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# The Riemann–Hilbert approach to a generalized sine kernel

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## Abstract

We investigate the asymptotic behavior of a generalized sine kernel acting on a finite size interval  $[-q; q]$ . We determine its asymptotic resolvent as well as the first terms in the asymptotic expansion of its Fredholm determinant. Further, we apply our results to build the resolvent of truncated Wiener–Hopf operators generated by holomorphic symbols. Finally, the leading asymptotics of the Fredholm determinant allows us to establish the asymptotic estimates of certain oscillatory multidimensional coupled integrals.

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# 1 Introduction

The sine kernel

$$S(\lambda, \mu) = \frac{\sin \frac{\pi}{2} (\lambda - \mu)}{\pi (\lambda - \mu)}, \quad (1.1)$$

is a very important object in mathematical physics. In particular, the Fredholm determinant of the integral operator  $I - S$  acting on some interval  $J \subset \mathbb{R}$  appears in random matrix theory [17]. In the bulk scaling limit,  $\det_J [I - S]$  stands for the probability [18] that a matrix belonging to the Gaussian unitary ensemble has no eigenvalues in  $xJ$ . The kernel (1.1) also appears in the theory of quantum integrable systems. In particular, the determinant  $\det_J [I + \gamma S]$ ,  $\gamma$  being a parameter, describes various zero temperature correlation functions of the impenetrable Bose gaz [35, 28].

In all these interpretations of the sine kernel, one is interested in the large  $x$  behavior of its Fredholm determinant. The first attempt to analyze the  $x \rightarrow +\infty$  asymptotics of  $\det_J [I - S]$  goes back to Gaudin and Mehta [17, 18]. In 1973, Des Cloizeaux and Mehta [14] showed that

$$\log \det_{[-1;1]} [I - S] = -\frac{x^2}{8} - \frac{1}{4} \log x + O(1), \quad x \rightarrow +\infty. \quad (1.2)$$

Three years later, using Widom's formula [39] for the asymptotics of Toeplitz determinants supported on an arc, Dyson [15] gave a heuristic derivation of the constant terms  $c_0$  and proposed a recursive method for computing the subleading coefficients  $c_1, c_2, \dots$  in the asymptotic expansion:

$$\log \det_{[-1;1]} [I - S] = -\frac{x^2}{8} - \frac{1}{4} \log x + c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \dots \quad (1.3)$$

However, the forementioned results were somehow heuristic. It was only in 1994 that Widom [40] managed to prove rigorously the first term in the asymptotics (1.2):

$$\frac{d}{dx} \log \det_{[-1;1]} [I - S] = -\frac{x}{4} + o(1). \quad (1.4)$$

One year later, this analysis was extended to the multiple interval case [41]. While Widom studied the Fredholm determinant's asymptotics by operator techniques, Deift, Its and Zhou applied the Riemann–Hilbert problem (RHP) formulation for integrable operators [26] to the sine kernel acting on a union of intervals  $\cup_\ell J_\ell$  and proved the existence of the asymptotic expansion (1.3). However, their method did not allow them to obtain an estimate for the constant  $c_0$ , as they inferred the asymptotic expansion of  $\log \det_J [I - S]$  from those of

$$P_x = x \frac{d}{dx} \log \det [I - S]. \quad (1.5)$$

The first proofs of Dyson's heuristic formula for  $c_0$  appeared in the independent, and based on two completely different methods, works of Ehrhardt [16] and Krasovsky [33].

We would like to point out that there is a very nice connection of the sine kernel to the Painlevé V equation [28], as  $P_x$  solves this equation. The link between Painlevé V and  $P_x$  was also investigated in [12] in the framework of RHP. It was shown that one can deduce this Painlevé equation directly from the RHP data.

This article is devoted to the study a generalization of the sine kernel (1.1). This kernel, that we will refer to as the generalized sine kernel (GSK), is of the form

$$V(\lambda, \mu) = \frac{\gamma \sqrt{F(\lambda) F(\mu)}}{2i\pi(\lambda - \mu)} [e_+(\lambda) e_-(\mu) - e_-(\lambda) e_+(\mu)], \quad (1.6)$$

where

$$e_{\pm}(\lambda) = e^{\pm[ixp(\lambda)+g(\lambda)]/2}. \quad (1.7)$$

We will be more specific about the functions  $F$ ,  $p$  and  $g$  later on.

Various particular cases of the kernel (1.6) already appeared in the literature. These particular kernels were mostly used for the description of correlation functions of matrix models or quantum integrable models equivalent to free fermions (see e.g. [38, 36, 23, 24, 31, 8, 9, 25, 27]). In the present paper we consider a rather general case and only based on the analytic properties of the functions  $F$ ,  $p$  and  $g$ . The GSK (1.6) plays a crucial role in the study of correlation functions of (non free-fermionic) quantum integrable systems [30]. It is also useful for the asymptotic analysis of truncated Wiener–Hopf operators with Fischer–Hartwig singularities [32].

We investigate here the large  $x$  asymptotic behavior of the Fredholm determinant of the GSK in the framework of RHP. Our work is a natural extension of an unpublished analysis by Deift, Its and Zhou of the pure sine kernel  $I + \gamma S$  by RHP.

This article is organized as follows: in Section 2, we announce the main results of the paper, namely,

- the large  $x$  asymptotic behavior of the Fredholm determinant of the integral operator  $I + V$ , *cf.* (1.6);
- the asymptotic resolvent of some Wiener–Hopf operators connected to (1.6);
- the asymptotic behavior of coupled multiple integrals involving a cycle of kernels  $V$  (1.6), versus some holomorphic symmetric functions.

The proof of the asymptotic behavior of  $\log \det[I + V]$  is given in the core of the paper (Sections 3, 4, 5 and 6). More precisely, in Section 3, we recast the problem into a certain RHP. In Section 4, we transform this initial RHP into a RHP that can easily be solved asymptotically. This asymptotic solution is presented in Section 5 and used in Section 6 to obtain the leading and the first subleading terms of  $\log \det[I + V]$  in the  $x \rightarrow +\infty$  limit.

Section 7 is devoted to several applications of these results. We first show how one can use the asymptotic resolvent of the generalized sine kernel to construct asymptotic

resolvents of truncated Wiener–Hopf operators acting on  $[-x; x]$ , with  $x$  large. This asymptotic resolvent is used to reproduce the low magnetic field behavior of the so-called dressed charge arising in the theory of quantum integrable models solvable by the Bethe ansatz [6]. We then apply the asymptotics of  $\log \det [I + V]$  to derive the asymptotic behavior of coupled multiple integrals closely related to the ones appearing in the description of correlation functions of quantum integrable models. In this context we also consider the case of further modifications of the GSK useful for the correlation functions of the integrable Heisenberg spin chains [30].

Some properties of confluent hypergeometric functions and proofs of several Lemmas are gathered in the Appendices.

## 2 Problem to solve and main results

### 2.1 Generalized Sine Kernel: assumptions and notations

Let  $I + V$  be the integral operator with kernel (1.6) and acting on  $L^2([-q; q])$ .

We assume that there exists some open relatively compact neighborhood  $U$  of  $[-q; q]$  such that the functions  $p$ ,  $F$  and  $g$ , as well as the parameter  $\gamma$ , satisfy the following properties:

- $F$  and  $g$  are holomorphic on  $\overline{U}$ , the closure of  $U$ ;
- $p$  is holomorphic and injective on  $\overline{U}$ ,  $p([-q; q]) \subset \mathbb{R}$ , and  $p$  stabilizes the upper half plane  $\mathcal{H}_+$  (resp. the lower half plane  $\mathcal{H}_-$ ), i.e.  $p(U \cap \mathcal{H}_\pm) \subset \mathcal{H}_\pm$ ;
- $\gamma \in D_{0,r} = \{\lambda \in \mathbb{C} : |\lambda| < r\}$ , where  $r$  is such that  $|rF| < 1$  and  $\arg(1 + \gamma F) \in ]-\pi; \pi[$  on  $\overline{U}$ .

We study the large  $x$  expansion of the Fredholm determinant of  $I + V$  under these assumptions. This will be done by asymptotically solving a certain matrix RHP. It will become clear in the next section that the assumption  $p([-q; q]) \subset \mathbb{R}$  is tantamount to imposing the associated RHP to be of oscillatory nature. Moreover, the case  $p(U \cap \mathcal{H}_\pm) \subset \mathcal{H}_\mp$  is obtained by the negation  $(\gamma, g(\lambda)) \mapsto (-\gamma, -g(\lambda))$ .

Before presenting the main result of this article, let us introduce some convenient notations.

First, we define two auxiliary functions used all along the article:

$$\nu(\lambda) = \frac{-1}{2i\pi} \log(1 + \gamma F(\lambda)) , \quad (2.1)$$

$$\kappa(\lambda; q) \equiv \kappa(\lambda) = \exp \left\{ \int_{-q}^q \frac{\nu(\lambda) - \nu(\mu)}{\lambda - \mu} d\mu \right\} . \quad (2.2)$$

Note that  $\kappa$  is a function of the two parameters  $\lambda$  and  $q$ , although we will sometimes omit the dependence on the second parameter.

Finally, we will use the following simplified notations for the values of the functions  $p$  and  $\nu$  and of their derivatives at the points  $\pm q$ :

$$p_{\pm} = p(\lambda) \Big|_{\lambda=\pm q}, \quad p'_{\pm} = p'(\lambda) \Big|_{\lambda=\pm q}, \quad \text{etc.} \quad (2.3)$$

$$\nu_{\pm} = \nu(\lambda) \Big|_{\lambda=\pm q}, \quad \nu'_{\pm} = \nu'(\lambda) \Big|_{\lambda=\pm q}, \quad \text{etc.} \quad (2.4)$$

## 2.2 The main results

We now give the asymptotic behavior of the Fredholm determinant in the  $x \rightarrow +\infty$  limit:

**Theorem 2.1** *Let  $V$  be the GSK (1.6) with  $p$ ,  $g$ ,  $F$  and  $\gamma$  satisfying the assumptions of Section 2.1. Then, in the  $x \rightarrow +\infty$  limit,  $\log \det[I + V]$  behaves as*

$$\log \det[I + V] = \log \det[I + V]^{(0)} + o(1), \quad (2.5)$$

with

$$\begin{aligned} \log \det[I + V]^{(0)} = & -ix \int_{-q}^q \nu(\lambda) p'(\lambda) d\lambda - (\nu_+^2 + \nu_-^2) \log x - \int_{-q}^q \nu(\lambda) g'(\lambda) d\lambda \\ & + \log \left[ \frac{G(1, \nu_+) G(1, \nu_-) \kappa^{\nu_+}(q; q)}{(2qp'_+)^{\nu_+^2} (2qp'_-)^{\nu_-^2} \kappa^{\nu_-}(-q; q)} \right] + \frac{1}{2} \int_{-q}^q d\lambda d\mu \frac{\nu'(\lambda)\nu(\mu) - \nu(\lambda)\nu'(\mu)}{\lambda - \mu}, \end{aligned} \quad (2.6)$$

in which we have used the notations of Section 2.1. The Barnes  $G$ -function [3, 2] admits the integral representation:

$$G(z+1) = (2\pi)^{\frac{z}{2}} \exp \left\{ -\frac{z(z-1)}{2} + \int_0^z t\psi(t) dt \right\}, \quad \Re(z) > -1, \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad (2.7)$$

and we denote  $G(1, z) \equiv G(1+z)G(1-z)$ .

Using the perturbation theory for singular integral equations one can refine the theorem and obtain sub-leading corrections. Although, in principle, nothing opposes to derive the next sub-leading corrections, the computations become more and more involved. We have proved the structure of the first corrections to the equation (2.6).

**Proposition 2.1** *Let  $V$  be the GSK (1.6) with the conditions of Section 2.1. The leading asymptotics  $\log \det [I + V]^{(0)}$  of  $\log \det [I + V]$  as defined in Theorem 2.1 has non-oscillating and oscillating corrections.*

Let  $\delta > 0$  be such that the disks  $D_{\pm q, \delta}$  of radius  $\delta$  centered at  $\pm q$  fulfill  $D_{\pm q, \delta} \subset \bar{U}$ . Let  $\bar{\varepsilon} = 2 \sup_{\partial D_{q, \delta} \cup \partial D_{-q, \delta}} |\Re(\nu)|$ . Then the first non-oscillating corrections are of the form

$$\frac{N_1}{x} + \mathcal{O}\left(\frac{1}{x^{2(1-\bar{\varepsilon})}}\right), \quad (2.8)$$

with

$$N_1 = i \sum_{\sigma=\pm} \frac{\nu_\sigma^2}{p'_\sigma} \left\{ 2\sigma\nu'_\sigma \log x + \sigma \frac{d}{dq} \log u_\sigma + p'_\sigma \frac{d}{dq} \left( \frac{\nu_\sigma}{p'_\sigma} \right) - \frac{\nu_\sigma}{q} \right\}. \quad (2.9)$$

The first oscillating corrections are of the form

$$\frac{O_1}{x^2} + \mathcal{O}\left(\frac{1}{x^{3(1-\bar{\varepsilon})}}\right). \quad (2.10)$$

and the leading oscillating coefficient is given by

$$O_1 = \frac{\nu_- \nu_+}{(2q)^2 p'_+ p'_-} \sum_{\sigma=\pm 1} \left( \frac{u_+}{u_-} \right)^\sigma x^{2\sigma(\nu_+ + \nu_-)} e^{i\sigma x(p_+ - p_-)}. \quad (2.11)$$

We have introduced

$$u_+ = e^{g(q)} \frac{\Gamma(1 - \nu_+)}{\Gamma(1 + \nu_+)} \left\{ \frac{(2qp'_+)^{\nu_+}}{\kappa(q; q)} \right\}^2, \quad (2.12)$$

$$u_- = e^{g(-q)} \frac{\Gamma(1 + \nu_-)}{\Gamma(1 - \nu_-)} \left\{ (2qp'_-)^{\nu_-} \kappa(-q; q) \right\}^{-2}. \quad (2.13)$$

*Remark.* The GSK depends only on the combination  $ixp(\lambda) + g(\lambda)$  (see (1.7)). Therefore the Fredholm determinant and its asymptotics can only depend on this combination. This observation allows us to obtain the complete asymptotic expansion depending on the function  $g(\lambda)$  from the asymptotic expansion of the Fredholm determinant  $I + V$  corresponding to  $g = 0$ . Namely, it is enough to replace in the obtained formulae  $p(\lambda)$  by  $p(\lambda) - \frac{i}{x}g(\lambda)$  and then expand into negative powers of  $x$ .

It is quite interesting to apply the latter proposition in order to obtain the first few terms of the asymptotic expansion of  $\det [I + V]$ . The reason why we draw the reader's attention to these asymptotics is because they present a very interesting structure: the leading oscillating terms in the asymptotic expansion are just given by the sum of the leading asymptotics evaluated at  $\nu$  shifted by 1 or  $-1$ . This structure of the asymptotics seems to restore, at least partly, the original periodicity  $\nu \rightarrow \nu + n$ ,  $n \in \mathbb{Z}$ , of the Fredholm determinant of  $I + V$ .

**Corollary 2.1** *Let  $I + V$  be the GSK as above,  $\det [I + V]^{(0)}[\nu]$  the leading asymptotics of this determinant just as in Theorem 2.1,  $N_1$  and  $O_1$  as in Proposition 2.1. Note that we have emphasized the structure of  $\det [I + V]^{(0)}[\nu]$  as a functional of  $\nu$ . Then the*



oscillating corrections  $O_1$  can be reproduced from the non-oscillating part via the shift of  $\nu$  by  $\pm 1$ :

$$\det [I + V]^{(0)} [\nu] \frac{O_1}{x^2} = \det [I + V]^{(0)} [\nu + 1] + \det [I + V]^{(0)} [\nu - 1] . \quad (2.14)$$

This structure of the first terms of the large  $x$  asymptotic expansion for  $\det [I + V]$  leads us to raise the following conjecture on the structure of the asymptotic series :

**Conjecture 2.1** *The asymptotic expansion of the Fredholm determinant  $\det [I + V]$  of the GSK restores the periodicity  $\nu \rightarrow \nu + n$ ,  $n \in \mathbb{Z}$ , of the determinant. In particular, this asymptotic expansion contains all the  $\mathbb{Z}$ -periodized terms with respect to  $\nu$  of the leading asymptotics  $\det [I + V]^{(0)} [\nu]$ . Thus, all the oscillating terms can be deduced from the non-oscillating ones. More precisely let*

$$\mathcal{A}[\nu](x) \sim \det [I + V]^{(0)} [\nu] \left( 1 + \frac{C_1(\log x)[\nu]}{x} + \dots + \frac{C_M(\log x)[\nu]}{x^M} + \dots \right) \quad (2.15)$$

stand for the formal asymptotic series corresponding to the non-oscillating part of the asymptotic series for  $\log \det [I + V]$ . There  $C_k(X)[\nu]$  are polynomials of degree  $k$  in  $X$  whose coefficients are functionals in  $\nu$ . Moreover each of the  $C_k$ 's has no oscillating exponents of the type  $e^{\pm i x p \pm}$ . Then the formal asymptotic series for  $\det [I + V]$  is given by

$$\det [I + V] \sim \sum_{n \in \mathbb{Z}} \mathcal{A}[\nu + n](x) . \quad (2.16)$$

This conjecture is supported by (2.14) and also by the results of [37] where several sub-leading corrections to the asymptotics of the Fredholm determinant of the pure sine-kernel were computed.

Our first motivation to study the asymptotics of  $\log \det [I + V]$  comes from the theory of one dimensional quantum integrable models. Indeed, the generating function of the zero temperature two-point correlation functions (at distance  $x$ ) of different quantum integrable models [30] has a series expansion in terms of cycle integrals of the type

$$\mathcal{I}_n[\mathcal{F}_n] = \oint_{\Gamma([-q; q])} \frac{d^n z}{(2i\pi)^n} \int_{-q}^q \frac{d^n \lambda}{(2i\pi)^n} \mathcal{F}_n \left( \begin{matrix} \{\lambda\} \\ \{z\} \end{matrix} \right) \frac{e^{ix(p(z_j) - p(\lambda_j))}}{(z_j - \lambda_j)(z_j - \lambda_{j+1})} . \quad (2.17)$$

There  $\Gamma([-q; q])$  is a closed contour around  $[-q; q]$  whereas the function  $\mathcal{F}_n$  is holomorphic in some open neighborhood of  $[-q; q]^{2n}$  and symmetric in the  $n$  variables  $\{\lambda\}$  (we set  $\lambda_{n+1} \equiv \lambda_1$ ) and  $n$  variables  $\{z\}$ .

In Section 7, using the above results for the GSK, we prove the following asymptotic expansion of  $\mathcal{I}_n[\mathcal{F}_n]$  in the  $x \rightarrow +\infty$  limit :

**Proposition 2.2** *Let  $\mathcal{F}_n$  and  $\mathcal{I}_n[\mathcal{F}_n]$  be as above. Then for  $x \rightarrow +\infty$ ,*

$$\begin{aligned}
\mathcal{I}_n[\mathcal{F}_n] &= \frac{1}{2i\pi} \int_{-q}^q d\lambda \{ixp'(\lambda) + \partial_\epsilon\} \mathcal{F}_n \left( \begin{array}{c} \{\lambda\}^n \\ \{\lambda + \epsilon\}, \{\lambda\}^{n-1} \end{array} \right) \Big|_{\epsilon=0} \\
&+ \sum_{\sigma=\pm} (b_n - c_n \log(2qp'_\sigma x)) \mathcal{F}_n \left( \begin{array}{c} \{\sigma q\}^n \\ \{\sigma q\}^n \end{array} \right) \\
&+ \frac{n}{(2\pi)^2} \sum_{\sigma=\pm} \sum_{p=1}^{n-1} \int_{-q}^q d\lambda \frac{\mathcal{F}_n \left( \begin{array}{c} \{\sigma q\}^n \\ \{\sigma q\}^n \end{array} \right) - \mathcal{F}_n \left( \begin{array}{c} \{\sigma q\}^p, \{\lambda\}^{n-p} \\ \{\sigma q\}^p, \{\lambda\}^{n-p} \end{array} \right)}{p(n-p)(q-\sigma\lambda)} \\
&+ \frac{n}{2(2\pi)^2} \sum_{p=1}^{n-1} \int_{-q}^q \frac{d\lambda d\mu}{(n-p)(\lambda-\mu)} \left\{ \partial_\epsilon \mathcal{F}_n \left( \begin{array}{c} \{\lambda + \epsilon\}, \{\lambda\}^{p-1}, \{\mu\}^{n-p} \\ \{\lambda + \epsilon\}, \{\lambda\}^{p-1}, \{\mu\}^{n-p} \end{array} \right) \right. \\
&\quad \left. - \partial_\epsilon \mathcal{F}_n \left( \begin{array}{c} \{\mu + \epsilon\}, \{\mu\}^{p-1}, \{\lambda\}^{n-p} \\ \{\mu + \epsilon\}, \{\mu\}^{p-1}, \{\lambda\}^{n-p} \end{array} \right) \right\} \Big|_{\epsilon=0} + o(1), \quad (2.18)
\end{aligned}$$

with

$$b_n = \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial^n \nu_0^2}{\partial \gamma^n} \Big|_{\gamma=0}, \quad c_n = \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial^n \log G(1, \nu_0)}{\partial \gamma^n} \Big|_{\gamma=0}, \quad \nu_0 = \frac{i}{2\pi} \log(1+\gamma), \quad (2.19)$$

and where  $\{\lambda\}^n$  denotes the set formed by  $n$  copies of the same parameter  $\lambda$ .

Such a result seems hardly obtainable by a direct analysis of the multiple integrals (2.17).

Another result of this article concerns the asymptotic inversion of truncated Wiener–Hopf operators. We will prove the following proposition:

**Proposition 2.3** *Let  $I + K$  be a truncated Wiener–Hopf operator on  $] -x; x [$ , acting on functions  $g \in L^2(\mathbb{R})$  as*

$$[(I + K) \cdot g](t) = g(t) + \int_{-x}^x K(t-t') g(t') dt'. \quad (2.20)$$

The kernel  $K$  is defined by its Fourier transform  $F$ :

$$K(t) = \mathcal{F}^{-1}[F](t), \quad (2.21)$$

and we suppose that there exists  $\delta > 0$  such that

- $F$  admits an analytic continuation to  $\{z : |\Im(z)| \leq \delta\}$ ;
- $\xi \mapsto F(\xi \pm i\delta) \in L^1(\mathbb{R})$ ;

- the analytic continuation of  $1 + F$  never vanishes for  $|\Im(z)| \leq \delta$ .

Then the resolvent  $I - R$  of  $I + K$  fulfills

$$R(\lambda, \mu) = \int_{\mathbb{R}} \frac{d\xi d\eta}{i(2\pi)^2} F(\xi) \left\{ \frac{\alpha_+(\eta)}{\alpha_-(\xi)} e^{ix(\xi-\eta)} - \frac{\alpha_+(\xi)}{\alpha_-(\eta)} e^{-ix(\xi-\eta)} \right\} \frac{e^{i(\mu\eta-\lambda\xi)}}{\xi-\eta} + O\left(e^{-2\delta x}\right), \quad (2.22)$$

where  $\alpha(\lambda)$  is given by

$$\alpha(\lambda) = \exp \left\{ -\frac{1}{2i\pi} \int_{\mathbb{R}} \log(1 + F(\mu)) \frac{d\mu}{\mu - \lambda} \right\}. \quad (2.23)$$

### 2.3 Comparison with known results

There are several results in the literature concerning the asymptotic behavior of the Fredholm determinant  $\det[I + \gamma S]$ . This determinant corresponds to the GSK with  $p = \text{id}$ ,  $F = 1$  and  $g = 0$ . It is clear that we reproduce the answer concerning the leading asymptotics of  $\det[I + \gamma S]$  analyzed in [7] and [4].

As observed in [28],  $x \frac{d}{dx} \log \det[I + \gamma S]$  satisfies the fifth Painlevé equation. The authors of [28] used this property to obtain an asymptotic expansion of  $\log \det[I + \gamma S]$ . This fact was also exploited by the authors of [37] in order to derive the first few terms in the sub-leading asymptotics of the latter quantity. Their result reads

$$x \frac{d}{dx} \log \det[I + \gamma S] = -4ix\nu_0 - 2\nu_0^2 - i \frac{\nu_0^3}{x} + i \frac{\nu_0^2}{4x} \left\{ \left( \frac{\Gamma(-\nu_0)}{\Gamma(\nu_0)} \right)^2 (4x)^{4\nu_0} e^{4ix} - \left( \frac{\Gamma(\nu_0)}{\Gamma(-\nu_0)} \right)^2 \frac{e^{-4ix}}{(4x)^{4\nu_0}} \right\}, \quad (2.24)$$

with  $\nu_0$  given in (2.19) and  $q = 2$ . It is straightforward to see that in such a limit  $N_1 = i\nu_0^3$  and

$$O_1 \rightarrow \frac{\nu_0^2}{(2q)^2} \left\{ e^{2iqx} (2qx)^{4\nu_0} \left( \frac{\Gamma(-\nu_0)}{\Gamma(\nu_0)} \right)^2 + \frac{e^{-2iqx}}{(2qx)^{4\nu_0}} \left( \frac{\Gamma(\nu_0)}{\Gamma(-\nu_0)} \right)^2 \right\}, \quad (2.25)$$

which reproduces the oscillating terms (2.24) after setting  $q = 2$  and taking the  $q$  derivative.

## 3 The initial Riemann–Hilbert problem

The GSK (1.6) belongs to a special algebra of integral operators, the so-called integrable integral operators. This algebra was first singled out in [26] and then studied more

thoroughly in [12]. It is well known that many properties of these integrable operators can be obtained from the solution of a certain RHP.

In this Section, we formulate our problem in terms of a RHP that we then asymptotically solve.

### 3.1 Notations

An important property of completely integrable integral operators is that their resolvent still lies in the same algebra. However, before presenting the formula for the resolvent we introduce some quite useful vector notations. Namely, let

$$|E^R(\lambda)\rangle = \frac{\gamma\sqrt{F(\lambda)}}{2i\pi} \begin{pmatrix} e_+(\lambda) \\ e_-(\lambda) \end{pmatrix}, \quad \langle E^L(\lambda)| = \sqrt{F(\lambda)}(-e_-(\lambda), e_+(\lambda)), \quad (3.1)$$

so that the kernel  $V$  has a simple expression in terms of  $|E^R(\lambda)\rangle$  and  $\langle E^L(\lambda)|$ :

$$V(\lambda, \mu) = \frac{\langle E^L(\lambda)|E^R(\mu)\rangle}{\lambda - \mu}. \quad (3.2)$$

Observe that

$$\langle E^L(\lambda)|E^R(\lambda)\rangle = 0, \quad (3.3)$$

and, hence, the kernel  $V$  is not singular at  $\lambda = \mu$ .

Let  $|F^R(\lambda)\rangle$  be the solution to the integral equation:

$$|F^R(\mu)\rangle + \int_{-q}^q V(\lambda, \mu)|F^R(\lambda)\rangle d\lambda = |E^R(\mu)\rangle, \quad (3.4)$$

and  $\langle F^L(\lambda)|$  be the solution to the corresponding dual equation. It is convenient to write  $|F^R(\lambda)\rangle$  as well as its dual  $\langle F^L(\lambda)|$  in a form similar to  $|E^R(\lambda)\rangle$  and  $\langle E^L(\lambda)|$ :

$$|F^R(\lambda)\rangle = \frac{\gamma\sqrt{F(\lambda)}}{2i\pi} \begin{pmatrix} f_+(\lambda) \\ f_-(\lambda) \end{pmatrix}, \quad \langle F^L(\lambda)| = \sqrt{F(\lambda)}(-f_-(\lambda), f_+(\lambda)). \quad (3.5)$$

Then the resolvent reads:

$$R(\lambda, \mu) = \frac{\langle F^L(\lambda)|F^R(\mu)\rangle}{\lambda - \mu} = \frac{\gamma\sqrt{F(\lambda)F(\mu)}}{2i\pi(\lambda - \mu)} [f_+(\lambda)f_-(\mu) - f_+(\mu)f_-(\lambda)]. \quad (3.6)$$

### 3.2 The Riemann–Hilbert problem associated to a GSK

**Proposition 3.1** *Let  $V$  be the GSK (1.6) understood as acting on  $L^2([-q; q])$ , and such that  $\det[I + V] \neq 0$ . Then, there exists a  $2 \times 2$  matrix  $\chi(\lambda)$  such that*

$$|F^R(\lambda)\rangle = \chi(\lambda)|E^R(\lambda)\rangle, \quad \langle F^L(\lambda)| = \langle E^L(\lambda)|\chi^{-1}(\lambda). \quad (3.7)$$

*The matrix  $\chi(\lambda)$  is the unique solution of the RHP:*

- $\chi$  is analytic on  $\mathbb{C} \setminus [-q; q]$  ;
- $\chi(\lambda) = O\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \log|\lambda^2 - q^2|$  for  $\lambda \rightarrow \pm q$ ;
- $\chi(\lambda) \xrightarrow{\lambda \rightarrow \infty} I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  ;
- $\chi_+(\lambda) G_\chi(\lambda) = \chi_-(\lambda)$  for  $\lambda \in ]-q; q[$  .

The jump matrix  $G_\chi$  for this RHP reads

$$G_\chi(\lambda) = \begin{pmatrix} 1 - \gamma F(\lambda) & \gamma F(\lambda) e_+^2(\lambda) \\ -\gamma F(\lambda) e_-^2(\lambda) & 1 + \gamma F(\lambda) \end{pmatrix} = I + 2i\pi |E^R(\lambda)\rangle \langle E^L(\lambda)| . \quad (3.8)$$

Finally,  $\chi$  and its inverse can be expressed in terms of  $|F^R(\lambda)\rangle$  and of its dual  $\langle F^L(\lambda)|$ :

$$\chi(\lambda) = I_2 - \int_{-q}^q \frac{|F^R(\mu)\rangle \langle E^L(\mu)|}{\mu - \lambda} d\mu, \quad \chi^{-1}(\lambda) = I_2 + \int_{-q}^q \frac{|E^R(\mu)\rangle \langle F^L(\mu)|}{\mu - \lambda} d\mu. \quad (3.9)$$

We emphasize that the big  $O$  symbol,  $O\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ , is to be understood entrywise. Moreover,  $\chi_\pm(\mu)$  stands for the non-tangential limit of  $\chi(\lambda)$  when  $\lambda$  approaches a point  $\mu$  belonging to the jump curve from the left, resp. right, side of the contour (see Fig. 1).

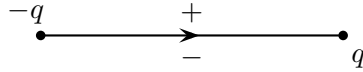


Figure 1: Original contour for the RHP.

*Proof* — The unicity of solutions to this RHP is proved along the same line as in [34]. The proof of existence of solutions is based on the equivalence between RHP and singular integral equations which, in the case of the above RHP, implies

$$\chi(\lambda) = I_2 + \int_{-q}^q \frac{d\mu}{\lambda - \mu} \chi_+(\mu) |E^R(\mu)\rangle \langle E^L(\mu)|, \quad \lambda \in \mathbb{C} \setminus [-q; q]. \quad (3.10)$$

The solution to this equation can be expressed in terms of the resolvent kernel  $I - R$  of  $I + V$

$$\chi(\lambda) = I_2 + \int_{-q}^q \frac{d\mu}{\lambda - \mu} \{|E^R\rangle \cdot (I - R)\}(\mu) \langle E^L(\mu)|. \quad (3.11)$$

In its turn, the resolvent kernel exists as  $\det [I + V] \neq 0$ . Moreover, the explicit construction of the resolvent through a Fredholm series, shows that  $(\lambda, \mu) \mapsto R(\lambda, \mu)$  is analytic in  $\overline{U} \times \overline{U}$ . Hence, so is  $|F^R(\mu)\rangle = [|E^R\rangle \cdot (I - R)](\mu)$ . The estimate  $|\chi| = O(\log |\lambda^2 - q^2|)$ ;  $\lambda \rightarrow \pm q$  follows from the integral representation (3.11) supplemented with the fact that both  $\langle E^L |$  and  $|F^R\rangle$  are smooth on  $[-q; q]$ .

Applying (3.9) to  $|E^R(\lambda)\rangle$  and  $\langle E^L(\lambda)|$  we obtain the equations (3.7). Hereby one can easily check that due to the orthogonality condition (3.3) the transform (3.7) is continuous across  $[-q, q]$ .  $\square$

It is also possible to express logarithmic derivatives of  $\det [I + V]$  either in terms of the resolvent  $R$  of  $I + V$  or in terms of  $\chi$ . Indeed, we have the

**Lemma 3.1** *The derivative of  $\log \det [I + V]$  with respect to  $x$  is related to the following trace involving the matrix  $\chi$*

$$\partial_x \log \det [I + V] = \oint_{\Gamma([-q; q])} \frac{d\lambda}{4\pi} p(\lambda) \operatorname{tr} [\partial_\lambda \chi(\lambda) \sigma_3 \chi^{-1}(\lambda)], \quad (3.12)$$

with  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , whereas its derivatives with respect to  $\gamma$  and  $q$  are expressed in terms of the resolvent as

$$\partial_\gamma \log \det [I + V] = \int_{-q}^q \frac{d\lambda}{\gamma} R(\lambda, \lambda), \quad \partial_q \log \det [I + V] = R(q, q) + R(-q, -q). \quad (3.13)$$

*Proof* — The last two equations are easily proved by the multiple integral series expansion of  $\log \det [I + V]$ . We shall only focus on the equation relating the  $x$  derivative of  $\log \det [I + V]$  to  $\chi$ . Clearly

$$\partial_x \log \det [I + V] = \int_{-q}^q [\partial_x V \cdot (I - R)](\lambda, \lambda) d\lambda, \quad (3.14)$$

with

$$\partial_x V(\lambda, \mu) = - \oint_{\Gamma([-q; q])} \frac{dz}{4\pi} \frac{p(z)}{(z - \lambda)(z - \mu)} \langle E^L(\lambda) | \sigma_3 | E^R(\mu) \rangle. \quad (3.15)$$

So that, using the representation (3.6) of the resolvent  $R$  in terms of  $\langle F^L |$  and  $|F^R\rangle$

and the fact that  $\langle F^L(\mu) | \sigma_3 | F^R(\lambda) \rangle = \text{tr} [\sigma_3 | F^R(\lambda) \rangle \langle F^L(\mu) |]$ , we get

$$\begin{aligned} \partial_x \log \det [I + V] = & - \oint_{\Gamma([-q;q])} \frac{dz}{4\pi} p(z) \int_{-q}^q d\lambda \frac{\langle E^L(\lambda) | \sigma_3 | E^R(\lambda) \rangle}{(z - \lambda)^2} \\ & + \text{tr} \left\{ \oint_{\Gamma([-q;q])} \frac{dz}{4\pi} p(z) \int_{-q}^q d\lambda d\mu | F^R(\lambda) \rangle \langle E^L(\lambda) | \right. \\ & \left. \times \left( \frac{1}{\lambda - z} - \frac{1}{\lambda - \mu} \right) \sigma_3 \frac{| E^R(\mu) \rangle \langle F^L(\mu) |}{(\mu - z)^2} \right\}. \end{aligned} \quad (3.16)$$

Using the integral expressions (3.9) for  $\chi$  and  $\chi^{-1}$ , we obtain

$$\begin{aligned} \partial_x \log \det [I + V] = & - \oint_{\Gamma([-q;q])} \frac{dz}{4\pi} p(z) \int_{-q}^q d\lambda \frac{\langle E^L(\lambda) | \sigma_3 | E^R(\lambda) \rangle}{(z - \lambda)^2} \\ & + \text{tr} \left\{ \oint_{\Gamma([-q;q])} \frac{dz}{4\pi} p(z) \int_{-q}^q d\mu (\chi(\mu) - \chi(z)) \sigma_3 \frac{| E^R(\mu) \rangle \langle F^L(\mu) |}{(\mu - z)^2} \right\} \\ = & \oint_{\Gamma([-q;q])} \frac{dz}{4\pi} p(z) \text{tr} \{ \partial_z \chi(z) \sigma_3 \chi^{-1}(z) \}, \end{aligned} \quad (3.17)$$

where we used (3.7).  $\square$

It is worth noticing that formula (3.12) is particularly effective when  $p$  is a rational function as then the contour of integration can be deformed to the poles of  $p$  (including the pole at  $\infty$ ). The integrals can be then easily calculated. In particular, in the case  $p(\lambda) = \lambda$ , we have the following result:

**Corollary 3.1** *Let  $\chi_1$  be the first non-trivial coefficient of the expansion of  $\chi$  around  $\infty$ , i.e.*

$$\chi(\lambda) = I_2 + \frac{\chi_1}{\lambda} + o\left(\frac{1}{\lambda}\right). \quad (3.18)$$

Then

$$\partial_x \log \det [I + V] |_{p=\text{id}} = -\frac{i}{2} \text{tr} \{ \chi_1 \sigma_3 \}. \quad (3.19)$$

*Proof* —  $-\text{tr}(\sigma_3 \chi_1)$  is the residue of the pole at infinity of  $\lambda \mapsto \lambda \text{tr} \{ \partial_\lambda \chi(\lambda) \sigma_3 \chi^{-1}(\lambda) \}$ .

$\square$

In this way, we recover one of the formulae derived for the pure sine kernel [12], but also for more general kernels as in [23, 24, 27]. We emphasize that (3.19) is valid not only for the pure sine kernel as it was originally derived, but also for the generalized sine kernel with  $p = \text{id}$ .

## 4 Transformations of the original RHP

In this Section we perform several transformations on the RHP for  $\chi$  so as to implement Deift–Zhou’s steepest descent method [13]. The first substitution maps the RHP for the matrix  $\chi$  into a RHP for a matrix  $\Xi$  whose jump matrix has 1 on its lower diagonal entry. This jump matrix is then easily upper/lower factorized. This factorization allows us to define another RHP for an unknown matrix  $\Upsilon$  whose jump matrices are already exponentially close to identity uniformly away from the endpoints  $\pm q$ . It remains to construct the parametrices at  $q$  and  $-q$ . These parametrices allow to define a matrix  $\Pi$  satisfying a RHP with jump matrices uniformly  $I_2 + o(1)$  when  $x \rightarrow +\infty$ .

### 4.1 The first step $\chi \rightarrow \Xi$

Let

$$\alpha(\lambda) = \exp \left\{ \int_{-q}^q \frac{\nu(\mu)}{\mu - \lambda} d\mu \right\} = \kappa(\lambda) \left( \frac{\lambda - q}{\lambda + q} \right)^{\nu(\lambda)}. \quad (4.1)$$

Then clearly,  $\alpha(\lambda)$  solves the scalar RHP

$$\alpha_-(\lambda) = \alpha_+(\lambda) (1 + \gamma F(\lambda)), \quad \lambda \in [-q; q], \quad \alpha(\lambda) \rightarrow 1 \quad \text{at } \lambda \rightarrow \infty. \quad (4.2)$$

The functions  $\kappa(\lambda)$  and  $\nu(\lambda)$  were already introduced in the introduction (2.2) and (2.1). In the following, we shall also use another representation for the function  $\alpha(\lambda)$

$$\alpha(\lambda) = \kappa_p(\lambda) \left[ \frac{p(\lambda) - p(q)}{p(\lambda) - p(-q)} \right]^{\nu(\lambda)}, \quad (4.3)$$

where  $\kappa_p$  is defined as

$$\log \kappa_p(\lambda; q) \equiv \log \kappa_p(\lambda) = \int_{-q}^q \left( \nu(\lambda) \frac{p'(\mu)}{p(\lambda) - p(\mu)} - \frac{\nu(\mu)}{\lambda - \mu} \right) d\mu. \quad (4.4)$$

We specify that we chose the principal branch of the logarithm, i.e.  $\arg \in ]-\pi; \pi[$ . Due to our assumptions on  $\gamma$ ,  $F$  and  $p$ , Morera theorem implies that the functions  $\nu$ ,  $\log \kappa$  and  $\log \kappa_p$  are holomorphic on  $\overline{U}$ . Moreover we have  $|\Re(\nu(\lambda))| < 1/2 \forall \lambda \in \overline{U}$ . Indeed

$$\nu(\lambda) = \frac{i}{2\pi} \log |1 + \gamma F(\lambda)| - \frac{1}{2\pi} \arg(1 + \gamma F(\lambda)), \quad (4.5)$$

and we have assumed that  $\arg(1 + \gamma F) \in ]-\pi; \pi[$ .

We use the function  $\alpha$  to transform the RHP for  $\chi$ . Let us define the matrix  $\Xi(\lambda)$  according to

$$\Xi(\lambda) = \chi(\lambda) \begin{pmatrix} \alpha(\lambda) & 0 \\ 0 & \alpha^{-1}(\lambda) \end{pmatrix}. \quad (4.6)$$

This new matrix  $\Xi(\lambda)$  satisfies the following RHP:



- $\Xi$  is analytic on  $\mathbb{C} \setminus [-q; q]$ ;
- $|\Xi(\lambda)| = O\left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right) |\lambda^2 - q^2|^{\pm\sigma_3 \Re(\nu_{\pm})} \log |\lambda^2 - q^2|$  for  $\lambda \rightarrow \pm q$ ;
- $\Xi(\lambda) \xrightarrow{\lambda \rightarrow \infty} I_2$ ;
- $\Xi_+(\lambda) G_{\Xi}(\lambda) = \Xi_-(\lambda)$  for  $\lambda \in [-q; q]$ .

Here the new jump matrix  $G_{\Xi}$  reads

$$G_{\Xi}(\lambda) = \begin{pmatrix} 1 + P(\lambda)Q(\lambda) & P(\lambda)e^{ixp(\lambda)} \\ Q(\lambda)e^{-ixp(\lambda)} & 1 \end{pmatrix}, \quad (4.7)$$

and

$$P(\lambda) = \frac{\gamma F(\lambda)}{1 + \gamma F(\lambda)} \alpha_+^{-2}(\lambda) e^{g(\lambda)} = -2ie^{i\pi\nu(\lambda)} \frac{\sin \pi\nu(\lambda)}{\alpha_+^2(\lambda)} e^{g(\lambda)}, \quad (4.8)$$

$$Q(\lambda) = -\frac{\gamma F(\lambda)}{1 + \gamma F(\lambda)} \alpha_-^2(\lambda) e^{-g(\lambda)} = 2ie^{i\pi\nu(\lambda)} \frac{\sin \pi\nu(\lambda)}{e^{g(\lambda)}} \alpha_-^2(\lambda). \quad (4.9)$$

The solution of this RHP for  $\Xi$  exists as it can be constructed from  $\chi$ . Moreover it is unique as seen by arguments similar to those providing uniqueness of the solution to the RHP for  $\chi$ .

## 4.2 The second step $\Xi \rightarrow \Upsilon$

As already mentioned, the jump matrix  $G_{\Xi}$  admits an explicit factorization into a product of upper and lower triangular matrices:

$$G_{\Xi} = M_+ M_- . \quad (4.10)$$

The matrices  $M_{\pm}$  are given by

$$M_+(\lambda) = \begin{pmatrix} 1 & P(\lambda)e^{ixp(\lambda)} \\ 0 & 1 \end{pmatrix}, \quad M_-(\lambda) = \begin{pmatrix} 1 & 0 \\ Q(\lambda)e^{-ixp(\lambda)} & 1 \end{pmatrix}, \quad (4.11)$$

and can be continued to  $\overline{U} \cap \mathcal{H}_+$ , resp.  $U \cap \mathcal{H}_-$ , where we recall that  $\mathcal{H}_{\pm}$  is the upper/lower half plane and  $\overline{U}$  is the domain of holomorphy of all the functions appearing in the RHP. Then we draw two new contours  $\Gamma_{\pm}$  in  $p(U)$  and define a new matrix  $\Upsilon(\lambda)$  according to Fig. 2. As readily checked,  $\Upsilon(\lambda)$  is continuous across  $] -q; q[$  and thus holomorphic in the interior of  $\Gamma_+ \cup \Gamma_-$ . We have thus removed the cut along  $[-q; q]$  and replaced it with cuts along  $\Gamma_+ \cup \Gamma_-$ .

The matrix  $\Upsilon$  solves the following RHP:

- $\Upsilon$  is analytic in  $\mathbb{C} \setminus \Gamma_+ \cup \Gamma_-$ ;
- $\Upsilon(\lambda) = O\left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right) \begin{pmatrix} |\lambda \mp q|^{\pm\Re(\nu_{\pm})} & |\lambda \mp q|^{\mp\Re(\nu_{\pm})} \\ 0 & |\lambda \mp q|^{\mp\Re(\nu_{\pm})} \end{pmatrix} \log |\lambda \mp q|$ ,  $\lambda \xrightarrow{\lambda \in H_I} \pm q$ ;

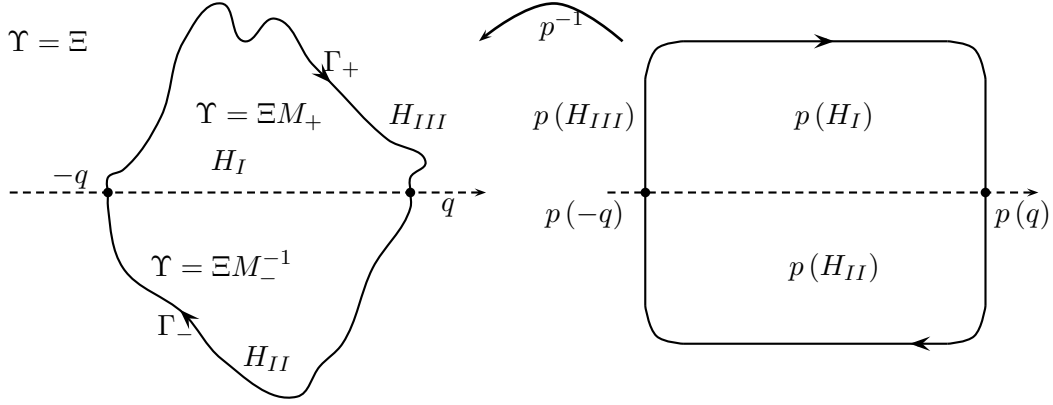


Figure 2: Contours  $\Gamma_+$  and  $\Gamma_-$  associated with the RHP for  $\Upsilon$ .

- $\Upsilon(\lambda) = \text{O} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |\lambda \mp q|^{\pm \Re(\nu_{\pm})} & 0 \\ |\lambda \mp q|^{\mp \Re(\nu_{\pm})} & |\lambda \mp q|^{\mp \Re(\nu_{\pm})} \end{pmatrix} \log |\lambda \mp q|, \quad \lambda \xrightarrow[\lambda \in H_{II}]{} \pm q;$
- $\Upsilon(\lambda) = \text{O} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} |\lambda \mp q|^{\pm \sigma_3 \Re(\nu_{\pm})} \log |\lambda \mp q|, \quad \lambda \xrightarrow[\lambda \in H_{III}]{} \pm q;$
- $\Upsilon(\lambda) \xrightarrow[\lambda \rightarrow \infty]{} I_2;$
- $\begin{cases} \Upsilon_+(\lambda) M_+(\lambda) = \Upsilon_-(\lambda) & \text{for } \lambda \in \Gamma_+, \\ \Upsilon_+(\lambda) M_-^{-1}(\lambda) = \Upsilon_-(\lambda) & \text{for } \lambda \in \Gamma_-, \end{cases}$

where the domains  $H_I, H_{II}, H_{III}$  are shown on the Figure 2.

Clearly, the solution of the RHP for  $\Upsilon$  exists and is unique. Hence, the matrices  $\Upsilon$  and  $\chi$  are in a one to one correspondence.

Note that, except in some vicinities of  $q$  and  $-q$ , the jump matrices  $M_+$  and  $M_-^{-1}$  for  $\Upsilon$  are exponentially close to the identity matrix. Therefore, to study the asymptotic solution of the RHP, it is enough to study the local problems in the vicinities of  $q$  and  $-q$ .

### 4.3 Parametrix around $-q$

We first present the parametrix  $\mathcal{P}$  on a small disk  $D_{-q,\delta} \subset U$  of radius  $\delta$  and centered at  $-q$ , that is an exact solution of the RHP:

- $\mathcal{P}$  is analytic on  $D_{-q,\delta} \setminus \{\Gamma_+ \cup \Gamma_-\}$  ;

- $\mathcal{P}(\lambda) = O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |\lambda + q|^{-\Re(\nu_-)} & |\lambda + q|^{\Re(\nu_-)} \\ 0 & |\lambda + q|^{\Re(\nu_-)} \end{pmatrix} \log |\lambda + q|, \lambda \xrightarrow{\lambda \in H_I} -q;$
- $\mathcal{P}(\lambda) = O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |\lambda + q|^{-\Re(\nu_-)} & 0 \\ |\lambda + q|^{+\Re(\nu_-)} & |\lambda + q|^{\Re(\nu_-)} \end{pmatrix} \log |\lambda + q|, \lambda \xrightarrow{\lambda \in H_{II}} -q;$
- $\mathcal{P}(\lambda) = O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} |\lambda + q|^{-\sigma_3 \Re(\nu_-)} \log |\lambda + q|, \lambda \xrightarrow{\lambda \in H_{III}} -q;$
- $\mathcal{P}(\lambda) = I_2 + O \left( \frac{1}{x^{1-\varepsilon}} \right),$  uniformly for  $\lambda \in \partial D_{-q, \delta},$
- $\begin{cases} \mathcal{P}_+(\lambda) M_+(\lambda) = \mathcal{P}_-(\lambda) & \text{for } \lambda \in \Gamma_+ \cap D_{-q, \delta}, \\ \mathcal{P}_+(\lambda) M_-^{-1}(\lambda) = \mathcal{P}_-(\lambda) & \text{for } \lambda \in \Gamma_- \cap D_{-q, \delta}. \end{cases}$

Here  $\varepsilon = 2 \sup_{\partial D_{-q, \delta}} |\Re(\nu)| < 1.$  The canonically oriented contour  $\partial D_{-q, \delta}$  is depicted in Fig. 3. The RHP for  $\mathcal{P}$  admits a class of solutions. Each element of this class is related

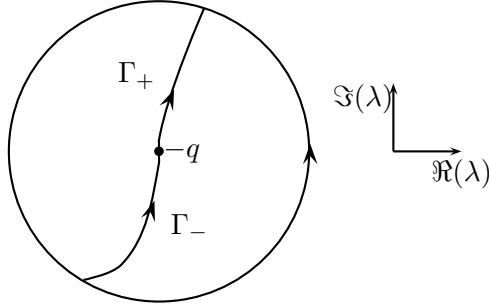


Figure 3: Contours in the RHP for  $\mathcal{P}.$

to another one through a left multiplication by a holomorphic matrix that is uniformly  $I_2 + O(1/x^{1-\varepsilon})$  on  $\partial D_{-q, \delta}.$  In order to construct the solution  $\mathcal{P}$  to this problem, we first focus on the simpler case where the functions  $F, g$  and  $\kappa_p$  are constant. Then the solution to the RHP for  $\mathcal{P}_{const}$  can be obtained by the differential equation method [22, 10, 11]. This leads to the solution

$$\mathcal{P}_{const}(\lambda) = \Psi(\lambda) L(\lambda) [-i\zeta_{-q}]^{-\nu\sigma_3} e^{\frac{i\pi\nu}{2}}. \quad (4.12)$$

Here  $\zeta_{-q} = x(p(\lambda) - p(-q)), \nu = i \log(1 + \gamma F)/2\pi,$

$$\Psi(\lambda) = \begin{pmatrix} \Psi(-\nu, 1; -i\zeta_{-q}) & ib_{12} \Psi(1 + \nu, 1; i\zeta_{-q}) \\ -ib_{21} \Psi(1 - \nu, 1; -i\zeta_{-q}) & \Psi(\nu, 1; i\zeta_{-q}) \end{pmatrix}, \quad (4.13)$$

and finally

$$b_{12}(\lambda) = -i \frac{\sin[\pi\nu] \Gamma^2(1+\nu)}{\pi \kappa_p^2 [x(p(q) - p(\lambda))]^{2\nu}} e^{ixp(q)+g}, \quad (4.14)$$

$$b_{21}(\lambda) = -i \frac{\pi \kappa_p^2 [x(p(q) - p(\lambda))]^{2\nu}}{\sin[\pi\nu] \Gamma^2(\nu)} e^{-ixp(q)-g}. \quad (4.15)$$

$\Psi(a, c; z)$  denotes Tricomi confluent hypergeometric function (CHF) of the second kind (see Appendix A). It solves the differential equation

$$zy'' + (c - z)y' - ay = 0. \quad (4.16)$$

Recall that  $\Psi$  has a cut along  $\mathbb{R}_-$ . Note that the choice to have  $\Psi$ 's cut on  $\mathbb{R}^-$  implies the use of the principal branch of the logarithm:  $-\pi < \arg(z) < \pi$ . The expression for the piecewise constant matrix  $L$  depends on the region of the complex plane. Namely

$$L(\lambda) = \begin{cases} I_2 & -\pi/2 < \arg[p(\lambda) - p(-q)] < \pi/2, \\ \begin{pmatrix} 1 & 0 \\ 0 & e^{-2i\pi\nu} \end{pmatrix} & \pi/2 < \arg[p(\lambda) - p(-q)] < \pi, \\ \begin{pmatrix} e^{-2i\pi\nu} & 0 \\ 0 & 1 \end{pmatrix} & -\pi < \arg[p(\lambda) - p(-q)] < -\pi/2. \end{cases} \quad (4.17)$$

The reader can check using the monodromy properties of Tricomi CHF (A.4) and (A.5) that the jump condition for constant functions  $F$  and  $g$  are satisfied by the matrix  $\mathcal{P}_{const}$ . Moreover the asymptotic expansion for  $\Psi(a, c; z)$  at  $z \rightarrow \infty$  allows one to check that  $\mathcal{P}_{const}$  has the correct behavior at infinity.

In order to extend to the case of arbitrary holomorphic functions  $F(\lambda)$ ,  $g(\lambda)$  and  $\kappa_p(\lambda)$  it is enough to add the  $\lambda$  dependency in all places where these functions appear. One ends up with the following solution to the RHP for  $\mathcal{P}$

$$\mathcal{P}(\lambda) = \Psi(\lambda) L(\lambda) [-i\zeta_{-q}]^{-\nu(\lambda)\sigma_3} e^{\frac{i\pi\nu(\lambda)}{2}}. \quad (4.18)$$

Here  $\zeta_{-q} = x(p(\lambda) - p(-q))$ ,

$$\Psi(\lambda) = \begin{pmatrix} \Psi(-\nu(\lambda), 1; -i\zeta_{-q}) & ib_{12}(\lambda) \Psi(1+\nu(\lambda), 1; i\zeta_{-q}) \\ -ib_{21}(\lambda) \Psi(1-\nu(\lambda), 1; -i\zeta_{-q}) & \Psi(\nu(\lambda), 1; i\zeta_{-q}) \end{pmatrix}, \quad (4.19)$$

with

$$b_{12}(\lambda) = -i \frac{\sin[\pi\nu(\lambda)] \Gamma^2(1+\nu(\lambda))}{\pi \kappa_p^2(\lambda) [x(p(q) - p(\lambda))]^{2\nu(\lambda)}} e^{g(\lambda)+ixp(q)} = -i\nu(\lambda) u(\lambda; x), \quad (4.20)$$

$$b_{21}(\lambda) = -i \frac{\pi \kappa_p^2(\lambda) [x(p(q) - p(\lambda))]^{2\nu(\lambda)}}{\sin[\pi\nu(\lambda)] \Gamma^2(\nu(\lambda))} e^{-ixp(q)-g(\lambda)} = -i \frac{\nu(\lambda)}{u(\lambda; x)}, \quad (4.21)$$

and finally

$$u(\lambda; x) = \frac{\Gamma(1 + \nu(\lambda))}{\Gamma(1 - \nu(\lambda))} \left\{ \kappa_p(\lambda) x^{\nu(\lambda)} [p(q) - p(\lambda)]^{\nu(\lambda)} \right\}^{-2} e^{ixp - +g(\lambda)}. \quad (4.22)$$

In the above formulae we have explicitly stressed the dependence of the functions  $b_{12}$ ,  $b_{21}$  and  $\nu$  on  $\lambda$ . Finally, the matrix  $L(\lambda)$  is given by (4.17) with  $\nu$  replaced by the function  $\nu(\lambda)$ .

This construction originates from the observation that the replacements  $F \mapsto F(\lambda)$ ,  $g \mapsto g(\lambda)$  and  $\kappa_p \mapsto \kappa_p(\lambda)$  preserve the jump conditions as the latter hold pointwise. Of course, once the parametrix  $\mathcal{P}$  is guessed it is not a problem to check directly that it solves the RHP in question. The asymptotic behavior is inferred from (A.6), whereas the jump conditions can be verified thanks to (A.4) and (A.5). Furthermore, due to the definition of the matrix  $L$ , the solution is continuous across the line  $\arg[p(\lambda) - p(-q)] = \pi$  and thus analytic in the whole domain  $\{\lambda \in \mathbb{C}; \Re[p(\lambda) - p(-q)] < 0\}$ .

#### 4.4 Parametrix around $q$

The RHP for the parametrix  $\tilde{\mathcal{P}}$  around  $q$  reads

- $\tilde{\mathcal{P}}$  is analytic on  $D_{q,\delta} \setminus \{\Gamma_+ \cup \Gamma_-\}$  ;
- $\tilde{\mathcal{P}}(\lambda) = O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |\lambda - q|^{+\Re(\nu_+)} & |\lambda - q|^{-\Re(\nu_+)} \\ 0 & |\lambda - q|^{-\Re(\nu_+)} \end{pmatrix} \log |\lambda - q|, \quad \lambda \xrightarrow{\lambda \in H_I} q;$
- $\tilde{\mathcal{P}}(\lambda) = O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |\lambda - q|^{+\Re(\nu_+)} & 0 \\ |\lambda - q|^{-\Re(\nu_+)} & |\lambda - q|^{-\Re(\nu_+)} \end{pmatrix} \log |\lambda - q|, \quad \lambda \xrightarrow{\lambda \in H_{II}} q;$
- $\tilde{\mathcal{P}}(\lambda) = O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} |\lambda - q|^{\sigma_3 \Re(\nu_+)} \log |\lambda - q|, \quad \lambda \xrightarrow{\lambda \in H_{III}} q;$
- $\tilde{\mathcal{P}}(\lambda) = I_2 + O \left( \frac{1}{x^{1-\tilde{\varepsilon}}} \right)$  uniformly for  $\lambda \in \partial D_{q,\delta}$ ;
- $\begin{cases} \tilde{\mathcal{P}}_+(\lambda) M_+(\lambda) = \tilde{\mathcal{P}}_-(\lambda) & \text{for } \lambda \in \Gamma_+ \cap D_{q,\delta}, \\ \tilde{\mathcal{P}}_+(\lambda) M_-^{-1}(\lambda) = \tilde{\mathcal{P}}_-(\lambda) & \text{for } \lambda \in \Gamma_- \cap D_{q,\delta}. \end{cases}$

and  $\tilde{\varepsilon} = 2 \sup_{\partial D_{q,\delta}} |\Re(\nu)| < 1$ . The solution to the RHP for the parametrix  $\tilde{\mathcal{P}}$  around  $q$  can be formally obtained from the one at  $-q$  through the transformation  $q \rightarrow -q$  and  $\nu \rightarrow -\nu$  on the solution to the RHP for  $\mathcal{P}$ . Indeed, the two RHP are identical modulo this negation.

Just as in the RHP for the parametrix around  $-q$ , we focus on the solution

$$\tilde{\mathcal{P}}(\lambda) = \tilde{\Psi}(\lambda) \tilde{L}(\lambda) \zeta_q^{\nu(\lambda) \sigma_3} e^{-\frac{i\pi\nu(\lambda)}{2}}, \quad (4.23)$$

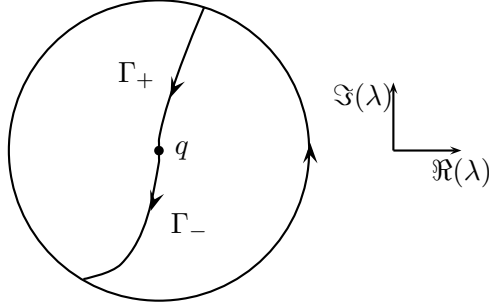


Figure 4: Contours in the RHP for  $\tilde{\mathcal{P}}$ .

where  $\zeta_q = x [p(\lambda) - p(q)]$ , and

$$\tilde{\Psi}(\lambda) = \begin{pmatrix} \Psi(\nu(\lambda), 1; -i\zeta_q) & i\tilde{b}_{12}(\lambda) \Psi(1 - \nu(\lambda), 1; i\zeta_q) \\ -i\tilde{b}_{21}(\lambda) \Psi(1 + \nu(\lambda), 1; -i\zeta_q) & \Psi(-\nu(\lambda), 1; i\zeta_q) \end{pmatrix}. \quad (4.24)$$

Here

$$\tilde{b}_{12}(\lambda) = i \frac{\sin[\pi\nu(\lambda)] \Gamma^2(1 - \nu(\lambda))}{\pi \kappa_p^2(\lambda)} [x(p(\lambda) - p(-q))]^{2\nu(\lambda)} e^{g(\lambda) + ixp(q)} = i\nu(\lambda) \tilde{u}(\lambda; x),$$

$$\tilde{b}_{21}(\lambda) = i \frac{\pi \kappa_p^2(\lambda) e^{-g(\lambda) - ixp(q)}}{\sin[\pi\nu(\lambda)] \Gamma^2(-\nu(\lambda)) [x(p(\lambda) - p(-q))]^{2\nu(\lambda)}} = i \frac{\nu(\lambda)}{\tilde{u}(\lambda; x)},$$

and

$$\tilde{u}(\lambda; x) = \frac{\Gamma(1 - \nu(\lambda))}{\Gamma(1 + \nu(\lambda))} \left\{ \frac{x^{\nu(\lambda)} [p(\lambda) - p(-q)]^{\nu(\lambda)}}{\kappa_p(\lambda)} \right\}^2 e^{ixp + g(\lambda)}. \quad (4.25)$$

Just as for the parametrix around  $-q$ , the matrix  $\tilde{L}(\lambda)$  depends on the quadrant of the complex plane:

$$\tilde{L}(\lambda) = \begin{cases} I_2 & -\pi/2 < \arg [p(\lambda) - p(q)] < \pi/2, \\ \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\pi\nu(\lambda)} \end{pmatrix} & \pi/2 < \arg [p(\lambda) - p(q)] < \pi, \\ \begin{pmatrix} e^{2i\pi\nu(\lambda)} & 0 \\ 0 & 1 \end{pmatrix} & -\pi < \arg [p(\lambda) - p(q)] < -\pi/2. \end{cases} \quad (4.26)$$

#### 4.5 The last transformation $\Upsilon \rightarrow \Pi$

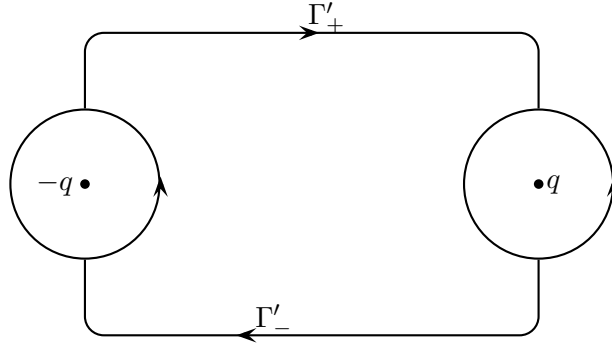
Let

$$\Pi(\lambda) = \begin{cases} \Upsilon(\lambda) \tilde{\mathcal{P}}^{-1}(\lambda) & \text{for } \lambda \in D_{q,\delta}, \\ \Upsilon(\lambda) \mathcal{P}^{-1}(\lambda) & \text{for } \lambda \in D_{-q,\delta}, \\ \Upsilon(\lambda) & \text{for } \lambda \in \mathbb{C} \setminus \{\overline{D}_{q,\delta} \cup \overline{D}_{-q,\delta}\}. \end{cases} \quad (4.27)$$

Introduce the curve  $\mathcal{C} = \{\Gamma_+ \cup \Gamma_-\} \cap \{\overline{D}_{q,\delta} \cup \overline{D}_{-q,\delta}\}$ . Then  $\Pi$  is continuous across  $\mathcal{C} \setminus \{q, -q\}$ . Since  $\Pi$  is holomorphic in a vicinity of  $\mathcal{C}$ , we have that  $\Pi$  is holomorphic in  $D_{q,\delta} \cup D_{-q,\delta} \setminus \{q, -q\}$ . This, in turn, due to the estimates for  $\mathcal{P}$ ,  $\tilde{\mathcal{P}}$  and  $\Upsilon$  around the points  $\pm q$ , ensures that the singularities at these points are of a removable type. Hence  $\Pi$  is holomorphic on the disks  $D_{q,\delta} \cup D_{-q,\delta}$ . Finally, we see that  $\Pi$  satisfies the following RHP:

- $\Pi$  is analytic in  $\mathbb{C} \setminus \Sigma_\Pi$  (cf Fig. 5) ;
- $\Pi(\lambda) = I_2 + O(1/\lambda)$  for  $\lambda \rightarrow \infty$ ;
- $\begin{cases} \Pi_+(\lambda) M_+(\lambda) = \Pi_-(\lambda) & \text{for } \lambda \in \Gamma'_+, \\ \Pi_+(\lambda) M_-^{-1}(\lambda) = \Pi_-(\lambda) & \text{for } \lambda \in \Gamma'_-, \\ \Pi_+(\lambda) \mathcal{P}(\lambda) = \Pi_-(\lambda) & \text{for } \lambda \in \partial D_{-q,\delta}, \\ \Pi_+(\lambda) \tilde{\mathcal{P}}(\lambda) = \Pi_-(\lambda) & \text{for } \lambda \in \partial D_{q,\delta}. \end{cases}$

The solution to the RHP for  $\Pi$ , exists and is unique as seen by standard arguments.



$$\Sigma_\Pi = \Gamma'_- \cup \Gamma'_+ \cup \partial D_{-q,\delta} \cup \partial D_{q,\delta}$$

Figure 5: Contour  $\Sigma_\Pi$  appearing in the RHP for  $\Pi$ .

The jump matrices for  $\Pi$  are uniformly exponentially close to  $I_2$  in  $x$  on  $\Gamma'_- \cup \Gamma'_+$  and uniformly  $I_2 + O(x^{\bar{\varepsilon}-1})$  on  $\partial D_{q,\delta} \cup \partial D_{-q,\delta}$ , with  $\bar{\varepsilon} = 2 \sup_{\partial D_{q,\delta} \cup \partial D_{-q,\delta}} |\Re(\nu)|$ . As a consequence,  $I_2$  is the unique solution of the RHP, up to uniformly  $O(x^{\bar{\varepsilon}-1})$  corrections. In addition, using the equivalence between singular integral equations and RHP, the asymptotic expansion of  $\Pi$  can be obtained by a Neumann series. This will be done in the upcoming section.

## 5 Asymptotic solution of the RHP

In this Section we asymptotically solve the above RHP for  $\Pi$ .

We derive an asymptotic expansion into negative powers of  $x$  for the jump matrices for  $\Pi$ , and use it to prove the existence of an asymptotic series for  $\Pi$ . The corresponding asymptotic series for  $\chi$  follows readily. One can finally infer the asymptotic behavior of the resolvent of the GSK up to any order in  $1/x$ .

## 5.1 Asymptotics of the jump matrices

Denote the jump matrices for  $\Pi$  by  $I_2 + \Delta(\lambda)$ . Then the matrix  $\Delta(\lambda)$  has the asymptotic expansion in the limit  $x \rightarrow +\infty$ :

$$\Delta(\lambda) = \sum_{m=1}^M \frac{\Delta^{(m)}(\lambda; x)}{x^m} + o(x^{-M-1+\bar{\varepsilon}}), \quad (5.1)$$

with  $\bar{\varepsilon} = 2 \sup_{\partial D_{q,\delta} \cup \partial D_{-q,\delta}} |\Re(\nu)|$ .

The explicit form of the matrices  $\Delta^{(n)}(\lambda; x)$  depends on the position of  $\lambda$  in the contour  $\Sigma_\Pi$ : they vanish to any order in  $1/x$  on  $\Gamma'_+ \cup \Gamma'_-$ , whereas the asymptotic expansion for  $\Delta$  on  $\partial D_{q,\delta} \cup D_{-q,\delta}$  follows promptly from the asymptotic expansion of Tricomi CHF (A.6). More explicitly, for any  $n \in \mathbb{N}^*$ ,

$$\Delta^{(n)}(\lambda; x) = \begin{cases} \frac{\Delta_{(-)}^{(n)}(\lambda; x)}{[p(\lambda) - p(-q)]^n} & \text{for } \lambda \in \partial D_{-q,\delta}, \\ \frac{\Delta_{(+)}^{(n)}(\lambda; x)}{[p(\lambda) - p(q)]^n} & \text{for } \lambda \in \partial D_{q,\delta}, \\ 0 & \text{for } \lambda \in \Gamma'_+ \cup \Gamma'_-. \end{cases} \quad (5.2)$$

We have separated the jump matrices into their pole parts  $[p(\lambda) - p_\pm]^{-n}$  and regular parts  $\Delta_{(-)}^{(n)}$  and  $\Delta_{(+)}^{(n)}$ , with

$$\Delta_{(-)}^{(n)}(\lambda; x) = \frac{i^n}{n!} \begin{pmatrix} 1 & -\frac{n u(\lambda; x)}{\nu(\lambda)} \\ n & 1 \end{pmatrix} \begin{pmatrix} (-1)^n (-\nu(\lambda))_n^2 & 0 \\ 0 & (\nu(\lambda))_n^2 \end{pmatrix} \quad (5.3)$$

for  $\lambda \in \partial D_{-q,\delta}$ , and

$$\Delta_{(+)}^{(n)}(\lambda; x) = \frac{i^n}{n!} \begin{pmatrix} 1 & \frac{n \tilde{u}(\lambda; x)}{\nu(\lambda)} \\ -\frac{n}{\nu(\lambda) \tilde{u}(\lambda; x)} & 1 \end{pmatrix} \begin{pmatrix} (-1)^n (\nu(\lambda))_n^2 & 0 \\ 0 & (-\nu(\lambda))_n^2 \end{pmatrix} \quad (5.4)$$

for  $\lambda \in \partial D_{q,\delta}$ . Here we use the standard notation  $(\nu)_n = \Gamma(\nu + n) / \Gamma(\nu)$  and  $u(\lambda; x)$ , resp.  $\tilde{u}(\lambda; x)$ , have been defined in (4.22), resp. (4.25). Thus, the matrices  $\Delta^{(n)}$  slightly depend on  $x$ , but their entries are a  $O(x^{\bar{\varepsilon}})$ .



## 5.2 Asymptotic expansion for $\Pi$

Using the equivalence between RHP and singular integral equations we can express  $\Pi$  in terms of its boundary value from the “+” side of the contour  $\Sigma_\Pi$

$$\Pi(\lambda) = I_2 + \frac{1}{2i\pi} \int_{\Sigma_\Pi} \Pi_+(s) \Delta(s) \frac{ds}{\lambda - s}. \quad (5.5)$$

In its turn  $\Pi_+(\lambda)$  belongs to  $L^2(\Sigma_\Pi)$  and fulfills the linear singular integral equation of Cauchy type

$$\Pi_+(z) = I_2 + C_{\Sigma_\Pi}^+ [\Pi_+ \Delta](z). \quad (5.6)$$

Recall that the Cauchy operator on  $L^2(\Sigma_\Pi)$  is defined as

$$C_{\Sigma_\Pi}^+[g](z) = \lim_{t \rightarrow z^+} C_{\Sigma_\Pi}[g](t) \quad \text{and} \quad C_{\Sigma_\Pi}[g](t) = \frac{1}{2i\pi} \int_{\Sigma_\Pi} \frac{g(s) ds}{t - s}, \quad t \notin \Sigma_\Pi. \quad (5.7)$$

The notation  $t \rightarrow z^+$  stands for the non-tangential limit of  $t$  approaching  $z$  from the “+” side of the contour  $\Sigma_\Pi$ . Recall that the Cauchy operator is bounded: i.e. there exists a constant  $c_2$  such that, for any function  $g \in L^2(\Sigma_\Pi)$ , one has  $\|C_{\Sigma_\Pi}^+[g]\| \leq c_2 \|g\|$ , where  $\|\cdot\|$  is the canonical  $L^2(\Sigma_\Pi)$  norm.

The matrix  $\Pi_+$  can be asymptotically approximated by the following series:

**Proposition 5.1** *Let  $\Pi_+^{(k)}$  be defined recursively according to*

$$\Pi_+^{(k)} = \sum_{p=1}^k C_{\Sigma_\Pi}^+ [\Pi_+^{(k-p)} \Delta^{(p)}] \quad \text{with} \quad \Pi_+^{(0)} = I_2. \quad (5.8)$$

*Then, for any integer  $M > 0$ , there exists a constant  $C(M) > 0$  such that*

$$\left\| \Pi_+ - \sum_{p=0}^{M-1} x^{-p} \Pi_+^{(p)} \right\| \leq \frac{C(M)}{x^{M(1-\bar{\varepsilon})}}. \quad (5.9)$$

*Proof* — Let us prove this statement by induction on  $M$ . For  $M = 1$  we have that

$$\|\Pi_+ - I_2\| = \left\| C_{\Sigma_\Pi}^+ [(\Pi_+ - I_2) \Delta] + C_{\Sigma_\Pi}^+ [\Delta] \right\| \leq c_2 \|\Pi_+ - I_2\| \|\Delta\| + c_2 \|\Delta\|. \quad (5.10)$$

Therefore, for  $x$  large,

$$\|\Pi_+ - I_2\| \leq \frac{c_2 \|\Delta\|}{1 - c_2 \|\Delta\|} \leq \frac{C(1)}{x^{1-\bar{\varepsilon}}}. \quad (5.11)$$

Let us now suppose that the result holds up to  $M$ . Then,

$$\begin{aligned}
\left\| \Pi_+ - \sum_{k=0}^M x^{-k} \Pi_+^{(k)} \right\| &= \left\| C_{\Sigma_{\Pi}}^+ \left[ \Pi_+ \Delta - \sum_{k=1}^M x^{-k} \sum_{p=1}^k \Pi_+^{(k-p)} \Delta^{(p)} \right] \right\| \\
&= \left\| C_{\Sigma_{\Pi}}^+ \left[ \Pi_+ \left( \Delta - \sum_{k=1}^M x^{-k} \Delta^{(k)} \right) + \sum_{p=1}^M \left( \Pi_+ - \sum_{k=0}^{M-p} \Pi_+^{(k)} x^{-k} \right) \Delta^{(p)} x^{-p} \right] \right\| \\
&\leq c_2 \|\Pi_+\| C_{\Delta_M} x^{-M-1+\bar{\varepsilon}} + \sum_{p=1}^M c_2 \|\Delta^{(p)}\| x^{-p} C(M-p) x^{(p-M-1)(1-\bar{\varepsilon})} \\
&\leq \frac{C(M+1)}{x^{(M+1)(1-\bar{\varepsilon})}}, \tag{5.12}
\end{aligned}$$

for some constants  $C_{\Delta_M}$  and  $C(M+1)$ . We used the fact that all  $\Delta^{(p)}$  are in  $L^2(\Sigma_{\Pi})$  and that  $\|\Pi_+\|$  is bounded in virtue of (5.11).  $\square$

Let us now extend this result for points  $\lambda$  being uniformly away from the contour  $\Sigma_{\Pi}$ . Define the matrices

$$\Pi^{(0)}(z) = I_2, \quad \Pi^{(p)}(z) = \sum_{k=1}^p C_{\Sigma_{\Pi}} \left[ \Pi_+^{(p-k)} \Delta^{(k)} \right](z), \quad p > 0, \tag{5.13}$$

$$\Pi(z; M) = \sum_{p=0}^M x^{-p} \Pi^{(p)}(z), \tag{5.14}$$

which are analytic away from  $\Sigma_{\Pi}$ . Then we have the following result:

**Proposition 5.2** *Let  $K$  be any compact subset of  $\mathbb{C} \setminus \Sigma_{\Pi}$ . Then,  $\forall k \in \mathbb{N}, \forall M \in \mathbb{N}^*$ ,*

$$\left| \partial_{\lambda}^k \Pi(\lambda) - \partial_{\lambda}^k \Pi(\lambda; M-1) \right| \leq \frac{k! C(M) \text{lgth}(\Sigma_{\Pi})}{d(K, \Sigma_{\Pi})^{k+1} x^{M(1-\bar{\varepsilon})}}, \quad \lambda \in K. \tag{5.15}$$

Here  $|\cdot|$  denotes the usual max norm  $|\Pi| \equiv \max_{i,j} |\Pi_{i,j}|$ ,  $d(\cdot, \cdot)$  is any distance on  $\mathbb{C}$  and  $\text{lgth}(\Sigma_{\Pi})$  is the length of the curve  $\Sigma_{\Pi}$ .

*Proof* — Let  $k \in \mathbb{N}, M \in \mathbb{N}^*$ , then

$$\begin{aligned}
&\left| \partial_{\lambda}^k \Pi(\lambda) - \sum_{p=0}^{M-1} x^{-p} \partial_{\lambda}^k \Pi^{(p)}(\lambda) \right| \\
&\leq \left| \frac{k!}{2i\pi} \int_{\Sigma_{\Pi}} \frac{ds}{(\lambda-s)^{k+1}} \left\{ \Pi_+(s) \Delta(s) - \sum_{p=1}^{M-1} x^{-p} \sum_{l=1}^p \Pi_+^{(p-l)}(s) \Delta^{(l)}(s) \right\} \right| \\
&\leq \frac{k! C(M) \text{lgth}(\Sigma_{\Pi})}{x^{M(1-\bar{\varepsilon})} d^{k+1}(K, \Sigma_{\Pi})}
\end{aligned}$$

due to (5.9).  $\square$

### 5.3 The functions $f_{\pm}$ to the leading order

We now perform the transformations from  $\Pi$  back to  $\chi$ .

The solution to the RHP of Proposition 3.1 reads

$$\chi(\lambda) = \Pi(\lambda) \chi^{(0)}(\lambda). \quad (5.16)$$

We call  $\chi^{(0)}$  the zero<sup>th</sup> order solution (i.e. obtained for  $\Pi = I_2$ ). In the vicinities of the endpoints of  $[-q; q]$ ,  $\chi^{(0)}$  is given as

$$\chi^{(0)}(\lambda) = \begin{cases} \mathcal{P}(\lambda) M_+^{-1}(\lambda) \alpha(\lambda)^{-\sigma_3}, & \lambda \in D_{-q,\delta} \cap \{0 < \arg[p(\lambda) - p_-] < \pi/2\}, \\ \tilde{\mathcal{P}}(\lambda) M_+^{-1}(\lambda) \alpha(\lambda)^{-\sigma_3}, & \lambda \in D_{q,\delta} \cap \{\pi/2 < \arg[p(\lambda) - p_+] < \pi\}. \end{cases}$$

Similarly, on  $[-q; q]$ , and uniformly away from the endpoints,

$$\chi^{(0)}(\lambda) = M_+^{-1}(\lambda) \alpha_+(\lambda)^{-\sigma_3}, \quad \lambda \in ]-q + \delta; q - \delta[.$$

In the  $\Im(\lambda) = 0^+$  limit and for  $\Re(\lambda) \in [-q; q]$ ,

$$M_+^{-1} \alpha_+^{-\sigma_3} \begin{pmatrix} e_+(\lambda) \\ e_-(\lambda) \end{pmatrix} = (\alpha_+ e_-)^{-\sigma_3} \begin{pmatrix} e^{2i\pi\nu} \\ 1 \end{pmatrix}, \quad (5.17)$$

so that, for  $\lambda \in ]-q + \delta; q + \delta[$ ,

$$\begin{pmatrix} f_+^{(0)}(\lambda) \\ f_-^{(0)}(\lambda) \end{pmatrix} = e^{i\pi\nu(\lambda)} \left[ \alpha_+^{-1}(\lambda) e_+(\lambda) e^{i\pi\nu(\lambda)} \right]^{\sigma_3} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (5.18)$$

where we have explicitly written all the dependence on  $\lambda$ .

When  $\lambda \in [-q; -q + \delta]$ , we should multiply the latter expression by  $\mathcal{P}$ . Using the decomposition (A.7) of Humbert CHF into a sum of two Tricomi CHF we get

$$\begin{pmatrix} f_+^{(0)}(\lambda) \\ f_-^{(0)}(\lambda) \end{pmatrix} = e^{\frac{i\pi\nu}{2}} \left( \frac{e_+(\lambda)}{\kappa_p(\lambda) \zeta_q^\nu} \right)^{\sigma_3} \begin{pmatrix} \Gamma(1+\nu) \Phi(-\nu, 1; -i\zeta_{-q}) \\ \Gamma(1-\nu) \Phi(\nu, 1; i\zeta_{-q}) \end{pmatrix}, \quad (5.19)$$

with  $\zeta_q = x[p(q) - p(\lambda)]$  and  $\zeta_{-q} = x[p(\lambda) - p(-q)]$ .

Analogously, for  $\lambda \in [q - \delta; q]$ ,

$$\begin{pmatrix} f_+^{(0)}(\lambda) \\ f_-^{(0)}(\lambda) \end{pmatrix} = e^{\frac{i\pi\nu}{2}} \left( \frac{e_+(\lambda) \zeta_{-q}^\nu}{\kappa_p(\lambda)} \right)^{\sigma_3} \begin{pmatrix} \Gamma(1-\nu) \Phi(\nu, 1; i\zeta_q) \\ \Gamma(1+\nu) \Phi(-\nu, 1; -i\zeta_q) \end{pmatrix}. \quad (5.20)$$

Note that the piecewise expressions for the functions  $f_{\pm}^{(0)}(\lambda)$  are in fact analytic in a vicinity of their respective domain of validity, regardless that they have been obtained by taking the limit of  $\lambda$  approaching a point in  $[-q; q]$  from the upper half plane. More precisely, the formula (5.19) holds on  $D_{-q,\delta}$ , (5.20) on  $D_{q,\delta}$ , and (5.18) on the connected component of the interior of  $\Sigma_\pi$  containing  $[\delta - q; q - \delta]$ . This observation follows from (3.3), but of course it can be checked by a direct computation based on the expression for the matrix  $\chi$  in the lower half plane.

## 5.4 Integral bounds for the resolvent.

We now introduce a function  $R^{(0)}(\lambda, \mu)$  and show that it is a good approximate of the resolvent in the sense that

$$\mathrm{tr} \left( R - R^{(0)} \right) = O \left( x^{\bar{\varepsilon}-1} \right). \quad (5.21)$$

Such estimates are necessary for the integration of the  $\gamma$  derivative of  $\log \det [I + V]$ .

**Definition 5.1** *Let  $\Pi_\tau(\lambda)$  denote the solution of the RHP given in Subsection 4.5 whose jumps are on circles of radius  $\tau$  and on the corresponding curves  $\Gamma'_+$  and  $\Gamma'_-$ .*

We can then write the solution of the RHP for  $\chi$  as  $\chi(\lambda) = \Pi_\tau \chi_\tau^{(0)}$ . There  $\chi_\tau^{(0)}$  do not depend explicitly on  $\tau$ . The radius  $\tau$  only determines which patch we should use for the definition of the matrix  $\chi_\tau^{(0)}$ . Moreover the whole combination  $\Pi_\tau \chi_\tau^{(0)}$  does not depend on the radius  $\tau$  at all. Hence, we can represent the exact resolvent as

$$R(\lambda, \mu) = \langle E^L(\lambda) | \left[ \chi_\tau^{(0)}(\lambda) \right]^{-1} \frac{\Pi_\tau^{-1}(\lambda) \Pi_\tau(\mu)}{\lambda - \mu} \chi_\tau^{(0)}(\mu) | E^R(\mu) \rangle. \quad (5.22)$$

There, without altering the value of  $R(\lambda, \mu)$ , we can chose different values of  $\tau$  depending on the point we consider. This is quite useful as we can take one value of the radius  $\tau$  in order to have estimates around  $\pm q$  and another one to perform estimates in the bulk  $[\delta - q; q - \delta]$ . This will become clearer during the proof of the proposition below.

**Definition 5.2** *Let us fix  $\delta > 0$  and define what we call the diagonal zero<sup>th</sup> order resolvent*

$$R^{(0)}(\lambda, \lambda) = \frac{\gamma F(\lambda)}{2i\pi} \left( \partial_\lambda f_+^{(0)}(\lambda) f_-^{(0)}(\lambda) - \partial_\lambda f_-^{(0)}(\lambda) f_+^{(0)}(\lambda) \right), \quad (5.23)$$

where the functions  $f_\pm^{(0)}(\lambda)$  are given by (5.18) for  $\lambda \in [\delta - q; q - \delta]$ , (5.19) for  $\lambda \in [-q; \delta - q[$  and (5.20) for  $\lambda \in ]q - \delta; q]$ . Similarly,  $|F^{R;(0)}(\lambda)\rangle$  and  $\langle F^{L;(0)}(\lambda)|$  are defined in terms of the same functions  $f_\pm^{(0)}$ .

We stress that the radius  $\tau$  previously introduced to build the exact solution  $\Pi_\tau(\lambda) \chi_\tau^{(0)}(\lambda)$  and  $\delta > 0$  appearing in the definition are, a priori, unrelated.

**Proposition 5.3** *Let  $R(\lambda, \mu)$  be the exact resolvent of the generalized sine kernel. Then*

$$\mathrm{tr} \left( R - R^{(0)} \right) = O \left( x^{\bar{\varepsilon}-1} \right), \quad (5.24)$$

where the  $O$  is uniform in  $\gamma \in D_{0,r}$ .

*Proof* — According to the preceding observations we have, for  $\lambda \in [-q; -q + \delta[ \cup ]q - \delta; q]$ ,

$$R(\lambda, \lambda) = R^{(0)}(\lambda, \lambda) + \langle F^{L;(0)}(\lambda) | \Pi_{2\delta}(\lambda) \partial_\lambda \Pi_{2\delta}(\lambda) | F^{R;(0)}(\lambda) \rangle, \quad (5.25)$$

and

$$R(\lambda, \lambda) = R^{(0)}(\lambda, \lambda) + \langle F^{L;(0)}(\lambda) | \Pi_{\delta/2}(\lambda) \partial_\lambda \Pi_{\delta/2}(\lambda) | F^{R;(0)}(\lambda) \rangle, \quad (5.26)$$

for  $\lambda \in [\delta - q; q - \delta]$ . The advantage of using two different matrices  $\Pi$  for the corrections of  $R(\lambda, \lambda)$  with respect to the zero<sup>th</sup> order resolvent  $R^{(0)}(\lambda, \lambda)$  is that the corrections are always analytic on the whole domain where they are considered. One does not need to take into account that  $\Pi_\delta(\lambda)$  has a jump across  $\lambda = \pm(q - \delta)$ . This might be problematic as, for instance, the integral of  $\partial_\lambda \Pi(\lambda)$  on  $[-q; \delta - q]$  might be ill-defined. Moreover the uniform estimates that we have derived for the matrix  $\Pi(\lambda)$  only hold uniformly away from the jump contour. As we will only integrate the terms containing  $\Pi_{2\delta}$  on  $[-q; \delta - q] \cup [q - \delta; q]$ , we will be in this situation. The same holds for the terms involving  $\Pi_{\delta/2}$ . However, we would not be able to use the uniform estimates (5.2) for  $\partial_\lambda \Pi_\delta$  when integrating it on  $[-q; \delta - q]$ , as we would not always be uniformly away from the boundary of the jump contour for  $\Pi_\delta$ .

With this way of understanding the corrections we have

$$\begin{aligned} \text{tr} \left( R - R^{(0)} \right) &= \left( \int_{-q}^{\delta-q} + \int_{q-\delta}^q \right) d\lambda \langle F^{L;(0)}(\lambda) | \Pi_{2\delta}(\lambda) \partial_\lambda \Pi_{2\delta}(\lambda) | F^{R;(0)}(\lambda) \rangle \\ &\quad + \int_{\delta-q}^{q-\delta} d\lambda \langle F^{L;(0)}(\lambda) | \Pi_{\delta/2}(\lambda) \partial_\lambda \Pi_{\delta/2}(\lambda) | F^{R;(0)}(\lambda) \rangle. \end{aligned} \quad (5.27)$$

Let us start by the bulk part of integral, i.e. the part on  $[\delta - q; q - \delta]$ . From the explicit form for  $f_\pm^{(0)}$  on  $[\delta - q; q - \delta]$  given in (5.18) we see that these functions are uniformly  $O(1)$ . Moreover, the uniform estimates for the matrices  $\Pi_{\delta/2}(\lambda)$  for  $\lambda$  uniformly away from the jump contour guarantee that

$$\langle F^{L;(0)}(\lambda) | \Pi_{\delta/2}(\lambda) \partial_\lambda \Pi_{\delta/2}(\lambda) | F^{R;(0)}(\lambda) \rangle = O(x^{\bar{\varepsilon}-1}), \quad (5.28)$$

the  $O(x^{\bar{\varepsilon}-1})$  being uniform in  $\gamma$ , at least for  $\gamma$  small enough.

The situation at the boundaries is a little more complex. We only consider the right boundary as the other case is treated similarly. We still have that  $\Pi_{2\delta}(\lambda) = I_2 + O(x^{\bar{\varepsilon}-1})$  and  $\partial_\lambda \Pi_{2\delta}(\lambda) = O(x^{\bar{\varepsilon}-1})$  uniformly on  $[q - \delta; q]$ . However the functions  $f_\pm^{(0)}(\lambda)$  are no longer uniformly a  $O(1)$  on this interval. We should thus estimate the following integral

$$\sum_{\sigma, \sigma' = \pm} \int_{q-\delta}^q f_\sigma^{(0)}(\lambda) f_{\sigma'}^{(0)}(\lambda) G_{\sigma, \sigma'}(\lambda) d\lambda \quad (5.29)$$

with  $G_{\sigma, \sigma'}(\lambda) = O(x^{\bar{\varepsilon}-1})$  being related to the entries of  $\Pi_{\delta/2}(\lambda) \partial_\lambda \Pi_{\delta/2}(\lambda)$ . The situation being similar for all the possible choices of  $\sigma$  and  $\sigma'$ , we explain the mechanism for  $(\sigma, \sigma') = (+, +)$ . The asymptotics of Humbert CHF guarantees that

$$\Phi(a, 1; \pm it) = \frac{c_\pm}{|t|^a} (1 + o(1)) \quad t \rightarrow +\infty \quad (5.30)$$

for some computable constants  $c_{\pm}$  depending on  $a$ . These constants are continuous with respect to  $a$  belonging to an open neighborhood of  $\nu([q - \delta; q])$ , and so is the  $o(1)$  term. Hence, there exist an  $a$  independent constant  $C$  such that

$$\left| (1 + |t|)^{\Re(a)} \Phi(a, 1; \pm it) \right| \leq C . \quad (5.31)$$

Indeed the latter function is continuous on  $\mathbb{R}$  and has a finite limit at  $\infty$ . Moreover the constant  $C$  can be chosen in such a way that the estimate holds for  $a$  belonging to some small vicinity of  $\nu([q - \delta; q])$ . Thence, by explicitly extracting the  $x^{\bar{\varepsilon}-1}$  factor coming from  $G_{+,+}(\lambda)$  we get that for some constant  $C'$

$$\left| f_+^{(0)}(\lambda) f_+^{(0)}(\lambda) G_{+,+}(\lambda) \right| \leq C' x^{\bar{\varepsilon}-1} \varphi_x(p(\lambda) - p(q)) \quad (5.32)$$

With  $\varphi_x(t) = x^{2\Re(\nu(\lambda))} (1 + x|p(\lambda) - p(q)|)^{-2\Re(\nu(\lambda))}$ . The function  $\varphi_x(t)$  fulfills

$$|\varphi_x(p(\lambda) - p(q))| \leq \tilde{C} |p(\lambda) - p(q)|^{-2\Re(\nu(\lambda))} \quad (5.33)$$

as for any  $\alpha \in \mathbb{R}$   $t \mapsto t^\alpha / (1 + t^\alpha)$  is bounded. The latter function is integrable on  $[q - \delta; q]$  (we consider the case  $|\Re(\nu)| < 1/2$ ). Thus the integrals in (5.29) do eventually yield  $O(x^{\bar{\varepsilon}-1})$  contributions.  $\square$

One can prove, in a very similar way, the estimates for the Hilbert–Schmidt norm of the resolvent. Namely,

**Proposition 5.4** *Under the assumptions of the previous proposition,*

$$\left\| R - R^{(0)} \right\|_2 = O(x^{\bar{\varepsilon}-1}) \quad (5.34)$$

with  $\|\cdot\|_2$  being the Hilbert–Schmidt norm.

## 5.5 Asymptotic expansion of the resolvent

We now prove that the asymptotic expansion for  $\Pi$  can be used to obtain an asymptotics expansion for the diagonal of the resolvent  $R(\lambda, \lambda)$ . We derive point–wise bounds for the diagonal resolvent as this quantity appears in the  $q$ -derivative of the Fredholm determinant:

$$\partial_q \log \det [I + V] = R(q, q) + R(-q, -q) . \quad (5.35)$$

We need to estimate the error when we replace the exact resolvent  $R$  by the approximate one  $R^{(0)}$ . The magnitude for the error term follows from the

**Proposition 5.5** *Let  $\chi^{(0)}$  be the solution of the RHP for  $\chi$  up to the leading order in  $x$  that is to say the one obtained from  $\Pi = I_2$  and corresponding to the contour  $\Sigma_\Pi$  with disks  $D_{\pm q, \delta}$  having radius  $\delta$ . Define the leading vectors  $\langle F^{L;(0)} |$  and  $| F^{R;(0)} \rangle$  as*

$$\langle F^{L;(0)}(\lambda) | = \langle E^L(\lambda) | \chi^{(0)}(\lambda)^{-1}, \quad | F^{R;(0)}(\lambda) \rangle = \chi^{(0)}(\lambda) | E^R(\lambda) \rangle, \quad (5.36)$$

and the leading order of the resolvent by

$$R^{(0)}(\lambda, \mu) = \frac{\langle F^{L;(0)}(\lambda) | F^{R;(0)}(\mu) \rangle}{\lambda - \mu}. \quad (5.37)$$

Then

$$R(\lambda, \lambda) = R^{(0)}(\lambda, \lambda) + \sum_{p=1}^k \frac{R^{(p)}(\lambda, \lambda)}{x^p} + O\left(\frac{x^{\bar{\varepsilon}}}{x^{(k+1)(1-\bar{\varepsilon})}}\right) \quad (5.38)$$

for  $\lambda$  uniformly away from  $\Sigma_{\Pi}$  and belonging to  $[-q; q]$ . Here,

$$R^{(0)}(\lambda, \lambda) = -\langle F^{L;(0)}(\lambda) | \partial_{\lambda} F^{R;(0)}(\lambda) \rangle, \quad (5.39)$$

$$R^{(p)}(\lambda, \lambda) = -\langle F^{L;(0)}(\lambda) | \hat{\Pi}^{(p)}(\lambda) | F^{R;(0)}(\lambda) \rangle, \quad p > 0, \quad (5.40)$$

in which

$$\Pi^{-1}(\lambda; k) \partial_{\lambda} \Pi(\lambda; k) = \sum_{p=1}^k \hat{\Pi}^{(p)}(\lambda) x^{-p} + O\left(\frac{1}{x^{(k+1)(1-\bar{\varepsilon})}}\right). \quad (5.41)$$

*Proof* — Clearly,

$$\begin{aligned} R(\lambda, \mu) &= \frac{\langle F^{L;(0)}(\lambda) | F^{R;(0)}(\mu) \rangle}{\lambda - \mu} + \langle F^{L;(0)}(\lambda) | \frac{\Pi^{-1}(\lambda) \Pi(\mu) - I_2}{\lambda - \mu} | F^{R;(0)}(\mu) \rangle, \\ &\xrightarrow{\lambda \rightarrow \mu} -\langle F^{L;(0)}(\lambda) | \partial_{\lambda} F^{R;(0)}(\lambda) \rangle - \langle F^{L;(0)}(\lambda) | \Pi^{-1}(\lambda) \partial_{\lambda} \Pi(\lambda) | F^{R;(0)}(\mu) \rangle. \end{aligned}$$

The corrections to the leading order for the resolvent are given here by the second term.

The inversion operator on  $\mathcal{M}_2(\mathbb{C})$ :  $u \mapsto u^{-1}$  is continuously differentiable around the identity  $I_2$ . Thus there exists an open neighborhood  $W$  of the identity matrix  $I_2$  and a constant  $C > 0$  such that  $\forall A, B \in W$  one has  $\|A^{-1} - B^{-1}\| \leq C \|A - B\|$ . Here  $\|\cdot\|$  denotes any matrix norm. The matrices  $\Pi(\lambda)$  and  $\Pi(\lambda; k)$  belong to  $W$  for  $x$  sufficiently large, as they both go to  $I_2$  in the  $x \rightarrow +\infty$  limit for  $\lambda$  uniformly away from  $\Sigma_{\Pi}$ , and we get, from Proposition 5.2,

$$\begin{aligned} &\|\Pi^{-1}(\lambda) \partial_{\lambda} \Pi(\lambda) - \Pi^{-1}(\lambda; k) \partial_{\lambda} \Pi(\lambda; k)\| \\ &\leq C \|\Pi(\lambda) - \Pi(\lambda; k)\| \|\partial_{\lambda} \Pi(\lambda)\| \\ &\quad + C (\|\Pi(\lambda)\| + \|\Pi(\lambda) - \Pi(\lambda; k)\|) \|\partial_{\lambda} \Pi(\lambda) - \partial_{\lambda} \Pi(\lambda; k)\| \\ &\leq \frac{\tilde{C}(k)}{x^{(k+1)(1-\bar{\varepsilon})}}, \end{aligned}$$

for some constant  $\tilde{C}(k)$ . Thus, uniformly away from  $\Sigma_{\Pi}$  and on the real axis, one has

$$\begin{aligned} &\left| \langle F^{L;(0)}(\lambda) | \Pi^{-1}(\lambda) \partial_{\lambda} \Pi(\lambda) - \Pi^{-1}(\lambda; k) \partial_{\lambda} \Pi(\lambda; k) | F^{R;(0)}(\lambda) \rangle \right| \\ &= O\left(\frac{x^{\bar{\varepsilon}}}{x^{(k+1)(1-\bar{\varepsilon})}}\right). \quad (5.42) \end{aligned}$$

In the last equality, we used the fact that  $f_{\pm}^{(0)}$  are at most of order  $O(x^{\bar{\varepsilon}})$  on the real axis, as follows from their behavior around  $\pm q$ .  $\square$

## 6 Leading asymptotic behavior of $\log \det [I + V]$

In this Section, we prove the result of Theorem 2.1; that is to say, we compute  $\det[I + V]^{(0)}$  the leading asymptotic behavior of  $\det[I + V]$  up to  $o(1)$  corrections in the  $x \rightarrow +\infty$  limit. More precisely, we show that

$$\begin{aligned} \log \det [I + V]^{(0)} = & 2 \int_{-q}^q d\lambda \nu(\lambda) \log'[e_-(\lambda)] + \sum_{\sigma=\pm} \log \left[ \frac{G(1, \nu_\sigma) \kappa^{\sigma\nu_\sigma}(\sigma q; q)}{(2qp'_\sigma x)^{\nu_\sigma^2}} \right] \\ & + \frac{1}{2} \int_{-q}^q d\lambda d\mu \frac{\nu'(\lambda) \nu(\mu) - \nu(\lambda) \nu'(\mu)}{\lambda - \mu}. \end{aligned} \quad (6.1)$$

This result will be obtained by two different methods based on the integration of equations (3.13). The first one, which uses the derivative of the Fredholm determinant over  $\gamma$ , is based on the uniformness of the asymptotic expansion for the resolvent for  $\gamma$  small enough. It is worth mentioning that this way is technically quite involved. The second method deals with the derivative of the Fredholm determinant over  $q$ . Although we have not been able to provide a full rigorous proof for it, we would like to draw the reader's attention to this method as it is much more direct and simple.

### 6.1 The leading asymptotics from the $\gamma$ -derivative method

Due to Proposition 5.3 the proof of the leading asymptotics of the Fredholm determinant from the first equation (3.13),

$$\partial_\gamma \log \det [I + V] = \int_{-q}^q \frac{d\lambda}{\gamma} R(\lambda, \lambda), \quad (6.2)$$

only necessitates the use of  $R^{(0)}(\lambda, \lambda)$  defined in (5.37). Recall that  $R^{(0)}(\lambda, \lambda)$  has different leading asymptotics in the bulk  $]-q; q[$  and near the boundary. Let  $\delta > 0$  be sufficiently small. Then

$$\frac{R^{(0)}(\lambda, \lambda)}{\gamma} = \begin{cases} R_q^{(0)}(\lambda, \lambda) & \lambda \in [q - \delta; q], \\ R_{\text{bk}}^{(0)}(\lambda, \lambda) & \lambda \in [-q + \delta; q - \delta], \\ R_{-q}^{(0)}(\lambda, \lambda) & \lambda \in [-q; -q + \delta], \end{cases} \quad (6.3)$$

where

$$R_{\text{bk}}^{(0)}(\lambda, \lambda) = \frac{F(\lambda)}{2i\pi(1 + \gamma F(\lambda))} \left\{ 2\partial_\lambda \log e_+(\lambda) - 2\partial_\lambda \log \left[ \kappa_p(\lambda) \left( \frac{p_+ - p(\lambda)}{p(\lambda) - p_-} \right)^{\nu(\lambda)} \right] \right\},$$

$$\begin{aligned} R_{-q}^{(0)}(\lambda, \lambda) = & -\nu\varphi(\nu; x[p - p_-]) \left\{ 2\nu' \log x - 2\partial_\lambda [\nu \log(p_+ - p) - 2\partial_\lambda \log \kappa_p] \right. \\ & \left. + \nu' [\psi(1 + \nu) + \psi(1 - \nu)] + g' \right\} + ix\nu p' \tau(\nu; x[p - p_-]) \\ & + \nu\nu' \rho(\nu; x[p - p_-]), \end{aligned}$$



$$\begin{aligned}
R_q^{(0)}(\lambda, \lambda) &= -\nu\varphi(\nu; x[p_+ - p]) \left\{ 2\nu' \log x - 2\partial_\lambda [\nu \log(p - p_-) - 2\partial_\lambda \log \kappa_p] \right. \\
&\quad \left. + \partial_\lambda \nu [\psi(1 + \nu) + \psi(1 - \nu)] + g' \right\} + ix\nu p' \tau(\nu; x[p_+ - p]) \\
&\quad - \nu\nu' \rho(\nu; x[p_+ - p]).
\end{aligned}$$

Here  $\psi(z) = \frac{d}{dz} \log \Gamma(z)$  and we have introduced the shorthand notations

$$\begin{aligned}
\varphi(\nu; t) &= \Phi(-\nu, 1; -it) \Phi(\nu, 1; it), \\
\rho(\nu; t) &= (\partial_1 \Phi)(\nu, 1; it) \Phi(-\nu, 1; -it) + (\partial_1 \Phi)(-\nu, 1; -it) \Phi(\nu, 1; it), \\
\tau(\nu; t) &= -\Phi(-\nu, 1; -it) \Phi(\nu, 1; it) + (\partial_z \Phi)(-\nu, 1; -it) \Phi(\nu, 1; it) \\
&\quad + \Phi(-\nu, 1; -it) (\partial_z \Phi)(\nu, 1; it).
\end{aligned}$$

Moreover, in order to lighten the above expressions and similar ones in the following, we omit the explicit dependence on the argument  $\lambda$  of the different functions involved (like  $\nu$ ,  $p$ , and their derivatives  $\nu'$ ,  $p'$ , etc.).

We can now split the integration contour into three parts

$$\int_{-q}^q R^{(0)}(\lambda, \lambda) \frac{d\lambda}{\gamma} = \int_{-q}^{-q+\delta} \frac{d\lambda}{\gamma} R_{-q}^{(0)}(\lambda, \lambda) + \int_{-q+\delta}^{q-\delta} \frac{d\lambda}{\gamma} R_{\text{bk}}^{(0)}(\lambda, \lambda) + \int_{q-\delta}^q \frac{d\lambda}{\gamma} R_q^{(0)}(\lambda, \lambda). \quad (6.4)$$

The bulk integral is carried out straightforwardly. The integrals over the vicinities of the endpoints are more involved. Consider, for instance, the integration over  $[-q; -q + \delta]$ .

Using the asymptotic series for Humbert CHF  $\Phi$  (A.9) and the equations (A.10), (A.11) we get that

$$\begin{aligned}
\varphi(a; t) &= e^{i\pi a}, \\
\rho(a; t) &+ \frac{e^{i\pi a}}{\Gamma(1-a)\Gamma(1+a)} \left\{ 2 \log t - \psi(1-a) - \psi(1+a) - \frac{4ia}{1+t} \right\}, \\
\tau(a; t) &+ e^{i\pi a} \left( 1 - \frac{2ia}{1+t} \right),
\end{aligned}$$

are uniformly Riemann integrable on  $\mathbb{R}^+$  in the sense of the definition of Lemma B.1 (See Appendix B). Using the integration Lemma B.1 as well as the estimates for the

integrals of  $\tau$  and  $\varphi$  (A.10), (A.11), we find

$$\begin{aligned}
\int_{-q}^{-q+\delta} R^{(0)}(\lambda, \lambda) d\lambda &= - \int_{-q}^{-q+\delta} \frac{d\lambda e^{i\pi\nu}}{\Gamma(\nu)\Gamma(1-\nu)} \{2\nu' \log x - 2\partial_\lambda [\nu \log(p_+ - p)] \\
&\quad + \nu' (\psi(\nu) + \psi(-\nu)) + g'\} \\
+ \int_{-q}^{-q+\delta} \frac{d\lambda e^{i\pi\nu\nu'}}{\Gamma(\nu)\Gamma(1-\nu)} &\left\{ 2\log[x(p-p_-)] - \psi(-\nu) - \psi(\nu) + \frac{4i\nu}{1+x(p-p_-)} \right\} \\
+ ix \int_{-q}^{-q+\delta} d\lambda p'\nu &\left\{ \frac{2i\nu}{x(p-p_-)+1} - 1 \right\} \\
&+ \frac{\nu_- e^{i\pi\nu_-}}{\Gamma(1-\nu_-)\Gamma(\nu_-)} \{2 - \psi(\nu_-) - \psi(-\nu_-)\} + o(1) . \quad (6.5)
\end{aligned}$$

Here the  $o(1)$  is with respect to the successive limits  $x\delta \rightarrow +\infty$  and  $\delta \rightarrow 0$ . The two terms proportional to  $\log x$  compensate each other. The remaining part of the first three lines of (6.5) is an  $O(\delta)$  and can thus be dropped. The integral in the last two lines of (6.5) is evaluated thanks to the second integration Lemma B.2 (see Appendix B). We get,

$$\begin{aligned}
\int_{-q}^{-q+\delta} R^{(0)}(\lambda, \lambda) d\lambda &= -i \int_{-q}^{-q+\delta} \frac{e^{i\pi\nu} p'}{\Gamma(\nu)\Gamma(1-\nu)} d\lambda \\
+ \frac{e^{i\pi\nu_-} \nu_-}{\Gamma(\nu_-)\Gamma(1-\nu_-)} &\{-2\log[x(p(\delta-q)-p_-)] + 2 - \psi(\nu_-) - \psi(-\nu_-)\} + o(1) . \quad (6.6)
\end{aligned}$$

The integration over  $[q-\delta; q]$  can be treated similarly. The result reads

$$\begin{aligned}
\int_{q-\delta}^q R^{(0)}(\lambda, \lambda) d\lambda &= -i \int_{q-\delta}^q \frac{e^{i\pi\nu} p'}{\Gamma(\nu)\Gamma(1-\nu)} \\
+ \frac{\nu_+ e^{i\pi\nu_+}}{\Gamma(\nu_+)\Gamma(1-\nu_+)} &\{-2\log[x(p_+ - p(q-\delta))] + [2 - \psi(\nu_+) - \psi(-\nu_+)]\} + o(1) . \quad (6.7)
\end{aligned}$$

So that,

$$\begin{aligned}
\int_{-q}^q R^{(0)}(\lambda, \lambda) \, d\lambda &= \int_{-q}^q \frac{\gamma F}{2i\pi(1+\gamma F)} [ixp' + g' - 2\partial_\lambda \log \kappa_p] \, d\lambda \\
&+ \int_{\delta-q}^{q-\delta} \nu \log \left( \frac{p_+ - p}{p - p_-} \right) \partial_\lambda \left\{ \frac{\gamma F}{1 + \gamma F} \right\} \frac{d\lambda}{i\pi} - \left[ \frac{\gamma \nu F}{i\pi(1 + \gamma F)} \log \left( \frac{p_+ - p}{p - p_-} \right) \right]_{\delta-q}^{q-\delta} \\
&+ \frac{\gamma \nu_- F_-}{2i\pi(1 + \gamma F_-)} \{2 \log [x(p(\delta - q) - p_-)] - 2 + \psi(\nu_-) + \psi(-\nu_-)\} \\
&+ \frac{\gamma \nu_+ F_+}{2i\pi(1 + \gamma F_+)} \{2 \log [x(p(q - \delta) - p_+)] - 2 + \psi(\nu_+) + \psi(-\nu_+)\} + o(1),
\end{aligned} \tag{6.8}$$

where we used  $\frac{e^{i\pi\nu}}{\Gamma(\nu)\Gamma(1-\nu)} = -\frac{\gamma F}{2i\pi(1+\gamma F)}$ . Using the integral representation for the Barnes  $G$ -function (2.7), it is not a problem to see that

$$\begin{aligned}
\int_{-q}^q R^{(0)}(\lambda, \lambda) \frac{d\lambda}{\gamma} &= \partial_\gamma \left\{ \int_{-q}^q \nu \partial_\lambda \log e_- \, d\lambda + \log \left( \frac{G(1, \nu_+) G(1, \nu_-)}{[x(p_+ - p_-)]^{\nu_+^2 + \nu_-^2}} \right) \right\} \\
&+ 2 \int_{-q}^q \partial_\gamma \nu \partial_\lambda \log \kappa_p \, d\lambda - 2 \int_{-q}^q [\partial_\lambda \partial_\gamma \nu] \nu \log \left( \frac{p_+ - p}{p - p_-} \right) \, d\lambda. \tag{6.9}
\end{aligned}$$

Now we should recast the last line as a derivative with respect to  $\gamma$ . We have

$$\begin{aligned}
&2 \int_{-q}^q \left[ \partial_\gamma \nu \partial_\lambda \log \kappa_p - [\partial_\lambda \partial_\gamma \nu] \nu \log \left( \frac{p_+ - p}{p - p_-} \right) \right] \, d\lambda \\
&= \partial_\gamma \log \left\{ \frac{\kappa^{\nu_+}(q; q)}{\kappa^{\nu_-}(-q; q)} \left( \frac{p_+ - p_-}{2qp'_+} \right)^{2\nu_+^2} \left( \frac{p_+ - p_-}{2qp'_-} \right)^{2\nu_-^2} \right\} \\
&+ \sum_{\sigma=\pm} \left[ \sigma \partial_\gamma \nu_\sigma \log \kappa(\sigma q; q) - \sigma \nu_\sigma \partial_\gamma \log \kappa(\sigma q; q) \right] \\
&\quad - 2 \int_{-q}^q \left[ \partial_\gamma \partial_\lambda \nu \right] \left[ \log \kappa + \nu \log \left( \frac{q - \lambda}{q + \lambda} \right) \right] \, d\lambda. \tag{6.10}
\end{aligned}$$

It remains to apply the identity

$$\begin{aligned} \partial_\gamma \int_{-q}^q \frac{\nu'(\lambda) \nu(\mu) - \nu'(\mu) \nu(\lambda)}{2(\lambda - \mu)} d\lambda d\mu &= -2 \int_{-q}^q \left[ \partial_\gamma \partial_\lambda \nu \right] \left[ \log \kappa + \nu \log \left( \frac{q - \lambda}{q + \lambda} \right) \right] d\lambda \\ &+ \sum_{\sigma=\pm} \left[ \sigma \partial_\gamma \nu_\sigma \log \kappa(\sigma q; q) - \sigma \nu_\sigma \partial_\gamma \log \kappa(\sigma q; q) \right]. \end{aligned} \quad (6.11)$$

Indeed, we have for the r.h.s. of (6.11)

$$\begin{aligned} RHS &= \int_{-q}^q \left( \frac{\nu_+ \partial_\gamma \nu - \nu \partial_\gamma \nu_+}{q - \lambda} + \frac{\nu_- \partial_\gamma \nu - \nu \partial_\gamma \nu_-}{q + \lambda} \right) d\lambda \\ &+ \int_{-q}^q \nu(\mu) \partial_\gamma \partial_\lambda \nu(\lambda) \left\{ \frac{1}{\lambda - \mu + i0} + \frac{1}{\lambda - \mu - i0} \right\} d\lambda d\mu \\ &= \frac{1}{2} \int_{-q}^q \sum_{\sigma=\pm} (\nu_\sigma \partial_\gamma \nu + \nu \partial_\gamma \nu_\sigma) \left\{ \frac{1}{q - \sigma\lambda + i0} + \frac{1}{q - \sigma\lambda - i0} \right\} d\lambda \\ &+ \int_{-q}^q \nu(\mu) \partial_\gamma \nu(\lambda) \left\{ \frac{1}{(\lambda - \mu + i0)^2} + \frac{1}{(\lambda - \mu - i0)^2} \right\} d\lambda d\mu. \end{aligned} \quad (6.12)$$

There we have regularized all the integrands and then performed an integration by parts. On the other hand, one has for the l.h.s. of (6.11)

$$LHS = \int_{-q}^q [\partial_\gamma \nu(\mu) \partial_\lambda \nu(\lambda) + \nu(\mu) \partial_\gamma \partial_\lambda \nu(\lambda)] \left\{ \frac{1}{\lambda - \mu + i0} + \frac{1}{\lambda - \mu - i0} \right\} d\lambda d\mu. \quad (6.13)$$

Taking the last integral by parts we arrive at (6.12).

Thus, the l.h.s. of (6.8) is presented as the derivative with respect to  $\gamma$ . Since the asymptotic expansion is uniform in  $\gamma$  we can integrate this result from 0 to  $\gamma$ . As  $\log \det [I + V] |_{\gamma=0} = 0$  we get the desired result.

## 6.2 The leading asymptotics from the $q$ -derivative method

The method we use here is based on the second equation in (3.13),

$$\partial_q \log \det [I + V] = R(q, q) + R(-q, -q). \quad (6.14)$$

For the purpose of this sub-Section we assume that  $|\Re(\nu(\lambda))| < 1/4$ . Indeed we are then able to use the pointwise estimates for the resolvent established in Proposition

5.5. Such a restriction on  $|\Re(\nu(\lambda))|$  could be lightened by much more refined estimates. Recall that one has for  $\lambda$  uniformly away from the boundary  $\Sigma_{\Pi}$  corresponding to disks of radius  $\delta$ :

$$\left| R(\lambda, \lambda) - R^{(0)}(\lambda, \lambda) \right| \leq \frac{C(q)}{x^{1-2\epsilon}} \leq \frac{C(q)}{x^{1-2\epsilon}} \quad , \text{ with } \epsilon = \frac{2\sup |\Re \nu|}{\overline{U}} \quad . \quad (6.15)$$

So that the  $q$  anti-derivative of  $R(q, q) + R(-q, -q) - R^{(0)}(q, q) - R^{(0)}(-q, -q)$  will be a  $o(1)$  in the  $x \rightarrow +\infty$  limit.

Equation (6.3) allows to determine the value of

$$\begin{aligned} R^{(0)}(\lambda, \lambda) &= -\langle F^{L;(0)}(\lambda) \mid \partial_{\lambda} F^{R;(0)}(\lambda) \rangle \\ &= \frac{\gamma F(\lambda)}{2i\pi} f_+^{(0)}(\lambda) f_-^{(0)}(\lambda) \{ \partial_{\lambda} \log f_+ - \partial_{\lambda} \log f_- \} \end{aligned} \quad (6.16)$$

at both endpoints  $q$  and  $-q$ .

Consider, for instance,  $R^{(0)}(-q, -q)$ . We have for  $\lambda \in D_{-q, \delta}$ ,

$$\begin{aligned} R^{(0)}(\lambda, \lambda) &= -\nu(\lambda) \Phi(-\nu, 1; -i\zeta_{-q}) \Phi(\nu, 1; i\zeta_{-q}) \left\{ 2\partial_{\lambda} \log e_+(\lambda) \right. \\ &\quad - 2\partial_{\lambda} [\log \kappa_p(\lambda) + \nu(\lambda) \log \zeta_q] + \nu'(\lambda) [\psi(1+\nu) + \psi(1-\nu)] \\ &\quad - ix p'(\lambda) \left[ \frac{(\partial_z \Phi)(\nu, 1; i\zeta_{-q})}{\Phi(\nu, 1; i\zeta_{-q})} + \frac{(\partial_z \Phi)(-\nu, 1; -i\zeta_{-q})}{\Phi(-\nu, 1; -i\zeta_{-q})} \right] \\ &\quad \left. - \nu'(\lambda) \left[ \frac{(\partial_1 \Phi)(\nu, 1; i\zeta_{-q})}{\Phi(\nu, 1; i\zeta_{-q})} + \frac{(\partial_1 \Phi)(-\nu, 1; -i\zeta_{-q})}{\Phi(-\nu, 1; -i\zeta_{-q})} \right] \right\}, \end{aligned} \quad (6.17)$$

where, so as to lighten the formula, we have omitted the argument  $\lambda$  of  $\nu(\lambda)$  when  $\nu$  appears as an argument of another function (here  $\psi$  or  $\Phi$ ). The symbol  $\partial_z$  stands for the derivative of a CHF with respect to its variable, whereas  $\partial_1$  stands for the derivative with respect to its first argument. Recall also that  $\zeta_{-q} = x[p(\lambda) - p(-q)]$  and  $\zeta_q = x[p(q) - p(\lambda)]$ .

It is remarkable that the last two terms involving derivatives of CHF vanish in the  $\lambda \rightarrow -q$  limit. The resulting expression can be further simplified thanks to the identities:

$$\log \kappa_p(\lambda) = \log \kappa(\lambda) + \nu(\lambda) \left\{ \log \left( \frac{q - \lambda}{p_+ - p(\lambda)} \right) - \log \left( \frac{\lambda + q}{p(\lambda) - p_-} \right) \right\}, \quad (6.18)$$

$$\nu(\lambda) \nu'(\lambda) [\psi(1+\nu) + \psi(1-\nu)] = \partial_{\lambda} \log G(1, \nu) + 2\nu(\lambda) \nu'(\lambda), \quad (6.19)$$

Thus, we obtain

$$\begin{aligned} R^{(0)}(-q, -q) &= -2\nu_- [\partial_{\lambda} \log e_+(\lambda)] \Big|_{\lambda=-q} + 2\nu'_- \nu_- \log x \\ &\quad + 2\nu'_- \nu_- \log(2qp'_-) - \frac{\nu_-^2}{q} + \nu_-^2 \frac{p''_-}{p'_-} - 2\nu_- \nu'_- + 2\nu_- [\partial_{\lambda} \log \kappa(\lambda)] \Big|_{\lambda=-q}, \end{aligned} \quad (6.20)$$

where we have used the notations (2.3), (2.4).

The final aim is to integrate (6.14) over the variable  $q$ . One should keep in mind that the function  $\kappa(\lambda) \equiv \kappa(\lambda; q)$  is actually a function of the two parameters  $\lambda$  and  $q$ . Therefore, one should replace partial  $\lambda$  derivatives at  $\lambda = \pm q$  by total  $q$  derivatives thanks to

$$\frac{d}{dq} [\log \kappa(-q; q)] = -\partial_\lambda \log \kappa(\lambda; q) \Big|_{\lambda=-q} + \partial_q \log \kappa(\lambda; q) \Big|_{\lambda=-q}. \quad (6.21)$$

Then  $R^{(0)}(-q, -q)$  is almost a total  $q$  derivative:

$$\begin{aligned} R^{(0)}(-q, -q) &= -2\nu_- [\partial_\lambda \log e_+(\lambda)] \Big|_{\lambda=-q} + \frac{d}{dq} \log \left[ \frac{G(1, \nu_-)}{(2qp'_-x)^{\nu_-^2}} \right] \\ &\quad - 2\nu_- \frac{d}{dq} [\log \kappa(-q; q)] + \nu_- \frac{(\nu_+ - \nu_-)}{q}. \end{aligned} \quad (6.22)$$

Similar calculations based on the expressions (5.20) for  $f_\pm^{(0)}$  around  $q$  lead to

$$\begin{aligned} R^{(0)}(q, q) &= -2\nu_+ [\partial_\lambda \log e_+(\lambda)] \Big|_{\lambda=q} + \frac{d}{dq} \log \left[ \frac{G(1, \nu_+)}{(2qp'_+x)^{\nu_+^2}} \right] \\ &\quad + 2\nu_+ \frac{d}{dq} [\log \kappa(q; q)] - \nu_+ \frac{(\nu_+ - \nu_-)}{q}. \end{aligned} \quad (6.23)$$

Hence, we have

$$\begin{aligned} \partial_q \log \det [I + V] &= 2 \sum_{\sigma=\pm} \nu_\sigma [\partial_\lambda \log e_-(\lambda)] \Big|_{\lambda=\sigma q} + \frac{d}{dq} \log \left[ \frac{G(1, \nu_+) G(1, \nu_-)}{(2qp'_-x)^{\nu_-^2} (2qp'_+x)^{\nu_+^2}} \right] \\ &\quad + 2 \sum_{\sigma=\pm} \sigma \nu_\sigma \frac{d \log \kappa(\sigma q; q)}{dq} - \frac{(\nu_+ - \nu_-)^2}{q} + o(1) \end{aligned} \quad (6.24)$$

It remains to express the last line as a total  $q$  derivative thanks to Lemma B.3. After an integration with respect to  $q$  of (6.24) we arrive to

$$\begin{aligned} \log \det [I + V] &= 2 \int_{-q}^q d\lambda \nu(\lambda) \log' [e_-(\lambda)] + \log \left[ \frac{G(1, \nu_+) G(1, \nu_-) \kappa^{\nu_+}(q; q)}{(2qp'_+x)^{\nu_+^2} (2qp'_-x)^{\nu_-^2} \kappa^{\nu_-}(-q; q)} \right] \\ &\quad + \frac{1}{2} \int_{-q}^q d\lambda d\mu \frac{\nu'(\lambda) \nu(\mu) - \nu(\lambda) \nu'(\mu)}{\lambda - \mu} + C + o(1), \end{aligned} \quad (6.25)$$

where  $C$  is a  $q$ -independent integration constant still to be determined.

One can find arguments that this constant should be also  $\gamma$ -independent. Indeed, the asymptotic expansion of the Fredholm determinant, being a functional of the holomorphic function  $\gamma F(\lambda)$  in  $\overline{U}$ , can depend on this function either in the integral form with

integration over  $[-q; q]$ , or through the values of  $\gamma F$  and of its derivatives at the ends of the integration contour  $-q$  and  $q$ . In both cases the result should depend on  $q$ . Hence, the  $q$ -independent constant  $C$  can not depend on  $\gamma F(\lambda)$  and, thus, it is  $\gamma$ -independent. We can then fix the constant  $C$  by setting  $\gamma = 0$  in the asymptotic formula. This yields  $C = 0$ .

### 6.3 The first corrections to the leading asymptotics of the Fredholm determinant

We close this Section by deriving the sub-leading corrections from the  $x$  derivative of  $\log \det [I + V]$  (3.19). This will constitute the proof of Proposition 2.1.

In order to prove the claim of the Proposition 2.1, one has to derive the first two sub-leading corrections for the matrix  $\Pi$ . As one might expect the computations are, by far, simpler than those necessary to fix the constant. We also would like to point out that one can obtain the sub-leading asymptotic of  $\det [I + V]$  by the  $q$ -derivative method. However, the computations are quite involved, so we omit the presentation of this method.

We derive the first term in the  $1/x$  expansion of  $\log \det [I + V]$  thanks to (3.12):

$$\partial_x \log \det [I + V] = \oint_{\Gamma([-q; q])} \frac{d\lambda}{4\pi} p(\lambda) \operatorname{tr} [\partial_\lambda \chi(\lambda) \sigma_3 \chi^{-1}(\lambda)] , \quad (6.26)$$

where we chose the contour  $\Gamma([-q; q])$  to lie outside of the contour  $\Sigma_\Pi$  but still in the region of holomorphy for  $p$ . There the solution for the RHP for  $\chi$  has a simple form:

$$\chi(\lambda) = \Pi(\lambda) \alpha^{-\sigma_3}(\lambda) . \quad (6.27)$$

In order to derive the first correction to the leading asymptotics, it is enough to consider the first two terms in the asymptotic expansion for  $\Pi(\lambda)$ :

$$\Pi(\lambda) = I_2 + \frac{\Pi^{(1)}(\lambda)}{x} + \frac{\Pi^{(2)}(\lambda)}{x^2} + \mathcal{O}\left(x^{3(\bar{\varepsilon}-1)}\right) . \quad (6.28)$$

There, as follows from (5.15), the  $\mathcal{O}$  is uniform on the whole contour  $\Gamma([-q; q])$ . Thus

$$\begin{aligned} \partial_x \log \det [I + V] &= - \oint_{\Gamma([-q; q])} \frac{d\lambda}{2\pi} p(\lambda) \frac{\partial_\lambda \alpha(\lambda)}{\alpha(\lambda)} + \frac{1}{x} \oint_{\Gamma([-q; q])} \frac{d\lambda}{4\pi} p(\lambda) \operatorname{tr} \left[ \sigma_3 \partial_\lambda \Pi^{(1)}(\lambda) \right] \\ &+ \frac{1}{x^2} \oint_{\Gamma([-q; q])} \frac{d\lambda}{4\pi} p(\lambda) \operatorname{tr} \left\{ \sigma_3 \left[ \partial_\lambda \Pi^{(2)}(\lambda) - \Pi^{(1)}(\lambda) \partial_\lambda \Pi^{(1)}(\lambda) \right] \right\} + \mathcal{O}\left(x^{3(\bar{\varepsilon}-1)}\right) . \end{aligned} \quad (6.29)$$

The first term in this expansion will yield the leading correction. Indeed,

$$- \oint_{\Gamma([-q; q])} \frac{d\lambda}{2\pi} p(\lambda) \frac{\partial_\lambda \alpha(\lambda)}{\alpha(\lambda)} = \oint_{\Gamma([-q; q])} \frac{d\lambda}{2\pi} p'(\lambda) \log \alpha(\lambda) = -i \int_{-q}^q d\lambda p'(\lambda) \nu(\lambda) . \quad (6.30)$$

Here we shrank the contour to  $[-q; q]$  and used the jump condition for  $\alpha$ .

In order to evaluate the higher order corrections in (6.29) we need to derive the expressions for the matrices  $\Pi^{(1)} = C_{\Sigma_{\Pi}}[\Delta^{(1)}]$  and  $\Pi^{(2)} = C_{\Sigma_{\Pi}}[\Pi_+^{(1)}\Delta^{(1)} + \Delta^{(2)}]$  outside of  $\Sigma_{\Pi}$ . An elementary computation of residues yields:

$$\partial_{\lambda}\Pi^{(1)}(\lambda) = -\frac{\Delta_{(+)}^{(1)}(q; x)}{(\lambda - q)^2 p'_+} - \frac{\Delta_{(-)}^{(1)}(-q; x)}{(\lambda + q)^2 p'_-}, \quad (6.31)$$

as well as

$$\begin{aligned} & \partial_{\lambda}\Pi^{(2)}(\lambda) - \Pi^{(1)}(\lambda) \partial_{\lambda}\Pi^{(1)}(\lambda) \\ &= -\sum_{\sigma=\pm} \frac{\left(\partial_{\lambda}\Delta_{(\sigma)}^{(2)}\right)(\sigma q; x) + \Delta_{(\sigma)}^{(1)}(\sigma q; x) \left(\partial_{\lambda}\Delta_{(\sigma)}^{(1)}\right)(\sigma q; x)}{[(\lambda - \sigma q) p'_{\sigma}]^2} \\ &+ \sum_{\sigma=\pm} \frac{2\Delta_{(\sigma)}^{(2)}(\sigma q; x) + \left[\Delta_{(\sigma)}^{(1)}(\sigma q; x)\right]^2}{2(p'_{\sigma})^2} \left\{ \frac{p''_{\sigma}}{p'_{\sigma}(\lambda - \sigma q)^2} - \frac{2}{(\lambda - \sigma q)^3} \right\} \\ &+ \frac{\left[\Delta_{(+)}^{(1)}(q; x), \Delta_{(-)}^{(1)}(-q; x)\right]}{2q p'_+ p'_- (\lambda - q)(\lambda + q)}. \end{aligned} \quad (6.32)$$

Thus the  $1/x$  term in (6.29) gives the coefficient in front of  $\log x$  in (2.6). Indeed,

$$\begin{aligned} \oint_{\Gamma([-q; q])} \frac{d\lambda}{4\pi} p(\lambda) \operatorname{tr} \left[ \sigma_3 \partial_{\lambda}\Pi^{(1)}(\lambda) \right] &= \frac{1}{2i} \operatorname{tr} \left[ \sigma_3 \Delta_{(-)}^{(1)}(-q; x) + \sigma_3 \Delta_{(+)}^{(1)}(q; x) \right] \\ &= -(\nu_+^2 + \nu_-^2). \end{aligned} \quad (6.33)$$

We now focus on the last term in (6.29). It yields, after an  $x$  integration, the first correction to (2.6). A straightforward computation leads to

$$\begin{aligned} & \oint_{\Gamma([-q; q])} \frac{d\lambda}{2i\pi} p(\lambda) \operatorname{tr} \left\{ \sigma_3 \left[ \partial_{\lambda}\Pi^{(2)}(\lambda) - \Pi^{(1)}(\lambda) \partial_{\lambda}\Pi^{(1)}(\lambda) \right] \right\} \\ &= \frac{p_+ - p_-}{4q^2 p'_- p'_+} \operatorname{tr} \left\{ \sigma_3 \left[ \Delta_{(+)}^{(1)}(q; x), \Delta_{(-)}^{(1)}(-q; x) \right] \right\} \\ &- \sum_{\sigma=\pm} \frac{1}{p'_{\sigma}} \operatorname{tr} \left\{ \sigma_3 \left[ \left(\partial_{\lambda}\Delta_{(\sigma)}^{(2)}\right)(\sigma q; x) + \Delta_{(\sigma)}^{(1)}(\sigma q; x) \left(\partial_{\lambda}\Delta_{(\sigma)}^{(1)}\right)(\sigma q; x) \right] \right\}. \end{aligned} \quad (6.34)$$

The first term corresponds to the oscillating correction:

$$\operatorname{tr} \left\{ \sigma_3 \left[ \Delta_{(+)}^{(1)}(q; x), \Delta_{(-)}^{(1)}(-q; x) \right] \right\} = 2\nu_+ \nu_- \left\{ \frac{\tilde{u}(q; x)}{u(-q; x)} - \frac{u(-q; x)}{\tilde{u}(q; x)} \right\}. \quad (6.35)$$

The last term gives the non-oscillating one:

$$\operatorname{tr} \left\{ \sigma_3 \left(\partial_{\lambda}\Delta_{(\sigma)}^{(2)}\right)(\sigma q; x) \right\} = -2 \frac{d\nu_{\sigma}^3}{dq}, \quad (6.36)$$



and

$$\begin{aligned}
& \sum_{\sigma=\pm} \frac{1}{p'_\sigma} \operatorname{tr} \left\{ \sigma_3 \Delta_{(\sigma)}^{(1)}(\sigma q; x) \left( \partial_\lambda \Delta_{(\sigma)}^{(1)}(\sigma q; x) \right) \right\} \\
&= \frac{2}{p'_+} \nu_+^2 \partial_\lambda (\log \tilde{u}(\lambda; x)) \Big|_{\lambda=q} + \frac{2}{p'_-} \nu_-^2 \partial_\lambda (\log u(\lambda; x)) \Big|_{\lambda=-q} \\
&= \sum_{\sigma=\pm} \frac{2\nu_\sigma^2}{p'_\sigma} \left\{ 2\sigma\nu'_\sigma \log x + \sigma \frac{d}{dq} \log u_\sigma + \frac{p'_\sigma}{\nu_\sigma} \frac{d}{dq} \left( \frac{\nu_\sigma^2}{p'_\sigma} \right) - \frac{\nu_{-\sigma}}{p'_-} \right\}, \quad (6.37)
\end{aligned}$$

with

$$u_+ = e^{g(q)} \frac{\Gamma(1 - \nu_+)}{\Gamma(1 + \nu_+)} \left\{ \frac{(2qp'_+)^{\nu_+}}{\kappa(q; q)} \right\}^2, \quad (6.38)$$

$$u_- = e^{g(-q)} \frac{\Gamma(1 + \nu_-)}{\Gamma(1 - \nu_-)} \left\{ (2qp'_-)^{\nu_-} \kappa(-q; q) \right\}^{-2}. \quad (6.39)$$

Putting all this together we obtain

$$\begin{aligned}
\partial_x \log \det [I + V] &= -i \int_{-q}^q d\lambda \nu(\lambda) p'(\lambda) - \frac{\nu_+^2 + \nu_-^2}{x} - 2i \frac{\log x - 1}{x^2} \sum_{\sigma=\pm} \frac{\nu_\sigma^2}{p'_\sigma} \frac{d\nu_\sigma}{dq} \\
&- \frac{i}{x^2} \sum_{\sigma=\pm} \frac{\nu_\sigma^2}{p'_\sigma} \left\{ \sigma \frac{d}{dq} \log u_\sigma + p'_\sigma \frac{d}{dq} \left( \frac{\nu_\sigma}{p'_\sigma} \right) - \frac{\nu_{-\sigma}}{q} \right\} \\
&+ \frac{i(p_+ - p_-) \nu_- \nu_+}{(2q)^2 p'_+ p'_- x^2} \left\{ \frac{u_+}{u_-} x^{2(\nu_+ + \nu_-)} e^{ix(p_+ - p_-)} - \frac{u_-}{u_+} x^{-2(\nu_+ + \nu_-)} e^{ix(p_- - p_+)} \right\} \\
&+ O \left( \frac{e^{ix(p_+ - p_-)}}{x^{3(1-\bar{\varepsilon})}}, \frac{1}{x^{3(1-\bar{\varepsilon})}} \right). \quad (6.40)
\end{aligned}$$

The first two terms reproduce the already known answer for the leading asymptotics. The remaining ones reproduce the first oscillating and non-oscillating corrections as given in Proposition 2.1. Note that for the oscillating terms, one only should integrate the exponent with respect to  $x$  as all the other terms will give subdominant contributions.

## 7 Applications

### 7.1 The Akhiezer–Kac formula

Truncated Wiener–Hopf operators appear in many domains of mathematical physics such as scattering or diffusion processes. Moreover, many observables (dressed energy, momentum or dressed charge) related to quantum integrable models are solutions of integral equations of truncated Wiener–Hopf type (2.20).

Let us recall that a truncated Wiener–Hopf operator can be interpreted as an integral operator  $I + K$  on  $L^2(\mathbb{R})$  such that it acts on  $L^2(\mathbb{R})$  functions according to

$$(I + K) \cdot \varphi(t) = \varphi(t) + \int_0^x dt' K(t - t') \varphi(t') dt'. \quad (7.1)$$

The kernel  $K$  is characterized in terms of its Fourier transform  $F$ :

$$K(t) = \mathcal{F}^{-1}[F](t), \quad \text{with} \quad \mathcal{F}^{-1}[F](t) = \frac{1}{2\pi} \int_{\mathbb{R}} d\xi F(\xi) e^{-it\xi}, \quad \forall F \in L^1(\mathbb{R}). \quad (7.2)$$

The study of truncated Wiener–Hopf operators is equivalent to a  $2 \times 2$  matrix RHP. Another facet of this equivalence is the correspondence between a truncated Wiener–Hopf operator and the GSK acting on  $\mathbb{R}$  in which  $p = \text{id}$  and  $g = 0$ . Indeed, it is easy to see that

$$K \cdot \varphi = \mathcal{F}^{-1} \circ \tilde{V} \circ \mathcal{F}[\varphi], \quad (7.3)$$

where  $\tilde{V}$  acts in  $L^2(\mathbb{R})$  with a kernel

$$\tilde{V}(\xi, \eta) = F(\xi) \frac{e^{ix(\xi - \eta)} - 1}{2i\pi(\xi - \eta)}. \quad (7.4)$$

The operator identity

$$I + K = \mathcal{F}^{-1} \cdot (I + \tilde{V}) \mathcal{F}, \quad (7.5)$$

together with the facts that  $\tilde{V}$  is trace-class and  $\mathcal{F}^{\pm 1}$  are continuous, ensures the equality between the Fredholm determinants:

$$\det[I + K] = \det[I + \tilde{V}]. \quad (7.6)$$

The kernel  $\tilde{V}$  is related to

$$V(\xi, \eta) = \sqrt{F(\xi)F(\eta)} \frac{\sin(x(\xi - \eta)/2)}{\pi(\xi - \eta)} \quad (7.7)$$

by a similarity transformation. Hence,

$$\det[I + K]_{L^2(0,x)} = \det[I + V]_{L^2(\mathbb{R})}. \quad (7.8)$$

Our study of the generalized sine kernel allows us to recover the Akhiezer–Kac formula describing the large  $x$  behavior of Fredholm determinants of truncated Wiener–Hopf operators:

**Theorem 7.1 (Akhiezer–Kac [1, 29])** *Let  $I + K$  be a truncated Wiener–Hopf operator as above and such that*

- $F$  is analytic in an open neighborhood  $U$  of  $\mathbb{R}$  ;
- $F$  goes sufficiently fast to 0 at  $\pm\infty$  ;
- $1 + F(\xi)$  does not vanish on  $U$ .

Then

$$\log \det [I + K] = x \tau(0) + E[F] + o(1), \quad \text{with} \quad E[F] = \int_0^{\infty} t \tau(t) \tau(-t) dt, \quad (7.9)$$

in which

$$\tau(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \log(F(\xi) + 1) e^{-it\xi} d\xi. \quad (7.10)$$

*Proof* — The large  $x$  asymptotics of  $\det [I + K]$  follows from (7.8) after taking the  $q \rightarrow +\infty$  limit in the leading asymptotics for the corresponding generalized sine kernel (2.6). This limit may seem a little heuristic as we did not specify any estimates in  $q$  for the small  $o$  terms with respect to the leading asymptotics. However, the validity of such a limit may either be seen by refining all the estimates obtained in the previous Section or by considering the RHP for  $\chi$  (3.1) on the whole real line from the very beginning. We shall make the second approach more explicit in the forthcoming subsection 7.2. Here we formally take the  $q \rightarrow +\infty$  limit in the leading asymptotics of Theorem 2.1.

One should notice that, in the asymptotic formula (2.6), all the terms evaluated at the endpoints vanish due to the fact that  $\nu(\pm q) \log q \rightarrow 0$  when  $q \rightarrow +\infty$ , which is a consequence of the sufficiently fast decrease of  $F$  at infinity. Hence, the only constant contribution  $E[F]$  to the asymptotics of  $\log \det [I + K]$  is given by the integral

$$E[F] = \lim_{q \rightarrow +\infty} \frac{1}{2} \int_{-q}^q \frac{\nu'(\lambda) \nu(\mu) - \nu(\lambda) \nu'(\mu)}{\lambda - \mu} d\lambda d\mu. \quad (7.11)$$

Let us recast the constant  $E[F]$  in a more standard form. We have

$$\begin{aligned} E[F] &= -\frac{1}{8\pi^2} \int_{\mathbb{R}} d\xi d\eta \frac{\log'(F(\xi) + 1) \log(F(\eta) + 1) - \log'(F(\eta) + 1) \log(F(\xi) + 1)}{\xi - \eta} \\ &= \frac{i}{16\pi^2} \int_{\mathbb{R}} d\xi d\eta dx dy \left\{ \frac{1}{\xi - \eta + i0} + \frac{1}{\xi - \eta - i0} \right\} \tau(x) \tau(y) (x - y) e^{ix\eta + iy\xi}. \end{aligned}$$

Let  $H$  be the Heaviside function, then

$$\begin{aligned}
E[F] &= -\frac{1}{8\pi} \int_{\mathbb{R}} d\eta dx dy \tau(x) \tau(y) (x-y) \left( e^{i\eta(x+y)} H(y) - e^{i\eta(x+y)} H(-y) \right) \\
&= -\frac{1}{4} \int_0^{+\infty} dy \tau(y) \tau(-y) (-2y) + \frac{1}{4} \int_{-\infty}^0 dy \tau(y) \tau(-y) (-2y) \\
&= \int_0^{+\infty} dy \tau(y) \tau(-y) y \quad .
\end{aligned}$$

□

It happens that this correspondence between truncated Wiener–Hopf operators and generalized sine kernels can be pushed further so as to obtain the asymptotic behavior of Fredholm determinants of truncated Wiener–Hopf operators with symbols having Fischer–Hartwig type discontinuities. Considering the GSK for finite  $q$  somehow corresponds to the asymptotic behavior of a determinant whose symbol has two jumps. The case of symbols having general Fischer–Hartwig type singularities is studied in [32].

## 7.2 The resolvent of truncated Wiener–Hopf operators

**Proposition 7.1** *Let  $I + K$  be a truncated Wiener–Hopf operator on  $] -x ; x [$ ,*

$$[(I + K).g](t) = g(t) + \int_{-x}^x K(t-t') g(t') dt', \quad \text{with } K(t) = \mathcal{F}^{-1}[F](t). \quad (7.12)$$

*Suppose that there exists  $\delta > 0$  such that*

- *$F$  admits an analytic continuation to  $\{z : |\Im(z)| \leq \delta\}$ ;*
- *$\xi \mapsto F(\xi \pm i\delta) \in L^1(\mathbb{R})$ ;*
- *the analytic continuation of  $1 + F$  does not vanish on  $U$ .*

*Then the resolvent  $I - R$  of  $I + K$  fulfills*

$$\begin{aligned}
R(\lambda, \mu) &= \int_{\mathbb{R}} \frac{d\xi d\eta}{4i\pi^2} F(\xi) \left\{ \frac{\alpha_+(\eta)}{\alpha_-(\xi)} e^{ix(\xi-\eta)} - \frac{\alpha_+(\xi)}{\alpha_-(\eta)} e^{-ix(\xi-\eta)} \right\} \frac{e^{i(\mu\eta-\lambda\xi)}}{\xi-\eta} \\
&\quad + O(e^{-2\delta x}), \quad (7.13)
\end{aligned}$$

*where*

$$\alpha(\lambda) = \exp \left\{ \int_{\mathbb{R}} \frac{\nu(\mu)}{\mu-\lambda} d\mu \right\}, \quad \text{and } \nu(\lambda) = \frac{i}{2\pi} \log(1 + F(\lambda)). \quad (7.14)$$

*Proof* — The GSK associated to  $I + K$  through the transformation  $\mathcal{F}^{-1} \circ [I + V] \circ \mathcal{F} = I + K$  acts on the whole real axis with the kernel

$$V(\xi, \eta) = F(\xi) \frac{e^{i(\xi-\eta)x} - e^{i(\eta-\xi)x}}{2i\pi(\xi - \eta)}. \quad (7.15)$$

Just as for the leading asymptotics of  $\log \det [I + K]$  (see Section 7.2), one can obtain the leading asymptotic of the resolvent of  $V$  just by taking formally the limit  $q \rightarrow +\infty$  in all the expressions derived in the former part of the article. Note that in this process all power law corrections vanish: they are computed as contour integrals around  $\pm q$  and, since  $F$  approaches 0 sufficiently fast at infinity, the residues at  $\pm q$  vanish in the  $q \rightarrow +\infty$  limit. However, in order to justify this limit, one should also check that all the uniform estimates still hold for  $q \rightarrow +\infty$ .

An alternative way is to consider from the very beginning a RHP for  $\chi$  on the whole real axis  $\mathbb{R}$ . This is actually much simpler, than the RHP on a finite interval. Then it is enough to perform the first two transformations described in Section 4 so as to obtain jump matrices that are already uniformly close to  $I_2$  up to exponentially small corrections in  $x$ . Moreover, the jump matrices for this RHP are given by  $M_+$  and  $M_-^{-1}$  (4.11), so that they approach the identity matrix at  $\lambda \rightarrow \infty$  just as fast as  $F$  goes to zero at infinity. As expected, there is no need for parametrices, and the corrections are immediately exponentially decreasing with  $x$ . It means that, up to uniformly exponentially small corrections, the resolvent  $R_V$  of  $V$  is given by

$$R_V^{(0)}(\xi, \eta) = \frac{F(\xi)}{2i\pi(\xi - \eta)} \left\{ \frac{\alpha_+(\eta)}{\alpha_-(\xi)} e^{ix(\xi-\eta)} - \frac{\alpha_+(\xi)}{\alpha_-(\eta)} e^{-ix(\xi-\eta)} \right\}, \quad (7.16)$$

where as usual  $\alpha(\lambda)$  is given by (7.14). Note that the integral in (7.14) is well defined in virtue of the assumptions made on  $F$ .

We should now take the Fourier/inverse Fourier of  $R_V$  in order to get  $R$ . To this end, we must justify that the sub-leading corrections do admit a Fourier transform in two variables. Recall the exact expression for the resolvent:

$$R_V(\lambda, \mu) = R_V^{(0)}(\lambda, \mu) + \langle F^{L;(0)}(\lambda) | \frac{\Pi^{-1}(\lambda)\Pi(\mu) - I_2}{\lambda - \mu} | F^{R;(0)}(\mu) \rangle. \quad (7.17)$$

Thereby, the matrix  $\Pi$  is defined in terms of  $\Pi_+(\lambda)$ , the limiting value of  $\Pi$  on  $\Sigma_\Pi$  when  $\lambda$  approaches a point of  $\Sigma_\Pi$  from the “+” side of the contour:

$$\begin{aligned} \Pi(\lambda) = I_2 + \int_{\Gamma_+} \frac{dz}{\lambda - z} \Pi_+(z) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{F(z) \alpha_+^{-2}(z)}{1 + F(z)} e^{2ixz} \\ + \int_{\Gamma_-} \frac{dz}{\lambda - z} \Pi_+(z) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{F(z) \alpha_-^2(z)}{1 + F(z)} e^{2ixz}. \end{aligned} \quad (7.18)$$

The  $L^1$  integrability of  $F$  as well as the asymptotic condition  $\Pi(\lambda) \xrightarrow{\lambda \rightarrow +\infty} I_2$  guarantee that the integrals are well defined. Thus one readily infers from (7.18) the asymptotics

of  $\Pi$  on the real axis:

$$\Pi(\lambda) = I_2 + \frac{e^{-2\delta x} C}{\lambda} + \mathcal{O}\left(\frac{e^{-2\delta x}}{\lambda^2}\right), \quad (7.19)$$

where  $C$  is some constant matrix and where we have explicitly extracted the exponential decay in  $x$  of the matrix  $C$ . Hence using the boundedness of  $f_{\pm}^{(0)}$  on the real axis we obtain that

$$R_V(\lambda, \mu) = R_V^{(0)}(\lambda, \mu) + e^{-2\delta x} C \frac{F(\lambda)}{\lambda\mu} + o\left(\frac{e^{-2\delta x} F(\lambda)}{\lambda\mu}\right). \quad (7.20)$$

The corrections thus admit a Fourier transform in  $\lambda$  and an inverse one in  $\mu$  as oscillatory integrals. Therefore, taking the Fourier transform does not change the nature of the corrections.  $\square$

It is clear that, up to a similarity transformation, a Wiener–Hopf operator on  $[-a; b]$  has the same generalized sine kernel than the same operator acting on  $[0; a + b]$ . Therefore our method works for any interval, of course up to a similarity transformation on the resolvent (7.16) of  $V$ . We chose here to present this less standard form of Wiener–Hopf operators as it fits better the forthcoming application.

We apply our asymptotic inversion formula for truncated Wiener–Hopf operators acting on a symmetric interval  $[-x; x]$  to re-derive some formulas concerning the low magnetic field behavior of the so-called dressed charge [6]. This function, traditionally denoted  $Z(\lambda)$ , describes the intrinsic magnetic moment of an elementary excitation above the ground state in the XXZ spin-1/2 model. It satisfies the integral equation:

$$Z(\lambda) + \int_{-x}^x d\mu K(\lambda - \mu) Z(\mu) = 1, \quad \text{with } K(\lambda) = \frac{\sin 2\zeta}{2\pi \sinh(\lambda + i\zeta) \sinh(\lambda - i\zeta)}. \quad (7.21)$$

Thereby  $K$  is often called the Lieb kernel and  $\zeta \in ]0; \pi[$  is some real parameter describing the coupling of the model. The large parameter  $x$  is a function of the external longitudinal magnetic field; it goes to infinity when the magnetic field vanishes.

For the study of  $Z$ , one should distinguish two domains in the interval  $[-x; x]$ : the bulk, i.e. the region  $|\lambda| \ll x$ , and the boundaries  $\lambda \sim \pm x$ . While the asymptotic value of  $Z(\lambda)$  in the bulk ( $|\lambda| \ll x$ ) is enough to describe the intrinsic magnetic moment of elementary excitations, the value of  $Z$  at the boundaries ( $\pm x$ ) determines the critical exponents of the two-point functions of the model [19, 20, 21, 5]. As we will see, the bulk and the boundary behavior of the dressed charge differ fundamentally.

First, let us note that, setting directly  $x = +\infty$  in (7.21), one can solve explicitly the integral equation for  $Z$  by taking the Fourier transform: one obtains in this case that  $Z(\lambda)$  is equal to a constant value  $Z(\lambda) = \pi / [2(\pi - \zeta)]$  on the whole real axis.

Let us now consider the limit  $x \rightarrow \infty$  in (7.21) in a more accurate way, namely, taking  $x$  large but finite, and use the method described above. Let  $\widehat{K}$  be the Fourier transform of  $K$ ,

$$\widehat{K}(\xi) \equiv \mathcal{F}[K](\xi) = \frac{\sinh[\xi(\zeta - \pi/2)]}{\sinh(\xi\zeta/2)}. \quad (7.22)$$

Then in virtue of Proposition 7.1,

$$Z(\lambda) = 1 - \int_{-x}^x R(\lambda, \mu) d\mu \quad (7.23)$$

$$= 1 - \int_{\mathbb{R}} \frac{d\xi}{2i\pi} \frac{\widehat{K}(\xi)}{\xi} \left\{ \frac{\alpha_+(0)}{\alpha_-(\xi)} e^{i(x-\lambda)\xi} - \frac{\alpha_+(\xi)}{\alpha_-(0)} e^{-i(x+\lambda)\xi} \right\}. \quad (7.24)$$

Let us first study the bulk limit i.e.  $|\lambda| \ll x$ . Using the jump equation satisfied by  $\alpha_{\pm}$ :  $[1 + \widehat{K}(\lambda)]\alpha_+(\lambda) = \alpha_-(\lambda)$ , we recast the integrand as

$$\begin{aligned} Z(\lambda) &= 1 - \int_{\mathbb{R}} \frac{d\xi}{2i\pi} \frac{\widehat{K}(\xi)}{[1 + \widehat{K}(\xi)](\xi - i0^+)} \left\{ \frac{\alpha_+(0)}{\alpha_+(\xi)} e^{i(x-\lambda)\xi} - \frac{\alpha_-(\xi)}{\alpha_-(0)} e^{-i(x+\lambda)\xi} \right\} \\ &= \frac{1}{1 + \widehat{K}(0)} - \int_{\mathbb{R}} \frac{d\xi}{2i\pi} \left\{ \frac{\widehat{K}(\xi + i\zeta/2)}{1 + \widehat{K}(\xi + i\zeta/2)} \frac{\alpha_+(0)}{\alpha_+(\xi + i\zeta/2)} \frac{e^{(x-\lambda)(i\xi - \zeta/2)}}{\xi + \zeta/2} \right. \\ &\quad \left. - \frac{\widehat{K}(\xi - i\zeta/2)}{1 + \widehat{K}(\xi - i\zeta/2)} \frac{\alpha_-(0)}{\alpha_-(\xi - i\zeta/2)} \frac{e^{-(x+\lambda)(i\xi + \zeta/2)}}{\xi - \zeta/2} \right\}. \quad (7.25) \end{aligned}$$

Here we have separated the integral into two parts and then moved the contour to the upper/lower half-plane. This gives a pole contribution from  $\xi = i0$ . The integral appearing in (7.25) is clearly a  $O(e^{-(x-|\lambda)|\zeta/2})$ . So that, in the bulk,

$$Z(\lambda) \sim \frac{1}{1 + \widehat{K}(0)} = \frac{\pi}{2(\pi - \zeta)}, \quad (7.26)$$

up to exponentially small corrections. As expected, we recover the value of  $Z$  obtained in the case of an infinite interval. Note that the corrections become larger and larger as we approach any of the endpoints  $\pm x$ .

Let us now study the behavior of the dressed charge at the boundaries. Since the kernel  $K$  is even, so is  $Z$ . We can thus focus on a single boundary, say  $\lambda = x$ . We have,

$$Z(x) = 1 - \int_{\mathbb{R}} \frac{d\xi}{2i\pi} \frac{1}{(\xi - i0^+)} \left\{ \widehat{K}(\xi) \frac{\alpha_+(0)}{\alpha_-(\xi)} - \frac{\widehat{K}(\xi)}{1 + \widehat{K}(\xi)} \frac{\alpha_-(\xi)}{\alpha_-(0)} e^{-2ix\xi} \right\}. \quad (7.27)$$

As before, the integral of the second term gives an exponentially small contribution  $O(e^{-x\zeta})$ . The integral of the first term is explicitly computable. Using once again the jump equation satisfied by  $\alpha_{\pm}(\xi)$ , we have,

$$\begin{aligned} Z(x) &= 1 - \alpha_+(0) \int_{\mathbb{R}} \frac{d\xi}{2i\pi} \frac{\alpha_+^{-1}(\xi) - 1 + 1 - \alpha_-^{-1}(\xi)}{\xi - i0} + O(e^{-x\zeta}) \\ &= 1 - \alpha_+(0) \int_{\mathbb{R}} \frac{d\xi}{2i\pi} \frac{\alpha_+^{-1}(\xi) - 1}{\xi - i0} + O(e^{-x\zeta}) \\ &= \alpha_+(0) + O(e^{-x\zeta}). \quad (7.28) \end{aligned}$$

We have computed the remaining integral by residues, since  $\alpha_{\pm}^{-1}(\xi) - 1 = O(\xi^{-1})$  for  $\xi \rightarrow \infty$  in the respective half plane of holomorphy.

For an even kernel like the Lieb one  $\widehat{K}(\xi) = \widehat{K}(-\xi)$ , and then it follows from the integral representation (7.14) of  $\alpha$  that  $\alpha_+(\xi) = \alpha_-^{-1}(-\xi)$ . This means that  $1 + \widehat{K}(0) = \alpha_+^{-2}(0)$ . Hence, for  $x$  large enough

$$Z(x) \sim \sqrt{\frac{1}{1 + \widehat{K}(0)}} = \sqrt{\frac{\pi}{2(\pi - \zeta)}}, \quad (7.29)$$

and the value of  $Z(\lambda)$  at the boundary is the square root of its value in the bulk up to exponentially small corrections. In the limit  $x \rightarrow +\infty$  this correspondence becomes exact.

### 7.3 Asymptotics of multiple integrals.

We have already mentioned in Section 2 that the asymptotic expansion of the Fredholm determinant of the GSK can be used for the asymptotic analysis of correlation functions of quantum integrable models. For a relatively wide class of integrable systems, the correlation functions can be presented as series of multiple integrals of a special type [30]. These series can be summed up to Fredholm determinants in the case of models equivalent to free fermions. In the general case, such a reduction to determinants is not known. However, the asymptotic behavior of separate terms of the series can be derived from the asymptotics of the Fredholm determinant of the GSK. In the present Section we consider this problem.

More precisely, we derive the large  $x$  asymptotic behavior of the following cycle integrals:

$$\mathcal{I}_n[\mathcal{F}_n] = \oint_{\Gamma([-q; q])} \frac{d^n z}{(2i\pi)^n} \int_{-q}^q \frac{d^n \lambda}{(2i\pi)^n} \mathcal{F}_n \left( \begin{array}{c} \{\lambda\} \\ \{z\} \end{array} \right) \frac{e^{ix(p(z_j) - p(\lambda_j))}}{(z_j - \lambda_j)(z_j - \lambda_{j+1})}. \quad (7.30)$$

$\Gamma([-q; q])$  is a loop surrounding  $[-q; q]$  with index 1 whereas  $\mathcal{F}_n$  is a holomorphic function in the  $2n$  variables  $\{\lambda_1, \dots, \lambda_n, z_1, \dots, z_n\} \in U^n \times W^n$ . There  $U$  and  $W$  are open neighborhoods of  $[-q; q]$ . We at last assume that  $\mathcal{F}_n$  is symmetric separately in the  $n$  variables  $\lambda_1, \dots, \lambda_n$  and in the  $n$  variables  $z_1, \dots, z_n$ . Finally, we agree upon  $\lambda_{n+1} \equiv \lambda_1$ .

If the function  $\mathcal{F}_n$  is a pure factor of one variable holomorphic functions  $\varphi \in \mathcal{H}(U)$  and  $\phi \in \mathcal{H}(W)$  where  $\phi$  is non-vanishing on  $W$

$$\mathcal{F}_n^{(\varphi, \phi)} \left( \begin{array}{c} \{\lambda\} \\ \{z\} \end{array} \right) = \prod_{i=1}^n [\varphi(\lambda_i) \phi(z_i)], \quad (7.31)$$

then the integral (7.30) can be expressed in terms  $\log \det [I + V^{(\varphi, \phi)}]$  where kernel



$V^{(\varphi, \phi)}$  given by (1.6) with  $F(\lambda) = \varphi(\lambda)\phi(\lambda)$  and  $e^{g(z)} = \phi(z)$ . Indeed, we have,

$$\begin{aligned} \mathcal{I}_n[\mathcal{F}_n^{(\varphi, \phi)}] &= \int_{-q}^q d^n \lambda \prod_{k=1}^n V^{(\varphi, \phi)}(\lambda_k, \lambda_{k+1}) \\ &= \frac{(-1)^{n-1}}{(n-1)!} \partial_\gamma^n \log \det [I + V^{(\varphi, \phi)}] \Big|_{\gamma=0}. \end{aligned} \quad (7.32)$$

In such a special case, it is straightforward to obtain the leading asymptotic behavior of the multiple integral (7.32) in the large  $x$  limit thanks to the results of the previous Section.

**Definition 7.1** *Let  $U, W$  be two open neighborhoods of  $[-q; q]$ , and let  $\mathcal{H}(U)$  (resp.  $\mathcal{H}(W)$ ) be the set of holomorphic functions on  $U$  (resp. on  $W$ ). Then we set*

$$\tilde{\mathcal{S}}_n^{U, W} = \left\{ \sum_{\ell=1}^p \mathcal{F}_n^{(\varphi_\ell, \phi_\ell)} ; p \in \mathbb{N}, (\varphi_\ell, \phi_\ell) \in \mathcal{H}(U) \times \mathcal{H}(W) \text{ and } \phi_\ell \Big|_{[-q; q]} \neq 0 \right\}. \quad (7.33)$$

There  $\mathcal{F}_n^{(\varphi, \phi)}$  denotes a function of  $2n$  variables defined in terms of  $\varphi$  and  $\phi$  as in (7.31). We define the linear form  $H_n^{(0)}$  on  $\tilde{\mathcal{S}}_n^{U, W}$  as

$$H_n^{(0)}[\mathcal{F}_n^{(\varphi, \phi)}] = \frac{(-1)^{n-1}}{(n-1)!} \partial_\gamma^n \log \det [I + V^{(\varphi, \phi)}]^{(0)} \Big|_{\gamma=0}, \quad (7.34)$$

and by imposing linearity on functions  $\sum_{\ell=1}^p \mathcal{F}_n^{(\varphi_\ell, \phi_\ell)}$ . Here  $V^{(\varphi, \phi)}$  denotes the generalized sine kernel (1.6) with  $F(\lambda) = \varphi(\lambda)\phi(\lambda)$  and  $e^{g(z)} = \phi(z)$ , and  $\log \det [I + V^{(\varphi, \phi)}]^{(0)}$  denotes the leading asymptotics of the Fredholm determinant  $\log \det [I + V^{(\varphi, \phi)}]$  as in Theorem 2.1.

It is easy, using the leading asymptotics of  $\log \det [I + V^{(\varphi, \phi)}]$  given in Theorem 2.1, to obtain the explicit expression of  $H_n^{(0)}[\mathcal{F}_n^{(\varphi, \phi)}]$ :

$$\begin{aligned} H_n^{(0)}[\mathcal{F}_n^{(\varphi, \phi)}] &= \int_{-q}^q \frac{d\lambda}{2i\pi} \varphi^n(\lambda) \phi^{n-1}(\lambda) (ixp'(\lambda)\phi(\lambda) + \phi'(\lambda)) \\ &\quad + \sum_{\sigma=\pm} (b_n - c_n \log(2qp'_\sigma x)) [\varphi(\sigma q)\phi(\sigma q)]^n + \sum_{p=1}^{n-1} H_{n,p}^{(0)}[\mathcal{F}_n^{(\varphi, \phi)}], \end{aligned} \quad (7.35)$$

where  $b_n$  and  $c_n$  are given by (2.19) and

$$\begin{aligned}
H_{n,p}^{(0)}[\mathcal{F}_n^{(\varphi,\phi)}] &= \frac{n}{4\pi^2} \sum_{\sigma=\pm} \int_{-q}^q \frac{[\varphi(\sigma q) \phi(\sigma q)]^n - [\varphi(\sigma q) \phi(\sigma q)]^p [\varphi(\lambda) \phi(\lambda)]^{n-p}}{p(n-p)(q-\sigma\lambda)} d\lambda \\
&+ \frac{n}{8\pi^2} \int_{-q}^q \frac{d\lambda d\mu}{(n-p)(\lambda-\mu)} \left\{ \partial_\lambda [\varphi(\lambda) \phi(\lambda)] [\varphi(\lambda) \phi(\lambda)]^{p-1} [\varphi(\mu) \phi(\mu)]^{n-p} \right. \\
&\quad \left. - \partial_\mu [\varphi(\mu) \phi(\mu)] [\varphi(\mu) \phi(\mu)]^{p-1} [\varphi(\lambda) \phi(\lambda)]^{n-p} \right\}. \quad (7.36)
\end{aligned}$$

The  $x \rightarrow +\infty$  asymptotics of the Fredholm determinant are uniform in  $\gamma$  to an arbitrary but fixed order  $n$  in  $\partial_\gamma^n$ . This means that

$$\mathcal{I}_n[\mathcal{F}_n^{(\varphi,\phi)}] = H_n^{(0)}[\mathcal{F}_n^{(\varphi,\phi)}] + o\left(\frac{1}{x^{1-\varepsilon}}\right). \quad (7.37)$$

In the next proposition we show that  $H_n^{(0)}$  can be extended into a linear functional on the space of holomorphic function  $\mathcal{F}_n$  (not necessarily of the form  $\mathcal{F}_n^{(\varphi,\phi)}$ ) that are symmetric in the  $n$  variables  $\lambda$  and the  $n$  variables  $z$ . This extension of  $H_n^{(0)}$ , as we prove in the following, is the good approximate way to evaluate cycle integrals (7.30) involving arbitrary symmetric functions  $\mathcal{F}_n$ .

**Proposition 7.2** *Let  $U$  and  $W$  be open neighborhoods of  $[-q; q]$ , and let  $\text{Sym}_n(U, W)$  be the set of holomorphic functions  $\mathcal{F}_n$  on  $U^n \times W^n$  of  $2n$  variables  $\lambda_1, \dots, \lambda_n, z_1, \dots, z_n$ , symmetric in the  $n$  variables  $\lambda_1, \dots, \lambda_n$  and in the  $n$  variables  $z_1, \dots, z_n$  separately. Then  $H_n^{(0)}$  extends to a continuous linear form on  $\text{Sym}_n(U, W)$  endowed with the topology of sup norm convergence on compact sets.*

*Proof* —  $H_n^{(0)}[\mathcal{F}_n^{(\varphi,\phi)}]$  contains at most first order derivatives of the functions  $\varphi$  and  $\phi$ . Now recall that, for any compact  $K \subset U$ ,

$$\forall k \in \mathbb{N}, \exists c_k \text{ such that } \forall \phi \in \mathcal{H}(U), \left\| \phi^{(k)} \right\|_{0;K} \leq c_k \|\phi\|_{0;K}, \quad (7.38)$$

where  $\|\cdot\|_{0;K} = \sup_{z \in K} |\cdot|$  is the sup norm with support on the compact  $K$ . In consequence,  $H_n^{(0)}$  is continuous on  $\tilde{\mathcal{S}}_n^{U,W}$ . The latter is dense in  $\mathcal{S}_n^{U,W}$ , with

$$\mathcal{S}_n^{U,W} = \left\{ \sum_{\ell=1}^p \mathcal{F}_n^{(\varphi_\ell, \phi_\ell)} ; p \in \mathbb{N}, (\varphi_\ell, \phi_\ell) \in \mathcal{H}(U) \times \mathcal{H}(W) \right\}. \quad (7.39)$$

Hence  $H_n^{(0)}$  extends by density to a continuous linear form on  $\mathcal{S}_n^{U,W}$ . Due to the density Theorem C.1 (See Appendix C), we have that  $\mathcal{S}_n^{U,W}$  is dense in  $\text{Sym}_n(U, W)$ . Therefore  $H_n^{(0)}$  extends to a linear form on  $\text{Sym}_n(U, W)$ .  $\square$

**Corollary 7.1** *Let  $U$  and  $W$  be open neighborhoods of  $[-q; q]$ , and let a function  $\mathcal{F}_n \in \text{Sym}_n(U, W)$ . Then,*

$$\begin{aligned}
H_n^{(0)}[\mathcal{F}_n] &= \frac{1}{2i\pi} \int_{-q}^q d\lambda \{ixp'(\lambda) + \partial_\epsilon\} \mathcal{F}_n \left( \begin{array}{c} \{\lambda\}^n \\ \{\lambda + \epsilon\}, \{\lambda\}^{n-1} \end{array} \right) \Big|_{\epsilon=0} \\
&+ \sum_{\sigma=\pm} (b_n - c_n \log(2qp'_\sigma x)) \mathcal{F}_n \left( \begin{array}{c} \{\sigma q\}^n \\ \{\sigma q\}^n \end{array} \right) \\
&+ \frac{n}{(2\pi)^2} \sum_{\sigma=\pm} \sum_{p=1}^{n-1} \int_{-q}^q d\lambda \frac{\mathcal{F}_n \left( \begin{array}{c} \{\sigma q\}^n \\ \{\sigma q\}^n \end{array} \right) - \mathcal{F}_n \left( \begin{array}{c} \{\sigma q\}^p, \{\lambda\}^{n-p} \\ \{\sigma q\}^p, \{\lambda\}^{n-p} \end{array} \right)}{p(n-p)(q-\sigma\lambda)} \\
&+ \frac{n}{2(2\pi)^2} \sum_{p=1}^{n-1} \int_{-q}^q \frac{d\lambda d\mu}{(n-p)(\lambda-\mu)} \left\{ \partial_\epsilon \mathcal{F}_n \left( \begin{array}{c} \{\lambda + \epsilon\}, \{\lambda\}^{p-1}, \{\mu\}^{n-p} \\ \{\lambda + \epsilon\}, \{\lambda\}^{p-1}, \{\mu\}^{n-p} \end{array} \right) \right. \\
&\quad \left. - \partial_\epsilon \mathcal{F}_n \left( \begin{array}{c} \{\mu + \epsilon\}, \{\mu\}^{p-1}, \{\lambda\}^{n-p} \\ \{\mu + \epsilon\}, \{\mu\}^{p-1}, \{\lambda\}^{n-p} \end{array} \right) \right\} \Big|_{\epsilon=0}.
\end{aligned}$$

There  $\{\lambda\}^n$  denotes the set formed by  $n$  copies of the same parameter  $\lambda$ .

*Proof* — Apply Theorem C.1 to (7.35). □

Finally, we prove the following large  $x$  asymptotic behavior for integrals of the form (7.30) (which seems hardly attainable through a direct analysis of the multiple integrals):

**Theorem 7.2** *Let  $U$  and  $W$  be open neighborhoods of  $[-q; q]$ , and let a function  $\mathcal{F}_n \in \text{Sym}_n(U, W)$ . Then, when  $x \rightarrow +\infty$ , the integral  $I_n[\mathcal{F}_n]$  (7.30) behaves as*

$$\mathcal{I}_n[\mathcal{F}_n] = H_n^{(0)}[\mathcal{F}_n] + \mathcal{O}\left(\frac{\log^n x}{x}\right), \tag{7.40}$$

$H_n^{(0)}[\mathcal{F}_n]$  being given in Corollary 7.1.

Moreover, for any positive integer  $m$ , there exists a continuous linear functional  $H_n^{(m)}$  such that:

$$\mathcal{I}_n[\mathcal{F}_n] = H_n^{(m)}[\mathcal{F}_n] + \mathcal{O}\left(\frac{\log^n x}{x^{m+1}}\right). \tag{7.41}$$

Note that the explicit expression for  $H_n^{(m)}$  can be obtain by some perturbative computations.

*Proof* — We start by proving the first part of the theorem, the second part being a mere generalization.

The whole difficulty of the proof is to show that the small  $o(1)$  in (2.6) is preserved by the procedure formulated in Theorem C.1. This is nontrivial since the series converging to  $\mathcal{F}_n$  may not converge absolutely.

Let us recast (3.13) as

$$\partial_\gamma \log \det [I + V] = \partial_\gamma \log \det [I + V]^{(0)} - \int_{-q}^q \frac{d\lambda}{\gamma} \langle F^{L;(0)}(\lambda) |\Pi^{-1}(\lambda) \Pi'(\lambda)| F^{R;(0)}(\lambda) \rangle.$$

The first term, after the density procedure is applied, yields  $H_n^{(0)}[\mathcal{F}_n]$  as follows from the above Corollary. Therefore it remains to analyze the second term in the RHS.

Recall that  $\Pi_+(\lambda)$  satisfies the following integral equation on  $\mathcal{M}_2(L^2(\Sigma_\Pi))$ :

$$(I - C_{\Sigma_\Pi}^\Delta) [\Pi_+] = I_2. \quad (7.42)$$

The matrix Cauchy operator  $C_{\Sigma_\Pi}^\Delta$  is defined by

$$C_{\Sigma_\Pi}^\Delta [M] \equiv C_{\Sigma_\Pi}^+ [M\Delta], \quad \forall M \in \mathcal{M}_2(L^2(\Sigma_\Pi)). \quad (7.43)$$

This matrix Cauchy operator is invertible, at least for  $x$  large enough. Indeed, using the continuity of the scalar Cauchy operator,

$$\exists c_2 > 0 \text{ such that, } \forall g, \quad \left\| C_{\Sigma_\Pi}^+ [g] \right\|_{L^2(\Sigma_\Pi)} \leq c_2 \|g\|_{L^2(\Sigma_\Pi)}, \quad (7.44)$$

one gets that the operator norm  $\left\| C_{\Sigma_\Pi}^\Delta \right\|$  fulfills:

$$\left\| C_{\Sigma_\Pi}^\Delta \right\| \leq c_2 \|\Delta\|_{L^2(\Sigma_\Pi)} \xrightarrow{x \rightarrow +\infty} 0. \quad (7.45)$$

As in addition  $C_{\Sigma_\Pi}^\Delta$  is holomorphic in  $\gamma$  we have that, for  $x$  large enough  $(I - C_{\Sigma_\Pi}^\Delta)$  is invertible and that this inverse is also holomorphic in  $\gamma$ . In particular, (7.42) implies

$$\partial_\gamma \Pi_+ = (I - C_{\Sigma_\Pi}^\Delta)^{-1} \circ (\partial_\gamma C_{\Sigma_\Pi}^\Delta) [\Pi_+]. \quad (7.46)$$

A straightforward induction shows that there exist some coefficients  $c_{n;p_1, \dots, p_l} \in \mathbb{Z}$  such that

$$\begin{aligned} \partial_\gamma^n \Pi_+ &= \sum_{k=1}^n \sum_{\Sigma_{i=1}^k p_i = n} c_{n;p_1, \dots, p_k} (I - C_{\Sigma_\Pi}^\Delta)^{-1} \circ (\partial_\gamma^{p_1} C_{\Sigma_\Pi}^\Delta) \\ &\quad \circ (I - C_{\Sigma_\Pi}^\Delta)^{-1} \circ (\partial_\gamma^{p_2} C_{\Sigma_\Pi}^\Delta) \circ \dots \circ (I - C_{\Sigma_\Pi}^\Delta)^{-1} \circ (\partial_\gamma^{p_k} C_{\Sigma_\Pi}^\Delta) [\Pi_+]. \end{aligned} \quad (7.47)$$

There is a slight simplification when  $\gamma = 0$  since then  $\Delta|_{\gamma=0} = 0$  and  $\Pi_+ = I_2$ . Hence,

$$\partial_\gamma^n \Pi_+ \Big|_{\gamma=0} = \sum_{k=1}^n \sum_{\Sigma_{i=1}^k p_i = n} c_{n;p_1, \dots, p_k} (\partial_\gamma^{p_1} C_{\Sigma_\Pi}^\Delta) \circ (\partial_\gamma^{p_2} C_{\Sigma_\Pi}^\Delta) \circ \dots \circ (\partial_\gamma^{p_k} C_{\Sigma_\Pi}^\Delta) [I_2]. \quad (7.48)$$

It is also quite easy to see that

$$\partial_\gamma^p \Delta(z) \Big|_{\gamma=0} = \begin{pmatrix} \delta_{11}^{(p)}(z; x) & \delta_{12}^{(p)}(z; x) e^{g(z)} \\ \delta_{21}^{(p)}(z; x) e^{-g(z)} & \delta_{22}^{(p)}(z; x) \end{pmatrix} (F(z))^p, \quad (7.49)$$

where the matrix  $\delta^{(p)}(z; x)$  is such that  $\delta^{(p)}(z; x) = O(\log^p x/x)$  uniformly on the contour  $\Sigma_\Pi$ , and  $\delta^{(0)} = 0$ . Finally one infers from

$$\Pi(\lambda) = I_2 + \int_{\Sigma_\Pi} \frac{dz}{\lambda - z} \Pi_+(z) \Delta(z) \quad (7.50)$$

that, for some recursively computable coefficients  $\tilde{c}_{(k;p_i)}$ ,

$$\begin{aligned} \partial_\gamma^n \Pi'(\lambda) &= \sum_{k=1}^n \sum_{\Sigma_1^k p_i = n} \tilde{c}_{(k;p_i)} \int_{\Sigma_\Pi^{(i)} < \Sigma_\Pi^{(i-1)}} \frac{d^k z}{(\lambda - z_1)^2} \frac{\prod_{\ell=1}^k F^{p_\ell}(z_\ell)}{\prod_{\ell=2}^k (z_{\ell-1} - z_\ell)} \\ &\quad \times \prod_{\ell=1}^k \begin{pmatrix} \delta_{11}^{(p_\ell)}(z_\ell; x) & \delta_{11}^{(p_\ell)}(z_\ell; x) e^{g(z)} \\ \delta_{21}^{(p_\ell)}(z_\ell; x) e^{-g(z)} & \delta_{22}^{(p_\ell)}(z_\ell; x) \end{pmatrix}. \end{aligned} \quad (7.51)$$

There the integration is performed over slightly deformed contours  $\Sigma_\Pi^{(i)}$  such that  $\Sigma_\Pi^{(i-1)}$  is inside of  $\Sigma_\Pi^{(i)}$ , with an exception of a finite number of points. Such a deformation of the original contour  $\Sigma_\Pi$  is licit in virtue of the holomorphy of the jump matrix. All the above observations also hold for the inverse matrix  $\Pi^{-1}(\lambda)$ . Indeed  $\Pi_+^{-1}$  satisfies

$$\Pi_+^{-1}(\lambda) = I_2 + {}^t C_{\Sigma_\Pi}^\nabla [\Pi_+^{-1}], \quad (7.52)$$

in which

$${}^t C_{\Sigma_\Pi}^\nabla [M] \equiv C_{\Sigma_\Pi}^+ [\nabla M], \quad \forall M \in \mathcal{M}_2(L^2(\Sigma_\Pi)), \quad (7.53)$$

and the matrix  $\nabla$  is defined<sup>1</sup> as  $I_2 + \nabla = (I_2 + \Delta)^{-1}$ , and fulfills

$$\partial_\gamma^p \nabla(z) \Big|_{\gamma=0} = \begin{pmatrix} \mathfrak{d}_{11}^{(p)}(z; x) & \mathfrak{d}_{12}^{(p)}(z; x) e^{g(z)} \\ \mathfrak{d}_{21}^{(p)}(z; x) e^{-g(z)} & \mathfrak{d}_{22}^{(p)}(z; x) \end{pmatrix} F^p(z), \quad (7.54)$$

where  $\mathfrak{d}^{(p)}(z; x)$  is such that  $\mathfrak{d}^{(p)}(z; x) = O(\log^p x/x)$  uniformly on the contour  $\Sigma_\Pi$ . Also, there exists recursively computable coefficients  $\bar{c}_{k;p_i}$  such that

$$\begin{aligned} \partial_\gamma^n \Pi^{-1}(\lambda) &= \sum_{k=1}^n \sum_{\Sigma_1^k p_i = n} \bar{c}_{(k;p_i)} \int_{\Sigma_\Pi^{(i)} < \Sigma_\Pi^{(i-1)}} \frac{d^k z}{(\lambda - z_1)} \frac{\prod_{\ell=1}^k F^{p_\ell}(z_\ell)}{\prod_{\ell=2}^k (z_{\ell-1} - z_\ell)} \\ &\quad \times \prod_{\ell=1}^k \begin{pmatrix} \mathfrak{d}_{11}^{(p_\ell)}(z_\ell; x) & \mathfrak{d}_{11}^{(p_\ell)}(z_\ell; x) e^{g(z)} \\ \mathfrak{d}_{21}^{(p_\ell)}(z_\ell; x) e^{-g(z)} & \mathfrak{d}_{22}^{(p_\ell)}(z_\ell; x) \end{pmatrix}. \end{aligned} \quad (7.55)$$

<sup>1</sup>we stress that  $\nabla$  has nothing to do with the differential operator usually denoted by  $\nabla$

Finally, a direct inspection of the local expressions for the functions  $f_{\pm}$  leads to

$$\partial_{\gamma}^n (\gamma^{-1} | F^{R;(0)}(\lambda) \rangle) \Big|_{\gamma=0} = \frac{\sqrt{F(\lambda)}}{2i\pi} e^{\sigma_3 \frac{g(\lambda)}{2}} \begin{pmatrix} a_n^+ \\ a_n^- \end{pmatrix} F^n(\lambda). \quad (7.56)$$

Having such local expressions for the multiple  $\gamma$  derivatives, it is not a problem to see that there exist coefficients  $C_{\{p_{\ell}; q_{\ell}; r_{\ell}\}}$  such that

$$\begin{aligned} & \partial_{\gamma}^n \left[ \gamma^{-1} \langle F^{L;(0)}(\lambda) | \Pi^{-1}(\lambda) \Pi'(\lambda) | F^{R;(0)}(\lambda) \rangle \right] \Big|_{\gamma=0} \\ &= \sum_{\Sigma p_i = n} \sum_{\substack{\Sigma q_{\ell} = p_2 \\ \Sigma r_{\ell} = p_3}} C_{\{p_{\ell}; q_{\ell}; r_{\ell}\}} \frac{F^{p_1 + p_4 + 1}(\lambda)}{2i\pi} \\ & \times (-a_{p_1}^- a_{p_1}^-) \int_{\Sigma_{\Pi}} \frac{d^{\#q_{\ell} + r_{\ell}} F^{q_1}(z_1) F^{r_1}(z_{\#q_i + 1})}{(\lambda - z_1)(\lambda - z_{\#q_{\ell} + 1})^2} \prod_{\ell=2}^{\#q_i} \frac{F^{q_{\ell}}(z_{\ell})}{z_{\ell-1} - z_{\ell}} \prod_{\ell=2+\#q_i}^{\#q_i + r_i} \frac{F^{r_{\ell}}(z_{\ell})}{z_{\ell-1} - z_{\ell}} \\ & \times e^{-\frac{g(\lambda)\sigma_3}{2}} \prod_{\ell=1}^{\#q_i} \begin{pmatrix} \mathfrak{d}_{11}^{(p_{\ell})}(z_{\ell}; x) & \mathfrak{d}_{11}^{(p_{\ell})}(z_{\ell}; x) e^{g(z_{\ell})} \\ \mathfrak{d}_{21}^{(p_{\ell})}(z_{\ell}; x) e^{-g(z_{\ell})} & \mathfrak{d}_{22}^{(p_{\ell})}(z_{\ell}; x) \end{pmatrix} \\ & \times \prod_{\ell=1+\#q_i}^{\#r_i + q_i} \begin{pmatrix} \delta_{11}^{(p_{\ell})}(z_{\ell}; x) & \delta_{11}^{(p_{\ell})}(z_{\ell}; x) e^{g(z_{\ell})} \\ \delta_{21}^{(p_{\ell})}(z_{\ell}; x) e^{-g(z_{\ell})} & \delta_{22}^{(p_{\ell})}(z_{\ell}; x) \end{pmatrix} e^{\frac{g(\lambda)\sigma_3}{2}} \begin{pmatrix} a_{p_4}^+ \\ a_{p_4}^- \end{pmatrix}. \quad (7.57) \end{aligned}$$

Eventually, once all the matrix products are taken, we get that each entry of the final matrix contains terms  $e^{\Sigma \epsilon_i g(z_i)}$ . There  $\epsilon_i \in \{\pm 1, 0\}$  and the sum contains as many pluses as minuses. So there exist some functions  $D(\epsilon_i, p_i, q_i, r_i; z_i)$  on a vicinity of  $\Sigma_{\Pi}$  such that they all are uniformly an  $O(\log^n x/x)$  and that

$$\begin{aligned} & \partial_{\gamma}^n \int_{-q}^q \frac{d\lambda}{\gamma} \left[ \langle F^{L;(0)}(\lambda) | \Pi^{-1}(\lambda) \Pi'(\lambda) | F^{R;(0)}(\lambda) \rangle \right] \Big|_{\gamma=0} \\ &= \sum_{\Sigma p_i = n} \sum_{\substack{\Sigma q_{\ell} = p_2 \\ \Sigma r_{\ell} = p_3}} \sum_{\substack{\epsilon_i = \pm 1, 0 \\ \Sigma \epsilon_i = 0}} \int_{-q}^q \frac{d\lambda}{2i\pi} \int_{\Sigma_{\Pi}} \frac{d^{\#q_{\ell} + r_{\ell}} F^{q_1}(z_1) F^{r_1}(z_{\#q_i + 1})}{(\lambda - z_1)(\lambda - z_{\#q_{\ell} + 1})^2} D(\epsilon_i, p_i, q_i, r_i; z_i) \\ & \times F^{p_1 + p_4 + 1}(\lambda) \prod_{\ell=2}^{\#q_i} \frac{F^{q_{\ell}}(z_{\ell})}{z_{\ell-1} - z_{\ell}} \prod_{\ell=2+\#q_i}^{\#q_i + r_i} \frac{F^{r_{\ell}}(z_{\ell})}{z_{\ell-1} - z_{\ell}} e^{\epsilon_0 g(\lambda) + \Sigma \epsilon_i g(z_i)}. \quad (7.58) \end{aligned}$$

We stress that  $\epsilon_i \neq 0 \Rightarrow \exists \ell : q_{\ell} > 0$  or  $r_{\ell} > 0$ . In other words,  $e^{\pm g(z)}$  can only appear in combination with  $F(z)$ :  $F(z) e^{\pm g(z)}$ . This guarantees that the functional above is continuous with respect to the sup norm on the space of symmetric functions in  $n$  variables  $z$  and  $n$  variables  $\lambda$ . It follows immediately from the procedure formulated in Theorem C.1 that there exist functions  $G(\epsilon_i, p_i; z_i, x)$  that are uniformly  $O(\log^n x/x)$

on a vicinity of  $\Sigma_{\Pi}$  and such that

$$\begin{aligned} \mathcal{I}_n[\mathcal{F}_n] &= H_n^{(0)}[\mathcal{F}_n] \\ &+ \sum_{\Sigma_{p_i=n}} \sum_{\substack{\epsilon_i=\pm,0 \\ \Sigma_{\epsilon_i}=0}} \int_{-q}^q dz_0 \int_{\Sigma_{\Pi}^{(i)} < \Sigma_{\Pi}^{(i-1)}} d^{\#p_i-2} z G(\epsilon_i, p_i; z_i, x) \mathcal{F}_n \left( \begin{array}{c} \{z_i\}_{p_i} \\ \{z_i\}_{p_i+\epsilon_i} \end{array} \right), \end{aligned} \quad (7.59)$$

for functions  $\mathcal{F}_n$  holomorphic on  $U^n \times W^n$ , with  $U$  and  $W$  open neighborhoods of  $[-q; q]$ . We also precise that  $z_{n-1} \equiv \lambda$  and that  $\epsilon_{n-1}$  is always set to 0. The sum in the second line being finite, (7.40) follows readily.

Let us now modify this approach so as to include the corrections. We have the asymptotic expansion for  $\log \det [I + V]$  thanks to the one of  $\Pi$ :

$$\begin{aligned} \partial_{\gamma} \log \det [I + V] &= \partial_{\gamma} \log \det [I + V]^{(m)} \\ &- \int_{-q}^q \frac{d\lambda}{\gamma} \langle F^{L;(0)}(\lambda) | \Pi^{-1}(\lambda) \Pi'(\lambda) - \Pi^{-1}(\lambda; m) \Pi'(\lambda; m) | F^{R;(0)}(\lambda) \rangle, \end{aligned}$$

with

$$\begin{aligned} \log \det [I + V]^{(m)} &= \log \det [I + V]^{(0)} \\ &- \int_0^{\gamma} \frac{d\gamma'}{\gamma'} \int_{-q}^q d\lambda \langle F^{L;(0)}(\lambda) | \Pi^{-1}(\lambda; m) \Pi'(\lambda; m) | F^{R;(0)}(\lambda) \rangle, \end{aligned} \quad (7.60)$$

and the matrices  $\Pi(\lambda; m)$  are defined in (5.14). This time we have to show that the correction to  $\log \det [I + V]^{(m)}$  has an expansion similar to (7.58) with the only difference that the functions  $D(\epsilon_i, p_i, q_i, r_i; z_i)$  are now a  $O(\log^n x/x^{m+1})$ . It is easy to establish the equations

$$\Pi_+ - \sum_{p=0}^{m-1} \frac{\Pi_+^{(p)}}{x^p} = (I - C_{\Sigma_{\Pi}}^{\Delta})^{-1} \circ C_{\Sigma_{\Pi}}^{\Delta} \left[ \sum_{p=0}^{m-1} \frac{\Pi_+^{(p)}}{x^p} \left( \Delta - \sum_{k=1}^{m-1-p} \frac{\Delta^{(k)}}{x^k} \right) \right]. \quad (7.61)$$

The matrices  $\Pi_+^{(p)}(\lambda)$ , defined recursively by  $\Pi_+^{(p)} = \sum_{k=1}^p C_{\Sigma_{\Pi}}^+ [\Pi_+^{(p-k)} \Delta^{(k)}]$ , are holomorphic in  $\gamma$ . They are also holomorphic in  $\lambda$  belonging to some open neighborhood of  $\Sigma_{\Pi}$  due to the holomorphicity of the matrices  $\Delta^{(k)}$ . Finally the entries of  $\Pi_+^{(p)}$  are a  $O(x^{p\bar{\epsilon}})$  uniformly on  $\Sigma_{\Pi}$ . These in turn will give some logarithmic contributions  $(\log x)^n$  after the  $n^{\text{th}}$   $\gamma$  derivative at  $\gamma = 0$  is taken.

Of course an analogous version exists for  $\Pi_+^{-1}$ , namely,

$$\Pi_+^{-1} - \sum_{p=0}^{m-1} \frac{\overline{\Pi}_+^{(p)}}{x^p} = (I - {}^t C_{\Sigma_{\Pi}}^{\nabla})^{-1} \circ {}^t C_{\Sigma_{\Pi}}^{\nabla} \left[ \sum_{p=0}^{m-1} \left( \nabla - \sum_{k=1}^{m-1-p} \frac{\nabla^{(k)}}{x^k} \right) \frac{\overline{\Pi}_+^{(p)}}{x^p} \right]. \quad (7.62)$$

The matrices  $\nabla^{(k)}$  are defined similarly to  $\Delta^{(k)}$ , namely

$$\nabla = \sum_{p=1}^M \nabla^{(k)} x^{-k} + o(x^{\bar{\varepsilon}}/x^M) \quad (7.63)$$

uniformly on the contour  $\Sigma_{\Pi}$ . The matrices  $\bar{\Pi}_+^{(p)}$

$$\bar{\Pi}_+^{(p)} = \sum_{k=1}^p C_{\Sigma_{\Pi}}^+ \left[ \bar{\Pi}_+^{(p-k)} \nabla^{(k)} \right], \quad (7.64)$$

are holomorphic in  $\gamma$  and in  $\lambda$  belonging to some open neighborhood of  $\Sigma_{\Pi}$  due to the analyticity of the matrices  $\nabla^{(k)}$ . Finally the entries of  $\bar{\Pi}_+^{(p)}$  are a  $O(x^{p\bar{\varepsilon}})$  uniformly on  $\Sigma_{\Pi}$ . These in turn will give some logarithmic contributions  $(\log x)^n$  after the  $n^{\text{th}}$   $\gamma$  derivative at  $\gamma = 0$  is taken. We can now obtain a representation for  $\Pi^{-1}(\lambda) - \Pi^{-1}(\lambda; m)$  in terms of Cauchy operators

$$\begin{aligned} \Pi^{-1}(\lambda) - \Pi^{-1}(\lambda; m) &= C_{\Sigma_{\Pi}} \left[ \left( \nabla - \sum_{p=1}^m \nabla^{(p)} x^{-p} \right) \Pi_+^{-1} \right] \\ &\quad + C_{\Sigma_{\Pi}} \left[ \sum_{p=1}^m \nabla^{(p)} x^{-p} \left( \Pi_+^{-1} - \sum_{k=0}^{n-p} \bar{\Pi}_+^{(k)} x^{-k} \right) \right]. \quad (7.65) \end{aligned}$$

Similar formulas holds for  $\Pi'(\lambda) - \Pi'(\lambda; m)$ . Then applying the same steps as for the leading asymptotics part, we get that there exist a function  $D(\epsilon_i, p_i, q_i, r_i; z_i)$  that is uniformly  $O(\log^n x/x^{m+1})$  and such that

$$\begin{aligned} &\partial_{\gamma}^n \int_{-q}^q \frac{d\lambda}{\gamma} \left[ \langle F^{L;(0)}(\lambda) | \Pi^{-1}(\lambda) \Pi'(\lambda) - \Pi^{-1}(\lambda; m) \Pi'(\lambda; m) | F^{R;(0)}(\lambda) \rangle \right] \Big|_{\gamma=0} \\ &= \sum_{\Sigma p_i=n} \sum_{\substack{\Sigma q_{\ell}=p_2 \\ \Sigma r_{\ell}=p_3}} \sum_{\substack{\epsilon_i=\pm,0 \\ \Sigma \epsilon_i=0}} \int_{-q}^q \frac{d\lambda}{2i\pi} \int_{\Sigma_{\Pi}} \frac{d^{\#q_{\ell}+r_{\ell}} z F^{q_1}(z_1) F^{r_1}(z_{\#q_i+1})}{(\lambda - z_1)(\lambda - z_{\#q_{\ell}+1})^2} D(\epsilon_i, p_i, q_i, r_i; z_i) \\ &\quad \times F^{p_1+p_4+1}(\lambda) \prod_{\ell=2}^{\#q_i} \frac{F^{q_{\ell}}(z_{\ell})}{z_{\ell-1} - z_{\ell}} \prod_{\ell=2+\#q_i}^{\#q_i+r_i} \frac{F^{r_{\ell}}(z_{\ell})}{z_{\ell-1} - z_{\ell}} e^{\epsilon_0 g(\lambda) + \Sigma \epsilon_i g(z_i)}, \quad (7.66) \end{aligned}$$

It remains to compute the  $n^{\text{th}}$   $\gamma$ -derivative of (7.60). The fact that the resulting functional will be continuous with respect to the sup norm on  $\text{Sym}_n(U, W)$  follows from the already stated remark that  $e^{-g(z)}$  only appear in combinations with  $F(z)$ , i.e.  $F(z) e^{-g(z)}$ . Applying the density procedure, we immediately get

$$\begin{aligned} \mathcal{I}_n[\mathcal{F}_n] &= H_n^{(m)}[\mathcal{F}_n] \\ &\quad + \sum_{\Sigma p_i=n} \sum_{\substack{\epsilon_i=\pm,0 \\ \Sigma \epsilon_i=0}} \int_{-q}^q dz_0 \int_{\Sigma_{\Pi}^{(i)} < \Sigma_{\Pi}^{(i-1)}} d^{\#p_i-2} z G(\epsilon_i, p_i; z_i, x) \mathcal{F}_n \left( \begin{array}{c} \{z_i\}_{p_i} \\ \{z_i\}_{p_i+\epsilon_i} \end{array} \right), \quad (7.67) \end{aligned}$$



where  $\mathcal{F}_n$  is as above and we remind that  $z_{n-1} \equiv \lambda$  and  $\epsilon_{n-1} = 0$ . The functions  $G$  are all uniformly  $O(\log^n x/x^{m+1})$  on the contour of integration. The sums being finite we get the second claim of theorem follows.  $\square$

## 7.4 More general kernels

Actually, in the applications to quantum integrable models, one sometimes needs to use certain modified versions of the GSK.

Consider the operator  $I + V_\theta$  acting on  $[-q; q]$  with kernel

$$V_\theta(\lambda, \mu) = \sqrt{F(\lambda)F(\mu)\theta'(\lambda)\theta'(\mu)} \frac{e_+(\lambda)e_-(\mu) - e_-(\lambda)e_+(\mu)}{2i\pi[\theta(\lambda) - \theta(\mu)]}, \quad (7.68)$$

where  $e_\pm$  and  $F$  are defined as in (1.7). We assume in addition that  $\theta$  is a biholomorphism of  $U$  onto its image, that  $\theta([-q; q]) \subset \mathbb{R}$  and  $\theta(U \cap \mathcal{H}_\pm) \subset \mathcal{H}_\pm$ .

Then the asymptotic behavior of  $\log \det[I + V_\theta]$  when  $x \rightarrow \infty$  follows from Theorem 2.1. More precisely, we have the following

**Corollary 7.2** *Let  $V_\theta$  be as above. Then*

$$\begin{aligned} \log \det [I + V_\theta] &= 2 \int_{-q}^q d\lambda \nu(\lambda) \log' [e_-(\lambda)] + \sum_{\sigma=\pm} \log \left[ \frac{G(1, \nu_\sigma) \theta'(\sigma q)^{\nu_\sigma^2} \mathcal{K}^{\sigma\nu_\sigma}(\sigma q; q)}{[(\theta(q) - \theta(-q)) p'_\sigma x]^{\nu_\sigma^2}} \right] \\ &\quad + \frac{1}{2} \int_{-q}^q d\lambda d\mu \frac{\nu'(\lambda) \theta'(\mu) \nu(\mu) - \nu(\lambda) \theta'(\lambda) \nu'(\mu)}{\theta(\lambda) - \theta(\mu)} + o(1), \end{aligned} \quad (7.69)$$

where

$$\nu(\lambda) = \frac{-1}{2i\pi} \log(1 + \gamma F(\lambda)), \quad \mathcal{K}(\lambda; q) = \exp \left\{ \int_{-q}^q \frac{\nu(\lambda) - \nu(\mu)}{\theta(\lambda) - \theta(\mu)} \theta'(\mu) d\mu \right\}, \quad (7.70)$$

and, as before,  $p_\pm = [\partial_\lambda p(\lambda)]|_{\lambda=\pm q}$ ,  $\nu_\pm = \nu(\pm q)$ .

*Proof* — The change of variables  $\theta(\lambda) = \xi$  maps the kernel  $V_\theta$  on the one of the GSK

$$V(\xi, \eta) = \sqrt{F \circ \theta^{-1}(\xi) F \circ \theta^{-1}(\eta)} \frac{e_+ \circ \theta^{-1}(\xi) e_- \circ \theta^{-1}(\eta) - e_- \circ \theta^{-1}(\xi) e_+ \circ \theta^{-1}(\eta)}{2i\pi(\xi - \eta)}.$$

This kernel acts on  $[\theta(-q); \theta(q)]$  which is, a priori, a non symmetric interval. However, it is enough to apply the transformation  $\lambda \mapsto \lambda - (\theta(q) + \theta(-q))/2$  so as to recover the symmetry of the interval. Then, it remains to enforce the inverse transformations on the asymptotic formula for the Fredholm determinant of  $V$ .  $\square$

Let us write explicitly the asymptotics (7.69) in the case of the kernel

$$V_{\text{sh}}(\lambda, \mu) = \gamma \sqrt{F(\lambda) F(\mu)} \frac{e_+(\lambda) e_-(\mu) - e_-(\lambda) e_+(\mu)}{2i\pi \sinh(\lambda - \mu)}, \quad e_{\pm}(\lambda) = e^{\pm[ixp(\lambda)+g(\lambda)]/2},$$

as it plays a crucial role in the analysis of the asymptotic behavior of the two-point functions in massless phase of the  $XXZ$  Heisenberg chain [30]. In this case, equation (7.69) reads

$$\begin{aligned} \log \det [I + V_{\text{sh}}] &= 2 \int_{-q}^q d\lambda \nu(\lambda) \log'[e_-(\lambda)] + \frac{1}{2} \int_{-q}^q d\lambda d\mu \frac{\nu'(\lambda) \nu(\mu) - \nu(\lambda) \nu'(\mu)}{\tanh(\lambda - \mu)} \\ &+ \sum_{\sigma=\pm} \left\{ \log \frac{G(1, \nu_{\sigma})}{[\sinh(2q) p'_{\sigma} x]^{\nu_{\sigma}^2}} + \sigma \nu_{\sigma} \int_{-q}^q d\lambda \frac{\nu_{\sigma} - \nu(\lambda)}{\tanh(\sigma q - \lambda)} \right\} + o(1). \end{aligned} \quad (7.71)$$

It is clear that the last equation can be used in order to obtain an analog of the asymptotic expansion for multiple integrals of the type (7.30) where the rational functions  $z - \lambda$  are replaced by the hyperbolic  $\sinh(z - \lambda)$ . Namely, let

$$\mathcal{I}_n^{\text{sh}}[\mathcal{F}_n] = \oint_{\Gamma([-q; q])} \frac{d^n z}{(2i\pi)^n} \int_{-q}^q \frac{d^n \lambda}{(2i\pi)^n} \mathcal{F}_n \left( \begin{matrix} \{\lambda\} \\ \{z\} \end{matrix} \right) \frac{e^{ix(p(z_j) - p(\lambda_j))}}{\sinh(z_j - \lambda_j) \sinh(z_j - \lambda_{j+1})}. \quad (7.72)$$

Then under the conditions of Corollary 7.1 one has the following asymptotic estimate

$$\begin{aligned} \mathcal{I}_n^{\text{sh}}[\mathcal{F}_n] &= \frac{1}{2i\pi} \int_{-q}^q d\lambda \{ixp'(\lambda) + \partial_{\epsilon}\} \mathcal{F}_n \left( \begin{matrix} \{\lambda\}^n \\ \{\lambda + \epsilon\}, \{\lambda\}^{n-1} \end{matrix} \right) \Big|_{\epsilon=0} \\ &+ \sum_{\sigma=\pm} (b_n - c_n \log(\sinh(2q) p'_{\sigma} x)) \mathcal{F}_n \left( \begin{matrix} \{\sigma q\}^n \\ \{\sigma q\}^n \end{matrix} \right) \\ &+ \frac{n}{(2\pi)^2} \sum_{\sigma=\pm} \sum_{p=1}^{n-1} \int_{-q}^q d\lambda \frac{\mathcal{F}_n \left( \begin{matrix} \{\sigma q\}^n \\ \{\sigma q\}^n \end{matrix} \right) - \mathcal{F}_n \left( \begin{matrix} \{\sigma q\}^p, \{\lambda\}^{n-p} \\ \{\sigma q\}^p, \{\lambda\}^{n-p} \end{matrix} \right)}{p(n-p) \tanh(q - \sigma\lambda)} \\ &+ \frac{n}{2(2\pi)^2} \sum_{p=1}^{n-1} \int_{-q}^q \frac{d\lambda d\mu}{(n-p) \tanh(\lambda - \mu)} \left\{ \partial_{\epsilon} \mathcal{F}_n \left( \begin{matrix} \{\lambda + \epsilon\}, \{\lambda\}^{p-1}, \{\mu\}^{n-p} \\ \{\lambda + \epsilon\}, \{\lambda\}^{p-1}, \{\mu\}^{n-p} \end{matrix} \right) \right. \\ &\quad \left. - \partial_{\epsilon} \mathcal{F}_n \left( \begin{matrix} \{\mu + \epsilon\}, \{\mu\}^{p-1}, \{\lambda\}^{n-p} \\ \{\mu + \epsilon\}, \{\mu\}^{p-1}, \{\lambda\}^{n-p} \end{matrix} \right) \right\} \Big|_{\epsilon=0} + o(1). \end{aligned} \quad (7.73)$$

## Conclusion

We have obtained the asymptotic expansion of the Fredholm determinant of the GSK. As we have mentioned, our main motivation is to apply this result to the asymptotic

analysis of the correlation functions of quantum integrable models that will appear in a forthcoming publication [30].

Another development was to extend the above analysis so as to handle truncated Wiener–Hopf operators with symbols having Fischer–Hartwig type discontinuities. The corresponding results are published in [32].

Let us also point out some unsolved problems. One of them concerns the derivation of the asymptotics of the Fredholm determinant of the GSK via the method based on its derivative over endpoint  $q$ . It would be important to obtain a complete justification of this method, since it is rather powerful and at the same time relatively simple.

Another problem is to prove the conjecture on  $\mathbb{Z} - \nu$  periodicity the asymptotic expansion for the Fredholm determinant. If this property does hold, then all oscillating corrections can be obtained from the non-oscillating ones via simple shift of  $\nu$  by integer numbers. This could lead to a much simpler way to compute sub-leading corrections for such determinants.

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## A Some properties of confluent hypergeometric function

For generic parameters  $(a, c)$  the Tricomi confluent hypergeometric function  $\Psi(a, c; z)$  is one of the solutions to the differential equation

$$zy'' + (c - z)y' - ay = 0. \quad (\text{A.1})$$

It satisfies the properties:

- Differentiation:

$$\begin{aligned} \Psi'(a, c; z) &= \frac{a}{z} \left[ (a - c + 1)\Psi(a + 1, c; z) - \Psi(a, c; z) \right] \\ &= \frac{1}{z} \left[ (a - c + z)\Psi(a, c; z) - \Psi(a - 1, c; z) \right]. \end{aligned} \quad (\text{A.2})$$

- Monodromy:

$$\begin{aligned} \Psi(a, 1; ze^{2mi\pi}) &= \Psi(a, 1; z) \left( 1 - me^{i\pi a(\epsilon+1)} + me^{i\pi a(\epsilon-1)} \right) \\ &\quad + \frac{2\pi ime^{i\pi a\epsilon+z}}{\Gamma^2(a)} \Psi(1 - a, 1; -z), \end{aligned} \quad (\text{A.3})$$

where  $\epsilon = \text{sgn}(\Im(z))$ . In particular,

$$\Psi(a, 1; ze^{2i\pi}) = \Psi(a, 1; z)e^{-2i\pi a} + \frac{2\pi i e^{-i\pi a + z}}{\Gamma^2(a)} \Psi(1 - a, 1; -z), \quad \Im(z) < 0, \quad (\text{A.4})$$

$$\Psi(a, 1; ze^{-2i\pi}) = \Psi(a, 1; z)e^{2i\pi a} - \frac{2\pi i e^{i\pi a + z}}{\Gamma^2(a)} \Psi(1 - a, 1; -z), \quad \Im(z) > 0. \quad (\text{A.5})$$

• Asymptotic expansion:

$$\Psi(a, c; z) \sim \sum_{n=0}^{\infty} (-1)^n \frac{(a)_n (a - c + 1)_n}{n!} z^{-a-n}, \quad z \rightarrow \infty, \quad -\frac{3\pi}{2} < \arg(z) < \frac{3\pi}{2}, \quad (\text{A.6})$$

with  $(a)_n = \Gamma(a + n) / \Gamma(a)$ .

We have the following recombination between the Tricomi CHF  $\Psi(a, c; z)$  and the Humbert CHF  $\Phi(a, c; z)$

$$\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(c - a)} e^{i\epsilon a \pi} \Psi(a, c; z) + \frac{\Gamma(c)}{\Gamma(a)} e^{i\epsilon \pi (a - c) + z} \Psi(c - a, c; -z), \quad (\text{A.7})$$

where  $\epsilon = \text{sgn}(\Im(z))$ , and

$$\Phi(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!}. \quad (\text{A.8})$$

Such a recombination formula allows to obtain the asymptotic expansion of the Humbert CHF:

$$\begin{aligned} \Phi(a, c; z) &= \frac{\Gamma(c)}{\Gamma(c - a)} \left( \frac{e^{i\pi \epsilon}}{z} \right)^a \sum_{n=0}^M \frac{(a)_n (a - c + 1)_n}{n! (-z)^n} + \mathcal{O}(|z|^{-a-M-1}) \\ &\quad + \frac{\Gamma(c)}{\Gamma(a)} e^z z^{a-c} \sum_{n=0}^N \frac{(c - a)_n (1 - a)_n}{n! z^n} + \mathcal{O}(|e^z z^{a-1-c-N}|). \end{aligned} \quad (\text{A.9})$$

One can estimate integrals involving a product of two CHF as below, either by using Laplace-type integral representations for the functions  $\Phi(a, c; z)$  and  $\Psi(a, c; z)$  or applying the method given in [32]. The latter uses Erdelyi's representation of Laplace transforms of products of CHF in terms of Lauricella function adjoint to some asymptotic expansion of Lauricella function. In any case, the result reads:

$$\int_0^{+\infty} dt \{ e^{-i\pi a} \varphi(a; t) - 1 \} = -2ia, \quad (\text{A.10})$$

$$\int_0^{+\infty} dt \left\{ e^{-i\pi a} \tau(a; t) + 1 + \frac{2ia}{t+1} \right\} = 2ia - a[\psi(a) + \psi(-a)], \quad (\text{A.11})$$

and the Riemann integrability of the integrands is part of the conclusion. We remind the definition of the functions  $\tau(a; t)$  and  $\varphi(a; t)$ :

$$\varphi(\nu; t) = \Phi(-\nu, 1; -it) \Phi(\nu, 1; it), \quad (\text{A.12})$$

$$\begin{aligned} \tau(\nu; t) = & -\Phi(-\nu, 1; -it) \Phi(\nu, 1; it) + (\partial_z \Phi)(-\nu, 1; -it) \Phi(\nu, 1; it) \\ & + \Phi(-\nu, 1; -it) (\partial_z \Phi)(\nu, 1; it). \end{aligned} \quad (\text{A.13})$$

## B Three preparatory Lemmas

Here we prove three preparatory integration lemmas used in Section 6.

**Lemma B.1** *Let  $\mathcal{R}(u, t)$  be a function of two variables defined on  $I \times \mathbb{R}^+$ , where  $I$  is an open interval of  $\mathbb{R}$  containing 0. Suppose that the partial applications  $u \rightarrow \mathcal{R}(u, t)$  are  $\mathcal{C}^1(I)$  for all but finitely many  $t$ 's and that  $t \rightarrow \mathcal{R}(u, t)$  is Riemann integrable uniformly in  $u$ , i.e.:*

$$\begin{aligned} & \forall \rho > 0, \forall M > 0, \forall u_0 \in I, \exists v > 0 \quad \text{such that} \\ & u \in ]-v + u_0; v + u_0[ \cap I, k \in \{0, 1\} \Rightarrow \left| \int_M^{+\infty} dt \left[ \partial_1^k \mathcal{R}(u, t) - \partial_1^k \mathcal{R}(u_0, t) \right] \right| \leq \rho. \end{aligned} \quad (\text{B.1})$$

Then for  $g \in \mathcal{C}^1(I)$

$$\int_0^\delta xg(t) \mathcal{R}(t, xt) dt = g(0) \int_0^{+\infty} \mathcal{R}(0, t) dt + o(1) \quad (\text{B.2})$$

where the small  $o(1)$  is with respect to the successive limit  $x\delta \rightarrow +\infty$  and  $\delta \rightarrow 0$ .

*Proof* — One has

$$\int_0^\delta x(g(t) \mathcal{R}(t, xt) - g(0) \mathcal{R}(0, xt)) dt = \int_0^\delta dy \int_{xy}^{x\delta} \partial_y [g(y) \mathcal{R}(y, t)] dt. \quad (\text{B.3})$$

Consider a function

$$g : (y, a, b) \mapsto \int_a^b dt \partial_y [g(y) \mathcal{R}(y, t)] \quad (\text{B.4})$$

on the compact set  $[0; \delta] \times \overline{\mathbb{R}^+} \times \overline{\mathbb{R}^+}$ .  $g$  is clearly continuous on the interior and the uniform Riemann-integrability of  $\mathcal{R}(y, t)$  guarantees that it is continuous in an

neighborhood of  $(*, +\infty, *)$ ,  $(*, *, +\infty)$  and  $(*, +\infty, +\infty)$ . Hence,  $|g|$  is bounded, say by  $B$ , as continuous function on a compact set. Thus,

$$\left| \int_0^\delta dy \int_{xy}^{x\delta} \partial_y [g(y) \mathcal{R}(y, t)] dt \right| \leq \delta B, \quad (\text{B.5})$$

which ends the proof of Lemma B.1.  $\square$

**Lemma B.2** *Let  $g \in \mathcal{C}^1(I)$  for some open interval  $I$  containing 0, then*

$$\int_0^\delta \frac{g(t) x dt}{1 + xt} = g(0) \log x\delta + o(1), \quad (\text{B.6})$$

where  $o(1)$  stands with respect to the successive limits  $x\delta \rightarrow +\infty$  and  $\delta \rightarrow 0$ .

*Proof* — We have

$$\int_0^\delta g(t) \frac{x dt}{1 + xt} = g(0) \log(x\delta + 1) + \int_0^\delta dt \int_0^t dy \frac{g'(y) x}{1 + xt} \quad (\text{B.7})$$

$$= g(0) \log \delta x + o(1) + \int_0^\delta dy g'(y) \log \left( \frac{x\delta + 1}{xy + 1} \right). \quad (\text{B.8})$$

But,

$$\left| \int_0^\delta dy g'(y) \log \left( \frac{\delta + 1/x}{y + 1/x} \right) \right| \leq \sup_{[0; \delta]} |g'| \times (\delta - \log(\delta + 1/x)/x) \xrightarrow{\delta \rightarrow 0} 0, \quad (\text{B.9})$$

which ends the proof of Lemma B.2.  $\square$

**Lemma B.3** *Let  $\kappa$  be defined in terms of  $\nu$  as in (2.2), and set*

$$H(q) = \frac{1}{2} \int_{-q}^q d\lambda d\mu \frac{\nu'(\lambda) \nu(\mu) - \nu'(\mu) \nu(\lambda)}{\lambda - \mu} + \sum_{\epsilon=\pm} \epsilon \nu_\epsilon \log \kappa(\epsilon q; q). \quad (\text{B.10})$$

*Then,*

$$2\nu_+ \frac{d}{dq} [\log \kappa(q; q)] - 2\nu_- \frac{d}{dq} [\log \kappa(-q; q)] - \frac{(\nu_+ - \nu_-)^2}{q} = \frac{dH(q)}{dq}. \quad (\text{B.11})$$

*Proof* — Using (2.2), one can express the derivative of  $H(q)$  as

$$\frac{dH(q)}{dq} = \sum_{\epsilon=\pm} \nu(\epsilon q) \int_{-q}^q d\mu \frac{\nu'(\epsilon q) - \nu'(\mu)}{\epsilon q - \mu} + \sum_{\epsilon=\pm} \epsilon \nu(\epsilon q) \frac{d}{dq} [\log \kappa(\epsilon q; q)]. \quad (\text{B.12})$$

Thus, proving (B.11) amounts to establishing the equality

$$\sum_{\epsilon=\pm} \epsilon \nu(\epsilon q) \frac{d}{dq} [\log \kappa(\epsilon q; q)] - \frac{(\nu_+ - \nu_-)^2}{q} = \sum_{\epsilon=\pm} \nu(\epsilon q) \int_{-q}^q d\mu \frac{\nu'(\epsilon q) - \nu'(\mu)}{\epsilon q - \mu}. \quad (\text{B.13})$$

The latter follows from an integration by parts:

$$\begin{aligned} & \sum_{\epsilon=\pm} \epsilon \nu(\epsilon q) \frac{d}{dq} [\log \kappa(\epsilon q; q)] \\ &= \sum_{\epsilon=\pm} \epsilon \nu_\epsilon \left\{ \nu'_\epsilon + \frac{\nu_+ - \nu_-}{2q} + \epsilon \int_{-q}^q d\mu \frac{\nu(\mu) - \nu_\epsilon - \nu'_\epsilon(\mu - \epsilon q)}{(\mu - \epsilon q)^2} \right\} \\ &= \sum_{\epsilon=\pm} \epsilon \nu_\epsilon \left\{ \nu'_\epsilon + \frac{\nu_+ - \nu_-}{2q} + \frac{\nu_{-\epsilon} - \nu_\epsilon + 2\epsilon q \nu'_\epsilon}{-2\epsilon q} + \epsilon \int_{-q}^q d\mu \frac{\nu'(\mu) - \nu'_\epsilon}{\mu - \epsilon q} \right\} \\ &= \frac{(\nu_+ - \nu_-)^2}{q} + \sum_{\epsilon=\pm} \nu_\epsilon \int_{-q}^q d\mu \frac{\nu'(\mu) - \nu'_\epsilon}{\mu - \epsilon q}. \end{aligned}$$

This ends the proof. □

## C The density Theorem

**Theorem C.1** *Let  $U, W$  be two open neighborhoods of  $[-q; q]$ ,  $\mathcal{F}_n \left( \begin{smallmatrix} \{\lambda\} \\ \{z\} \end{smallmatrix} \right)$  a symmetric function in  $n$  variables  $\lambda$  and  $n$  variables  $z$  that is holomorphic on  $U^n \times W^n$ . Then, for any compact subsets  $K$  (resp.  $P$ ) of  $U$  (resp.  $W$ ) there exists a sequence  $(\varphi_p, \phi_p)_{p \in \mathbb{N}} \in \mathcal{H}(K_\epsilon) \times \mathcal{H}(P_\epsilon)$  such that*

$$\mathcal{F}_n \left( \begin{smallmatrix} \{\lambda\} \\ \{z\} \end{smallmatrix} \right) = \sum_{p=0}^{+\infty} \prod_{i=1}^n \varphi_p(\lambda_i) \phi_p(z_i) \quad \text{uniformly on } K^n \times P^n. \quad (\text{C.1})$$

*There  $\text{int}[K_\epsilon]$  (resp.  $\text{int}[P_\epsilon]$ ) is an open neighborhood in  $U$  (resp.  $W$ ) of  $K$  (resp.  $P$ ).*

*Proof* — Let  $K$  and  $P$  be as above.

We define  $X = K^n \times P^n / \sim$ , where  $(\lambda, z) \sim (\lambda', z')$  if there exists  $(\sigma, \pi) \in \mathfrak{S}_n \times \mathfrak{S}_n$  such that  $(\lambda^\sigma, z^\pi) = (\lambda', z')$ , and  $\lambda^\sigma$  stands for  $(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)})$ . Since  $\mathfrak{S}_n \times \mathfrak{S}_n$  is a discrete group, its action on  $K^n \times P^n$  is by definition proper i.e.  $\forall L \supset K^n \times P^n$

$$\{(\sigma, \pi) \in \mathfrak{S}_n \times \mathfrak{S}_n : L^{\sigma, \pi} \cap L = \emptyset\} \quad \text{is discrete.} \quad (\text{C.2})$$

This ensures that  $X$  is a compact Hausdorff topological space. Moreover the space  $\mathcal{C}(X, \mathbb{C})$  of continuous functions on  $X$  is canonically identified with the space of continuous functions on  $K^n \times P^n$  that are symmetric in the first or the last  $n$  variables.

Define the subspace  $S$  of  $\mathcal{C}(X, \mathbb{C})$  as the subset of functions  $\mathcal{F}_n^{(\varphi, \phi)}$  of the form

$$\mathcal{F}_n^{(\varphi, \phi)} \left( \begin{array}{c} \{\lambda\} \\ \{z\} \end{array} \right) = \prod_{i=1}^n f(\lambda_i) g(z_i), \quad (\text{C.3})$$

where  $(\varphi, \phi) \in \mathcal{H}(K) \times \mathcal{H}(P)$ , and let  $\mathcal{S}$  be the  $C^*$ -algebra generated by  $S$ . We have that  $S$  and hence  $\mathcal{S}$  separates points in  $X$ . Indeed, let  $(\lambda, z)$  and  $(\mu, y)$  be any two representatives in  $K^n \times P^n$  of two distinct points in  $X$ . Thus

- there exists  $\lambda_i \in K$  such that exactly  $p$  of the  $n$  coordinates of the  $n$ -tuple  $\lambda$  are equal to  $\lambda_i$ , whereas exactly  $q$  of the  $n$  coordinates of the  $n$ -tuple  $\mu$  are equal to  $\lambda_i$ , with  $p \neq q$ ;
- or there exists  $z_i \in P$  such that exactly  $p$  of the  $n$  coordinates of the  $n$ -tuple  $z$  are equal to  $z_i$ , whereas exactly  $q$  of the  $n$  coordinates of the  $n$ -tuple  $y$  are equal to  $z_i$ , with  $p \neq q$ .

The situation is similar in the case of the first  $n$  and last  $n$  variables, therefore we only treat the first case. By Lagrange interpolation there exist a polynomial  $Q$  such that, for any coordinate  $\lambda_k$  of  $\lambda$  and any coordinate  $\mu_k$  of  $\mu$  satisfying  $\lambda_k \neq \lambda_i$  and  $\mu_k \neq \lambda_i$ ,

$$Q(\lambda_k) = Q(\mu_k) = 1 \quad \text{and} \quad Q(\lambda_i) = 2. \quad (\text{C.4})$$

The function

$$\mathcal{F}_n^{(Q, 1)} \left( \begin{array}{c} \{\lambda\} \\ \{z\} \end{array} \right) = \prod_{p=1}^n Q(\lambda_p) \in \mathcal{S} \quad (\text{C.5})$$

separates the projections of  $(\lambda, z)$  and  $(\mu, y)$  on  $X$ . Thus  $\mathcal{S}$  is a  $C^*$ -subalgebra of  $\mathcal{C}(X; \mathbb{C})$  that separates points. It then follows by the Stone-Weierstrass theorem that  $\mathcal{S} = \mathcal{C}(X; \mathbb{C})$ .

Let  $\mathcal{F}_n$  be holomorphic on  $U^n \times W^n$  and symmetric in the first and the last  $n$  variables. There exists compact sets  $K_\epsilon \subset U$  and  $P_\epsilon \subset W$  such that  $K \subset \overset{\circ}{K}_\epsilon$  and  $P \subset \overset{\circ}{P}_\epsilon$ . Here,  $\overset{\circ}{K}_\epsilon$  stands for the interior of  $K_\epsilon$ . Thus the restriction of  $\mathcal{F}_n$  to  $K_\epsilon^n \times P_\epsilon^n$  also belongs to  $\mathcal{C}(X_\epsilon; \mathbb{C})$ , with  $X_\epsilon = K_\epsilon^n \times P_\epsilon^n / \sim$ , and therefore there exists  $(\tilde{\varphi}_p, \tilde{\phi}_p)_{p \in \mathbb{N}}$  in  $\mathcal{C}(K_\epsilon; \mathbb{C}) \times \mathcal{C}(P_\epsilon; \mathbb{C})$  such that

$$\mathcal{F}_n \left( \begin{array}{c} \{\lambda\} \\ \{z\} \end{array} \right) = \sum_{p=0}^{+\infty} \prod_{i=1}^n \tilde{\varphi}_p(\lambda_i) \tilde{\phi}_p(z_i) \quad \text{uniformly on } K_\epsilon^n \times P_\epsilon^n. \quad (\text{C.6})$$



In particular the sequence converges uniformly to  $\mathcal{F}_n$  on  $(\partial K_\epsilon)^n \times (\partial P_\epsilon)^n$ , the latter set being compact. Therefore we have

$$\begin{aligned} \sum_{p=0}^N \int_{\partial K_\epsilon} \frac{d^n \mu}{(2i\pi)^n} \int_{\partial P_\epsilon} \frac{d^n y}{(2i\pi)^n} \prod_{i=1}^n \frac{\tilde{\varphi}_p(\mu_i) \tilde{\phi}_p(y_i)}{(\mu_i - \lambda_i)(y_i - z_i)} &= \sum_{p=0}^N \prod_{i=1}^n \varphi_p(\lambda_i) \phi(z_i) \\ \xrightarrow{N \rightarrow +\infty} \int_{\partial K_\epsilon} \frac{d^n \mu}{(2i\pi)^n} \int_{\partial P_\epsilon} \frac{d^n y}{(2i\pi)^n} \frac{\mathcal{F}_n(\{\mu\} | \{y\})}{\prod_{i=1}^n (\mu_i - \lambda_i)(y_i - z_i)} &= \mathcal{F}_n(\{\lambda\} | \{z\}), \end{aligned} \quad (\text{C.7})$$

uniformly in  $(\lambda, z) \in K^n \times P^n$ . Moreover,

$$\varphi_p(\lambda) = \int_{\partial K_\epsilon} \frac{d\mu}{2i\pi} \frac{\tilde{\varphi}_p(\mu)}{\mu - \lambda} \quad \text{and} \quad \phi_p(z) = \int_{\partial P_\epsilon} \frac{dy}{2i\pi} \frac{\tilde{\phi}_p(y)}{y - z} \quad (\text{C.8})$$

are holomorphic in  $K$ , resp.  $P$ . □

□

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