Conversion/Preference Games

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Abstract

We introduce the concept of *Conversion/Preference Games*, or CP games for short. CP games generalize the standard notion of *strategic games*. First we exemplify the use of CP games. Second we formally introduce and define the CP-games formalism. Then we sketch two 'real-life' applications, namely a connection between CP games and gene regulation networks, and the use of CP games to formalize implied information in Chinese Wall security. We end with a study of a particular fixed-point construction over CP games and of the resulting existence of equilibria in possibly infinite games.

1 Introduction

We give a stand-alone account of Conversion/Preference games or CP games, as originally used in [9]. CP games are built from a set of *players* and a set of (game) *situations*. The ability of the players to change a situation to another is formalised in *conversion* relations. A *preference* relation dictates how the players compare the different situations against each other. The three main aims of this article are to show i) that discrete Nash-style game theory is possible and natural, ii) that the two basic CP concepts of *Conversion* and *Preference* are of wider interest, and iii) that game-theoretic notions are both applicable and relevant in situations where no payoff function need exist, or where the payoff concept would dramatically alter what aspects of the game are being considered.

2 Basic concepts

To start with, let us give the two main notions of games. First, a game involves *players*. Second, a game is characterized by situations. In CP games these situations will be called *situations* or sometimes *synopses* or *game situations*. A player can move from one situation to another, but she[1] does that under some constraints as she has no total freedom to perform her moves, therefore a relation called *conversion* is defined for each player; it tells what moves a player is allowed to perform. Conversion of player *Alice* will be written $\dashrightarrow_{Alice}$. As such, conversion tells basically the rules of the game. In chess it would say *"a player can move her bishop along a diagonal"*, but it does not tell the game line of the player. In other words it does not tell why the player chooses to move or to "convert" her situation. Another relation called *preference* compares situations in order for a player to choose a better move or to perform a better conversion. Preference of player *Beth* will be written $\dashrightarrow_{Beth}$ and when we write $s \dashrightarrow_{Beth} s'$ we mean that *Beth* prefers s' to s or, rather than s, she chooses s' or in situation s she is attracted toward situation s'. Preference (or choice) is somewhat disconnected from conversion, a player can clearly prefer a situation she cannot move to and vice versa she can move to a situation she does not prefer. Moreover players may share the same conversion relation, but this not a rule and the may share or not the same preference relation or not. Those different situation sill be illustrated by examples throughout the article.

A key concept in games is this of *equilibrium*. As a player can convert a situation, she can convert it into a situation she likes better, in the sense that she prefers the new situation she converted to. A player is *happy* in a situation, if there is no situation she can convert into and she prefers. A situation is an equilibrium if each player is happy with this situation. We will see that this concept of equilibrium captures and generalizes the concept known as *Nash equilibrium* in strategic games, hence the name *abstract Nash equilibrium*.

3 Some examples

Let us present the above concepts of conversion, preference and equilibrium through examples. We will introduce a new concept called *change of mind*.

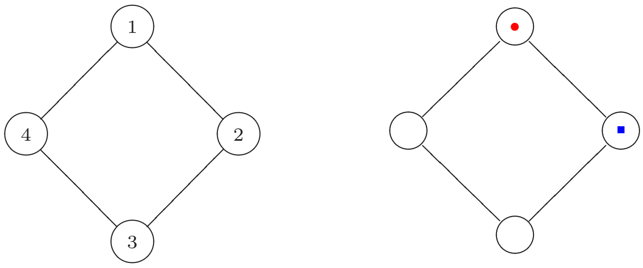
3.1 A simple game on a square

As an introduction, we will look at variations of a simple game on a board.

[1]See the preface of [5] for the use of personal pronouns

3.1.1 A first version

Imagine a simple game where *Alice* and *Beth* play using tokens on a square. We number the four positions as 1, 2, 3 and 4.



Assume that player *Alice* has a red round token and that player *Beth* has a blue squared token. The two players place their tokens on vertices and then they move along edges. They can also decide not to move. Assume that *Alice* and *Beth* never put their token on a vertex taken by the other player and a position further than this impossible situation is better than a position closer. In other words, a position with Alice on vertex i and Beth on vertex j with $i-j$ even is preferred to a position with $i-j$ odd.

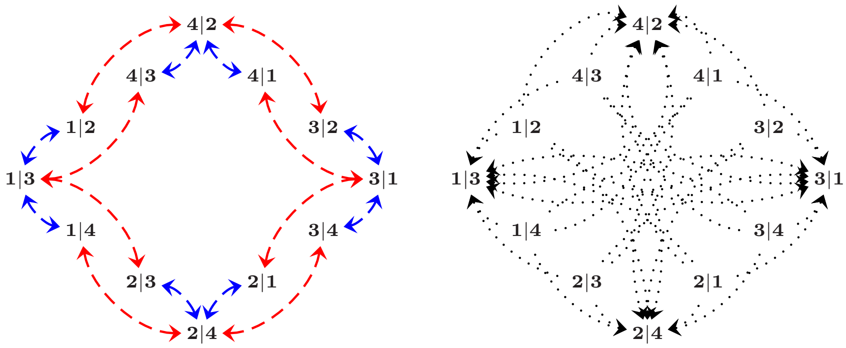


Figure 1: Conversion and preference for the square game

The game has 12 situations, which we write $i|j$ for $1 \leq i, j \leq 4$ and $i \neq j$. The above pictured situation corresponds to $1|2$. The two conversions are described by Figure 1 left. In this figure, $\dashrightarrow$ is *Alice*'s conversion and $\dashrightarrow$ is *Beth*'s conversion.

In this game, both players share the same preference, namely the following: since a player does not want her token on a position next to the other token, she prefers a situation where her token is on the opposite corner of the other token. This gives the preference given in Figure 1 right. The arrow from $1|2$ to $1|3$ means players prefer $1|3$ to $1|2$.

From the conversion and the preference we build a relation that we call *change of mind*. *Alice* can change her mind from a situation s to a new one s', if she can convert s into the new situation s' and rather than s she chooses s'.

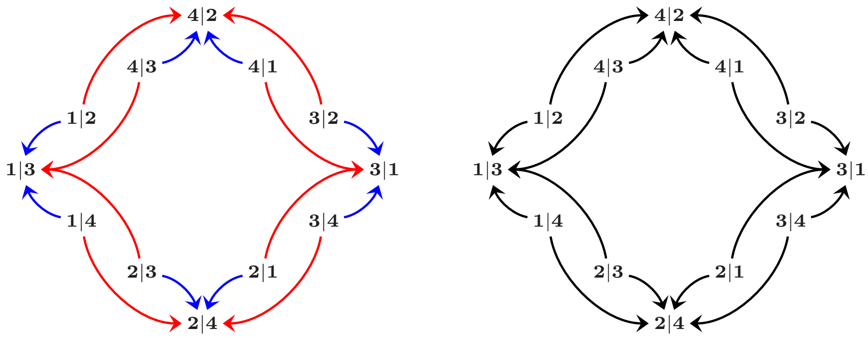


Figure 2: Agent changes of mind (on the left) and (general) change of mind (on the right) for the square game

Changes of mind for *Alice* and *Beth* are given in Figure 2 on the left. In this figure, $\longrightarrow$ is *Alice*'s change of mind and $\longrightarrow$ is *Beth*'s conversion. The *(general) change of mind* is the union of the *agent change of mind*, it is given by Figure 2 on the right. The equilibria are the end points (or "minimal point") for that relation, namely 1|3, 4|2, 3|1 and 2|4. This means that no change of mind arrows leave those nodes. In these situations players have their tokens on opposite corners and they do not move. An equilibrium like 1|3 which is an end point is called an *Abstract Nash Equilibrium*.

3.1.2 A second version

We propose a second version of the game, where moves of the token can only be made clockwise. This implies to change the conversion changes, but also the preference, as a player does want not to be threatened by another token placed before hers clockwise and prefers a situation that places this token as far as possible. The conversions, the preferences and the changes of mind are given in Figure 3 (page 5). If one looks at the equilibrium, one sees that there is no fixed position where players are happy. To be happy the players have to move around for ever, one chasing the other. It is not really a cycle, but a perpetual move. We also call that an equilibrium. It is sometimes called a *dynamic equilibrium* or a *stationary state*.

3.1.3 A third version

The third version is meant to present an interesting feature of the change of mind. In this version, we use the same rules as the second one, except that we suppose that the game does not start with both token on the board. Actually it starts as follows. *Alice* has put her token on node 1 (this game positions is described as $1|\omega$). Then *Beth* chooses a position among 2, 3 or 4. The conversion

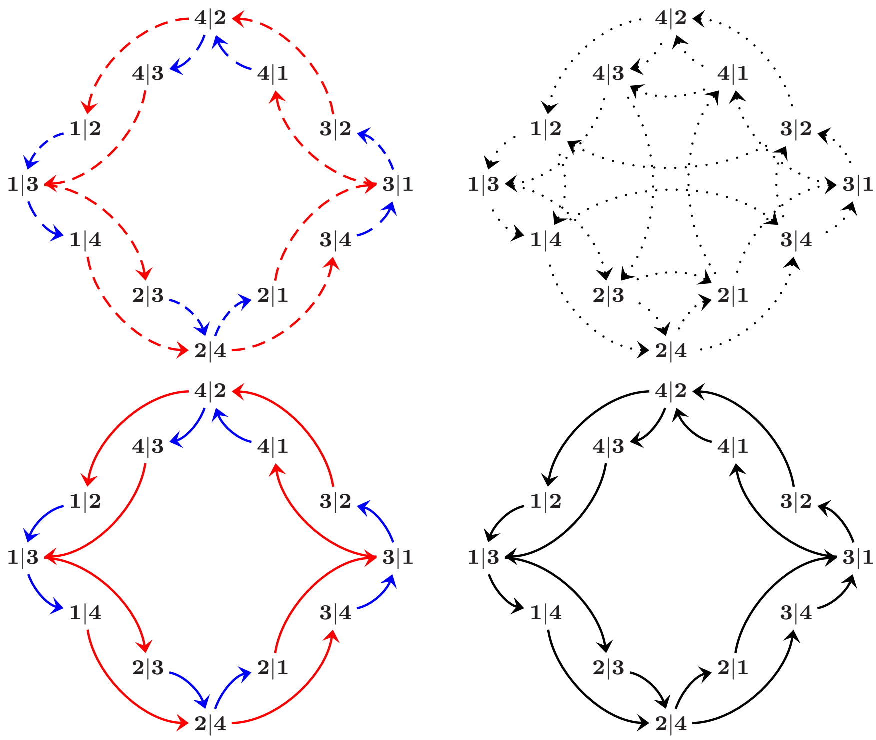


Figure 3: Conversion, preference and change of mind for the second version of the square game

is given in Figure 4 left. *Beth* may choose not to play, but in this case she loses, in other words, she prefers any position to $1|\omega$. We do not draw the preference relation, as it would make for an entangled picture. The change of mind is given on Figure 4 right (page 6). There is again a dynamic equilibrium and one sees that this dynamic equilibrium is not the whole game, indeed one enters the perpetual move after at least one step in the game.

3.2 Strategic games

In this presentation of strategic games we do not use payoff functions, but directly a preference relation (See Section 1.1.2 of [5] for a discussion) and we present several games.

3.2.1 The Prisoner's Dilemma

The problem is stated usually as follows

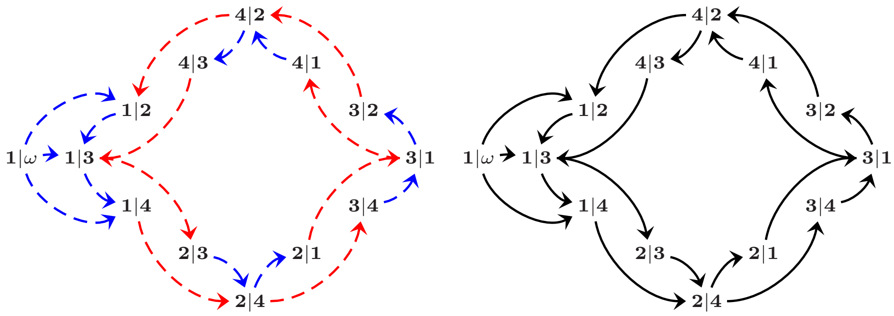


Figure 4: Conversion and change of mind for the third version of the square game

> Two suspects, A and B, are arrested by the police. The police have insufficient evidence for a conviction, and, having separated both prisoners, visit each of them to offer the same deal: if one acts as an informer against the other (*finks*) and the other remains *quiet*, the betrayer goes free and the quiet accomplice receives the full sentence. If both stay quiet, the police can sentence both prisoners to a reduced sentence in jail for a minor charge. If each finks, each will receive a similar intermediate sentence. Each prisoner must make the choice of whether to fink or to remain quiet. However, neither prisoner knows for sure what choice the other prisoner will make. So the question this dilemma poses is: What will happen? How will the prisoners act?

Each prisoner can be into two states, either *fink* (F) or be *quiet* (Q). Each prisoner can go from Q to F and vice-versa, hence the following conversion, where is prisoner A conversion and is prisoner B conversion (Figure 5 left). Each prisoner prefers to go free over being sentenced and prefers a light sentence to a full sentence. Hence the preference are given in Figure 5 right, where is prisoner A preference and is prisoner B preference.

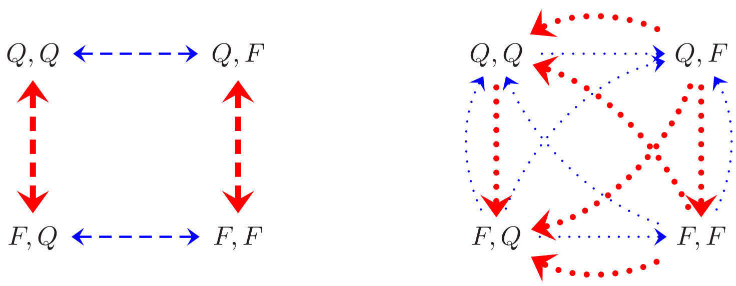


Figure 5: Conversion and preference in the prisoner's dilemma

From this we get the change of mind of Figure 6. One sees clearly that the only equilibrium is F, F despite both prefer Q, Q as shown on Figure 5 right.

$$\begin{tikzcd}
Q,Q \arrow[r] \arrow[d] & Q,F \arrow[d] \\
F,Q \arrow[r] & F,F
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Q,Q \arrow[r] \arrow[d] & Q,F \arrow[d] \\
F,Q \arrow[r] & F,F
\end{tikzcd}$$

Figure 6: Agent and (general) change of mind in the prisoner's dilemma

Such an equilibrium is called a Nash equilibrium in strategic game theory.

The paradox comes from the fact that F, F is an equilibrium despite the fact one has: $\begin{tikzcd} Q,Q & F,F \arrow[l, dotted, shift right] \arrow[l, dotted, shift left] \end{tikzcd}$ in the preference.

3.2.2 Matching Pennies

This second example is also classic. This is a simple example of strategic game where there is no singleton equilibrium. As an equilibrium can contain more than one situation, we call singleton equilibrium a CP equilibrium which contains only one situation. This boils down to the kind of equilibrium we have introduced so far.

> The game is played between two players, Player A and Player B. Each player has a penny and must secretly turn the penny to heads (H) or tails (T). The players then reveal their choices simultaneously. If the pennies match (both heads or both tails), Player A wins. If the pennies do not match (one heads and one tails), Player B wins.

The conversion is similar to this of the prisoner's dilemma (Figure 7 left) and the preference is given by who wins (Figure 7 center).

Change of mind for matching pennies is in Figure 7 right. One notices that there is a cycle. This cycle is the equilibrium. No player has clear mind of what to play and changes her minds each time she loses.

3.2.3 Scissors, Paper, Stone

Here we present the famous game known as *scissors, paper, stone*. It involves two players, *Alice* and *Beth* who announce either *scissors* (C) or *paper* (P) or *stone* (T) with the rules that *stone beats scissors, scissors beat paper, and paper beats stone*. There are nine situations (see below), one sees that *Alice* may convert her situation C, P into P, P or T, P and the same for the other

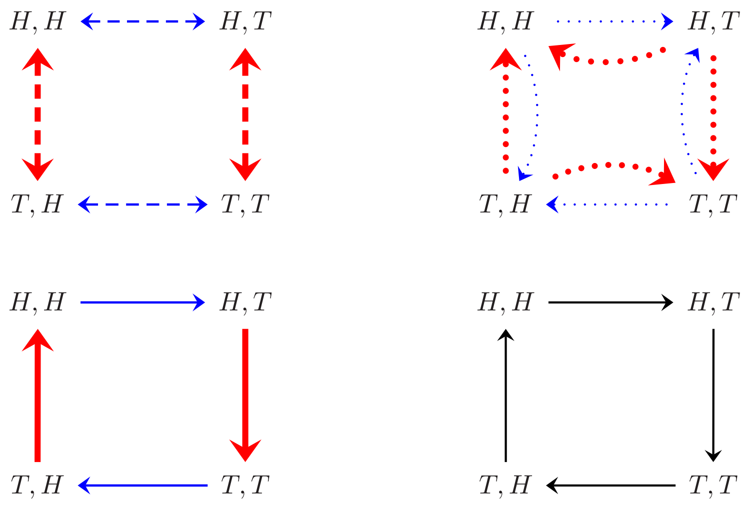
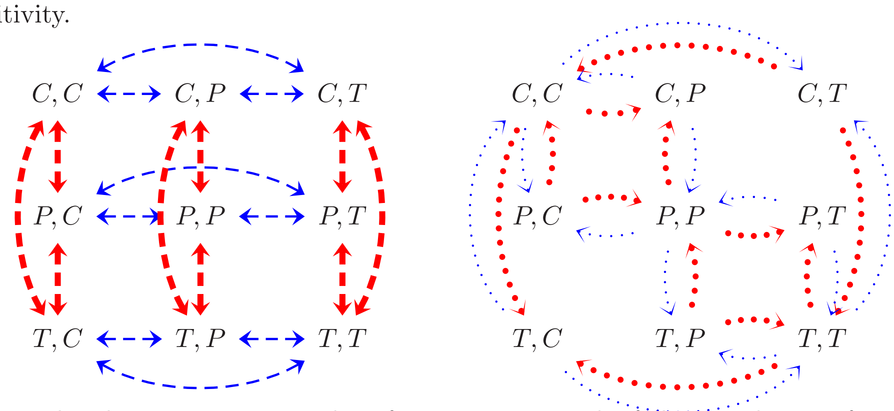


Figure 7: Conversion, preference and (general) change of mind in Matching Pennies

situations. The conversion is given below left. Since the rules, it seems clear that *Alice* prefers P,P to T,P and C,P to P,P, hence the preference given below right with $\cdots\!\rightarrow$ is *Alice*'s preference and $\cdots\!\rightarrow$ is *Beth*'s preference. To avoid a cumbersome diagram, in the preference we do not put the arrows deduced by transitivity.



From the above conversion and preference, one gets the following change of mind.

$$\begin{tikzcd}
C,C \arrow[rr, bend left] \arrow[dd, bend right] & C,P \arrow[l] \arrow[r] & C,T \arrow[d] \arrow[dd, bend left] \\
P,C \arrow[u] \arrow[d] & P,P \arrow[l] \arrow[u] & P,T \arrow[l] \arrow[ll, bend right] \\
T,C \arrow[r] \arrow[rr, bend right] & T,P \arrow[u] \arrow[uu, bend left] & T,T \arrow[l] \arrow[u]
\end{tikzcd}$$

One sees also perpetual moves as in the matching pennies of which it is a generalization.

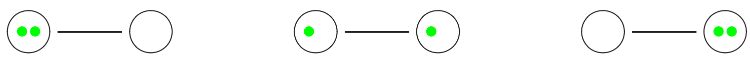
3.2.4 Strategic games as CP games

A strategic game is a specific kind of CP games. To be a strategic game, a CP game has to fulfill the following conditions.

1. Each situation is a n-Cartesian product, where n is the number of players. The constituents of the Cartesian product are called *strategies*.
2. Conversion for player a, written $\dashrightarrow_a$, is any change along the a-th dimension, i.e., $(s_1, ..., s_a, ..., s_n) \dashrightarrow_a (s_1, ..., s'_a, ..., s_n)$. Hence in strategic games, conversion is an equivalent relation, namely
   - symmetric, ($s \dashrightarrow_a s'$ implies $s' \dashrightarrow_a s$),
   - transitive, ($s \dashrightarrow_a s'$ and $s' \dashrightarrow_a s''$ imply $s \dashrightarrow_a s''$),
   - and reflexive ($s \dashrightarrow_a s$).

3.3 Blink and you lose

Blink and you lose is a game played on a simple graph with two undifferentiated tokens. There are three positions:



There are two players, *Left* and *Right*. The leftmost position above is the winning position for *Left* and the rightmost position is the winning position for *Right*. In other words, the one who owns both token is the winner. Let us call the positions L, C, and R respectively. One plays by taking a token on the opposite node.

3.3.1 A first tactic: Foresight

A player realizes that she can win by taking the opponent's token faster than the opponent can react, i.e., player *Left* can convert C into L by outpacing player *Right*. Player *Right*, in turn, can convert C into R. This version of the game has two singleton equilibria: L and R. This is described by the following conversion

$$\begin{tikzcd} L & C \arrow[l, dashed] \arrow[r, dashed] & R \end{tikzcd}$$

preference is

$$\begin{tikzcd} L \arrow[r, bend left, dotted] & C \arrow[l, bend left, dotted] \arrow[r, bend left, dotted] & R \arrow[l, bend left, dotted] \end{tikzcd}$$

where $\dashrightarrow$ is the preference for *Left* and $\dashrightarrow$ is the preference for *Right*. The change of mind is then:

$$L \longleftarrow C \longrightarrow R$$

and one sees that there are two equilibria: namely L and R, which means that players have taken both token and keep them.

3.3.2 A second tactic: Hindsight

A player, say *Left*, analyzes what would happen if she does not act. In case *Right* acts, the game would end up in R and *Left* loses. As we all know, people hate to lose so they have an aversion for a losing position. Actually *Left* concludes that she could have prevented the R outcome by acting. In other words, it is within *Left*'s power to convert R into C. Similarly for player *Right* from L to C.

$$\begin{tikzcd} L \arrow[r, dashed] & C & R \arrow[l, dashed] \end{tikzcd}$$

We call naturally *aversion* the relation that escapes from positions a player does not want to be, especially a losing position. Aversion deserves its name as it works like conversion, but flies from bad position. We get the following change of mind:

$$L \longrightarrow C \longleftarrow R$$

where C is singleton equilibrium or an Abstract Nash Equilibrium.

3.3.3 A third tactic: Omnisight

The players have both hindsight and foresight, resulting in a CP game

$$\begin{tikzcd} L \arrow[r, bend left, dashed] & C \arrow[l, bend left, dashed] \arrow[r, bend left, dashed] & R \arrow[l, bend left, dashed] \end{tikzcd}$$

with one change-of-mind equilibrium covering all outcomes thus, no singleton equilibrium (or Abstract Nash Equilibrium) exists.

$$\begin{tikzcd} L \arrow[r, bend left] & C \arrow[l, bend left] \arrow[r, bend left] & R \arrow[l, bend left] \end{tikzcd}$$

3.3.4 A four tactic: Defeatism

One of the player, say *Left*, acknowledges that she will be outperformed by the other, *Right* in this case. She is so terrified by her opponent that she returns the token when she has it. This yields the following conversion:

$$L \dashrightarrow C \dashrightarrow R$$

We get the following change of mind:

$$L \longrightarrow C \longrightarrow R$$

where R is a singleton equilibrium or an Abstract Nash Equilibrium.

3.3.5 Relation with evolutionary games

In [4] (Fig. 2, p. 795), Nowak and Sigmund comment a similar situation in evolutionary games. They call the first tactic, *bistability*, the second tactic, *coexistence*, the third tactic, *neutrality* and the fourth tactic, *dominance* and exhibit the same pictures.

The changes of mind corresponding to the four tactics and their correspondence with evolutionary games with two strategies can be summarized as follows

Blink you lose	**Change of Mind**	**Evol. games**
Foresight	$\mathrm{L} \longleftarrow \mathrm{C} \longrightarrow \mathrm{R}$	Bistablity
Hindsight	$\mathrm{L} \longrightarrow \mathrm{C} \longleftarrow \mathrm{R}$	Coexistence
Omnisight	$\mathrm{L} \rightleftarrows \mathrm{C} \rightleftarrows \mathrm{R}$	Neutrality
Defeatism	$\mathrm{L} \longrightarrow \mathrm{C} \longrightarrow \mathrm{R}$	Dominance

3.4 The λ phage as a CP game

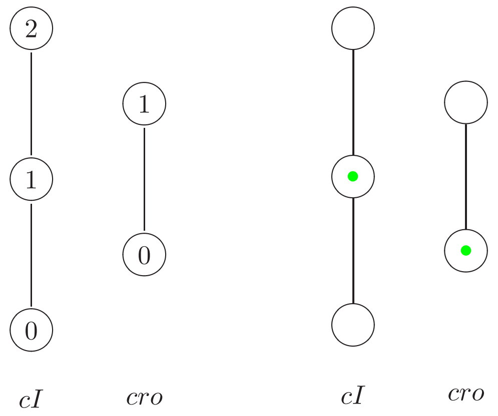
The λ phage is a game inspired from biology [6, 7]. The origin of the game will be given in Section 6, here we give just *the rules of the game.*

There are three players cI, cro and Env. The game can be seen as a game with two tokens moving on two graphs where each player may choose to move one of the two tokens[2]. Env moves one token from the bottom position. The conversion is therefore the same for the three players[3] and is given by the fol-

[2] In the asynchronous version.

[3] Note the difference with the square game where players had different conversions and the same preference. The fact that the conversion is the same for everybody seems to be a feature of biologic game. Moreover notice also that, unlike in strategic games, the conversion is not transitive.

lowing rightmost diagram:



$$\begin{tikzcd}
\langle cI_2, cro_0\rangle \arrow[r, dashed, leftrightarrow] \arrow[d, dashed, leftrightarrow] & \langle cI_2, cro_1\rangle \arrow[d, dashed, leftrightarrow] \\
\langle cI_1, cro_0\rangle \arrow[r, dashed, leftrightarrow] \arrow[d, dashed, leftrightarrow] & \langle cI_1, cro_1\rangle \arrow[d, dashed, leftrightarrow] \\
\langle cI_0, cro_0\rangle \arrow[r, dashed, leftrightarrow] & \langle cI_0, cro_1\rangle
\end{tikzcd}$$

The preference is difficult to describe as an actual game to be played, it comes from the genetics and is specific to each player. The philosophy is as follows: a gene prefers a position if it is "pushed forward" that position.

$$\begin{tikzcd}
\langle cI_2, cro_0\rangle \arrow[d, dotted, shift right] \arrow[dd, dotted, bend right] & \langle cI_2, cro_1\rangle \arrow[l, dotted] \arrow[d, dotted, shift right] \arrow[dd, dotted, bend left] \\
\langle cI_1, cro_0\rangle \arrow[u, dotted, shift right] & \langle cI_1, cro_1\rangle \arrow[l, dotted] \arrow[u, dotted, shift right] \\
\langle cI_0, cro_0\rangle & \langle cI_0, cro_1\rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle cI_2, cro_0\rangle & \langle cI_2, cro_1\rangle \arrow[d, dotted] \arrow[dd, dotted, bend left] \\
\langle cI_1, cro_0\rangle & \langle cI_1, cro_1\rangle \arrow[d, dotted] \\
\langle cI_0, cro_0\rangle & \langle cI_0, cro_1\rangle
\end{tikzcd}$$

$$\begin{tikzcd}
\langle cI_2, cro_0\rangle & \langle cI_2, cro_1\rangle \\
\langle cI_1, cro_0\rangle & \langle cI_1, cro_1\rangle \\
\langle cI_0, cro_0\rangle \arrow[u, dotted] \arrow[r, dotted] & \langle cI_0, cro_1\rangle
\end{tikzcd}$$

From the conversion and the preferences one deduces three changes of mind.

$$\begin{tikzcd}
\langle cI_2, cro_0\rangle \arrow[d, shift right] & \langle cI_2, cro_1\rangle \arrow[l] \arrow[d, shift right] \\
\langle cI_1, cro_0\rangle \arrow[u, shift right] & \langle cI_1, cro_1\rangle \arrow[l] \arrow[u, shift right] \\
\langle cI_0, cro_0\rangle & \langle cI_0, cro_1\rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle cI_2, cro_0\rangle & \langle cI_2, cro_1\rangle \arrow[d] \\
\langle cI_1, cro_0\rangle & \langle cI_1, cro_1\rangle \arrow[d] \\
\langle cI_0, cro_0\rangle & \langle cI_0, cro_1\rangle
\end{tikzcd}$$

$$
\begin{tikzcd}
\langle cI_2, cro_0\rangle & \langle cI_2, cro_1\rangle \\
\langle cI_1, cro_0\rangle & \langle cI_1, cro_1\rangle \\
\langle cI_0, cro_0\rangle \arrow[u, "\triangle" description, dash] \arrow[r, "\triangleright" description, dash] & \langle cI_0, cro_1\rangle
\end{tikzcd}
$$

from which we deduce the (general) change of mind of the game:

$$
\begin{tikzcd}
\langle cI_2, cro_0\rangle \arrow[d, bend right] & \langle cI_2, cro_1\rangle \arrow[l] \arrow[d, bend right] \\
\langle cI_1, cro_0\rangle \arrow[u, bend right] & \langle cI_1, cro_1\rangle \arrow[l] \arrow[u, bend right] \arrow[d] \\
\langle cI_0, cro_0\rangle \arrow[u] \arrow[r] & \langle cI_0, cro_1\rangle
\end{tikzcd}
$$

One sees one singleton equilibrium namely $\langle cI_0, cro_1\rangle$ (called the *lyse*) and one dynamic equilibrium namely $\{\langle cI_2, cro_0\rangle, \langle cI_1, cro_0\rangle\}$ (called the *lysogen*).

4 Formal presentation of CP games

To define a CP game we have to define four concepts:

- a set $\mathcal{A}$ of *agents*,
- a set $\mathcal{S}$ of *situations*,
- for every agent a a relation $\dashrightarrow_a$ on $\mathcal{S}$, called *conversion*,
- for every agent a a relation $\cdots\!\!\rightarrow_a$ on $\mathcal{S}$, called *preference*.

From these relations we are going to define a relation called *change of mind*. Before let us introduce formally what a game is.

Definition 1 (Game) *A game is a 4-uple* $\langle \mathcal{A}, \mathcal{S}, (\dashrightarrow_a)_{a\in\mathcal{A}}, (\cdots\!\!\rightarrow_a)_{a\in\mathcal{A}}\rangle$.

Example 1 (Square game 1rst version) *For the first version of the square game we have:*

- $\mathcal{A} = \{Alice, Beth\}$,
- $\mathcal{S} = \{1|2, 1|3, 1|4, 2|3, 2|4, 2|1, 3|4, 3|1, 3|2, 4|1, 4|2, 4|3\}$,
- *Conversions* $\dashrightarrow_{Alice}$ *and* $\dashrightarrow_{Beth}$ *are given by Figure 1 left,*
- $\cdots\!\!\rightarrow_{Alice}$ *is the same as* $\cdots\!\!\rightarrow_{Beth}$ *and this relation is given by Figure 1 right.*

4.1 Abstract Nash equilibrium or singleton equilibrium

Let us look at a first kind of equilibria.

Definition 2 (Abstract Nash equilibrium or singleton equilibrium) *A* singleton equilibrium *is a situation s such that:*

$$\forall a \in \mathcal{A}, s' \in \mathcal{S} \quad . \quad (s \dashrightarrow_a s') \implies \neg(s \mathrel{\cdots\!\!\!\rightarrow}_a s').$$

We write $\mathrm{Eq}_{\mathrm{G}}^{aN}(s)$ *(aN stands for* abstract Nash*).*

In the previous paragraphs, we have seen examples of singleton equilibria. If we are at such an equilibrium, this is fine, but if not, we may wonder how to reach an equilibrium. If s is not an equilibrium, this means that s fulfills

$$\exists s' \in \mathcal{S} \quad . \quad s \dashrightarrow_a s' \wedge s \mathrel{\cdots\!\!\!\rightarrow}_a s'$$

which is the negation of

$$\forall s' \in \mathcal{S} \quad . \quad (s \dashrightarrow_a s') \implies \neg(s \mathrel{\cdots\!\!\!\rightarrow}_a s').$$

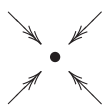
The relation $s \dashrightarrow_a s' \wedge s \mathrel{\cdots\!\!\!\rightarrow}_a s'$ between s and s' is a derived one. Let us call it *change of mind for a* and write it $\longrightarrow_a$. We say that a changes her mind, because she is not happy with s and hopes that following $\longrightarrow_a$ she will reach not necessary the equilibrium, but at least a better situation. Actually since we want to make everyone happy, we have to progress along all the $\longrightarrow_a$'s. Thus we consider a more general relation which we call just *change of mind* and which is the union of the $\longrightarrow_a$'s. We define this new relation as the union of the changes of mind of the agents.

$$\longrightarrow \;\triangleq\; \bigcup_{a \in \mathcal{A}} \longrightarrow_a.$$

Now suppose that we progress along $\longrightarrow$. What happens if we reach an s from which we cannot progress further? This means

$$\forall a \in \mathcal{A}, s' \in \mathcal{S} \quad . \quad \neg(s \dashrightarrow_a s' \wedge s \mathrel{\cdots\!\!\!\rightarrow}_a s')$$

in other words, s is an equilibrium. Hence to reach an equilibrium, we progress along $\longrightarrow$ until we are stuck. In graph theory, a vertex from which there is no outgoing arrow is called an *end point* or a *sink*. In relation theory it is called a *minimal element*:



Thus we look for end points in the graph.

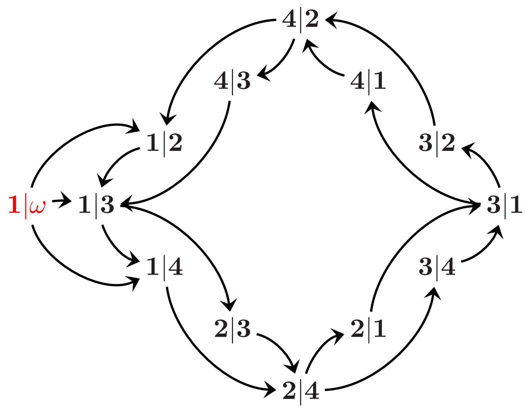


Figure 8: A game with two SCC's

4.2 Dynamic equilibrium

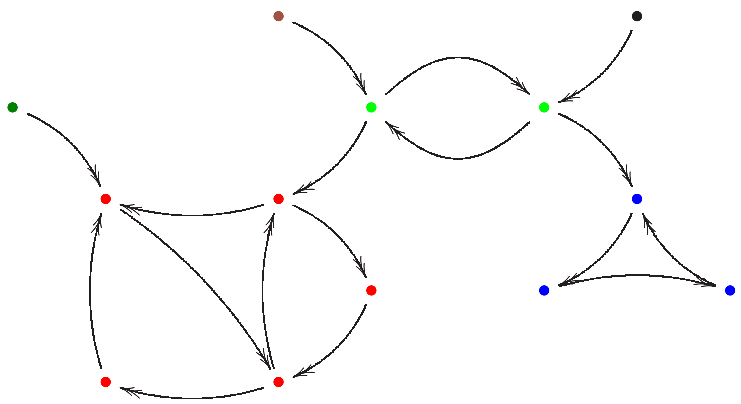
Actually this progression along $\longrightarrow$ is not the panacea to reach an equilibrium. Indeed it could be the case that this progression never ends, since we enter a perpetual move (think at the square game 2nd version, Figure 4). Actually we identify this perpetual move as a second kind of equilibrium.

4.2.1 Strongly connected components

Here it is relevant to give some concepts of graph theory. A graph[4] is *strongly connected*, if given two nodes n_1 and n_2 there is always a path going from n_1 to n_2 and a path going from n_2 to n_1. Not all the graphs are strongly connected, but they may contain some maximal subgraphs that are strongly connected; "maximal" means that one cannot add nodes without breaking the strong connectedness. Such a strongly connected subgraph is called a *strongly connected components*, *SCC* in short.

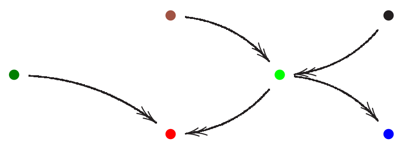
[4] In this paper, when we say "graph" , we mean always "oriented graph" or "digraph".

The graph below has six SCC's:



The graph of Figure 4 has two SCC's (Figure 8).

From a graph, we can deduce a new graph, which we call the *reduced graph* (or *condensation* [1]), whose nodes are the SCC's and the arcs are given as follows: there is an arc from an SCC S_1 to an SCC S_2 (assuming that S_1 is different from S_2), if there exists a node n_1 in S_1, a node s_2 in S_2 and an arc between n_1 and n_2. By construction the reduced graph has no cycle and its strongly connected components are singletons. The reduced graphs associated with the graphs given above are as follows:



$\{1|\omega\} \dashrightarrow \{1|3, 1|2, 1|4, ...\}$

4.2.2 Dynamic equilibria as strongly connected components

At the price of extending the notion of equilibrium, we can prove that there is always an equilibrium in finite non degenerated games, i.e., in games with a finite non zero number of game situations. Indeed given a graph, we compute its reduced graph. Then in this reduced graph, we look for end points. There is always such an end point since in a finite acyclic graph (the reduced graph is always acyclic) there exists always at least an end point.

CP Equilibria *are* end points *in the reduced graph.*

We write $\mathrm{Eq}_{\mathrm{G}}^{CP}(A)$ to say that the subset A of situations is a CP equilibrium. We can now split equilibria into two categories?

1. *CP Equilibria* (i.e., *Dynamic equilibria*) are equilibria associated with an SCC and may contain many situations.

2. *Abstract Nash Equilibria* (aka *Singleton equilibria*) are equilibria associated with an SCC that contains exactly one situation, i.e., associated with an SCC which is a singleton, hence the name singleton equilibrium. There are specific dynamic equilibria.

Tarjan [10] has shown that the reduced graph can be computed in linear time w.r.t. the numbers of nodes and edges of the original graph. Therefore CP equilibria can be computed in linear time in the number of game situations and edges in the change of mind relation, which provides an efficient algorithm to compute CP equilibria.

5 What are CP good for?

After the success of strategic games over years, one may wonder why we introduce a new concept, namely CP games. The first nice feature is a theorem that says that *there always exists an equilibrium*. We know that pure strategic games do not enjoy that property and that to obtain such equilibria, Nash had to extend the concept of strategic game to this of probabilistic games. Similarly we have relaxed the notion of equilibrium to this of CP equilibrium.

Beside abstract Nash equilibria that are really like those of strategic games. CP games have other equilibria that biologists called *dynamic equilibria* and that correspond to phenomena they actually consider. Physicists speak about *stationary states* in that case.

Economists know that the concept of payoff is somewhat artificial[5]. In CP games *the concept of payoff is completely abandoned*, no number are attached to situations and a general relation between situations is proposed instead.

In normal form strategic games, moves from one situation to another are tightly ruled and strong restrictions are imposed, namely right and left moves for one player, back and forth moves for another player and up and down moves for a third, etc., unlike CP games where *very general moves between situations ruled by the conversions are allowed*, like diagonal moves for instance or on the opposite more restricted moves like horizontal or vertical moves to a neighbor situation only (see the λ phage). The flexibility of the conversions and the preferences makes possible to formalize many situations, like some that occur in biology.

It is known that games are a good framework to analyze models where the principle of causality fails. CP games allow analyzing a larger class of models.

6 Gene regulation networks as CP games

In the λ phage, levels $0, 1, 2$, for a gene, correspond to levels of activation or levels of concentration of the corresponding protein. Thus cI has three levels. 0 corresponds to the gene being inactive (the protein is absent), 1 corresponds

[5] See for instance Osborne's introduction of his textbook [5].

to the gene being moderately active (the protein is present but moderately concentrated), 2 corresponds to the gene being highly active (the protein is concentrated). On the other hand, *cro* has two levels of activation, corresponding to the gene being inactive or active. *Env* has only one level, it is always active. A gene can move from one level at a time, as translated by the conversion relation on page 11. It has been shown that cI is a repressor for *cro* and a repressor for itself at level 2 and an activator for itself at level 1. This leads to the preference $\cdots\succ\cdots$ for cI on the left of diagram on page 12. On the other hand, it has been shown that *cro* is a repressor for cI, this leads to the preference and an activator for itself at level 1. This leads to the preference $\cdots\supset\cdots$ for *cro* on the right of diagram on page 12. Moreover when both genes are inactive, the *environment* may lead to activate either cI or *cro*, this leads to the preference $\cdots\triangleright\cdots$ of the diagram on page 12.

The two equilibria correspond to two well-known states of the λ phage: the *lyse* and the *lysogen*, which the phage always reaches. In particular the lysogen $\{\langle cI_2, cro_0\rangle, \langle cI_1, cro_0\rangle\}$ is a relatively stable state, where the phage seems inactive (dormant state). This is due to the fact that the concentration of the protein associated to cI is controlled: if it is too concentrated, a repression process makes the concentration to decrease and vice-versa if the concentration is too low an activation process makes it to increase. These antagonistic actions maintain the concentration at an intermediate level and the associated state is stable. The state $\langle cI_0, cro_1\rangle$ corresponds to what is called the *lyse* of the λ phage.

What is amazing in the presentation as CP games is that these states are actually computed as CP equilibria. Somewhat connected approaches are [13, 12].

7 Chinese Wall information security and corporate liability

A main claim of this article is that CP games, simple as they are, is a natural formalism whose conversion/preference distinction is of wider relevance. We shall further justify the claim in this section, with an example chosen because of the succinctness of its CP-game presentation and because the conversion/preference distinction is of stand-alone interest in the context of the example, without any consideration of equilibrium analysis.

The concept of Chinese Wall information security pertains to the prevention of insider trading, and more generally the insulation of insider knowledge [2]. Chinese Wall requirements are codified in laws in many countries, and are interesting to informaticians in part because Chinese Wall security is different from military-style *need to know*. In particular, any information is in principle accessible to the subjects in question, but access is only granted if the subject is not already in possession of information that could create a conflict of interest. Formally, we consider a set of *subjects*, P (for people), a set of *interests classes*,

I, a set of *companies*, C, with *interest classification* function $\mathcal{I} : C \to I$, and a set of *objects*, O, with *ownership* function $\mathcal{C} : O \to C$. The typical scenario is that the subjects are consultants and the interest classes consists, for example, of `bank`, `oil company`, etc., with the requirement that no consultant handles objects, i.e., information, for more than one, e.g., `bank`. In other words, we are considering a game played by the subjects, $\mathcal{A} = P$, over complete accounts of what objects each subject has had access to, $\mathcal{S} = \otimes_{p \in P} 2^O$. Writing s_p for the p-projection of an $s \in \mathcal{S}$, [2, Axiom 2] that governs when a subject, p, is allowed to gain access to an object amounts to the conversion relation where $s \dashrightarrow_p s'$ iff

$$\begin{array}{l} \forall p' \,.\, p \neq p' \Rightarrow s_{p'} = s'_{p'} \\ \wedge \\ \exists o \,.\, s'_p = s_p \cup \{o\} \wedge (\forall o' \in s_p \,.\, \mathcal{I}(\mathcal{C}(o)) \neq \mathcal{I}(\mathcal{C}(o')) \vee \mathcal{C}(o) = \mathcal{C}(o')) \end{array}$$

In words, subject p may convert s to s' if the situations only differ by some object, o, being added to the p-projection and, for all other objects in s_p, o either belongs to a different interest class or hails from the same company. By [2, Axiom 3], we are only interested in situations that can be reached from the situation where no subject has had access to any object, $\overrightarrow{\emptyset}$. With this, [2, Theorem 2] says: "A subject can at most have access to one company dataset in each conflict of interest class". Formally, we have the following.

Definition 3 *A state, $s \in \mathcal{S}$, has* no insider trading *if*

$$\mathrm{NIT}(s) \quad \triangleq \quad \forall p \in P \,.\, \forall o_1, o_2 \in s_p \,.\, \mathcal{I}(\mathcal{C}(o_1)) \neq \mathcal{I}(\mathcal{C}(o_2)) \vee \mathcal{C}(o_1) = \mathcal{C}(o_2)$$

Theorem 1 *Given a* Chinese Wall CP game form*, $\langle P, \otimes_{p \in P} 2^O, (\dashrightarrow_p)_{p \in P} \rangle$, no derived* Chinese Wall CP game *can reach a situation with insider trading.*

$$\forall (\cdots\!\!\!\rightarrow_p)_{p \in P} \,.\, \forall s \,.\, \overrightarrow{\emptyset} \longrightarrow^* s \Rightarrow \mathrm{NIT}(s)$$

While logically straightforward, the point of the CP-game version of the theorem is that it is universally quantified over the family of preference relations. In other words, the theorem explicitly states that for a company that implements Chinese Wall regulations for its subjects (the conversion relations), no matter what those employees may be tempted to do (the preference relations), insider trading can only take place if one of the employees breaks the company's rules. This means that the CP-game version of the result formalizes a notion of corporate liability protection, which is directly relevant to the study of Chinese Wall information security.

8 Conversion or preference, how to choose?

The attentive reader may have noticed that what counts to compute equilibria is the *change of mind* and that keeping the same set of equilibria there is some freedom on the conversion and the preference provided one keeps the same change of mind. More precisely, we have

$$
\begin{array}{rcll}
\longrightarrow_a & = & \dashrightarrow_a \cap \mathrel{\cdots\!\!\!\rightarrow}_a & \\
 & = & (\dashrightarrow_a \cup R) \cap \mathrel{\cdots\!\!\!\rightarrow}_a & \text{when } R \cap \mathrel{\cdots\!\!\!\rightarrow}_a = \emptyset \\
 & = & \dashrightarrow_a \cap (\mathrel{\cdots\!\!\!\rightarrow}_a \cup T) & \text{when } T \cap \dashrightarrow_a = \emptyset
\end{array}
$$

On another hand, one notices that in some examples, the preference is independent of the agent whereas, in others, the conversion is independent of the agent. It seems that this is correlated with the domain of application. In particular, we may emit the following hypothesis. In biology, conversion is physics and chemistry, whereas preference is the part that cannot be explained by physics and chemistry, then we may induce that change of mind (combination of physics and true biology) is life. Indeed since physics and chemistry is the same for everyone, it makes sense to say that conversion is the same for everybody, whereas, due to evolution and biological effects, preference, changes with agents.

9 Fixed point construction, and equilibria in infinite games

For proving the existence of a fixed point for every probabilistic game, Nash [3] used Kakutani's fixed point theorem. Since we deal with discrete games, we present in this section a proof of the existence of equilibrium based on a Tarski fixed-point theorem [11]. Recall that Tarski's theorem uses an *update function*, say f, on a lattice and builds a fixed point starting from an element, say a, by iteration, a, $f(a)$, ..., $f^n(a)$, ... Here the lattice is the powerset $\mathscr{P}(\mathcal{S})$ of situations ordered by the subset order.

In analogy to Nash's update function, the function takes a subset of situations and creates a new subset based on how the agents would like to improve upon the old subset.

Definition 4 (Update) *Given a game* G *and a subset* $\mathrm{C} \subseteq \mathcal{S}$ *of the set of situations, let* $\mathscr{U}(\mathrm{C}) \triangleq \bigcup_{\mathrm{s}\in\mathrm{C}}\{\mathrm{s}' \mid \mathrm{s}\rightarrow^*\mathrm{s}'\}$.

With this, we have the following result, covering all CP games.

Lemma 1 *Given (any)* G*,* $\mathscr{U}$ *has a complete lattice of fixed points.*[6]

Not all fixed points will correspond to equilibria but the equilibria are the least, non-empty fixed points of the update function.

Lemma 2 *Given* G*, with change-of-mind relation* $\rightarrow$*.*

$$
\begin{array}{c}
\mathrm{Eq}^{CP}_{\mathrm{G}}(\mathrm{C}) \\
\Updownarrow \\
\mathscr{U}(\mathrm{C}) = \mathrm{C} \ \wedge \ \mathrm{C} \neq \emptyset \ \wedge \ (\forall \mathrm{C}' . \emptyset \subsetneq \mathrm{C}' \subsetneq \mathrm{C} \Longrightarrow \mathscr{U}(\mathrm{C}') \not\subseteq \mathrm{C}')
\end{array}
$$

[6] We note that complete lattices are non-empty by definition.

Proof By two direct arguments. The only interesting step is from bottom to top and showing that, for any two $s_1, s_2 \in C$, we have $s_1 \rightarrow^* s_2$. We first note that $\mathscr{U}$ is post-fixpointed: $C \subseteq \mathscr{U}(C)$, idempotent: $\mathscr{U}(\mathscr{U}(C)) = \mathscr{U}(C)$, and order-preserving: $C_1 \subseteq C_2 \implies \mathscr{U}(C_1) \subseteq \mathscr{U}(C_2)$. By order-preservation and $\mathscr{U}(C) = C$, we have $\mathscr{U}(\{s_1\}) \subseteq C$. If $\neg(s_1 \rightarrow^* s_2)$, then $s_2 \in C \setminus \mathscr{U}(\{s_1\})$, i.e., $\mathscr{U}(\{s_1\}) \subsetneq C$. By post-fixpointed-ness, $\mathscr{U}(\{s_1\})$ is non-empty and, by assumption of least-ness, we may therefore conclude $\mathscr{U}(\mathscr{U}(\{s_1\})) \not\subseteq \mathscr{U}(\{s_1\})$. This contradicts idempotency, and thus $s_1 \rightarrow^* s_2$. $\square$

CP equilibria are therefore *atomic*, in the sense that neither anything smaller nor anything bigger will have the same defining properties. For finite G, a counting argument shows that the complete lattice of $\mathscr{U}$-fixed point will have least, non-empty elements, thus guaranteeing existence. For the infinite case, e.g., the following unbounded change-of-mind relation will not lead to the existence of least, non-empty elements in the fixed-point lattice because all tails are fixed points.

However there are infinite cases where the existence of a CP equilibrium can be guaranteed, namely when there exists an SCC C, which is extremal for the reduced change of mind.

More generally, a sufficient condition for the existence of Eq_G^{CP} is that some s can reach only finitely many other elements in $\mathcal{S}$ using $\rightarrow^*$. The condition is also necessary if we restrict attention to finite Eq_G^{CP}. In particular, finite games have Eq_G^{CP}. Because of the role played by reduced graphs above, also games with finite reduced graphs have Eq_G^{CP}.

10 Conclusion

We have presented conversion preference games as a strict extension of strategic games and we have proved that in a finite CP game an equilibrium always exists and under some conditions in infinite games as well. This theory is infancy and we expect it to generate as many theorems as the classical Nash game theory. See for instance [8].

References

[1] Sara Baase. *Computer Algorithms: Introduction to Design and Analysis.* Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 1978.

[2] D. F. C. Brewer and M. J. Nash. The Chinese Wall security policy. In *IEEE Symposium on Security and Privacy*, pages 206–214, 1989.

[3] John F. Nash, Jr. Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences*, 36:48–49, 1950.

[4] M. A. Nowak and K. Sigmund. Evolutionary dynamics of biological games. *Science*, 3003:793–798, February 2004.

[5] Martin J. Osborne. *An Introduction to Game Theory*. Oxford, 2004.

[6] Mark Ptashne. *Genetic Switch: Phage Lambda Revisited*. Cold Spring Harbor Laboratory Press, April 2004. 3rd edition.

[7] Mark Ptashne and Alexander Gann. *Genes and Signals*. Cold Spring Harbor Laboratory Press, November 2001.

[8] Stéphane Le Roux. *Abstraction and Formalization in Game Theory*. PhD thesis, École normale supérieure de Lyon (France), January 2008.

[9] Stéphane Le Roux, Pierre Lescanne, and René Vestergaard. A discrete Nash theorem with quadratic complexity and dynamic equilibria. Research Report IS-RR-2006-006, JAIST, May 2006.

[10] Robert E. Tarjan. Depth first search and linear graph algorithms. *SIAM Journal on computing*, pages 146–160, Januar 1972.

[11] Alfred Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5:285–309, 1955.

[12] Denis Thieffry and René Thomas. Dynamical behaviour of biological regulatory networks II: Immunity control in bacteriophage lambda. *Bulletin of Mathematical Biology*, 57:277–297, 1995.

[13] René Thomas. Boolean formalization of genetic control circuits. *Journal of Theoretical Biology*, 42(3):563–585, 1973.