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Concentration for Young measures and conditioned Boolean model

Julien Michel *

January 30, 2007

Abstract

We present a method for determining the probability that a given point lies in the vacant region of a Boolean model subject to the condition that this probability takes imposed values on certain points. This method is based on a characterisation of the intensity of the Boolean model as a Young measure, such that after discretisation a large deviations principle holds; this induces a concentration property which can be transposed for the Boolean model, yielding the maximum of entropy state satisfying the conditioning. The numerical computation of this Gibbs state gives the parameters of the conditioned Boolean model, and thus one can answer both numerically and theoretically to the problem of finding the vacancy function at any point.

MSC 2000: primary 60G55, secondary 60D05, 60K40.
KEYWORDS: Point processes, Young measures, simulation, maximum entropy principle, Boolean model.

1 Introduction

The motivation for this work comes from the following question: assume that one has drilled a few wells searching for oil, and has thus obtained some information from those test wells: how can one predict the outcome of a new well at some other position? The answer to this question depends strongly on the underlying model: in this paper we propose to use a concentration property of Young measures associated to the Boolean model of stochastic geometry. The oil reservoir is given by some domain $D \times \mathbb{R}_-$ where D is a compact subset of $\mathbb{R}^2$, and we denote by (x, z) the variables. In this domain oil is assumed to be contained in a subset which is the occupied phase of a Boolean model, for convenience we assume that this model is driven by the following facts:

- the random shape is a ball with random radius R with law μ_0,
- the centers of those balls are located at the points of a Poisson Point Process $\mathbf{X}$ with intensity measure Λ (for instance λ times the Lebesgue measure) on $\mathbb{R}^2 \times \mathbb{R}$,

and thus

$$\text{Oil} = \bigcup_{y \in \mathbf{X}} B(y, R_y) \cap D \times \mathbb{R}_-.$$

Remark 1 *We use the following notations for point processes (taken from [vL00]): a point process on a complete separable metric space $\mathscr{X}$ is a measurable mapping from a probability space into the set N^{lf} of locally finite point configurations $\mathbf{x}$ on $\mathscr{X}$ endowed with the smallest σ-algebra, denoted by $\mathcal{N}^{lf}$, such that for all bounded Borel subset A of $\mathscr{X}$ the mapping*

$$\begin{array}{rcl} N^{lf} & \to & \mathbb{N}, \\ \mathbf{x} & \mapsto & N_{\mathbf{x}}(A) := \sharp\{\mathbf{x} \cap A\}, \end{array}$$

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is measurable.

Drilling a well at some point $(x, 0)$ one can gain the information $\mathbf{1}_{\text{Oil}}(x, z)$ for $z \in [-\text{Depth}, 0]$, let us assume that this information is summarised in the following quantity (outcome)

$$Q(x) = \frac{1}{\text{Depth}} \int_{-\text{Depth}}^{0} \mathbf{1}_{\text{Oil}}(x, z)\, dz,$$

then if the depth of the well is sufficiently large one can state that if μ is for instance compactly supported and Λ is invariant under vertical translations:

$$Q(x) \simeq P(x \in \mathscr{O}_h),$$

where $\mathscr{O}_h$ denotes the occupied region for any horizontal slice of the Boolean model: this comes from the ergodic theorem as

$$\mathbf{1}_{\text{Oil}}(x, z) = \mathbf{1}_{\text{Oil}_{h(z)}}(x),$$

where

$$\text{Oil}_h(z) = \bigcup_{y \in \mathbf{X}} B(y, R_y) \cap \{(x, z) \ : \ x \in \mathbb{R}^2\}$$

is a Boolean model the parameters of which can be fully determined from Λ and μ (see section 5). Actually one has

$$P(x \in \mathscr{O}_h) = P((x, 0) \in \text{Oil}),$$

so that one has the following set of conditions:

$$\mathscr{C} = \{P((x_i, 0) \in \text{Oil}) = q_i, \ i = 1, \ldots, N\},$$

where x_i is the location of the i-th well, and q_i the observed measure of the outcome of this well.
The computation of the occupancy function is a classical fact for the Boolean model [SKM87, Mol96]:

$$P((x, z) \in \text{Oil}) = 1 - \exp\left(-E\left[\sum_{y \in \mathbf{X}} \mathbf{1}_{B(y, R_y)}(x, z)\right]\right),$$

the expectation on the right hand side is obtained thanks to Campbell's formula [Kin93]:

$$E\left[\sum_{y \in \mathbf{X}} \mathbf{1}_{B(y, R_y)}(x)\right] = \int_{\mathbb{R}^3 \times \mathbb{R}_+} \mathbf{1}_{B(y, R)}(x, z)\, d\Lambda(y)\, d\mu(R),$$

so that

$$P((x, z) \in \text{Oil}) = 1 - \exp\left(-\int_{\mathbb{R}^3 \times \mathbb{R}_+} \mathbf{1}_{B(y, R)}(x, z)\, d\Lambda(y)\, d\mu(R)\right).$$

One observes here that this quantity is expressed thanks to the intensity measure $\Lambda \otimes \mu$ of the marked Poisson Point Process of the couples (position, radius): this remark is at the core of our approach here, indeed such a measure may be seen as a special case of a Young measure (parametrised probability measure) on the space $\mathbb{R}^3 \times \mathbb{R}_+$, such a measure determines the law of the Boolean model, and the condition on the Boolean model writes as a condition on this Young measure. The problem at hand now becomes: how does the condition force the behaviour of the Young measure, and what does it imply for the Boolean model?
The answer to this question will come from the following developments: in section 2 we propose a discretisation of the Young measures that approximates the actual intensity thanks to laws of large numbers and central limit theorems, those results are a generalisation of results presented in [MP98]. In section 3 we

prove a large deviation principle for the discretised Young measures and the associated concentration property, which yields the maximum of entropy states presented in section 4 for the Boolean model. Numerical methods and results for those maximum of entropy states are presented in section 5. From section 2 on we shall assume the following hypotheses:

(**H0**) the intensity measure Λ is invariant under vertical translations.

(**H1**) Λ is absolutely continuous with respect to the Lebesgue measure $\mathscr{L}_3$, and there exists a constant $C > 0$ such that

$$\forall x \in \mathbb{R}^3,\ \frac{1}{C} \leq \frac{d\Lambda}{d\mathscr{L}_3}(x) \leq C.$$

(**H2**) The laws $(\mu_x)_{x\in\mathbb{R}^3}$ of the radius of the balls located at points x are compactly supported: there exists R_0 such that $\mu_x([0, R_0]) = 1$ for all $x \in \mathbb{R}^3$.

We shall use the notation μ for the Young measure with disintegration $(\mu_x)_{x\in\mathbb{R}^3}$ with respect to Λ.

2 Discretised Young measures

2.1 Young measures

Young measures first appeared as a way to generalize the notion of measurable function, where the deterministic values at the points x of the source space were replaced by probability measures on the goal space, for instance Dirac masses represent deterministic functions. Later Young measures were developed as a more general mathematical object, one can cite [Val90] or [CRdFV04] for a precise presentation of Young measures in general frameworks.

In all this section we shall consider Young measures defined on a product space $\mathscr{X} \times \mathscr{M}$, where $\mathscr{X}$ is a compact subset of $\mathbb{R}^d$ with smooth boundary, and $\mathscr{M}$ is a compact subset of $\mathbb{R}^e$, with $d, e \geq 1$. We assume that

(**H1′**) $\mathscr{X}$ is endowed with a finite non negative measure Λ such that $\Lambda \ll \mathscr{L}_d$, the d-dimensional Lebesgue measure, with a density λ uniformly bounded on $\mathscr{X}$:

$$0 < \frac{1}{C} \leq \lambda(x) = \frac{d\Lambda}{d\mathscr{L}_d}(x) \leq C < +\infty,\ \forall x \in \mathscr{X}.$$

Definition 1 *The Young measure with base Λ and disintegration $(\mu_x)_{x\in\mathscr{X}}$ is the measure μ on $\mathscr{X} \times \mathscr{M}$ defined by*

$$\forall \phi \text{ non negative measurable on } \mathscr{X} \times \mathscr{M},\ \langle \mu, \phi \rangle = \int_{\mathscr{X}} \langle \mu_x, \phi(x, \cdot) \rangle \, d\Lambda(x),$$

where the application $x \mapsto \mu_x \in M_1^+(\mathscr{M})$ is supposed to be measurable, $M_1^+(\mathscr{M})$ being the space of probability measures on $\mathscr{M}$.

Remark 2 *The narrow and vague topology induce the same topology on the set of Young measures on $\mathscr{X} \times \mathscr{M}$.*

As one can deduce from Jirina's theorem [Jiř59], one has the equivalence

$$\mu \text{ is a Young measure} \Longleftrightarrow \forall f \in \mathscr{C}_b(\mathscr{X}), \langle \mu, f \rangle = \int_{\mathscr{X}} f(x) \, d\Lambda(x).$$

2.2 Discretisations with respect to a uniform mesh

In [MR94] or [MP98] the discretisations were made either according to regular or random partitions of the set $\mathscr{X}$ and there was no dependency on x for the laws at each point, in this paper we will use only the regular partition, though one may try to adapt the random case, but will accept more general Young measures. We define the regular mesh of step $1/n$ of $\mathscr{X}$ as the collection of the intersections of $\mathscr{X}$ with the cubes of side-length $1/n$ with vertices on the lattice $(1/n)\mathbb{Z}^d$, the cube with 'lower-left' vertex at $i = (i_1, \ldots, i_d)$ will be denoted by C_i^n, and we set

$$\mathscr{X}_i^n = C_i^n \cap \mathscr{X}.$$

We will introduce two different discretisations of a Young measure μ:

Definition 2 *Let μ be a Young measure as in definition 1, and $n \geq 1$, for all $i \in \mathbb{Z}^d$ such that $\Lambda(\mathscr{X}_i^n) > 0$ define*

$$\mu_{(i)}^n = \frac{1}{\Lambda(\mathscr{X}_i^n)} \int_{\mathscr{X}_i^n} \mu_x \, d\Lambda(x), \;\; \mu_{(i)}^n = \delta_0 \text{ otherwise,}$$

then $\overline{\mu}^n$ is defined as the Young measure with locally constant disintegration:

$$\forall x \in \mathscr{X}_i^n, \;\; \overline{\mu}_x^n = \mu_{(i)}^n.$$

We can also define:

Definition 3 *Let μ be a Young measure as above, and $(\Omega, \mathscr{F}, P)$ some complete probability space, let $\left(M_{(i)}^n\right)_{i \in \mathbb{Z}^d}$ be a sequence of independent random variables defined on this probability space with respective laws $\left(\mu_{(i)}^n\right)_{i \in \mathbb{Z}^d}$, then $\widetilde{\mu}^n$ is defined as the* random *Young measure with locally constant disintegration:*

$$\forall x \in \mathscr{X}_i^n, \;\; \forall \omega \in \Omega, \;\; \widetilde{\mu}_x^n(\omega) = \delta_{M_{(i)}^n(\omega)}.$$

If we denote by $\mathbf{X}$ a marked Poisson Point Process with intensity μ on $\mathscr{X} \times \mathscr{M}$, we will also denote by $\overline{\mathbf{X}}^n$ and $\omega \mapsto \widetilde{\mathbf{X}}^n(\omega)$ the (random) marked Poisson Point Processes associated to those (random) Young measures. In the following paragraph we shall examine the convergence of those approximations towards the original Young measure, as well as the convergence in law of the associated point processes.

2.3 Laws of large numbers and coupling results

The results of this section are straightforward, their purpose is to justify the chosen discretisations of the intensity measure. The transcription in terms of marked Poisson Point Processes is also a classical result on convergence of point processes: the main part of this paper lies in the concentration property of section 3 whereas all the results here are only presented for the sake of completeness.

One checks easily:

Proposition 1 *As n tends to infinity, one has the vague convergence of $\overline{\mu}^n$ towards μ.*

The proof is straightforward, one has for all compactly supported continuous function ϕ on $\mathscr{X} \times \mathscr{M}$:

$$\langle \overline{\mu}^n, \phi \rangle - \langle \mu, \phi \rangle \;\; = \;\; \sum_{i \in \mathbb{Z}^d} \int_{\mathscr{X}_i^n} \int_{\mathscr{M}} \Big(\frac{1}{\Lambda(\mathscr{X}_i^n)} \int_{\mathscr{X}_i^n} (\phi(z,m) - \phi(x,m)) \, d\Lambda(z) \Big) \, d\mu x(m) \, d\Lambda(x),$$

and one sees easily that each integrand converges to 0 thanks to the uniform continuity of ϕ with respect to its first variable in the bounded sets $\mathscr{X}_i^n$,

$$|\langle \overline{\mu}^n, \phi \rangle - \langle \mu, \phi \rangle| \leq \Lambda(\mathscr{X}) \omega_\phi(\sqrt{d} n^{-1}),$$

where ω_ϕ is the continuity modulus of ϕ, thus the vague convergence of proposition 1.

One may refine the previous statement by computing the exact rate of convergence in a metric associated to the vague convergence. We use here the Dudley metric for bounded measures:

$$d_{\mathscr{D}}(\mu,\nu) = \sup_{\|\phi\|_\infty \leq 1,\ L(\phi)\leq 1} |\langle \mu-\nu,\phi\rangle|,$$

where $L(\phi)$ is the Lipschitz constant of ϕ. Now it becomes obvious that one obtains

Theorem 1 *One has*

$$d_{\mathscr{D}}(\mu,\overline{\mu}^n) \leq A\varepsilon,$$

where A depends only on $\mathscr{X}$ and Λ.

As for the random discretised Young measures $\widetilde{\mu}^n$, in order to show their almost sure vague convergence, one only has to show that for each compactly supported continuous function ϕ, one has the almost sure convergence of the sequence $(\langle\widetilde{\mu}^n,\phi\rangle)_{n\geq 1}$, let indeed $(\phi_m)_{m\in\mathbb{N}}$ be a denumerable dense subset of $\mathscr{C}(\mathscr{X}\times\mathscr{M})$, if one has the convergence of $(\langle\widetilde{\mu}^n,\phi_m\rangle)_{n\geq 1}$ towards $\langle\mu,\phi_m\rangle$ outside a neglectable set $N_m\subset\Omega$ of $\mathscr{F}$, then for all $\omega\in\Omega\setminus\bigcup_{m\in\mathbb{N}}N_m$ one has for any continuous function ϕ on $\mathscr{X}\times\mathscr{M}$

$$\limsup_{\varepsilon\to 0} |\langle\widetilde{\mu}^n(\omega)-\mu,\phi\rangle| \leq \inf_{m\in\mathbb{N}}(2\Lambda(\mathscr{X})\|\phi-\phi_m\|_\infty) = 0.$$

Theorem 2 *One has*

$$\widetilde{\mu}^n \overset{vaguely}{\underset{a.s.}{\longrightarrow}} \mu, \text{ as } n \text{ tends to } +\infty.$$

The proof of this theorem comes from the following rate of convergence in L^p norm for the difference $\widetilde{\mu}^n-\mu$ applied to any continuous function ϕ, as is done in [MP98].

Proposition 2 *Let ϕ be any continuous function on $\mathscr{X}\times\mathscr{M}$, then for any $p\geq 1$ there exists a constant $C_{p,\phi}$ such that*

$$E\left[|\langle\widetilde{\mu}^n-\mu,\phi\rangle|^p\right] \leq C_{p,\phi}n^{-pd/2}.$$

The first step is to replace ϕ be the function ψ defined by

$$\forall(x,m)\in\mathscr{X}\times\mathscr{M},\ \psi(x,m)=\phi(x,m)-\int_{\mathscr{M}}\phi(x,q)\,d\mu x(q),$$

so that $\|\psi\|_\infty\leq 2\|\phi\|_\infty$ and the quantity to estimate is now

$$E\left[|\langle\widetilde{\mu}^n-\mu,\psi\rangle|^p\right] = E\left[\left|\sum_{i\in\mathbb{Z}^d}\int_{\mathscr{X}_i^n}\psi(x,M^n_{(i)})\,d\Lambda(x)\right|^p\right],$$

this is a power of a sum of independent centered random variables. Let us suppose that p is even, the right hand side can be expanded with a multinomial formula, and the expectations of the terms of this expansion are zero by independence as soon as there is a term to the power 1 in the product, there remains the terms of the form

$$\sum_{|k|=p,\ k_i\geq 2 \text{ or } k_i=0} E\left[\prod_{i\in\mathbb{Z}^d}\left(\int_{\mathscr{X}_i^n}\psi(x,M^n_{(i)})\,d\Lambda(x)\right)^{k_i}\right],$$

where $k=(k_i)_{i\in\mathbb{Z}^d}$ is a multi-index whose length $|k|$ is equal to $\sum_{i\in\mathbb{Z}^d}k_i$. All those terms are uniformly bouded by $C\|\psi\|_\infty^p n^{-pd}$, and the most numerous terms are those for which the k_i's are either 0 or 2, the number of which can be estimated by $n^{pd/2}$, so that one has

$$E\left[|\langle\widetilde{\mu}^n-\mu,\psi\rangle|^p\right] \leq Bn^{-pd/2},$$

and the estimate of the theorem is obtained for even p's. The general estimate for p is obtained by using Hölder's inequality for $p \leq 2k$ with exponents $2k/p$ and $(2k/p)'$.

From proposition 2 it is easy to deduce theorem 2: for each positive η one has

$$P\left(|\langle \widetilde{\mu}^n - \mu, \psi\rangle| > \eta\right) \quad \leq \quad \frac{B}{\eta^p n^{pd/2}},$$

if one takes p sufficiently large ($p > 2/d$) Borel-Cantelli's lemma gives the desired result as the sum of those terms is finite.

The natural correspondence between intensity measures and point processes is summarised in the following result [Nev77]:

Theorem 3 *Let $(\mu_m)_{m\in\mathbb{N}}$ be a sequence of Λ-based Young measures on $\mathscr{X} \times \mathscr{M}$ vaguely converging towards μ, then the associated marked Poisson Point Processes $(\mathbf{X}_m)_{m\in\mathbb{N}}$ converge in law towards $\mathbf{X}$.*

One can easily obtain coupling results for the point processes associated to the discretised Young measures (note that this coupling ignores the notion of Young measure), with obvious notations one has

Proposition 3 *There exists a coupling of $(\overline{\mathbf{X}}^n)_{n\geq 1}$ and $\mathbf{X}$ marked Poisson processes with intensity measures $(\overline{\mu}^n)$ and μ such that $\overline{\mathbf{X}}^n$ converges almost surely towards $\mathbf{X}$. Furthermore there exists a coupling of $(\overline{\mathbf{X}}^n)_{n\geq 1}$ and $(\widetilde{\mathbf{X}}^n)_{n\geq 1}$ and a constant K such that*

$$P(\widetilde{\mathbf{X}}^n \neq \overline{\mathbf{X}}^n) \leq K n^{-d}.$$

Let us describe those couplings: all the processes are based on the same source Poisson Point Process $\mathbf{Y}$ with intensity measure Λ on $\mathscr{X}$, this process is classically defined as the identity mapping on

$$\Omega_{\mathscr{X}} = \bigcup_{m\geq 0} (\mathscr{X}, \mathscr{B}(\mathscr{X}), \Lambda/\Lambda(\mathscr{X}))^m,$$

endowed with the probability measure

$$P_{\mathscr{X}} = \sum_{m\geq 0} \exp(-\Lambda(\mathscr{X}))\frac{1}{m!}\,\Lambda^{\otimes m}.$$

Since the support $\{x_1, \ldots, x_m\} \subset \mathscr{X}$ of this point process is of Lebesgue measure 0, one may use the following elementary lemma (Lebesgue points):

Lemma 1 *Let $C_n(x)$ denote the cube containing $x \in \mathscr{X}$ with side of length $1/n$ and vertices on the lattice $(1/n)\mathbb{Z}^d$, then one has*

$$\frac{1}{\Lambda(C_n(x)\cap\mathscr{X})}\int_{C_n(x)\cap\mathscr{X}} \mu_z\, d\Lambda(z) \overset{vaguely}{\longrightarrow} \mu_x, \ \mathscr{L}_d \text{ almost everywhere.}$$

Let us denote by $N \subset \mathscr{X}$ the set of points where the conclusion of the lemma does not hold. If we denote for $x \notin N$ by $M_x^{(n)}$ a random variable with law $\Lambda(C_n(x)\cap\mathscr{X})^{-1}\int_{C_n(x)\cap\mathscr{X}} \mu_z\, d\Lambda(z)$, and M_x with law μ_x, then there exists a probability space $(\Omega_x, \mathscr{F}_x, \mathbf{P}_x)$ such that one has the almost sure convergence of the random variables $M_x^{(n)}$ towards M_x on this probability space. For the points in N take any probability space $(\Omega_x, \mathscr{F}_x, \mathbf{P}_x)$. Then consider the product probability space

$$\begin{aligned}\Omega &= \Omega_{\mathscr{X}} \times \prod_{x\in\mathscr{X}} \Omega_x,\\ \mathbf{P} &= P_{\mathscr{X}} \otimes \bigotimes_{x\in\mathscr{X}} \mathbf{P}_x,\end{aligned}$$

and the random variables defined for $\omega \in \Omega$

$$\begin{aligned}
\omega &= (x_1, \ldots, x_m, (\omega_x)_{x\in\mathscr{X}}), \\
\overline{\mathbf{X}}^n(\omega) &= \left\{(x_i, M_{x_i}^{(n)}(\omega_{x_i})),\ i = 1, \ldots, m\right\}, \\
\mathbf{X}(\omega) &= \{(x_i, M_{x_i}(\omega_{x_i})),\ i = 1, \ldots, m\}.
\end{aligned}$$

Then one checks easily the almost sure convergence of those random variables.

Now if we denote by $Z(x, n)$ the center of $C_n(x)$, one defines $\widetilde{\mathbf{X}}^n$ and a new version of $\overline{\mathbf{X}}^n$ on the same probability space by

$$\begin{aligned}
\overline{\mathbf{X}}^n(\omega) &= \left\{(x_i, M_{Z(x_i,n)}^{\{j(x_i\}})(\omega_{Z(x_i,n)})),\ i = 1, \ldots, m\right\}, \\
\widetilde{\mathbf{X}}^n(\omega) &= \left\{(x_i, M_{Z(x_i,n)}^{\{0\}}(\omega_{Z(x_i,n)})),\ i = 1, \ldots, m\right\},
\end{aligned}$$

where $j(x_i)$ is the number of points x_j up to x_{i-1} belonging to the same cube as x_i, and $(M_{Z(x,n)}^{\{m\}})_{m\in\mathbb{N}}$ are independent identically distributed random variables with common law $\Lambda(C_n(x) \cap \mathscr{X})^{-1} \int_{C_n(x)\cap\mathscr{X}} \mu_z \, d\Lambda(z)$. Obviously the probability that two points belong to the same cube is of order less than n^{-d}, so that with probability greater than $1 - O(n^{-d})$ those two processes coincide.

2.4 Central limit theorems

As in [MP98] one can obtain central limit theorems for sequences $(\widetilde{\mu}^n)_{n\geq 1}$:

Theorem 4 *Let ϕ be a continuous function on $\mathscr{X} \times \mathscr{M}$, one has*

$$n^{d/2} \langle \widetilde{\mu}^n - \mu, \phi\rangle \underset{n\to+\infty}{\overset{law}{\longrightarrow}} \mathscr{N}(0, \sigma_\mu^2(\phi)),$$

where $\mathscr{N}(0, \sigma_\mu^2(\phi))$ denotes the Gaussian centered random variable with variance

$$\sigma_\mu^2(\phi) = \int_{\mathscr{X}} \int_{\mathscr{M}} \left(\phi(x, m) - \int_{\mathscr{M}} \phi(x, q)\, d\mu_x(q)\right)^2 d\mu_x(m)\, \lambda(x)\, d\Lambda(x).$$

The proof of this theorem is similar to the one in [MP98], except for the variance, where the density λ appears because of the choice of the regular discretisation of space into cubes, and not cells of constant Λ measure.

Let us use once again the function ψ defined by

$$\psi(x, m) = \phi(x, m) - \int_{\mathscr{M}} \phi(x, n)\, d\mu_x(n),$$

so that

$$\langle \widetilde{\mu}^n - \mu, \phi\rangle = \langle \widetilde{\mu}^n, \psi\rangle.$$

One has

$$n^{d/2}\langle \widetilde{\mu}^n, \psi\rangle = \sum_{i\in\mathbb{Z}^d} \underbrace{n^{d/2} \int_{\mathscr{X}_i^n} \psi(z, M_n^{(i)})\, d\Lambda(z)}_{X_{n,i}(\psi)},$$

which is a sum of independent, uniformely bounded by $n^{-d/2}$, centered random variables, the number N_n of which is of order n^d.
Let us compute the variance of those random variables: let us introduce the discretised function

$$\psi_{\mathscr{X}_i^n}(x,m) = \frac{1}{\Lambda(\mathscr{X}_i^n)} \int_{\mathscr{X}_i^n} \psi(z,m)\, d\Lambda(z),\ \forall (x,m) \in \mathscr{X}_i^n \times \mathscr{M}.$$

Then

$$|X_{n,i}(\psi) - X_{n,i}(\psi_{\mathscr{X}_i^n})| \leq n^{d/2}\Lambda(\mathscr{X}_i^n)\omega_\psi(\sqrt{d}/n),$$

where ω_ψ is the continuity modulus of ψ. Thanks to the fact that $E[X_{n,i}(\psi)] = 0$ one has

$$\begin{aligned}
\mathrm{var}(X_{n,i}(\psi)) &= E[X_{n,i}(\psi)^2], \\
&= E[X_{n,i}(\psi_{\mathscr{X}_i^n})^2] + o_\psi(n^{-d}), \\
&= n^d \Lambda(\mathscr{X}_i^n)^2 E\left[\left(\psi_{\mathscr{X}_i^n}(M_n^{(i)})\right)^2\right] + o_\psi(n^{-d}),
\end{aligned}$$

One is in the context of a variant of the Lindeberg central limit theorem ([Bil95], Theorem 27.2), if we denote by s_n^2 the sum of those variances, one has:

$$\begin{aligned}
s_n^2 &= \sum_{i\in\mathbb{Z}^d} \left(n^d \Lambda(\mathscr{X}_i^n)^2 E\left[\left(\psi_{\mathscr{X}_i^n}(M_n^{(i)})\right)^2\right] + o_\psi(n^{-d})\right), \\
&= \sum_{i\in\mathbb{Z}^d} \frac{\Lambda(\mathscr{X}_i^n)}{n^{-d}} \left(\Lambda(\mathscr{X}_i^n) E\left[\left(\psi_{\mathscr{X}_i^n}(M_n^{(i)})\right)^2\right]\right) + o_\psi(1), \\
&= \sum_{i\in\mathbb{Z}^d} \int_{\mathscr{X}_i^n} \left(\int_{\mathscr{M}} (\psi_{\mathscr{X}_i^n}(m))^2\, d\mu_x(m)\right) \frac{\Lambda(\mathscr{X}_i^n)}{n^{-d}}\, d\Lambda(x) + o_\psi(1), \\
&\to_{n\to+\infty} \int_{\mathscr{X}} \int_{\mathscr{M}} \psi(x,m)^2\, d\mu_x(m)\, \lambda(x)\, d\Lambda(x),
\end{aligned}$$

where this last convergence is achieved thanks to the fact that the boundary terms, which are not exactly a discretisation of this integral (whereas the interior terms fit exactly), are few since the boundary is regular. The other hypothesis for the Lindeberg theorem is achieved as for instance

$$\begin{aligned}
\frac{\sum_{i\in\mathbb{Z}^d} E[X_{n,i}^{2+\delta}]}{s_n^{2+\delta}} &\leq \frac{N_n (n^{-d/2})^{2+\delta}}{s_n^{2+\delta}},\ \delta > 0, \\
&\to_{n\to+\infty} 0,
\end{aligned}$$

so that one may conclude that

$$\frac{n^{d/2}\langle \widetilde{\mu}^n, \psi\rangle}{s_n} \underset{n\to+\infty}{\overset{\text{law}}{\longrightarrow}} \mathscr{N}(0,1).$$

One just has then to use the asymptotics of s_n to conclude the proof of the central limit theorem 4.

3 Large deviations and concentration

3.1 Large deviations for the discretised Young measures

Random Young measures are infinite dimensional random variables so that in order to prove large deviations properties we will use the abstract Gärtner-Ellis theorem [DZ98] or its version known as Baldi's theorem, first used for stochastic homogenisation in [Bal88]. The starting point is the computation of the log-Laplace

transform: let ϕ be a continuous function on $\mathscr{X} \times \mathscr{M}$, then if one defines a discretisation of the function ϕ on each $\mathscr{X}_i^n$ by

$$\phi_n(x,m) = \frac{1}{\Lambda(\mathscr{X}_i^n)} \int_{\mathscr{X}_i^n} \phi(y,m)\, d\Lambda(y), \forall x \in \mathscr{X}_i^n,$$

then

$$\begin{aligned} E\left[\exp\langle n^d \widetilde{\mu}^n, \phi\rangle\right] &= E\left[\exp\langle n^d \widetilde{\mu}^n, \phi_n\rangle + o_\phi(n^d)\right], \\ &= \prod_{i\in\mathbb{Z}^d} E\left[\exp\left(n^d \int_{\mathscr{X}_i^n} \phi_n(x, M_n^{(i)})\, d\Lambda(x)\right)\right] \times \exp(o_\phi(n^d)), \\ &= \prod_{i\in\mathbb{Z}^d} E\left[\exp\left(\Lambda(\mathscr{X}_i^n) n^d \phi_n(Z_{i,n}, M_n^{(i)})\right)\right] \times \exp(o_\phi(n^d)), \end{aligned}$$

where $Z_{i,n}$ denotes any point in $\mathscr{X}_i^n$. By taking the logarithm of this expression, one obtains

$$\begin{aligned} & n^{-d} \log E\left[\exp\langle n^d \widetilde{\mu}^n, \phi\rangle\right] \\ = \quad & n^{-d} \sum_{i\in\mathbb{Z}^d} \log E\left[\exp\left(\Lambda(\mathscr{X}_i^n) n^d \phi_n(Z_{i,n}, M_n^{(i)})\right)\right] + o_\phi(1), \\ \to_{n\to+\infty} \quad & \int_{\mathscr{X}} \log \int_{\mathscr{M}} \exp\left(\phi(x,m)\lambda(x)\right)\, d\mu_x(m)\, \lambda(x)^{-1}\, d\Lambda(x). \end{aligned}$$

If one denotes this limit by $\ell_\mu(\phi)$, one has:

Proposition 4 *The function ℓ_μ is finite convex and differentiable on the set of continuous functions on $\mathscr{X} \times \mathscr{M}$, and its Legendre transform is given by*

- *if ν is not absolutely continuous with respect to μ or if ν is not a Young measure with base Λ, then*

$$I_\mu(\nu) = +\infty,$$

- *otherwise*

$$I_\mu(\nu) = \int_{\mathscr{X}} \int_{\mathscr{M}} \log \frac{d\nu_x}{d\mu_x}(m)\, d\nu_x(m)\, dx.$$

Hence using for instance Baldi's theorem [Bal88] one has the following result:

Theorem 5 *The sequence of random Young measures $(\widetilde{\mu}^n)_{n\geq 1}$ satisfies a large deviation principle with respect to the vague/narrow topology in the space $M_b(\mathscr{X} \times \mathscr{M})$ of bounded measures on $\mathscr{X} \times \mathscr{M}$ with speed n^d and rate function I_μ.*

3.2 Proof of proposition 4

The proof follows the lines of [MP98] with only minor changes. The regularity and boundedness of ℓ_μ are straightforward. Recall that its Legendre transform $I(\nu)$ is defined by

$$I_\mu(\nu) = \sup_\phi \left(\langle \nu, \phi\rangle - \ell_\mu(\phi)\right),$$

where the supremum is taken on all continuous functions ϕ on $\mathscr{X} \times \mathscr{M}$. Then one checks that:

1. if ν is not a non negative measure $I_\mu(\nu) = +\infty$, as there exists a function ϕ which is non positive with $\langle \nu, \phi\rangle \geq 0$ so that $\ell_\mu(\phi) \leq 0$. Then $I_\mu(t\phi) \geq t\langle \nu, \phi\rangle$ which tends to $+\infty$ with t.

2. If $I_\mu(\nu)$ is finite, so must be ν. One checks easily that for all continuous function ϕ on $\mathscr{X}$ and real number t,

$$\begin{aligned} I_\mu(\nu) &\geq \langle \nu, t\phi \otimes \mathbf{1}\rangle - \ell(t\phi \otimes \mathbf{1}), \\ &\geq t\left(\langle \nu, \phi \otimes \mathbf{1}\rangle - \int_{\mathscr{X}} \phi(x)\, d\Lambda(x)\right), \end{aligned}$$

so that

$$\langle \nu, \phi \otimes \mathbf{1}\rangle - \int_{\mathscr{X}} \phi(x)\, d\Lambda(x) = 0,$$

using [Jiř59] one sees that this is equivalent to ν being a Λ-based Young measure.

3. Let us recall that the Kullback information for probability measures on $\mathscr{M}$ is given by

$$\begin{aligned} \mathbf{K}(\mu|\pi) &= \int_{\mathscr{M}} \log \frac{d\mu}{d\nu}(m)\, d\mu(m), \\ &= \sup_\psi \left(\langle \mu, \phi\rangle - \log \int_{\mathscr{M}} \exp \psi(m)\, d\pi(m)\right), \end{aligned}$$

where the supremum is taken either over all continuous functions ψ on $\mathscr{M}$ or over all bounded measurable functions. Then

$$\begin{aligned} \langle \nu, \phi\rangle - \ell_\mu(\phi) &= \int_{\mathscr{X}} \Big\{ \int_{\mathscr{M}} \phi(x,m)\lambda(x)\, d\nu_x(m) \\ &\quad - \log\left[\int_{\mathscr{M}} \exp\left(\phi(x,m)\lambda(x)\right)\, d\mu_x(m)\right]\Big\} \frac{1}{\lambda(x)}\, d\Lambda(x), \\ &\leq \int_{\mathscr{X}} \mathbf{K}(\nu_x|\mu_x)\, dx, \end{aligned}$$

the inverse inequality comes from Jensen's inequality:

$$\begin{aligned} \ell_\mu(\phi) &= \int_{\mathscr{X}} \log\left[\int_{\mathscr{M}} \exp\left(\phi(x,m)\lambda(x)\right)\, d\mu_x(m)\right] dx, \\ &\leq \mathscr{L}_d(\mathscr{X}) \log\left(\frac{1}{\mathscr{L}_d(\mathscr{X})} \int_{\mathscr{X}} \int_{\mathscr{M}} \exp\left(\phi(x,m)\lambda(x)\right)\, d\mu_x(m)\, dx\right), \end{aligned}$$

so that

$$\begin{aligned} \langle \nu, \phi\rangle - \ell_\mu(\phi) &\geq \mathscr{L}_d(\mathscr{X}) \int_{\mathscr{X}} \langle \nu_x, \phi(x,\cdot)\rangle \frac{1}{\mathscr{L}_d(\mathscr{X})}\, d\Lambda(x) - \\ &\quad \mathscr{L}_d(\mathscr{X}) \log\left(\frac{1}{\mathscr{L}_d(\mathscr{X})} \int_{\mathscr{X}} \int_{\mathscr{M}} \exp\left(\phi(x,m)\lambda(x)\right)\, d\mu_x(m)\, dx\right), \\ &\geq \mathscr{L}_d(\mathscr{X}) \Big[\int_{\mathscr{X}} \langle \nu_x, \lambda(x)\phi(x,\cdot)\rangle \frac{dx}{\mathscr{L}_d(\mathscr{X})} - \\ &\quad \log\left(\int_{\mathscr{X}} \int_{\mathscr{M}} \exp\left(\phi(x,m)\lambda(x)\right)\, d\mu_x(m) \frac{dx}{\mathscr{L}_d(\mathscr{X})}\right)\Big], \end{aligned}$$

and by taking the supremum over all bounded measurable functions ϕ one has

$$\sup_\phi \left(\langle \nu, \phi\rangle - \ell_\mu(\nu)\right) \geq \mathscr{L}_d(\mathscr{X}) \mathbf{K}(\nu'|\mu'),$$

where the measure ν' and μ' are the probability measures on $\mathscr{X} \times \mathscr{M}$ given by

$$\langle \mu', \phi\rangle = \left\langle \mu, (\mathscr{L}_d(\mathscr{X})\lambda)^{-1}\phi\right\rangle, \text{ and } \langle \nu', \phi\rangle = \left\langle \nu, (\mathscr{L}_d(\mathscr{X})\lambda)^{-1}\phi\right\rangle.$$

Hence this lower bound becomes

$$\begin{aligned}
& \mathscr{L}_d(\mathscr{X}) \int_{\mathscr{X}\times\mathscr{M}} \log \frac{d\nu'}{d\mu'}(x,m)\, d\nu'(x,m) \\
= \; & \mathscr{L}_d(\mathscr{X}) \int_{\mathscr{X}\times\mathscr{M}} \log \frac{d\nu_x}{d\mu_x}(m)\, d\mu_x(m)\, \frac{1}{\Lambda(\mathscr{X})\lambda(x)}\, d\Lambda(x), \\
= \; & \int_{\mathscr{X}} \mathbf{K}(\nu_x|\mu_x)\, dx,
\end{aligned}$$

so that one obtains the desired result by combining the two bounds.

Remark 3 *We have thoroughly used the fact that the supremum defining the Legendre transform may be taken with measurable functions such as $\lambda\phi$ (see lemma 6.2.13 in [DZ98]) as we are in a finite measure space.*

3.3 Concentration principle

The large deviation property implies a concentration property as may be seen in [MR94]. Let us recall the meaning of such a property in our context:

Proposition 5 *Let $\mathscr{E}$ be a closed subset of the set $M_b(\mathscr{X}\times\mathscr{M})$ of non negative bounded measures on $\mathscr{X}\times\mathscr{M}$, and set $\mathscr{E}^\star$ the subset of $\mathscr{E}$ where I_μ achieves its minimum value, then*

1. *for any open neighbourhood W' of 0 in $M_b(\mathscr{X}\times\mathscr{M})$,*

$$\liminf_{n\to+\infty} n^{-d} \log \mathbf{P}(\widetilde{\mu}^n \in \mathscr{E} + W') > -\infty,$$

2. *for any open neighbourhood $W^\star$ of 0 in $M_b(\mathscr{X}\times\mathscr{M})$, there exist $\alpha > 0$ and W open neighbourhood of 0 such that*

$$\forall W', \; \frac{\mathbf{P}(\widetilde{\mu}^n \in (\mathscr{E}+W)\setminus(\mathscr{E}^\star+W^\star))}{\mathbf{P}(\widetilde{\mu}^n \in (\mathscr{E}+W'))} \le \exp\left(-n^d\alpha\right),$$

for n large enough.

Roughly speaking, this means that when n tends to infinity, under some set of constraints on the realisations of the Young measure, one has the concentration of the measure around a measure maximizing an entropy. The laws of the marks tend to fit that of maximum entropy, those measures are the Gibbs states associated with the constraints.

The proof of this proposition follows exactly the lines of [MR94].

4 Maximum of entropy states for the Boolean model

The aim of this section is now to apply the concentration result to the initial problem of finding the most probable state for the local laws of the radius in the Boolean model to satisfy the desired constraints. This procedure will be achieved thanks to proposition 5 by maximizing the entropy under the constraints. In this context the entropy of a Λ-based Young measure ν will be finite only if one has the absolute continuity of ν_x with respect to μ_x for Λ almost every point $x \in \mathscr{X}$. If we denote by $\rho(x,\cdot)$ its density, finding the Gibbs state amounts to maximizing

$$I_\mu(\nu) = \int_{\mathscr{X}} \int_{\mathscr{M}} \rho(x.m) \log \rho(x,m)\, d\mu_x(m)\, dx,$$

under the constraints $\rho\mu \in \mathscr{C}$, and

$$\int_{\mathscr{M}} \rho(x,m)\, d\mu_x(m) = 1, \text{ for } \Lambda \text{ almost every } x.$$

This can be achieved as in [Rob91] by Lagrange multipliers (even if the number of constraints is infinite).

We recall in a few lines the context: $\mathscr{X} = D \times [-\text{Depth}, 0]$ is a compact subset of $\mathbb{R}^3$, the set of marks is a compact set $[0, R_0] \subset \mathbb{R}_+$ and the constraints are

$$\mathscr{C} = \{P((x_i, 0) \in \text{Oil}) = q_i,\ i = 1, \dots, N\},$$

where x_i is the location of the i-th well, and q_i the actual measure of the outcome of this well. This set of constraint rewrites as

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \mathbf{1}_{B(y,R)}(x_i, 0)\, d\mu_y(R)\, d\Lambda(y) = -\log(1 - q_i),\ i = 1, \dots, N,$$

where the *a priori* law of the radius of a ball centered at y is μ_y, so that one has to maximize the entropy

$$I_\mu(\nu) = \int_{D\times[-\text{Depth},0]} \int_0^{R_0} \rho(x,m) \log \rho(x,m)\, d\mu_x(m)\, dx,$$

subject to

$$\begin{aligned} \int_0^{R_0} \rho(x,m)\, d\mu_x(m) &= 1, \text{ for } \Lambda \text{ almost every } x, \\ \int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \mathbf{1}_{B(y,R)}(x_i, 0) \rho(y,R)\, d\mu_y(R)\, d\Lambda(y) &= -\log(1 - q_i),\ i = 1, \dots, N. \end{aligned}$$

This gives the following solution:

$$\rho^\star(y,R) = \frac{1}{Z_\beta(y)} \exp\left(-\sum_{i=1}^N \beta_i \mathbf{1}_{B(y,R)}(x_i, 0) \right),$$

where $Z_\beta(x)$ is the partition function

$$Z_\beta(x) = \int_0^{R_0} \exp\left(-\sum_{i=1}^N \beta_i \mathbf{1}_{B(y,R)}(x_i, 0) \right) d\mu_y(R),$$

and $\beta_1, \dots, \beta_N$ are the Lagrange multipliers associated to the constraints at $x_1, \dots, x_N$.
The computation of the Lagrange multipliers comes from the inverse problem: given $\beta_1, \dots, \beta_N$ the computation of the constraints is given by the following numerical integration:

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}_+} \mathbf{1}_{B(y,R)}(x_i, 0) \frac{1}{Z_\beta(y)} \exp\left(-\sum_{j=1}^N \beta_j \mathbf{1}_{B(y,R)}(x_j, 0) \right) d\mu_y(R)\, d\Lambda(y) = c_i,$$

so that one can deduce q_i from this integral. The resolution of the inverse problem would give the Lagrange multipliers as functions of the q_i's.

5 Numerical results

Let us start this section with a remark: in the introduction we saw that the probability to be in the occupied phase was the same according to whether we were talking about the 3-dimensional Boolean model or the 2-dimensional one: this enables us to reduce the dimension of the problem by one and to consider the problem in a compact subset of $\mathbb{R}^2$. We will use the following notations:

- $\lambda > 0$ is the constant intensity measure of the point process,
- μ_0 is the law of the radius on $[0, R_0]$.

The transformation from 3D to 2D gives the following intensity:

- $\tilde{\lambda} = 2E_{\mu_0}[R]\lambda$ is the intensity in the plane,
- $\tilde{\mu}_0$ the induced law of the radius is given by:

$$\tilde{\mu}_0([\tilde{R}, R_0]) = \frac{1}{2E_{\mu_0}[R]} \int_{\tilde{R}}^{R_0} \sqrt{R^2 - \tilde{R}^2}\, d\mu_0(R).$$

5.1 Algorithms for the determination of the Gibbs states

The first question is to give an algorithm for the direct problem: given the Lagrange multipliers can we compute easily the probability to be in the occupied region?
It seems that the most tractable algorithm, in absence of an exact integration which could be possible for toy models only, is to use Monte-Carlo techniques.
Let $x_1, \ldots, x_N$ be the positions in $\mathbb{R}^2$ of the wells, and set $G = \{x_1, \ldots, x_N\} \oplus B(0, R_0)$, alternatively denote by $G(x)$ the set $G \cup \{x\} \oplus B(0, R_0)$ for $x \in \mathbb{R}^2$. The integral to compute is either

$$\tilde{\lambda} \int_{\mathbb{R}^2} \int_{\mathbb{R}_+} \mathbf{1}_{|x_i - y| \le R} \frac{1}{Z_\beta(y)} \exp\left(-\sum_{j=1}^N \beta_j \mathbf{1}_{|x_j - y| \le R} \right) d\tilde{\mu}_0(R)\, dy,$$

$$\text{or} \qquad \tilde{\lambda} \int_{\mathbb{R}^2} \int_{\mathbb{R}_+} \mathbf{1}_{|x - y| \le R} \frac{1}{Z_\beta(y)} \exp\left(-\sum_{j=1}^N \beta_j \mathbf{1}_{|x_j - y| \le R} \right) d\tilde{\mu}_0(R)\, dy.$$

To compute the approximate probability that the point $z = x_i$ (resp. x) lies in the occupied phase one will perform a Monte-Carlo simulation on G (resp. $G(x)$): let $X_1, \ldots, X_M$ be M independent uniformly distributed points on this set (obtained for instance by a rejection method).

$$\begin{aligned} -\log(1 - q(z)) &= \tilde{\lambda} \int_{\mathbb{R}^2} \int_{\mathbb{R}_+} \mathbf{1}_{|z - y| \le R} \frac{1}{Z_\beta(y)} \exp\left(-\sum_{j=1}^N \beta_j \mathbf{1}_{|x_j - y| \le R} \right) d\tilde{\mu}_0(R)\, dy \\ &\simeq \frac{\tilde{\lambda} \mathscr{L}_2(G(x))}{M} \sum_{k=1}^M \mathbf{1}_{|z - X_k| \le R_k} \frac{1}{\tilde{Z}_\beta(X_k)} \exp\left(-\sum_{j=1}^N \beta_j \mathbf{1}_{|x_j - X_k| \le R_k} \right), \end{aligned}$$

where $R_1, \ldots, R_M$ are sampled according to $\tilde{\mu}_0$ and $\tilde{Z}_\beta(X_k)$ is a Monte-Carlo approximation of $Z_\beta(X_k)$. The proposed estimated probability $\hat{q}(z)$ is then given by

$$\hat{q}(z) \simeq 1 - \exp\left(-\frac{\tilde{\lambda} \mathscr{L}_2(G(x))}{M} \sum_{k=1}^M \mathbf{1}_{|z - X_k| \le R_k} \frac{1}{\tilde{Z}_\beta(X_k)} \exp\left(-\sum_{j=1}^N \beta_j \mathbf{1}_{|x_j - X_k| \le R_k} \right) \right).$$

The second question is to solve the inverse problem and deduce the correct Lagrange multipliers from the observed occupation probabilies: we have not yet found how to manage this problem in an efficient way.

5.2 Numerical results

We give below a numerical result for the toy model in one space dimension and for the exponential law μ_0. The choice of this law, though contradicting hypothesis (**H2**) gives closed formulæ for the partition function and can be easily implemented. With two points one obtains the following results:

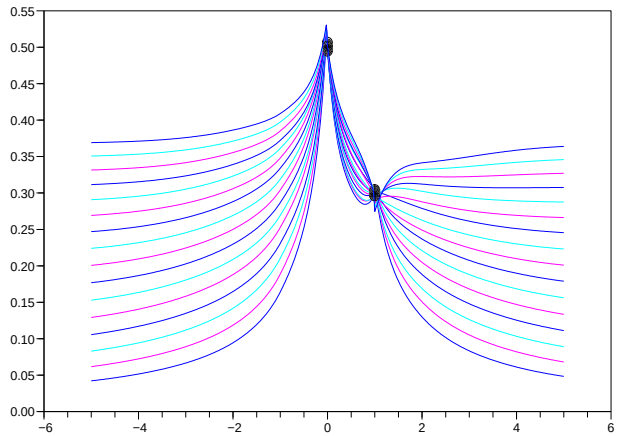

Figure 1: Constraints at points 0 and 1 for an exponential law of radius, with different parameters ranging from 0.5 (bottom) to 2 (top)

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