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ON THE CONTINUITY SET OF AN OMEGA RATIONAL FUNCTION

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Olivier Carton[1], Olivier Finkel[2] and Pierre Simonnet[3]

Abstract. In this paper, we study the continuity of rational functions realized by Büchi finite state transducers. It has been shown by Prieur that it can be decided whether such a function is continuous. We prove here that surprisingly, it cannot be decided whether such a function f has at least one point of continuity and that its continuity set $C(f)$ cannot be computed.

In the case of a synchronous rational function, we show that its continuity set is rational and that it can be computed. Furthermore we prove that any rational $\mathbf{\Pi}^0_2$-subset of Σ^ω for some alphabet Σ is the continuity set $C(f)$ of an ω-rational synchronous function f defined on Σ^ω.

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[1] LIAFA, Université Paris 7 et CNRS
2 Place Jussieu 75251 Paris cedex 05, France.
e-mail: `Olivier.Carton@liafa.jussieu.fr`

[2] Equipe Modèles de Calcul et Complexité
Laboratoire de l'Informatique du Parallélisme
(UMR 5668 - CNRS - ENS Lyon - UCB Lyon - INRIA)
CNRS et Ecole Normale Supérieure de Lyon
46, Allée d'Italie 69364 Lyon Cedex 07, France.
e-mail: `Olivier.Finkel@ens-lyon.fr`

[3] UMR 6134-Systèmes Physiques de l'Environnement
Faculté des Sciences, Université de Corse
Quartier Grossetti BP52 20250, Corte, France
e-mail: `simonnet@univ-corse.fr`.

1. Introduction

Acceptance of infinite words by finite automata was firstly considered in the sixties by Büchi in order to study decidability of the monadic second order theory of one successor over the integers [Büc62]. Then the so called ω-regular languages have been intensively studied and many applications have been found. We refer the reader to [Tho90, Sta97, PP04] for many results and references.

Gire and Nivat studied infinitary rational relations accepted by Büchi transducers in [Gir81, GN84]. Infinitary rational relations are subsets of $\Sigma^\omega \times \Gamma^\omega$, where Σ and Γ are finite alphabets, which are accepted by 2-tape finite Büchi automata with two asynchronous reading heads. They have been much studied, in particular in connection with the rational functions they may define, see for example [CG99, BCPS00, Sim92, Sta97, Pri00].

Gire proved in [Gir83] that one can decide whether an infinitary rational relation $R \subseteq \Sigma^\omega \times \Gamma^\omega$ recognized by a given Büchi transducer $\mathcal{T}$ is the graph of a function $f : \Sigma^\omega \to \Gamma^\omega$, (respectively, is the graph of a function $f : \Sigma^\omega \to \Gamma^\omega$ recognized by a synchronous Büchi transducer). Such a function is called an ω-rational function (respectively, a synchronous ω-rational function).

The continuity of ω-rational functions is an important issue since it is related to many aspects. Let us mention two of them. First, sequential functions that may be realized by input deterministic automata are continuous but the converse is not true. Second, continuous functions define a reduction between subsets of a topological space that yields a hierarchy called the Wadge hierarchy. The restriction of this hierarchy to rational sets gives the Wagner hierarchy.

This paper is focused on the continuity sets of rational functions. Prieur proved in [Pri00, Pri01] that it can be decided whether a given ω-rational function is continuous. This means that it can be decided whether the continuity set is equal to the domain of the function. We show however that it cannot be decided whether a rational function has at least one point of continuity. We show that in general the continuity set of a rational function is not rational and even not context-free. Furthermore, we prove that it cannot be decided whether this continuity set is rational.

We pursue this study with synchronous rational functions. These functions are accepted by Büchi transducers in which the two heads move synchronously. Contrary to the general case, the continuity set of synchronous rational function is always rational and it can be effectively computed. We also give a characterization of continuity sets of synchronous functions. It is well known that a continuity set is a $\mathbf{\Pi}^0_2$-set. We prove conversely that any rational $\mathbf{\Pi}^0_2$-set is the continuity set of some synchronous rational function.

The paper is organized as follows. In section 2 we recall the notions of infinitary rational relation, of ω-rational function, of synchronous or asynchronous ω-rational function, of topology and continuity; we recall also some recent results on the topological complexity of infinitary rational relations. In section 3 we study the continuity sets of ω-rational functions in the general case, stating some undecidability results. Finally we study the case of synchronous ω-rational functions in section 4.

2. Recall on ω-rational functions and topology

2.1. Infinitary rational relations and ω-rational functions

Let Σ be a finite alphabet whose elements are called letters. A non-empty finite word over Σ is a finite sequence of letters: $x = a_1 a_2 \ldots a_n$ where $\forall i \in [1; n]$ $a_i \in \Sigma$. We shall denote $x(i) = a_i$ the i^{th} letter of x and $x[i] = x(1) \ldots x(i)$ for $i \leq n$. The length of x is $|x| = n$. The empty word will be denoted by λ and has 0 letter. Its length is 0. The set of finite words over Σ is denoted $\Sigma^\star$. $\Sigma^+ = \Sigma^\star - \{\lambda\}$ is the set of non empty words over Σ. A (finitary) language L over Σ is a subset of $\Sigma^\star$. The usual concatenation product of u and v will be denoted by $u.v$ or just uv. For $V \subseteq \Sigma^\star$, we denote by $V^\star$ the set $R = \{v_1 \ldots v_n \mid n \in \mathbb{N}$ and $\forall i \in [1; n]$ $v_i \in V\}$.

The first infinite ordinal is ω. An ω-word over Σ is an ω -sequence $a_1 a_2 \ldots a_n \ldots$, where for all integers $i \geq 1$ $a_i \in \Sigma$. When σ is an ω-word over Σ, we write $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)\ldots$ and $\sigma[n] = \sigma(1)\sigma(2)\ldots\sigma(n)$ the finite word of length n, prefix of σ. The set of ω-words over the alphabet Σ is denoted by Σ^ω. An ω-language over an alphabet Σ is a subset of Σ^ω. For $V \subseteq \Sigma^\star$, $V^\omega = \{\sigma = u_1 \ldots u_n \ldots \in \Sigma^\omega \mid \forall i \geq 1$ $u_i \in V\}$ is the ω-power of V. The concatenation product is extended to the product of a finite word u and an ω-word v: the infinite word $u.v$ is then the ω-word such that: $(u.v)(k) = u(k)$ if $k \leq |u|$, and $(u.v)(k) = v(k - |u|)$ if $k > |u|$.

We assume the reader to be familiar with the theory of formal languages and of ω-regular languages, see [Büc62, Tho90, EH93, Sta97, PP04] for many results and references. We recall that ω-regular languages form the class of ω-languages accepted by finite automata with a Büchi acceptance condition and this class, denoted by RAT, is the omega Kleene closure of the class of regular finitary languages.

We are going now to recall the notion of infinitary rational relation which extends the notion of ω-regular language, via definition by Büchi transducers:

Definition 2.1. A Büchi transducer is a sextuple $\mathcal{T} = (K, \Sigma, \Gamma, \Delta, q_0, F)$, where K is a finite set of states, Σ and Γ are finite sets called the input and the output alphabets, Δ is a finite subset of $K \times \Sigma^\star \times \Gamma^\star \times K$ called the set of transitions, q_0 is the initial state, and $F \subseteq K$ is the set of accepting states.
A computation $\mathcal{C}$ of the transducer $\mathcal{T}$ is an infinite sequence of consecutive transitions

$$(q_0, u_1, v_1, q_1), (q_1, u_2, v_2, q_2), \ldots (q_{i-1}, u_i, v_i, q_i), (q_i, u_{i+1}, v_{i+1}, q_{i+1}), \ldots$$

The computation is said to be successful iff there exists a final state $q_f \in F$ and infinitely many integers $i \geq 0$ such that $q_i = q_f$. The input word and output word of the computation are respectively $u = u_1.u_2.u_3\ldots$ and $v = v_1.v_2.v_3\ldots$ The input and the output words may be finite or infinite. The infinitary rational relation $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$ accepted by the Büchi transducer $\mathcal{T}$ is the set of couples $(u, v) \in \Sigma^\omega \times \Gamma^\omega$ such that u and v are the input and the output words of some successful computation $\mathcal{C}$ of $\mathcal{T}$. The set of infinitary rational relations will be denoted RAT_2.

If $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$ is an infinitary rational relation recognized by the Büchi transducer $\mathcal{T}$ then we denote $\mathrm{Dom}(R(\mathcal{T})) = \{u \in \Sigma^\omega \mid \exists v \in \Gamma^\omega \ (u, v) \in R(\mathcal{T})\}$ and $\mathrm{Im}(R(\mathcal{T})) = \{v \in \Gamma^\omega \mid \exists u \in \Sigma^\omega (u, v) \in R(\mathcal{T})\}$.

It is well known that, for each infinitary rational relation $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$, the sets $\mathrm{Dom}(R(\mathcal{T}))$ and $\mathrm{Im}(R(\mathcal{T}))$ are regular ω-languages.

The Büchi transducer $\mathcal{T} = (K, \Sigma, \Gamma, \Delta, q_0, F)$ is said to be synchronous if the set of transitions Δ is a finite subset of $K \times \Sigma \times \Gamma \times K$, i.e. if each transition is labelled with a pair $(a, b) \in \Sigma \times \Gamma$. An infinitary rational relation recognized by a synchronous Büchi transducer is in fact an ω-language over the product alphabet $\Sigma \times \Gamma$ which is accepted by a Büchi automaton. It is called a synchronous infinitary rational relation. An infinitary rational relation is said to be asynchronous if it can not be recognized by any synchronous Büchi transducer. Recall now the following undecidability result of C. Frougny and J. Sakarovitch.

Theorem 2.2 ([FS93]). *One cannot decide whether a given infinitary rational relation is synchronous.*

A Büchi transducer $\mathcal{T} = (K, \Sigma, \Gamma, \Delta, q_0, F)$ is said to be functional if for each $u \in \mathrm{Dom}(R(\mathcal{T}))$ there is a unique $v \in \mathrm{Im}(R(\mathcal{T}))$ such that $(u, v) \in R(\mathcal{T})$. The infinitary rational relation recognized by $\mathcal{T}$ is then a functional relation and it defines an ω-rational (partial) function $f_{\mathcal{T}} : \mathrm{Dom}(R(\mathcal{T})) \subseteq \Sigma^\omega \to \Gamma^\omega$ by:
for each $u \in \mathrm{Dom}(R(\mathcal{T}))$, $f_{\mathcal{T}}(u)$ is the unique $v \in \Gamma^\omega$ such that $(u, v) \in R(\mathcal{T})$.
An ω-rational (partial) function $f : \Sigma^\omega \to \Gamma^\omega$ is said to be synchronous if there is a synchronous Büchi transducer $\mathcal{T}$ such that $f = f_{\mathcal{T}}$.
An ω-rational (partial) function $f : \Sigma^\omega \to \Gamma^\omega$ is said to be asynchronous if there is no synchronous Büchi transducer $\mathcal{T}$ such that $f = f_{\mathcal{T}}$.

Theorem 2.3 ([Gir83]). *One can decide whether an infinitary rational relation recognized by a given Büchi transducer $\mathcal{T}$ is a functional infinitary rational relation (respectively, a synchronous functional infinitary rational relation).*

2.2. Topology

We assume the reader to be familiar with basic notions of topology which may be found in [Mos80, Kec95, LT94, Sta97, PP04]. There is a natural metric on the set Σ^ω of infinite words over a finite alphabet Σ which is called the prefix metric and defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $d(u, v) = 2^{-l_{pref(u,v)}}$ where $l_{pref(u,v)}$ is the least integer n such that the $(n+1)^{th}$ letter of u is different from the $(n+1)^{th}$ letter of v. This metric induces on Σ^ω the usual Cantor topology for which open subsets of Σ^ω are in the form $W.\Sigma^\omega$, where $W \subseteq \Sigma^\star$. Recall that a set

$L \subseteq \Sigma^\omega$ is a closed set iff its complement $\Sigma^\omega - L$ is an open set. We define now the next classes of the Borel Hierarchy:

Definition 2.4. The classes $\mathbf{\Sigma}^0_n$ and $\mathbf{\Pi}^0_n$ of the Borel Hierarchy on the topological space Σ^ω are defined as follows:

- $\mathbf{\Sigma}^0_1$ is the class of open sets of Σ^ω.
- $\mathbf{\Pi}^0_1$ is the class of closed sets of Σ^ω.

And for any integer $n \geq 1$:

- $\mathbf{\Sigma}^0_{n+1}$ is the class of countable unions of $\mathbf{\Pi}^0_n$-subsets of Σ^ω.
- $\mathbf{\Pi}^0_{n+1}$ is the class of countable intersections of $\mathbf{\Sigma}^0_n$-subsets of Σ^ω.

The Borel Hierarchy is also defined for transfinite levels: The classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$, for a non-null countable ordinal α, are defined in the following way.

- $\mathbf{\Sigma}^0_\alpha$ is the class of countable unions of subsets of Σ^ω in $\cup_{\gamma<\alpha}\mathbf{\Pi}^0_\gamma$.
- $\mathbf{\Pi}^0_\alpha$ is the class of countable intersections of subsets of Σ^ω in $\cup_{\gamma<\alpha}\mathbf{\Sigma}^0_\gamma$.

Let us recall the characterization of rational $\mathbf{\Pi}^0_2$-subsets of Σ^ω, due to Landweber [Lan69]. This characterization will be used in the proof that any rational $\mathbf{\Pi}^0_2$-subset is the continuity set of some rational synchronous function.

Theorem 2.5 (Landweber)**.** *A rational subset of Σ^ω is $\mathbf{\Pi}^0_2$ if and only if it can be recognized by a deterministic Büchi automaton.*

There are some subsets of the Cantor set, (hence also of the topological space Σ^ω, for a finite alphabet Σ having at least two elements) which are not Borel sets. There exists another hierarchy beyond the Borel hierarchy, called the projective hierarchy. The first class of the projective hierarchy is the class $\mathbf{\Sigma}^1_1$ of **analytic** sets. A set $A \subseteq \Sigma^\omega$ is analytic iff there exists a Borel set $B \subseteq (\Sigma \times Y)^\omega$, with Y a finite alphabet, such that $x \in A \leftrightarrow \exists y \in Y^\omega$ such that $(x, y) \in B$, where $(x, y) \in (\Sigma \times Y)^\omega$ is defined by: $(x, y)(i) = (x(i), y(i))$ for all integers $i \geq 1$.

Remark 2.6. An infinitary rational relation is a subset of $\Sigma^\omega \times \Gamma^\omega$ for two finite alphabets Σ and Γ. One can also consider that it is an ω-language over the finite alphabet $\Sigma \times \Gamma$. If $(u, v) \in \Sigma^\omega \times \Gamma^\omega$, one can consider this couple of infinite words as a single infinite word $(u(1), v(1)).(u(2), v(2)).(u(3), v(3))\ldots$ over the alphabet $\Sigma \times \Gamma$. Since the set $(\Sigma \times \Gamma)^\omega$ of infinite words over the finite alphabet $\Sigma \times \Gamma$ is naturally equipped with the Cantor topology, it is natural to investigate the topological complexity of infinitary rational relations as ω-languages, and to locate them with regard to the Borel and projective hierarchies. Every infinitary rational relation is an analytic set and there exist some $\mathbf{\Sigma}^1_1$-complete, hence non-Borel, infinitary rational relations [Fin03a]. The second author has recently proved the following very surprising result: infinitary rational relations have the same topological complexity as ω-languages accepted by Büchi Turing machines [Fin06b, Fin06a]. In particular, for every recursive non-null ordinal α there exist some $\mathbf{\Pi}^0_\alpha$-complete and some $\mathbf{\Sigma}^0_\alpha$-complete infinitary rational relations, and the supremum of the set of Borel ranks of infinitary rational relations is the ordinal γ^1_2. This ordinal is defined by A.S. Kechris, D. Marker, and R.L. Sami in [KMS89] and it is proved to be strictly greater than the ordinal δ_2^1 which is the first non Δ_2^1 ordinal. Thus the ordinal γ_2^1 is also strictly greater than the first non-recursive ordinal ω_1^{CK}, usually called the Church-kleene ordinal. Notice that amazingly the exact value of the ordinal γ_2^1 may depend on axioms of set theory, see [KMS89, Fin06b].

Remark 2.7. Infinitary rational relations recognized by synchronous Büchi transducers are regular ω-languages thus they are boolean combinations of $\mathbf{\Pi}_2^0$-sets hence $\mathbf{\Delta}_3^0$-sets [PP04]. So we can see that there is a great difference between the cases of synchronous and of asynchronous infinitary rational relations. We shall see in the sequel that these two cases are also very different when we investigate the continuity sets of ω-rational functions.

2.3. Continuity

We have already seen that the Cantor topology of a space Σ^ω can be defined by a distance d. We recall that a function $f : \mathrm{Dom}(f) \subseteq \Sigma^\omega \to \Gamma^\omega$, whose domain is $\mathrm{Dom}(f)$, is said to be continuous at point $x \in \mathrm{Dom}(f)$ if :

$$\forall n \geq 1 \ \exists k \geq 1 \ \forall y \in \mathrm{Dom}(f) \ [\ d(x, y) < 2^{-k} \Rightarrow d(f(x), f(y)) < 2^{-n} \ ]$$

The function f is said to be continuous if it is continuous at every point $x \in \Sigma^\omega$.

The continuity set $C(f)$ of the function f is the set of points of continuity of f.

Recall that if X is a subset of Σ^ω, it is also a topological space whose topology is induced by the topology of Σ^ω. Open sets of X are traces on X of open sets of Σ^ω and the same result holds for closed sets. Then one can easily show by induction that for every integer $n \geq 1$, $\mathbf{\Pi}_n^0$-subsets (resp. $\mathbf{\Sigma}_n^0$-subsets) of X are traces on X of $\mathbf{\Pi}_n^0$-subsets (resp. $\mathbf{\Sigma}_n^0$-subsets) of Σ^ω, i.e. are intersections with X of $\mathbf{\Pi}_n^0$-subsets (resp. $\mathbf{\Sigma}_n^0$-subsets) of Σ^ω.

We recall now the following well known result.

Theorem 2.8 (see [Kec95])**.** *Let f be a function from* $\mathrm{Dom}(f) \subseteq \Sigma^\omega$ *into* Γ^ω*. Then the continuity set $C(f)$ of f is always a $\mathbf{\Pi}_2^0$-subset of* $\mathrm{Dom}(f)$*.*

Proof. Let f be a function from $Dom(f) \subseteq \Sigma^\omega$ into Γ^ω. For some integers $n, k \geq 1$, we consider the set

$$X_{k,n} = \{x \in Dom(f) \mid \forall y \in Dom(f) \ [\ d(x, y) < 2^{-k} \Rightarrow d(f(x), f(y)) < 2^{-n} \ ]\}$$

We know, from the definition of the distance d, that for two ω-words x and y over Σ, the inequality $d(x, y) < 2^{-k}$ simply means that x and y have the same $(k + 1)$ first letters.

Then it is easy to see that the set $X_{k,n}$ is an open subset of $Dom(f)$, because for each $x \in X_{k,n}$, the set $X_{k,n}$ contains the open ball (in $Dom(f)$) of all $y \in Dom(f)$ such that $d(x, y) < 2^{-k}$.

By union we can infer that $X_n = \bigcup_{k\geq 1} X_{k,n}$ is an open subset of $Dom(f)$ and then the countable intersection $C(f) = \bigcap_{n\geq 1} X_n$ is a $\mathbf{\Pi}_2^0$-subset of $Dom(f)$.

□

In the sequel we are going to investigate the continuity sets of ω-rational functions, firstly in the general case and next in the case of synchronous ω-rational functions.

3. Continuity set of ω-rational functions

Recall that C. Prieur proved the following result.

Theorem 3.1 ([Pri00,Pri01]). *One can decide whether a given ω-rational function is continuous.*

Prieur showed that the closure (in the topological sense) of the graph of a rational relation is still a rational relation that can be effectively computed. From this closure, it is quite easy to decide whether a given ω-rational function is continuous.

So one can decide whether the continuity set of an ω-rational function f is equal to its domain Dom(f). We shall prove below some undecidability results, using the undecidability of the *Post Correspondence Problem* which we now recall.

Theorem 3.2 (Post). *Let Γ be an alphabet having at least two elements . Then it is undecidable to determine, for arbitrary n-tuples $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ of non-empty words in $\Gamma^\star$, whether there exists a non-empty sequence of indices $i_1, \ldots, i_k$ such that $u_{i_1} \ldots u_{i_k} = v_{i_1} \ldots v_{i_k}$.*

We now state our first undecidability result.

Theorem 3.3. *One cannot decide whether the continuity set $C(f)$ of a given ω-rational function f is empty.*

Proof. Let Γ be an alphabet having at least two elements and $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be two sequences of n non-empty words in $\Gamma^\star$. Let $A = \{a, b\}$ and $C = \{c_1, \ldots, c_n\}$ such that $A \cap C = \emptyset$ and $A \cap \Gamma = \emptyset$.

We define the ω-rational function f of domain Dom$(f) = C^+.A^\omega$ by:

- $f(x) = u_{i_1} \ldots u_{i_k}.z$ if $x = c_{i_1} \ldots c_{i_k}.z$ and $z \in (A^\star.a)^\omega$.
- $f(x) = v_{i_1} \ldots v_{i_k}.z$ if $x = c_{i_1} \ldots c_{i_k}.z$ and $z \in A^\star.b^\omega$.

Notice that $(A^\star.a)^\omega$ is simply the set of ω-words over the alphabet A having an infinite number of occurrences of the letter a. And $A^\star.b^\omega$ is the complement of $(A^\star.a)^\omega$ in A^ω, i.e. it is the set of ω-words over the alphabet A containing only a finite number of letters a.

The two ω-languages $(A^\star.a)^\omega$ and $A^\star.b^\omega$ are ω-regular, so they are accepted by Büchi automata. It is then easy to see that the function f is ω-rational and we can construct a Büchi transducer $\mathcal{T}$ that accepts the graph of f.

We are going to prove firstly that if $x = c_{i_1} \ldots c_{i_k}.z \in C^+.A^\omega$ is a point of continuity of the function f then the Post Correspondence Problem of instances

$(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ would have a solution $i_1, \ldots, i_k$ such that $u_{i_1} \ldots u_{i_k} = v_{i_1} \ldots v_{i_k}$.

We distinguish two cases.

First Case. Assume firstly that $z \in (A^\star.a)^\omega$. Then by definition of f it holds that $f(x) = u_{i_1} \ldots u_{i_k}.z$. Notice that there is a sequence of elements $z_n \in A^\star.b^\omega$, $n \geq 1$, such that the sequence $(z_n)_{n\geq 1}$ is convergent and $lim(z_n) = z$. This is due to the fact that $A^\star.b^\omega$ is dense in A^ω. We set $x_n = c_{i_1} \ldots c_{i_k}.z_n$. So we have also $lim(x_n) = x$.

By definition of f, it holds that $f(x_n) = f(c_{i_1} \ldots c_{i_k}.z_n) = v_{i_1} \ldots v_{i_k}.z_n$.

If $x = c_{i_1} \ldots c_{i_k}.z$ is a point of continuity of f then we must have $lim(f(x_n)) = f(x) = u_{i_1} \ldots u_{i_k}.z$. But $f(x_n) = v_{i_1} \ldots v_{i_k}.z_n$ converges to $v_{i_1} \ldots v_{i_k}.z$. Thus this would imply that $u_{i_1} \ldots u_{i_k} = v_{i_1} \ldots v_{i_k}$ and the Post Correspondence Problem of instances $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ would have a solution.

Second Case. Assume now that $z \in A^\star.b^\omega$. Notice that $(A^\star.a)^\omega$ is also dense in A^ω. Then reasoning as in the case of $z \in (A^\star.a)^\omega$, we can prove that if $x = c_{i_1} \ldots c_{i_k}.z$ is a point of continuity of f then $u_{i_1} \ldots u_{i_k} = v_{i_1} \ldots v_{i_k}$ so the Post Correspondence Problem of instances $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ would have a solution.

Conversely assume that the Post Correspondence Problem of instances $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ has a solution, i.e. a non-empty sequence of indices $i_1, \ldots, i_k$ such that $u_{i_1} \ldots u_{i_k} = v_{i_1} \ldots v_{i_k}$.

Consider now the function f defined above. We have:

$$f(c_{i_1} \ldots c_{i_k}.z) = u_{i_1} \ldots u_{i_k}.z = v_{i_1} \ldots v_{i_k}.z \quad \text{for every } z \in A^\omega$$

So it is easy to see that the function f is continuous at point $c_{i_1} \ldots c_{i_k}.z$ for every $z \in A^\omega$.

Finally we have proved that the function f is continuous at point $c_{i_1} \ldots c_{i_k}.z$, for $z \in A^\omega$, if and only if the non-empty sequence of indices $i_1, \ldots, i_k$ is a solution of the Post Correspondence Problem of instances $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$. Thus one cannot decide whether the function f has (at least) one point of continuity.

□

Theorem 3.4. *One cannot decide whether the continuity set $C(f)$ of a given ω-rational function f is a regular ω-language (respectively, a context-free ω-language).*

Proof. We shall use a particular instance of Post Correspondence Problem. For two letters c, d, let PCP_1 be the Post Correspondence Problem of instances (t_1, t_2, t_3) and (w_1, w_2, w_3), where $t_1 = c^2$, $t_2 = t_3 = d$ and $w_1 = w_2 = c$, $w_3 = d^2$. It is easy to see that its solutions are the sequences of indices in $\{1^i.2^i.3^i \mid i \geq 1\} \cup \{3^i.2^i.1^i \mid i \geq 1\}$. In particular this language over the alphabet $\{1, 2, 3\}$ is not context-free and this will be useful in the sequel.

Let now Γ be an alphabet having at least two elements such that $\Gamma \cap \{c, d\} = \emptyset$, and $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be two sequences of n non-empty words in $\Gamma^\star$. Let PCP be the Post Correspondence Problem of instances $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$.

Let $A = \{a, b\}$ and $C = \{c_1, \ldots, c_n\}$ and $D = \{d_1, d_2, d_3\}$ be three alphabets two by two disjoints. We assume also that $A \cap \{c, d\} = \emptyset$.

We define the ω-rational function f of domain $\mathrm{Dom}(f) = C^+.D^+.A^\omega$ by :

- $f(x) = u_{i_1} \ldots u_{i_k}.t_{j_1} \ldots t_{j_p}.z$ if $x = c_{i_1} \ldots c_{i_k}.d_{j_1} \ldots d_{j_p}.z$ and $z \in (A^\star.a)^\omega$.
- $f(x) = v_{i_1} \ldots v_{i_k}.w_{j_1} \ldots w_{j_p}.z$ if $x = c_{i_1} \ldots c_{i_k}.d_{j_1} \ldots d_{j_p}.z$ and $z \in A^\star.b^\omega$.

Reasoning as in the preceding proof and using the fact that $(A^\star.a)^\omega$ and $A^\star.b^\omega$ are both dense in A^ω, we can prove that the function f is continuous at point $x = c_{i_1} \ldots c_{i_k}.d_{j_1} \ldots d_{j_p}.z$, where $z \in A^\omega$, if and only if the sequence $i_1, \ldots, i_k$ is a solution of the Post Correspondence Problem PCP of instances $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ and the sequence $j_1, \ldots, j_p$ is a solution of the Post Correspondence Problem PCP_1. There are now two cases.

First Case. The Post Correspondence Problem PCP has not any solution. Thus the function f has no points of continuity, i.e. $C(f) = \emptyset$.

Second Case. The Post Correspondence Problem PCP has at least one solution $i_1, \ldots, i_k$. We now prove that in that case the continuity set $C(f)$ is not a context-free ω-language, i.e. is not accepted by any Büchi pushdown automaton. Towards a contradiction, assume on the contrary that $C(f)$ is a context-free ω-language. Consider now the intersection $C(f) \cap R$ where R is the regular ω-language $c_{i_1} \ldots c_{i_k}.(d_1^+.d_2^+.d_3^+).A^\omega$. The class CF_ω of context-free ω-languages is closed under intersection with regular ω-languages, [Sta97], thus the language $C(f) \cap R$ would be also context-free. But $C(f) \cap R = c_{i_1} \ldots c_{i_k}.\{d_1^i.d_2^i.d_3^i \mid i \geq 1\}.A^\omega$ and this ω-language is not context-free because the finitary language $\{d_1^i.d_2^i.d_3^i \mid i \geq 1\}$ is not context-free. So we have proved that $C(f)$ is not a context-free ω-language.

In the first case $C(f) = \emptyset$ so $C(f)$ is a regular hence also context-free ω-language. In the second case $C(f)$ is not a context-free ω-language so it is not ω-regular. But one cannot decide which case holds because one cannot decide whether the Post Correspondence Problem PCP has at least one solution $i_1, \ldots, i_k$. □

4. Continuity set of synchronous ω-rational functions

We have shown that for non-synchronous rational functions, the continuity set can be very complex. In this section, we show that the landscape is quite different for synchronous rational functions. Their continuity set is always rational. Furthermore we show that any $\mathbf{\Pi}_2^0$ rational set is the continuity set of some rational function.

Theorem 4.1. *Let $f : A^\omega \to B^\omega$ be a rational synchronous function. The continuity set $C(f)$ of f is rational.*

Proof. We actually prove that the complement $A^\omega \setminus C(f)$ is rational. Since the inclusion $C(f) \subseteq \mathrm{Dom}(f)$ holds and the set $\mathrm{Dom}(f)$ is rational, it suffices to prove that $\mathrm{Dom}(f) \setminus C(f)$ is rational.

Suppose that f is realized by the synchronous transducer $\mathcal{T}$. Without loss of generality, it may be assumed that $\mathcal{T}$ is trim, that is, any state q appears in an accepting path. Let x be an element of the domain of f. We claim that f is not continuous at x if there are two infinite paths γ and γ' in $\mathcal{T}$ such that the following properties hold,

i) the path γ is accepting and $y = f(x)$.
ii) the labels of γ and γ' are (x, y) and (x, y') with $y \neq y'$.

The path γ exists since x belongs to the domain of f. Remark that the path γ' cannot be accepting since $\mathcal{T}$ realizes a function. It is clear that if such a path γ' exists, the function f cannot be continuous at x.

Suppose that f is not continuous at x. There is a sequence $(x_n)_{n\geq 0}$ of elements from the domain of f converging to x such that $d(f(x), f(x_n)) > 2^{-k}$ for some integer k. Since each x_n belongs to the domain of f, there is a path γ_n whose label is the pair $(x_n, f(x_n))$. Since the set of infinite paths is a compact space, it can be extracted from the sequence $(x_n)_{n\geq 0}$ another sequence $(x_{s(n)})_{n\geq 0}$ such that the sequence $(\gamma_{s(n)})_{n\geq 0}$ converges to a path γ'. Let (x', y') be the label of this path γ'. Since $(x_n)_{n\geq 0}$ converges to x, x' is equal to x and since $d(f(x), f(x_n)) > 2^{-k}$, y' is different from y. This proves the claim.

From the claim, it is easy to build an automaton that accepts infinite words x such that f is not continuous at x. Roughly speaking, the automaton checks whether there are two paths γ and γ' as above. Let $\mathcal{T}$ be the transducer (Q, A, B, E, q_0, F). We build a non deterministic Büchi automaton $\mathcal{A}$. The state set of $\mathcal{A}$ is $Q \times Q \times \{0, 1\}$. The initial state is $(q_0, q_0, 0)$ and the set of final states is $F \times Q \times \{1\}$. The set of transitions of this automaton is

$$\begin{aligned} G = \{(p, p', 0) \xrightarrow{a} (q, q', 0) \mid \exists b \in B\ p \xrightarrow{a|b} q \in E \text{ and } p' \xrightarrow{a|b} q' \in E\} \\ \cup \{(p, p', 0) \xrightarrow{a} (q, q', 1) \mid \exists b, b' \in B\ p \xrightarrow{a|b} q \in E, p' \xrightarrow{a|b'} q' \in E \text{ and } b \neq b'\} \\ \cup \{(p, p', 1) \xrightarrow{a} (q, q', 1) \mid \exists b, b' \in B\ p \xrightarrow{a|b} q \in E \text{ and } p' \xrightarrow{a|b'} q' \in E\} \end{aligned}$$

□

Theorem 4.2. *Let X be a rational $\mathbf{\Pi}^0_2$ subset of A^ω. Then X is the continuity set $C(f)$ of some rational synchronous function f of domain A^ω.*

Proof. If A only contains one symbol a, the result is trivial since A^ω only contains the infinite word a^ω. We now assume that A contains at least two symbols. Let b be a distinguished symbol in A let c a new symbol not belonging to A.

We define a synchronous function f that is of the following form :

$$f(x) = \begin{cases} x & \text{if } x \in X \\ wc^\omega \text{ for some prefix } w \text{ of } x & \text{if } x \in \overline{X} \setminus X \\ wb^\omega \text{ for some prefix } w \text{ of } x & \text{if } x \in (A^*b)^\omega \setminus \overline{X} \\ wc^\omega \text{ for some prefix } w \text{ of } x & \text{otherwise,} \end{cases}$$

where w is a word precised below.

By Theorem 2.5, there is a deterministic Büchi automaton $\mathcal{A} = (Q, A, E, \{q_0\}, F)$ accepting X. We assume that $\mathcal{A}$ is trim.

The function f is now defined as follows. If x belongs to X, then $f(x)$ is equal to x. If x belongs to the closure $\overline{X}$ of X but not to X, let w be the longest prefix of x which is the label of a path in $\mathcal{A}$ from q_0 to a final state. Then $f(x)$ is equal to wc^ω. If x does not belong to the closure of X, let w be the longest prefix which is the label of a path in $\mathcal{A}$ from q_0. Then $f(x)$ is equal to wb^ω if b occurs infinitely many times in x and it is equal to wc^ω otherwise. It is easy to verify that the continuity set of f is exactly X.

We now give a synchronous transducer $\mathcal{T}$ realizing the function f. Let R be the set of pairs (q, a) such that there is no transition $q \xrightarrow{a} p$ in $\mathcal{A}$. The state set of $\mathcal{T}$ is $Q \times \{0\} \cup (Q \setminus F) \times \{1\} \cup \{q_1, q_2, q_3, q_4\}$. The initial state is q_0 and the set of final states is $F \times \{0\} \cup (Q \setminus F) \times \{1\} \cup \{q_2, q_4\}$. The set of transitions is defined as follows.

$$
\begin{aligned}
G = {} & \{(p,0) \xrightarrow{a|a} (q,0) \mid p \xrightarrow{a} q \in E\} \\
& \cup \{(p,0) \xrightarrow{a|c} (q,1) \mid p \xrightarrow{a} q \in E \text{ and } q \notin F\} \\
& \cup \{(p,1) \xrightarrow{a|c} (q,1) \mid p \xrightarrow{a} q \in E \text{ and } p, q \notin F\} \\
& \cup \{(p,0) \xrightarrow{a|b} q_1 \mid (p,a) \in R\} \\
& \cup \{(p,0) \xrightarrow{a|c} q_3 \mid (p,a) \in R\} \\
& \cup \{p \xrightarrow{a|b} q_1 \mid p \in \{q_1, q_2\} \text{ and } a \neq b\} \\
& \cup \{p \xrightarrow{b|b} q_2 \mid p \in \{q_1, q_2\}\} \\
& \cup \{q_3 \xrightarrow{a|c} q_3 \mid a \in A\} \\
& \cup \{p \xrightarrow{a|c} q_4 \mid p \in \{q_3, q_4\} \text{ and } a \neq b\}
\end{aligned}
$$

□

Recall that a point x of a subset D of a topological space X is *isolated* if there is a neighborhood of x whose intersection with D is equal to $\{x\}$. Isolated points have the following property with regard to continuity. Any function from a domain D is continuous at any isolated point of D. Therefore, if X is the continuity set of some function of domain D, X must contain all isolated points of D. The following theorem states that for rational $\mathbf{\Pi}^0_2$ sets, this condition is also sufficient.

Theorem 4.3. *Let D and X be two rational subsets of A^ω such that $X \subseteq D$. If there exists a rational $\mathbf{\Pi}^0_2$-subset X' of A^ω such that $X = X' \cap D$, and if X contains all isolated points of D, then it is the continuity set $C(f)$ of some synchronous rational function f of domain D.*

In the proof of Theorem 4.2, the complement of the set $\overline{X}$ has been partitioned into two dense sets. The following lemma extends this result to any rational set of infinite words.

Lemma 4.4. *Let X be a rational set containing no isolated points. Then, the set X can be partitioned into two rational sets X' and X'' such that both X' and X'' are dense in X.*

Proof. Let X be a rational set of infinite words with no isolated points and let $\mathcal{A}$ be a deterministic and trim Muller automaton accepting X. We refer the reader for instance to [Tho90,PP04] for the definition and properties of Muller automata. From any state q, either there is no accepting path starting in q or there are at least two accepting paths with different labels starting from q.

We first consider the case where the table $\mathcal{T}$ of $\mathcal{A}$ only contains one accepting set F. Since the automaton is trim, all states of F belong to the same strongly connected component of $\mathcal{A}$. We consider two cases depending on whether the set F contains all states of its connected component.

Suppose first that the state q does not belong to F but is in the same strongly connected component as F. Let X' and X'' the sets of words which respectively label an accepting path which goes an odd or an even number of times through the state q. It is clear that X' and X'' have the required property.

Suppose now that F contains all the states in its strongly connected component. Since X has no isolated point, there must be a state q of F with two outgoing edges e and e'. Let X' be the set of words which label a path of the following form. The trace of this path over the two edges e and e' is an infinite sequence of the form $(e+e')^*(ee')^\omega$.

We now come back to the general case where the table $\mathcal{T}$ of $\mathcal{A}$ may contain several accepting sets $\{F_1,\ldots,F_n\}$. Let X_i be the set of words accepted by the table $\{F_i\}$. Note first that if the both sets X_i and X_j can be partitioned into dense rational sets into X'_i, X''_i, X'_j and X''_j, the both sets $X'_i \cup X'_j$ and $X''_i \cup X''_j$ are dense in $X_i \cup X_j$.

Note also that if F_i is accessible from F_j, then any set dense in X_i is also dense in X_j and therefore in $X_i \cup X_j$. It follows that if the set X_i can be partitioned into two dense rational sets X'_i and X''_i, the set $X_i \cup X_j$ can be partitioned into $X'_i \cup X_j$ and X''_i.

From the previous two remarks, it suffices to partition independently each X_i such that the corresponding F_i is maximal for accessibility. By maximal, we mean that F_i is maximal whenever if F_j is accessible from F_i, then F_i is also accessible from F_j and both sets F_i and F_j are in the same strongly connected component. This can be done using the method described above. □

We now come to the proof of the previous theorem.

Proof. The proof is similar to the proof of Theorem 4.2 but the domain D has to be taken into account. By hypothesis there exists a rational $\mathbf{\Pi}^0_2$-subset X' of A^ω such that $X = X' \cap D$. Let $\mathcal{A}$ a trim and deterministic Büchi automaton accepting X'.

We define the function f of domain D as follows. For any x in X, $f(x)$ is still equal to x. If x belongs to $\overline{X} \cap D \setminus X$, $f(x)$ is equal to wc^ω where w is the longest prefix of x which is the label in $\mathcal{A}$ of a path form the initial state to a final state.

By Lemma 4.4, the set $Z = D \setminus \overline{X}$ can be partitioned into two rational subsets Z_1 and Z_2 such that both Z_1 and Z_2 are dense into Z. If x belongs to $D \setminus \overline{X}$, then $f(x)$ is defined as follows. Let w be the longest prefix of x which is the label in $\mathcal{A}$ of a path from the initial state. Then $f(x)$ is equal to wb^ω if $x \in Z_1$ and $f(x) = wc^\omega$ if $x \in Z_2$. $\square$

The following corollary provides a complete characterization of sets of continuity of synchronous rational functions of domain $D \subseteq A^\omega$ when D is the intersection of a rational $\mathbf{\Sigma}^0_2$-subset and of a $\mathbf{\Pi}^0_2$-subset of A^ω. This is in particular the case if D is simply a $\mathbf{\Sigma}^0_2$-subset or a $\mathbf{\Pi}^0_2$-subset of A^ω.

Corollary 4.5. *Let D and X be two rational subsets of A^ω such that X is a $\mathbf{\Pi}^0_2$-subset of D and $D = Y_1 \cap Y_2$, where Y_1 is a rational $\mathbf{\Sigma}^0_2$-subset of A^ω and Y_2 is a $\mathbf{\Pi}^0_2$-subset of A^ω. If X contains all isolated points of D, then it is the continuity set $C(f)$ of some synchronous rational function f of domain D.*

Proof. Let D and X satisfying the hypotheses of the corollary.

Assume firstly that $D = Y_1$ is a $\mathbf{\Sigma}^0_2$-subset of A^ω. Then it is easy to see that $X' = X \cup (A^\omega - D)$ is a rational $\mathbf{\Pi}^0_2$-subset of A^ω such that $X = X' \cap D$. Thus in this case Corollary 4.5 follows from Theorem 4.3.

Assume now that $D = Y_1 \cap Y_2$, where Y_1 is a rational $\mathbf{\Sigma}^0_2$-subset of A^ω and Y_2 is a $\mathbf{\Pi}^0_2$-subset of A^ω. By hypothesis X is a $\mathbf{\Pi}^0_2$-subset of D thus there is a $\mathbf{\Pi}^0_2$-subset X_1 of A^ω such that $X = X_1 \cap D = X_1 \cap (Y_1 \cap Y_2) = (X_1 \cap Y_2) \cap Y_1$. This implies that X is also a $\mathbf{\Pi}^0_2$-subset of Y_1 because $X_1 \cap Y_2$ is a $\mathbf{\Pi}^0_2$-subset of A^ω as intersection of two $\mathbf{\Pi}^0_2$-subsets of A^ω. From the first case we can infer that there is a rational $\mathbf{\Pi}^0_2$-subset X' of A^ω such that $X = X' \cap Y_1$. Now we have also $X = X' \cap (Y_1 \cap Y_2) = X' \cap D$ because $X \subseteq Y_2$. Thus in this case again Corollary 4.5 follows from Theorem 4.3. $\square$

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