Quasi-Fuchsian AdS representations are Anosov

Thierry Barbot

QUASI-FUCHSIAN ADS REPRESENTATIONS ARE ANOSOV

THIERRY BARBOT

Abstract. Let Γ be a cocompact lattice in $\mathrm{SO}(1,n)$. In [Mér07] Q. Mérigot proved that if a representation $\rho:\Gamma\to\mathrm{SO}(2,n)$ is Anosov in the sense of Labourie ([Lab06]), then it is *quasi-Fuchsian*, *ie.* it is faithfull, discrete, and preserves an acausal subset in the boundary of anti-de Sitter space. In the present paper, we prove the reverse implication. It also includes:

– A construction of Dirichlet domains in the context of anti-de Sitter geometry,

– A proof that spatially compact globally hyperbolic anti-de Sitter spacetimes with acausal limit set admit locally CAT(-1) Cauchy hypersurfaces.

1. Introduction

This paper is a complement to the preceding one by Q. Mérigot [Mér07]. We will use all the notations, definitions and results therein. Let's just remind that a representation $\rho:\Gamma\to\mathrm{SO}_0(2,n)$ is quasi-Fuchsian, or GHC-regular if it is faithfull, dicrete and preserves an achronal topological $(n-1)$-dimensional sphere Λ_ρ in the conformal boundary Ein_n of AdS_{n_1}, the so-called limit set. Except in § 5.3.2 we assume that Λ_ρ is acausal. Our main purpose is to prove the reverse of Theorem 1.1 in [Mér07], namely:

Theorem 1.1. *Let Γ be a lattice in $\mathrm{SO}_0(1,n)$. Then, any quasi-Fuchsian representation in $\mathrm{SO}_0(2,n)$ with acausal limit set is $(\mathrm{SO}_0(2,n),\mathcal{Y})$-Anosov.*

As in [Mér07] the proof deeply involves anti-de Sitter geometry, and is based on the fact that quasi-Fuchsian representations are precisely holonomy representations of spatially compact, globally hyperbolic Lorentzian manifolds locally modelled on AdS_{n+1}. We will also consider the case where no special hypothesis is made on the group Γ (see § 5.3.1). We give arguments in favor of the idea that Anosov representations coincide with "quasi-Fuchsian" representations even if Γ is not *a priori* assumed to be isomorphic to a lattice in $\mathrm{SO}_0(1,n)$. A crucial point is the fact that GHC-regular AdS-spacetimes with acausal limit set admit Cauchy surfaces which are CAT(-1), implying that the associated holonomy representation is still Anosov, but now in a weaker, non-differentiable sense. More generally, in the last § we give a list of groups admitting quasi-Fuchsian representations into $\mathrm{SO}_0(2,n)$ that we expect to be exhaustive.

2. A criteria for Anosov representations

A technical difficulty arising when one wants to prove that a representation is Anosov is to ensure the exponential decay. The following lemma shows that this feature follows from the compactness of the ambient manifold N of the Anosov flow and a weaker contraction property somewhere along the orbit. Compare with [Mér07, § 5.1.1].

Proposition 2.1. *Let $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ be a representation, and assume the existence of continuous maps $\ell^{\pm}_{\rho} : \widetilde{N} \to \mathrm{Ein}_n$ and of a continuous and Γ-equivariant family of Riemannian metrics g^p defined in a neighborhood of $\ell^{\pm}_{\rho}(p)$ in Ein_n such that, for all p in $\widetilde{N}$, there is $t > 0$ such that, for all w in $\mathrm{T}_{\ell^+_\rho(p)}\mathrm{Ein}_n$ (respectively in $\mathrm{T}_{\ell^-_\rho(p)}\mathrm{Ein}_n$), we have $g^{\tilde{\Phi}^t(p)}(w,w) \geq 2g^p(w,w)$ (respectively $g^{\tilde{\Phi}^t(p)}(w,w) \leq g^p(w,w)/2$). Then, ρ is $(\mathrm{SO}_0(2,n),\mathcal{Y})$-Anosov.*

Proof. We simply prove that the hypothesis above imply the exponential decay and exponential expansion expressed in Remark 5.1.2 in [Mér07]. Let $\pi_\rho : E_\rho \to N$ be the flat Ein_n-bundle associated to ρ, and let $s^{\pm} : N \to E_\rho$ be the sections induced by $\ell^{\pm}_{\rho}$. Since the the family $(g^p)_{(p\in\widetilde{N})}$ is Γ-equivariant, it induces for every p in N a metric $g^p_{\pm}$ on the fiber $\pi^{-1}(p)$ near $s^-(p)$ and $s^+(p)$. Denote by $V^{\pm}(p)$ the vertical tangent bundle at $s^{\pm}(p)$. For every p in N and every t define:

$$\begin{aligned}\alpha^-(p,t) &= \sup_{w\in V^-(p)} \frac{g_-^{\Phi^t(p)}(w,w)}{g^p_-(w,w)}\\ \alpha^+(p,t) &= \inf_{w\in V^-(p)} \frac{g_+^{\Phi^t(p)}(w,w)}{g^p_+(w,w)}\end{aligned}$$

Obviously, for $s,t > 0$:

$$\begin{aligned}\alpha^-(p,t+s) &\leq \alpha^-(p,s)\alpha^-(\Phi^s(p),t)\\ \alpha^+(p,t+s) &\geq \alpha^+(p,s)\alpha^+(\Phi^s(p),t)\end{aligned}$$

By hypothesis, and since N is compact, there is a finite covering $(U_i)_{(1\leq i\leq k)}$ of N, and a sequence $(T_i)_{(1\leq i\leq k)}$ such that for any i in $\{1,\dots,k\}$ and any p in U_i we have $\alpha^-(p,T_i) \leq 1/2$. Put $T = \sup\{T_i | 1 \leq i \leq k\}$ and $a = \sup\{\alpha^-(p,t) | t \in [0,T], p \in N\}$. For any p in N there exist sequences $(t_j)_{(0\leq j\leq J)}$ and $(i_j)_{(0\leq j\leq J)}$ such that $t_0 = 0$, $t_{J-1} \leq t \leq t_J$, $\Phi^{t_j}(p)$ lies in U_{i_j} and $t_{j+1} = t_j + T_{i_j}$. Then:

$$\begin{aligned}\alpha^-(p,t) &\leq \alpha^-(p,T_{i_0})\alpha^-(\Phi^{t_1}(p),T_{i_1})\dots\alpha^-(\Phi^{t_{J-1}}(p),t-t_{J-1})\\ &\leq (1/2)^{J-1}a\\ &\leq a(1/2)^{t/T-1}\end{aligned}$$

since $t \leq JT$. It follows that $\alpha^-(p,t)$ decreases exponentially with t. Similarly, $\alpha^+(p,t)$ increases exponentially with t. The proposition follows. □

3. Dynamical properties

In this § we consider a GHC-regular representation $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ with acausal limit set Λ_ρ. We don't assume that the group Γ is isomorphic to a lattice in $\mathrm{SO}_0(1,n)$. Let $(\gamma_n)_{(n\in\mathbb{N})}$ be a sequence in Γ escaping to infinity. It will be convenient to consider the image sequence $(\rho(\gamma_n))_{(n\in\mathbb{N})}$ as a sequence in $\mathrm{SO}_0(2,n+1)$ through the inclusion $\mathrm{SO}_0(2,n) \subset \mathrm{SO}_0(2,n+1)$ so that our dynamical study applies in Ein_{n+1}, and hence in the $\rho(\Gamma)$-invariant conformal copy of AdS_{n+1} inside Ein_{n+1}.

In [Fra05] (see also [BCD$^+$07, § 7]), C. Frances studied the dynamical behavior in Ein_{n+1} of $(\rho(\gamma_n))_{(n\in\mathbb{N})}$. Up to a subsequence, one the following holds (we will just afterwards remind ingredients of the proof):

(1) *Balanced distortion:* There are two lightlike geodesics Δ^+, Δ^- in Ein_{n+1}, called attracting and repelling photons, and two continuous applications $\pi_+ : \mathrm{Ein}_{n+1} \setminus \Delta^- \to \Delta^+$ and $\pi_- : \mathrm{Ein}_{n+1} \setminus \Delta^+ \to \Delta^-$ such that:
   – the fibers of π_+ (respectively π_-) are past lightcones $C^-(x)$ of points in Δ^- (respectively of points in Δ^+),
   – for every compact subset $K \subset \mathrm{Ein}_{n+1} \setminus \Delta^-$, the sequence $\rho(\gamma_n)$ uniformly converges on K to π_+,
   – for every compact subset $K \subset \mathrm{Ein}_{n+1} \setminus \Delta^+$, the sequence $\rho(\gamma_n^{-1})$ uniformly converges on K to π_-.

(2) *Unbalanced distortion:* There are two points x^+, x^- in Ein_{n+1}, called attracting and repelling poles, such that:
   – $\langle x^+ | x^- \rangle \leq 0$,
   – for every compact subset K of Ein_{n+1} contained in $\Omega^-(x^-) := \{x \in \mathrm{Ein}_{n+1} / \langle x | x^- \rangle < 0\}$ (resp. $\Omega^+(x^-) := \{x \in \mathrm{Ein}_{n+1} / \langle x | x^- \rangle > 0\}$) the sequence $\rho(\gamma_n)$ uniformly converges on K to the constant map x^+ (resp. $(x^+)^*$),
   – for every compact subset K of Ein_{n+1} contained in $\Omega^-(x^+) := \{x \in \mathrm{Ein}_{n+1} / \langle x | x^+ \rangle < 0\}$ (resp. $\Omega^+(x^+) := \{x \in \mathrm{Ein}_{n+1} / \langle x | x^+ \rangle > 0\}$) the sequence $\rho(\gamma_n^{-1})$ uniformly converges on K to the constant map x^- (resp. $(x^-)^*$).

Remark 3.1. Our presentation differs from Frances formulation. Indeed, we consider the dynamic in Ein_{n+1}, which is the double covering of the Einstein universe as defined in [Fra05] *ie.* as the projection of $\mathcal{C}_{n+1}$ in the projective space $\mathbb{P}(\mathbb{R}^{n+3})$, and not the projection in the sphere of rays $\mathbb{S}(\mathbb{R}^{n+3})$. C. Frances had no need to distinguish future cones from past cones since they have the same projection in $\mathbb{P}(\mathbb{R}^{n+3})$.

A nuisance of the option to consider the double covering is the non-uniqueness of pairs of attracting/repelling poles. Indeed, the opposite pair

$(-x^+,-x^-)$ is also convenient. Moreover, if $\langle x^-|x^+\rangle = 0$, we have four choices $(\pm x^+,\pm x^-)$ of pairs of attracting/repelling poles.

Remark 3.2. Every $\rho(\gamma_n)$ belongs to the subgroup $\mathrm{SO}_0(2,n)$ of $\mathrm{SO}_0(2,n+1)$, *ie.* preserves the conformal embedding $\mathrm{AdS}_{n+1} \subset \mathrm{Ein}_{n+1}$ and its boundary $\partial\,\mathrm{AdS}_{n+1} \approx \mathrm{Ein}_n$. In that situation, all the limit objects $\Delta^\pm$, $x^\pm$ involved in the various cases in the description of the asymptotic behavior of $(\rho(\gamma_{n_k}))_{(k\in\mathbb{N})}$ are contained in this boundary. In particular, they avoid AdS_{n+1}.

The dichotomy balanced/unbalanced is based on the Cartan decomposition of $\mathrm{SO}_0(2,n+1)$. More precisely, consider the quadratic form $\mathfrak{q}_{2,n+1} := -4a_1b_1 - 4a_2b_2 + x_1^2 + \ldots + x_{n-1}^2$ on $\mathbb{R}^{n+3}$: observe that $(\mathbb{R}^{n+3},\mathfrak{q}_{2,n+1})$ and $(\mathbb{R}^{2,n+1},\mathrm{q}_{2,n+1})$ are isometric (the isometry is $(a_1,b_1,a_2,b_2,x_1,\ldots,x_{n-1}) \to ((a_1+b_1)/2,(a_2+b_2)/2,x_1,\ldots,x_{n-1},(a_1-b_1)/2,(a_2-b_2)/2)$). Let $\mathfrak{A}$ be the free abelian subgroup of rank 2 of $\mathrm{SO}_0(2,n+1)$ comprising elements $a(\lambda,\mu)$ acting diagonally on $\mathbb{R}^{n+3}$ in the coordinates $(a_1,a_2,b_1,b_2,x_1,\ldots,x_{n-1})$, so that every x_i is unchanged, the coordinates a_1, a_2 are multiplied by $\exp(\lambda)$, $\exp(\mu)$, and the coordinates b_1, b_2 are multiplied by $\exp(-\lambda)$, $\exp(-\mu)$. It is a real split Cartan subgroup of $\mathrm{SO}_0(2,n+1)$, and we consider the Weyl chamber $\mathfrak{A}^+ \subset \mathfrak{A}$ comprising $a(\lambda,\mu)$ such that $0 \leq \mu \leq \lambda$. The Cartan decomposition Theorem ensures that every $\rho(\gamma_n)$ can be written in the form $\rho(\gamma_n) = \mathfrak{k}_n a_n \mathfrak{l}_n$ such that $a_n = a(\lambda_n,\mu_n)$ belongs to $\mathfrak{A}^+$ and $\mathfrak{k}_n$, $\mathfrak{l}_n$ belong to the stabilizer $\mathfrak{K}$ of the negative definite 2-plane $\{a_1 = b_1, a_2 = b_2, x_1 = \ldots = x_{n-1} = 0\}$ (it is a maximal compact subgroup). Observe that elements of $\mathfrak{K}$ are isometries of the Euclidean norm $\|(u,v,x_1,\ldots,x_n)\|_0^2 := u^2+v^2+x_1^2+\ldots+x_n^2$. Since $(\rho(\gamma_n))_{(n\in\mathbb{N})}$ escapes from any compact, the sequence $(\lambda_n)_{n\in\mathbb{N})}$ is not bounded from above. By compactness of $\mathfrak{K}$ there is a *converging subsequence, ie.* a subsequence $(\gamma_{n_k})_{(k\in\mathbb{N})}$ such that $\mathfrak{k}_{n_k}$, $\mathfrak{l}_{n_k}$ converge to some elements $\mathfrak{k}_\infty$, $\mathfrak{l}_\infty$ of $\mathfrak{K}$, and such that $\lim\limits_{k\to+\infty} \lambda_{n_k} = +\infty$, and $\lim\limits_{k\to+\infty} \exp(\mu_{n_k}-\lambda_{n_k}) = \nu$ with $0 \leq \nu \leq 1$.

3.0.1. *Balanced distortion.* The balanced distortion case is the case $\nu > 0$. Denote by P^-, P^+ the codimension two subspaces $\{a_1 = a_2 = 0\}$ and $\{b_1 = b_2 = 0\}$ respectively. Consider the following linear endomorphisms of $\mathbb{R}^{n+3}$:

$$\begin{aligned}\Pi_0^+(a_1,b_1,a_2,b_2,x_1,\ldots,x_{n-1}) &= (a_1,0,\nu a_2,0,\ldots,0)\\ \Pi_0^-(a_1,b_1,a_2,b_2,x_1,\ldots,x_{n-1}) &= (0,b_1,0,\nu b_2,0,\ldots,0)\end{aligned}$$

They induce maps $\pi_0^+ : \mathbb{S}(\mathbb{R}^{n+3}) \setminus \mathbb{S}(P^-)$ and $\pi_0^- : \mathbb{S}(\mathbb{R}^{n+3}) \setminus \mathbb{S}(P^+)$. Clearly, as a sequence of transformations of $\mathbb{S}(\mathbb{R}^{n+3})$, $(a_{n_k})_{(k\in\mathbb{N})}$ converges uniformly on compacts of $\mathbb{S}(\mathbb{R}^{n+3}) \setminus \mathbb{S}(P^-)$ to the map induced by π_0^+, and a similar remark applies for the inverse sequence $(a_{n_k}^{-1})_{(k\in\mathbb{N})}$. It follows that the sequence $(\rho(\gamma_{n_k}))_{(nk\in\mathbb{N})}$ converges uniformly on compacts of

$\mathbb{S}(\mathbb{R}^{n+3}) \setminus \mathbb{S}(\mathfrak{l}_\infty^{-1} P^-)$ towards $\mathfrak{k}_\infty \circ \pi_0^+ \circ \mathfrak{l}_\infty$ and that $(\rho(\gamma_{n_k}^{-1}))_{(nk \in \mathbb{N})}$ converges uniformly on compacts of $\mathbb{S}(\mathbb{R}^{n+3}) \setminus \mathbb{S}(\mathfrak{k}_\infty P^+)$ towards $\mathfrak{l}_\infty^{-1} \circ \pi_0^- \circ \mathfrak{k}_\infty^{-1}$. The description of the dynamic in Ein_{n+1} given above follows by observing that the intersections $P^\pm \cap \mathcal{C}_{n+1}$ are isotropic 2-planes.

3.0.2. *Unbalanced distortion.* It is the case $\nu = 0$. Identify the sphere $\mathbb{S}(\mathbb{R}^{n+3})$ of rays with the $\|_0$-unit sphere. The attracting fixed points of the action of a_n in $\mathbb{S}(\mathbb{R}^{n+3})$ are $\pm x_0^+$ where $x_0^+ = (1, 0, \dots, 0)$, and the repelling fixed points are $\pm x_0^-$ where $x_0^- = (0, 0, 1, 0, \dots, 0)$. Observe that the $\mathfrak{q}_{2,n+1}$-orthogonal $(x_0^+)^\perp$ is the hyperplane $\{b_1 = 0\}$: it is also the orthogonal of x_0^- for the Euclidean norm $\|_0$. Similarly, $(x_0^-)^\perp = \{b_2 = 0\}$ is the $\|_0$-orthogonal of x_0^+.

For every $\epsilon > 0$ let $C_0^-(\epsilon)$ be the spherical ball in $\mathbb{S}(\mathbb{R}^{n+3})$ of radius $\pi/2 - \epsilon$ centered at x_0^+. It can also be defined as the connected component containing x_0^+ of the complement in $\mathbb{S}(\mathbb{R}^{n+3})$ of the ϵ-neighborhood of $(x_0^-)^\perp$.

Every vector in $\mathbb{R}^{n+3}$ splits as a sum $rx_0^+ + y$ with y in $(x_0^-)^\perp$. Under the action of $a(\lambda_n, \mu_n)$ the component rx_0^+ is multiplied by $\exp(\lambda_n)$ whereas the norm of the component y is multiplied by at most $\exp(\mu_n)$. It follows easily:

Lemma 3.3. *Let $a(\lambda_n, \mu_n)$ be a sequence in $\mathfrak{A}^+$ with no balanced distortion. For any $\epsilon > 0$ and any $\eta > 0$ there is $N > 0$ such that, for every $n > N$, the restriction of $a(\lambda_n, \mu_n)$ to $C_0^-(\epsilon)$ is η-Lipschitz, with image contained in $C_0^+(\pi - \eta)$.* □

The description of the dynamic of unbalanced converging subsequences on Ein_{n+1} given above follows easily; the attracting pole x^+ is simply the image of x_0^+ by $\mathfrak{k}_\infty$, and the repelling pole is $x^- = \mathfrak{l}_\infty^{-1} x_0^-$. We entered in such a detail that the next lemma is now obvious. Consider the hemisphere $\mathcal{D}^- = \{x \in \mathbb{S}(\mathbb{R}^{n+3}) / \langle x | x^- \rangle < 0\}$. For every $\epsilon > 0$ let $C^-(\epsilon)$ be the set of points in $\mathcal{D}^-$ at distance $\geq \epsilon$ from $(x^-)^\perp \cap \mathbb{S}(\mathbb{R}^{n+3})$. Since $\mathfrak{k}_n$, $\mathfrak{l}_n$ are isometries for $\|_0$:

Lemma 3.4. *For any $\epsilon > 0$ and any $\eta > 0$ there is $N > 0$ such that, for every $k > N$, the restriction of $\rho(\gamma_{n_k})$ to $C^-(\epsilon)$ is η-Lipschitz, with image contained in the ball centered at x^+ and of radius η.* □

The statement we actually need is:

Corollary 3.5. *Assume that x^+ belongs to $\rho(\gamma_{n_k})\mathcal{D}^-$ for k sufficiently big. Then the differential at of the inverse of $\rho(\gamma_{n_k})$, as a transformation of the unit sphere $\mathbb{S}(\mathbb{R}^{2,n+1})$ expands all the vectors tangent to the sphere at x^+ by at least a factor ν_k, such that $\nu_k \to +\infty$ when $k \to +\infty$.* □

Balanced distortion is the typical behavior of converging subsequences $(\rho(\gamma_{n_k}))_{(k \in \mathbb{N})}$ when $\rho(\Gamma)$ acts properly discontinuously on AdS_{n+1}. But our situation here is different: by hypothesis, the group $\rho(\Gamma)$ preserves an achronal limit set Λ_ρ, which is not pure lightlike since $E(\Lambda_\rho) \neq \emptyset$ ([Mér07, Lemma 3.6]).

Proposition 3.6. *No sequence in $\rho(\Gamma)$ has balanced distortion.*

Proof. Assume *a contrario* that some sequence $(\rho(\gamma_n))_{(n\in\mathbb{N})}$ has balanced distortion. Denote by $\Delta^{\pm}$ the repelling and attracting photons. Since Λ_ρ is an acausal topological sphere, it intersects Δ^+ at an unique point x^+ Since Λ_ρ is $\rho(\Gamma)$-invariant, the image by π^+ of $\Lambda_\rho \setminus \Delta^-$ is x^+. The fibers of π^+ are past cones of elements of Δ^-. Hence, $\Lambda_\rho \setminus \Delta^-$ is contained in the past cone $C^-(x^+)$ Since $\Delta^- \cap \Lambda_\rho$ is a compact embedded segment, $\Lambda_\rho \setminus \Delta^-$ is dense in Λ_ρ (this argument is correct when the dimension of Λ_ρ is ≥ 2. For the case where Λ_ρ is a topological circle, see [Mes07, ABB$^+$07], or [BBZ07, § 6.2]). Hence Λ_ρ is contained in $C^-(x^+)$. It is impossible since Λ_ρ is not pure lightlike. □

Remark 3.7. The ambiguity on the definition of pairs of attracting/repelling poles, mentioned in Remark 3.1, can be removed for GHC-regular representations by selecting as poles the ones contained in Λ_ρ. Indeed:

– Λ_ρ contains an attracting pole. Indeed, since it is not contained in a cone $C(\pm x^-)$, Λ_ρ intersects $\Omega^+(x^-)$ or $\Omega^-(x^-)$, and the $\rho(\gamma_n)$-orbit of a point in this intersection accumulates on $\pm x^+$, that therefore belongs to Λ_ρ. Similarly, Λ_ρ contains a repelling pole.

– Λ_ρ contains one and only one attracting pole. Indeed, x^+ and $-x^+$ cannot both belong to Λ_ρ since Λ_ρ is not pure lightlike. Similarly, Λ_ρ contains one and only one repelling pole.

Observe that the condition $\langle x^-|x^+\rangle \leq 0$ is fulfilled since Λ_ρ is achronal.

4. Convex hull of GHC-representations

4.1. **The convex hull.** According to [Mér07, Lemma 3.9] the limit set Λ_ρ and the regular domain $E(\Lambda_\rho)$ are contained in $U \cup \partial U$ where U is an affine domain of AdS_{n+1}. In particular, it is contained in an affine chard V of $\mathbb{S}(\mathbb{R}^{2,n})$. We can consider the convex hull $\mathrm{Conv}(\Lambda_\rho)$ of Λ_ρ in this affine chard. This convex hull does not depend on the choice of V. Moreover, since $E(\Lambda_\rho)$ is convex, it contains $\mathrm{Conv}(\Lambda_\rho)$ (cf. [Mér07, Remark 3.11]). For more details, see for example [Bar05a].

Alternatively, we also can define $\mathrm{Conv}(\Lambda_\rho)$ as the projection $\mathbb{S}(C)$ where C is the set of barycentric combinations $t_1x_1 + \dots t_kx_k$ where t_i are positive real numbers such that $t_1 + \dots t_k = 1$ and x_i elements of $\mathcal{C}_n \subset \mathbb{R}^{2,n}$ the projections $\mathbb{S}(x_i)$ of which belong to Λ_ρ.

Lemma 4.1. *The convex hull $\mathrm{Conv}(\Lambda_\rho)$ is compact; its intersection with $\partial\,\mathrm{AdS}_{n+1}$ is Λ_ρ, and the "finite part" $\mathrm{Conv}(\Lambda_\rho) \cap \mathrm{AdS}_{n+1} = \mathrm{Conv}(\Lambda_\rho) \setminus \Lambda_\rho$ is contained in $E(\Lambda_\rho)$.Fix a $\rho(\Gamma)$-invariant future oriented timelike vector field V on $E(\Lambda_\rho)$.*

Proof. The compactness of $\mathrm{Conv}(\Lambda_\rho)$ arises from the compactness of Λ_ρ. Let $x = t_1x_1 + \dots t_kx_k$ be an element of $\mathbb{R}^{2,n}$ projecting in $\mathbb{S}(\mathbb{R}^{2,n})$ on an element of $\mathrm{Conv}(\Lambda_\rho)$. For every y such that $\mathbb{S}(y)$ belongs to Λ_ρ, according to [Mér07, Corollary 2.11]:

$$\langle x|y\rangle = \sum_{i=1}^{k} t_i \langle x_i|y\rangle \leq 0$$

Moreover, if $\langle x|y\rangle$ vanishes, then every $\langle y|x_i\rangle$ vanishes. But since Λ_ρ is acausal, $\langle y|x_i\rangle = 0$ implies $y = x_i$: according to [Mér07, Proposition 3.10] $\mathrm{Conv}(\Lambda_\rho) \setminus \Lambda_\rho$ is contained in $E(\Lambda_\rho)$. The lemma follows since $E(\Lambda_\rho)$ is contained in AdS_{n+1}. □

Lemma 4.2. *If* $\mathrm{Conv}(\Lambda_\rho)$ *has empty interior, then* ρ *is Fuchsian.*

Proof. If $\mathrm{Conv}(\Lambda_\rho)$ has empty interior, it is contained in a projective hyperplane $\mathbb{S}(v^\perp)$. If $\mathrm{q}_{2,n}(v) > 0$ then $\mathbb{S}(v^\perp) \cap \mathrm{AdS}_{n+1}$ is an isometric, totally geodesic embedding of AdS_n. In a well-chosen conformal chard $\mathrm{AdS}_{n+1} \approx \mathbb{S}^1 \times \mathbb{D}^n$ this AdS-wall is $\{x_n = 0\}$. It is a contradiction since its closure should contain Λ_ρ whereas Λ_ρ is a graph over $\partial\mathbb{D}^n$. Similarly, if $\mathrm{q}_{2,n}(v) = 0$ then Λ_ρ would be pure lightlike.

Hence, up to renormalization, v lies in AdS_{n+1}. If $v' \neq v$ is another element of AdS_{n+1} then the intersection $\mathbb{S}(v^\perp) \cap \mathbb{S}((v')^\perp) \cap \mathrm{AdS}_{n+1}$, if not empty, is contained in a totally geodesic hypersurface in $\mathbb{S}(v^\perp) \cap \mathrm{AdS}_{n+1}$: its closure cannot contain the topological $(n-1)$-sphere Λ_ρ. Therefore, v is unique: it is a global fixed point for $\rho(\Gamma)$. □

Since we already know that Fuchsian representations are Anosov ([Mér07, § 5.2]), *we assume from now that* $\mathrm{Conv}(\Lambda_\rho)$ *has non-empty interior.* The limit set Λ_ρ is the projection of an acausal closed subset $\widetilde{\Lambda}_\rho$ in $\widetilde{\mathrm{Ein}}_n$ and $E(\Lambda_\rho)$ is the 1-1 projection of a domain $\widetilde{E}(\widetilde{\Lambda}_\rho)$ in $\widetilde{\mathrm{AdS}}_{n+1} \approx \mathbb{R} \times \mathbb{D}^n$. Recall that there are two maps f_ρ^-, f_ρ^+ such that $\widetilde{E}(\widetilde{\Lambda}_\rho) = \{(\theta, \mathrm{x}) / f_\rho^-(\mathrm{x}) < \theta < f_\rho^+(\mathrm{x})\}$ (cf. [Mér07, Remark 3.3]).

Proposition 4.3. *The complement of* Λ_ρ *in the boundary* $\partial\,\mathrm{Conv}(\Lambda_\rho)$ *has two connected components. Both are closed edgeless achronal subsets of* AdS_{n+1}. *More precisely, in the conformal model their lifting in* $\widetilde{\mathrm{AdS}}_{n+1}$ *are graphs of 1-Lipschitz maps* F_ρ^+, F_ρ^- *from* $\mathbb{D}^n$ *into* $\mathbb{R}$ *such that*

$$f_\rho^- < F_\rho^- < F_\rho^+ < f_\rho^+$$

For a similar study when Λ_ρ is not necessarily a topological sphere but in the case $n = 2$, see [Bar05a, § 8.10]. For the proof of this proposition, we need a few lemmas.

Lemma 4.4. *Every timelike geodesic of* AdS_{n+1} *intersects* $\mathrm{Conv}(\Lambda_\rho)$.

Proof. Let D be a timelike geodesic in AdS_{n+1}. It is contained in a totally geodesic embedding A of AdS_2 in AdS_{n+1}, and the intersection $A \cap \mathrm{Conv}(\Lambda_\rho)$ contains the convex hull in A of $\mathrm{Cl}(A) \cap \Lambda_\rho$. We are thus reduced to the (easy) case $n = 2$. In that case, $A \setminus D$ has two connected components, and each of them contains a connected component of ∂A. The boundary ∂A has

two connected components l_1, l_2, and each of these connected components is an inextendible timelike curve in $\mathrm{Ein}_1 \subset \mathrm{Ein}_{n+1}$, which therefore intersects Λ_ρ at an unique point x_i. Then, the segment $[x_1, x_2]$ intersects D. □

Lemma 4.5. *Support hyperplanes in* $\mathbb{S}(\mathbb{R}^{2,n})$ *to* $\mathrm{Conv}(\Lambda_\rho)$ *at points inside* AdS_{n+1} *are spacelike.*

Proof. Let x be a point in $\mathrm{AdS}_{n+1} \cap \mathrm{Conv}(\Lambda_\rho)$, and let P be a support (projective) hyperplane at x to $\mathrm{Conv}(\Lambda_\rho)$. This support hyperplane is a projection $\mathbb{S}(v^\perp)$for some v in $\mathbb{R}^{2,n}$. If $\mathrm{q}_{2,n}(v) > 0$, then $\mathbb{S}(v^\perp)$ disconnects any affine domain, in particular, the affine domain V containing $E(\Lambda_\rho) \cup \Lambda_\rho$, and it follows easily, since Λ_ρ is a topological sphere, that the affine hyperplane $\mathbb{S}(v^\perp) \cap V$ disconnects Λ_ρ. It is a contradiction since this affine hyperplane is a support hyperplane in V and hence cannot disconnect the convex hull.

If $\mathrm{q}_{2,n}(v) = 0$, then the affine hyperplane $V \cap \mathbb{S}(v^\perp)$ is tangent to the hyperboloid ∂U at $\mathbb{S}(v)$ (up to a slight change of affine domain V, we can always assume that $\mathbb{S}(v)$ belongs to V). If it disconnects Λ_ρ, we obtain a contradiction as above. If not, it means that $\mathbb{S}(v)$ belongs to Λ_ρ. Write x as a sum $t_1x_1+\dots t_kx_k$ where x_i belongs to Λ_ρ: $0 = \langle v|x\rangle = t_1\langle v|x_1\rangle+\dots t_k\langle v|x_k\rangle$. Since every $\langle v|x_i\rangle$ is a nonpositive number, they all vanish, and it implies that $v = x_i$ for every i. Hence $x = v$; it is a contradiction since x is assumed in AdS_{n+1}. □

Proof of Proposition 4.3. Lift $\mathrm{Conv}(\Lambda_\rho)$ in $\widetilde{\mathrm{AdS}}_{n+1} \approx \mathbb{R}\times\overline{\mathbb{D}^n}$ as a subdomain $\mathrm{Conv}(\widetilde{\Lambda}_\rho)$ in $\widetilde{E}(\widetilde{\Lambda}_\rho)\cup\widetilde{\Lambda}_\rho$. For every x in $\mathbb{D}^n$, the line $\mathbb{R}\times\{\mathrm{x}\}$ is a timelike geodesic. According to Lemma 4.4 it intersects $\mathrm{Conv}(\widetilde{\Lambda}_\rho)$. Moreover, since this intersection is convex, it contains a geodesic segment $[F_\rho^-(\mathrm{x}), F_\rho^+(\mathrm{x})]\times\{\mathrm{x}\}$. If an element y in $]F_\rho^-(\mathrm{x}), F_\rho^+(\mathrm{x})[\times\{\mathrm{x}\}$ lies on the boundary of $\mathrm{Conv}(\widetilde{\Lambda}_\rho)$, then every support hyperplane to the convex hull at the projection of this point must contain the projection of the timelike segment $[F_\rho^-(\mathrm{x}), F_\rho^+(\mathrm{x})]\times\{\mathrm{x}\}$: it contradicts Lemma 4.5. Therefore, the boundary of $\mathrm{Conv}(\widetilde{\Lambda}_\rho)$ is the union of the graphs of F_ρ^- and F_ρ^+. It follows quite easily that these graphs are closed, hence, F_ρ^+ and F_ρ^- are continuous.

Consider the closed subset $\mathcal{E} := \{F_\rho^- = F_\rho^+\}$ in $\mathbb{D}^n$. For every x in $\mathbb{D}^n$, take a small chard in the Klein model around $(F_\rho^+(\mathrm{x}), \mathrm{x})$ such that F_ρ^- and F_ρ^+ expresses locally as graphs of maps from an affine hyperplane into $\mathbb{R}$. Since $\mathrm{Conv}(\Lambda_\rho)$ is convex, F_ρ^+ is convex and F_ρ^- is concave. It follows that the coincidence locus $\mathcal{E}$ is also open. Since $\mathbb{D}^n$ is connected, if $\mathcal{E}$ is not empty we get the equality $F_\rho^- = F_\rho^+$. It is impossible since the interior of $E(\Lambda_\rho)$ is not empty. Therefore, according to Lemma 4.1:

$$f_\rho^- < F_\rho^- < F_\rho^+ < f_\rho^+$$

Finally, for every x in $\mathbb{D}^n$, let $\mathbb{S}(v^\perp)$ be a support hyperplane to $\mathrm{Conv}(\Lambda_\rho)$ at the projection of $(F^+_\rho(\mathrm{x}), \mathrm{x})$. According to Lemma 4.5, $\mathbb{S}(v^\perp)$ is a totally geodesic embedding of $\mathbb{H}^n$. In particular, it lifts as the graph of a 1-Lipschitz map $\varphi^+_v : \mathbb{D}^n \to \mathbb{R}$. One of the region $\{(\theta, \mathrm{y})/\theta > \varphi^+_v(\mathrm{y})\}$, $\{(\theta, \mathrm{y})/\theta < \varphi^+_v(\mathrm{y})\}$ is disjoint from $\mathrm{Conv}(\widetilde{\Lambda}_\rho)$, and since $F^-_\rho(\mathrm{x}) < F^+_\rho(\mathrm{x}) = \varphi^+_v(\mathrm{x})$, it is the former. Hence on $\mathbb{D}^n$ we have $F^+_\rho \leq \varphi^+_p$. But since convex domains are intersections of half-spaces containing them, we get:

$$F^+_\rho = \min_v \varphi^+_v$$

Since every φ^+_v is 1-Lipschitz, the same is true for F^+_ρ. Similarly for F^-_ρ. □

We denote the components of $\partial\,\mathrm{Conv}(\Lambda_\rho)$ as $\widetilde{S}^+_\rho$, $\widetilde{S}^-_\rho$. Denote by S^+_ρ, S^-_ρ their projections in $M = \rho(\Gamma)\backslash E(\Lambda_\rho)$.

Lemma 4.6. *$S^\pm_\rho$ are Cauchy hypersurfaces in M.*

Proof. Since $\widetilde{S}^\pm_\rho$ is homeomorphic to $\mathbb{R}^n$ the quotient $S^\pm_\rho = \widetilde{S}^\pm_\rho$ is a $K(\Gamma, 1)$ space, as $\Gamma\backslash\mathbb{H}^n$. The cohomology groups $H^n(S^\pm_\rho, \mathbb{R})$ and $H^n(\Gamma\backslash\mathbb{H}^n, \mathbb{R})$ are therefore isomorphic. Since the later is non zero, the former is non zero: the compactness of $S^\pm_\rho$, and thus the Lemma, follows. □

As an immediate corollary, we get that the projection of $\mathrm{Conv}(\Lambda_\rho) \setminus \Lambda_\rho$ is the compact domain of M, bounded by the two disjoint Cauchy hypersurfaces $S^\pm_\rho$. We denote it $C(M)$, and call it the *convex core* of M.

Remark 4.7. It can be easily infered from Lemma 4.5 that $S^\pm_\rho$ are furthermore acausal, *ie.* that $F^\pm_\rho$ are contracting.

Remark 4.8. Let v be an element v such that $\mathrm{q}_{2,n}(v) > 0$. They are totally geodesic embeddings of AdS_n. We call AdS-*wall* the intersections of AdS_{n+1} with the orthogonal $v^\perp$. The *half* AdS-*spaces* defined by v are the domains $\mathrm{H}^+(v) = \{x \in \mathrm{AdS}_{n+1}/\langle v|x\rangle \geq 0\}$ and $\mathrm{H}^-(v) = \{x \in \mathrm{AdS}_{n+1}/\langle v|x\rangle \leq 0\} = \mathrm{H}^+(-v)$. According to Lemma 4.4, the intersection between any AdS-wall $\partial\,\mathrm{H}(v)$, and Λ_ρ is a topological $(n-2)$-sphere. Moreover, in a suitable conformal chard $\mathrm{H}^+(v)$ is the domain $\{(\theta, x_1, \dots x_n) \in \mathbb{S}^1 \times \mathbb{D}^n / x_n > 0\}$. It follows that $\mathfrak{S}(v) = \partial\,\mathrm{H}^\pm(v) \cap \mathrm{Conv}(\Lambda_\rho)$ is a topological n-dimensional disk, in particular compact, and cuts $\mathrm{Conv}(\Lambda_\rho)$ in two parts $\mathfrak{H}^\pm(v) = \mathrm{H}^\pm(v) \cap \mathrm{Conv}(\Lambda_\rho)$, that we call *convex caps.*

4.2. **Metric on the convex hull.** In the sequel we need to define a $\rho(\Gamma)$-metric on $\mathrm{Conv}(\Lambda_\rho)$. Since the action is cocompact, all these metrics are quasi-isometric one to the other (see § 4.4) and the choice is not important. However, in order to sustain our argumentation, we choose a specific metric.

Let Ω be a bounded open domain in $\mathbb{P}(\mathbb{R}^n)$, *ie.* an open domain contained in an affine chard and such that the closure $\overline{\Omega}$ in this affine chard is compact. The *Hilbert distance* between two points x, y in Ω is:

$$d^H(x,y) := \log(a,b,x,y)$$

where a, b are the two intersections between $\partial\Omega$ and the projective line containing x and y, and where (a,b,x,y) is the cross-ratio. It is a distance function, and the associated metric is proper, geodesic and every projective transformation preserving Ω preserves the Hilbert distance of points. Moreover, geodesics are intersections between projective lines and Ω (see [BK53]).

The interior of the convex hull $\mathrm{Conv}(\Lambda_\rho)$ is a bounded open domain, hence admits a well-defined $\rho(\Gamma)$-invariant Hilbert metric. However, in the sequel we will need metrics defined on $\mathrm{Conv}(\Lambda_\rho)\setminus\Lambda_\rho$ and not only on its interior. Hence we have to enlarge $\mathrm{Conv}(\Lambda_\rho)$ to another convex domain, still bounded and $\rho(\Gamma)$-invariant, but containing the boundaries $\widetilde{S}^{\pm}_\rho$.

A suitable solution is to consider, for $\epsilon > 0$ small enough, the domain $\mathrm{Conv}(\Lambda_\rho)_\epsilon$ in AdS_{n+1} made of points x such that every causal curve in AdS_{n+1} joining x to an element of $\mathrm{Conv}(\Lambda_\rho)$ is of Lorentzian length $\leq \epsilon$. It follows quite easily from the compactness of $\rho(\Gamma)\backslash\mathrm{Conv}(\Lambda_\rho)$ that for ϵ small enough $\mathrm{Conv}(\Lambda_\rho)_\epsilon$ is contained in $E(\Lambda_\rho)$. The proof that $\mathrm{Conv}(\Lambda_\rho)_\epsilon$ is still convex is straightforward, we refer to [BBZ07, Proposition 6.31] for a proof formulated in dimension $2+1$, but valid in any dimension. Observe also that $\mathrm{Conv}(\Lambda_\rho)_\epsilon$ is still bounded, and that its interior contains $\mathrm{Conv}(\Lambda_\rho)\setminus\Lambda_\rho$.

In the sequel, we fix once for all ϵ and denote by d^H the restriction to $\mathrm{Conv}(\Lambda_\rho)$ of the Hilbert metric of $\mathrm{Conv}(\Lambda_\rho)_\epsilon$.

Remark 4.9. If $]a,b[$ is a spacelike geodesic joining two points in Ein_n then for any x, y in $]a,b[$ the *AdS*-length of the piece of geodesic between x and y is $\log(a,b,x,y)$ (see e.g. [Sal99, Theorem 2.2.1.11], it is a generalization of the well-known fact that the Hilbert metric on the Klein model of the hyperbolic space is isometric to the hyperbolic metric). It follows that in the case where a, b lies on Λ_ρ this length is the Hilbert distance $d^H(x,y)$.

4.3. **Dirichlet domains.** If Γ acts freely and properly discontinuously on a proper complete metric space X, there is a well-known way to construct a fundamental domain of its action: the *Dirichlet domain* (see [Rat06, pp 243–245]). Here the action we consider does not preserve a Riemannian metric, but the construction of Dirichlet domain extends easily in our situation:

Definition 4.10. Fix an element x_0 of $\mathrm{Conv}(\Lambda_\rho)$. For every γ in Γ, let $D(\gamma)$ be the domain $\{x \in E(\Lambda_\rho)/\ \langle x|x_0\rangle > \langle x|\rho(\gamma)x_0\rangle\}$ (here we consider $E(\Lambda_\rho)$ as a subset of $\mathrm{AdS}_{n+1}\subset\mathbb{R}^{2,n}$). The Dirichlet domain $D(\Gamma)$ is the intersection $\cap_{\gamma\in\Gamma}D(\gamma)$.

Remark 4.11. Since the quotient M is globally hyperbolic, it admits no closed causal curve. Therefore, x_0 and $\rho(\gamma)x_0$ are not causally related: the $\mathrm{q}_{2,n}$-norm of $(\rho(\gamma)x_0 - x_0)$ is positive. The domain $D(\gamma)$ is the interior of the intersection between $E(\Lambda_\rho)$ and the half AdS-space $\mathrm{H}^-(\rho(\gamma)x_0 - x_0)$.

Lemma 4.12. *The complements $H(\gamma) = E(\Lambda_\rho) \setminus D(\gamma)$ form a locally finite family of subsets of $E(\Lambda_\rho)$.*

Proof. Assume by contradiction that a compact K of $E(\Lambda_\rho)$ intersects infinitely many $H(\gamma_n)$. According to Proposition 3.6 and Remark 3.7 we can assume, up to a subsequence, that the action induced in the Klein model by $\rho(\gamma_n^{-1})$ converges uniformly on K to a point x^- in Λ_ρ.

On the other hand, there is a sequence of points $(x_n)_{(n\in\mathbb{N})}$ in K, converging to some x, and such that for every n:

$$\langle x_n|x_0\rangle \leq \langle x_n|\rho(\gamma_n)x_0\rangle = \langle \rho(\gamma_n^{-1})x_n|x_0\rangle$$

Since the $\rho(\gamma_n^{-1})x_n$ has $\mathrm{q}_{2,n}$-norm -1, the convergence in the Klein model towards x^- means that for some sequence $\lambda_n \to 0$ the $\lambda_n\rho(\gamma_n^{-1})(x_n)$ converges to a representant $\hat{x}^-$ in $\mathcal{C}_n$ of x^-.

Hence

$$\langle x_n|x_0\rangle \leq \frac{1}{\lambda_n}\langle x_0|\lambda_n\rho(\gamma_n^{-1})x_n\rangle$$

The left term converges to $\langle x|x_0\rangle$, and since $\dfrac{1}{\lambda_n}$ converges to $+\infty$ and $\langle x_0|\lambda_n\rho(\gamma_n^{-1})x_0\rangle$ converges to the negative number $\langle x_0|\hat{x}^-\rangle$, the right term converges to $-\infty$. Contradiction. □

A first corollary of this lemma is that $D(\Gamma)$ is open, and its closure $\mathrm{Cl}(D(\Gamma))$ is the intersection of the closures of the $D(\gamma)$.

Lemma 4.13. *The Γ-iterates of $Cl(D(\Gamma))$ covers $E(\Lambda_\rho)$,* ie. *:*

$$E(\Lambda_\rho) = \cup_{\gamma\in\Gamma}\rho(\gamma)\,Cl(D(\Gamma))$$

Proof. Let x be in $E(\Lambda_\rho)$; consider the map $\xi : \Gamma \to \mathbb{R}$ defined by $\xi(\gamma) = \langle x|\rho(\gamma)x_0\rangle$. If there is a sequence γ_n such that $\xi(\gamma_n)$ increases, the argument used in the proof above with the constant sequence $x_n = x$ leads to a contradiction. Hence ξ attains its maximum at some γ_0, *ie.* $\langle \rho(\gamma_0)x_0|x\rangle \geq \langle \rho(\gamma)x_0|x\rangle$ for every γ in Γ. It follows that $\rho(\gamma_0)^{-1}x$ belongs to $\mathrm{Cl}(D(\Gamma))$. □

Lemma 4.14. *The iterates $\rho(\gamma)D(\Gamma)$ are disjoint one from the other.*

Proof. If x lies in $\rho(\gamma)D(\Gamma)$, then for every h in $\rho(\Gamma) \setminus \rho(\gamma)$ we have:

$$\langle x|\rho(\gamma)x_0\rangle > \langle x|hx_0\rangle$$

If moreover x lies in $\rho(\gamma')D(\Gamma)$ with $\rho(\gamma') \neq \rho(\gamma)$ then:

$$\langle x|\rho(\gamma')x_0\rangle > \langle x|\rho(\gamma)x_0\rangle$$

We obtain a contradiction with the above in the case $h = \rho(\gamma')$. □

The two lemmas above proves that $\mathrm{Cl}(D(\Gamma))$ is a fundamental domain for the action of $\rho(\Gamma)$ on $E(\Lambda_\rho)$. From now we restrict to the intersection $\mathrm{Cl}(D(\Gamma)) \cap \mathrm{Conv}(\Lambda_\rho)$ and denote it $\bar{D}_{\mathrm{conv}}(\Gamma)$. Since the quotient $C(M) = \rho(\Gamma)\backslash \mathrm{Conv}(\Lambda_\rho)$ is compact:

Proposition 4.15. *$\bar{D}_{\mathrm{conv}}(\Gamma)$ is a compact fundamental domain for the action of $\rho(\Gamma)$ on $\mathrm{Conv}(\Lambda_\rho)$.* □

This compactness implies that $\bar{D}_{\mathrm{conv}}(\Gamma)$ is the intersection between the convex hull and a finite sided convex polyhedron. Hence $\bar{D}_{\mathrm{conv}}(\Gamma)$ itself is also convex.

4.4. **Quasi-isometry between the group and the convex hull.** A map $f : X \to X'$ between two metric spaces (X, d), (X', d') is a *quasi-isometry* if for some $a > 0$, $b > 0$ we have $(1/a)d(x, y)-b < d'(f(x), f(y)) < ad(x, y)+b$, and if moreover any point in X' is at distance at most b from the image of f.

According to Lemma 4.12 the set S made of elements γ of Γ such that $\rho(\gamma)\bar{D}_{\mathrm{conv}}(\Gamma) \cap \bar{D}_{\mathrm{conv}}(\Gamma) \neq \emptyset$ is finite, and according to Lemma 4.13, 4.14 S is a generating set of Γ. We consider the Cayley graph (Γ_S, d_S), *ie.* the simplicial metric space admitting as vertices the elements of Γ, and such that two vertices γ, γ' are connected by an edge of length 1 if and only if $\gamma'\gamma^{-1}$ lies in S.

Since Γ acts cocompactly on $\mathrm{Conv}(\Lambda_\rho)$, the map:,

$$\hat{\jmath} : (\Gamma_S, d_S) \to (\mathrm{Conv}(\Lambda_\rho), d^H)$$

associating to any vertex γ the element $\rho(\gamma)x_0$ of $\rho(\gamma)\bar{D}_{\mathrm{conv}}(\Gamma)$ is a quasi-isometry.

A key feature is that the group Γ we consider is (Gromov) *hyperbolic;* for definitions and properties of hyperbolic spaces or groups, we refer to [Gro, GdLH$^+$90]. By definition, the Gromov boundary of a hyperbolic geodesic space (X, d) is the space of complete geodesic rays modulo the equivalence relation identifying two rays staying at bounded distance one from the other. Any quasi-isometry between hyperbolic spaces extends as a homeomorphism between their Gromov boundary: the image by a quasi-isometry of a geodesic ray is *quasi-geodesic*, *ie.* a map $c : [0, +\infty[\to X$ such that, for some $a, b > 0$:

$$1/a|t - s| - b \leq d(c(t), c(s)) \leq a|s - t| + b$$

Moreover, for every $a, b > 0$, there is a constant D such that for every (a, b)-quasi-geodesic ray $c : [0, +\infty[\to X$ there is a geodesic ray $c : [0, +\infty[$ such that, for every t, the distance $c(t)$ to the image of c_0 is less than D, and the distance of $c_0(t)$ to the image of c is less than D. We say that c *is at bounded distance* $\leq D$ *from* c_0.

It follows that the quasi-isometry between Γ_S and $\mathbb{H}^n$ extends to a homeomorphism between $\partial\Gamma$ and the conformal sphere $\partial\mathbb{H}^n$.

Proposition 4.16. *$\hat{\jmath}$ extends as a homeomorphism $\jmath$ between the Gromov boundary $\partial\Gamma \approx \partial\mathbb{H}^n$ and the limit set Λ_ρ.*

Proof. Let $(\gamma_n)_{(n\in\mathbb{N})}$ be the sequence of vertices of Γ_S visited by a complete geodesic ray r_0 in (Γ_S, d_S). According to the above, there is a constant

$D \geq 0$ such that the image $\hat{\jmath}(r_0)$ is at bounded d^H-distance $\leq D$ from a geodesic ray in $(\mathrm{Conv}(\Lambda_\rho), d^H)$, *ie.* a projective segment $[x, y^+[$ where x lies in $\mathrm{Conv}(\Lambda_\rho)$ and y^+ an element in $\partial\,\mathrm{Conv}(\Lambda_\rho)_\epsilon$. Since this geodesic ray, of infinite d^H-length, is contained in $\mathrm{Conv}(\Lambda_\rho)$ the limit point y^+ actually lies in Λ_ρ.

On the other hand, according to Proposition 3.6 every subsequence of $(\gamma_n)_{(n\in\mathbb{N})}$ admits a subsequence $(\gamma_{n_k})_{(k\in\mathbb{N})}$ with mixed or bounded distortion: there is an attracting pole x^+ in Λ_ρ such that $(\rho(\gamma_{n_k}))_{(k\in\mathbb{N})}$ converges uniformly on compacts of $E(\Lambda_\rho)$ to the constant map x^+. In particular, $x_k = \hat{\jmath}(\gamma_{n_k}) = \rho(\gamma_{n_k})(x_0)$ converges to x^+.

If $x^+ \neq y^+$ then $]x^+, y^+[$ is a complete geodesic in $(\mathrm{Conv}(\Lambda_\rho), d^H)$ of infinite length. Hence there is a complete geodesic c in Γ_S such that $\hat{\jmath}(c)$ is a quasi-geodesic at bounded distance from $]x^+, y^+[$. Therefore the geodesic ray r_0 alternatively approximates both ends of c: it is a contradiction since these ends are distinct whereas a geodesic ray admits ony one accumulation point in $\partial\Gamma$.

Therefore $x^+ = y^+$. It follows that x^+ does not depend on the subsequence, and that y^+ is the extremity of any d^H-geodesic ray at bounded distance from $\hat{\jmath}(r_0)$. Hence the map $\jmath : [r_0] \in \partial\Gamma \to x^+ \in \Lambda_\rho$ is well-defined.

We now prove the continuity of $\jmath$. Let V be a neighborhood of x^+ in Λ_ρ. Let U be a neighborhood of x^+ in Ein_{n+1} disjoint from x_0, such that $U \cap \Lambda_\rho \subset V$ and that $U \cap \mathrm{AdS}_{n+1}$ is convex. Finally, let $\mathfrak{H}^+(v)$ be a convex cap contained in U such that x^+ is in the interior of the topological disk $\mathfrak{H}^+(v) \cap \partial\,\mathrm{Conv}(\Lambda_\rho)$. The geodesic segment $[x_0, x^+[$ crosses $\mathfrak{S}(v)$ at some point x_1. Let x_2 be another point of that segment sufficiently close to x^+ so that the Hilbert distance between x_2 and $\mathfrak{H}^-(v)$ is bigger that $2D$, where D is the constant such that for every geodesic ray in Γ_S there is a d^H-geodesic in $\mathrm{Conv}(\Lambda_\rho)$ at uniform distance D from $\hat{\jmath}(r)$.

The point x_2 is at distance D from an element $\rho(\gamma_k)x_0$ of $\hat{\jmath}(r_0)$. Let now W be the neighborhood of $[r_0]$ such that every element $[r]$ of W is represented by a geodesic ray r starting from id and containing γ_k. Then, $\hat{\jmath}(r)$ is at bounded distance D from the geodesic segment $[x_0, \jmath([r])[$. Hence $[x_0, \jmath([r])[$ contains a point y_2 at distance $\leq D$ from $\rho(\gamma_k)x_0$, hence at distance $\leq 2D$ from x_2. According to our choice of x_2, this point y_2 lies on the same side of the wall $\mathfrak{S}(v)$ than x_2, *ie.* in $\mathfrak{H}^+(v)$. Hence $[x_0, \jmath([r])[$ crosses $\mathfrak{S}(v)$ before reaching $\jmath([r])$. Since U is convex, it follows that $\jmath([r])$ lies in U, hence, in V. The continuity of $\hat{\jmath}$ is proved.

If $[r]$ and $[r']$ are two distinct elements in $\partial\Gamma$, there is complete geodesic $c : \mathbb{R} \to \Gamma$ asymptotic to r near $-\infty$ and to r' near $+\infty$. The quasi-geodesic $\hat{\jmath}(c)$ is at bounded distance from a geodesic $]\jmath([r]), \jmath([r'])[$ in $\mathrm{Conv}(\Lambda_\rho)$. It follows that $\jmath([r]) \neq \jmath([r'])$.

Finally, for any x in Λ_ρ, the d^H-geodesic ray $[x_0, x[$ is at bounded distance from the image by $\hat{\jmath}$ of a quasi-geodesic ray in Γ_S, hence from the image by $\hat{\jmath}$

of a geodesic ray. It follows that $\jmath$ is onto. Since $\partial\Gamma$ is compact, the bijective map $\jmath$ is an homeomorphism. The proposition is proved. □

Remark 4.17. It was convenient for the proof above to consider (Γ_S, d_S). But this metric space is quasi-isometric in a Γ equivariant way to $\mathbb{H}^n$ and also $\mathrm{T}^1\,\mathbb{H}^n$. Hence, a corollary of Proposition 4.16 is that any quasi-isometry $\hat{\jmath}_c : \mathrm{T}^1\,\mathbb{H}^n \to \mathrm{Conv}(\Lambda_\rho)$ extends as a homeomorphism $\jmath_c$ between the Gromov boundary $\partial\,\mathrm{T}^1\,\mathbb{H}^n$ and Λ_ρ.

4.5. **The geodesic flow of the GHC-regular spacetime.**

Definition 4.18. The *non-wandering subset,* denoted $\mathcal{N}(\Lambda_\rho)$, is the subset of $\mathcal{E}^1\,\mathrm{AdS}_{n+1}$ comprising elements (x, v) such that the two extremities $\ell^\pm(x, v)$ lie in Λ_ρ. The *geodesic flow* on $\mathcal{N}(\Lambda_\rho)$ is the flow $\tilde{\phi}^t_{\mathcal{N}}$ such that $\tilde{\phi}^t_{\mathcal{N}}(x, v) = (x^t, v^t)$ where x^t is the point on the geodesic tangent to (x, v) at distance t (along the geodesic) from x, and v^t the vector tangent at x^t to this geodesic.

This definition is $\rho(\Gamma)$-equivariant, we denote by $\mathcal{N}(\rho)$ the quotient of $\mathcal{N}(\Lambda_\rho)$ by $\rho(\Gamma)$ and $\phi^t_{\mathcal{N}}$ the flow on $\mathcal{N}(\rho)$ induce by $\tilde{\phi}^t_{\mathcal{N}}$.

The projection $\mathrm{p}(\mathcal{N}(\Lambda_\rho))$ is obviously contained in $\mathrm{Conv}(\Lambda_\rho)$. Since $\ell^\pm$ are continuous, and since Λ_ρ, $C(M)$ are compact, the quotient $\mathcal{N}(\rho)$ is compact.

Proposition 4.19. *There is a Γ-equivariant homeomorphism $\mathfrak{f} : \mathrm{T}^1\,\mathbb{H}^n \to \mathcal{N}(\Lambda_\rho)$ mapping orbits of the geodesic flow $\tilde{\phi}^t$ on orbits of $\tilde{\phi}^t_{\mathcal{N}}$.*

Proof. The orbit space of $\tilde{\phi}^t$ is $\partial\mathbb{H}^n \times \partial\mathbb{H}^n \setminus \mathcal{D}$, whereas the orbit space of $\tilde{\phi}^t_{\mathcal{N}}$ is $\Lambda_\rho \times \Lambda_\rho \setminus \mathcal{D}$ (where $\mathcal{D}$ denotes the diagonal in both cases). Moreover, the quotient maps $p_\phi : \mathrm{T}^1\,\mathbb{H}^n \to \partial\mathbb{H}^n \times \partial\mathbb{H}^n \setminus \mathcal{D}$ and $p_{\mathcal{N}} : \mathcal{N}(\Lambda_\rho) \to \Lambda_\rho \times \Lambda_\rho \setminus \mathcal{D}$ are locally trivial $\mathbb{R}$-fibrations. By proposition 4.16, there is an equivariant homeomorphism $\jmath \times \jmath$ between the orbit spaces; the question is to lift this homeomorphism in a Γ-equivariant way to a map $\mathfrak{f}$ so that:

$$p_{\mathcal{N}} \circ \mathfrak{f} = (\jmath \times \jmath) \circ p_\phi$$

The way to perform such a lift is quite well-known. Take a finite collection $(T_i)_{1\leq i\leq l}$ of small transversals to $\tilde{\phi}^t$ in $\mathrm{T}^1\,\mathbb{H}^n$ so that for any p in $\mathrm{T}^1\,\mathbb{H}^n$ there is a positive real number t in $]-1, +1[$ such that $\phi^t(p)$ lies on γT_i for some γ in Γ. Observe that such a family is locally finite: given x, there are only finitely many γ fulfilling this condition. Now, since $p_{\mathcal{N}}$ is a fibration, and if the T_i are chosen sufficiently small, for every i, the restriction of $(\jmath \times \jmath) \circ p_\phi$ to T_i lifts to a map $\mathfrak{f}_i : T_i \to \mathcal{N}(\Lambda_\rho)$ such that, on T_i:

$$p_{\mathcal{N}} \circ \mathfrak{f}_i = (\jmath \times \jmath) \circ p_\phi$$

For every p in $\mathrm{T}^1\,\mathbb{H}^n$, for every triple $\alpha = (i, \gamma, t_i)$ with $-1 \leq t_i \leq 1$ such that $\tilde{\phi}^t_i(p)$ lies in γT_i define $x_\alpha(p) = \rho(\gamma)\mathfrak{f}_i(\tilde{\phi}^{t_i}(p))$. All these points lie on the same $\tilde{\phi}_{\mathcal{N}}$-orbit. Now select a partition of unity $(f_i)_{1\leq i\leq l}$ on $N = \Gamma\backslash\,\mathrm{T}^1\,\mathbb{H}^n$ subordinate to the covering $(U_i)_{1\leq i\leq l}$ where $U_i = \{\phi^t(p)/-$

$1 < t < 1, p \in T_i\}$. It associates to every x_α a weight, namely the value of f_i at the projection in N of p. Define $\mathfrak{f}(p)$ as the barycenter of x_α with respect to these weights. It defines a continuous Γ-equivariant map $\mathfrak{f}$ mapping orbits of $\tilde{\phi}^t$ into orbits of $\tilde{\phi}^t_{\mathcal{N}}$. Now it follows from the hyperbolicity of $\mathbb{H}^n$ that a diffusion process along the orbits transform this map to another map, that we still denote $\mathfrak{f}$, which is injective along the orbits (see [Ghy84, Gro00]). This map obviously satisfies the condition $p_{\mathcal{N}} \circ \mathfrak{f} = (\jmath \times \jmath) \circ p_\phi$ and is Γ-equivariant. It follows that it is injective. An homological argument ensures that it is a homeomorphism. □

We can now improve the content of Proposition 4.16.:

Proposition 4.20. *For any complete geodesic ray $[x_0, x^+[$ in $\mathrm{Conv}(\Lambda_\rho)$ there is a sequence $(\gamma_n)_{(n\geq 1)}$ in Γ and a convex cap $\mathfrak{H}^+$ such that:*

1. *the convex caps $\mathfrak{H}^+_n := \rho(\gamma_n)\mathfrak{H}^+$ shrink uniformly to x^+,*
2. *the repelling pole x^- belongs to $\mathfrak{d}^- := \partial\,\mathrm{Conv}(\Lambda_\rho) \cap \mathfrak{H}^-$,*
3. *the attracting pole x^+ belongs to every $\mathfrak{d}^+_n := \rho(\gamma_n)\mathfrak{d}^+$ where $\mathfrak{d}^+ := \partial\,\mathrm{Conv}(\Lambda_\rho) \cap \mathfrak{H}^+$.*

Proof. For every x in $[x_0, x^+[$, let $v(x)$ be the velocity, *ie.* the unit vector tangent to $[x_0, x^+[$ and oriented towards x^+. Since $\mathcal{N}(\rho)$ is compact, the $\phi^t_{\mathcal{N}}$-orbit of the projection of (x_0, v_0) (where $v_0 = v(x_0)$) admits an accumulation point. Let (x_∞, v_∞) be a lifting in $\mathcal{N}(\Lambda_\rho)$ of this accumulation point, and let $\mathfrak{H}^+$ be a convex cap such that the wall $\mathfrak{S}$ intersects $[x_0, x_+[$ and such that the final extremity $\ell^+(x_\infty, v_\infty)$ lies in the interior of $\mathfrak{d}^+$. Fix also a positive real number ϵ, and let W be a small neighborhood of (x_∞, v_∞) in $\mathcal{N}(\Lambda_\rho)$ made of points of the form $\tilde{\phi}^t_{\mathcal{N}}(y, w)$ where:

– $-\epsilon < t < \epsilon$,

– y lies in $\mathfrak{S}$,

– the tangent vector w points in the direction of $\mathfrak{H}^+$, *ie.* the final extremity of the $\tilde{\phi}^t_{\mathcal{N}}$-orbit of (y, w) lies in the interior of $\mathfrak{d}^+$.

By construction, there is a sequence $(\gamma_n)_{(n\in\mathbb{N})}$ and a sequence of points x_n in $[x_0, x^+[$ converging to the final extremity x^+ such that (x_n, v_n) (where $v_n := v(x_n)$) intersects $\rho(\gamma_n)W$. By replacing γ_n by $\gamma_n\gamma_1^{-1}$ and (x_∞, v_∞), $\mathfrak{H}^+$ and W by their images by $\rho(\gamma_1)$ we can assume that γ_1 is trivial and that x_1 belongs to W. Hence $x^+ = \ell^+(x_1, v_1)$ lies in the interior of $\mathfrak{d}^+$. Moreover, $\rho(\gamma_n)^{-1}v_n$ points in the direction of $\mathfrak{H}^+$: it follows that x^+ belongs to every $\mathfrak{H}^+_n$, and that x_0 belongs to $\mathfrak{H}^-_n$.

Up to a subsequence, we can assume that $(\rho(\gamma_n))_{(n\in\mathbb{N})}$ is a converging subsequence with unbalanced distortion. The $\rho(\gamma_n)x_\infty$ stay at uniformly bounded distance form x_n; it follows that they converge to x^+ and that x^+ is the attracting pole of $(\rho(\gamma_n))_{(n\in\mathbb{N})}$. On the other hand, every $\mathfrak{H}^-_n$ contains x_0: therefore, these convex caps do not shrink to a point. The repelling pole x^- lies in $\mathfrak{d}^-$. Hence the positive convex caps $\mathfrak{H}^+_n$ shrink to the attracting pole x^+. The proposition is proved. □

4.6. **End of the proof of Theorem 1.1.** Let $\ell^{\pm}_{\rho} : \mathrm{T}^1\,\mathbb{H}^n \to \Lambda_\rho$ be the composition of $\jmath : \partial\Gamma \to \Lambda_\rho$ with $\ell^{\pm} : \mathrm{T}^1\,\mathbb{H}^n \to \partial\Gamma$: they together define a map $(\ell^+_\rho, \ell^-_\rho) : \mathrm{T}^1\,\mathbb{H}^n \to \mathcal{Y}$. In order to achieve the proof of the main Theorem we just have to construct the metrics g^p satisfying the hypothesis of Proposition 2.1.

Fix a $\rho(\Gamma)$-invariant future oriented timelike vector field V on $E(\Lambda_\rho)$. For every x in $\mathrm{Conv}(\Lambda_\rho)$ we simplify the notations by denoting simply h^x the metric $\tilde{g}^{x,V(x)}$ on $\partial U(x) \subset \mathrm{Ein}_n$ introduced in [Mér07, § 5.2.2]. We define g^x as the metric $h^{\rho(\gamma)x_0}$ where γ is an element of Γ such that $\rho(\gamma)\bar{D}_{\mathrm{conv}}(\Gamma)$ contains x. This family of metrics has a drawback: it is not continuous.

A way to construct a continuous family of metrics is the following: Let $\varsigma : \mathrm{T}^1\,\mathbb{H}^n \to \widetilde{S}^+_\rho$ be the composition of the homeomorphism $\mathfrak{f}$ of Proposition 4.19 with the projection $\pi : \mathcal{N}(\Lambda_\rho) \to \mathrm{Conv}(\Lambda_\rho)$: it is a Γ-invariant homeomorphism. For $p = (x, v)$ in $\mathrm{T}^1\,\mathbb{H}^n$ define the metric g^p_0 as the metric $h^{\varsigma(p)}$ on the open neighborhood $\partial U(\varsigma(p))$ of $\ell^+_\rho(p)$ and $\ell^-_\rho(p)$. These metrics vary continuously with p.

Now the key observation is that to check the expanding property for g^p_0 is equivalent to check the same property for g^p. Indeed:

Lemma 4.21. *For every $\delta > 0$, there is a constant $C_\delta > 1$ such that for every x and y in $\mathrm{Conv}(\Lambda_\rho)$ such that $d^H(x, y) < \delta$, and for every vector w tangent to Ein_n at a point of Λ_ρ the following inequalities hold:*

$$C_\delta^{-1} h^y(w, w) \leq h^x(w, w) \leq C_\delta h^y(w, w)$$

Sketch of proof. When y is fixed, for example, $y = x_0$, the lemma follows from the compactness of the d^H-ball centered at x_0 and the continuity of $x \to h^x$. The general case follows by $\rho(\Gamma)$-equivariance. □

Hence, g^p_0 and g^p only differ by a factor C_δ where δ is the diameter of $\bar{D}_{\mathrm{conv}}(\Gamma)$. Therefore, the last step in the proof of Theorem 1.1 is:

Proposition 4.22. *Let $p = (x, v)$ be an element of $\mathrm{T}^1\,\mathbb{H}^n$. Then for every $C > 0$, there is a time $t > 0$ such that for every tangent vector w to Ein_n at $\ell^+_\rho(p)$ the inequality $g^{\tilde{\phi}^t(p)}(w, w) \geq C g^{\tilde{p}}(w, w)$ holds.*

Proof. Let $r_0 = [x_0, x^+[$ be the π-projection of the image by $\mathfrak{f}$ of the positive $\tilde{\phi}^t$-orbit of p. Observe that $x^+ = \ell^+_\rho(p)$. Let $\mathfrak{H}^+$ be the convex cap and $(\gamma_n)_{(n\geq 1)}$ be the sequence obeying the conclusion of Proposition 4.20.

According to Lemma 4.21, it is enough to prove that for every $C > 0$ there is a positive integer n such that the $h^{\gamma_n x_0}$-norm of any w in $\mathrm{T}_{x_+}\,\mathrm{Ein}_n$ is bounded from below by its h^{x_0}-norm multiplied by C. Since the metrics are $\rho(\Gamma)$-equivariant, we have to prove:

$$h^{x_0}(d_{x^+}\rho(\gamma_n)^{-1}w, d_{x^+}\rho(\gamma_n)^{-1}w) \geq C h^{x_0}(w, w)$$

This inequality only involves the metric h^{x_0}. But since Λ_ρ is a compact subset of $\partial U(x_0)$, the h^{x_0}-norm of vectors tangent to points in Λ_ρ is equivalent to their $\|_0$-norm - here by $\|_0$ we mean the restriction to Ein_n of the spherical metric on $\mathbb{S}(\mathbb{R}^{2,n})$ induced by the Euclidean norm. Hence, to achieve the proof, we just have to check that Corollary 3.5 applies, *ie.* , with the notations introduced in § 4.5, that the attracting pole x^+ belongs to $\rho(\gamma_n)D^-$.

The repelling pole x^- belongs to $\mathfrak{d}^-$ (Item (2) of Proposition 4.20). Hence, the positive convex cap $\mathfrak{H}^+$ is at positive distance from $(x^-)^\perp$ in the unit sphere $\mathbb{S}(\mathbb{R}^{2,n})$, *ie.* is contained in D_ϵ for ϵ sufficiently small. Hence $\rho(\gamma_n)D^-$ contains $\mathfrak{d}_n^+$. Since x^+ lies in $\mathfrak{d}_n^+$ (Item (3) of Proposition 4.20), we obtain as required that x^+ belongs to $\rho(\gamma_n)D^-$. □

5. Conclusion

5.1. **Closure of the set of quasi-Fuchsian representations.** In the Riemannian context, the set of quasi-Fuchsian representations is not closed. But the situation for quasi-Fuchsian representations in $\mathrm{SO}_0(2,n)$ of lattices in $\mathrm{SO}_0(1,n)$ is different. Whereas quasi-spheres in $\partial\mathbb{H}^{n+1}$ may degenerate, the limit sets of a sequence of quasi-Fuchsian representations $(\rho_k)_{(k\in\mathbb{N})}$ in $\mathrm{SO}_0(2,n)$ always converge, up to a subsequence, to a closed achronal topological sphere Λ in Ein_n, since the space of 1-Lipschitz maps $f:\mathbb{S}^n\to\mathbb{S}^1$ is compact. It is easy to see that if the representations ρ_k converge to some representation ρ, then Λ is preserved by $\rho(\Gamma)$.

Question 5.1. Is Λ acausal?

If this question admits a positive answer, the limit representation ρ is Anosov (faithfullness and discreteness follow from classical arguments). In other words, Anosov representations would form an entire component of $\mathrm{Rep}(\Gamma,\mathrm{SO}_0(2,n))$.

An element in favor of a positive answer is the (2+1)-dimensional case: up to finite index, $\mathrm{SO}_0(2,2)$ is isomorphic to $\mathrm{SO}_0(1,2)\times\mathrm{SO}_0(1,2)$, and quasi-Fuchsian representations (*ie.* GHC-regular representations) decomposes as a pair (ρ_L,ρ_R) of cocompact Fuchsian representations the surface group Γ into $\mathrm{SO}_0(1,2)$. Since Fuchsian representations form a connected component of $\mathrm{Rep}(\Gamma,\mathrm{SO}_0(1,2))$, our assertion follows. Moreover, Einstein space Ein_2 is homeorphic to a double covering of $\mathbb{P}(\mathbb{R}^2)\times\mathbb{P}(\mathbb{R}^2)$ so that the limit set is a lifting of the graph of a topological conjugacy between the projective actions of Γ on the projective line induced by ρ_L and ρ_R. This topological conjugacy is a homeomorphism, meaning that Λ is acausal. For more details, see [Mes07, Bar05a, Bar05b].

5.2. **Convex cocompact lattices.** Theorem 1.1 extends, *mutatis mutandis,* to the case where Γ is a non elementary convex cocompact subgroup of $\mathrm{SO}_0(1,n)$, *ie.* a discrete subgroup acting cocompactly on the convex hull in $\mathbb{H}^n$ of its limit set in $\partial\mathbb{H}^n$ (the non elementary hypothesis meaning that we

require that the cardinal of this limit set is infinite). The definition of Anosov representation extends in this context by taking as dynamical system (N, ϕ^t) not the entire $\Gamma\backslash \mathrm{T}^1\,\mathbb{H}^n$, but the non-wandering subset of the geodesic flow in $\Gamma\backslash \mathrm{T}^1\,\mathbb{H}^n$: it is not anymore a manifold, but a compact lamination with a flow (the restriction of the geodesic flow). The set of $(\mathrm{SO}_0(2,n), \mathcal{Y})$-Anosov representations is open, and it is still true that they correspond to faithfull, discrete representations admitting as limit set a closed *acausal* subset in Ein_n, but which now is not a topological sphere.

The main difference is that the associated domains $E(\Lambda_\rho)$ in AdS_{n+1} are not globally hyperbolic. However, the action of $\rho(\Gamma)$ on $E(\Lambda_\rho)$ is still free, properly discontinuous and *strongly causal, ie.* the quotient spacetime $\rho(\Gamma)\backslash E(\Lambda_\rho)$ is strongly causal. In dimension $2+1$ (when $n=2$) these spacetimes are the so-called *BTZ multi-black holes* (see [BTZ92, Bar05b]).

5.3. **Other MGHC spacetimes.** In this paper, we focused on the case where Γ is a lattice in $\mathrm{SO}_0(1,n)$. But observe that Theorem 4.7 in [Mér07] (GHC-spacetimes are GH-regular), Proposition 3.6 (no balanced distortion) and § 4.1 (definition of the convex hull and the boundary surfaces $S^\pm_\rho$) remains true without this hypothesis.

5.3.1. *GHC-representations with acausal limit set are weakly Anosov.* In this § we consider a GHC-regular representation $\rho : \Gamma \to \mathrm{SO}_0(2,n)$, but with no other assumption on the group Γ. However we assume that Λ_ρ is acausal so that Lemma 4.1 holds.

Define the length of Lipschitz curves $c : I \to \widetilde{S}^\pm_\rho$ as the integral over I of the Lorentzian norm of the tangent vector (defined everywhere), and then the distance $\tilde{d}^\pm(x,y)$ between two points x, y in $\widetilde{S}^\pm_\rho$ as the infimum of the length of Lipschitz curves joining x to y. It is not hard to see that $\tilde{d}^\pm$ is indeed a distance, providing to $\widetilde{S}^\pm_\rho$ a length space structure.

Observe that $(\widetilde{S}^\pm_\rho, \tilde{d}^\pm)$ is not in general a Riemannian space, neither Finslerian. However, this metric structure induces the manifold topology on $\widetilde{S}^\pm_\rho$, which admits a compact quotient: it is therefore a complete, proper metric space. By generalized Hopf-Rinow Theorem ([BH99, Proposition I.3.7]) $(\widetilde{S}^\pm, \tilde{d}^\pm_\rho)$ is geodesic: between two points x, y, there is always a curve joining the two points realizing the distance.

Proposition 5.2. $(\widetilde{S}^\pm_\rho, \tilde{d}^\pm)$ *are complete* $\mathrm{CAT}(-1)$ *spaces.*

For definition of $\mathrm{CAT}(-1)$ spaces, we refer to [BH99, § 2.1] or [Bal95].

Proof. We only consider the upper convex boundary $\widetilde{S}^-_\rho$, the case of $\widetilde{S}^+_\rho$ is similar (or obtained by reversing the time orientation). According to the Cartan-Hadamard Theorem (see e.g. [BH99, Theorem 4.1]) to be a $\mathrm{CAT}(-1)$ space is a local property: since $\widetilde{S}^-_\rho$ is simply connected, we just have to prove that every point x admits a neighborhood where the $\tilde{d}^-$ is metric of curvature ≤ -1 (in the sense of [BH99, Definition II.1.2]).

In the Klein model $\widetilde{S}^-_\rho$ is locally the graph of a convex function from an open domain of $\mathbb{R}^n$ into $\mathbb{R}$. More precisely, there is a coordinate system $(t, \bar{x}_1, \dots, \bar{x}_n)$, $-\epsilon < x_i < \epsilon$, $-\eta < t < \eta$ on a neighborhood U of x so that:

– x has coordinates $(0, \dots, 0)$,

– $U \cap \widetilde{S}^-_\rho$ is the graph of a convex map $\psi :]-\epsilon, \epsilon[^n \to]-\eta, \eta[$,

– $\{t = 0\}$ is a support hyperplane for ψ,

– every tangent vector with negative norm for $-d\eta^2 + dx_1^2 + ... + dx_n^2$ has negative norm for the AdS metric.

Shrinking ϵ if necessary, we moreover can assume that the gradient of ψ has almost everywhere $dx_1^2 + ... + dx_n^2$-norm less than 1. By convolution, we obtain smooth convex maps ψ_ν which uniformly converge to ψ when the parameter $\nu > 0$ converges to 0. Moreover, the norm of their gradient is bounded from above by 1, it follows that the graphs S_ν of ψ_ν are spacelike. Finally, this uniform convergence implies that for any Lipschitz curve $c : I \to]-\epsilon, \epsilon[^n$, the AdS length of $s \to (c(s), \psi_\nu(c(s))$ uniformly converges to the AdS-length of $s \to (c(s), \psi(c(s))$. Hence the graphs S_ν, equipped with their induced (Riemannian) length metric, converge in the Gromov-Hausdorff topology to the restriction of $\tilde{d}^-$ to $U \cap \widetilde{S}^-_\rho$ (cf. [BH99, Definition I.5.33]).

We can compute the sectional curvatures of the smooth hypersurfaces S_ν. Since ψ_ν is convex, its second fundamental form is positive definite, and since the ambient AdS metric has sectionnal curvatures -1 the Gauss equation implies that S_ν have sectional curvatures -1. They are therefore of curvature ≤ -1. The proposition follows since Gromov-Hausdorff limits of length spaces of curvature ≤ -1 have curvature ≤ -1 ([BH99, Theorem II.3.9]). □

CAT(-1) spaces enjoy many nice properties. For example, they are hyperbolic in the Gromov sense; hence the group Γ is Gromov hyperbolic. Furthermore:

Corollary 5.3 (Proposition II.2.2 in [BH99])**.** *$(\widetilde{S}^\pm_\rho, \tilde{d}^\pm_\rho)$ are uniquely geodesic: given two points x, y there is an unique geodesic joining them.* □

We therefore can define the *geodesic flow* of $S^\pm_\rho$, even if $S^\pm_\rho$ has no unit tangent bundle.

Definition 5.4. Let $\widetilde{\mathfrak{G}}^\pm_\rho$ denote the space of complete unit speed geodesics of $\widetilde{S}^\pm_\rho$, *ie.* isometries $c : \mathbb{R} \to \widetilde{S}^\pm_\rho$, endowed with the topology of uniform convergence on compact subsets. The geodesic flow $\tilde{\phi}^t_\pm$ is the flow defined by:

$$\tilde{\phi}^t_\pm(c)(s) = c(s + t)$$

The group $\rho(\Gamma)$ acts naturally, freely and properly discontinuously on $\widetilde{\mathfrak{G}}^\pm_\rho$. We denote by $\mathfrak{G}^\pm_\rho$ the quotient space, and by $\phi^t_\pm$ the flow on $\mathfrak{G}^\pm_\rho$ induced by $\tilde{\phi}^t_\pm$.

This flow is not differentiable but *weakly* (or topologically) Anosov: there are two continuous Γ-invariant foliations $\widetilde{\mathcal{F}}^s_\pm$, $\widetilde{\mathcal{F}}^u_\pm$ on $\widetilde{\mathfrak{G}}^\pm_\rho$, invariant by the geodesic flow such that for every pair p, q of points in the same leaf of $\widetilde{\mathcal{F}}^s$ (respectively $\widetilde{\mathcal{F}}^u$) there is a real number t_0 such that the distance between $\tilde\phi^{t+t_0}_\pm(p)$ and $\tilde\phi^t_\pm(q)$ decreases (respectively increases) exponentially with t. This claim follows quite easily from the CAT(-1) property - it is actually a general property of Gromov hyperbolic spaces admitting compact quotients: see [Gro, § 8.3], and for more details, [Cha94], [Mat]. The fact that the spaces we consider are CAT(-1) greatly simplifies the definition of the geodesic flow.

It should be clear to the reader that the methods used in the present paper prove that the GHC-regular representation ρ satisfies the $(\mathrm{SO}_0(2,n))$-Anosov property as defined in [Mér07, Remark 5.4] or appearing as hypothesis in Proposition 2.1 - observe that in these formulations the differential of the flow is not involved. The arguments in [Mér07, § 5.3] still apply for this non-differentiable version of (G,Y)-Anosov property. In other words, we can state that *GHC-regular representations with acausal limit sets are precisely weakly (G,Y)-Anosov representations.* Moreover, we guess that weakly Anosov representations form an open subset of $\mathrm{Rep}(\Gamma,G)$: the differentiable setting should be avoided through arguments in [Sul85]. Anyway, for the pair $(\mathrm{SO}_0(2,n),\mathcal{Y})$, it comes through the discussion above - a representation is GHC-regular with acausal limit set if and only if it is $(\mathrm{SO}_0(2,n))$-Anosov - and the fact that GHC-regular representations form an open domain: using the arguments in [Mes07] one can show that small deformations of holonomy representations of MGHC AdS-spacetimes are still holonomy representations of MGHC spacetimes (see also the introduction of [Bon05] for more details).

Anyway, we don't discuss or justify furthermore this notion of weaky Anosov representations because we believe that weakly (G,Y)-Anosov representations are (differentially) (G,Y)-Anosov in the sense of Labourie. This statement would be a corollary of a positive answer to the following question:

Question 5.5. Let $\rho:\Gamma\to\mathrm{SO}_0(2,n)$ be a GH-regular representation with acausal limit set. Is there a $\rho(\Gamma)$-invariant *smooth* (*ie.* C^r with $r\geq 3$) convex Cauchy hypersurface?

Indeed, we could replace the Cauchy hypersurfaces $\widetilde{S}^\pm$ in the discussion above by this smooth convex one, *ie.* with curvature ≤ -1, hence, with differentiable Anosov geodesic flow. Concerning this question, observe that the main task in [BBZ07] was to give a positive answer to this question in dimension $2+1$.

Finally, as before, we can address the question:

Question 5.6. Is the space of (weakly) $(\mathrm{SO}_0(2,n),\mathcal{Y})$-Anosov representations closed?

which, as in the case where Γ is a lattice of $\mathrm{SO}_0(1,n)$, essentially reduces to the proof that the limit set of a sequence of $(\mathrm{SO}_0(2,n),\mathcal{Y})$-Anosov representations is acausal.

5.3.2. *Classification of MGHC spacetimes of constant curvature* -1.

Question 5.7. Let $\rho:\Gamma\to\mathrm{SO}_0(2,n)$ a GHC-representation with acausal limit set. Is Γ isomorphic to a lattice of $\mathrm{SO}_0(1,n)$?

A natural way to find a positive answer to this question is to exhibit in the associated MGHC spacetime a Cauchy hypersurface with constant Gauss curvature -1: one of the main results of [BBZ] is precisely that such a Cauchy hypersurface exists in the $(2+1)$-dimensional case. Of course, in this low dimension, this kind of argument is sophisticated, since it is only a matter to prove that the genus of the Cauchy surfaces is ≥ 2, which can be obtained with more elementary arguments. However, this last idea does not extends in higher dimension, whereas most part of the content of [BBZ] applies in any dimension.

Another way to give a positive answer would be to study the functional on $\mathrm{Anos}_{\mathcal{Y}}(\Gamma,\mathrm{SO}_0(2,n))$ associating to a representation the volume of the convex core in the associated spacetime. Indeed, according to Lemma 4.2, this functional vanishes only on Fuchsian representations.

Finally, it is easy to produce GHC-regular representations with non-acausal limit set: let (p,q) be a pair of positive integers such that $p+q=n$, and let Γ be a cocompact lattice of $\mathrm{SO}_0(1,p)\times\mathrm{SO}_0(1,q)$. There is a natural inclusion of $\mathrm{SO}_0(1,p)\times\mathrm{SO}_0(1,q)$ into $\mathrm{SO}_0(2,n)$ arising from the orthogonal splitting $\mathbb{R}^{2,n}=\mathbb{R}^{1,p}\oplus\mathbb{R}^{1,p}$. The isotropic cone of $\mathbb{R}^{1,p}$ (respectively $\mathbb{R}^{1,q}$) is contained in $\mathcal{C}_n$ and projects in Ein_n on the union of two spacelike spheres $\Lambda_p^{\pm}\approx\mathbb{S}^{p-1}$ (respectively $\Lambda_q^{\pm}\approx\mathbb{S}^{q-1}$). Every point in $\Lambda_p^{\pm}$ is joined to every point in $\Lambda_q^{\pm}$ by a lightlike geodesic segment in Ein_n: let Λ be the union of lightlike geodesic segments joining a point of Λ_p^+ to a point in Λ_q^+ and avoiding $\Lambda_p^-\cup\Lambda_q^-$. We leave to the reader the proofs of the following facts:

– Λ is a non pure lightlike achronal topological sphere,

– The convex hull $\mathrm{Conv}(\Lambda)$ of Λ in AdS_{n+1} coincide with the regular domain $E(\Lambda)$.

The group $\Gamma\subset\mathrm{SO}_0(1,p)\times\mathrm{SO}_0(1,q)\subset\mathrm{SO}_0(2,n)$ preserves $\mathrm{Conv}(\Lambda)=E(\Lambda)$; the quotient spacetime $M(\Gamma)=\Gamma\backslash E(\Lambda)$ is MGH. Moreover, it is spatially compact: indeed, the set of orthogonal sums $u+v$ where u (respectively v) is an element of $\mathbb{R}^{1,p}$ such that $\mathrm{q}_{1,p}(u)=-1/2$ (respectively an element of $\mathbb{R}^{1,q}$ of $\mathrm{q}_{1,q}$-norm $-1/2$) admits two components in AdS_{n+1}, one lying in $E(\Lambda)$. This component is a spacelike hypersurface isometric to $\mathbb{H}^p\times\mathbb{H}^q$ and Γ-invariant. Its projection in the quotient $M(\Lambda)$ is a compact spacelike hypersurface, hence a Cauchy hypersurface.

Remark 5.8. By Margulis superrigidity Theorem ([Mar91]), if $p,q\geq 2$ every Γ into $\mathrm{SO}_0(2,n)$ either has finite image, or conjugate in $\mathrm{SO}_0(2,n)$

to the inclusion $\Gamma \subset \mathrm{SO}_0(1,p) \times \mathrm{SO}_0(1,q) \subset \mathrm{SO}_0(2,n)$. It follows that every MGHC spacetime of constant curvature -1 with fundamental group isomorphic to a lattice Γ in $\mathrm{SO}_0(1,p) \times \mathrm{SO}_0(1,q)$ is isometric to a spacetime $M(\Gamma)$ described above.

Remark 5.9. When $n = 2$, the only possibility is $p = q = 1$. It is the case of *Torus universe* (see [BBZ07, § 7], [Car03, § 3.3]).

Question 5.10. Let $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ be a GHC-regular representation with non acausal limit set. Is Γ isomorphic to a lattice of some product $\mathrm{SO}_0(1,p) \times \mathrm{SO}_0(1,q)$?

Our personal guess is that all the questions reported above admit a positive answer.

Conjecture 5.11. Every GHC-regular representation into $\mathrm{SO}_0(2,n)$ is either a quasi-Fuchsian representation of a lattice in $\mathrm{SO}_0(1,n)$, or a representation of a lattice in $\mathrm{SO}_0(1,p) \times \mathrm{SO}_0(1,q)$ with $p+q=n$, $p \geq 1$, $q \geq 1$.

References

[ABB+07] T. Andersson, T. Barbot, R. Benedetti, F. Bonsante, W.M. Goldman, F. Labourie, K.P. Scannell, and J.M. Schlenker. Notes on a paper of Mess. *Geom. Dedicata*, 126:47–70, 2007.

[Bal95] Werner Ballmann. *Lectures on spaces of nonpositive curvature*, volume 25 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1995. With an appendix by Misha Brin.

[Bar05a] T. Barbot. Causal properties of AdS-isometry groups I: Causal actions and limit sets. *Arxiv preprint math.GT/0509552*, 2005. to appear in Advances in Theoretical and Mathematical Physics, Volume 12, Number 1.

[Bar05b] T. Barbot. Causal properties of AdS-isometry groups II: BTZ multi black-holes. *Arxiv preprint math.GT/0510065*, 2005.

[BBZ] T. Barbot, F. Béguin, and A. Zeghib. Prescribing Gauss curvature of surfaces in 3-dimensional spacetimes, application to Minkowski problem in the 3-dimensional Minkowski space. *In preparation.*

[BBZ07] T. Barbot, F. Béguin, and A. Zeghib. Constant mean curvature foliations of globally hyperbolic spacetimes locally modelled on AdS. *Geom. Dedicata*, 126:71–129, 2007.

[BCD+07] T. Barbot, V. Charette, T Drumm, W.M. Goldman, and K. Melnick. A primer on the (2+1) einstein universe. *Arxiv preprint math.DG/0706.3055*, 2007.

[BH99] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[BK53] H. Busemann and P.J. Kelly. *Projective geometry and projective metrics*. Academic Press New York, 1953.

[Bon05] F. Bonsante. *Deforming the Minkowskian cone of a closed hyperbolic manifold*. PhD thesis, Pisa, 2005.

[BTZ92] M. Bañados, C. Teitelboim, and J. Zanelli. Black hole in three-dimensional spacetime. *Physical Review Letters*, 69(13):1849–1851, 1992.

[Car03] S. Carlip. *Quantum Gravity in 2+ 1 Dimensions*. Cambridge University Press, 2003.

[Cha94] C. Champetier. Petite simplification dans les groupes hyperboliques. *Annales de la faculté des sciences de Toulouse Sér. 6*, 3(2):161–221, 1994.

[Fra05] C. Frances. Lorentzian Kleinian groups. *Comment. Math. Helv.*, 80(4):883–910, 2005.

[GdLH+90] E. Ghys, P. de La Harpe, et al. *Sur les groupes hyperboliques d'après Mikhael Gromov*. Birkhäuser, 1990.

[Ghy84] E. Ghys. Flots d'Anosov sur les 3-variétés fibrées en cercles. *Ergodic Theory Dynam. Systems*, 4(1):67–80, 1984.

[Gro] M. Gromov. Hyperbolic groups. *Essays in group theory*, 8:75–263.

[Gro00] M. Gromov. Three remarks on geodesic dynamics and fundamental group. *Enseign. Math. (2)*, 46(3-4):391–402, 2000.

[Lab06] F. Labourie. Anosov flows, surface groups and curves in projective space. *Inventiones Mathematicae*, 165(1):51–114, 2006.

[Mar91] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991.

[Mat] F. Mathéus. Flot géodésique et groupes hyperboliques d'après M. *Gromov, from: Séminaire de Théorie Spectrale et Géométrie*, pages 1990–1991.

[Mér07] Q. Mérigot. Anosov AdS representations are quasi-Fuchsian. *Arxiv preprint math.DG/0710.0618*, 2007.

[Mes07] G. Mess. Lorentz Spacetimes of Constant Curvature. *Geom. Dedicata*, 126:3–45, 2007.

[Rat06] J. G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.

[Sal99] F. Salein. *Variétés anti-de Sitter de dimension 3*. PhD thesis, École Normale Sup. de Lyon, 1999.

[Sul85] D. Sullivan. Quasiconformal homeomorphisms and dynamics II: Structural stability implies hyperbolicity for Kleinian groups. *Acta Mathematica*, 155(1):243–260, 1985.

†CNRS, UMPA, École Normale Supérieure de Lyon
E-mail address: Thierry.Barbot@umpa.ens-lyon.fr