



Electromagnetic Coulomb Gas with Vector Charges and "Elastic" Potentials: Renormalization Group Equations

David Carpentier, Pierre Le Doussal

► To cite this version:

David Carpentier, Pierre Le Doussal. Electromagnetic Coulomb Gas with Vector Charges and "Elastic" Potentials: Renormalization Group Equations. 2007. ensl-00163660

HAL Id: ensl-00163660

<https://ens-lyon.hal.science/ensl-00163660>

Preprint submitted on 17 Jul 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Electromagnetic Coulomb Gas with Vector Charges and “Elastic” Potentials : Renormalization Group Equations

David Carpentier

*Laboratoire de Physique de l'Ecole Normale Supérieure de Lyon - CNRS,
46, Allée d'Italie, 69364 Lyon Cedex 07, France*

Pierre Le Doussal

*Laboratoire de Physique Théorique de l'Ecole Normale Supérieure - CNRS,
24, Rue Lhomond 75005 Paris, France*

Abstract

We present a detailed derivation of the renormalization group equations for two dimensional electromagnetic Coulomb gases whose charges lie on a triangular lattice (magnetic charges) and its dual (electric charges). The interactions between the charges involve both angular couplings and a new electromagnetic potential. This motivates the denomination of “elastic” Coulomb gas. Such elastic Coulomb gases arise naturally in the study of the continuous melting transition of two dimensional solids coupled to a substrate, either commensurate or with quenched disorder.

1 Introduction

The understanding of defect-mediated phase transitions in two dimensions relies on the renormalization group study of Coulomb gases (CG). In the simplest examples of the $O(2)$ or XY model, the criticality of the Kosterlitz-Thouless phase transition is described using the scalar Coulomb Gas[12]. In this case, the charges correspond to the integer topological charges of the XY vortices, which interact *via* the 2D Coulomb (\ln) potential. If we perturb the XY model by a p -fold symmetry breaking potential (the so called clock model), the previous scalar CG has to be extended : the clock potential translates into magnetic scalar charges[10]. These magnetic charges mutually interact *via* the same Coulomb potential, and their coupling with electric charges is a Aharonov-Bohm potential [11]. This scalar electromagnetic CG has been studied using the real-space renormalization techniques [10,17], which provide

the critical properties of the initial clock model. Moreover, the phase transitions of various two dimensional models, such as the Ashkin-Teller model, the q -state Potts model, and the $O(n)$ model can be studied using these scalar CG techniques[11,17].

An extension of the scalar (electric) Coulomb gas is required in the study of the continuous melting transition of a two dimensional solid[15]. This extension is twofold : (i) the topological charges of two dimensional dislocations are Burgers vectors instead of integers and (ii) the interaction between these vector charges consist of the usual 2D Coulomb potential (*i.e* \ln interaction), and an *angular interaction* which couples the charges to the vector \mathbf{r}_{12} joining the two defects¹. This angular interaction spoils the conformal invariance of the \ln CG. The renormalization group study of the conformally invariant case was achieved in ref. [14]. Studying the melting transition in the general case amounts to consider the perturbation by marginal conformal (rotation) symmetry-breaking operators of the previous conformal fixed point. The study of the corresponding vector CG was performed in [16,20]. The natural extension of this vector CG to the electromagnetic case arises in the study of two dimensional melting in the presence of a translation symmetry breaking potential, *e.g* a coupling to a substrate via a periodic modulation of the density, as in Ref. [16]. Such a general vector electromagnetic CG has never been studied to our knowledge, and it is the purpose of the present paper to derive the RG equations describing its scaling behavior to lowest order. A preliminary study, motivated by the problem of a substrate with quenched disorder [6] was published some time ago, and involved a replicated VECG [7]. The present study provides a complete and general derivation of the RG equations valid for any type of substrate (periodic and/or disordered). The VECG studied here can be viewed as an extension to the vector/elastic case of the scalar electromagnetic CG [17], and an extension to the electromagnetic case of the vector CG of [16,20]. As we will see, the elasticity manifests itself not only in the angular interactions of the electric/electric and magnetic/magnetic potentials, but also into the electric/magnetic interaction which is no longer a simpler Aharonov-Bohm potential.

Before turning to a more precise definition of our model, let us mention the field theoretical approach to the CG problem. The scalar electromagnetic CG admits an equivalent Sine-Gordon field theoretical formulation [19]. Its scaling behavior in the electric case was derived in Ref. [1]. Extension to the electromagnetic CG case were considered in [2,3] (see also [5]), which included in

¹ This angular interaction is a manifestation of the microscopic nature of the dislocations, which can be viewed as additional half-line of atoms inserted in the lattice[13]. A pair of dislocations of opposite Burgers vectors, which is an extra segment of atoms, has obviously some preferred orientation with respect to the initial regular lattice.

particular parafermionic operators[9]. This electromagnetic In-CG was extended to consider charges in higher groups[4], as well as relations with string theory models. In these generalized Toda field theories, the CG charges appear as root vectors of Lie algebra, and the charges of the $SU(3)$ Toda field theory can be identified with Burgers vectors of a triangular lattice. In this perspective, our present study corresponds to an extension to the non-conformal case where angular interactions are included of the $SU(3)$ study of Boyanovsky and Holman .

The paper is organized as follows : in section 2, we derive the CG formulation of an elastic solid coupled to a substrate. We consider explicitly two important cases : the case of a periodic commensurate substrate, and the case of a random pinning substrate. This allows to define the general vector “elastic” CG which is the subject of this paper. In section 3, the renormalization group equations for this general CG are derived to order one loop, using a real space procedure similar in spirit to the method described in [17]. The results are summarized in Section 4. Finally, in section 5 these equations are restricted to the original elastic models. Due to the complexity of the present derivation we have deferred to a separate publication the study of these RG equations for the various models.

Notations

Throughout this paper, we use the notations $\int_{\vec{r}} = \int d^2\vec{r} = \int_{-\infty}^{+\infty} dx dy$ and $\int_{\vec{q}} = \int d^2\vec{q}/(2\pi)^2$. The notation \vec{r} corresponds to vectors in the two dimensional plane, originating from either the direct or dual lattice, while boldfaces \mathbf{A} denote vectors in the replica space. Vectors both in replica and two dimensional plane $A_{i,a}$ are denoted $\vec{\mathbf{A}}$. The sum over repeated (real space or replica) indices will be assumed :

$$A_{i,a}B_{i,a} = \sum_{i=1,2} \sum_{a=1}^n A_{i,a}B_{i,a} \quad (1)$$

and we use the convolution notation

$$[A * B](\vec{r}) = \int_{\vec{r}'} A(\vec{r}') B(\vec{r} - \vec{r}') \quad (2)$$

which for a density of charges $\vec{b}(\vec{r}) = \sum_{\alpha} \vec{b}_{\alpha} \delta(\vec{r} - \vec{r}_{\alpha})$ reduces to

$$b_i * V_{ij} * b_j = \sum_{i,j=1,2} \sum_{\alpha\beta} b_{\alpha,i} V_{ij}(\vec{r}_{\alpha} - \vec{r}_{\beta}) b_{\beta,j} \quad (3)$$

Unless otherwise stated, the indices i, j, k, l will correspond to real space indices $i = 1, 2$; a, b, c, d to replica indices between 1 and n ; and greek indices α, β label the charges in a collection of charges. The notation \hat{e} corresponds to the unit vector $\vec{e}/|\vec{e}|$.

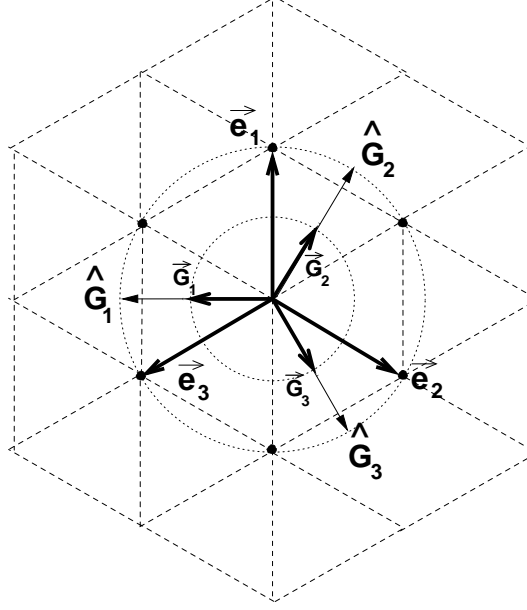


Fig. 1. Representation of a hexagonal lattice we will consider in this paper. The 6 vectors $\pm\vec{e}_{i=1,2,3}$ are the unit vectors of the original lattices (here the lattice spacing has been set to $a_0 = 1$), and the 6 unit vectors $\pm\hat{G}_{i=1,2,3}$ lies on the dual lattice.

2 The model

2.1 Elastic description of a pinned two dimensional crystal

2.1.1 Two dimensional elastic energy

In this paper, we will consider a crystal with hexagonal symmetry (see Fig.1). For such a lattice, the elasticity is isotropic, and the elastic energy is given by the harmonic hamiltonian[13]

$$H_0[\vec{u}] = \frac{1}{2} \int d^2\vec{r} u_{ij}(\vec{r}) C_{ijkl} u_{kl}(\vec{r}) = \frac{1}{2} \int d^2\vec{r} (2\mu u_{ij}^2 + \lambda u_{kk}^2) \quad (4)$$

$$= \frac{1}{2} \int \frac{d^2\vec{q}}{(2\pi)^2} u_i(\vec{q}) \Phi_{ij}(\vec{q}) u_j(-\vec{q}) \quad (5)$$

with $C_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}$ where λ, μ are Lamé coefficients, and the tensor u_{ij} is defined by² $u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. For later convenience, it is useful to define the local stress tensor $\sigma_{ij} = C_{ijkl} u_{kl} = 2\mu u_{ij} + \lambda\delta_{ij} u_{kk}$. The elastic matrix $\Phi_{ij}(\vec{q})$ is given by

$$\Phi_{ij}(\vec{q}) = c_{11} q^2 P_{ij}^L(\vec{q}) + c_{66} q^2 P_{ij}^T(\vec{q}) \quad ; \quad c_{11} = 2\mu + \lambda ; c_{66} = \mu \quad (6)$$

² Note that we have neglected the nonlinear component of u_{ij} .

where c_{11}, c_{66} are respectively the compression and shear moduli, and λ, μ the Lamé coefficients of the crystal. We have used the projectors $P_{ij}^L(\vec{q}) = \hat{q}_i \hat{q}_j$, $P_{ij}^T(\vec{q}) = \hat{q}_i^\perp \hat{q}_j^\perp = \delta_{ij} - \hat{q}_i \hat{q}_j$. In this expression, \vec{u} is a smooth displacement field, which corresponds to the long wavelength distortions of the original lattice. Within the context of elasticity, it must satisfy the condition $|\vec{u}(\vec{r}) - \vec{u}(\vec{r} + \vec{e}_i)| \ll a_0$, where \vec{e}_i is one of the unit vectors of the original lattice, and a_0 the lattice spacing. To go beyond this elastic description of the lattice distortions, one must allow for dislocations, which are the topological excitations of this elastic model.

2.1.2 Two dimensional dislocations

A two dimensional (edge) dislocation located in \vec{r}_α is characterised by its topological charge called the Burgers vector \vec{b}_α . This Burgers vector lies on the original lattice, and for most of our purpose, we will restrict ourselves to unit Burgers vectors corresponding to one of the six $\vec{e}_i, i = 1, \dots, 6$. By definition, this Burgers vector corresponds to the increment of the displacement field when surrounding the dislocation :

$$\oint \vec{u}(\vec{r}) \, dl = a_0 \vec{b} \quad (7)$$

where the contour integral circles around \vec{r}_α , and we choose to consider dimensionless Burgers vectors \vec{b} . A collection of dislocations can be described by the Burgers vector density

$$\vec{b}(\vec{r}) = \sum_{\alpha} \vec{b}_\alpha \delta(\vec{r} - \vec{r}_\alpha) \quad (8)$$

This density of dislocations induces a density of strain relaxed by a displacement field $\vec{u}_d(\vec{r})$, derived in appendix A, and given by[18]

$$u_{d,i}(\vec{r}) = \frac{a_0}{2\pi} [\mathcal{G}_{ij} * b_j](\vec{r}) = \frac{a_0}{2\pi} \sum_{\alpha} \tilde{\mathcal{G}}_{ij}(\vec{r} - \vec{r}_\alpha) b_{\alpha,j} \quad (9)$$

$$\text{with } \tilde{\mathcal{G}}_{ij}(\vec{r}) = \delta_{ij} \Phi(\vec{r}) + \frac{c_{66}}{c_{11}} \epsilon_{ij} \tilde{G}(\vec{r}) + \frac{c_{11} - c_{66}}{c_{11}} \epsilon_{jk} H_{ik}(\vec{r}) \quad (10)$$

The potential $\Phi(\vec{r})$ gives the angle between the vector \vec{r} and *e.g* the \vec{e}_1 vector, $\tilde{G}(\vec{r})$ corresponds to the usual (*e.g* lattice) Coulomb potential and $H_{ij}(\vec{r})$ is an angular potential. We regularize these potentials with a hard cut off : using $\theta(|\vec{r}| - a_0) = 1$ if $|\vec{r}| > a_0$ and 0 otherwise, they are defined as

$$\tilde{G}(\vec{r}) = \left(\ln \left(\frac{|\vec{r}|}{a_0} \right) + c \right) \theta(|\vec{r}| - a_0) \quad ; \quad G(\vec{r}) + i\Phi(\vec{r}) = \ln \left(\frac{z}{a_0} \right) \theta(|\vec{r}| - a_0) \quad (11)$$

$$H_{ij}(\vec{r}) = \left(\frac{r_i r_j}{r^2} - \frac{1}{2} \delta_{ij} \right) \theta(|\vec{r}| - a_0) \quad (12)$$

where $z = r_x + ir_y$, c is an arbitrary constant, and we have defined for later convenience the logarithmic potential $G(\vec{r})$. In the presence of dislocations, the displacement field splits into the above component $\vec{u}_d(\vec{r})$ induced by the dislocations themselves, and an independent smooth phonons part $\vec{u}_{ph}(\vec{r})$: $\vec{u}(\vec{r}) = \vec{u}_{ph}(\vec{r}) + \vec{u}_d(\vec{r})$. Without any perturbation, the usual melting transition is studied by performing explicitly the integral over the smooth phonons field in the partition function. One is left with the partition function of Coulomb gas with vector charges \vec{b}_α , whose scaling behavior describes the KTHNY melting transition. However, with translation symmetry breaking perturbations, this usual (magnetic) Coulomb gas must be extended to an electromagnetic gas, as explained below.

2.2 Breaking the translation symmetry

In this paper, we will consider a two dimensional crystal coupled to a substrate modeled by a potential $V(\vec{r})$ coupling directly to the density $\rho(\vec{r})$ of the lattice. This coupling adds to the elastic Hamiltonian (4) an energy

$$H_V = \int_{\vec{r}} \rho(\vec{r}) V(\vec{r}) \quad (13)$$

which explicitly depends on \vec{u} instead of u_{ij} , reflecting the breaking of the translation symmetry. This symmetry breaking corresponds to the situation where the density $\rho(\vec{r})$ and the potential $V(\vec{r})$ have some harmonics in common corresponding to a reciprocal lattice vector \vec{G} . In the following, we will consider either the case of a periodic potential *commensurate* with the lattice, or a random pinning potential. In both cases, we can consider that $\int_{\vec{r}} V(\vec{r}) = 0$. We decompose the lattice density as

$$\rho(\vec{r}) = \rho_0 \left(1 - \partial_i u_i(\vec{r}) + \sum_{\vec{G} \neq 0} e^{i\vec{G} \cdot (\vec{r} - \vec{u}(\vec{r}))} \right) + h.o.t. \quad (14)$$

where the \vec{G} are reciprocal lattice vectors. Similarly, the coupling (13) reads :

$$\int_{\vec{r}} \rho(\vec{r}) V(\vec{r}) = - \int_{\vec{r}} (\rho_0 V(\vec{r})) \partial_i u_i(\vec{r}) + \frac{1}{2} \int_{\vec{r}} \sum_{\vec{G} \neq 0} \left(V_{\vec{G}} e^{-i\vec{G} \cdot \vec{u}(\vec{r})} + V_{-\vec{G}} e^{i\vec{G} \cdot \vec{u}(\vec{r})} \right) \quad (15)$$

where we have defined $V_{\vec{G}} = \rho_0 V(\vec{r}) e^{i\vec{G} \cdot \vec{r}}$. Upon coarse graining (or in an effective long wavelength hamiltonian), only the reciprocal lattice vectors common to $V(\vec{r})$ and $\rho(\vec{r})$ will survive. In the above equation, the primed sum is on these common reciprocal lattice vectors corresponding to a non vanishing $V_{\vec{G}}$, which exists in the cases considered. In the following, we will restrict ourselves only to these vectors in common \vec{G} of minimum length. They correspond to the most relevant perturbations near the pure melting transition.

2.2.1 Periodic commensurate substrate

In the case of a periodic and commensurate substrate, we can use the symmetry $V_{\vec{G}} = V_{-\vec{G}}^*$ to rewrite the second term of (15) as

$$\frac{H_V}{T} = \int_{\vec{r}} \sum_{\vec{G}} 2 \frac{|V_{\vec{G}}|}{2T} \cos(\vec{G} \cdot \vec{u}(\vec{r})) \quad (16)$$

As previously mentionned, we will restrict ourselves to the three reciprocal lattice vectors $\vec{G}_{\alpha=1,2,3}$ of minimal length $|\vec{G}_1|$ arising in this sum. We will also use below the unit vectors $\hat{G}_{\alpha=1,2,3} = \vec{G}_{\alpha}/|\vec{G}_{\alpha}|$. In addition, a periodic substrate modifies the elastic part of the energy by generating a new term coupling the orientations of the lattice to the substrate. Defining the local orientation $\theta(\vec{r}) = \frac{1}{2}(\partial_x u_y - \partial_y u_x)$, this new term can be written as

$$\delta H_0 = \frac{\gamma}{2} \int_{\vec{r}} \theta^2(\vec{r}) \quad (17)$$

where γ is a new elastic constant. This term is non zero even in the floating solid phase where the direct coupling (16) is irrelevant, and must thus be included.

Finally, we focus on the case of weak perturbations : to first order in $V_{\vec{G}}$, we can expand the cosine coupling into

$$\begin{aligned} \exp \left(2 \frac{|V_{\vec{G}_\alpha}|}{2T} \sum_{\alpha=1,2,3} \cos(\vec{G}_\alpha \cdot \vec{u}(\vec{r})) \right) &\simeq 1 + \frac{|V_{\vec{G}_\alpha}|}{2T} \sum_{\vec{m}(\vec{r})=\pm\hat{G}_1, \pm\hat{G}_2, \pm\hat{G}_3} e^{i|\vec{G}_1|\vec{m}(\vec{r}) \cdot \vec{u}(\vec{r})} \\ &= \sum_{\vec{m}(\vec{r})=0, \pm\hat{G}_{i=1,2,3}} \left(\frac{|V_{\hat{G}_\alpha}|}{2T} \right)^{\vec{m}(\vec{r}) \cdot \vec{m}(\vec{r})} e^{i|\vec{G}_1|\vec{m}(\vec{r}) \cdot \vec{u}(\vec{r})} \end{aligned} \quad (18)$$

Defining a fugacity $Y[0, \vec{m}]$ for the formal charges $\vec{m}(\vec{r})$ as

$$Y[0, \vec{m}] = y^{\vec{m} \cdot \vec{m}} \text{ with } y = \frac{|V_{\vec{G}_1}|}{2T} \quad (19)$$

we rewrite the partition function of the perturbed lattice as

$$\begin{aligned} Z &= \int d[\vec{u}_{ph}(\vec{r})] \int d[\vec{u}_d(\vec{r})] \left(\prod_{\vec{r}} \sum_{\vec{m}(\vec{r})} Y[0, \vec{m}(\vec{r})] \right) \\ &\exp \left(-\frac{1}{2T} \int_{\vec{r}} \left[u_{ij}^{(d)}(\vec{r}) C_{ijkl} u_{kl}^{(d)}(\vec{r}) + u_{ij}^{(ph)}(\vec{r}) C_{ijkl} u_{kl}^{(ph)}(\vec{r}) + u_{ij}^{(d)}(\vec{r}) C_{ijkl} u_{kl}^{(ph)}(\vec{r}) \right] \right) \\ &\exp \left(-\frac{1}{2T} \int_{\vec{r}} \gamma \theta^2 \right) \exp \left(i|\vec{G}_1| \int_{\vec{r}} \vec{m}(\vec{r}) \cdot (\vec{u}^{(d)} + \vec{u}^{(ph)}) (\vec{r}) \right) \end{aligned} \quad (20)$$

Plugging the expression (9) for $\vec{u}^{(d)}$, and integrating over the gaussian displacement field $\vec{u}^{(ph)}$, we obtain three contributions to the remaining action:

$$Z = \sum_{\{\vec{b}(\vec{r})\}} \left(\prod_{\vec{r}} \sum_{\vec{m}(\vec{r})} Y[0, \vec{m}(\vec{r})] \right) \exp \left(S[\vec{b}/\vec{b}] + S[\vec{b}/\vec{m}] + S[\vec{m}/\vec{m}] \right) \quad (21)$$

The dislocation interaction is given by the usual form extended to include the γ coupling (see appendix A) :

$$S[\vec{b}/\vec{b}] = -\frac{1}{2T} \int_{\vec{r}} u_{ij}^{(d)}(\vec{r}) C_{ijkl} u_{kl}^{(d)}(\vec{r}) \quad (22)$$

$$= -\frac{a_0^2}{2T} \int_{\vec{q}} b_i(\vec{q}) b_j(-\vec{q}) \frac{1}{q^2} \left(\frac{4c_{66}\gamma}{c_{66} + \gamma} P_{ij}^L(\vec{q}) + \frac{4c_{66}(c_{11} - c_{66})}{c_{11}} P_{ij}^T(\vec{q}) \right) \quad (23)$$

$$= \frac{1}{2} \sum_{\alpha \neq \beta} \left(K_1 \vec{b}_\alpha \cdot \vec{b}_\beta G(\vec{r}_\alpha - \vec{r}_\beta) - K_2 b_{\alpha,i} b_{\beta,j} H_{ij}(\vec{r}_\alpha - \vec{r}_\beta) \right) - \frac{E_c}{T} \sum_{\alpha} \vec{b}_\alpha \cdot \vec{b}_\alpha \quad (24)$$

where the inverse Fourier transform of

$$f_{ij}(\vec{q}) = q^{-2} \left(A P_{ij}^L + B P_{ij}^T \right) \quad (25)$$

was determined as

$$f_{ij}(\vec{r}) = \int_{\vec{q}} (1 - e^{i\vec{q}\cdot\vec{r}}) f_{ij}(\vec{q}) = \delta_{ij} \frac{A+B}{4\pi} \left(\ln \frac{r}{a} + cte \right) + \frac{A-B}{4\pi} H_{ij}(\vec{r}) \quad (26)$$

providing the following expressions for the coupling constants

$$K_{1/2} = \frac{a_0^2}{\pi T} \left(\frac{c_{66}(c_{11} - c_{66})}{c_{11}} \pm \frac{c_{66}\gamma}{c_{66} + \gamma} \right) = \frac{a_0^2}{\pi T} \left(\frac{\mu(\mu + \lambda)}{2\mu + \lambda} \pm \frac{\mu\gamma}{\mu + \gamma} \right) \quad (27)$$

Note that the dislocation core energy E_c in (24) arises from the standard continuum approximation (11) of the lattice Coulomb interaction $\tilde{G}(r)$, and the use of the neutrality condition $\int_{\vec{r}} \vec{b}(\vec{r}) = 0$. From now on, the core energy E_c will be incorporated in a fugacity for the \vec{b} charges:

$$Y[\vec{b}, \vec{0}] = \tilde{y}^{\vec{b} \cdot \vec{b}} \quad \text{with} \quad \tilde{y} = e^{-\frac{E_c}{T}}. \quad (28)$$

The interaction between the \vec{m} charges follows from the gaussian integration

over $\vec{u}^{(ph)}$:

$$S[\vec{m}/\vec{m}] = -\frac{|\vec{G}_1|^2}{2} \int_{\vec{q}} m_i(\vec{q}) \Phi_{ij}^{-1} m_j(\vec{q}) \quad (29)$$

$$= -\frac{1}{2} \int_{\vec{q}} m_i(\vec{q}) \left(\frac{|\vec{G}_1|^2 T}{c_{11} q^2} P_{ij}^L + \frac{|\vec{G}_1|^2 T}{(c_{66} + \gamma) q^2} P_{ij}^T \right) m_j(\vec{q}) \quad (30)$$

$$= \frac{1}{2} \sum_{\alpha \neq \beta} (K_3 \vec{m}_\alpha \cdot \vec{m}_\beta G(\vec{r}_\alpha - \vec{r}_\beta) - K_4 m_{\alpha,i} m_{\beta,j} H_{ij}(\vec{r}_\alpha - \vec{r}_\beta)) \\ - \frac{\tilde{E}_c}{T} \sum_{\alpha} \vec{m}_\alpha \cdot \vec{m}_\alpha \quad (31)$$

with the coupling constants

$$K_{3/4} = \frac{T|\vec{G}_1|^2}{4\pi} \left(\frac{1}{c_{66} + \gamma} \pm \frac{1}{c_{11}} \right) = \frac{T|\vec{G}_1|^2}{4\pi} \left(\frac{1}{\mu + \gamma} \pm \frac{1}{2\mu + \lambda} \right) \quad (32)$$

The core energy \tilde{E}_c will be incorporated from now on into the bare fugacity $Y[\vec{0}, \vec{m}]$. Finally the cross coupling comes from the last term in ³ (20) :

$$S[\vec{b}/\vec{m}] = \frac{ia_0|\vec{G}_1|}{2\pi} \sum_{\alpha, \beta} m_i(\vec{r}_\alpha) \mathcal{G}_{ij}(\vec{r}_\alpha - \vec{r}_\beta) b_j(\vec{r}_\beta) \quad (33)$$

$$= i \sum_{\alpha, \beta} m_i(\vec{r}_\alpha) \left(\delta_{ij} \frac{a_0|\vec{G}_1|}{2\pi} \Phi(\vec{r}_\alpha - \vec{r}_\beta) + K_5 \epsilon_{ij} G(\vec{r}_\alpha - \vec{r}_\beta) \right. \\ \left. + K_6 \epsilon_{jk} H_{ik}(\vec{r}_\alpha - \vec{r}_\beta) \right) b_j(\vec{r}_\beta) \quad (34)$$

with

$$K_5 = \frac{a_0|\vec{G}_1|}{2\pi} \left(\frac{c_{66}}{c_{11}} - \frac{\gamma}{\gamma + c_{66}} \right) \quad ; \quad K_6 = \frac{a_0|\vec{G}_1|}{2\pi} \left(\frac{c_{11} - c_{66}}{c_{11}} - \frac{\gamma}{\gamma + c_{66}} \right) \quad (35)$$

Defining the potential

$$V_{ij}(K_1, K_2, \vec{r}) = K_1 \delta_{ij} G(\vec{r}) - K_2 H_{ij}(\vec{r}) \quad (36)$$

we can rewrite the above partition function as that of a Coulomb gas with both electric and magnetic vector charges :

$$Z = \sum_{\{\vec{r}_\alpha, \vec{b}(\vec{r}_\alpha), \vec{m}(\vec{r}_\alpha)\}} \prod_{\alpha} Y[\vec{b}_\alpha, \vec{m}_\alpha] \exp S[\vec{b}(\vec{r}_\alpha), \vec{m}(\vec{r}_\alpha)] \quad (37)$$

³ Note that, as a consequence of the hard-core regularization of the potentials (11), one can use indifferently a sum over distinct charges ($\sum_{\alpha \neq \beta}$) or not ($\sum_{\alpha, \beta}$) in the expression below.

with the action

$$\begin{aligned}
S[\vec{b}(\vec{r}_\alpha), \vec{m}(\vec{r}_\alpha)] = & \frac{1}{2} \sum_{\alpha \neq \beta} b_{\alpha,i} V_{ij}(K_1, K_2, \vec{r}_\alpha - \vec{r}_\beta) b_{\beta,j} \\
& + \frac{1}{2} \sum_{\alpha \neq \beta} m_{\alpha,i} V_{ij}(K_3, K_4, \vec{r}_\alpha - \vec{r}_\beta) m_{\beta,j} \\
& + i \sum_{\alpha \neq \beta} m_{\alpha,i} \left(\delta_{ij} \frac{a_0 |\vec{G}_1|}{2\pi} \Phi(\vec{r}_\alpha - \vec{r}_\beta) + V_{ik}(K_5, K_6) \epsilon_{kj} \right) b_{\beta,j}
\end{aligned} \tag{38}$$

2.2.2 Substrate Disorder

In the case of a substrate disorder, the potential $V(\vec{r})$ which couples to the local density of atoms of the crystal is random : its distribution will be taken as gaussian, with variance

$$\overline{V(\vec{r})V(\vec{r}')} = h(\vec{r} - \vec{r}') \tag{39}$$

where $h(\vec{r} - \vec{r}')$ is a short range correlator and here and below $\overline{}$ denotes an average over the disorder V . The two first contributions from the Fourier decomposition (15) are

$$\frac{H_V[\vec{u}]}{T} = \int_{\vec{r}} \left(\frac{1}{T} \sigma_{ij} u_{ij} + 2\sqrt{y_m} \sum_{\nu=1,2,3} \cos(\vec{G}_\nu \cdot \vec{u}(\vec{r}) + \phi_\nu(\vec{r})) \right) \tag{40}$$

where σ_{ij} is a random stress field, arising from the long wavelength part of the disorder potential $V(\vec{r})$. It induces local random compression/dilation and shear stress. Its correlator is parametrized as

$$\overline{\sigma_{ij}(\vec{r})\sigma_{kl}(\vec{r}')} = \delta(\vec{r} - \vec{r}') [(\Delta_{11} - 2\Delta_{66})\delta_{ij}\delta_{kl} + \Delta_{66}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})] \tag{41}$$

whose bare values, derived from (15) are:

$$\Delta_{11} = \rho_0^2 h_{\vec{K}=\vec{0}} \quad ; \quad \Delta_{66} = 0 \quad ; \quad y_m = \rho_0^2 h_{\vec{K}=\vec{G}_1} / T^2 \tag{42}$$

where ρ_0 is the mean density. The second part of the disorder comes from the first harmonic of $V(\vec{r})$ with almost the same periodicity as the lattice, i.e. it is proportional to $\sqrt{y_m}$ the amplitude of the $\vec{q} \simeq \vec{G}_1$ component of $V(\vec{r})$ occuring in (15). Since it is not invariant under a uniform shift of \vec{u} it is usually called the *pinning disorder*. The random phase field in (41) is uniformly distributed over $[0, 2\pi]$ and satisfies

$$\overline{\langle e^{i(\phi_\nu(\vec{r}) - \phi_{\nu'}(\vec{r}'))} \rangle} = \delta_{\nu,\nu'} \delta^2(\vec{r} - \vec{r}'). \tag{43}$$

The \vec{G}_ν are the first reciprocal lattice vectors (of modulus $G_1^2 = 16\pi^2/3a_0^2$).

The average over the disorder fields $\phi_\nu(\vec{r})$ and $\sigma_{ij}(\vec{r})$ is performed using the replica trick introducing the replicated field $\vec{u}^a(\vec{r})$, $a = 1, \dots, n$, and the corresponding replicated Burgers charge $\vec{b}^a(\vec{r})$. One defines:

$$Z = \overline{Z_V^n} = \prod_{a=1}^n \int d[\vec{u}_a] \exp \left(\frac{H_0[\vec{u}_a] + H_V[\vec{u}_a]}{T} \right) \quad (44)$$

and consider the limit $n = 0$. We focus on the case of weak pinning disorder y_m , and expand the exponential of the cosine coupling in (40) as in (18) and perform the disorder average in (44):

$$\overline{\exp \left(-2\sqrt{y_m} \sum_{\nu=1,2,3} \sum_{a=1}^n \cos \left(\vec{G}_\nu \cdot \vec{u}^a(\vec{r}) + \phi_\nu(\vec{r}) \right) \right)} \quad (45)$$

$$= 1 + y_m \sum_{a,b=1}^n \sum_{\nu=1,2,3} e^{i\vec{G}_\nu \cdot (\vec{u}^a(\vec{r}) - \vec{u}^b(\vec{r}))} + \mathcal{O}(y_m^2) \quad (46)$$

$$= ny_m + \sum_{\vec{m}^a(\vec{r})} Y[0, \vec{m}^a] e^{-i|\vec{G}_1| \sum_a \vec{m}^a \cdot \vec{u}^a} \quad (47)$$

The replicated \vec{m} charges have initially two opposite non zero components:

$$\vec{m}^a = \hat{G}_\nu (\delta_{a,b_1} - \delta_{a,b_2}) \quad \text{with } b_1 \neq b_2, 1 \leq b_1, b_2 \leq n, \nu = 1, 2, 3 \quad (48)$$

However, under the fusion process of the renormalization procedure, we will have to consider charges obtained as the sum of these initial charges. These general charges will be characterized by the property $\sum_a \vec{m}^a = \vec{0}$. Their bare fugacity, introduced in the above formula, reads

$$Y[0, \vec{m}^a] = \sqrt{y_m}^{\sum_a \vec{m}^a \cdot \vec{m}^a} \quad (49)$$

To introduce dislocations one can now follow the same steps as in Section 2.2.1 splitting $\vec{u}_a = \vec{u}_a^{(ph)} + \vec{u}_a^{(d)}$. The average over the random stress tensor (41) leads to the replicated elastic matrices

$$c_{11}^{ab} = c_{11} \delta^{ab} - \Delta_{11} \quad ; \quad c_{66}^{ab} = c_{66} \delta^{ab} - \Delta_{66} \quad ; \quad \gamma^{ab} = \gamma \delta^{ab} - \Delta_\gamma \quad (50)$$

Hence, by the same technique as in the case of the commensurate regular substrate, we obtain a Coulomb gas description (38) of the random model, albeit with coupling constant $K_{1,\dots,6}$ which are now replica matrices involving products and inverses of the replica elastic matrices (50)

$$K_{1/2} = \frac{a_0^2}{\pi T} \left(c_{66}(c_{11} - c_{66})c_{11}^{-1} \pm c_{66}\gamma(c_{66} + \gamma)^{-1} \right) \quad (51a)$$

$$K_{3/4} = \frac{T|\vec{G}_1|^2}{4\pi} \left((c_{66} + \gamma)^{-1} \pm c_{11}^{-1} \right) \quad (51b)$$

$$K_5 = \frac{a_0|\vec{G}_1|}{2\pi} (c_{66}c_{11}^{-1} - \gamma(\gamma + c_{66})^{-1}) \quad (51c)$$

$$K_6 = \frac{a_0|\vec{G}_1|}{2\pi} ((c_{11} - c_{66})c_{11}^{-1} - \gamma(\gamma + c_{66})^{-1}) \quad (51d)$$

and thus contain information both about elastic constants and longwavelength disorder. The only other modification is the nature of the \vec{m} charges, detailed above.

2.3 Electromagnetic Coulomb gas with vector charges

2.3.1 Definition

To study the scaling behaviour of the two above models with and without disorder, it appears necessary to consider a general electromagnetic Coulomb gas with vector charges. In full generality, we will consider replicated charges \vec{b}^a, \vec{m}^a of n components. Each component of the Burgers charges \vec{b}^a lies on the direct lattice, while components of the \vec{m}^a charges are reciprocal lattice vectors. Any additional condition on the allowed charges, specific to the model considered, will be detailed at a later stage of the study. Our derivation of the renormalization equations will stick to the most general model. The partition function of this Coulomb gas is defined by

$$Z = \sum_{\{\vec{r}_\alpha, \vec{b}_\alpha^a(\vec{r}_\alpha), \vec{m}_\alpha^a(\vec{r}_\alpha)\}} \prod_{\alpha} Y[\vec{b}_\alpha, \vec{m}_\alpha] \exp S[\vec{b}_\alpha^a(\vec{r}_\alpha), \vec{m}_\alpha^a(\vec{r}_\alpha)] \quad (52)$$

where the sum counts each configuration of indistinguishable charges only once. These configurations correspond to electromagnetic charges $\vec{b}_\alpha^a, \vec{m}_\alpha^a$, labelled by the index α , both located in \vec{r}_α which belongs either to a lattice (lattice Coulomb gas) or to the continuum plane with a hard core constraint (see eq. (11)). These configurations satisfy a neutrality condition :

$$\sum_{\alpha} \vec{b}_\alpha^a = \sum_{\alpha} \vec{m}_\alpha^a = \vec{0} \text{ for each } a = 1, \dots, n \quad (53)$$

The action of this Coulomb gas reads

$$\begin{aligned}
S[\vec{b}_\alpha(\vec{r}_\alpha), \vec{m}_\alpha(\vec{r}_\alpha)] &= \frac{1}{2} \sum_{\alpha \neq \beta} b_{\alpha,i}^a V_{ij}(K_1^{ab}, K_2^{ab}, \vec{r}_\alpha - \vec{r}_\beta) b_{\beta,j}^b \\
&+ \frac{1}{2} \sum_{\alpha \neq \beta} m_{\alpha,i}^a V_{ij}(K_3^{ab}, K_4^{ab}, \vec{r}_\alpha - \vec{r}_\beta) m_{\beta,j}^b \\
&+ i \sum_{\alpha \neq \beta} m_{\alpha,i}^a \left(\delta_{ij} \delta^{ab} \frac{\lambda_\Phi}{2\pi} \Phi(\vec{r}_\alpha - \vec{r}_\beta) + V_{ik}(K_5^{ab}, K_6^{ab}) \epsilon_{kj} \right) b_{\beta,j}^b
\end{aligned} \tag{54}$$

where the interaction potentials V_{ij} has been defined in (36), and the coupling matrices K_i in section (2.2.1) for the commensurate potential, and in (51) for the pinning random potential. We also define the geometrical factor

$$\lambda_\Phi = a_0 |\vec{G}_1|. \tag{55}$$

where $\lambda_\Phi = 4\pi/\sqrt{3}$ for the triangular lattice, and $\lambda_\Phi = 2\pi$ for the square lattice. Defining charge densities as

$$\vec{b}^a(\vec{r}) = \sum_{\alpha} \vec{b}_{\alpha}^a \delta(\vec{r} - \vec{r}_{\alpha}) \quad ; \quad \vec{m}^a(\vec{r}) = \sum_{\alpha} \vec{m}_{\alpha}^a \delta(\vec{r} - \vec{r}_{\alpha}) \tag{56}$$

we can express this partition function as

$$Z = \sum_{\{\vec{b}^a(\vec{r}), \vec{m}^a(\vec{r})\}} \exp \left(\int \frac{d^2 \vec{r}}{a_0^2} \ln Y[\vec{b}^a(\vec{r}), \vec{m}^a(\vec{r})] \right) \exp S[\vec{b}^a(\vec{r}), \vec{m}^a(\vec{r})] \tag{57}$$

with ⁴

$$\begin{aligned}
S[\vec{b}(\vec{r}), \vec{m}(\vec{r})] &= \frac{1}{2} b_i^a * V_{ij}(K_1^{ab}, K_2^{ab}) * b_j^b + \frac{1}{2} m_i^a * V_{ij}(K_3^{ab}, K_4^{ab}) * m_j^b \\
&+ i m_i^a * \left(\delta_{ij} \delta^{ab} \frac{\lambda_\Phi}{2\pi} \Phi + V_{ik}(K_5^{ab}, K_6^{ab}) \epsilon_{kj} \right) * b_j^b
\end{aligned} \tag{58}$$

This is a vector generalisation of the 2D scalar electromagnetic coulomb gas and of the electric vector coulomb gas which enter the standard study of melting.

⁴ Note that the angle Φ being defined up to a constant, the model is defined for configurations satisfying $\sum_{\alpha} \sum_a \vec{b}_{\alpha}^a \cdot \vec{m}_{\alpha}^a = 0$. This condition is satisfied in a bare model consisting of a collection of purely electric ($\vec{m} = \vec{0}$) and purely magnetic ($\vec{b}_{\alpha} = \vec{0}$) charges. Without this condition, a change of definition of the angle $\Phi \rightarrow \Phi + \theta_0$ is accompanied by a redefinition of the fugacities for composites charges : $Y[b, m] \rightarrow Y[b, m] \exp[-i\theta_0 b \cdot m]$.

2.3.2 Electromagnetic duality

In 2D coulomb gas, the Kramers-Wannier duality corresponds to the interchange of electric and magnetic charges : $\vec{b} \leftrightarrow \vec{m}$. In the usual scalar ECG, this corresponds to the interchange of strong and weak coupling regimes of the theory : $g \leftrightarrow 1/g$ where g is the coupling constant of the ECG. For the present general VECG, this duality transformation can be inferred by writing explicitly the action (58) as

$$\begin{aligned} S[\vec{b}(\vec{r}), \vec{m}(\vec{r})] = & \frac{1}{2} \sum_{\alpha \neq \beta} \left[K_1^{ac} (\vec{b}_\alpha^a \cdot \vec{b}_\beta^c) G(r_{\alpha\beta}) - K_2^{ac} \left((\vec{b}_\alpha^a \cdot \hat{r}_{\alpha\beta}) (\vec{b}_\beta^c \cdot \hat{r}_{\alpha\beta}) - \frac{1}{2} (\vec{b}_\alpha^a \cdot \vec{b}_\beta^c) \right) \right] \\ & + \frac{1}{2} \sum_{\alpha \neq \beta} \left[K_3^{ac} (\vec{m}_\alpha^a \cdot \vec{m}_\beta^c) G(r_{\alpha\beta}) - K_4^{ac} \left((\vec{m}_\alpha^a \cdot \hat{r}_{\alpha\beta}) (\vec{m}_\beta^c \cdot \hat{r}_{\alpha\beta}) - \frac{1}{2} (\vec{m}_\alpha^a \cdot \vec{m}_\beta^c) \right) \right] \\ & + i \sum_{\alpha \neq \beta} \left[(\vec{m}_\alpha^a \cdot \vec{b}_\beta^a) \frac{\lambda_\Phi}{2\pi} \Phi(\vec{r}_{\alpha\beta}) \right. \\ & \left. + K_5^{ac} (\vec{m}_\alpha^a \cdot (\vec{b}_\beta^\perp)^c) G(\vec{r}_{\alpha\beta}) - K_6^{ac} \left((\vec{m}_\alpha^a \cdot \hat{r}_{\alpha\beta}) ((\vec{b}_\beta^\perp)^c \cdot \hat{r}_{\alpha\beta}) - \frac{1}{2} (\vec{m}_\alpha^a \cdot (\vec{b}_\beta^\perp)^c) \right) \right] \end{aligned}$$

with the convention $a_i^\perp = \epsilon_{ij} a_j$. Inspection of the above expression, and the relation⁵ $\hat{r}_i \hat{r}_j + \hat{r}_i^\perp \hat{r}_j^\perp = \delta_{ij}$ (or $H_{ij}(\hat{r}^\perp) = -H_{ij}(\hat{r})$), shows that performing the simultaneous change:

$$(\vec{b}_\alpha, \vec{m}_\alpha) \rightarrow (\vec{b}'_\alpha = \vec{m}_\alpha^\perp, \vec{m}'_\alpha = \vec{b}_\alpha^\perp) \quad (59)$$

and

$$K_1 \rightarrow K'_1 = K_3 \quad , \quad K_3 \rightarrow K'_3 = K_1 \quad (60)$$

$$K_2 \rightarrow K'_2 = -K_4 \quad , \quad K_4 \rightarrow K'_4 = -K_2 \quad (61)$$

$$K_5 \rightarrow K'_5 = -K_5 \quad , \quad K_6 \rightarrow K'_6 = -K_6 \quad (62)$$

leaves the action unchanged. This is the duality transformation. Note that the symmetry by orientation change $B \rightarrow -B$ (or time reversal) corresponds to $i \rightarrow -i$. It affects only the \vec{b}/\vec{m} interaction.

3 Renormalization of the Coulomb gas

The renormalization of this electromagnetic Coulomb gas goes along the lines of the Coulomb gas with scalar charges (Nienhuis) : upon increasing the real space cut-off $a_0 \rightarrow a_0 e^{dl}$ (corresponding to the size of the charges), we have

⁵ Note also the useful relation $\hat{r}_i \hat{r}_j^\perp - \hat{r}_j \hat{r}_i^\perp = -\epsilon_{ij}$

to consider three different processes : (i) the simple rescaling of the partition functions's integration measures and the Coulomb interaction, (ii) the screening or annihilation of charges, corresponding to the modification of the Coulomb interaction of distant charges by two opposite charges distant by less than the new cut-off $a_0 e^{dl}$, and (iii) the fusion of charges when two non-opposite charges distant by less than the new cut-off have to be considered as a new single charge at the new scale. We will consider successively this three processes.

3.1 Reparametrization

Simple rescaling of the cut-off $a_0 \rightarrow a_0 e^{dl}$ into the integration measure ($d^2\vec{r}/a_0^2$) and the Coulomb interaction (from the terms containing $\ln(r/a_0)$) results in the eigenvalue

$$\partial_l Y[\vec{\mathbf{b}}, \vec{\mathbf{m}}] = \left(2 - \frac{1}{2} \left(\vec{b}^a \cdot \vec{b}^b K_1^{ab} + \vec{m}^a \cdot \vec{m}^b K_3^{ab} + 2im_i^a \epsilon_{ij} b_j^b K_5^{ab} \right) \right) Y[\vec{\mathbf{b}}, \vec{\mathbf{m}}] \quad (63)$$

3.2 Fusion of charges

We consider the situation where two charges $(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1)$ and $(\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2)$ located in \vec{r}_1 and \vec{r}_2 are distant by less than the rescaled cutoff : $a_0 < |\vec{\rho}| < a_0 e^{dl}$ where we define $\vec{\rho} = \vec{r}_1 - \vec{r}_2$. The part \tilde{S}_{12} of the action (58) involving these two charges can be decomposed into their mutual interaction and the interaction with the rest of the charge configuration $\tilde{S}_{12} = S_{1,2} + \sum_{\alpha \neq 1,2} S_{1,2/\alpha}$. From now on, we will use the notation

$$V_{(1),ij}^{ab} = V_{ij}(K_1^{ab}, K_2^{ab}) \quad ; \quad V_{(3),ij}^{ab} = V_{ij}(K_3^{ab}, K_4^{ab}) \quad (64)$$

$$\mathcal{G}_{ij}^{ab} = \delta_{ij} \delta^{ab} \frac{\lambda \Phi}{2\pi} + \epsilon_{kj} V_{ik}(K_5^{ab}, K_6^{ab}) \quad (65)$$

With this notation, the mutual interaction between charges 1 and 2 reads

$$S_{1,2}(\vec{\rho}) = b_{1,i}^a V_{(1),ij}^{ab}(\vec{\rho}) b_{2,j}^b + m_{1,i}^a V_{(3),ij}^{ab}(\vec{\rho}) m_{2,j}^b + i \left(m_{1,i}^a \mathcal{G}_{ij}^{ab}(\vec{\rho}) b_{2,j}^b + m_{2,i}^a \mathcal{G}_{ij}^{ab}(-\vec{\rho}) b_{1,j}^b \right) \quad (66)$$

Similarly the interaction between this pair and another charge α is written as

$$S_{1,2/\alpha} = b_{1,i}^a V_{(1),ij}^{ab}(\vec{r}_1 - \vec{r}_\alpha) b_{\alpha,j}^b + m_{1,i}^a V_{(3),ij}^{ab}(\vec{r}_1 - \vec{r}_\alpha) m_{\alpha,j}^b + i \left(m_{1,i}^a \mathcal{G}_{ij}^{ab}(\vec{r}_1 - \vec{r}_\alpha) b_{\alpha,j}^b + m_{\alpha,i}^a \mathcal{G}_{ij}^{ab}(\vec{r}_\alpha - \vec{r}_1) b_{1,j}^b \right) + (1 \leftrightarrow 2) \quad (67)$$

The part of the partition function involving the two charges $(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1)$ and $(\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2)$ can be written as⁶

$$Z_{1,2} = \sum_{(\vec{\mathbf{b}}_{1/2}, \vec{\mathbf{m}}_{1/2}) \in \{\vec{\mathbf{b}}_\alpha, \vec{\mathbf{m}}_\alpha\}} \left(\prod_\alpha \int \frac{d^2 \vec{r}_\alpha}{a_0^2} \right) \prod_{\alpha \neq 1,2} Y[\vec{\mathbf{b}}_\alpha, \vec{\mathbf{m}}_\alpha] Y[\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1] Y[\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2] e^{S_{1,2} + \sum_{\alpha \neq 1,2} S_{1,2/\alpha}} \quad (68)$$

We are interested in the correction of order dl coming from this partial partition function. To proceed, two cases must be distinguished : either the total charge is non zero, or $\vec{\mathbf{b}}_1 + \vec{\mathbf{b}}_2 = \vec{\mathbf{m}}_1 + \vec{\mathbf{m}}_2 = \vec{0}$. The first case corresponds to the fusion of charges considered below, and the second to the annihilation of charges (or Debye screening of the interactions), which will be considered in the next section.

In the first case we have $\vec{\mathbf{b}}_1 + \vec{\mathbf{b}}_2 \neq 0$ or/and $\vec{\mathbf{m}}_1 + \vec{\mathbf{m}}_2 \neq 0$. This gives after coarse graining a non zero effective charge located in $\vec{R} = (\vec{r}_1 + \vec{r}_2)/2$. To proceed, we assume a low density for the Coulomb gas, which amounts to consider that all interdistances $\vec{r}_\alpha - \vec{r}_\beta$ between the remaining charges are much larger than a_0 . This allows to perform a gradient expansion of the integrand $\exp(S_{1,2} + \sum_{\alpha \neq 1,2} S_\alpha)$. The first non-vanishing term of this expansion is simply the term of order 0 for the fusion of charges. To this order, the correction (68) simply reads

$$Z_{1,2} = dl \sum_{\substack{(\vec{\mathbf{b}}_{1/2}, \vec{\mathbf{m}}_{1/2}) \in \{\vec{\mathbf{b}}_\alpha, \vec{\mathbf{m}}_\alpha\} \\ (\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1) + (\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2) \neq (\vec{0}, \vec{0})}} \left(\prod_{\alpha \neq 1,2} \int \frac{d^2 \vec{r}_\alpha}{a_0^2} Y[\vec{\mathbf{b}}_\alpha, \vec{\mathbf{m}}_\alpha] \right) \int \frac{d^2 \vec{R}}{a^2} Y[\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1] Y[\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2] \left(\int d\hat{\rho} e^{S_{1,2}} \right) e^{\sum_{\alpha \neq 1,2} S_{1,2/\alpha}} + \mathcal{O}(dl^2) \quad (69)$$

where we have used the notation $\int d\hat{\rho}$ for the integral on the unit circle $\int_0^{2\pi} d\theta_{\hat{\rho}}$. The term (69) will correct the partition function over the same final configuration of charges, including the new effective charge in \vec{R} . To order 0 in the gradient expansion, $\sum_{\alpha \neq 1,2} S_{1,2/\alpha}$ provides exactly the correct interaction between the new charge and the rest of the configuration. Thus the above partition function can be absorbed into a correction to the fugacity for non zero charges

$$\partial_l Y[\vec{\mathbf{b}}, \vec{\mathbf{m}}] = \sum_{(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1) + (\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2) = (\vec{\mathbf{b}}, \vec{\mathbf{m}})} A_{(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1); (\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2)} Y[\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1] Y[\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2] \quad (70)$$

where the numerical factor

$$A_{(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1); (\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2)} = \int d\hat{\rho} \exp(S[(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1); (\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2)]) \quad (71)$$

⁶ Note that the multiple integral should be restricted to the domain $|\vec{r}_\alpha - \vec{r}_\beta| \geq a_0$

with the action $S[(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1); (\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2)]$ given by (66) with $\rho = a_0$:

$$\begin{aligned} S[(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1); (\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2)] = & -(\vec{\mathbf{b}}_1)_i^a (\vec{\mathbf{b}}_2)_j^b K_2^{ab} H_{ij}(\hat{\rho}) - (\vec{\mathbf{m}}_1)_i^a (\vec{\mathbf{m}}_2)_j^b K_4^{ab} H_{ij}(\hat{\rho}) \\ & + i(\vec{\mathbf{m}}_1)_i^a (\vec{\mathbf{b}}_2)_i^a \frac{\lambda_\Phi}{2\pi} \Phi(\hat{\rho}) - i(\vec{\mathbf{m}}_1)_i^a (\vec{\mathbf{b}}_2)_j^b K_6^{ab} \epsilon_{kj} H_{ik}(\hat{\rho}) \\ & + i(\vec{\mathbf{m}}_2)_i^a (\vec{\mathbf{b}}_1)_i^a \frac{\lambda_\Phi}{2\pi} \Phi(\hat{\rho}) - i(\vec{\mathbf{m}}_2)_i^a (\vec{\mathbf{b}}_1)_j^b K_6^{ab} \epsilon_{kj} H_{ik}(\hat{\rho}) \end{aligned} \quad (72)$$

In the case $K_2 = K_4 = K_6 = 0$, the angular integration (71) provides the constraint $\sum_{a,i} ((\vec{\mathbf{m}}_1)_i^a (\vec{\mathbf{b}}_2)_i^a + (\vec{\mathbf{m}}_2)_i^a (\vec{\mathbf{b}}_1)_i^a) = 0$ upon fusion, implying that the condition $\sum_a \vec{\mathbf{m}}^a \cdot \vec{\mathbf{b}}^a = 0$ is preserved. Unlike the scalar case, this is not sufficient to forbid the generation of composite charges. For arbitrary K_2, K_4, K_6 , these composite charges will certainly be generated upon coarse-graining.

3.3 Annihilation of charges : the screening

Now we consider the situation of two opposite charges $\vec{\mathbf{b}}_1 + \vec{\mathbf{b}}_2 = \vec{\mathbf{m}}_1 + \vec{\mathbf{m}}_2 = \vec{0}$. The correction to the partition function coming from the configurations with these opposite charges still take the form of (68), with the condition $\vec{\mathbf{b}}_1 = -\vec{\mathbf{b}}_2; \vec{\mathbf{m}}_1 = -\vec{\mathbf{m}}_2$. This condition implies that the first term of the gradient expansion, considered in (69), now only provides a constant term to the free energy, which we will neglect. To get the first non-trivial corrections to the system's thermodynamics, we have to consider this gradient expansion up to second order. To this purpose, we expand the action $S_{1,2/\alpha}$ in powers of ρ , *i.e.* of a_0 , with charges $\vec{\mathbf{b}}_{1/2}, \vec{\mathbf{m}}_{1/2}$ now located in \vec{R} . In the present case the terms of order 0 and 2 vanish as the pair 1, 2 is neutral, and we obtain

$$\begin{aligned} S_{1,2/\alpha} = & b_{1,i}^a \rho_\eta \partial_\eta V_{(1),ij}^{ab}(\vec{R} - \vec{r}_\alpha) b_{\alpha,j}^b + m_{1,i}^a \rho_\eta \partial_\eta V_{(3),ij}^{ab}(\vec{R} - \vec{r}_\alpha) m_{\alpha,j}^b \\ & + i \left(m_{1,i}^a \rho_\eta \partial_\eta \mathcal{G}_{ij}^{ab}(\vec{R} - \vec{r}_\alpha) b_{\alpha,j}^b - m_{\alpha,i}^a \rho_\eta \partial_\eta \mathcal{G}_{ij}^{ab}(\vec{r}_\alpha - \vec{R}) b_{1,j}^b \right) + \mathcal{O}(a_0^3) \end{aligned} \quad (73)$$

Expanding the second exponential to second order in a_0 , the correction (68) takes the form⁷

$$\begin{aligned} Z_{1,2} = & \sum_{\{\vec{\mathbf{b}}_\alpha, \vec{\mathbf{m}}_\alpha\}, \alpha \neq 1,2} \left(\prod_{\alpha \neq 1,2} \int \frac{d^2 \vec{r}_\alpha}{a_0^2} Y[\vec{\mathbf{b}}_\alpha, \vec{\mathbf{m}}_\alpha] \right) \frac{1}{2} \sum_{(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1)} Y[\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1] Y[-\vec{\mathbf{b}}_1, -\vec{\mathbf{m}}_1] \\ & \int \frac{d^2 \vec{R}}{a_0^2} \int_{a_0 \leq |\vec{\rho}| \leq a_0 e^{dL}} \frac{d^2 \vec{\rho}}{a_0^2} \left(1 + \sum_\alpha S_{1,2/\alpha} + \frac{1}{2} \sum_{\alpha,\beta} S_{1,2/\alpha} S_{1,2/\beta} \right) e^{\tilde{S}[\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1]} \end{aligned} \quad (74)$$

⁷ Notice the $\frac{1}{2}$ factor in front of the sum over $(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1)$, which accounts for the indiscernability of the charges 1 and 2.

with $S_{1,2/\alpha}$ given by (73) and $\tilde{S}[\vec{b}_1, \vec{m}_1]$ by (66) with $\vec{b}_2 = -\vec{b}_1, \vec{m}_2 = -\vec{m}_1$:

$$\begin{aligned} \tilde{S}[\vec{b}_1, \vec{m}_1] = & -b_{1,i}^a V_{(1),ij}^{ab}(\vec{\rho}) b_{1,j}^b - m_{1,i}^a V_{(3)}^{ab}(\vec{\rho}) m_{1,j}^b \\ & - i \left(m_{1,i}^a \mathcal{G}_{ij}^{ab}(\vec{\rho}) b_{1,j}^b + m_{1,i}^a \mathcal{G}_{ij}^{ab}(-\vec{\rho}) b_{1,j}^b \right) \end{aligned} \quad (75)$$

As explained above, the first term can be neglected as it renormalizes by a constant the free energy. The second term vanishes by the symmetry $\vec{\rho} \rightarrow -\vec{\rho}$ of the integral. Using $\int d^2\vec{\rho}/a_0^2 = dl \int \hat{\rho}$ (where the last integral runs over the unit circle), the correction from (74) that we will focus on can be written explicitly as⁸

$$Z_{1,2} = \sum_{\{\vec{b}_\alpha, \vec{m}_\alpha\}, \alpha \neq 1,2} \left(\prod_{\alpha \neq 1,2} \int \frac{d^2\vec{r}_\alpha}{a_0^2} Y[\vec{b}_\alpha, \vec{m}_\alpha] \right) \frac{1}{2} \sum_{\alpha, \beta} dS[(\vec{b}_\alpha, \vec{m}_\alpha); (\vec{b}_\beta, \vec{m}_\beta)] \quad (76)$$

with the (correction to the) action

$$\begin{aligned} dS[(\vec{b}_\alpha, \vec{m}_\alpha); (\vec{b}_\beta, \vec{m}_\beta)] = & \quad (77) \\ dl \frac{1}{2} \sum_{(\vec{b}_1, \vec{m}_1)} & Y[\vec{b}_1, \vec{m}_1] Y[-\vec{b}_1, -\vec{m}_1] \int d^2\vec{R} \int d\hat{\rho} e^{\tilde{S}[\vec{b}_1, \vec{m}_1]} \hat{\rho}_s \hat{\rho}_t \\ & \left[b_{1,i}^a \partial_s V_{(1),ij}^{ab} b_{\alpha,j}^b + m_{1,i}^a \partial_s V_{(3),ij}^{ab} m_{\alpha,j}^b + i \left(m_{1,i}^a \partial_s \mathcal{G}_{ij}^{ab} b_{\alpha,j}^b + m_{\alpha,j}^b \partial_s \mathcal{G}_{ji}^{ba} b_{1,i}^a \right) \right] \\ & \times \left[b_{1,k}^c \partial_t V_{(1),kl}^{cd} b_{\beta,l}^d + m_{1,k}^c \partial_t V_{(3),kl}^{cd} m_{\beta,l}^d + i \left(m_{1,k}^c \partial_t \mathcal{G}_{kl}^{cd} b_{\beta,l}^d + m_{\beta,l}^d \partial_t \mathcal{G}_{lk}^{dc} b_{1,k}^c \right) \right] \end{aligned}$$

⁸ Note that similarly to the case of the scalar Coulomb gas, the term $\alpha = \beta$ in this sum generates a renormalisation of order Y^3 to the fugacity $Y[\vec{b}, \vec{m}]$, which will be neglected in the present study.

This correction to the action between two charges can be rewritten as

$$\begin{aligned}
dS[(\vec{\mathbf{b}}_\alpha, \vec{\mathbf{m}}_\alpha); (\vec{\mathbf{b}}_\beta, \vec{\mathbf{m}}_\beta)] = & \quad (78) \\
& b_{\alpha,j}^b b_{\beta,l}^d \left[[M_1]_{st,ik}^{ac} [I_1^{(1,1)}]_{s,ij;t,kl}^{ab;cd}(\vec{r}_{\alpha\beta}) + i[M_2]_{st,ik}^{ac} [I_2^{(1)}]_{s,ij;t,kl}^{ab;cd}(\vec{r}_{\alpha\beta}) \right. \\
& \quad \left. + i[M_2]_{st,ki}^{ca} [I_2^{(1)}]_{t,kl;s,ij}^{cd;ab}(\vec{r}_{\alpha\beta}) - [M_3]_{st,ik}^{ac} [I_3]_{s,ij;t,kl}^{ab;cd}(\vec{r}_{\alpha\beta}) \right] \\
& + m_{\alpha,j}^b m_{\beta,l}^d \left[[M_3]_{st,ik}^{ac} [I_1^{(3,3)}]_{s,ij;t,kl}^{ab;cd}(\vec{r}_{\alpha\beta}) + i[M_2]_{st,ki}^{ca} [I_2^{(3)}]_{s,ij;t,kl}^{ab;dc}(\vec{r}_{\alpha\beta}) \right. \\
& \quad \left. + i[M_2]_{st,ik}^{ac} [I_2^{(3)}]_{t,kl;s,ji}^{cd;ba}(\vec{r}_{\alpha\beta}) - [M_1]_{st,ik}^{ac} [I_3]_{s,ji;t,kl}^{ba;dc}(\vec{r}_{\alpha\beta}) \right] \\
& + b_{\alpha,j}^b m_{\beta,l}^d \left[[M_2]_{st,ik}^{ac} [I_1^{(1,3)}]_{s,ij;t,kl}^{ab;cd}(\vec{r}_{\alpha\beta}) + i[M_1]_{st,ik}^{ac} [I_2^{(1)}]_{s,ij;t,kl}^{ab;dc}(\vec{r}_{\alpha\beta}) \right. \\
& \quad \left. + i[M_3]_{st,ik}^{ac} [I_2^{(3)}]_{t,kl;s,ij}^{cd;ab}(\vec{r}_{\alpha\beta}) - [M_2]_{st,ki}^{ca} [I_3]_{s,ij;t,kl}^{ab;dc}(\vec{r}_{\alpha\beta}) \right] \\
& + m_{\alpha,j}^b b_{\beta,l}^d \left[[M_2]_{st,ki}^{ca} [I_1^{(3,1)}]_{s,ij;t,kl}^{ab;cd}(\vec{r}_{\alpha\beta}) + i[M_1]_{st,ik}^{ac} [I_2^{(1)}]_{t,kl;s,ji}^{cd;ba}(\vec{r}_{\alpha\beta}) \right. \\
& \quad \left. + i[M_3]_{st,ik}^{ac} [I_2^{(3)}]_{s,ij;t,kl}^{ab;cd}(\vec{r}_{\alpha\beta}) - [M_2]_{st,ik}^{ac} [I_3]_{s,ji;t,kl}^{ba;cd}(\vec{r}_{\alpha\beta}) \right]
\end{aligned}$$

where we define the tensors relative respectively to the integration over $\vec{\rho}$ and \vec{R} :

$$[M_1]_{st,ik}^{ac} = \frac{dl}{2} \sum_{(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1)} Y[\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1] Y[-\vec{\mathbf{b}}_1, -\vec{\mathbf{m}}_1] \int d\hat{\rho} e^{\tilde{S}[\vec{b}_1, \vec{m}_1]} \hat{\rho}_s \hat{\rho}_t b_{1,i}^a b_{1,k}^c \quad (79a)$$

$$[M_2]_{st,ik}^{ac} = \frac{dl}{2} \sum_{(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1)} Y[\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1] Y[-\vec{\mathbf{b}}_1, -\vec{\mathbf{m}}_1] \int d\hat{\rho} e^{\tilde{S}[\vec{b}_1, \vec{m}_1]} \hat{\rho}_s \hat{\rho}_t b_{1,i}^a m_{1,k}^c \quad (79b)$$

$$[M_3]_{st,ik}^{ac} = \frac{dl}{2} \sum_{(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1)} Y[\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1] Y[-\vec{\mathbf{b}}_1, -\vec{\mathbf{m}}_1] \int d\hat{\rho} e^{\tilde{S}[\vec{b}_1, \vec{m}_1]} \hat{\rho}_s \hat{\rho}_t m_{1,i}^a m_{1,k}^c \quad (79c)$$

$$[I_1^{(1,3)}]_{s,ij;t,kl}^{ab;cd}(\vec{r}_\alpha - \vec{r}_\beta) = \int d^2 \vec{R} \partial_s V_{(1),ij}^{ab}(\vec{R} - \vec{r}_\alpha) \partial_t V_{(3),kl}^{cd}(\vec{R} - \vec{r}_\beta) \quad (80a)$$

$$[I_2^{(1)}]_{s,ij;t,kl}^{ab;cd}(\vec{r}_\alpha - \vec{r}_\beta) = \int d^2 \vec{R} \partial_s V_{(1),ij}^{ab}(\vec{R} - \vec{r}_\alpha) \partial_t \mathcal{G}_{kl}^{cd}(\vec{R} - \vec{r}_\beta) \quad (80b)$$

$$[I_3]_{s,ij;t,kl}^{ab;cd}(\vec{r}_\alpha - \vec{r}_\beta) = \int d^2 \vec{R} \partial_s \mathcal{G}_{ij}^{ab}(\vec{R} - \vec{r}_\alpha) \partial_t \mathcal{G}_{kl}^{cd}(\vec{R} - \vec{r}_\beta) \quad (80c)$$

where all the above expressions are symmetric in α, β . We have used that all derivatives of the potentials V and \mathcal{G} are odd.

To proceed, we thus have to (i) perform the integral over \vec{R} , *i.e* calculate explicitly the tensors $I_{1,2,3}$ (ii) perform the integral over $\vec{\rho}$, *i.e* calculate explicitly the tensors $M_{1,2,3}$, and finally (iii) contract all the tensors in (78). If this final contraction can be cast into contributions to the initial potential $V_{ij}^{ab}(\vec{r}), \mathcal{G}_{ij}^{ab}(\vec{r})$, this will prove the renormalizability of the present vector Coulomb gaz to one loop.

3.3.1 Integration over \vec{R}

We focus on the tensors $I_{1,2,3}$, which are all integral of double products of gradients of V, \mathcal{G} . These integrations are conveniently done in Fourier space, and we start by obtaining Fourier representation of these potential's gradients : with the definition of the projectors $P_{ij}^L(\hat{q}) = \hat{q}_i \hat{q}_j$ and $P_{ij}^T(\hat{q}) = \delta_{ij} - \hat{q}_i \hat{q}_j = \epsilon_{ik} \epsilon_{jl} P_{kl}^L(\hat{q})$, we obtain, from the definition

$$V_{ij}^{ab}(K_1, K_2)(\vec{r}) = \int \frac{d^2 \vec{q}}{(2\pi)^2} (1 - e^{i \vec{q} \cdot \vec{r}}) \frac{2\pi}{q^2} [(K_1 - K_2)^{ab} P_{ij}^L + (K_1 + K_2)^{ab} P_{ij}^T] \quad (81)$$

the expression or its gradient

$$\begin{aligned} \partial_s V_{(1),ij}^{ab}(\vec{r}) &= -2\pi i [(K_1 - K_2)^{ab} \delta_{ik} \delta_{jl} + (K_1 + K_2)^{ab} \epsilon_{ik} \epsilon_{jl}] \int \frac{d^2 \vec{q}}{(2\pi)^2} e^{i \vec{q} \cdot \vec{r}} \frac{q_s}{q^2} P_{kl}^L(\hat{q}) \\ &\equiv -2\pi i \mathcal{C}_{ijkl}^{ab}(K_1, K_2) \int \frac{d^2 \vec{q}}{(2\pi)^2} e^{i \vec{q} \cdot \vec{r}} \frac{q_s q_k q_l}{q^4} \end{aligned} \quad (82)$$

Similarly, using the equality (65) the second gradient reads

$$\partial_s \mathcal{G}_{ij}^{ab}(\vec{r}) = \frac{-\lambda_\Phi}{2\pi} \delta^{ab} \epsilon_{st} \partial_t V_{ij}(1, 0) + \epsilon_{mj} \partial_s V_{im}^{ab}(K_5, K_6) \quad (83)$$

$$= -2i\pi \left[\frac{-\lambda_\Phi}{2\pi} \delta^{ab} \epsilon_{st} \mathcal{C}_{ijkl}(1, 0) + \epsilon_{mj} \delta_{st} \mathcal{C}_{imkl}^{ab}(K_5, K_6) \right] \int \frac{d^2 \vec{q}}{(2\pi)^2} e^{i \vec{q} \cdot \vec{r}} \frac{q_t}{q^2} P_{kl}^L \quad (84)$$

$$\equiv -2i D_{ijkl,st}^{ab} \int \frac{d^2 \vec{q}}{(2\pi)^2} e^{i \vec{q} \cdot \vec{r}} \frac{q_t q_k q_l}{q^4} \quad (85)$$

where we have defined

$$\mathcal{C}_{ijkl}^{ab}(K_1, K_2) = (K_1 - K_2)^{ab} \delta_{ik} \delta_{jl} + (K_1 + K_2)^{ab} (\delta_{ij} \delta_{kl} - \delta_{il} \delta_{jk}) \quad (86)$$

$$\begin{aligned} D_{ijkl,st}^{ab} &= -\frac{\lambda_\Phi}{2\pi} \delta^{ab} \epsilon_{st} [\delta_{ik} \delta_{jl} + \delta_{ij} \delta_{kl} - \delta_{il} \delta_{jk}] \\ &\quad + \epsilon_{mj} \delta_{st} [(K_5 - K_6)^{ab} \delta_{ik} \delta_{ml} + (K_5 + K_6)^{ab} (\delta_{im} \delta_{kl} - \delta_{il} \delta_{mk})] \end{aligned} \quad (87)$$

With this representation, the integrals $I_{1,2,3}$ are expressed as

$$[I_1^{(1,3)}]_{s,ij;t,kl}^{ab;cd}(\vec{r}_\alpha - \vec{r}_\beta) = -4\pi^2 \mathcal{C}_{ijmn}^{ab}(K_1, K_2) \mathcal{C}_{klpq}^{cd}(K_3, K_4) Q_{mnpqst}(\vec{r}_\alpha - \vec{r}_\beta) \quad (88)$$

$$[I_2^{(1)}]_{s,ij;t,kl}^{ab;cd}(\vec{r}_\alpha - \vec{r}_\beta) = -4\pi^2 \mathcal{C}_{ijmn}^{ab}(K_1, K_2) D_{klpq,tu}^{cd} Q_{mnpqsu}(\vec{r}_\alpha - \vec{r}_\beta) \quad (89)$$

$$[I_3]_{s,ij;t,kl}^{ab;cd}(\vec{r}_\alpha - \vec{r}_\beta) = -4\pi^2 D_{ijmn,su}^{ab} D_{klpq,tv}^{cd} Q_{mnpq,uv}(\vec{r}_\alpha - \vec{r}_\beta) \quad (90)$$

where we have defined the integral

$$\begin{aligned} Q_{klmnst}(\vec{r}_\alpha - \vec{r}_\beta) &= \int d^2 \vec{R} \int \frac{d^2 \vec{q}}{(2\pi)^2} \int \frac{d^2 \vec{q}'}{(2\pi)^2} e^{i(\vec{q} \cdot (\vec{R} - \vec{r}_\alpha) + \vec{q}' \cdot (\vec{R} - \vec{r}_\beta))} \frac{q_s q_k q_l q'_t q'_m q'_n}{q^4 (q')^4} \\ &= \frac{\partial^6}{\partial_s \partial_t \partial_k \partial_l \partial_m \partial_n} \int \frac{d^2 \vec{q}}{(2\pi)^2} \frac{1}{q^8} e^{i\vec{q} \cdot (\vec{r}_\alpha - \vec{r}_\beta)} \end{aligned} \quad (91)$$

Using the Schwinger representation, the last integral yields :

$$\int \frac{d^2 \vec{q}}{(2\pi)^2} \frac{1}{q^8} e^{i\vec{q} \cdot \vec{r}} = \frac{1}{6} \int \frac{d^2 \vec{q}}{(2\pi)^2} \int_0^\infty du u^3 e^{-uq^2 + i\vec{q} \cdot \vec{r}} = \frac{1}{12} \int_0^\infty \frac{du}{2\pi u} u^3 e^{-\frac{r^2}{4u}} \quad (92)$$

The differentiation of the gaussian up to order 6 is now straightforward :

$$\begin{aligned} \frac{\partial^6}{\partial_s \partial_t \partial_k \partial_l \partial_m \partial_n} \left(e^{-\frac{A}{2} r^2} \right) &= \left[-A^3 \delta_{st} \delta_{kl} \delta_{mn} + c.p.(15\text{terms}) \right. \\ &\quad + A^4 r_s r_t \delta_{kl} \delta_{mn} + c.p.(45\text{terms}) \\ &\quad - A^5 r_s r_t r_k r_l \delta_{mn} + c.p.(15\text{terms}) \\ &\quad \left. + A^6 r_s r_t r_k r_l r_m r_n \right] e^{-\frac{A}{2} r^2} \end{aligned} \quad (93)$$

where *c.p.* means circular permutation of the indices (the number of corresponding permuted terms is indicated). Finally, using $\int_0^\infty du u^\beta e^{-u} = \Gamma(\beta + 1)$, we find :

$$\begin{aligned} 12 \times Q_{stklmn}(\vec{r}) &= \frac{1}{16\pi} Ei \left(-\frac{r^2}{4L^2} \right) \delta_{st} \delta_{kl} \delta_{mn} + \frac{1}{8\pi} \hat{r}_s \hat{r}_t \delta_{kl} \delta_{mn} \\ &\quad - \frac{1}{4\pi} \hat{r}_s \hat{r}_t \hat{r}_k \hat{r}_l \delta_{mn} + \frac{1}{\pi} \hat{r}_s \hat{r}_t \hat{r}_k \hat{r}_l \hat{r}_m \hat{r}_n + (c.p.) \end{aligned} \quad (94)$$

In this expression L stands for an IR cut-off. We will use the following asymptotic limit for the exponential integral[6] : $Ei(-x) \simeq \gamma + \ln(x)$ in the limit $x \rightarrow 0$. The expressions (86,87,94), together with the contractions formula (88,89,90) constitute our final explicit expressions for the integrals $I_{1,2,3}$.

3.3.2 Integration over $\vec{\rho}$

The invariance under $2\pi/3$ rotations of the integrals in $M_{1,2,3}$, defined in eq. (79), ensures that these tensors are isotropic, provided that the fugacity of a vector charge $Y[\vec{\mathbf{b}}, \vec{\mathbf{m}}]$ is constant under any rotation of the charge $\vec{\mathbf{b}}, \vec{\mathbf{m}}$ (in particular, this implies $Y[\vec{\mathbf{b}}, \vec{\mathbf{m}}] = Y[-\vec{\mathbf{b}}, -\vec{\mathbf{m}}]$). Using this isotropy, we decompose the tensors $M_{1,2,3}$ according to⁹

$$[M_w]_{st,ik}^{ac} = dl \left(\left(\Gamma_w^{ac} - \tilde{\Gamma}_w^{ac} \right) T_{st,ik} + \tilde{\Gamma}_w^{ac} \tilde{T}_{st,ik} \right) \quad ; \quad w = 1, 3 \quad (95)$$

$$i[M_2]_{st,ik}^{ac} = dl \left(\left(\Gamma_2^{ac} - \tilde{\Gamma}_2^{ac} \right) U_{st,ik} + \tilde{\Gamma}_2^{ac} \tilde{U}_{st,ik} \right) \quad (96)$$

and where we used the definitions of the symmetric and antisymmetric tensors

$$T_{st,ik} = \delta_{st}\delta_{ik} \quad ; \quad \tilde{T}_{st,ik} = \delta_{si}\delta_{tk} + \delta_{ti}\delta_{sk} \quad (97)$$

$$U_{st,ik} = \delta_{st}\epsilon_{ik} \quad ; \quad \tilde{U}_{st,ik} = \delta_{sk}\epsilon_{it} + \delta_{tk}\epsilon_{is} \quad (98)$$

By using (note the unusual definition of the trace) :

$$\text{Tr}(AB) \equiv \sum_{st,ik} A_{st,ik} B_{st,ik}, \quad (99)$$

$$\text{Tr}(T^2) = 4, \text{Tr}(T\tilde{T}) = 4, \text{Tr}(\tilde{T}\tilde{T}) = 12, \quad (100)$$

$$\text{Tr}(U^2) = 4, \text{Tr}(\tilde{U}^2) = 12, \text{Tr}(U\tilde{U}) = 4, \quad (101)$$

we obtain the formal expression for the coefficients $\Gamma_w^{ac}, \tilde{\Gamma}_w^{ac}$:

$$dl\Gamma_w^{ac} = \frac{1}{4}\text{Tr}(TM_w^{ac}) \quad ; \quad w = 1, 3 \quad (102a)$$

$$dl\tilde{\Gamma}_w^{ac} = -\frac{1}{8}\text{Tr}(TM_w^{ac}) + \frac{1}{8}\text{Tr}(\tilde{T}M_w^{ac}) \quad ; \quad w = 1, 3 \quad (102b)$$

$$dl\Gamma_2^{ac} = \frac{i}{4}\text{Tr}(UM_2^{ac}) \quad (102c)$$

$$dl\tilde{\Gamma}_2^{ac} = -\frac{i}{8}\text{Tr}(UM_2^{ac}) + \frac{i}{8}\text{Tr}(\tilde{U}M_2^{ac}) \quad (102d)$$

⁹ The tensor U and \tilde{U} arises as can be seen e.g. by expanding the definition of $M^{(2)}$ to first order in K_6 which yields tensors of the form (in the case $m.b = 0$)

$$\sum_{b,m} \int d\rho \hat{\rho}_s \hat{\rho}_t b_i m_k \left((\rho.m)(\rho.b^\perp) - \frac{1}{2} m.b^\perp \right)$$

with

$$\text{Tr}(TM_1^{ac}) = \frac{dl}{2} \sum_{(\vec{\mathbf{b}}, \vec{\mathbf{m}})} Y^2[\vec{\mathbf{b}}, \vec{\mathbf{m}}] (\vec{b}^a \cdot \vec{b}^c) \int d\hat{\rho} e^{\tilde{S}[\vec{b}, \vec{m}]} \quad (103a)$$

$$\text{Tr}(\tilde{T}M_1^{ac}) = dl \sum_{(\vec{\mathbf{b}}, \vec{\mathbf{m}})} Y^2[\vec{\mathbf{b}}, \vec{\mathbf{m}}] \int d\hat{\rho} e^{\tilde{S}[\vec{b}, \vec{m}]} (\hat{\rho} \cdot \vec{b}^a) (\hat{\rho} \cdot \vec{b}^c) \quad (103b)$$

$$\text{Tr}(UM_2^{ac}) = -\frac{dl}{2} \sum_{(\vec{\mathbf{b}}, \vec{\mathbf{m}})} Y^2[\vec{\mathbf{b}}, \vec{\mathbf{m}}] (\vec{b}^{a,\perp} \cdot \vec{m}^c) \int d\hat{\rho} e^{\tilde{S}[\vec{b}, \vec{m}]} \quad (103c)$$

$$\text{Tr}(\tilde{U}M_2^{ac}) = -dl \sum_{(\vec{\mathbf{b}}, \vec{\mathbf{m}})} Y^2[\vec{\mathbf{b}}, \vec{\mathbf{m}}] \int d\hat{\rho} e^{\tilde{S}[\vec{b}, \vec{m}]} [(\hat{\rho} \cdot \vec{b}^{a,\perp}) (\hat{\rho} \cdot \vec{m}^c)] \quad (103d)$$

$$\text{Tr}(TM_3^{ac}) = \frac{dl}{2} \sum_{(\vec{\mathbf{b}}, \vec{\mathbf{m}})} Y^2[\vec{\mathbf{b}}, \vec{\mathbf{m}}] (\vec{m}^a \cdot \vec{m}^c) \int d\hat{\rho} e^{\tilde{S}[\vec{b}, \vec{m}]} \quad (103e)$$

$$\text{Tr}(\tilde{T}M_3^{ac}) = dl \sum_{(\vec{\mathbf{b}}, \vec{\mathbf{m}})} Y^2[\vec{\mathbf{b}}, \vec{\mathbf{m}}] \int d\hat{\rho} e^{\tilde{S}[\vec{b}, \vec{m}]} (\hat{\rho} \cdot \vec{m}^a) (\hat{\rho} \cdot \vec{m}^c) \quad (103f)$$

Note the following useful relations :

$$\text{Tr}(\tilde{T}M_1^{ac}) - \text{Tr}(TM_1^{ac}) = \frac{\partial}{\partial K_2^{ac}} \left(dl \sum_{(\vec{\mathbf{b}}, \vec{\mathbf{m}})} Y^2[\vec{\mathbf{b}}, \vec{\mathbf{m}}] \int d\hat{\rho} e^{\tilde{S}[\vec{b}, \vec{m}]} \right) \quad (104)$$

$$\text{Tr}(\tilde{T}M_3^{ac}) - \text{Tr}(TM_3^{ac}) = \frac{\partial}{\partial K_4^{ac}} \left(dl \sum_{(\vec{\mathbf{b}}, \vec{\mathbf{m}})} Y^2[\vec{\mathbf{b}}, \vec{\mathbf{m}}] \int d\hat{\rho} e^{\tilde{S}[\vec{b}, \vec{m}]} \right) \quad (105)$$

$$\text{Tr}(\tilde{U}M_2^{ac}) - \text{Tr}(UM_2^{ac}) = -\frac{\partial}{\partial K_6^{ac}} \left(\frac{dl}{2} \sum_{(\vec{\mathbf{b}}, \vec{\mathbf{m}})} Y^2[\vec{\mathbf{b}}, \vec{\mathbf{m}}] \int d\hat{\rho} e^{\tilde{S}[\vec{b}, \vec{m}]} \right) \quad (106)$$

3.3.3 Final contraction of tensors

With the above expressions for the M and I tensors, we can now explicitly perform the contractions of eq. (78). This tedious task is performed using mathematica. We find that the result can be cast in the same form as the original interaction with changes dK_i in the couplings : this proves the renormalizability of the model to order Y^2 . Additional constants are produced which correct fugacities to cubic order in Y . The result of these contractions is presented in the following section.

4 Resulting RG equations for the general model

In this Section we collect and analyze the RG equations for the fugacity variables $Y[\vec{\mathbf{b}}, \vec{\mathbf{m}}]$ and the matrices K_i , $i = 1, \dots, 6$, which parameterize the general VECG model defined by the action (58).

4.1 Scaling equations for the fugacities

The equations (63,70) provide the full equations for the fugacities :

$$\begin{aligned} \partial_t Y[\vec{\mathbf{b}}, \vec{\mathbf{m}}] = & \left(2 - \frac{1}{2} \left(\vec{b}^a \cdot \vec{b}^b K_1^{ab} + \vec{m}^a \cdot \vec{m}^b K_3^{ab} + 2im_i^a \epsilon_{ij} b_j^b K_5^{ab} \right) \right) Y[\vec{\mathbf{b}}, \vec{\mathbf{m}}] \\ & + \sum_{(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1) + (\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2) = (\vec{\mathbf{b}}, \vec{\mathbf{m}})} A_{(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1); (\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2)} Y[\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1] Y[\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2] \end{aligned} \quad (107)$$

where the numerical factor $A_{(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1); (\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2)}$ is defined in eqs. (71) and (72)

$$A_{(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1); (\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2)} = \int d\hat{\rho} \exp(S[(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1); (\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2)]) \quad (108)$$

with the action $S[(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1); (\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2)]$ given by (66) with $\rho = a_0$:

$$\begin{aligned} S[(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1); (\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2)] = & -K_2^{ab} \vec{\mathbf{b}}_1^a \cdot H(\hat{\rho}) \cdot \vec{\mathbf{b}}_2^b - K_4^{ab} \vec{\mathbf{m}}_1^a \cdot H(\hat{\rho}) \cdot \vec{\mathbf{m}}_2^b \\ & + i(\vec{\mathbf{m}}_1^a \cdot \vec{\mathbf{b}}_2^a + \vec{\mathbf{m}}_2^a \cdot \vec{\mathbf{b}}_1^a) \frac{\lambda_\Phi}{2\pi} \Phi(\hat{\rho}) \\ & - iK_6^{ab} \left(\vec{\mathbf{m}}_1^a \cdot H(\hat{\rho}) \cdot \vec{\mathbf{b}}_2^{b\perp} + \vec{\mathbf{m}}_2^a \cdot H(\hat{\rho}) \cdot \vec{\mathbf{b}}_1^{b\perp} \right) \end{aligned} \quad (109)$$

The evaluation of the coefficients $A_{(\vec{\mathbf{b}}_1, \vec{\mathbf{m}}_1); (\vec{\mathbf{b}}_2, \vec{\mathbf{m}}_2)}$ is model dependent. For the models considered here, it will be performed in subsequent publication.

4.2 Scaling equations for the couplings matrices

The RG equations for the coupling constants K_i are obtained by performing the tensors contractions of eq. (78). The resulting expression is displayed in the appendix B. Here we show that their structure can be further simplified by introducing the new couplings p_i defined by:

$$p_1 = 2\pi (K_1 + K_2) \quad ; \quad p_2 = 2\pi (K_1 - K_2) \quad (110)$$

$$p_3 = 2\pi (K_3 + K_4) \quad ; \quad p_4 = 2\pi (K_3 - K_4) \quad (111)$$

$$p_5 = 2\pi (K_5 + K_6) \quad ; \quad p_6 = 2\pi (K_5 - K_6) . \quad (112)$$

In the general case the p_i (and the Γ_i and $\tilde{\Gamma}_i$ below) are commuting replica matrices. Quite remarkably, the 6 scaling equations (B.1) decouple into two independent set of 3 equations for the groups p_1, p_4, p_6 , and p_2, p_3, p_5 :

$$\begin{aligned} \partial_l p_1 = & -\Gamma_1 p_1^2 + \tilde{\Gamma}_1 p_1^2 + 2\Gamma_2 p_1 p_6 + 2\tilde{\Gamma}_2 p_1 (\lambda_\phi + p_6) \\ & + \Gamma_3 (\lambda_\phi^2 + p_6^2) + \tilde{\Gamma}_3 (\lambda_\phi + p_6)^2 \end{aligned} \quad (113a)$$

$$\begin{aligned} \partial_l p_4 = & +\Gamma_1 (\lambda_\phi^2 + p_6^2) - \tilde{\Gamma}_1 (\lambda_\phi - p_6)^2 + 2\Gamma_2 p_4 p_6 - 2\tilde{\Gamma}_2 (\lambda_\phi - p_6) p_4 \\ & - \Gamma_3 p_4^2 - \tilde{\Gamma}_3 p_4^2 \end{aligned} \quad (113b)$$

$$\begin{aligned} \partial_l p_6 = & -\Gamma_1 p_1 p_6 + \tilde{\Gamma}_1 p_1 (p_6 - \lambda_\phi) + (\Gamma_2 + \tilde{\Gamma}_2) (-\lambda_\phi^2 + p_6^2 - p_1 p_4) \\ & - \Gamma_3 p_4 p_6 - \tilde{\Gamma}_3 p_4 (\lambda_\phi + p_6) \end{aligned} \quad (113c)$$

and

$$\begin{aligned} \partial_l p_2 = & -\Gamma_1 p_2^2 - \tilde{\Gamma}_1 p_2^2 + 2\Gamma_2 p_5 p_2 - 2\tilde{\Gamma}_2 (\lambda_\phi + p_5) p_2 \\ & + \Gamma_3 (\lambda_\phi^2 + p_5^2) - \tilde{\Gamma}_3 (\lambda_\phi + p_5)^2 \end{aligned} \quad (114a)$$

$$\begin{aligned} \partial_l p_3 = & +\Gamma_1 (\lambda_\phi^2 + p_5^2) + \tilde{\Gamma}_1 (\lambda_\phi^2 - p_5)^2 + 2\Gamma_2 p_3 p_5 + 2\tilde{\Gamma}_2 (\lambda_\phi - p_5) p_3 \\ & - \Gamma_3 p_3^2 + \tilde{\Gamma}_3 p_3^2 \end{aligned} \quad (114b)$$

$$\begin{aligned} \partial_l p_5 = & -\Gamma_1 p_2 p_5 + \tilde{\Gamma}_1 p_2 (\lambda_\phi - p_5) + (\Gamma_2 - \tilde{\Gamma}_2) (-\lambda_\phi^2 + p_5^2 - p_2 p_3) \\ & - \Gamma_3 p_3 p_5 + \tilde{\Gamma}_3 (\lambda_\phi + p_5) p_3. \end{aligned} \quad (114c)$$

where the Γ_i and $\tilde{\Gamma}_i$ were defined in (102, 103). Their flow equation can be deduced from the fugacity RG equation given in the previous section.

In addition these equations possess remarkable symmetries. The following transformation:

$$p_1 \leftrightarrow p_2, p_3 \leftrightarrow p_4, p_5 \leftrightarrow p_6 \quad (115a)$$

$$\Gamma_i \rightarrow \Gamma_i; \tilde{\Gamma}_i \rightarrow -\tilde{\Gamma}_i, i = 1, \dots, 3. \quad (115b)$$

exchanges these two groups. In terms of the Coulomb gas couplings, it corresponds to $K_2 \rightarrow -K_2; K_4 \rightarrow -K_4; K_6 \rightarrow -K_6$. It can be viewed formally as a $\pi/2$ charge rotation $(\vec{b}, \vec{m}) \rightarrow (\vec{b}^\perp, \vec{m}^\perp)$ in the original action. This means that a model where the signs of K_2, K_4, K_6 are simultaneously changed is the same (up to an immaterial global rotation) with the same fugacities.

The second symmetry is the previously discussed electromagnetic duality. It operates inside each of these groups, i.e the RG equations are invariant under:

$$p'_1 = p_4 \quad ; \quad p'_4 = p_1 \quad ; \quad p'_6 = -p_6 \quad (116a)$$

$$p'_2 = p_3 \quad ; \quad p'_3 = p_2 \quad ; \quad p'_5 = -p_5 \quad (116b)$$

$$\Gamma'_1 = \Gamma_3; \tilde{\Gamma}'_1 = -\tilde{\Gamma}_3; \Gamma'_2 = -\Gamma_2; \tilde{\Gamma}'_2 = -\tilde{\Gamma}_2; \Gamma'_3 = \Gamma_1; \tilde{\Gamma}'_3 = -\tilde{\Gamma}_1 \quad (116c)$$

5 Resulting RG equations for the Elastic Models

We now focus on the models defined at the beginning of the paper, i.e. an elastic lattice with dislocations in presence of a substrate, which can include a periodic modulation and/or a substrate with quenched disorder. At the bare level these models do not span the whole space of the six K_i (considered in the previous Section) but only a "3 dimensional" subspace of Coulomb gases (called below the "elastic sub-manifold"). Indeed these models correspond to the same definitions (51) of the couplings constants K_i (resp. replica matrices) in terms of the elastic constants (resp. matrices) c_{11}, c_{66}, γ . We find, and this is one of the main results of the paper, that this sub-manifold, i.e. the definitions (51), is preserved by the RG flow. We emphasize that this property is far from obvious, and cannot be easily inferred from the structure of the RG equations (B.1) of the full Coulomb gas, without any knowledge of the definitions (51).

5.1 Stable elastic sub-manifold

Let us start by expressing the coupling constants/matrices p_i in terms of the elastic constants/matrices. The constants from the first group read

$$p_1 = 2\pi(K_1 + K_2) = \frac{4a_0^2}{T}c_{66}(c_{11} - c_{66})c_{11}^{-1}, \quad (117)$$

$$p_4 = 2\pi(K_3 - K_4) = T|\vec{G}_1|^2c_{11}^{-1}, \quad (118)$$

$$p_6 = 2\pi(K_5 - K_6) = a_0|\vec{G}_1| \left(2\frac{c_{66}}{c_{11}} - 1 \right). \quad (119)$$

Note that these 3 constants depend only on c_{11}, c_{66} , and not on γ . These equations can be inverted into

$$c_{11} = T|\vec{G}_1|^2p_4^{-1} \quad ; \quad c_{66} = \frac{T|\vec{G}_1|}{2a_0} \frac{p_1}{\lambda_\phi - p_6}. \quad (120)$$

We recall that $\lambda_\phi = a_0|\vec{G}_1|$.

The scaling of γ (together with c_{66}) is described by the second group of couplings :

$$p_2 = 2\pi(K_1 - K_2) = \frac{4a_0^2}{T}c_{66}\gamma(c_{66} + \gamma)^{-1}, \quad (121)$$

$$p_3 = 2\pi(K_3 + K_4) = T|\vec{G}_1|^2(c_{66} + \gamma)^{-1}, \quad (122)$$

$$p_5 = 2\pi(K_5 + K_6) = a_0|\vec{G}_1|(c_{66} - \gamma)(c_{66} + \gamma)^{-1}, \quad (123)$$

which are inverted into

$$c_{66} = \frac{T|\vec{G}_1|}{2a_0} \frac{\lambda_\phi + p_5}{p_3}, \quad ; \quad \gamma = \frac{T|\vec{G}_1|}{2a_0} \frac{\lambda_\phi - p_5}{p_3}. \quad (124)$$

From these considerations we find the equations defining the "elastic sub-manifold".

$$\lambda_\phi^2 - p_6^2 = p_1 p_4. \quad (125a)$$

$$\lambda_\phi^2 - p_5^2 = p_2 p_3. \quad (125b)$$

$$(\lambda_\phi + p_5)(\lambda_\phi - p_6) = p_1 p_3, \quad (125c)$$

The last relation is obtained by equating the relation (124) with (120). The second is nothing but the first, after the $\pi/2$ rotation symmetry $(\vec{b}, \vec{m}) \rightarrow (\vec{b}^\perp, \vec{m}^\perp)$. These equation also imply:

$$(\lambda_\phi + p_6)(\lambda_\phi - p_5) = p_2 p_4 \quad ; \quad (p_5 - p_6)^2 = (p_1 - p_2)(p_3 - p_4) \quad (126)$$

It is now simple to check that the "elastic manifold" (125) is preserved by the RG. For the two first conditions it is straightforward, and for the third one can show and use that:

$$p_1 \partial_l p_3 + p_3 \partial_l p_1 + (\lambda_\phi + p_5) \partial_l p_6 - (\lambda_\phi - p_6) \partial_l p_5 = 0 \quad (127)$$

We can now write the RG equations restricted to this subspace. Using the above expressions of the p_i in terms of the elastic matrices, we obtain the main result of the paper:

$$\partial_l (c_{11} - c_{66}) = - \left(\Gamma_1 - 2\tilde{\Gamma}_1 \right) \frac{2a_0^2}{T} (c_{11} - c_{66})^2 \quad (128a)$$

$$+ \left(\Gamma_2 - 2\tilde{\Gamma}_2 \right) 2a_0 |\vec{G}_1| (c_{11} - c_{66}) + \left(\Gamma_3 + 2\tilde{\Gamma}_3 \right) \frac{1}{2} T |\vec{G}_1|^2$$

$$\partial_l c_{66} = - \Gamma_1 \frac{2a_0^2}{T} c_{66}^2 - \Gamma_2 2a_0 |\vec{G}_1| c_{66} + \Gamma_3 \frac{|\vec{G}_1|^2 T}{2} \quad (128b)$$

$$\begin{aligned} \partial_l \gamma = & - \left(\Gamma_1 + 2\tilde{\Gamma}_1 \right) \frac{2a_0^2}{T} \gamma^2 \\ & + \left(\Gamma_2 + 2\tilde{\Gamma}_2 \right) 2a_0 |\vec{G}_1| \gamma + \left(\Gamma_3 - 2\tilde{\Gamma}_3 \right) \frac{1}{2} T |\vec{G}_1|^2. \end{aligned} \quad (128c)$$

where the Γ_i and $\tilde{\Gamma}_i$ were defined in (102, 103). Their explicit calculation and analysis of the resulting equations, in the specific models, go well beyond this paper.

5.2 symmetries

Let us comment the symmetries of these equations. They are invariant under the transformation:

$$c'_{11} - c'_{66} = \gamma \quad (129)$$

$$c'_{66} = c_{66} \quad (130)$$

$$\gamma' = c_{11} - c_{66}. \quad (131)$$

which, as noted above, results from invariance under a $\pi/2$ charge rotation (115). It means that if $(c_{11}(l), c_{66}(l), \gamma(l), Y_l[\vec{\mathbf{b}}, \vec{\mathbf{m}}])$ is a solution of the RG flow, then $(c'_{11}(l), c'_{66}(l), \gamma'(l), Y_l[\vec{\mathbf{b}}, \vec{\mathbf{m}}])$ is also a solution. The self-adjoint manifold corresponds to $K_2 = K_4 = K_6 = 0$, i.e. $\gamma = c_{11} - c_{66}$ which corresponds to an isotropic elastic energy and interaction between charges. It is a family of conformally invariant VECG. Examples have been studied in [14] (electric case) and in [4].

Similarly, the electromagnetic duality (116) is written as

$$c'_{11} - c'_{66} = \frac{T^2 |\vec{G}_1|^2}{4a_0^2} 1/(c_{11} - c_{66}) \quad (132)$$

$$c'_{66} = \frac{T^2 |\vec{G}_1|^2}{4a_0^2} 1/c_{66} \quad (133)$$

$$\gamma' = \frac{T^2 |\vec{G}_1|^2}{4a_0^2} 1/\gamma. \quad (134)$$

It means that if $(c_{11}(l), c_{66}(l), \gamma(l), Y_l[\vec{\mathbf{b}}, \vec{\mathbf{m}}])$ is a solution of the RG flow, then $(c'_{11}(l), c'_{66}(l), \gamma'(l), Y_l[\vec{\mathbf{m}}^\perp, \vec{\mathbf{b}}^\perp])$ is also a solution. Hence there is a self-dual submanifold invariant by the flow, defined by:

$$c_{11} - c_{66} = c_{66} = \gamma = T |\vec{G}_1| / (2a_0) \quad (135)$$

$$Y_l[\vec{\mathbf{b}}, \vec{\mathbf{m}}] = Y_l[\vec{\mathbf{m}}^\perp, \vec{\mathbf{b}}^\perp] \quad (136)$$

In the space of elastic constants this self-dual point forms a "line" as T varies. This manifold is clearly included in the "conformal submanifold" $c_{11} - c_{66} = \gamma$ defined above (it obeys $K_1 = K_3$, $K_2 = -K_4 = 0$, $K_5 = K_6 = 0$).

6 Conclusion

To conclude, we have shown how to derive the RG equations of pinned two dimensional defective solids from generalized "elastic" electromagnetic Coulomb

gases with vector charges, defined in (52,54). These RG equations were obtained to lowest order in the charge fugacity, and displayed in full generality in appendix B. They involve, in addition to charge fugacities, six elastic coefficients (or replica matrices in the disordered case). We found that they decouple in two independent sets of simpler equations (113) and (114) which obey two additional symmetry relations. We found that these general equations exhibit a restriction to only three scaling elastic coefficients, corresponding to the initial pinned elastic models, which we showed to be preserved under the RG flow. This provides our final result : eqs. (128) which is still sufficiently general to include all known cases, e.g. the scalar electromagnetic Coulomb gas[17], the scalar vector Coulomb gas describing the melting transition of 2D elastic solids[15], together with various extensions, e.g. the melting transition of pinned 2D solids [7]. Their detailed analysis is the subject of a separate publication.

Acknowledgments : We thank the Kavli Institute for Theoretical Physics for its hospitality where the very last stage of this work was completed, and PLD acknowledges support from ANR grant 05-BLAN-0099-01.

A Two dimensional dislocations

In this appendix, we derive the displacement field corresponding to a finite density of 2D edge dislocations (and of disclinations) in the presence of a coupling to a substrate. We present it here for sake of completeness, and to clarify the notations used in this paper.

We consider a 2D isotropic elastic lattice coupled to a periodic substrate according to $H = \frac{1}{2} \int d^2\vec{r} \, 2\mu u_{ij}^2 + \lambda u_{kk}^2 + \gamma(\epsilon_{ij}\partial_i u_j)^2$. Without dislocations, the phonon displacement field \mathbf{u} is single valued and satisfies $\epsilon_{ij}\partial_i\partial_j\mathbf{u} = 0$. Using this property, we can show that (apart from boundary terms)

$$\int_{\vec{r}} (\epsilon_{ij}\partial_i u_j)^2 = 2 \int_{\vec{r}} (u_{ij}^2 - u_{kk}^2)$$

which implies that the coupling constant to the substrate γ can be incorporated in new Lamé coefficients $\tilde{\lambda} = \lambda - 2\gamma$ and $\tilde{\mu} = \mu + \gamma$ and thus is not a new independant elastic constant of the lattice :

$$H = \frac{1}{2} \int d^2\vec{r} \, 2\mu u_{ij}^2 + \lambda u_{kk}^2 + \gamma\theta^2 = \frac{1}{2} \int d^2\vec{r} \, 2(\mu + \gamma)u_{ij}^2 + (\lambda - 2\gamma)u_{kk}^2 \quad (\text{A.1})$$

This transformation can also be written as $c_{11} \rightarrow \tilde{c}_{11} = c_{11}$, $c_{66} \rightarrow \tilde{c}_{66} = c_{66} + \gamma$. As we will see, the appearance of dislocations breaks this symmetry.

The local equilibrium condition for the hamiltonian (A.1) $H = \frac{1}{2}u_i * M_{ij} * u_j$ reads

$$\frac{\partial H}{\partial u_i} = 0 \Rightarrow M_{ij} * u_j = 2\mu \partial_j u_{ij} + \lambda \partial_i u_{kk} + \gamma \epsilon_{ji} \epsilon_{mn} \partial_j \partial_m u_n = 0 \quad (\text{A.2})$$

Only for non singular fields does the matrix M_{ij} reduce to : $M_{ij}(\mathbf{q}) = q^2[(2\tilde{\mu} + \tilde{\lambda})P_{ij}^L + \tilde{\mu}P_{ij}^T]$ where we have use the modified Lamé coefficients introduced above.

Since dislocations correspond to topological singularities of the lattice, they induce multi-valued displacement fields u_i . Hence if we want to formulate the problem of the determination of their displacement field as a classical elasticity problem, we need to split the displacement field u_i into a multi-valued part u_i^s and a smooth component $\tilde{u}_i : u_i = u_i^s + \tilde{u}_i$. The field u_i^s provides the necessary multi-valueness : $u_i^s = b_i * \frac{\Phi}{2\pi}$ where $\mathbf{b}(\vec{r}) = \sum_{\alpha} \delta(\mathbf{r} - \mathbf{r}_{\alpha}) \mathbf{b}_{\alpha}$ is the dislocations density. Using $\partial_i \Phi = -\epsilon_{ik} \partial_k G$, we find the contribution of u_i^s to (A.2) :

$$u_{ij}^s = -\frac{1}{4\pi}(b_j\epsilon_{ik} + b_i\epsilon_{jk})\partial_k G \quad (\text{A.3})$$

$$\Rightarrow f_i^s \equiv -M_{ij} * u_j^s = -2\mu \partial_j u_{ij}^s - \lambda \partial_i u_{kk}^s - \gamma \epsilon_{ji}\epsilon_{mn}\partial_j \partial_m u_n^s \quad (\text{A.4})$$

$$= \frac{1}{2\pi} b_j \left((\tilde{\mu} - 2\gamma)\epsilon_{ik}\partial_j \partial_k + (\tilde{\lambda} + 2\gamma)\epsilon_{jk}\partial_i \partial_k \right) G \quad (\text{A.5})$$

We are now facing a classical elasticity problem consisting of finding the response of an isotropic 2D lattice under a local force $f_i^s : M_{ik} * \tilde{u}_k = f_i^s$, which can be inverted in Fourier transform as

$$\tilde{u}_i(\mathbf{q}) = M_{ij}^{-1}(\mathbf{q}) f_j^s(\mathbf{q})$$

with

$$M_{ij}^{-1} = \frac{1}{q^2} \left(\frac{1}{2\tilde{\mu} + \tilde{\lambda}} P_{ij}^L + \frac{1}{\tilde{\mu}} P_{ij}^T \right)$$

and

$$f_i^s = -b_j \left((\tilde{\mu} - 2\gamma)\epsilon_{ik} P_{jk}^L + (\tilde{\lambda} + 2\gamma)\epsilon_{jk} P_{ik}^L \right).$$

We end up with a displacement field \tilde{u} given by

$$\tilde{u}_i(\mathbf{q}) = -b_j \frac{1}{q^2} \left(\frac{\tilde{\lambda} + 2\gamma}{2\tilde{\mu} + \tilde{\lambda}} \epsilon_{jk} P_{ik}^L + \frac{\tilde{\mu} - 2\gamma}{\tilde{\mu}} \epsilon_{ik} P_{jk}^L \right) \quad (\text{A.6})$$

Using the approximate Fourier transform

$$\int \frac{d^2 \mathbf{q}}{(2\pi)^2} e^{i\mathbf{q} \cdot \vec{r}} \frac{1}{q^2} P_{ij}^L = -\frac{1}{4\pi} \left(\delta_{ij} \ln r + \frac{r_i r_j}{r^2} - \frac{1}{2} \delta_{ij} + C(\phi) \right)$$

and the relation $\epsilon_{jk} \hat{r}_i \hat{r}_k = \epsilon_{ik} \hat{r}_j \hat{r}_k + \epsilon_{ij}$, we obtain the result :

$$\tilde{u}_i = \frac{b_j}{2\pi} \left(\frac{\tilde{\mu}^2 - \gamma(3\tilde{\mu} + \tilde{\lambda})}{\tilde{\mu}(2\tilde{\mu} + \tilde{\lambda})} \epsilon_{ij} \ln(r) + \frac{(\tilde{\mu} + \tilde{\lambda})(\tilde{\mu} - \gamma)}{\tilde{\mu}(2\tilde{\mu} + \tilde{\lambda})} \epsilon_{jk} H_{ik} \right). \quad (\text{A.7})$$

Thus the total displacement field due to a density of dislocations \mathbf{b} is $u_i = \frac{1}{2\pi} \mathcal{G}_{ij} * b_j$ where

$$\mathcal{G}_{ji}(r) = \delta_{ij} \Phi(r) + \frac{\tilde{c}_{66}^2 - \gamma(\tilde{c}_{11} + \tilde{c}_{66})}{\tilde{c}_{11} \tilde{c}_{66}} \epsilon_{ij} \ln(r) + \frac{(\tilde{c}_{11} - \tilde{c}_{66})(\tilde{c}_{66} - \gamma)}{\tilde{c}_{11} \tilde{c}_{66}} \epsilon_{jk} H_{ik}(r) \quad (\text{A.8})$$

This expression reduces to the known formula[8] without any coupling to the substrate $\gamma = 0$. We also realize that in the presence of dislocations, this coupling γ can no longer be incorporated into renormalized elastic coupling : its corresponds to a third independent constant.

To obtain the effective interaction between the dislocations, we first express the strain tensor corresponding to a collection of dislocations : $u_{ij}(\mathbf{q}) = u_{ij}^s(\mathbf{q}) +$

$\tilde{u}_{ij}(\mathbf{q})$ where, using (A.3)

$$u_{ij}^s(\mathbf{q}) = i(b_j \epsilon_{ik} + b_i \epsilon_{jk}) \frac{q_k}{2q^2} \quad (\text{A.9})$$

and from (A.6)

$$\tilde{u}_{ij}(\mathbf{q}) = -i \frac{b_l}{2q^2} \left(\frac{2\tilde{\lambda} + 4\gamma}{2\tilde{\mu} + \tilde{\lambda}} \epsilon_{lk} q_i P_{jk}^L + \frac{\tilde{\mu} - 2\gamma}{\tilde{\mu}} (\epsilon_{jk} q_i + \epsilon_{ik} q_j) P_{lk}^L \right). \quad (\text{A.10})$$

Now plugging this strain tensor into the elastic energy and using

$$\begin{aligned} u_{ij}^s(\mathbf{q}) u_{ij}^s(-\mathbf{q}) &= \frac{1}{2q^2} b_i(\mathbf{q}) b_j(-\mathbf{q}) (\delta_{ij} + P_{ij}^T), \\ \tilde{u}_{ij}(\mathbf{q}) \tilde{u}_{ij}(-\mathbf{q}) &= \frac{1}{4q^2} b_i(\mathbf{q}) b_j(-\mathbf{q}) \left[\left(\frac{2\tilde{\lambda} + 4\gamma}{2\tilde{\mu} + \tilde{\lambda}} \right)^2 P_{ij}^T + 2 \left(\frac{\tilde{\mu} - 2\gamma}{\tilde{\mu}} \right)^2 P_{ij}^L \right] \\ \tilde{u}_{ij}(\mathbf{q}) u_{ij}^s(-\mathbf{q}) &= -\frac{1}{2q^2} \left(\frac{\tilde{\mu} - 2\gamma}{\tilde{\mu}} \right) b_i(\mathbf{q}) b_j(-\mathbf{q}) P_{ij}^L \\ u_{kk}(\mathbf{q}) u_{kk}(-\mathbf{q}) &= \frac{1}{4q^2} \left(2 - \frac{2\tilde{\lambda} + 4\gamma}{2\tilde{\mu} + \tilde{\lambda}} \right)^2 b_i(\mathbf{q}) b_j(-\mathbf{q}) P_{ij}^T \end{aligned}$$

we obtain the desired result :

$$H_{b/b} = \int_q \frac{1}{2q^2} b_i(\mathbf{q}) b_j(-\mathbf{q}) \left[\frac{4\gamma^2}{\tilde{c}_{66}} P_{ij}^L + \frac{4(\tilde{\mu}(\tilde{\mu} + \tilde{\lambda}) + \gamma^2)}{2\tilde{\mu} + \tilde{\lambda}} P_{ij}^T \right] \quad (\text{A.11})$$

$$= \int_q \frac{1}{2q^2} b_i(\mathbf{q}) b_j(-\mathbf{q}) \left[\frac{4\gamma^2}{\tilde{\mu}} P_{ij}^L + \frac{4c_{66}(c_{11} - c_{66}) + 4\gamma^2}{c_{11}} P_{ij}^T \right] \quad (\text{A.12})$$

Note that another method to obtain this interaction, incorporating in particular the contribution of disclinations, is to use the so called Airy functions[8]. However it does not provide the displacement field, necessary in the present case.

B Renormalization Group Equations for the Full Model

In this appendix, we present the RG equations for the full VECG. In these expression, the couplings K_i , Γ_i and $\tilde{\Gamma}_i$ are commuting replica matrices.

$$\begin{aligned}
\partial_l K_1^{ab} = & -2\pi\Gamma_1 (K_1^2 + K_2^2) + 4\pi\tilde{\Gamma}_1 K_1 K_2 \\
& + 4\pi\Gamma_2 (K_1 K_5 - K_2 K_6) - 4\pi\tilde{\Gamma}_2 \left(K_1 K_6 - K_2 K_5 - K_2 \frac{\lambda_\phi}{(2\pi)} \right) \\
& + 2\pi\Gamma_3 \left(K_5^2 + K_6^2 + \left(\frac{\lambda_\phi}{(2\pi)} \right)^2 \right) - 4\pi\tilde{\Gamma}_3 \left(K_6 \frac{\lambda_\phi}{(2\pi)} K_5 K_6 \right) \quad (B.1a)
\end{aligned}$$

$$\begin{aligned}
\partial_l K_2^{ab} = & -4\pi\Gamma_1 K_1 K_2 + 2\pi\tilde{\Gamma}_1 (K_1^2 - K_2^2) \\
& + 4\pi\Gamma_2 (K_2 K_5 - K_1 K_6) + 4\pi\tilde{\Gamma}_2 \left(K_1 K_5 - K_2 K_6 + K_1 \frac{\lambda_\phi}{(2\pi)} \right) \\
& - 4\pi\Gamma_3 K_5 K_6 + 2\pi\tilde{\Gamma}_3 \left(K_5^2 + K_6^2 + 2K_5 \frac{\lambda_\phi}{(2\pi)} + \left(\frac{\lambda_\phi}{(2\pi)} \right)^2 \right) \quad (B.1b)
\end{aligned}$$

$$\begin{aligned}
\partial_l K_3^{ab} = & +2\pi\Gamma_1 \left(K_5^2 + K_6^2 + \left(\frac{\lambda_\phi}{(2\pi)} \right)^2 \right) + 4\pi\tilde{\Gamma}_1 \left(K_5 K_6 \right) - K_6 \frac{\lambda_\phi}{(2\pi)} \\
& + 4\pi\Gamma_2 (K_3 K_5 + K_4 K_6) - 4\pi\tilde{\Gamma}_2 \left(K_4 K_5 + K_3 K_6 - K_4 \frac{\lambda_\phi}{(2\pi)} \right) \\
& - 2\pi\Gamma_3 (K_3^2 + K_4^2) + 4\pi\tilde{\Gamma}_3 K_3 K_4 \quad (B.1c)
\end{aligned}$$

$$\begin{aligned}
\partial_l K_4^{ab} = & +4\pi\Gamma_1 K_5 K_6 + 2\pi\tilde{\Gamma}_1 \left(\left(\frac{\lambda_\phi}{(2\pi)} \right)^2 - 2K_5 \frac{\lambda_\phi}{(2\pi)} + K_5^2 + K_6^2 \right) \\
& + 4\pi\Gamma_2 (K_4 K_5 + K_3 K_6) - 4\pi\tilde{\Gamma}_2 \left(K_3 K_5 - K_3 \frac{\lambda_\phi}{(2\pi)} + K_4 K_6 \right) \\
& - 4\pi\Gamma_3 K_3 K_4 + 2\pi\tilde{\Gamma}_3 (K_3^2 - K_4^2) \quad (B.1d)
\end{aligned}$$

$$\begin{aligned}
\partial_l K_5^{ab} = & 2\pi\Gamma_1 (K_2 K_6 - K_1 K_5) + 2\pi\tilde{\Gamma}_1 \left(-\frac{\lambda_\phi}{(2\pi)} K_2 - K_1 K_6 + K_2 K_5 \right) \\
& + 2\pi\Gamma_2 \left(-\left(\frac{\lambda_\phi}{(2\pi)} \right)^2 + K_5^2 + K_6^2 - K_1 K_3 + K_2 K_4 \right) \\
& - 2\pi\tilde{\Gamma}_2 (K_2 K_3 - K_1 K_4 + 2K_5 K_6) \\
& - 2\pi\Gamma_3 (K_3 K_5 + K_4 K_6) + 2\pi\tilde{\Gamma}_3 \left(K_4 \frac{\lambda_\phi}{(2\pi)} + K_3 K_6 + K_4 K_5 \right) \quad (B.1e)
\end{aligned}$$

$$\begin{aligned}
\partial_l K_6^{ab} = & 2\pi\Gamma_1 (K_2 K_5 - K_1 K_6) + 2\pi\tilde{\Gamma}_1 \left(K_1 \frac{\lambda_\phi}{(2\pi)} - K_1 K_5 + K_2 K_6 \right) \\
& + 2\pi\Gamma_2 (K_2 K_3 - K_1 K_4 + 2K_5 K_6) \\
& - 2\pi\tilde{\Gamma}_2 \left(-\left(\frac{\lambda_\phi}{(2\pi)} \right)^2 + K_5^2 + K_6^2 - K_1 K_3 + K_2 K_4 \right) \\
& - 2\pi\Gamma_3 (K_4 K_5 + K_3 K_6) + 2\pi\tilde{\Gamma}_3 \left(K_3 \frac{\lambda_\phi}{(2\pi)} + K_3 K_5 + K_4 K_6 \right) \quad (B.1f)
\end{aligned}$$

References

- [1] D.J. Amit, Y.Y. Goldschmidt, and G. Grinstein. Renormalisation group analysis of the phase transition in the 2d coulomb gas, sine-gordon theory and xy-model. *J. Phys. A*, 13:585, 1980.
- [2] D. Boyanovsky. Field-theoretical renormalisation and fixed-point structure of a generalised coulomb gas. *J. Phys. A*, 22:2601–2614, 1989.
- [3] D. Boyanovsky and R. Holman. The coulomb gas, vertex operators and the c-theorem. *Nucl. Phys. B*, 332:641–665, 1990.
- [4] D. Boyanovsky and R. Holman. Critical behavior and duality in extended sine-gordon theories. *Nucl. Phys. B*, 358:619–653, 1991.
- [5] S.A. Bulgadaev. Phase transitions in ln-gases with generalized charges. *Phys. Letters*, 86A:213, 1981.
- [6] D. Carpentier and P. Le Doussal. Glass phase of two-dimensional triangular elastic lattices with disorder. *Phys. Rev. B*, 55, 1997.
- [7] D. Carpentier and P. Le Doussal. Melting of two-dimensional solids on disordered substrates. *Phys. Rev. Lett.*, 81:1881–1884, 1998.
- [8] P.M. Chaikin and T.C. Lubensky. *Principles of condensed matter physics*. Cambridge University Press, Cambridge, 1996.
- [9] E. Fradkin and L.P. Kadanoff. Disorder variables and para-fermions in two-dimensional statistical mechanics. *Nucl. Phys. B*, 170:1–15, 1980.
- [10] J.V. Jose, L.P. Kadanoff, S. Kirkpatrick, and D. Nelson. Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model. *Phys. Rev. B*, 16:1217–1241, 1977.
- [11] L.P. Kadanoff. Lattice coulomb gas representations of two-dimensional problems. *J. Phys. A*, 11:1399, 1978.
- [12] J.M. Kosterlitz and D.J. Thouless. Ordering, metastability and phase transitions in two-dimensional systems. *J. Phys. C*, 6:1181, 1973.
- [13] L.D. Landau and E.M. Lifshitz. *Theory of Elasticity*. Pergamon Press (London), 1959.
- [14] D.R. Nelson. Study of melting in two dimensions. *Phys. Rev. B*, 18:2318–2338, 1978.
- [15] D.R. Nelson. Defect-mediated phase transitions. In C. Domb and C. Leibowitz, editors, *Phase Transitions and Critical Phenomena*, volume 7, page 165. Academic Press, London, 1983.
- [16] D.R. Nelson and B.I. Halperin. Dislocation-mediated melting in two dimensions. *Phys. Rev. B*, 19:2457, 1979.

- [17] B. Nienhuis. Coulomb gas formulation of two-dimensional phase transitions. In C. Domb and C. Leibovitz, editors, *Phase Transitions and Critical Phenomena*, volume 11, pages 1–54. Academic Press, London, 1987.
- [18] A. Shi and A. Berlinsky. Pinning and i-v characteristics of a two-dimensionnal defective flux-line lattice. *Phys. Rev. Lett.*, 67:1926, 1991.
- [19] P.B. Wiegmann. One-dimensional fermi system and plane xy model. *J. Phys. C*, 11:1583, 1978.
- [20] P. Young. Melting and the vector coulomb gas in two dimensions. *Phys. Rev. B*, 19:1855–1866, 1979.