a nonlinear action of the sigma-model type. In the case considered here we should start from the action which is simultaneously invariant under the rescalings and the $U(1)$ transformations. The unique object invariant under both gauged isometries is constructed as

$$Y = \frac{q^{+a} c_{ab} q^{-b}}{(q^{+a} q_a^-)}, \tag{4.18}$$

where

$$q^{-a} \equiv D^{--} q^{+a} - V_1^{--} q^{+a} - V_2^{--} c_b^a q^{+b} \tag{4.19}$$

and $V_{1,2}^{--}$ were defined in (3.12). It is the true nonlinear analog of the invariant X defined in (3.35). It is easy to check that Y admits an equivalent representation in terms of the τ world objects, such that it is manifestly invariant under the τ gauge transformations (4.7)

$$Y = \frac{Q^{+a} c_{ab} Q^{-b}}{(Q^{+a} Q_a^-)}, \tag{4.20}$$

and satisfies the relations

$$(D^{++})^2 Y = (D^{--})^2 Y = 0, \quad D^{++} D^{--} Y = 2Y, \quad (D^{++} Y \, D^{--} Y) - Y^2 = 1. \tag{4.21}$$

In the gauge (4.13) and frame (3.18)

$$Y = \frac{2i}{\left(1 + \Phi\bar{\Phi}\right)} \left[\bar{\Phi} \, u_2^+ u_2^- - \Phi \, u_1^+ u_1^- + \frac{1}{2} \left(1 - \Phi\bar{\Phi}\right) \left(u_1^+ u_2^- + u_2^+ u_1^-\right) \right]. \tag{4.22}$$

A nonlinear analog of the relation (3.39) is

$$\frac{\Phi}{1 + \Phi\bar{\Phi}} = i \left(Y \, u_2^- u_2^+ - \frac{1}{2} D^{++} Y \, u_2^- u_2^- - \frac{1}{2} D^{--} Y \, u_2^+ u_2^+ \right). \tag{4.23}$$

From this relation and its conjugate it is easy to express Φ and $\bar{\Phi}$ through Y and its harmonic derivatives.

The general sigma-model type action of the nonlinear chiral multiplet corresponds to the following gauged subclass of the general q^{+a} actions

$$S_{chn} = \int \mu_H \, \mathcal{L}(u, Y, D^{++}Y, D^{--}Y). \tag{4.24}$$

After expressing Y in terms of the τ world objects, choosing the gauge (4.13), $SU(2)$ frame (3.18) and preforming the integration over harmonics, (4.24) becomes the general sigma model action of the nonlinear chiral multiplet $\Phi, \bar{\Phi}$ as it was given in [18].

Let us point out that this correspondence, as in other similar cases [4, 5], allows one to equivalently deal with the action in the original λ frame q^+ representation by choosing the appropriate Wess-Zumino gauge for the "topological" gauge superfields $V_{1,2}^{\pm\pm}$ and using the residual two-parameter gauge freedom to trade two out of four original physical bosonic fields

of q^{+a} for two $d=1$ "gauge fields". The latter become just two auxiliary fields of the nonlinear chiral multiplet. To work in the q^{+} representation is in some aspects easier than to use the τ frame where the same action looks as the general action of the superfields Φ and $\bar{\Phi}$.

It is interesting that in the case under consideration there also exists a unique $SU(2)_R$ invariant action of the WZW type which is gauge invariant up to a total derivative in the integral and in this sense is an analog of the free action of the linear chiral multiplet in the form (3.24), (3.25). The quantity

$$F = q^{+a}q_a^- = e^{2v_1}Q^{+a}Q_a^- \tag{4.25}$$

is manifestly $SU(2)_R$ invariant and invariant under the gauge Λ_2 transformations, while its Λ_1 transformation reads

$$\delta_{\Lambda_1}F = 2\Lambda_1\, F\,, \quad \delta_{\Lambda_1}\log F = 2\Lambda_1\,. \tag{4.26}$$

Then the action

$$S_{su(2)} = \int \mu_H\, \log F = \int \mu_H\, \left[2v_1 + \log(Q^{+a}Q_a^-)\right] \tag{4.27}$$

is gauge invariant since the full $\mathcal{N}{=}4$ superspace integral of the analytic parameter Λ_1 vanishes. Using (4.4), it is easy to check that the bridge v_1, modulo a constant and purely analytic term which vanish after integration, in the gauge (4.13) is reduced to

$$v_1 \Rightarrow \log\left(1 - u_1^+u_2^- + 2\Phi u_1^+u_1^-\right) + \log\left(1 + u_2^+u_1^- + 2\bar{\Phi}u_2^+u_2^-\right)\,. \tag{4.28}$$

It follows from the constraints (4.14) that the integral of any holomorphic or antiholomorphic function of $\Phi, \bar{\Phi}$ over the full $\mathcal{N}{=}4$ superspace vanishes. So the bridge v_1 drops out from (4.27) in the gauge (4.13) and, taking into account that in this gauge

$$Q^{+a}Q_a^- = 1 + \Phi\bar{\Phi}\,, \tag{4.29}$$

we obtain the simple final expression for the action (4.27)

$$S_{su(2)} = \int dt d^4\theta\, \log(1 + \Phi\bar{\Phi})\,. \tag{4.30}$$

It describes the $\mathcal{N}{=}4$ superextension of the $d=1$ sigma model on $S^2 \sim SU(2)_R/U(1)_R$ and was derived in this form for the first time in [13]. The Lagrangian in (4.30) is just the corresponding Kähler potential [3]. Actually, using eq. (4.23), we could write an equivalent representation for $S_{su(2)}$ through the universal tensorial invariant Y. However, the $SU(2)_R$ symmetry in such a representation is not manifest due to the presence of explicit harmonics over which one should integrate.

Finally, we discuss the structure of two FI terms in the present case. They are originally given by the same $V_{1,2}^{++}$ actions (3.41) as in the linear case. Further, passing to the integral over the full $\mathcal{N}{=}4$ superspace, using the relation

$$v_1 - iv_2 = \log\left[(q^{+a} + ic^a_{\ b}q^{+b})u_a^-\right] - \log\left[(Q^{+a} + ic^a_{\ b}Q^{+b})u_a^-\right] \tag{4.31}$$

---

[3]Formally, (4.30) looks also invariant under $SU(2)_{PG}$, however the constraints (4.14) do not respect this second $SU(2)$, so the latter is not a symmetry of (4.30).

and its conjugate (they follow from (4.4)), as well as the analyticity of q^{+a}, and, finally, performing the integration over harmonics in the gauge (4.13), the FI terms can be expressed through Φ, $\bar{\Phi}$ as follows

$$\tilde{S}_1^{FI} = -\frac{i}{2}\tilde{\xi}_1 \int dt d^4\theta \left(\theta_1\bar{\theta}^2\Phi - \theta_2\bar{\theta}^1\bar{\Phi}\right), \quad \tilde{S}_2^{FI} = \frac{1}{2}\tilde{\xi}_2 \int dt d^4\theta \left(\theta_1\bar{\theta}^2\Phi + \theta_2\bar{\theta}^1\bar{\Phi}\right). \qquad (4.32)$$

Surprisingly, they are still linear in Φ, $\bar{\Phi}$, like their customary chiral superfield analogs (3.43), (3.44). Note that the $SU(2)_R$ invariance of (4.32) can be checked with the help of the basic constraints (4.14), taking into account that $SU(2)_R$ transforms the explicit θs in (4.32) in the standard way (rotates them in the doublet index), while Φ and $\bar{\Phi}$ are transformed according to the law (4.17) and its conjugate.

5 Nonabelian gauge group

Let us now consider gauging of the last two-parameter group admitting a realization on the analytic superfield q^{+a}, the non-abelian solvable group (2.13) consisting of a dilatation and a shift of q^{+a}. The gauge transformation laws are

$$\delta q^{+a} = \Lambda_1 q^{+a} + \Lambda_2 u^{+a}, \qquad (5.1)$$

where as before Λ_1 and Λ_2 are charge-zero analytic superfields. Due to the nonabelian character of this group, its gauging is a little bit more tricky as compared to the previous two (abelian) cases. The commutation relations of the gauge transformations are given by

$$[\delta, \delta']q^{+a} = \delta''q^{+a}, \quad \Lambda_1'' = 0, \quad \Lambda_2'' = \Lambda_2\Lambda_1' - \Lambda_1\Lambda_2'. \qquad (5.2)$$

In order to covariantize the harmonic constraints, we need to introduce two analytic gauge superfields V^{++}, W^{++} with the transformation laws

$$\delta V^{++} = D^{++}\Lambda_1, \quad \delta W^{++} = D^{++}\Lambda_2 + \Lambda_1 W^{++} - \Lambda_2 V^{++}. \qquad (5.3)$$

It is easy to check that the Lie bracket of two such transformations has the form (5.2). The covariant harmonic constraint on the superfield q^{+a} now reads

$$D^{++}q^{+a} - V^{++}q^{+a} - W^{++}u^{+a} = 0. \qquad (5.4)$$

The gauge superfields V^{++} and W^{++} are expressed through the corresponding non-analytic bridge superfields v, w as

$$V^{++} = D^{++}v, \quad W^{++} = e^v D^{++}w. \qquad (5.5)$$

As in other cases, the bridges v, w transform under two types of gauge transformations, the original ones with the analytic parameters $\Lambda_1(\zeta, u)$ and $\Lambda_2(\zeta, u)$, and new ones with the parameters $\tau_1(t, \theta, \bar{\theta})$ and $\tau_2(t, \theta, \bar{\theta})$ which are independent of harmonic variables:

$$\delta v = \Lambda_1(\zeta, u) - \tau_1(t, \theta, \bar{\theta}), \quad \delta w = e^{-v}\Lambda_2(\zeta, u) - \tau_2(t, \theta, \bar{\theta}) + \tau_1(t, \theta, \bar{\theta})\,w. \qquad (5.6)$$

We also define the new non-analytic "τ-world" superfield

$$Q^{+a} = e^{-v}q^{+a} - wu^{+a}, \quad \delta Q^{+a} = \tau_1 Q^{+a} + \tau_2 u^{+a}. \qquad (5.7)$$

13

As a consequence of (5.4), the superfield Q^{+a} is homogeneous in the harmonic variables

$$D^{++}Q^{+a} = 0 \;\Rightarrow\; Q^{+a}(t,\theta,\bar\theta,u) = Q^{ba}(t,\theta,\bar\theta)u_b^+ \,, \quad \overline{(Q^{ab})} = Q_{ab} \,. \tag{5.8}$$

The harmonic-independent superfields Q^{ba} transform as

$$\delta Q^{ba} = \tau_1 Q^{ba} - \tau_2 \epsilon^{ba} . \tag{5.9}$$

The real parameter τ_2 may be used to gauge away the antisymmetric part of the tensor Q^{ba}:

$$Q^{ab} = Q^{(ab)} \,. \tag{5.10}$$

Then the target space scale invariance with the real parameter τ_1 may be used to fix the value of a component of this tensor. We choose the gauge

$$Q^{21} = Q^{12} = i \;\Rightarrow\; Q^{+1} = Q^{11}u_1^+ + iu_2^+ \,, \quad Q^{+2} = Q^{22}u_2^+ + iu_1^+ \,. \tag{5.11}$$

We now should take into account the analyticity constraints on the original superfields q^{+a}. When expressed in terms of the new superfields, these constraints become

$$\begin{aligned} D^+Q^{+a} + D^+v\,Q^{+a} + (D^+w + wD^+v)\,u^{+a} &= 0 \,, \\ \bar D^+Q^{+a} + \bar D^+v\,Q^{+a} + (\bar D^+w + w\bar D^+v)\,u^{+a} &= 0 \,. \end{aligned} \tag{5.12}$$

Due to the analyticity of the gauge superfields V^{++}, W^{++}, the fermionic connections D^+v, $D^+w + wD^+v$ have a simple dependence on harmonic variables

$$\begin{aligned} D^{++}D^+v &= 0 &\Rightarrow& \quad D^+v = A^a(t,\theta,\bar\theta)u_a^+ , \tag{5.13}\\ D^{++}(D^+w + wD^+v) &= 0 &\Rightarrow& \quad D^+w + w\,D^+v = B^a(t,\theta,\bar\theta)u_a^+ \,. \tag{5.14} \end{aligned}$$

Analogously, for the conjugate connections we have

$$\bar D^+v = -\bar A^a u_a^+ \,, \quad \bar D^+w + w\bar D^+v = -\bar B^a u_a^+ \,. \tag{5.15}$$

Then, the constraints (5.12) imply, in ordinary superspace,

$$\begin{aligned} D^2Q^{11} + A^2Q^{11} + iA^1 - B^1 &= 0 \,, & B^2 &= iA^2 \,, \\ D^1Q^{22} + A^1Q^{22} + iA^2 + B^2 &= 0 \,, & B^1 &= -iA^1 \,, \tag{5.16}\\ D^1Q^{11} + A^1Q^{11} &= 0 \,, & D^2Q^{22} + A^2Q^{22} &= 0 \,. \tag{5.17} \end{aligned}$$

From eq. (5.16) one expresses the gauge connections

$$\begin{aligned} A^1 = iB^1 &= \frac{1}{4 + Q^{11}Q^{22}}\left(2iD^2Q^{11} - Q^{11}D^1Q^{22}\right), \\ A^2 = -iB^2 &= \frac{1}{4 + Q^{11}Q^{22}}\left(2iD^1Q^{22} - Q^{22}D^2Q^{11}\right). \tag{5.18} \end{aligned}$$

Substituting these expressions into eqs. (5.17) and then using complex conjugation yield the full set of the nonlinear constraints on the superfields $Q^{11} \equiv \Phi, Q^{22} = \bar\Phi$

$$D^1\Phi + \frac{\Phi}{4 + \Phi\bar\Phi}\left(2iD^2\Phi - \Phi D^1\bar\Phi\right) = 0 \,, \; \bar D^1\Phi + \frac{\Phi}{4 + \Phi\bar\Phi}\left(2i\bar D^2\Phi - \Phi\bar D^1\bar\Phi\right) = 0 \,, \tag{5.19}$$

$$D^2\bar\Phi + \frac{\bar\Phi}{4 + \Phi\bar\Phi}\left(2iD^1\bar\Phi - \bar\Phi D^2\Phi\right) = 0 \,, \; \bar D^2\bar\Phi + \frac{\bar\Phi}{4 + \Phi\bar\Phi}\left(2i\bar D^1\bar\Phi - \bar\Phi\bar D^2\Phi\right) = 0 \,. \tag{5.20}$$

14

Equations (5.19) and (5.20) may be interpreted as (twisted) non-linear chirality constraints on the superfields Φ and its complex conjugate $\bar{\Phi}$.

Let us now define the subclass of the general q^+ actions which respects the invariance under the rigid transformations (2.13) and, after gauging, under their local counterparts (5.1). In the τ frame it should yield the general sigma-model type action of the nonlinear chiral multiplet in question.

The invariance under the shift transformations in (2.13) just means that the corresponding superfield Lagrangian cannot depend on the trace part in $q^{ia}u_i^+$, i.e. $q^{ia} \rightarrow q^{(ib)}$. Next, the invariance under the target space scale transformations constrains the action to depend only on two independent ratios of three components of $q^{(ab)}$, i.e. $q^{(ab)} \rightarrow q^{11}/q^{12}, q^{22}/q^{12}$. In other words, the appropriate general superfield q^{+a} Lagrangian should be an arbitrary function of the superfields $q^{11}/q^{12}, q^{22}/q^{12}$ which can be interpreted as projective coordinates of some two-sphere S^2. Being reformulated in terms of the superfields $q^{\pm a}$, this requirement amounts to the following particular choice of the q^+ lagrangian

$$\mathcal{L} = \mathcal{L}\left(u^\pm, \frac{\hat{q}^{\pm a}}{|\hat{q}|}\right), \tag{5.21}$$

where

$$\hat{q}^{\pm a} = q^{\pm a} - \frac{1}{2}u^{\pm a}\left(q^{+b}u_b^- - q^{-b}u_b^+\right),$$

$$|\hat{q}|^2 = \hat{q}^{+a}\hat{q}_a^- = (q^{+a}u_a^+)(q^{-b}u_b^-) - \frac{1}{4}(q^{+a}u_a^- + q^{-a}u_a^+)^2 = \frac{1}{2}q^{(ab)}q_{(ab)}. \tag{5.22}$$

As in the previous case, the standard free action of q^{+a} is not invariant under the target space rescalings. The Lagrangians from the subclass (5.21) are of the sigma-model type, with non-constant bosonic target metrics.

The gauging of these Lagrangians goes in the standard way, by subjecting q^{+a} to the co-variantized constraint (5.4) and defining q^{-a} in a gauge-covariant way as

$$q^{-a} = D^{--}q^{+a} - V^{--}q^{+a} - W^{--}u^{+a}, \quad \delta q^{-a} = \Lambda_1 q^{-a} + \Lambda_2 u^{-a}. \tag{5.23}$$

The final gauge-covariantized action has the same form as the rigidly invariant one (5.21) but with the superfields $q^{\pm a}$ defined in a gauge-covariant way. Just due to this covariance, the basic objects (5.22) admit the equivalent τ frame representation

$$\hat{q}^{\pm a} = e^v\left[Q^{\pm a} - \frac{1}{2}u^{\pm a}\left(Q^{+b}u_b^- - Q^{-b}u_b^+\right)\right] \equiv e^v\,\hat{Q}^{\pm a},$$

$$|\hat{q}|^2 = e^{2v}\left[(Q^{+a}u_a^+)(Q^{-b}u_b^-) - \frac{1}{4}\left(Q^{+a}u_a^- + Q^{-a}u_a^+\right)^2\right]$$

$$= \frac{1}{2}\,e^{2v}Q^{(ab)}Q_{(ab)} \equiv e^{2v}\,|\hat{Q}|^2. \tag{5.24}$$

The covariantized superfield argument in (5.21) does not depend on the bridge v, whence

$$\mathcal{L} = \mathcal{L}\left(u^\pm, \frac{\hat{q}^{\pm a}}{|\hat{q}|}\right) = \mathcal{L}\left(u^\pm, \frac{\hat{Q}^{\pm a}}{|\hat{Q}|}\right). \tag{5.25}$$

In the gauges (5.10) and (5.11):

$$\hat{Q}^{\pm 1} = Q^{\pm 1} = \Phi u_1^\pm + i u_2^\pm \,, \ \hat{Q}^{\pm 2} = Q^{\pm 2} = \bar{\Phi} u_2^\pm + i u_1^\pm \,, \quad |\hat{Q}| = \sqrt{1 + \Phi \bar{\Phi}} \,, \tag{5.26}$$

and the action corresponding to the Lagrangian (5.25), after performing the integration over harmonics, becomes the general off-shell action of the supermultiplet $\Phi, \bar{\Phi}$. Note that the relations (5.26), like analogous relations of the previous cases, are invertible:

$$\Phi = \hat{Q}^{+1} u_2^- - \hat{Q}^{-1} u_2^+ \,, \quad \bar{\Phi} = \hat{Q}^{-2} u_1^+ - \hat{Q}^{+2} u_1^- \,. \tag{5.27}$$

This ensures the possibility to express $\Phi, \bar{\Phi}$ through the basic gauge invariant object, $\hat{q}^{\pm a}/|\hat{q}| = \hat{Q}^{\pm a}/|\hat{Q}|$, and in fact proves the equivalence of the general $\mathcal{N}=4$ action of superfields $\Phi, \bar{\Phi}$ and the particular class of the gauged q^+ actions defined above.

It is worth noting that the building blocks of the λ world gauge invariants can be successively reproduced from the simplest invariant of the shift λ gauge transformation (with the parameter Λ_2)

$$q^{++} = q^{+a} u_a^+ = \hat{q}^{+a} u_a^+ \,, \quad \delta q^{++} = \Lambda_1 q^{++} \,. \tag{5.28}$$

Acting on (5.28) by the covariant derivative $D^{--} - V^{--}$, we can produce new non-analytic superfields which are invariant under the Λ_2 transformations and covariant with respect to the Λ_1 transformations:

$$q^{+-} = \frac{1}{2}(D^{--} - V^{--})q^{++}, \ q^{--} = (D^{--} - V^{--})q^{+-}, \ \delta q^{\pm -} = \Lambda_1 q^{\pm -} \,. \tag{5.29}$$

They are related to the superfields $q^{\pm a}$ and $\hat{q}^{\pm a}$ by

$$q^{+-} = \frac{1}{2}(q^{+a} u_a^- + q^{-a} u_a^+) = \hat{q}^{+a} u_a^- = \hat{q}^{-a} u_a^+ \,, \ q^{--} = q^{-a} u_a^- = \hat{q}^{-a} u_a^- \tag{5.30}$$

and can be used to form two independent gauge invariant ratios

$$X^{++} = \frac{q^{++}}{\sqrt{q^{++} q^{--} - (q^{+-})^2}}, \quad X^{--} = \frac{q^{--}}{\sqrt{q^{++} q^{--} - (q^{+-})^2}}, \tag{5.31}$$

which are just independent harmonic projections of the superfield argument in (5.25).

Let us now dwell on the peculiarities of the realization of $SU(2)_R$ and $SU(2)_{PG}$ symmetries on the superfields Φ and $\bar{\Phi}$ and the surprising relation to the nonlinear chiral multiplet discussed in the previous Section.

The basic gauge covariant constraint (5.4) clearly breaks the original $SU(2)_R \times SU(2)_{PG}$ symmetry realized on q^{+a}, Grassmann and harmonic coordinates down to the diagonal R-symmetry group $SU(2)_R'$ which uniformly rotates all doublet indices. The gauge (5.10) is $SU(2)_R'$ covariant, so the superfield $Q^{(ab)}$ is transformed as

$$\delta_{R'} Q^{(ab)} \simeq Q^{(ab)\prime}(t, \theta', u') - Q^{(ab)}(t, \theta, u) = \lambda^a{}_d Q^{(db)} + \lambda^b{}_d Q^{(ad)} \,, \quad \lambda^b{}_b = 0 \,. \tag{5.32}$$

The gauge (5.11) is not preserved under (5.32), and in order to restore this gauge one should accompany the $SU(2)_R'$ transformations by a compensating τ_1 transformation with

$$(\tau_1)_{comp} = i \left(\lambda^{22} \Phi - \lambda^{11} \bar{\Phi} \right) . \tag{5.33}$$

16

As a result, in this gauge the superfields Φ and $\bar{\bar{\Phi}}$ are nonlinearly transformed under $SU(2)'_R$

$$\delta_{R'}\Phi = 2\lambda^{12}\Phi - i\lambda^{11}\left(2 + \Phi\bar{\Phi}\right) + i\lambda^{22}(\Phi)^2\,, \quad \delta_{R'}\bar{\Phi} = \overline{(\delta_{R'}\Phi)}\,, \tag{5.34}$$

and so can be treated as coordinates of the coset $S^2 \sim SU(2)'_R/U(1)'_R$ in a particular parametrization. Obviously, there should exist an equivalence transformation to the stereographic projection parametrization in which the S^2 coordinates are transformed according to the holomorphic law (4.17). The precise form of this field redefinition is as follows

$$\chi = i\frac{\Phi}{1 + \sqrt{1 + \Phi\bar{\Phi}}}\,, \quad \Phi = -2i\frac{\chi}{1 - \chi\bar{\chi}}\,, \tag{5.35}$$

$$\delta_{R'}\chi = \lambda^{11} + 2\lambda^{12}\chi + \lambda^{22}(\chi)^2 \quad \text{and c.c.}. \tag{5.36}$$

The transformation law (5.36) coincides with (4.17), which suggests that in this new holomorphic parametrization the constraints (5.19), (5.20) take the form (4.14). Indeed, a simple calculation shows that after the field redefinition (5.35) the constraints (5.19), (5.20) are equivalently rewritten as

$$D^1\chi + \chi D^2\chi = 0\,, \quad \bar{D}^1\chi + \chi\bar{D}^2\chi = 0 \quad \text{(and c.c.)}\,. \tag{5.37}$$

Thus we see that the nonlinear chiral multiplet considered in this Section is in fact a disguised form of the nonlinear (twisted) chiral multiplet of ref.[13] rederived within the gauging procedure in the previous Section. This is rather surprising, because in the two cases we gauged two essentially different two-parameter groups, respectively, abelian and non-abelian ones (2.12b) and (2.13). The identity of these two multiplets amounts to the identity of their general actions, despite the fact that the classes of the appropriate q^+ actions one starts with in these two cases are essentially different. Here we again encounter the phenomenon of non-uniqueness of the inverse oxidation procedure as compared with the target space dimensional reduction [4, 5]: the same off-shell multiplet can be recovered by gauging some non-equivalent isometries of the "root" multiplet. For instance, the $\mathcal{N}=4, d=1$ multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ and its most general action can be obtained from the q^+ multiplet and the appropriate set of the q^+ actions by gauging either the non-abelian $SU(2)_{PG}$ group or the abelian group of three independent shift isometries of q^{+a} [5]. Basically, the difference between these two gauging procedures lies only in the fact that they start from different subclasses of the general set of q^+ actions. However, the final action of the reduced multiplet does not "remember" from which parent q^+ action it originated.

Taking for granted that all off-shell $\mathcal{N}=4, d=1$ superfields can be recovered from the q^{+a} superfield by gauging different symmetries realized on the latter and taking into account that only three independent two-parameter groups (defined in (2.12) and (2.13)) can be implemented on q^{+a}, we conclude that only two essentially different off-shell $\mathcal{N}=4, d=1$ multiplets with the content $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ exist: the standard linear chiral multiplet and the nonlinear chiral multiplet introduced in [13]. Any other version of the chiral multiplet should be reducible to one of these two via some field redefinition.

There is one more way to see that the constraints (5.19), (5.20) are equivalent to (4.14). After some algebra, using (5.19), (5.20) at the intermediate steps, the expressions for the spinor connections (5.18) can be cast in the following form

$$A^1 = D^2\left(\frac{i\Phi}{1 + \sqrt{1 + \Phi\bar{\Phi}}}\right) - D^1\log\left(1 + \sqrt{1 + \Phi\bar{\Phi}}\right),$$

$$A^2 = D^1\left(\frac{i\bar{\Phi}}{1 + \sqrt{1 + \Phi\bar{\Phi}}}\right) - D^2\log\left(1 + \sqrt{1 + \Phi\bar{\Phi}}\right). \tag{5.38}$$

Substituting these expressions and their complex conjugates into (5.17) and complex conjugates of (5.17), we recover (5.37), with χ being related to Φ just by eqs. (5.35).

Finally, as an instructive example, we present the $SU(2)'_R$ invariant action in terms of the original superfield variables, as well as the relevant FI term.

The $SU(2)'_R$ invariant action is given by an expression similar to (4.27)

$$S_{su(2)'} = \int \mu_H \, \log |\hat{q}| = \int \mu_H \left(v + \log |\hat{Q}| \right). \tag{5.39}$$

It is manifestly invariant under the gauge shift Λ_2 transformation (since $\hat{q}^{+a}$ is invariant), as well as under the scale and shift τ gauge transformations. It is also invariant under the scale gauge Λ_1 transformations since under the latter the Lagrangian in (5.39) is shifted by an analytic gauge parameter the integral of which over the full $\mathcal{N}{=}4$ superspace vanishes:

$$\delta_1 \log |\hat{q}| = \Lambda_1 \,, \quad \int \mu_H \, \Lambda_1 = 0 \,. \tag{5.40}$$

To find the precise form of the action in terms of the nonlinear chiral superfields $\Phi, \bar{\Phi}$, we should make use of eq. (5.26) and also compute the bridge part of the $\mathcal{N}{=}4$ superspace integral in (5.39):

$$\int \mu_H \, v \,. \tag{5.41}$$

This integral can be evaluated by taking one spinor derivative, say D^+, off the measure $d^4\theta$, throwing it on v, expressing D^+v as in (5.13), doing the harmonic integral du, substituting the gauge-fixed expressions (5.38) for the spinor connections A^1, A^2 and, finally, restoring the full Grassmann measure by taking the spinor derivatives D^1, D^2 off these expressions. It turns out that only the second terms in the expressions (5.38) contribute, and we obtain

$$\int \mu_H \, v = -\int dt d^4\theta \, \log \left(1 + \sqrt{1 + \Phi\bar{\Phi}} \right). \tag{5.42}$$

Using this in (5.39), we obtain

$$S_{su(2)'} = \int dt d^4\theta \left[\log \sqrt{1 + \Phi\bar{\Phi}} - \log \left(1 + \sqrt{1 + \Phi\bar{\Phi}} \right) \right]. \tag{5.43}$$

Now it is straightforward to check that, after passing to the superfields $\chi, \bar{\chi}$ via (5.35), the Lagrangian in (5.43) is reduced (modulo a constant shift) just to

$$\log \left(1 + \chi\bar{\chi} \right).$$

Thus we obtain the expected result that the action (5.39), (5.43) is in fact identical to the previously considered $SU(2)_R$ invariant action (4.30).

As for the FI terms, in the present case only the gauge superfield V^{++} possesses an abelian gauge transformation law, so one is able to construct only one FI term:

$$S_v^{FI} = i\xi_v \int du d\zeta^{(-2)} \, V^{++} = -i\xi_v \int \mu_H \, (\theta^+\bar{\theta}^- + \theta^-\bar{\theta}^+) v \,. \tag{5.44}$$

Inserting the identities $1 = D^+\theta^-$, $1 = -\bar{D}^+\bar{\theta}^-$ into the round brackets in the r.h.s. of (5.44), integrating by parts with respect to spinor derivatives, using the relations (5.13) with (5.38) and their conjugates, doing harmonic integral and, at the end, integrating by parts once again, one finally finds

$$S_v^{FI} = -i\xi_v \int dt d^4\theta \, (\theta_1\bar{\theta}^2\chi - \theta_2\bar{\theta}^1\bar{\chi}) \,, \tag{5.45}$$

which coincides with one of the FI terms in (4.32).

6 Conclusions

In this article and two previous papers [4, 5] we showed that all known off-shell $\mathcal{N}=4, d=1$ multiplets with 4 physical fermions can be reproduced from the basic ("root") multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ by gauging some symmetries, abelian or non-abelian, realized on this multiplet. The corresponding general $\mathcal{N}=4$ mechanics actions are recovered as the result of the proper gauge-fixing in the appropriate gauged subclasses of the general q^+ action, the subclasses which enjoy invariance under the symmetries just mentioned. Our gauging procedure uses the manifestly supersymmetric universal language of $\mathcal{N}=4$ superspace and does not require to resort to component considerations at all. Another merit of our approach is that it reduces the whole set of non-equivalent superfield actions of the $\mathcal{N}=4$ mechanics models to some particular cases of the generic q^+ action extended by non-propagating "topological" gauge superfields. Just the presence of the latter enables one to preserve the manifest supersymmetry at each step and to reveal the irreducible off-shell superfield contents of one or another model by choosing the appropriate superfield gauges and (in the cases considered in the present paper) by passing to the equivalent τ frame formulations. The alternative (and in many cases more technically feasible) way of doing suggested by the gauging approach is to always stay in the initial q^+ representation where the harmonic analyticity is manifest and to choose the WZ gauge for the relevant analytic non-propagating gauge superfields. Each "topological" gauge multiplet in the WZ gauge contributes just one scalar ("gauge") field which, after fully fixing the residual gauge freedom, becomes an auxiliary field of the new off-shell $\mathcal{N}=4$ multiplet related to the q^+ multiplet via linear or nonlinear versions of the "automorphic duality" [6]. Thus in the component formulation our approach automatically yields the explicit realization of this intrinsically one-dimensional off-shell duality. The distinctions between various types of this duality are related to the differences between the global symmetry groups subjected to gauging.

The basic peculiarity of the cases considered in this paper as compared to those treated in [4, 5] is that the superfields describing the $\mathcal{N}=4$ multiplets $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ do not "live" on the $\mathcal{N}=4$ analytic harmonic subspace (as distinct from the multiplets $(\mathbf{0}, \mathbf{4}, \mathbf{4})$, $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ and $(\mathbf{3}, \mathbf{4}, \mathbf{2})$). They are most naturally described after passing to the equivalent "τ frame" [16, 17], with the ordinary $\mathcal{N}=4$ superfield gauge parameters and the harmonic superfield bridges to the "λ frame" as the basic gauge objects. These bridges ensure the equivalence of the manifestly analytic λ frame picture one starts with and the picture in the τ frame. In the τ frame, the original gauge-covariantized analyticity-preserving harmonic constraints on the superfield q^{+a} amount to the harmonic independence of the involved superfields. The harmonic Grassmann analyticity, which is manifest in the λ frame, in the τ frame amounts to the covariant analyticity conditions. After properly fixing $\mathcal{N}=4$ supersymmetric τ gauges, these conditions become the linear or nonlinear $\mathcal{N}=4$ chirality conditions, depending on which two-parameter symmetry group realized on q^{+a} is subjected to gauging. There are only three such groups and they are listed in (2.12) and (2.13). We considered gauging of all these three groups and found that the gauging of the group (2.12a) leads to the linear chiral $\mathcal{N}=4$ multiplet, while gaugings of (2.12b) and (2.13) lead to the same nonlinear chiral multiplet [13], despite the obvious non-equivalence of these two groups. This non-uniqueness is a manifestation of the general non-uniqueness of the oxidation procedure as inverse to the automorphic duality. Since only three two-parameter symmetries can be realized on q^{+a}, from our results it follows, in particular, that no other non-equivalent nonlinear chiral $\mathcal{N}=4$ multiplet can be defined.

Interesting venues for further applications of our gauge approach are provided by models

of $\mathcal{N}{=}8$ supersymmetric mechanics (see e.g. [19, 20] and refs. therein). It was argued in [20], by considering a wide set of examples, that the off-shell $\mathcal{N}{=}8$ multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ is the true $\mathcal{N}{=}8$ analog of the "root" $\mathcal{N}{=}4$ multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and that the whole set of the component actions of the $\mathcal{N}{=}8$ mechanics models with 8 physical fermions (and finite numbers of auxiliary fields) follow from the general action of this basic $\mathcal{N}{=}8$ multiplet via a linear version of the automorphic duality. It would be interesting to apply our techniques to these cases. Recall that our approach is bound by the requirement that the symmetries to be gauged commute with supersymmetry. In the $\mathcal{N}{=}8$ case the target space scale and shift transformations still obey this criterion, so one can hope that the gauging would nicely work in this case too and could help to understand the relationships between the multiplet $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ and the rest of the $\mathcal{N}{=}8$ multiplets in a manifestly $\mathcal{N}{=}4$ supersymmetric superfield fashion. We can also hope to discover in this way new nonlinear $\mathcal{N}{=}8$ multiplets and the corresponding new $\mathcal{N}{=}8$ mechanics models, besides those already known [14, 15]. The primary question to be answered is how to define an $\mathcal{N}{=}8$ analog of the $\mathcal{N}{=}4$ topological gauge multiplet which plays a crucial role in our approach. Another possible way of extending our study is to construct "topological" $\mathcal{N}{=}4, d{=}1$ supergravity multiplets and to gauge, with their help, the R-symmetry $SU(2)$ groups of $\mathcal{N}{=}4$ supersymmetry, with new models of $\mathcal{N}{=}4$ mechanics as an outcome. Finally, let us note that the nonlinear chiral multiplets exist also in dimensions $d{>}1$ [13], e.g. in $d{=}3$ [21]. It would be interesting to inquire whether they can also be derived by gauging some symmetries realized on the appropriate analytic harmonic superfields, some analogs of q^{+a}, i.e. whether they are also a disguised τ frame form of the harmonic analyticity conditions.

Acknowledgements

References

[1] R. de Lima Rodrigues, *"The quantum mechanics SUSY algebra: an introductory review"*, [hep-th/0205017];
A.V. Smilga, *"Low dimensional sisters of Seiberg-Witten effective theory"*, [hep-th/0403294].

[2] S. Bellucci, S.J. Gates, Jr., E. Orazi, *"A Journey Through Garden Algebras"*, Lectures given at Winter School on Modern Trends in Supersymmetric Mechanics (SSM05), Frascati, Italy, 7-12 Mar 2005, [hep-th/0602259].

[3] Z. Kuznetsova, M. Rojas, F. Toppan, JHEP 0603 (2006) 098, [hep-th/0511274]; F. Toppan, *"Irreps and Off-shell Invariant Actions of the N-extended Supersymmetric Quantum Mechanics"*, Fifth International Conference on Mathematical Methods in Physics IC2006, April 24-28 2006, CBPF, Rio de Janeiro, Brazil, [hep-th/0610180].

[4] F. Delduc, E. Ivanov, Nucl. Phys. B753 (2006) 211, [hep-th/0605211].

[5] F. Delduc, E. Ivanov, Nucl. Phys. B770 (2007) 179, [hep-th/0611247].

[6] S.J. Gates, Jr., L. Rana, *"On Extended Supersymmetric Quantum Mechanics"*, Maryland Univ. Preprint # UMDPP 93-24, Oct. 1994; S.J. Gates, Jr., L. Rana, Phys. Lett. B342 (1995) 132, [hep-th/9410150]; B345 (1995) 233, [hep-th/9411091];
A. Pashnev, F. Toppan, J. Math. Phys. 42 (2001) 5257, [hep-th/0010135].

[7] S. Bellucci, S. Krivonos, A. Marrani, E. Orazi, Phys. Rev. D73 (2006) 025011, [hep-th/0511249].

[8] S. Hellerman, J. Polchinski, *"Supersymmetric quantum mechanics from light cone quantization"*, [hep-th/9908202].

[9] F. Delduc, S. Krivonos, unpublished, 2005.

[10] C. Burdik, S. Krivonos, A. Shcherbakov, Czech. J. Phys. 55 (2005) 1357, [hep-th/0508165].

[11] S. Krivonos, A. Shcherbakov, Phys. Lett. B637 (2006) 119, [hep-th/0602113].

[12] E. Ivanov, O. Lechtenfeld, JHEP 0309 (2003) 073, [hep-th/0307111].

[13] E. Ivanov, S. Krivonos, O. Lechtenfeld, Class. Quant. Grav. 21 (2004) 1031, [hep-th/0310299].

[14] E. Ivanov, Phys. Lett. B639 (2006) 579, [hep-th/0605194];
S. Bellucci, S. Krivonos, A. Marrani, Phys. Rev. D74 (2006) 045005, [hep-th/0605165].

[15] S. Bellucci, A. Beylin, S. Krivonos, A. Shcherbakov, Phys. Lett. B633 (2006) 382, [hep-th/0511054].

[16] A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Pis'ma ZhETF 40 (1984) 155 [JETP Lett. 40 (1984) 912];
A.S. Galperin, E.A. Ivanov, S. Kalitzin, V.I. Ogievetsky, E.S. Sokatchev,
Class. Quant. Grav. 1 (1984) 469.

[17] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky and E.S. Sokatchev, *"Harmonic Superspace"*, Cambridge University Press 2001, 306 p.

[18] S. Bellucci, A. Beylin, S. Krivonos, A. Nersessian, E. Orazi, Phys. Lett. B616 (2005) 228, [hep-th/0503244].

[19] S. Bellucci, E. Ivanov, S. Krivonos, O. Lechtenfeld, Nucl. Phys. B699 (2004) 226, [hep-th/0406015].

[20] E. Ivanov, O. Lechtenfeld, A. Sutulin, *"Hierarchy of N=8 Mechanics Models"*, Hannover University Preprint ITP-UH-11/07, arXiv:0705.3064 [hep-th].

[21] S. Bellucci, S. Krivonos, A. Shcherbakov, Phys. Rev. D74 (2006) 065016, [hep-th/0606052].