Stochastic invariances and Lamperti transformations for Stochastic Processes

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Scale invariances and Lamperti transformations for Stochastic Processes

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Abstract. Scale invariant processes, and hereafter processes with broken versions of this symmetry, are studied by means of the Lamperti transformation, a one-to-one transformation linking stationary and self-similar processes. A general overview of the use of the transformation, and of the stationary generators it builds, is given for modelling and analysis of scale invariance. We put an emphasis on generalisations to non strictly scale invariant situations. The examples of Discrete Scale Invariance and Finite Size Scale Invariance are developed by means of the Lamperti transformation framework, and some specific examples of processes with these generalised symmetries are given.

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1. Lamperti transformation: A new reading

1.1. Scale invariance and beyond

Scale invariance, once acknowledged as an important feature [39], has often been used as a fundamental property to handle physical phenomena. The idea that some quantity behaves the same at each scale, irrelevant to the scale at which it is observed, has made its way to study geometrical fractal sets [21], $1/f$ spectra, long memory [6], simple dimensional analysis [5] or involved analysis of critical systems in statistical physics [22], textures in geophysics [52] or image processing [38], turbulence of fluids [27], data of network traffic [45], and so on.

Common as this invariance may be in physics, it often eludes general and convenient methods or models. Even though there is no single definition of scale invariance [19], it is often described as a symmetry of the system relatively to a transformation of scale, that is mainly a dilation or a contraction (up to some renormalisation) of the system parameters. A first problem is that this symmetry is not always compatible with usual symmetries such that stationarity in space or time, or isotropy, short-term memories and so on. In those cases, the meaning of Fourier spectrum or correlation functions may be unclear or unproper, and simple models such as Langevin equations or auto-regressive systems are not sufficient. Another issue is that, as with all symmetries, one expects real systems to experience symmetry breakings: incomplete invariance under dilations because of some additive part when scale is changed (self-affinity in fractals [21], renormalisation equation of free energy in critical systems [22]), or the invariance holds only for some part of the meaningful scales (and sometimes infinite zooming is unmeaningful) [17, 43], else there exists some preferred scale ratios and the invariance stands true only for those scale ratios (the so-called Discrete Scale Invariance property) [51].

We propose here to give a fresh look on the methodology for scale invariances (exact or incomplete) of stochastic processes; instead of dealing directly with the scale-invariant signal, one may transform the signal in some image that has a better-known invariance such as stationarity. This approach comes from a generalisation of the Lamperti theorem [35] which relates stationarity and self-similarity, and it is close to the extension of the concept of what stationarity is, as proposed by Hannan [33]: one can study invariances such as stationarity or self-similarity in similar frameworks. We argue hereafter that there exist such stationarising transformations for exact or broken scale invariances and that effective methods for modelling and analysing scale invariances can be derived from this. We advocate the use of such a transform to define and study scale invariances, exact invariance or its many variations as broken symmetries.

This paper is then organized to cover two aspects of the Lamperti transformation: its usefulness for self-similarity, and new insights about its generalisation for broken scale invariances. By means of the Lamperti transform, we provide a new way of handling self-similar processes that leads to some methods of synthesis and analysis of exact self-similarity. The stationarisation of self-similar processes was studied piecewise for specific applications by different authors and we give here a general formulation of the method, with new comments on its numerical applicability. This is the scope of the current section. The idea of stationarisation is generalised in Section 2 to encompass various forms of broken scale invariance. The specific case of Discrete Scale Invariance [51] is studied in Section 3. The corresponding stochastic property is defined and elaborated here by means of the Lamperti transformation. Then a fourth section is devoted to broken scale invariances defined by a distorsion of the dilation operators; this will provide insights into generalised scaling laws, especially the property of Finite Size Scale Invariance [17] in the framework of the Lamperti transformation.

1.2. Definition and property of the Lamperti transformation

The roots of this work is the paper of J.W. Lamperti on scale invariance for stochastic processes [35], where he first pointed out a one-to-one correspondence between scale invariant processes and stationary processes. To define what a dilation is, and thus the precise meaning of scale invariance, we use here the framework of stochastic representation of signals (for instance fluctuations or noises).

The following is the proper formalism for stochastic processes [20]. A random process $\{X(t), t > 0\}$ is said to be self-similar of index H (or scale invariant, noted "H-ss") if for any $\lambda \in \mathbb{R}^+$,

$$\{(\mathcal{D}_{H,\lambda}X)(t) \hat{=} \lambda^{-H} X(\lambda t), t > 0\} \stackrel{d}{=} \{X(t), t > 0\}. \tag{1}$$

where $\stackrel{d}{=}$ stands for the "equality" of the stochastic processes, that is equality of all joint finite-dimensional distributions. This symmetry is an invariance under any renormalised dilation $\mathcal{D}_{H,\lambda}$ by a scale factor λ, $\mathcal{D}_{H,\lambda}$ being defined in the preceding equation. The theorem introduced by J.W. Lamperti in 1962 [35] uses the invertible transformation $\mathcal{L}_H{}^{-1}$, acting on $\{X(t), t > 0\}$:

$$\left(\mathcal{L}_H{}^{-1}X\right)(t) \hat{=} e^{-Ht}X(e^t) = Y(t),\ t \in \mathbb{R}, \tag{2}$$

Considering $\mathcal{L}_H{}^{-1}$ as an inverse transformation, the corresponding direct transformation $\mathcal{L}_H$ is called the Lamperti transformation and is given by:

$$\left(\mathcal{L}_H Y\right)(t) \hat{=} t^H Y(\log t) = X(t),\ t > 0. \tag{3}$$

The theorem states that a process $\{X(t), t > 0\}$ is H-ss if and only if $\{Y(t) = (\mathcal{L}_H{}^{-1}X)(t),\ t \in \mathbb{R}\}$ is stationary; we call $Y(t)$ the stationary generator of the process $X(t)$. We recall that stationarity is the invariance under any time-shift (or space-shift if the parameter is space). For any $\tau \in \mathbb{R}$, a process $\{Y(t) = (\mathcal{L}_H{}^{-1}X)(t),\ t \in \mathbb{R}\}$ has thus to satisfy $(\mathcal{S}_\tau Y)(t) \hat{=} Y(t+\tau) \stackrel{d}{=} Y(t)$. The core of the theorem is a mapping of operators such that for any $\lambda \in \mathbb{R}_*^+$, the Lamperti transformation guarantees that

$$\mathcal{L}_H{}^{-1}\, \mathcal{D}_{H,\lambda}\, \mathcal{L}_H = \mathcal{S}_{\log\lambda}. \tag{4}$$

Quoted in review books or articles (e.g., [16, 53, 48, 20]), the theorem is completed by the unicity result of the transformation mapping stationarity to self-similarity, demonstrated by Burnecki *et al.* [13].

1.3. Properties of stationary generators of H-ss processes

There has been few uses of the result of Lamperti beside mathematical works on self-similarity. Gray and Zhang [32] or Yazici and Kashyap [57] summoned some form of Lamperti's theorem to study specific classes of self-similar processes; Nuzman and Poor [44] and more recently Lim and Muniandy [36] used it extensively for fractional Brownian motions only. But those are among the few practical uses of the transformation. We have introduce general results on this subject in [11, 26], and we point out here their practical consequences for the modelling and analysis of scale invariance.

1.3.1. Covariance and spectrum of H-ss processes. The covariance of a scale invariant random process $\{X(t), t > 0\}$ admits necessarily the general form

$$R_X(t,s) \hat{=} \mathbb{E}\{X(t)X(s)\} = (st)^H C_X(t/s), \tag{5}$$

where $C_X(e^{\cdot})$ is some non-negative definite function. In this expression, we have written $\mathbb{E}$ for the probabilistic expectation. The property (5) comes from the stationary generator $(Y = \mathcal{L}_H{}^{-1}X)$ correlation function. The transformation assures that $R_X(t,s) = (st)^H R_Y(\log t, \log s)$. Hence, because of the stationarity of Y, its covariance depends only on the time difference, $R_Y(u,v) = c_Y(u-v)$. The central identity is therefore given for the correlation function $c_Y(\tau) = C_X(e^{\tau})$.

Our first and well known example is the Brownian motion $B(t)$ defined as an integral of independent stationary Gaussian increments. $B(t)$ is also self-similar of index $1/2$ and thus it has a stationary generator. Its covariance is: $R_B(t,s) = \sigma^2 \max(t,s)$, σ^2 being its variance. A simple calculus of the covariance of $(\mathcal{L}_H{}^{-1}B)(t)$ gives: $c_{\mathcal{L}_H{}^{-1}B}(\tau) = \sigma^2 e^{-|\tau|/2}$ and proves that this generator is the Ornstein-Uhlenbeck (OU) process, solution of the stationary linear Langevin equation driven by white noise. Consequently, properties of the Brownian motion derives from properties of the OU process [16, 20].

The power spectrum $\mathbf{\Gamma}_Y(f)$ of $Y(t)$ is the Fourier transform of the correlation function of a stationary process: $\mathbf{\Gamma}_Y(f) = (\mathbf{F}c_Y)(f)$. The Fourier transform is known as a suited representation for stationarity, but not for self-similarity. Starting from a H-ss process, the following algebra expresses the power spectrum of the stationary generator of a self-similar process:

$$\begin{aligned}(\mathbf{F}c_{\mathcal{L}_H{}^{-1}X})(f) &= \int_{-\infty}^{+\infty} C_X(e^{\tau}) e^{-i2\pi f \tau} \mathrm{d}\tau \\ &= \int_0^{\infty} C_X(u) u^{-i2\pi f - 1} \mathrm{d}u = (\mathbf{M}C_X)(i2\pi f),\end{aligned} \tag{6}$$

where $\mathbf{M}$ stands for the Mellin transform of the function. The definition of $\mathbf{M}$, for any function $g(u)$ and any variable $s \in \mathbb{C}$, is: $(\mathbf{M}g)(s) = \int_0^{\infty} g(u) u^{-s-1} \mathrm{d}u$.

The Mellin transform plays the same central role for self-similarity as the Fourier transform plays for stationarity because of the relation established in (6), which indeed is general for any quantity $g(u)$, given some index H:

$$(\mathbf{M}g)(H + i2\pi f) = \int_0^{\infty} g(u) u^{-i2\pi f - H - 1} \mathrm{d}u = (\mathbf{F}\mathcal{L}_H{}^{-1} g)(f). \tag{7}$$

The basis functions are the Mellin chirps $\{t^{H+i2\pi f}, t > 0\}$, with $f \in \mathbb{R}$. Using this equivalence, note that one can obtain a harmonic-like representation of a self-similar process $X(t)$ as an inverse Mellin transform, namely an integral of uncorrelated spectral increments $\mathrm{d}\xi_X(f)$ on the Mellin basis [10, 26]:

$$X(t) = \int_0^{\infty} t^{i2\pi f+H} \mathrm{d}\xi_X(f). \tag{8}$$

This is the Cramér representation for self-similar processes, mapped by $\mathcal{L}_H$ from the usual result known for stationary processes. It is valid under the assumption of the Loève condition, that is the summability of the integral representation for the second order statistics of the process [37].

Here is obtained a spectral representation of a H-ss process with no assumption on stationarity (which is not compatible with self-similarity), nor on stationarity of the increments. With the added hypothesis of stationarity of the increments of the process, defined for any τ as $\{X(t+\tau) - X(t), t \in \mathbb{R}\}$, a good method to model the process is to use the wavelet transform [15]. It was proved really suited to study self-similar processes with stationary increments [1]. But, lacking this property, one can not use the wavelet transform properly because there will not be convenient decorrelation between the coefficients. Also, because the wavelet transform with a wavelet ψ reads as $T(a,t) = \int \psi(u)X(t-au)\mathrm{d}u$, the scale a defined with the wavelet tranform is mainly based on the difference between two times. This is revealed by the underlying affine structure of the wavelet transform or by taking a look at one of the crudest wavelet: $\psi(u) = (\delta(u) - \delta(u-\tau_0))/2$. In this case, the wavelet transform is: $T(a,t) = (X(t) - X(t-a\tau_0))/2$ (an expression close to the increment of the process) and the scale a is probed by the difference of two times: $t_1 = t - a\tau_0$ and $t_2 = t$.

On the contrary, the variable f defined in (8) which is also a scale, named the Mellin scale (see for instance [24], page 210), is built on a ratio of times. It gives a simple description of how a process has changed between two times t_1 and t_2: each component with Mellin scale f has been multiplied by $(t_2/t_1)^{H+i2\pi f}$. As dilations are defined by changing t in λt, the decomposition in Mellin scale is well suited to probe the behaviour under those dilations by the importance given to the ratio of two times, $t_2/t_1 = \lambda$. The Mellin scale f is the spectral variable associated to the dilation ratio, in the same sense that a Fourier frequency allow to describe easily the effects of a time-shift. Thus this Mellin scale is adapted to self-similarity where invariant properties under dilations are expected. Moreover a spectral decomposition on Mellin scales will be an interesting tools for processing of self-similar processes.

1.3.2. Scale-invariant filters and models. A further consequence is that scale invariant linear systems are found as images by $\mathcal{L}_H$ of stationary (linear) filters. Applying $\mathcal{L}_H$ on a stationary filter $\mathcal{H}$ that has the usual action on $Y(t)$ as a convolution: $(\mathcal{H}Y)(t) = \int_{-\infty}^{+\infty} h(t-u)Y(u)\mathrm{d}u$, one defines systems having the form of a multiplicative convolution,

$$(\mathcal{G}X)(t) = \int_0^\infty g(t/s)X(s)\frac{\mathrm{d}s}{s} = \int_0^\infty g(s)X(t/s)\frac{\mathrm{d}s}{s}. \tag{9}$$

They are related to usual filters by means of $\mathcal{L}_H$ because we obtain this equation by setting $(\mathcal{L}_H h)(s) = g(s)$. Note that it is not a wavelet transform because it say nothing about time-shifts here (we only have one variable). We put together the defining property of stationary filters which is that they commute with any time-shift $\mathcal{S}_\tau$ (i.e., for any $\tau \in \mathbb{R}$, $\mathcal{H}\mathcal{S}_\tau = \mathcal{S}_\tau\mathcal{H}$) and the equivalence (4). Then one finds that a defining property of the scale-invariant filters $\mathcal{G}$ is that they commute with dilations (covariance with dilations), i.e., $\mathcal{G}\mathcal{D}_{H,\lambda} = \mathcal{D}_{H,\lambda}\mathcal{G}$ for any $\lambda \in \mathbb{R}_*^+$. As such they preserve self-similarity and are a good tool to process self-similar signals without disturbing this key property.

Taking the Mellin transform of (9) one formally obtains a transfer function for those systems: $\mathrm{d}\xi_{\mathcal{G}X}(f) = (\mathbf{M}g)(i2\pi f)\mathrm{d}\xi_X(f)$. From this, it is possible to design parametric models built on scale-invariant filters, by taking a rational function for the transfer function $(\mathbf{M}g)(s)$. The models are mapped by $\mathcal{L}_H$ from the ARMA models, and are found to follow Euler-Cauchy systems. A theory of parametric modelling for self-similarity may be written on these premises [11] and some issues were covered in [26] and in previous works [12, 56, 57] that did not use explicitly the Lamperti correspondence.

1.3.3. Fractional Brownian motions. From the above results, on can study a typical model of H-ss stochastic processes: the fractional Brownian motion (fBm) $\{B_H(t), t > 0\}$. The fBm is the only (up to a multiplicative constant) Gaussian process, that is H-ss and has stationary increments [20, 41]. When $H = 1/2$, the fBm collapses onto the usual Brownian motion, the increments being therefore independent in that case. Its covariance has the form:

$$R_{B_H}(t,s) = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} + |t-s|^{2H}) \tag{10}$$

$$= \frac{\sigma^2}{2}(st)^H(|t/s|^H + |s/t|^H + |\sqrt{t/s} - \sqrt{s/t}|^{2H}), \tag{11}$$

which has the expected form of (5). Using the inverse Lamperti transform, the stationary generator $\{Y_H(t), t \in \mathbb{R}\}$ of the fBm is shown to be a stationary Gaussian process with zero mean and covariance function given by [26, 36, 44]

$$R_{Y_H}(\tau) = \frac{\sigma^2}{2}[\cosh(H\tau) - (2\sinh(|\tau|/2))^{2H}/2]. \tag{12}$$

When $H = 1/2$, we recognise the covariance function of the Ornstein-Uhlenbeck process, else we obtain some kind of generalised OU process. For the point of view adopted here, the fBm has the following Mellin spectrum (or Fourier spectrum of its associated generator):

$$(\mathbf{M}C_{B_H})(i2\pi f) = \mathbf{\Gamma}_{Y_H}(f) = \frac{\sigma^2}{H^2 + 4\pi^2 f^2}\left|\frac{\Gamma(1/2 + i2\pi f)}{\Gamma(H + i2\pi f)}\right|^2, \tag{13}$$

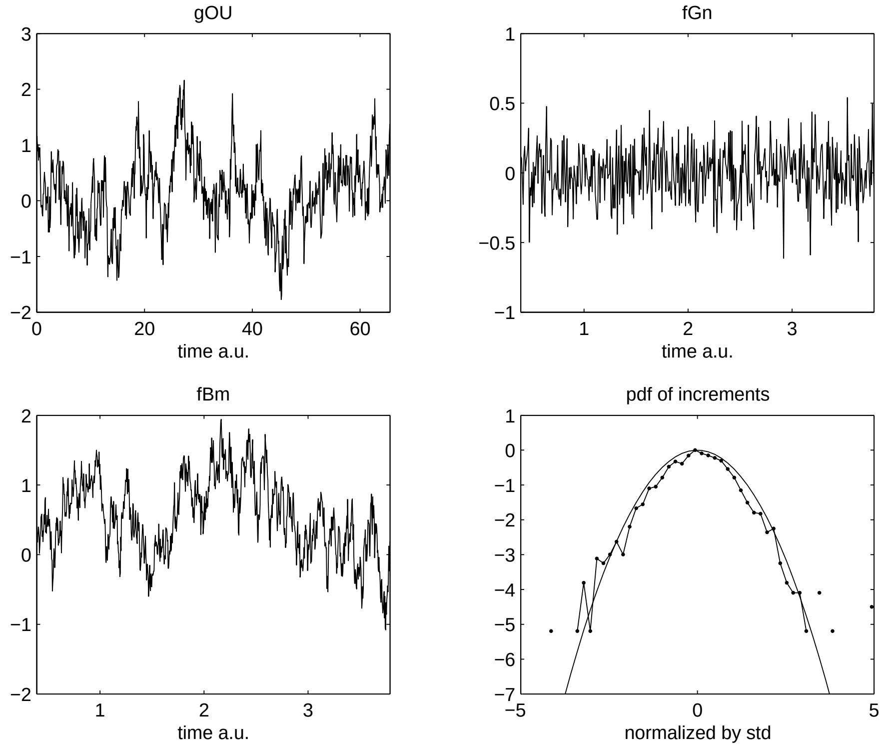

Figure 1. Example of a fractional Brownian motion, with $H = 0.3$, computed by means of its stationary generator. Top left: the generator $Y(t)$, synthetised from its stationary correlation function, equation (12) (using the method of the circulant matrix). Bottom left: the fBm deduced from the preceding process, by $B(t) = (\mathcal{L}_H Y)(t)$. Top right: the increments $B(t+1) - B(t)$ of the fBm, known to be stationary. Bottom right: the logarithm of histogram of the increments of the fBm, that shows they are Gaussian as it is expected (the curve is the Gaussian in this lin-log scale).

where Γ is the Euler Gamma function. Characterisations of the generalised OU process can be found in the references [25, 36, 44].

1.4. Use of Lamperti transform for numerical operations

The Lamperti transfor is twofold: a multiplication by a non-stationary factor t^H, and a change of the way the time variable is measured, from t to $\log t$. The first part captures the general growth of the statistics driven by the H exponent. The second part is a kind of clock-change in order to find the natural scale for measuring the evolution of the process (which here is $\log t$), and is referred as a warping of the time in signal processing [4].

It is then possible to use this feature for numerical analysis or synthesis of discrete-time sequences. Because of the stationarity of the generator, a good way of synthesis for self-similar processes is to compute a realisation of the generator by means of fast algorithms requiring stationarity (e.g., Levinson algorithm or the circulant matrix of

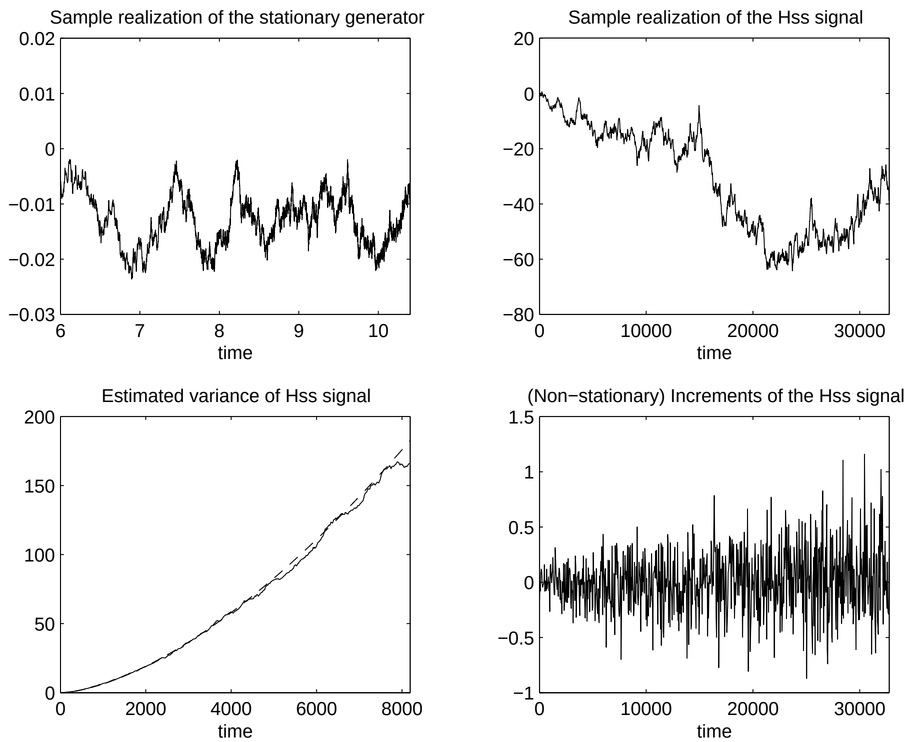

Figure 2. Example of a self-similar process, with $H = 0.8$, computed by means of its stationary generator having covariance $c_Y(\tau) = \exp(-\alpha|\tau|)$. Top left: the generator $Y(t)$, synthetised from its stationary correlation function. Top right: the H-ss process deduced from the preceding process, implementing $X(t) = (\mathcal{L}_H Y)(t)$. Bottom left: estimated variance of the process, $\mathbb{E}X(t)^2$, done empirically on 1024 realisations; the expected curve in $\sigma^2 t^{2H}$ is drawn in dash line. Bottom right: the increments $X(t+1) - X(t)$, which are not stationary (so cannot be used to synthetise conveniently the process). This process is also the output of an Euler-Cauchy system of order 1, and has been studied in [42, 26].

Wood and Chan [55]). The generator is obtained as a uniformally sampled sequence $Y(t = n\tau)$ with a fixed τ and the indices n in a subset of $\mathbb{Z}$. The transformed process is then known at times $t = e^{n\tau} = (e^\tau)^n$, thus with a geometrical sampling. By resampling the process, one can obtained a realization of the H-ss process with a uniform sampling. Figure 1 show a typical snapshot of a fBm obtained this way and its associated generalised OU generators. Figure 2 illustrates that this method may also be used for models with no stationary increments (whereas a fBm has stationary increments), and therefore in situations where a fast algorithm is lacking.

For the sake of analysis, we illustrate in figure 3 that the Mellin spectrum may be more meaningful than a Fourier spectrum for an unknown self-similar process. On the one hand, the Fourier spectrum hardly says a thing where there is no stationarity, and is usually broadband. For self-similar processes, with an added assumption of stationary increments, one may deduce the H exponent of the Fourier spectrum [6]. We indeed expects that it behaves as f^{-2H-1} over a large range of frequencies, but

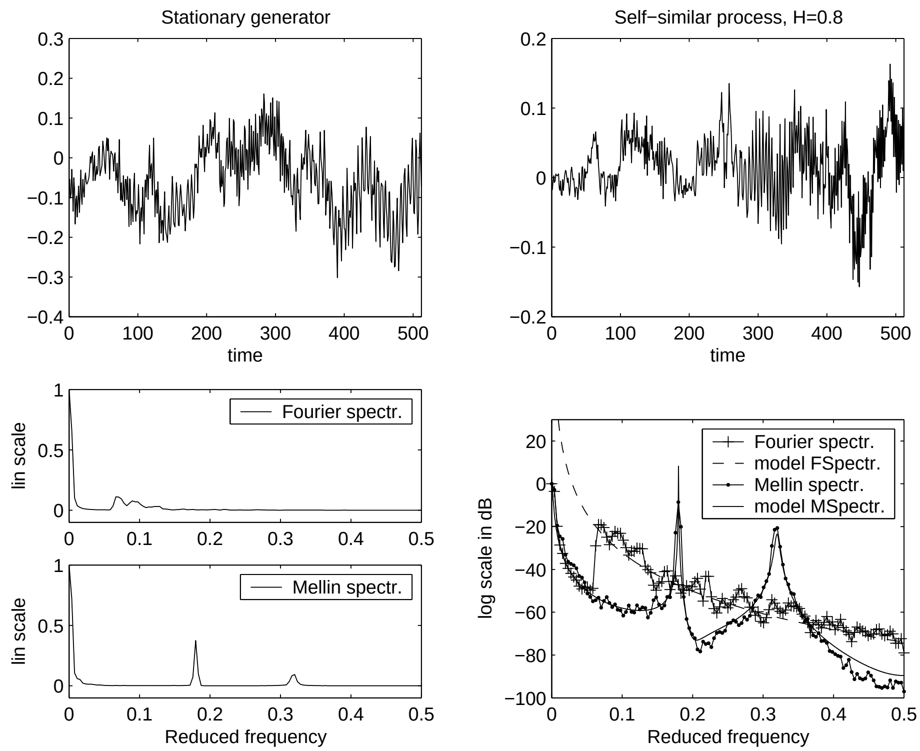

Figure 3. Modelling of a self-similar process with $H = 0.8$. Top left: the stationary generator $Y(t)$ is computed as an ARMA model of order (5,2), synthetised from the filtering of a white noise by the transfer function of the ARMA. Top right: the H-ss process deduced from the preceding process, implementing numerically $X(t) = (\mathcal{L}_H Y)(t)$. Bottom (left and right): estimated spectrum of X and of its inverse Lamperti transform $Y = {\mathcal{L}_H}^{-1} X$ (done also numerically); left is in linear scale, right in log scale. The Fourier spectrum of ${\mathcal{L}_H}^{-1} X$ is an estimation of the Mellin spectrum of X, according to equation (6). Here the spectrum of X (which is not properly defined because neither X nor its increments are stationary) appears complex because it has a rather large band and many small resonances and gives only an indication on H by being close to the model f^{-2H-1} (shown on the graph). On the other the estimated Mellin spectrum (that is the usual spectrum of Y) may be modelled well with more details, here an AR(5) whose accurate parameters may be found by a identification on the system (see the model on the curve on the right).

no other features are easy to understand with regards to self-similarity. On the other hand, one can expect H-ss processes to have Mellin spectrum that could be modelled by Euler-Cauchy systems (in the same way that stationary processes are well modelled by auto-regressive systems), involving a reduced number of parameters. An numerical inverse Lamperti transform may be used in practical implementation [11]. A central argument is the possibility of using a fast Mellin transform algorithm (based on FFT), as was studied in [8]. The exponent H is an external parameter of the transform and should be given a priori, estimated from a Fourier spectrum, or found by trying several exponents H until one matches. An other argument in favour of the practical use of the Lamperti transform is the not so great sensitivity of the spectral analysis to the choice of the renormalisation index H: being mistaken in the choice of H broadens a little the peaks in the Mellin transform. For instance, the numerical transform of $t^{H+i2\pi f_0}$ would not be the Dirac mass in f_0 but a lorenzian function of width $\sqrt{3}|H-r|/(\pi T)$ at half-height if the renormalization exponent is r instead of H, and T is the length of the signal analysed. It has usually small effects. Thus the relative robustness of the Mellin transform regarding H allows it to be used even if the proper H is not known with great accuracy.

Because using the Lamperti time warping from t to $\log t$ for discrete-time systems is equivalent to use geometric sampling $\{q^n, n \in \mathbb{Z}\}$ for the H-ss process (and thus usual arithmetic process for the stationary generator: $\{n \log q, n \in \mathbb{Z}\}$), designing procedures of estimation with geometrical sampling for self-similar processes is equivalent to designing procedures with arithmetic sampling for the stationary generator. This last thing is helped by all the amount of work done on digital signal processing. Consequences of this resampling were examined in specific situations in several works: direct estimation of correlation function C_X [32], estimation of the self-similarity index H [54], estimation of spectrum from random geometric sampling [30]. In all cases, the estimation was shown or proved to work. Finally, we have shown that a joint Fourier-Mellin analysis is also possible by means of time-frequency analysis. This method combines both kinds of informations: stationarity or non-stationarity in time and stationarity in scale [25].

2. Generalised Lamperti transformations for broken scale invariances

The equivalence (4) opens a window on generalisations of the Lamperti transformation for broken scale invariance. Indeed this usual Lamperti correspondence offers perspectives to study any property built on dilations as an image of some property built on shifts: homogeneity, time invariance, or even some nonstationarity.

There are two possibilities to extend the relation (4). First one can suppose some weakened symmetry for the generator besides exact stationarity. For instance discrete translation invariance (the one that occurs in crystal lattice) of the generator leads by $\mathcal{L}_H$ to discrete periodicity in scale, and we will elaborate in the next section on this symmetry, known as Discrete Scale Invariance (DSI). This solution draws upon classical results on some classes of nonstationary signals: properties of cyclostationary signals are mapped to DSI; local stationarity is mapped to local self-similarity (see below).

Another possibility is to make full use of the extended notion of stationarity proposed by Hannan [33]. If stationarity is understood as an invariance under any group of transformation $\{\mathcal{T}_g, g \in G\}$ where G is a group (in the case of self-similarity $\mathcal{T}_g$ is a dilation $\mathcal{D}_{H,\lambda}$, and G the multiplicative group $(\mathbb{R}_*^+, \times)$), one can ask whether there exists a invertible mapping $\mathcal{L}$ such that $\mathcal{L}^{-1}\mathcal{T}_g\mathcal{L} = \mathcal{S}_{\phi(g)}$. A first general answer is that the group G and the group $(\mathbb{R}, +)$ should have the same underlying structure, because this relation implies they are isomorphic. We examine the consequences of this remark to study scale invariance with finite size effects in section 4.

Finally, let us note that reducing any nonstationary process to its stationary generator is addressed from a mathematical point of view (with strong hypothesis of continuity on the correlation) in geostatistics [49] and statistics [46]. We point out here that the method has practical consequences for the modelling and analysis of scale invariance. Hereafter we deal only with one-dimensional signals, but the study of stationary generators of scale invariant two-dimensional fields is possible from the same point of view [9].

2.1. Nonstationary Time-Scale representations.

General nonstationary methods built on Fourier representation (related to time-shifts) lead to corresponding methods for Mellin representation. As an example, the time-dependent Wigner-Ville spectrum $W_Y(t,f)$ [24] where t is the local time and f the local frequency, given by the Fourier transform of the (nonstationary) covariance of the process,

$$W_Y(t,f) \hat{=} \int_{-\infty}^{+\infty} R_Y(t+\tau/2, t-\tau/2) e^{-i2\pi f\tau} \mathrm{d}\tau, \tag{14}$$

is mapped by $\mathcal{L}_H$ to a time-Mellin scale representation:

$$\begin{aligned} W_X^{Hss}(t,f) &= \int_0^{+\infty} R_X(t\sqrt{\tau}, t/\sqrt{\tau}) \tau^{-i2\pi f - 1} \mathrm{d}\tau \\ &= t^{2H} W_{\mathcal{L}_H{}^{-1}X}(\log t, f). \end{aligned} \tag{15}$$

The proof lies in equation (7) that shows that one has to change the Fourier transform in a Mellin one, and consequently that addition of time is replaced by a multiplication in scale. This recovers the scale-invariant Wigner spectrum $W_X^{Hss}(t,f)$ [23], and f has then the meaning of a Mellin scale, which is well adapted to describe invariant properties under dilations as argued before.

An interest of $W_Y(t,f)$ and $W_X^{Hss}(t,f)$ lies in the fact that for a stationary Y process, one recovers via the first one the stationary power spectrum $\mathbf{\Gamma}_Y(f)$,

$$W_Y(t,f) = \int_{-\infty}^{+\infty} c_Y(\tau) e^{-i2\pi f\tau} \mathrm{d}\tau = \mathbf{\Gamma}_Y(f), \tag{16}$$

and for self-similar X process, the corresponding property is that the scale-invariant Wigner spectrum factorizes simply as $W_X^{Hss}(t,f) = t^{2H} \mathbf{\Gamma}_{\mathcal{L}_H{}^{-1}X}(f)$. Note that the scale-invariant Wigner spectrum also has the property of invariance under dilations by a ratio λ for any process X (even if it is nonstationary and not H-ss):

$$W_{\mathcal{D}_{H,\lambda}X}^{Hss}(t,f) = \lambda^{-2H} W_X^{Hss}(\lambda t, f), \tag{17}$$

which could be used as a a defining property of $W_X^{Hss}(t,f)$.

This representation W^{Hss} gives the evolution in time of a process with respect to scale invariance; thus it disregards the part of nonstationary that is linked only to the t^H renormalisation term of scale invariance. If a signal is scale invariant, one obtains an time-invariant W^{Hss}; if not, the time evolution informs about the modifications of the scale composition of the process. Examples of such an evolution are given in the next paragraphs.

2.2. Local self-similarity.

General nonstationary models admit corresponding non scale invariant models through $\mathcal{L}_H$. A first example is the model for the correlation of locally stationary processes [50], that reads as $R_Y(t,s) = m_Y(\frac{t+s}{2})c_Y(t-s)$ with $m_Y(t) \geq 0$ and $c_Y(u)$ a non-negative definite function. The second function is an ordinary stationary covariance whereas R_Y fluctuates with the mean local time $(t+s)/2$ by means of the first function. By Lamperti mapping, a class of non-scale invariant processes is introduced that has the general covariance

$$R_X(t,s) = m_X(\log\sqrt{ts})(ts)^H C_X(t/s). \tag{18}$$

The mapping is obtained from $C_X(e^u) = c_Y(u)$ and $m_X(t) = m_Y(t)$. A comparison with (5) reveals that it is a generalised form allowing some evolution with the local mean time $\sqrt{ts}$ and the corresponding scale-invariant Wigner spectrum expresses as $W_X^{Hss}(t,f) = m_X(t)t^{2H}(\mathbf{M}C)(i2\pi f)$, illustrating both the mean evolution imposed by the function $m_X(t)$ and the scale behaviour given by the second term, a local scale spectrum.

We will not detail other aspects, except for Discrete Scale Invariance hereafter, and the reader will find elsewhere details on higher-order distributions that can be introduced on this grounding [3], on multiplicative harmonizability for non H-ss processes [10], or on the analysis of locally asymptotically self-similar processes by use of $\mathcal{L}_H$ [26, 11].

3. Discrete Scale Invariance

An application of the Lamperti correspondence has received more attention, namely the study of Discrete Scale Invariance, i.e., scale invariance for some preferred scale factors only. Some fractals such as the triadic Cantor set, or some simple signals such as the Mellin chirps of the form $t^H \exp(if_0 \log t)$, are naive examples of this symmetry which is the invariance under dilations of scale factors $\{\lambda_0^n, n \in \mathbb{N}\}$ ($\lambda_0 = 3$ for a triadic Cantor set and $\lambda_0 = \exp(2\pi/f_0)$ for Mellin chirps). This was advocated as a central concept in the study of many critical systems [47, 51, 52]. More attention has been given to the deterministic framework and we rely here on the stochastic extension of the property [10].

A process $\{X(t), t > 0\}$ is said to possess Discrete Scale Invariance (DSI) of index H and scaling factor $\lambda_0 > 0$ if $\{\mathcal{D}_{H,\lambda_0}X)(t), t > 0\} \stackrel{d}{=} \{X(t), t > 0\}$. Mapping this property back to $Y(t) = ({\mathcal{L}_H}^{-1}X)(t)$, it is straightforward to establish that, provided that $T_0 = \log\lambda_0$, we have $\{\mathcal{S}_{T_0}Y)(t), t \in \mathbb{R}\} \stackrel{d}{=} \{Y(t), t \in \mathbb{R}\}$. This property is a statistical periodicity of Y of period T_0 (and hence for any time-shift $nT_0, n \in \mathbb{N}$) and it defines cyclostationary (or periodically-correlated) processes [29, 31]. The Lamperti correspondence is such that a process has (H, λ_0)-DSI if and only if its inverse Lamperti transform is cyclostationary of period $T_0 = \log\lambda_0$. This is an extension to stochastic DSI of the Lamperti result on H-ss.

3.1. Characterization of DSI.

The theory of cyclostationary processes is well established [28]. A known characterisation uses the periodicity of the covariance, $R_Y(t+T, s+T) = R_Y(t, s)$ to write it as a Fourier series

$$R_Y(t, t+\tau) = \sum_{n=-\infty}^{+\infty} c_n(\tau) e^{i2\pi nt/T_0}. \tag{19}$$

A corresponding first characterisation of DSI is thus obtained on the covariance which has to read for a process $X(t)$ with (H, λ_0)-DSI

$$R_X(t, kt) = (kt)^H \sum_{n=-\infty}^{+\infty} c_n(\log k) t^{H+i2\pi n/\log\lambda_0}. \tag{20}$$

Furthermore, using time-Mellin scale analysis, one obtains a simple expansion of the covariance on Mellin chirps.

$$W_X^{Hss}(t, f) = \sum_n (\mathbf{F}c_n)(f - \frac{n}{\log\lambda_0}) t^{2H+i2\pi n/\log\lambda_0}. \tag{21}$$

This equation offers direct possibilities of studies of processes with DSI in a time-frequency space. Any DSI signal may be thus decomposed on a Mellin chirps expansion.

3.2. Discrete-time sequences with DSI.

An important feature of stochastic DSI is that one can analyse discrete-time sequences that might have this property by using standard cyclostationary tools on the stationarised process. Estimation of the cyclostationary period of this process, e.g. by means of the marginal of the cyclic periodogram, will give an estimate of the preferred scale ratio λ_0. From equation (21), a weighted time-average on the Mellin functions reads as:

$$\int_0^\infty W_X^{Hss}(t, f) t^{-2H-i2\pi\beta_c-1} \mathrm{d}t = \sum_n (\mathbf{F}c_n)(f - \frac{n}{\log\lambda_0}) \delta(\beta_c - n/\log\lambda_0). \tag{22}$$

Taking then the sum over all the scales f leads to the marginal cyclic spectrum that should show peaks on specific Mellin scales related to the preferred scale ratio:

$$S(\beta_c) = \int_{-\infty}^{+\infty} \mathrm{d}f \int_0^\infty W_X^{Hss}(t, f) t^{-2H-i2\pi\beta_c-1} \mathrm{d}t = \sum_n E_n \delta(\beta_c - n/\log\lambda_0). \tag{23}$$

Here $E_n = \int_{-\infty}^{+\infty} \mathbf{\Gamma}_n(f)\mathrm{d}f = \int_{-\infty}^{+\infty} (\mathbf{F}c_n)(f)\mathrm{d}f$ is, thanks to the Wiener-Khintchin relation, the total energy of a process having $c_n(\tau)$ as correlation function. Thus there are peaks in this cyclic spectrum, localized on Mellin scales $\beta_c = n/\log\lambda_0$.

In addition, there exist various models, parametric or not, of random sequences having DSI [11] that may serve as benchmarks. As an example, we plot in figure 4 one realisation of a stochastic process with DSI, with its estimated variance on the right. If one wants to resort to cyclostationary methods for the DSI problem for instance, a

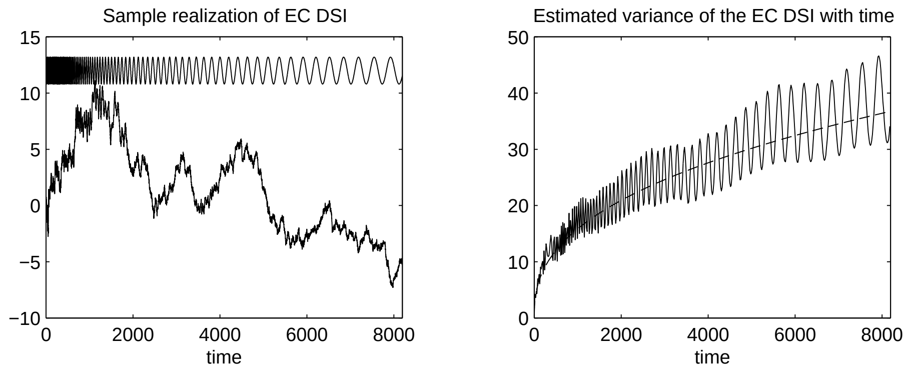

Figure 4. Example of parametric system with Discrete Scale Invariance. The model is based on a time-varying scale invariant filter (or Euler-Cauchy system), with a time representation as $tX'(t) + a(t)X(t) = \epsilon(t)$. $\epsilon(t)$ is white-noise with time-varying variance $(t^H\sigma(t))^2$ and we impose that $a(t)$ and $\sigma(t)$ has deterministic DSI with exponent 0, i.e., $a(\lambda_0 t) = a(t)$. This model, known as a DSI Euler-Cauchy system of order 1, is studied in details in [11], along with its proper discretisation in time. Left: sample path of the process with $H = 0.2$ and $\lambda_0 = 1.05$; a figuration of the oscillations of the coefficients is shown above it. Right: estimated variance of the process, done empirically with 1024 realisations. The log-periodic oscillations of the variance are a signature of DSI for this model.

statistical comparison of the estimates found for some real data and the results obtained on sequences with or without DSI will escape the difficulties and the possibilities of artefacts regarding DSI studied in [34].

3.3. DSI and stationary increments.

The property of DSI is compatible with the property of stationary increments. This is known since studies on the Weierstrass-Mandelbrot random function [7, 25]:

$$W(t) = \sum_{n=-\infty}^{+\infty} \lambda_0^{-nH}(1 - e^{i\lambda_0^n t})e^{i\phi_n}, \tag{24}$$

where the ϕ_n are i.i.d. random variables, uniform in $[0, 2\pi[$. An immediate property is the invariance of $W(t)$ under dilations $\mathcal{D}_{H,\lambda}$, but only of $\lambda = \lambda_0^m$ with $m \in \mathbb{Z}$, that is DSI. Analyzed as such, the Weierstrass-Mandelbrot random functions admit a decomposition on a Mellin basis. On figure 5 we illustrate this property by showing a sample realization of this random process and the corresponding cyclic Mellin spectrum that was defined in (23). A less known property concerns the increment process and reads as

$$\mathbb{E}\,|W(t+\tau) - W(t)|^2 = \sum_{n=-\infty}^{+\infty} \lambda_0^{-2nH} 2(1 - \cos\lambda_0^n \tau). \tag{25}$$

This expression is given by a straightforward calculus on the possible random phases. A striking feature is that the structure function of order 2 of $W(t)$ does not depend on t but only on τ; that is known as (second-order) stationary increments. This property

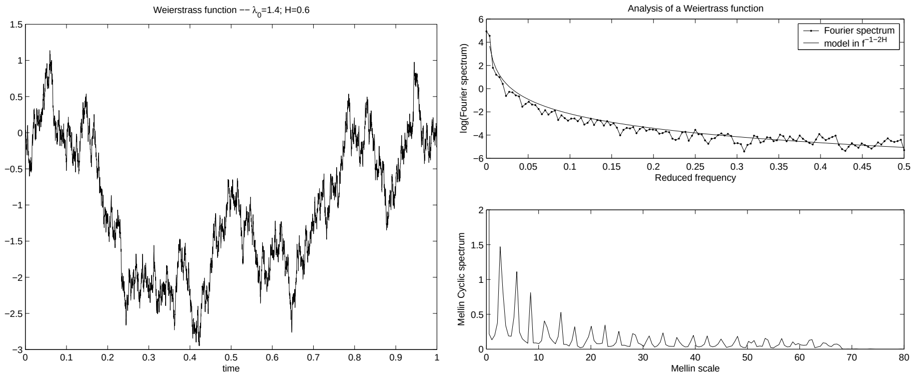

Figure 5. Example of a random Weierstrass-Mandelbrot function, as defined in equation (24) on the left, with $H = 0.4$, and $\lambda_0 = 1.4$. The simulation is made with a limited number of modes, the ones of higher frequency being $n = 50$. On the right are shown elements of analysis by means of $({\mathcal{L}_H}^{-1}W)(t)$. Upper part, an estimate of the Fourier spectrum of $({\mathcal{L}_H}^{-1}W)(t)$, surimposed with long-range dependance model for spectrum in f^{-1-2H} – this spectrum shows no striking feature of preferred scale ratio; lower part: an estimate of the cyclic spectrum $S(\beta_c)$ as defined in (23) is represented. The estimation was made first by doing numerically an inverse Lamperti transform, and then a double Fourier transform is made to estimate the quantity appearing in (22). Finally a sum is made on the frequency f. A average was also made by cutting the time series in several blocks and averaging the result over those blocks. The peaks are separated in Mellin scale by $1/\log\lambda \simeq 2.97$. For an exactly self-similar process, one would find only a peak around $\beta_c = 0$ so they are here relevant to measure the DSI.

is interesting because it allows to give a proper definiton to the Fourier spectrum of this otherwise non-stationary process.

This process can also be used as a starting model for DSI processes with stationary increments, changing $(1 - e^{i\lambda_0^n t})$ in its expression for a more general form $(g(0) - g(\lambda_0^n t))$, with any periodic function continuously differentiable at $t = 0$. DSI and some kind of stationarity might then coexist in physical models needing both. The reader is referred to [25] for a more detailed lecture of the properties on the Weiertrass-Mandelbrot function and the possibilities to uncover its properties by time-frequency methods.

4. Warped Lamperti Transformation for broken Self-similarity

We have told that Lamperti transform can be adapted to other forms of symmetry than proper scale invariance, by modifying $\mathcal{L}_H$ so that an equivalence similar to (4) remains valid. An interesting approach to tackle broken scale invariant signals is to postulate that the usual composition of scales $(\lambda \odot t = \lambda \times t)$ is no longer valid, and that another way of composing scales underlies the physics: this postulates an unusual law and action on scales. A symmetry of this kind has been first adopted by L. Nottale in his theory of scale relativity [43], and further developed by B. Dubrulle and F. Graner in another context [17, 18]. The symmetry in scale is broken because of the existence of bounds in scale and amplitudes of the studied processes.

We present here a general setting that allows to put in correspondence stationary stochastic processes with generalised scale invariant processes with finite size effects, or bounding. The setting is presented before constructing the generalised Lamperti transform associated to this law, and we give details on finite size scale invariant processes in section 4.3. Having this generalised stationarising transform gives new insights on this symmetry because it opens the subject to results coming from stationary modelling.

4.1. Generalised scale laws and the associated dilation operator

The stochastic processes $X(t)$ we consider describe the evolution in scale of some physical quantities. Therefore, the transformation of scales $\mathrm{e}_1 \odot \mathrm{e}_2 = \mathrm{e}_3$ and the transformation of the process must belong to a group of transformations (or at least to a semi-group, such as what happens in multiplicative cascades [14]).

We restrict the discussion here to the case for which scales belong to a group. Behind the hypothesis of an invertible law for the scales, the idea is that changing scale is an operation meaningful both as zooming out and zooming in so that each zooming ratio has to possess an inverse. The simplest choice is to take a group isomorphic to $(\mathbb{R}^+, \times)$. Let $\mathbb{A}$ be the set of scales that we consider and let $\odot$ be the law of composition of scales. Since we postulate that $(\mathbb{A}, \odot)$ is isomorphic to $(\mathbb{R}^+, \times)$, there exists a morphism $S_\odot : \mathbb{A} \longrightarrow \mathbb{R}^+$ such that $S_\odot(\mathrm{e}_1 \odot \mathrm{e}_2) = S_\odot(\mathrm{e}_1) \times S_\odot(\mathrm{e}_2)$. Any diffeormorphism from $\mathbb{A}$ onto $\mathbb{R}^+$ is a good candidate and for any $\alpha \in \mathbb{R}$, $(S_\odot(\cdot))^\alpha$ is also a convenient morphism.

A similar hypothesis is made on the amplitude of the process. For instance, instead of a process valued in $\mathbb{R}$, we may assume that it takes values in an interval $\mathbb{X} =]X_-, X_+[$ included in $\mathbb{R}^+$. A discussion concerning this restriction is provided below. We will show later that there exists a law $\otimes$ of composition on this interval that makes $(\mathbb{X}, \otimes)$ a group isomorphic to $(\mathbb{R}^+, \times)$. Let $S_\otimes : \mathbb{X} \longrightarrow \mathbb{R}^+$ be the associated morphism. Note here that the morphism $S_\otimes$ depends explicitly on the bounds of interval $\mathbb{X}$: in this formalism, we study finite size effects.

We can now define a generalised dilation operator following the construction of the usual dilation operator: time is dilated using the composition of scales ($\times$) and the process is renormalised properly using the composition law for amplitudes ($\times$). Let λ be a dilation factor in $\mathbb{A}$. We then define the generalised dilation operator $\mathcal{D}^g_{H,\lambda}$, acting on stochastic processes indexed by $\mathbb{A}$ with values in $\mathbb{X}$, as

$$(\mathcal{D}^g_{H,\lambda} X)(t) = g(\lambda) \otimes X(\lambda \odot t) \tag{26}$$

where $g(\lambda)$ is the renormalisation function. This function is not arbitrary and indeed depends on the morphisms $S_\odot$ and $S_\otimes$. To prove this, we note that going from time t to time $\lambda_1 \odot \lambda_2 \odot t$ can be done at least in two ways: directly by applying $\mathcal{D}^g_{H,\lambda_1\odot\lambda_2}$ or indirectly by applying successively $\mathcal{D}^g_{H,\lambda_1}$ and $\mathcal{D}^g_{H,\lambda_2}$. Mathematically this means that the generalised dilation operator is a representation of the group $(\mathbb{A}, \odot)$. It also implies that the function g satisfies $g(\lambda_1 \odot \lambda_2) = g(\lambda_1) \otimes g(\lambda_2)$, meaning that $S_\otimes \circ g = {S_\odot}^\alpha$ is an acceptable morphism for $\odot$. For reasons that will be clear later, we choose the exponent $\alpha = -H$ so that $g(\lambda) = S_\otimes^{-1}(S_\odot(\lambda)^{-H})$.

Given the notions introduced above, we will say that a stochastic process satisfies a generalised scale invariance property if

$$X(t) \stackrel{d}{=} (\mathcal{D}^g_{H,\lambda} X)(t) = g(\lambda) \otimes X(\lambda \odot t) \tag{27}$$

This equation is directly a generalisation of (1). Note that imposing the equality for a deterministic function $x(t)$, allows to obtain the form of the scale invariant function (analogy with power laws for the usual dilation operator). Indeed, a scale invariant function satisfies $x(t) = S_\otimes^{-1}(S_\odot(\lambda)^{-H}) \otimes x(\lambda \odot t), \forall \lambda \in \mathbb{A}$. Let e be the identity element for $\odot$. Then setting λ so that $\lambda \odot t = e$ (the group structure implies the existence of the inverse $\lambda = e \odot^{-1} t$) allows to write

$$\begin{aligned} x(t) &= x(e) \otimes S_\otimes^{-1}(S_\odot(e \odot^{-1} t)^{-H}) &(28)\\ &= x(e) \otimes S_\otimes^{-1}(S_\odot(t)^{+H}) = S_\otimes^{-1}\left[S_\otimes(x(e)) S_\odot(t)^H\right] &(29)\end{aligned}$$

This expression has two free parameters once the morphisms are fixed: the exponent H and the multiplicative constant $S_\otimes(x(e))$ that defines the initial value. In the usual case, $\odot = \times$ and the morphisms are the identity. We recover the usual power law as the deterministic scale invariant function and $g(\lambda) = \lambda^{-H}$. This recovers expression (1) for the dilation and this explains the choice $\alpha = -H$ previously adopted.

4.2. Generalised Lamperti transformation

We have shown in the first sections the usefulness of the Lamperti transformation to study scale invariant processes (and their broken versions). The fundamental fact behind this is that dilation and shift operators are equivalent through $\mathcal{L}_H$, according to (4). The same idea can be applied for generalised dilation operators. We hence seek an operator $\mathcal{L}^g_H$, acting on stationary signals Y indexed by $\mathbb{R}$ such that $(\mathcal{L}^g_H Y)(t)$ satisfies the generalised scale invariance property. This operator must be invertible, and generalised dilation and shift should be equivalent through its application. The expressions for the generalised Lamperti transformation and its inverse are easily shown to be

$$\begin{aligned} (\mathcal{L}^g_H Y)(t) &= S_\otimes^{-1}\left(Y(\log S_\odot(t))\right) \otimes S_\otimes^{-1}\left(S_\odot(t)^H\right) &(30)\\ &= S_\otimes^{-1}\left(S_\odot(t)^H Y(\log S_\odot(t))\right) &(31)\\ (\mathcal{L}^{g\,-1}_H X)(t) &= e^{-Ht} S_\otimes^{-1}\left(X(S_\odot^{-1}(e^t))\right) &(32)\end{aligned}$$

making $\mathcal{D}^g_{H,\lambda}$ and $\mathcal{S}_{\log S_\odot(\lambda)}$ equivalent operators, since

$$\mathcal{L}^{g\,-1}_H \mathcal{D}^g_{H,\lambda} \mathcal{L}^g_H = \mathcal{S}_{\log S_\odot(\lambda)} \tag{33}$$

The structure of the generalised Lamperti transformation is interesting, since it can also be written as a function of the usual Lamperti transformation. Indeed, we have

$$(\mathcal{L}^g_H Y)(t) = S_\otimes^{-1}\left((\mathcal{L}_H Y)(S_\odot(t))\right) \tag{34}$$

Therefore, a signal that has a generalised scale invariance property can be constructed from its stationary generator or can be obtained by time and amplitude warping of the associated H-ss process.

Note that the morphisms are indeed deterministic functions. We could imagine replacing in the definition above deterministic morphisms by random morphisms (Devil's staircase associated to a random measure for example). In this way, we recover approaches taken by processes with multifractal times [40] (see also [14] for more recent applications) in some definitions of multifractal processes. The link between the ideas developed here and multifractals remains however to be explored.

4.3. Finite Size Scale Invariant fractional Brownian motion

As an illustration of the ideas presented above, we consider the approach initiated by L. Notalle in his theory of scale relativity [43], and further developed by Dubrulle and Graner [18, 17]. In these works it is argued that the law for composition of scales may be more complicated than the usual product. Scale has the behaviour of a velocity when considered logarithmically, as $a = \log \lambda$. A natural generalisation is to consider that the logarithm of the scale does not follow the Galilean transformation of velocities but the Lorentz law. In this setting, scale is limited to a finite size range, and B. Dubrulle has developed the formalism of finite size scale invariance using this.

However, this scale invariance was applied directly on deterministic functions, or on the moments of random variables. As the invariance is only studied as a possible model for those moments, the method doed not give tools of synthesis or of analysis beyond testing the goodness-of-fit of the finite size scaling laws for some datas. Our approach is slightly different in that we impose scale invariance on the random variables themselves. This results in probabilistic scale invariance where the statistics will follow deterministic laws of finite-size scaling studied by Dubrulle; but as we work jointly on the scale evolution of the process and its probablity law by means of a Lamperti correspondance with a stationary generator, we obtain general methods to synthetise and analyse those processes. Our framework is more constrained, but gives better insights on the studied stochastic processes because of the correspondance with stationary processes. Thus there exists the possibility of adapting methods of stationary signal processing to finite-size scale invariance.

If scale is constrained to live in the finite size interval, let $\mathbb{A} =]a_-, a_+[\subseteq \mathbb{R}^+$. Then the Lorentz composition law for two scales reads

$$a_1 \odot a_2 = \exp \frac{\log a_1 + \log a_2 - \log a_1 \log a_2 (\frac{1}{\log a_-} + \frac{1}{\log a_+})}{1 - \frac{\log a_1 \log a_2}{\log a_- \log a_+}} \tag{35}$$

The associated morphism can be shown to be

$$S_{\odot}(a) = \begin{cases} \exp\left\{\frac{\log a_- \log a_+}{\log a_- - \log a_+} \log\left(\frac{1-\log a/\log a_-}{1-\log a/\log a_+}\right)\right\} \\ \exp\left\{\log\frac{1}{a_\pm}\log\left(1-\frac{\log a}{\log a_\pm}\right)\right\}, \text{if } \log a_\mp \to +\infty \\ a, \text{ if furthermore } \log a_\pm \longrightarrow -\infty \end{cases}$$

The same form of composition laws is adopted for the amplitudes of the signals, leading to the same form of morphism, replacing variables a by X in $\mathbb{X} =]X_-, X_+[$. There is however a difficulty concerning the application of this formalism to the amplitude of the signals: these laws are based on the equivalence between the multiplicative and the additive groups *via* a logarithm. Therefore they are restricted, as mentioned above, to the case of positive variables. This restriction is not a real problem for time, since we are satisfied to work with signals indexed by $\mathbb{R}^+$ only. It is more problematic when considering the amplitude of the signals that can be either positive or negative. To manipulate signed signals, the composition law has to separate the positive part from the negative part of the amplitude. Hence, the elements of the groups are represented as two parameter elements $X = (|X|, \mathrm{Sign}(X))$. In this case, the morphism is defined as

$$\begin{aligned} S_\otimes : (\mathbb{X}^*, \otimes) &\longrightarrow (\mathbb{R}^*, \times) \\ X &\longrightarrow S_\otimes(X) = \theta S_\theta(|X|) \text{ where } \theta = \mathrm{Sign}(X) \end{aligned}$$

and where S_{+1} (resp. S_{-1}) is a function between $]0, X_+[$ (resp. $]0, X_-[$) onto $\mathbb{R}^{+*}$. In the case of the special relativity-like laws, the details of the laws can be found in [2], and the functions S_θ are given by

$$\begin{cases} S_{+1}(X) = \exp\Big(-\log X_+ \log(1-\frac{\log|X|}{\log X_+})\Big) \text{if } X \geq 0 \\ \\ S_{-1}(X) = \exp\Big(-\log X_+ \log(\frac{\log X_- - \log|X|}{\log X_+})\Big) \text{if } X < 0 \end{cases}$$

The inverse of the morphism is given by

$$\begin{aligned} S_\otimes^{-1} : (\mathbb{R}^*, \times) &\longrightarrow (\mathbb{X}^*, \otimes) \\ x &\longrightarrow S_\otimes^{-1}(x) = \theta S_\theta^{-1}(|x|) \text{ where } \theta = \mathrm{Sign}(x) \end{aligned}$$

There is no real separation between the positive and the negative part; the morphism remains continuous in 0 and we may set $S_\otimes(0) = 0$. The separation was made to respect the group structure, and to obey the fact that 0 has a specific role with regards to the multiplication. The only specifity left is the fact that the valid scale ratios are taken positive only (because the time is here positive only) so there is no way for a dilation to change the sign of a function. But as the equality is imposed in law, a given realization of a process is allowed to explore without restriction both positive and negative values.

These results allow us to study a generalisation of the fractional Brownian motions; the stationary generator of the fBm (generalised Ornstein-Uhlenbeck process, gOU), whose correlation function is given by (12), can be used to create a fBm with a generalised scale invariance property, with finite size effects. Using the explicit forms of the morphisms corresponding to the finite size scale composition law provided above, we define the finite size scale invariant fractional Brownian motion as the generalised Lamperti transform of the gOU. We plot in figure 6 several examples depending on the finiteness of the bounds in scale (and thus in time) and/or in amplitude. According to

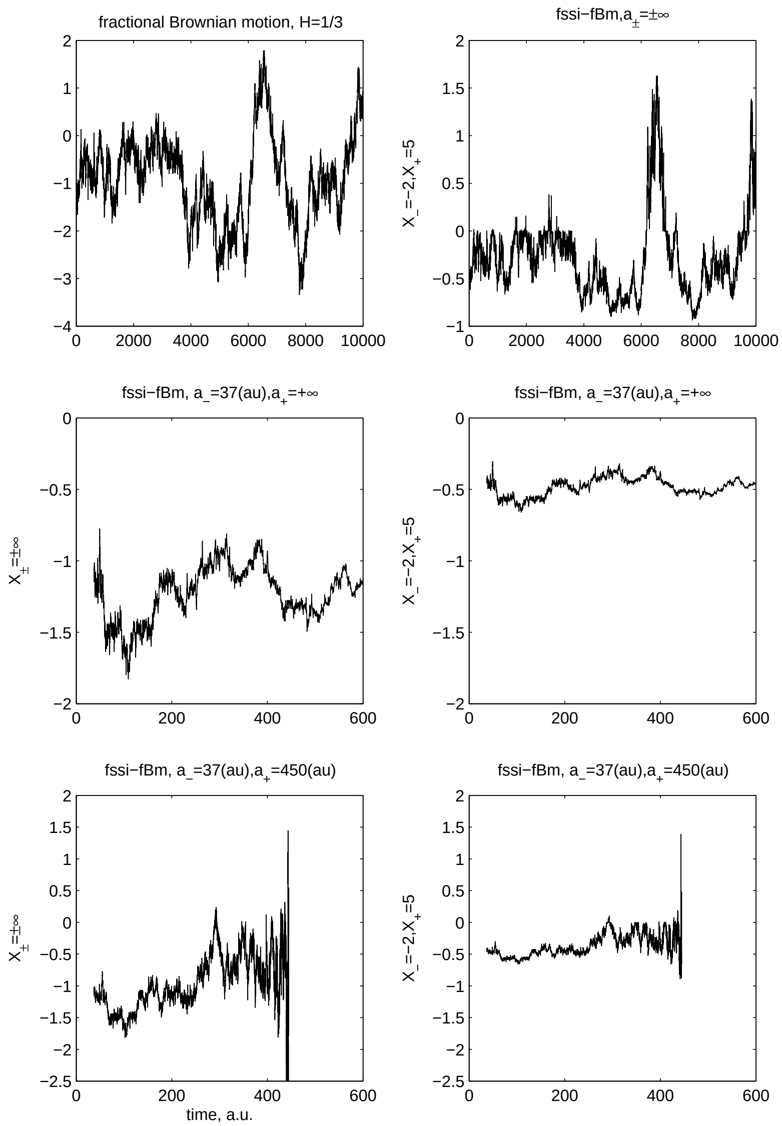

Figure 6. Example of fractional Brownian motions with a finite size scale invariance property. Top left: a fBm with $H = 1/3$, Hölder exponent for the velocity in fully developed turbulence. Middle left: same fBm after a time warping, time is bounded below. Bottom left: same fBm but warped to a finite size interval. Top right: same fBm but warped in amplitude between -2 and 5. Middle right: same fBm after an amplitude warping between -2 and 5, and a bounded below in time. Bottom right: same as middle right but living on a finite interval of time. The top left panels corresponds to usual self-similar processes, whereas the five others depict snapshots of processes that possess some kind of finite size scale-invariance (fssi).

equation (34), the finite size scale invariant process can be obtained directly from the scale invariant process sharing the same stationary generator. Therefore, the different snapshots in the figure are obtained by applying the warpings to numerically generated fBms (using the middle point displacement method, even if the obtained snapshots with this method are just approximations of fBm). Some of the snapshots have a strange looking, and their usefulness to describe real processes is of course under questions (and studies). To assess the usefulness of some of the model, we could imagine that the fssi-fBm bounded in time and unbounded in amplitude could model a critical phenomena presenting a rupture. In this view, the upper bound a_+ could represent the critical time.

The complete study of the processes is quite difficult to perform. When there is no warping in amplitude, the study is easy since the only transformation acts on time. Hence, the fBm with a generalised scale invariant property (restricted to scale) is a Gaussian process with covariance directly obtained from that of the fBm by properly warping the time lags of the covariance function. From equation (11), the covariance reads

$$R(t,s) = \frac{\sigma^2}{2}(|S_\odot(t)|^{2H} + |S_\odot(s)|^{2H} + |S_\odot(t) - S_\odot(s)|^{2H}), \tag{36}$$

and fully describes the process. In particular, the variance of the process reads $\sigma^2|S_\odot(t)|^{2H}$, close to the model studied in [18, 17] for moments of a finite-size process. When the scale is far from the bounds, the morphism is close to a power-law: $|S_\odot(t)|^{2H} \simeq t^{2H}$. We obtained thus a form of intermediate asymptotic [5] with a specific departure from the power-law given by the morphism law $S_\odot$.

When a warping of the amplitude is present, the probabilistic structure can be obtained but is practically restricted to a few points statistics. For example, let Z be a scale invariant process with stationary generator Y. Since Y is a stationary signal, its one point probability density function P_Y does not depend on time. Therefore, the one point probability density function of Z reads $P_Z(z,t) = P_Y(z/t^H)/|t|^H$. Therefore, the one point probability density function of X reads

$$P_X(x,t) = \frac{1}{|S_\odot(t)|^H} P_Y\left(\frac{\theta S_\theta(x)}{|S_\odot(t)|^H}\right) \left|\frac{dS_\theta(x)}{dx}\right|, \tag{37}$$

where again $\theta = \mathrm{Sign}(x)$. This departs from the framework of [17], because we characterise the probability law directly. As an illustration, we plot in figure 7 two of these functions for $H = 1/2$: In the first case, we choose $-X_- \neq X_+$, implying that the nonlinear distortion is asymmetric. The nonlinear function is depicted in the top left figure, whereas the Gaussian and the density of the transformed process are plotted in the top right figure. The symmetric case is depicted in the bottom row of the figure.

5. Conclusion

We have addressed here some reflexions to use the Lamperti correspondence and the stationary generators of self-similar processes, but a furthermore important feature is

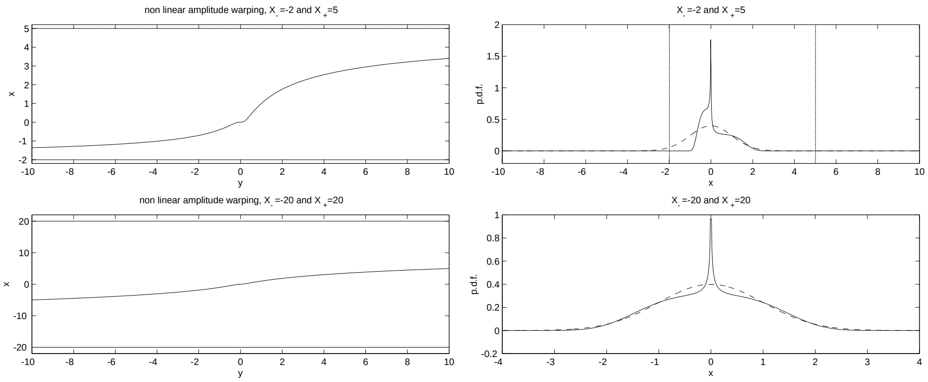

Figure 7. The figure on the left depicts the static nonlinearity used to warp the amplitude of the signals, for two cases. The figure on the right gives the corresponding probability density functions obtained, when the initial signal is Gaussian.

the existence of enlarged correspondences with broken or weakened self-similarities. An incentive to delve further on practical use of this framework is that some specific examples, namely Discrete Scale Invariance and Finite Size Scale Invariance, were put forward in previous works as relevant properties of physical systems: geophysics, fracture and growth problems for the former ([19, 52] and references herein), turbulence or fundamental physics for the latter [43, 17].

Our point of view was here from stochastic modelling and signal processing. We have paved the way by showing the general framework of stationary correspondence and generators, and given some detailed consequences. From this point of view, numerical models have been obtained and characterised that have the broken scale invariance envisioned here. The future of this work is to study specific physical systems and signals therefrom by means of the methods constructed with the Lamperti transformation, in order to find out, or rule out, the appearance of these broken scale invariance in problems of physics.

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