

# Intersecting Attractors

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## Abstract

We apply the entropy formalism to the study of the near-horizon geometry of extremal black  $p$ -brane intersections in  $D > 5$  dimensional supergravities.

The scalar flow towards the horizon is described in terms an effective potential given by the superposition of the kinetic energies of all the forms under which the brane is charged. At the horizon active scalars get fixed to the minima of the effective potential and the entropy function is given in terms of U-duality invariants built entirely out of the black  $p$ -brane charges.

The resulting entropy function reproduces the central charges of the dual boundary CFT and gives rise to a Bekenstein-Hawking like area law.

The results are illustrated in the case of black holes and black string intersections in  $D = 6, 7, 8$  supergravities where the effective potentials, attractor equations, moduli spaces and entropy/central charges are worked out in full detail.

# 1 Introduction

In  $D > 5$  dimensions, supergravity theories involve a rich variety of tensors fields of various rank (see *e.g.* [1, 2]). A single black hole solution is in general charged under different forms and can be thought of as the intersection on a timelike direction of extended branes of various types. More generally, branes intersecting on a  $(p + 1)$ -dimensional surface lead to a black  $p$ -brane intersecting configuration [3]. In complete analogy with what happens in the case of  $D = 4, 5$  black holes, one can think of the  $D > 5$  solutions as a scalar attractor flow from infinity to a horizon where a subset of the scalars becomes fixed to particular values depending exclusively on the black  $p$ -brane charges. The study of such flows requires a generalization of the attractor mechanism [4, 5, 6, 7, 8] in order to account for  $p$ -brane solutions carrying non-trivial charges under forms of various rank. In this paper we address the study of these general attractor flows.

We focus on static, asymptotically flat, spherically symmetric, extremal black  $p$ -brane solutions in supergravities at the two derivative level [9]. The analysis combines standard attractor techniques based on the extremization of the black hole central charge [4, 5, 6, 7, 8] and the so-called “entropy function formalism” introduced in [10] (see [11, 12] for reviews and complete lists of references). Like for black holes carrying vector-like charges, we define the entropy function for black  $p$ -branes as the Legendre transform with respect to the brane charges of the supergravity action evaluated at the near-horizon geometry (see [13] for previous investigations of black rings and non-extremal branes using the entropy formalism). The resulting entropy function can be written as a sum of a gravitational term and an effective potential  $V_{\text{eff}}$  given as a superposition of the kinetic energies of the forms under which the brane is charged. Extremization of this effective potential gives rise to the attractor equations which determine the values of the scalars at the horizon as functions of the brane charges. In particular, the entropy function itself can be expressed in terms of the U-duality invariants built from these charges and it is proportional to the central charge of the dual CFT living on the AdS boundary. The attractor flow can then be thought of as a  $c$ -flow towards the minimum of the *supergravity c-function* [14, 15]. Interestingly, the central charges for extremal black  $p$ -branes satisfy an area law formula generalizing the famous Bekenstein-Hawking result for black holes.

We will illustrate our results in the case of extremal black holes and black strings in  $D = 6, 7, 8$  supergravities. In each case we derive the entropy function  $F$  and the near-horizon geometry via extremization of  $F$ . At the extremum, the entropy function results into a U-duality invariant combination of the brane charges reproducing the black hole entropy and the black string central charge, respectively. Scalars fall into two classes: “fixed scalars” with strictly positive masses and “flat scalars” not fixed by the attractor equations, which span the moduli space of the solution. The moduli spaces will be given by symmetric product spaces that can be interpreted as the intersection of the charge orbits

of the various branes entering in the solution. In addition one finds extra “geometric moduli” (radii and Wilson lines) that are not fixed by the attractors.

The paper is organized as follows. In Sect. 2 we derive a Bekenstein-Hawking like area law for central charges associated to extremal black  $p$ -branes. In Sect. 3 the “entropy function” formalism is adapted to account for solutions charged under forms of different rank. In Sect. 4 we anticipate and summarize in a very universal form the results for the set of theories considered in detail in the rest of the paper, namely the two non-chiral (1, 1) and (2, 2) supergravities in  $D = 6$  (Sects. 5 and 6, respectively), and the maximal  $D = 7, 8$  supergravities (Sects. 7 and 8, respectively). In Sect. 9 the uplift of the previously discussed near-horizon geometries to  $D = 11$   $M$ -theory is briefly discussed. The concluding Sect. 10 contains some final remarks and comments.

## 2 Area Law for Central Charges

Before specifying to a particular supergravity theory, here we derive a universal Bekenstein-Hawking like formula underlying any gravity flow (supersymmetric or not) ending on an AdS point. Let  $AdS_d \times \Sigma_m$ , with  $\Sigma_m$  a product of Einstein spaces, be the near-horizon geometry of an extremal black  $(d - 2)$ -brane solution in  $D = d + m$  dimensions. After reduction along  $\Sigma_m$  this solution can be thought as the vacuum of a gauged gravity theory in  $d$  dimensions. To keep the discussion, as general as possible, we analyze the solution from its  $d$ -dimensional perspective. The only fields that can be turned on consistently with the  $AdS_d$  symmetries are constant scalar fields. Therefore we can describe the near-horizon dynamics in terms of a gravity theory coupled to scalars  $\varphi^i$  with a potential  $V_d$ . The potential  $V_d$  depends on the details of the higher-dimensional theory. The “entropy function” is given by evaluating this action at the  $AdS_d$  near horizon geometry (with constant scalars  $\varphi^i \approx u^i$ )

$$F = - \frac{1}{16\pi G_d} \int d^d x \sqrt{-g} (R - V_d) = \frac{\Omega_{AdS_d} r_{AdS}^d}{16\pi G_d} \left\{ \frac{d(d-1)}{r_{AdS}^2} + V_d \right\}, \quad (2.1)$$

with  $r_{AdS}$  the AdS radius and  $\Omega_{AdS_d}$  the regularized volume of an AdS slice of radius one. Following [16] we take for  $\Omega_{AdS_d}$  the finite part of the AdS volume integral when the cut off is sent to infinity. More precisely we write the AdS metric

$$ds^2 = r_{AdS}^d (d\rho^2 - \sinh^2 \rho d\tau^2 + \cosh^2 \rho d\Omega_{d-2}^2), \quad (2.2)$$

with  $\tau \in [0, 2\pi]$ ,  $0 \leq \rho \leq \cosh^{-1} r_0$  and  $d\Omega_{d-2}$  the volume form of a unitary  $(d-2)$ -dimensional sphere. The regularized volume  $\Omega_{AdS_d}$  is then defined as the (absolute value of the) finite part of the volume integral  $\int d^d x \sqrt{-g}$  in the limit  $r_0 \rightarrow \infty$ . This results into

$$\Omega_{AdS_d} = \frac{2\pi}{(d-1)} \Omega_{d-2}. \quad (2.3)$$

A different prescription for the volume regularization leads to a redefinition of the entropy function by a charge independent irrelevant constant. The “entropy” and near-horizon geometry follow from the extremization of the entropy function  $F$  with respect to the fixed scalars  $u^i$  and the radius  $r_{\text{AdS}}$

$$\begin{aligned} \frac{\partial F}{\partial u^i} &\propto \frac{\partial V_d}{\partial u^i} \stackrel{!}{=} 0, \\ \frac{\partial F}{\partial r_{\text{AdS}}} &\propto r_{\text{AdS}}^2 V_d + (d-1)(d-2) \stackrel{!}{=} 0. \end{aligned} \quad (2.4)$$

The first equation determines the values of the scalars at the horizon. The second equation determines the radius of AdS in terms of the value of the potential at the minimum. Notice that solutions exist only if the potential  $V_d$  is negative. Indeed, as we will see in the next section,  $V_d$  is always composed from a part proportional to a positive definite effective potential  $V_{\text{eff}}$  generated by the higher dimensional brane charges and a negative contribution  $-R_\Sigma$  related to the constant curvature of the internal space  $\Sigma$  (see eq.(3.25) below). The “entropy” is given by evaluating  $F$  at the extremum and can be written in the suggestive form

$$F = \frac{\Omega_{d-2} r_{\text{AdS}}^{d-2}}{4G_d} = \Omega_{d-2} r_{\text{AdS}}^{d-2} \frac{A}{4G_D}, \quad (2.5)$$

where  $A$  denotes the area of  $\Sigma_m$ ,  $\Omega_{d-2}$  is the volume of the unit  $(d-2)$ -sphere, and  $G_D = AG_d$  the  $D$ -dimensional Newton constant. For black holes ( $d = 2$ ), this formula is nothing than the well known Bekenstein-Hawking entropy formula  $S = \frac{A}{4G_D}$  and it shows that  $F$  can be identified with the black hole entropy. For black strings ( $d = 3$ ),  $\frac{3}{\pi}F = \frac{3r_{\text{AdS}}}{2G_3}$  reproduces the central charge  $c$  of the two-dimensional CFT living on the  $AdS_3$  boundary [17]. In general, the scaling of (2.5) with the AdS radius matches that of the supergravity  $c$ -function introduced in [14] and it suggests that  $F$  can be interpreted as the critical value of the central charge  $c$  reached at the end of the attractor flow.

In the remainder of this paper we will study the flows from the  $D$ -dimensional perspective where the black  $p$ -branes carry in general charges under forms of various rank.

### 3 The Entropy Function

The bosonic action of supergravity in  $D$ -dimensions can be written as

$$S_{\text{SUGRA}} = \int \left( R * \mathbb{1} - \frac{1}{2} g_{ij}(\phi) d\phi^i \wedge *d\phi^j - \frac{1}{2} N_{\Lambda_n \Sigma_n}(\phi^i) F_n^{\Lambda_n} \wedge *F_n^{\Sigma_n} + \mathcal{L}_{\text{WZ}} \right), \quad (3.1)$$

with  $F_n^{\Lambda_n}$ , denoting a set of  $n$ -form field strengths,  $\phi^i$  the scalar fields living on a manifold with metric  $g_{ij}(\phi)$  and  $\mathcal{L}_{\text{WZ}}$  some Wess-Zumino type couplings. The scalar-dependent

positive definite matrix  $N_{\Lambda_n \Sigma_n}(\phi^i)$  provides the metric for the kinetic term of the  $n$ -forms. The sum over  $n$  is understood. In the following we will omit the subscript  $n$  keeping in mind that both the rank of the forms and the range of the indices  $\Lambda$  depends on  $n$ . We will work in units where  $16\pi G_D = 1$ , and restore at the end the dependence on  $G_D$ . For simplicity we will restrict ourselves here to solutions with trivial Wess-Zumino contributions and this term will be discarded in the following.

We look for extremal black  $p$ -brane intersections with near-horizon geometry of topology  $M_D = AdS_{p+2} \times S^m \times T^q$ . Explicitly we look for solutions with near-horizon geometry

$$\begin{aligned} ds^2 &= r_{\text{AdS}}^2 ds_{\text{AdS}_{p+2}}^2 + r_S^2 ds_{S^m}^2 + \sum_{k=1}^q r_k^2 d\theta_k^2, \\ F^\Lambda &= p_a^\Lambda \alpha^a + e^{\Lambda r} \beta_r, \quad \phi^i = u^i, \end{aligned} \quad (3.2)$$

with  $\vec{r} = (r_{\text{AdS}}, r_S, r_k)$ , describing the  $AdS$  and sphere radii, and  $u^i$  denoting the fixed values of the scalar fields at the horizon.  $\alpha^a$  and  $\beta_r$  denote the volume forms of the compact  $\{\Sigma^a\}$  and non-compact  $\{\Sigma_r\}$  cycles, respectively, in  $M_D$ . The forms are normalized such as

$$\int_{\Sigma^a} \alpha^b = \delta_a^b, \quad \int_{\Sigma_r} \beta_s = \delta_s^r. \quad (3.3)$$

They define the volume dependent functions  $C^{ab}, C_{rs}$

$$\int_{M_D} \alpha^a \wedge * \alpha^b = C^{ab}, \quad \int_{M_D} \beta_r \wedge * \beta_s = C_{rs}, \quad (3.4)$$

describing the cycle intersections. In particular, for the factorized products of AdS space and spheres we consider here, these functions are diagonal matrices with entries

$$C^{ab} = \delta^{ab} \frac{v_D}{\text{vol}(\Sigma^a)^2}, \quad C_{rs} = \delta_{rs} \frac{v_D}{\text{vol}(\Sigma^r)^2}, \quad (3.5)$$

with  $v_D$  the volume of  $M_D$ . Integrals over AdS spaces are cut off to a finite volume, according to the discussion around (2.3).

The solutions will be labeled by their electric  $q_{I_r}$  and magnetic charges  $p_a^I$  defined as

$$\begin{aligned} p_a^\Lambda &= \int_{\Sigma^a} F^\Lambda, \\ q_{\Lambda r} &= \int_{*\Sigma^r} N_{\Lambda\Sigma} * F^\Sigma = C_{rs} N_{\Lambda\Sigma} e^{\Sigma s}, \end{aligned} \quad (3.6)$$

where we denote by  $*\Sigma^r$  the complementary cycle to  $\Sigma^r$  in  $M_D$ .

Let us now consider the “entropy function” associated to a black  $p$ -brane solution with near-horizon geometry (3.2). The entropy function  $F$  is defined as the Legendre transform in the electric charges  $q_{\Lambda r}$  of  $S_{\text{SUGRA}}$  evaluated at the near-horizon geometry

$$\begin{aligned} F &= e^{\Lambda r} q_{\Lambda r} - S_{\text{SUGRA}} \\ &= e^{\Lambda r} q_{\Lambda r} - R v_D + \frac{1}{2} N_{\Lambda\Sigma} p_a^\Lambda p_b^\Sigma C^{ab} - \frac{1}{2} N_{\Lambda\Sigma} e^{\Lambda r} e^{\Sigma s} C_{rs} , \end{aligned} \quad (3.7)$$

The fixed values of  $\vec{r}, u^i, e^{I r}$  at the horizon can be found via extremization of  $F$  with respect to  $\vec{r}, u^i$ , and  $e^{I r}$ :

$$\frac{\partial F}{\partial \vec{r}} = \frac{\partial F}{\partial u^i} = \frac{\partial F}{\partial e^{\Lambda r}} = 0 . \quad (3.8)$$

From the last equation one finds that

$$q_{\Lambda r} = N_{\Lambda\Sigma} e^{\Sigma s} C_{rs} , \quad (3.9)$$

in agreement with the definition of electric charges (3.6). Solving this set of equations for  $e^{\Lambda r}$  in favor of  $q_{\Lambda r}$  one finds

$$F(Q, \vec{r}, u^i) = -R(\vec{r}) v_D(\vec{r}) + \frac{1}{2} Q^T \cdot M(\vec{r}, u^i) \cdot Q , \quad (3.10)$$

with

$$M(\vec{r}, u^i) = \begin{pmatrix} N_{\Lambda\Sigma}(u^i) C^{ab}(\vec{r}) & 0 \\ 0 & N^{\Lambda\Sigma}(u^i) C^{rs}(\vec{r}) \end{pmatrix} , \quad Q \begin{pmatrix} p_a^\Sigma \\ q_{\Sigma r} \end{pmatrix} , \quad (3.11)$$

and  $N^{\Lambda\Sigma}, C^{rs}$  denoting the inverse of  $N_{\Lambda\Sigma}$  and  $C_{rs}$  respectively.

It is convenient to introduce the scalar and form intersection “vielbeine”  $\mathcal{V}_\Lambda^M, J^{ab}, J'^{rs}$  according to

$$N_{\Lambda\Sigma} = \mathcal{V}_\Lambda^M \mathcal{V}_\Sigma^N \delta_{MN} , \quad C^{ab} = J^{ac} J^{bc} , \quad C^{rs} = J'^{rt} J'^{st} . \quad (3.12)$$

From (3.5) one finds for the factorized products of AdS space and spheres

$$J^{ab} = \delta^{ab} \frac{v_D^{1/2}}{\text{vol}(\Sigma^a)} , \quad J'^{rs} = \delta^{rs} \frac{\text{vol}(\Sigma^r)}{v_D^{1/2}} . \quad (3.13)$$

The electric and magnetic central charges can be written in terms of these quantities as

$$Z_{\text{mag}}^{Ma} = \mathcal{V}_\Lambda^M J^{ba} p_b^\Lambda , \quad Z_{\text{el},M}^r = (\mathcal{V}^{-1})_M^\Lambda J'^{sr} q_{\Lambda s} . \quad (3.14)$$

Combining (3.12) and (3.14) one can rewrite the scalar dependent part of the entropy function as the effective potential

$$V_{\text{eff}} = \frac{1}{2} Q^T \cdot M(\vec{r}, u^i) \cdot Q = \frac{1}{2} Z_{\text{mag}}^{Ma} Z_{\text{mag}}^{Ma} + \frac{1}{2} Z_{\text{el},M}^r Z_{\text{el},M}^r . \quad (3.15)$$

For the  $n = D/2$ -forms in even dimensions the argument is similar, except for the possibility of an additional topological term

$$S_{\text{SUGRA}} = \int \left( R * \mathbb{1} - \frac{1}{2} \mathcal{I}_{\Lambda\Sigma}(\phi^i) F_n^\Lambda \wedge *F_n^\Sigma - \frac{1}{2} \mathcal{R}_{\Lambda\Sigma}(\phi^i) F_n^\Lambda \wedge F_n^\Sigma \right), \quad (3.16)$$

(note that  $\mathcal{R}_{\Lambda\Sigma} = \epsilon \mathcal{R}_{\Sigma\Lambda}$ , with  $\epsilon = (-1)^{[D/2]}$ ). Following the same steps as before one finds

$$V_{\text{eff}} \frac{1}{2} Q^T \cdot M(\vec{r}, u^i) \cdot Q, \quad (3.17)$$

with

$$M(\vec{r}, u^i) \equiv C^{ab} \begin{pmatrix} (\mathcal{I} + \epsilon \mathcal{R} \mathcal{I}^{-1} \mathcal{R})_{\Lambda\Sigma} & \epsilon (\mathcal{R} \mathcal{I}^{-1})_{\Lambda}{}^\Sigma \\ (\mathcal{I}^{-1} \mathcal{R})^{\Lambda\Sigma} & (\mathcal{I}^{-1})^{\Lambda\Sigma} \end{pmatrix}, \quad Q \begin{pmatrix} p_a^\Lambda \\ q_{\Lambda a} \end{pmatrix}. \quad (3.18)$$

For  $\mathcal{R} = 0$  we are back to the diagonal matrix (3.11). In general, thus we obtain for the  $D/2$ -forms an effective potential

$$V_{\text{eff}} = \frac{1}{2} Q^T \cdot M(\vec{r}, u^i) \cdot Q = \frac{1}{2} Z^{Ma} Z^{Ma}, \quad (3.19)$$

with

$$Z^{Ma} = J^{ba} (\mathcal{V}_\Lambda{}^M p_b^\Lambda + \mathcal{V}^{\Lambda M} q_{\Lambda b}), \quad (3.20)$$

where  $\mathcal{V}_I{}^M = (\mathcal{V}_\Lambda{}^M, \mathcal{V}^{\Lambda M})$  is the coset representative.

Summarizing, in the case of a general supergravity with bosonic action (3.1) the entropy function is given by

$$F(Q, \vec{r}, u^i) = -R(\vec{r}) v_D(\vec{r}) + V_{\text{eff}}(u^i, \vec{r}), \quad (3.21)$$

with the *intersecting-branes effective potential*

$$\begin{aligned} V_{\text{eff}} &= \frac{1}{2} \sum_n Q_n^T \cdot M_n(\vec{r}, u^i) \cdot Q_n \\ &= \frac{1}{2} Z^{Ma} Z^{Ma} + \frac{1}{2} \sum_{n \neq D/2} (Z_{\text{mag}}^{M_n a_n} Z_{\text{mag}}^{M_n a_n} + Z_{\text{el}, M_n}^{r_n} Z_{\text{el}, M_n}^{r_n}), \end{aligned} \quad (3.22)$$

where the first contribution in the second line comes from the  $n = D/2$  forms. Notice that there are two types of interference between the potentials coming from forms of different rank: First, they in general depend on a common set of scalar fields and second, they carry a non-trivial dependence on the *AdS* and the sphere radii. Besides this important difference the critical points of the effective potential can be studied with the standard attractor techniques for vector like charged black holes.

The near-horizon geometry follows from the extremization equations

$$\nabla V_{\text{eff}} \equiv \partial_{u^i} V_{\text{eff}} du^i \frac{1}{2} \sum_n Q_n^T \cdot \nabla M_n(\vec{r}, u^i) \cdot Q_n \stackrel{!}{=} 0, \quad (3.23)$$

$$\partial_{\vec{r}} [-R(\vec{r}) v_D(\vec{r}) + V_{\text{eff}}(u^i, \vec{r})] \stackrel{!}{=} 0. \quad (3.24)$$

We conclude this section by noticing that after reduction to  $AdS_d$ , the  $D$ -dimensional effective potential  $V_{\text{eff}}$  combines with the contribution coming from the scalar curvature  $R_\Sigma$  of the internal manifold into the  $d$ -dimensional scalar potential

$$V_d = \frac{1}{v_D} V_{\text{eff}} - R_\Sigma \quad (3.25)$$

appearing in (2.1). Notice that the resulting potential is not positive defined and therefore an AdS vacuum is supported.

## 4 Summary of Results

Before entering into the detailed analysis of the entropy function and its minima, here we summarize our main results in a universal form independent of the particular dimension  $D$  considered. We consider extremal black  $p$ -brane solutions with  $p = 0, 1$  in  $d = 6, 7, 8$  maximal supergravities and  $\mathcal{N} = (1, 1)$  supergravity in six dimensions. The attractor mechanism for black strings in  $\mathcal{N} = (1, 0)$  six-dimensional supergravity was studied in [18].

There are three classes of extremal black  $p$ -brane intersections. The corresponding near-horizon geometries, effective potentials  $V_{\text{eff}}$ , and entropy functions in each case are given as follows:

- $AdS_3 \times S^3 \times T^n$ :

$$V_{\text{eff}} = \frac{v_{AdS_3}}{v_{S^3}} |\mathbf{I}_2|, \quad r_{AdS} = r_S = \frac{|\mathbf{I}_2|^{1/4}}{2\pi v_{T^n}^{1/4}},$$

$$F = v_D \left( \frac{6}{r_{AdS}^2} - \frac{6}{r_S^2} \right) + V_{\text{eff}} = |\mathbf{I}_2|. \quad (4.1)$$

- $AdS_3 \times S^2 \times T^n$ :

$$V_{\text{eff}} = \frac{3}{2} \frac{v_{AdS_3}}{v_{S^2}} v_{T^n}^{1/3} |\mathbf{I}_3|^{\frac{2}{3}}, \quad r_{AdS} = 2r_S = \frac{|\mathbf{I}_2|^{1/3}}{2\pi v_{T^n}^{1/3}},$$

$$F = v_D \left( \frac{6}{r_{AdS}^2} - \frac{2}{r_S^2} \right) + V_{\text{eff}} = |\mathbf{I}_3|. \quad (4.2)$$

- $AdS_2 \times S^3 \times T^n$ :

$$V_{\text{eff}} = \frac{3}{2} \frac{v_{AdS_2}}{v_{S^3} v_{T^n}^{1/3}} |\mathbf{I}_3|^{\frac{2}{3}}, \quad r_{AdS} = \frac{1}{2} r_S = \frac{|\mathbf{I}_3|^{1/6}}{2\pi v_{T^n}^{1/3}},$$

$$F = v_D \left( \frac{2}{r_{AdS}^2} - \frac{6}{r_S^2} \right) + V_{\text{eff}} = |\mathbf{I}_3|^{1/2}. \quad (4.3)$$

where

$$v_D = v_{AdS_d} v_{S^m} v_{T^n}$$

$$v_{AdS_d} = \Omega_{AdS_d} r_{AdS}^d, \quad v_{S^n} = r_S^n \Omega_n, \quad v_{T^n} = \Omega_1^n \prod_{i=1}^n r_i,$$

$$\Omega_1 = 2\pi, \quad \Omega_2 = 4\pi, \quad \Omega_3 = 2\pi^2, \quad \Omega_{AdS_2} = 2\pi, \quad \Omega_{AdS_3} = 2\pi^2, \quad (4.4)$$

are the volumes of the near-horizon AdS/spheres and  $\mathbf{I}_{2,3}$  are the relevant quadratic and cubic U-duality invariants built out of the black  $p$ -brane charges. We stress that these invariants involve, in general, charges under forms of various ranks. This is also the case for the effective potential  $V_{\text{eff}}$  resulting from the interfering superpositions of the various form contributions.

We also note that in all cases the radii of the circles of the torus  $T^n$  are not fixed by the extremization equations but remain as free parameters.

The results (4.1)–(4.3) shows that the entropy function  $F$  can be related to the black hole entropy and black string central charges

$$S_{\text{black-hole}} = F = |\mathbf{I}_3|^{1/2} = \frac{v_{S^3} v_{T^n}}{4G_D},$$

$$c_{\text{black-string}} = \frac{3}{\pi} F = \frac{3}{\pi} |\mathbf{I}_{2,3}| = \frac{3r_{AdS}}{2G_3}. \quad (4.5)$$

In the following we will derive these results from the corresponding supergravities in various space-time dimensions.

## 5 $\mathcal{N} = (1, 1)$ in $D = 6$

### 5.1 $\mathcal{N} = (1, 1)$ , $D = 6$ Supersymmetry Algebra

The half-maximal  $(1, 1)$ ,  $D = 6$  Poincaré supersymmetry algebra has Weyl pseudo-Majorana supercharges and  $\mathcal{R}$ -symmetry  $SO(4) \sim SU(2)_L \times SU(2)_R$ . Its central extension reads as follows (see *e.g.* [19, 20, 21])

$$\{\mathcal{Q}_\gamma^A, \mathcal{Q}_\delta^B\} = \gamma_{\gamma\delta}^\mu Z_\mu^{[AB]} + \gamma_{\gamma\delta}^{\mu\nu\rho} Z_{\mu\nu\rho}^{(AB)}; \quad (5.1)$$

$$\{\mathcal{Q}_{\dot{\gamma}}^{\dot{A}}, \mathcal{Q}_{\dot{\delta}}^{\dot{B}}\} = \gamma_{\dot{\gamma}\dot{\delta}}^\mu Z_\mu^{[\dot{A}\dot{B}]} + \gamma_{\dot{\gamma}\dot{\delta}}^{\mu\nu\rho} Z_{\mu\nu\rho}^{(\dot{A}\dot{B})}; \quad (5.2)$$

$$\{\mathcal{Q}_\gamma^A, \mathcal{Q}_{\dot{\delta}}^{\dot{A}}\} = C_{\gamma\dot{\delta}} Z^{AA} + \gamma_{\gamma\dot{\delta}}^{\mu\nu} Z_{\mu\nu}^{AA}, \quad (5.3)$$

where  $A, \dot{A} = 1, 2$ , so that the (L,R)-chiral supercharges are  $SU(2)_{(L,R)}$ -doublets.

Notice that, in our analysis of both  $(1, 1)$  and  $(2, 2)$   $D = 6$  supergravities, it holds that  $Z_{\mu\nu\rho}^{(AB)} = Z_{\mu\nu\rho}^{(\dot{A}\dot{B})} = 0$ , because the presence of the term  $Z_{\mu\nu\rho}^{(AB)}$  is inconsistent with the bound  $p \leq D - 4$ , due to the assumed asymptotical flatness of the (intersecting) black  $p$ -brane space-time background.

Strings can be dyonic, and are associated to the central charges  $Z_\mu^{[AB]}, Z_\mu^{[\dot{A}\dot{B}]}$  in the  $(\mathbf{1}, \mathbf{1})$  of the  $\mathcal{R}$ -symmetry group. They are embedded in the  $\mathbf{1}_\pm$  (here and below the subscripts denote the weight of  $SO(1, 1)$ ) of the  $U$ -duality group  $SO(1, 1) \times SO(4, n_V)$ . On the other hand, black holes and their magnetic duals (black 2-branes) are associated to  $Z^{AA}, Z_{\mu\nu}^{AA}$  in the  $(\mathbf{2}, \mathbf{2}')$  of  $SO(4)$ , and they are embedded in the  $(\mathbf{n}_V + \mathbf{4})_{\pm\frac{1}{2}}$  of  $SO(1, 1) \times SO(4, n_V)$ .

In our analysis, the corresponding central charges are denoted respectively by  $Z_+$  and  $Z_-$  for dyonic strings, and by  $Z_{\text{el}, AA}$  and  $Z_{\text{mag}, AA}$  for black holes and their magnetic duals.

### 5.2 $\mathcal{N} = (1, 1)$ , $D = 6$ Supergravity

The bosonic field content of half-maximal  $\mathcal{N} = (1, 1)$  supergravity in  $D = 6$  dimensions coupled to  $n_V$  matter (*vector*) multiplets consists of a graviton,  $(n_V + 4)$  vector fields with field strengths  $F_2^M$ ,  $M = 1, \dots, (n_V + 4)$ , a three form field strength  $H_3$ , and  $4n_V + 1$  scalar fields parametrizing the scalar manifold

$$\mathcal{M} = SO(1, 1) \times \frac{SO(4, n_V)}{SO(4) \times SO(n_V)}, \quad \dim_{\mathbb{R}} \mathcal{M} = 4n_V + 1, \quad (5.4)$$

with the dilaton  $\phi$  spanning  $SO(1, 1)$ , and the  $4n_V$  real scalars  $z^i$  ( $i = 1, \dots, 4n_V$ ) parametrising the quaternionic manifold  $\frac{SO(4, n_V)}{SO(4) \times SO(n_V)}$ . The  $U$ -duality group is  $SO(1, 1) \times$

$SO(4, n_V)$  and the field strengths transform under this group in the representations

$$\begin{aligned} F_2^\Lambda &: \quad (\mathbf{n}_V + \mathbf{4})_{+\frac{1}{2}} , \\ H_3 &: \quad \mathbf{1}_{\pm 1} . \end{aligned} \tag{5.5}$$

The coset representative  $L_\Lambda^M$ ,  $\Lambda, M = 1, \dots, 4 + n_V$ , of  $\frac{SO(4, n_V)}{SO(4) \times SO(n_V)}$  sits in the  $(\mathbf{4}, \mathbf{n}_V)$  representation of the stabilizer  $H = SO(4) \times SO(n_V) \sim SU(2)_L \times SU(2)_R \times SO(n_V)$ , and satisfies the defining relations

$$L_\Lambda^M \eta_{MN} L_\Sigma^N = \eta_{\Lambda\Sigma} , \quad L_\Lambda^M \eta^{\Lambda\Sigma} L_\Sigma^N = \eta^{MN} , \tag{5.6}$$

with the  $SO(4, n_V)$  metric  $\eta_{\Lambda\Sigma}$ . It is related to the vielbein  $\mathcal{V}_\Lambda^M$  from (3.12) by

$$\mathcal{V}_\Lambda^M = e^{-\phi/2} L_\Lambda^M , \tag{5.7}$$

and its inverse is defined by  $L_M^\Lambda L_\Lambda^N = \delta_M^N$ . The Maurer-Cartan equations take the form

$$P_{MN} = L_M^\Lambda d_z L_{\Lambda N} = L_M^\Lambda \partial_i L_{\Lambda N} dz^i , \tag{5.8}$$

where  $P_{MN}$  is a symmetric off-diagonal block matrix with non-vanishing entries only in the  $(4 \times n_V)$ -blocks. Here and below we use  $\delta_{MN}$  to raise and lower the indices  $M, N$ .

The solutions will be specified by the electric and magnetic three-form charges  $q, p$ , and the two-form charges  $p^\Lambda, q_\Sigma$ . The quadratic and cubic U-duality invariants that can be built from these charges are

$$\mathcal{I}_2 = pq , \quad \mathcal{I}_3 = \frac{1}{2} \eta_{\Lambda\Sigma} p^\Lambda p^\Sigma p , \quad \mathcal{I}'_3 = \frac{1}{2} \eta^{\Lambda\Sigma} q_\Lambda q_\Sigma q . \tag{5.9}$$

The central charges (3.14), (3.20) are given by

$$\begin{aligned} Z_{\text{mag}, M} &= e^{-\phi/2} J_2 L_{\Lambda M} p^\Lambda , & Z_{\text{el}, M} &= e^{\phi/2} J'_2 L_M^\Lambda q_\Lambda , \\ Z_\pm &= \frac{1}{\sqrt{2}} J_3 (e^\phi p \pm e^{-\phi} q) . \end{aligned} \tag{5.10}$$

Using (5.6), the U-duality invariants (5.9) can be rewritten in terms of the central charges as

$$\begin{aligned} \frac{1}{2} (Z_+^2 - Z_-^2) &= J_3^2 \mathcal{I}_2 , \\ \frac{1}{2\sqrt{2}} \eta^{MN} Z_{\text{mag}, M} Z_{\text{mag}, N} (Z_+ + Z_-) &= (J_3 J_2^2) \mathcal{I}_3 , \\ \frac{1}{2\sqrt{2}} \eta^{MN} Z_{\text{el}, M} Z_{\text{el}, N} (Z_+ - Z_-) &= (J_3 J_2'^2) \mathcal{I}'_3 . \end{aligned} \tag{5.11}$$

The effective potential  $V_{\text{eff}}$  (3.22) for this theory is given by

$$V_{\text{eff}} = \frac{1}{2}Z_+^2 + \frac{1}{2}Z_-^2 + \frac{1}{2}Z_{\text{el},M}^2 + \frac{1}{2}Z_{\text{mag},M}^2 . \quad (5.12)$$

From the Maurer-Cartan equations (5.8) one derives

$$\begin{aligned} \nabla Z_{\text{mag},M} &= -P_{MN} Z_{\text{mag},N} - \frac{1}{2}P_\phi Z_{\text{mag},M} , \\ \nabla Z_{\text{el},M} &= P_{MN} Z_{\text{el},N} + \frac{1}{2}P_\phi Z_{\text{el},M} , \\ \nabla Z_\pm &= P_\phi Z_\mp . \end{aligned} \quad (5.13)$$

with  $P_\phi = d\phi$ . The attractor equations (3.23) thus translate into

$$P_{MN} (Z_{\text{el},M} Z_{\text{el},N} - Z_{\text{mag},M} Z_{\text{mag},N}) + P_\phi (2Z_+ Z_- - \frac{1}{2}Z_{\text{mag},M}^2 + \frac{1}{2}Z_{\text{el},M}^2) \stackrel{!}{=} 0 . \quad (5.14)$$

Splitting the index  $M$  into  $(A\dot{A}) = 1, \dots, 4$ ,  $(A, \dot{A} = 1, 2)$  (*central charges sector*) and  $I = 5, \dots, (n_V + 4)$  (*matter charges sector*), and using the fact that only the components  $P_{I,A\dot{A}} = P_{A\dot{A},I}$  are non-vanishing, the attractor equations can be written as

$$\begin{aligned} Z_{\text{el},A\dot{A}} Z_{\text{el},I} - Z_{\text{mag},A\dot{A}} Z_{\text{mag},I} &= 0 , \\ 4Z_+ Z_- - Z_{\text{mag},A\dot{A}} Z_{\text{mag}}^{A\dot{A}} + Z_{\text{el},A\dot{A}} Z_{\text{el}}^{A\dot{A}} - Z_{\text{mag},I}^2 + Z_{\text{el},I}^2 &= 0 . \end{aligned} \quad (5.15)$$

Indices  $A, \dot{A}$  are raised and lowered by  $\epsilon_{AB}, \epsilon_{\dot{A}\dot{B}}$ . We will study the solutions of these equations, their supersymmetry-preserving features, and the corresponding moduli spaces. BPS solutions correspond to the solutions of (5.15) satisfying

$$Z_{\text{mag},I} = Z_{\text{el},I} = 0 , \quad (5.16)$$

as follows from the Killing spinor equation  $\delta\lambda_A^I \sim T_{\mu\nu}^I \gamma^{\mu\nu} \epsilon_A = 0$  with  $T_{\mu\nu}^I$  the matter central charge densities.

Let us finally consider the *moduli space* of the attractor solutions, i.e. the scalar degrees of freedom which are *not* stabilized by the attractor mechanism at the classical level. For homogeneous scalar manifolds this space is spanned by the vanishing eigenvalues of the Hessian matrix  $\nabla\nabla V_{\text{eff}}$  at the critical point. Using the Maurer-Cartan equations (5.13) one can write  $\nabla\nabla V_{\text{eff}}$  at the critical point as

$$\begin{aligned} \nabla\nabla V_{\text{eff}} &= P_{I,A\dot{A}} P_J^{A\dot{A}} (2 Z_{\text{el},I} Z_{\text{el},J} + 2 Z_{\text{mag},I} Z_{\text{mag},J}) \\ &\quad + P^{I,A\dot{A}} P^{J,B\dot{B}} (2 Z_{\text{el},A\dot{A}} Z_{\text{el},B\dot{B}} + 2 Z_{\text{mag},A\dot{A}} Z_{\text{mag},B\dot{B}}) \\ &\quad + P_\phi P_\phi (2Z_+^2 + 2Z_-^2 + \frac{1}{2}Z_{\text{mag},M}^2 + \frac{1}{2}Z_{\text{el},M}^2) \\ &\quad + 2 P_\phi P^{I,A\dot{A}} (Z_{\text{el},I} Z_{\text{el},A\dot{A}} + Z_{\text{mag},I} Z_{\text{mag},A\dot{A}}) \\ &= H_{IA\dot{A},JB\dot{B}} P^{I,A\dot{A}} P^{J,B\dot{B}} + 2 H_{IA\dot{A},\phi} P^{I,A\dot{A}} P_\phi + H_{\phi,\phi} P_\phi P_\phi , \end{aligned} \quad (5.17)$$

which defines the Hessian symmetric matrix  $\mathbf{H}$  with components  $H_{IA\dot{A},JB\dot{B}}, H_{IA\dot{A},\phi}, H_{\phi,\phi}$ . By explicit evaluation of the Hessian matrix for both BPS and non-BPS solutions we will show that eigenvalues are always zero or positive implying the stability (at the classical level) of the solutions under consideration here. We will now specify to the different near-horizon geometries and study the BPS and non-BPS solutions of the attractor equations.

### 5.3 $AdS_3 \times S^3$

Let us start with an  $AdS_3 \times S^3$  near-horizon geometry, in which only the three-form charges (magnetic  $p$  and electric  $q$ ) are switched on (*dyonic black string*). There are no closed two-forms supported by this geometry and therefore two-form charges are not allowed. The near-horizon geometry ansatz can then be written as

$$ds^2 = r_{\text{AdS}}^2 ds_{\text{AdS}_3}^2 + r_S^2 ds_{S^3}^2, \quad H_3 = p \alpha_{S^3} + e \beta_{\text{AdS}_3}. \quad (5.18)$$

The attractor equations (5.15) are solved by

$$Z_{\text{mag},M} = Z_{\text{el}M} = Z_- = 0, \quad \text{or equivalently,} \quad (5.19)$$

$$Z_{\text{mag},M} = Z_{\text{el},M} = Z_+ = 0. \quad (5.20)$$

Solution (5.19) has  $\mathcal{I}_2 > 0$ , whereas solution (5.20) has  $\mathcal{I}_2 < 0$ ; they are both  $\frac{1}{4}$ -BPS, and they are equivalent, because the considered theory is non-chiral.

Plugging the solutions (5.19) or (5.20) into (5.12) one can write the effective potential at the horizon in the scalar independent form

$$V_{\text{eff}} = \frac{1}{2}Z_+^2 + \frac{1}{2}Z_-^2 = \frac{1}{2}|Z_+^2 - Z_-^2|J_3^2|\mathcal{I}_2| = \left(\frac{v_{\text{AdS}_3}}{v_{S^3}}\right)|\mathcal{I}_2|, \quad (5.21)$$

in agreement with the claimed formula (4.1). Extremizing  $F$  in  $\vec{r}$ , one finds the entropy function and near-horizon AdS and sphere radii (4.1).

Now let us consider the moduli space of the solutions. Plugging (5.19), (5.20) into (5.17) one finds that the only non-trivial component of the Hessian matrix is

$$H_{\phi\phi} = 2Z_+^2 + 2Z_-^2 = 4V_{\text{eff}} > 0. \quad (5.22)$$

Therefore, the Hessian matrix  $\mathbf{H}$  for the  $AdS_3 \times S^3$  solution has  $4n_V$  vanishing eigenvalues and one strictly positive eigenvalue, corresponding to the dilaton direction. Consequently, the moduli space of non-degenerate attractors with near-horizon geometry  $AdS_3 \times S^3$  is the quaternionic symmetric manifold

$$\mathcal{M}_{\text{BPS}} = \frac{SO(4, n_V)}{SO(4) \times SO(n_V)}. \quad (5.23)$$

This result is also evident from the explicit form of the attractor solution  $Z_- = 0$ : only the dilaton is stabilized, while all other scalars are not fixed since the remaining equations  $Z_{\text{el},M} = Z_{\text{mag},M} = 0$  are automatically satisfied for  $p^\Lambda = q_\Lambda = 0$ .

#### 5.4 $AdS_3 \times S^2 \times S^1$

For solutions with near-horizon geometry  $AdS_3 \times S^2 \times S^1$ , there is no support for electric two-form charges and therefore  $e^\Lambda = 0$ . We set also the electric three-form charge  $e$  to zero otherwise no solutions are found. The near-horizon ansatz becomes

$$\begin{aligned} ds^2 &= r_{\text{AdS}}^2 ds_{\text{AdS}_3}^2 + r_S^2 ds_{S^2}^2 + r_1^2 d\theta^2, \\ F_2^\Lambda &= p^\Lambda \alpha_{S^2}, \quad H_3 = p \alpha_{S^2 \times S^1}. \end{aligned} \quad (5.24)$$

The attractor equations (5.15) admit two types of solutions with non trivial central charges

$$\text{BPS :} \quad Z_+ = Z_-, \quad Z_{\text{mag},AA} Z_{\text{mag}}^{AA} = 4Z_+^2; \quad (5.25)$$

$$\text{non-BPS :} \quad Z_+ = Z_-, \quad Z_{\text{mag},I}^2 = 4Z_+^2. \quad (5.26)$$

Plugging the solution into (5.11) one finds the relation

$$|J_2^2 J_3 \mathcal{I}_3| = 2\sqrt{2} Z_+^3. \quad (5.27)$$

that allows us to write the effective potential (5.12) at the horizon in the scalar independent form

$$V_{\text{eff}} = 3Z_+^2 = \frac{3}{2} |J_2^2 J_3 \mathcal{I}_3|^{\frac{2}{3}}, \quad (5.28)$$

with

$$(J_2^2 J_3)^{\frac{2}{3}} = \frac{v_D}{(\text{vol}_{S^2}^2 \text{vol}_{S^2 \times S^1})^{\frac{2}{3}}} = \frac{v_{\text{AdS}_3} v_T^{\frac{1}{3}}}{v_S}, \quad (5.29)$$

in agreement with our proposed formula (4.2) upon taking  $\mathbf{I}_3 = \mathcal{I}_3$ . The black string central charge and the near-horizon radii follow from  $\vec{r}$ -extremization of the entropy function  $F$  and are given by (4.2). Note that the radius  $r_1$  of the extra  $S^1$  is not fixed by the extremization equations. Besides this geometric modulus the solutions can be also deformed by turning on Wilson lines for the vector field potentials  $A_5^\Lambda = c^\Lambda$ . This is in contrast with the more familiar case of black holes in  $D = 4, 5$  where the near-horizon geometry is completely fixed at the end of the attractor flow. As we shall see in the following, this will be always the case for extremal black  $p$ -branes with  $T^n$  factors where

the “geometric moduli” describing the shapes and volumes of the tori and constant values of field potentials along  $T^n$  remain unfixed at the horizon.

Now, let us consider the moduli spaces of the two solutions. The BPS solution (5.25) has remaining symmetry  $SO(3) \times SO(n_V)$ , because by using an  $SO(4)$  transformation this solution can be recast in the form

$$Z_{\text{mag},A\dot{A}} = 2z \delta_{A1} \delta_{\dot{A}1}, \quad Z_+ = Z_- = z, \quad Z_{\text{el},M} = 0. \quad (5.30)$$

Notice that both choices of sign satisfy the Killing spinor relations (5.16) and therefore correspond to supersymmetric solutions. Plugging (5.30) into the Hessian matrix (5.17) one finds

$$\mathbf{H} = z^2 \begin{pmatrix} 8\delta_{IJ}\delta_{A1}\delta_{B1}\delta_{\dot{A}1}\delta_{\dot{B}1} & 0_{4n_V \times 1} \\ 0_{1 \times 4n_V} & 6 \end{pmatrix}. \quad (5.31)$$

This matrix has  $3n_V$  vanishing eigenvalues and  $n_V + 1$  strictly positive eigenvalues, corresponding to the dilaton direction plus the  $n_V$  directions  $P_{I,11}$ . Consequently, the moduli space of the BPS attractor solution (5.25) with near-horizon geometry  $AdS_3 \times S^2 \times S^1$  is the symmetric manifold

$$\mathcal{M}_{\text{BPS}} = \frac{SO(3, n_V)}{SO(3) \times SO(n_V)}. \quad (5.32)$$

More precisely, the scalars along  $P_{I,A\dot{A}}$  in the  $(\mathbf{4}, \mathbf{n}_V)$  of the group  $H$  decompose with respect to the symmetry group  $SO(3) \times SO(n_V)$  as:

$$(\mathbf{4}, \mathbf{n}_V) \longrightarrow \underbrace{(\mathbf{3}, \mathbf{n}_V)}_{m^2=0} \oplus \underbrace{(\mathbf{1}, \mathbf{n}_V)}_{m^2>0}, \quad (5.33)$$

and only the  $(\mathbf{1}, \mathbf{n}_V)$  representation is massive, together with the dilaton. The  $(\mathbf{3}, \mathbf{n}_V)$  representation remains massless, and it contains all the massless Hessian modes of the attractor solutions.

The analysis of the moduli space for the non-BPS solution follows closely that for the BPS one. Now the symmetry is  $SO(4) \times SO(n_V - 1)$  and using an  $SO(n_V)$  transformation such a solution can be recast as follows:

$$Z_{\text{mag},I} = 2z \delta_{I1}, \quad Z_+ = Z_- = z, \quad Z_{\text{el},M} = Z_{\text{mag},A\dot{A}} = Z_{\text{el},A\dot{A}} = 0. \quad (5.34)$$

Plugging (5.34) into the Hessian matrix (5.17), now one finds

$$\mathbf{H} = z^2 \begin{pmatrix} 8\delta_{A\dot{A}}\delta_{B\dot{B}}\delta_{J1}\delta_{I1} & 0_{4n_V \times 1} \\ 0_{1 \times 4n_V} & 6 \end{pmatrix}. \quad (5.35)$$

This Hessian matrix has  $4(n_V - 1)$  vanishing eigenvalues and  $4+1$  strictly positive eigenvalues, corresponding to the dilaton direction plus the  $4 P_{1,AA}$  directions. Consequently, the *moduli space* of the non-BPS attractor solution with near-horizon geometry  $AdS_3 \times S^2 \times S^1$  is the symmetric manifold

$$\mathcal{M}_{\text{nonBPS}} = \frac{SO(4, n_V - 1)}{SO(4) \times SO(n_V - 1)} . \quad (5.36)$$

More precisely, the scalars along  $P_{I,AA}$  in the  $(\mathbf{4}, \mathbf{n}_V)$  of the group  $H$  decompose with respect to the symmetry group  $SO(4) \times SO(n_V - 1)$  as:

$$(\mathbf{4}, \mathbf{n}_V) \longrightarrow \underbrace{(\mathbf{4}, \mathbf{n}_V - \mathbf{1})}_{m^2=0} \oplus \underbrace{(\mathbf{4}, \mathbf{1})}_{m^2>0} , \quad (5.37)$$

and only the  $(\mathbf{4}, \mathbf{1})$  representation is massive, together with the dilaton. The  $(\mathbf{4}, \mathbf{n}_V - \mathbf{1})$  representation remains massless, and it contains all the massless Hessian modes of the attractor solution.

The BPS solution can be regarded as the intersection of one  $\frac{1}{2}$ -BPS black string (with  $pq = 0$ ) with one  $\frac{1}{4}$ -BPS black 2-brane (with  $p^\Lambda p^\Sigma \eta_{\Lambda\Sigma} > 0$ ). The latter is described by the charge orbit  $\frac{SO(4, n_V)}{SO(3, n_V)}$  [23]. The moduli space of the latter coincides with the moduli space of the whole considered intersection, and it is given by Eq. (5.32).

On the other hand, the non-BPS solution can be regarded as the intersection of one  $\frac{1}{2}$ -BPS black string (with  $pq = 0$ ) with one non-BPS black 2-brane (with  $p^\Lambda p^\Sigma \eta_{\Lambda\Sigma} < 0$ ). The latter is described by the charge orbit  $\frac{SO(4, n_V)}{SO(4, n_V - 1)}$  [23]. The moduli space of the latter coincides with the moduli space of the whole considered intersection, and it is given by the quaternionic manifold of Eq. (5.36).

A similar reasoning will be performed for the moduli spaces of the attractor solutions of the maximal non-chiral  $D = 6$  supergravity in Sect. 6.

## 5.5 $AdS_2 \times S^3 \times S^1$

For solutions with  $AdS_2 \times S^3 \times S^1$  near-horizon geometry, there is no support for magnetic two-form charges and therefore  $Z_{\text{mag}, M} = 0$ . The near-horizon ansatz becomes

$$\begin{aligned} ds^2 &= r_{\text{AdS}}^2 ds_{\text{AdS}_2}^2 + r_S^2 ds_{S^3}^2 + r_1^2 d\theta^2 , \\ F_2^\Lambda &= e^\Lambda \beta_{\text{AdS}_2} , \quad H_3 = e \beta_{\text{AdS}_2 \times S^1} . \end{aligned} \quad (5.38)$$

The fixed-scalar equations (5.14) admit two type of solutions

$$\text{BPS :} \quad Z_{\text{mag}, M} = Z_{\text{el}, I} = 0 , \quad Z_+ = -Z_- , \quad Z_{\text{el}, AA} Z_{\text{el}}^{AA} = 4Z_+^2 , \quad (5.39)$$

$$\text{non-BPS :} \quad Z_{\text{mag}, M} = Z_{\text{el}, AA} = 0 , \quad Z_+ = -Z_- , \quad Z_{\text{el}, I}^2 = 4Z_+^2 . \quad (5.40)$$

Now one finds

$$J_2'^2 J_3' \mathcal{I}_3' = 2\sqrt{2} Z_+^3, \quad (5.41)$$

and the effective potential (5.12) at the horizon can be written in the scalar independent form

$$V_{\text{eff}} = 3Z_+^2 = \frac{3}{2} |J_2'^2 J_3' \mathcal{I}_3'|^{\frac{2}{3}}, \quad (5.42)$$

with

$$(J_2'^2 J_3')^{\frac{2}{3}} = \frac{(\text{vol}_{\text{AdS}_2}^2 \text{vol}_{\text{AdS}_2 \times S^1})^{\frac{2}{3}}}{v_D} = \frac{v_{\text{AdS}_2}}{v_T^{1/3} v_S}, \quad (5.43)$$

in agreement with the proposed formula (4.3) upon taking  $\mathbf{I}_3 = \mathcal{I}_3'$ . Extremizing  $F$  in the radii  $\vec{r}$  one finds the result (4.3) for the black hole entropy and AdS and sphere radii. Again, the radius  $r_1$  of the extra  $S^1$  is not fixed by the extremization equations. The analysis of the moduli spaces follows *mutatis mutandis* that of the  $AdS_3 \times S^2$  attractors (replacing magnetic by electric charges) and the results are again given by the symmetric manifolds (5.32) and (5.36).

## 6 $\mathcal{N} = (2, 2)$ in $D = 6$

### 6.1 $\mathcal{N} = (2, 2)$ , $D = 6$ Supersymmetry Algebra

The maximal  $(2, 2)$ ,  $D = 6$  Poincaré supersymmetry algebra has Weyl pseudo-Majorana supercharges and  $\mathcal{R}$ -symmetry  $USp(4)_L \times USp(4)_R$  ( $USp(4) = Spin(5)$ ). Its central extension reads as follows (see *e.g.* [19, 20, 21])

$$\{\mathcal{Q}_\gamma^A, \mathcal{Q}_\delta^B\} = \gamma_{\gamma\delta}^\mu Z_\mu^{[AB]} + \gamma_{\gamma\delta}^{\mu\nu\rho} Z_{\mu\nu\rho}^{(AB)}; \quad (6.1)$$

$$\{\mathcal{Q}_{\dot{\gamma}}^{\dot{A}}, \mathcal{Q}_{\dot{\delta}}^{\dot{B}}\} = \gamma_{\dot{\gamma}\dot{\delta}}^\mu Z_\mu^{[\dot{A}\dot{B}]} + \gamma_{\dot{\gamma}\dot{\delta}}^{\mu\nu\rho} Z_{\mu\nu\rho}^{(\dot{A}\dot{B})}; \quad (6.2)$$

$$\{\mathcal{Q}_\gamma^A, \mathcal{Q}_{\dot{\delta}}^{\dot{A}}\} = C_{\gamma\dot{\delta}} Z^{A\dot{A}} + \gamma_{\gamma\dot{\delta}}^{\mu\nu} Z_{\mu\nu}^{A\dot{A}}, \quad (6.3)$$

where  $A, \dot{A} = 1, \dots, 4$ , so that the (L,R)-chiral supercharges are  $SO(5)_{(L,R)}$ -spinors.

Strings can be dyonic, and they are in the antisymmetric traceless  $(\mathbf{5}, \mathbf{1}) + (\mathbf{1}, \mathbf{5}')$  of the  $\mathcal{R}$ -symmetry group. They are embedded in the  $\mathbf{10}$  of the  $U$ -duality group  $SO(5, 5)$ . On the other hand, black holes and their magnetic duals (black 2-branes) sit in the  $(\mathbf{4}, \mathbf{4}')$  of  $USp(4)_L \times USp(4)_R$ , and they are embedded in the chiral spinor repr.  $\mathbf{16}_{(L)}$  of  $SO(5, 5)$ .

In our analysis, the corresponding central charges are denoted respectively by  $Z_a$  and  $Z_{\dot{a}}$  ( $a, \dot{a} = 1, \dots, 5$ ) for dyonic strings, and by  $Z_{\text{el}, A\dot{A}}$  and  $Z_{\text{mag}, A\dot{A}}$  for black holes and their magnetic duals.

## 6.2 $\mathcal{N} = (2, 2)$ , $D = 6$ Supergravity

The maximal  $\mathcal{N} = (2, 2)$  supergravity in  $D = 6$  dimensions [26] has bosonic field content given by the graviton, 25 scalar fields, 16 vectors and 5 two-form fields. Under the global symmetry group  $SO(5, 5)$  these fields organize as

$$\begin{aligned} \mathcal{V}_I^M &= \begin{pmatrix} \mathcal{V}_m^a & \mathcal{V}_m^{\dot{a}} \\ \mathcal{V}^{ma} & \mathcal{V}^{m\dot{a}} \end{pmatrix} : & \frac{SO(5, 5)}{SO(5) \times SO(5)} & \quad I, M = 1, \dots, 10, \\ & & & \quad a, \dot{a}, m = 1, \dots, 5, \\ F_2^\Lambda &: & \mathbf{16} & \quad \Lambda = 1, \dots, 16, \\ \{H_{3+}^a, H_{3-}^{\dot{a}}\} &: & \mathbf{10} & \quad a, \dot{a} = 1, \dots, 5. \end{aligned} \quad (6.4)$$

In particular, the scalar coset space is parametrized by the vielbein  $\mathcal{V}_I^M$  evaluated in the vector representation  $\mathbf{10}$  of  $SO(5, 5)$ , satisfying the defining relations

$$\mathcal{V}_I^a \mathcal{V}_J^a - \mathcal{V}_I^{\dot{a}} \mathcal{V}_J^{\dot{a}} = \eta_{IJ} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{V}_m^M \mathcal{V}^{mN} + \mathcal{V}^{mM} \mathcal{V}_m^N = \eta^{MN} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

i.e. the splits of basis  $\mathcal{V}_I^M \rightarrow (\mathcal{V}_m^M, \mathcal{V}^{mM})$  and  $\mathcal{V}_I^M \rightarrow (\mathcal{V}_I^a, \mathcal{V}_I^{\dot{a}})$  refer to the decompositions  $SO(5, 5) \rightarrow GL(5)$  and  $SO(5, 5) \rightarrow SO(5) \times SO(5)$ , respectively. They are relevant for splitting the two-forms into electric and magnetic potentials and for coupling them to the fermionic fields, respectively. The scalar coset space can equivalently be described by a scalar vielbein  $\mathcal{V}_\Lambda^{A\dot{A}}$  ( $A, \dot{A} = 1, \dots, 4$ ) evaluated in the  $\mathbf{16}$  spinor representation of  $SO(5, 5)$ . The Maurer-Cartan equations are given by

$$\nabla \mathcal{V}_I^a = -P_{a\dot{a}} \mathcal{V}_I^{\dot{a}}, \quad \nabla \mathcal{V}_I^{\dot{a}} = -P_{a\dot{a}} \mathcal{V}_I^a, \quad \nabla \mathcal{V}_\Lambda^{A\dot{A}} = -\frac{1}{2} P_{a\dot{a}} \gamma_a^{AB} \gamma_{\dot{a}}^{\dot{A}\dot{B}} \mathcal{V}_{\Lambda B\dot{B}}, \quad (6.5)$$

with the  $SO(5) \times SO(5)$  Gamma matrices  $\gamma_a^{AB}, \gamma_{\dot{a}}^{\dot{A}\dot{B}}$ , and the vector and spinorial indices raised and lowered by the  $SO(5)$  invariant symmetric tensors  $\delta_{ab}, \delta_{\dot{a}\dot{b}}$  and antisymmetric tensors  $\Omega_{AB}, \Omega_{\dot{A}\dot{B}}$ , respectively.

The Lagrangian involves the 5 two-forms  $B^m$ , whose field strengths are related to the selfdual  $H_{3+}^a$  and antiselfdual  $H_{3-}^{\dot{a}}$  by

$$dB^m = H^m \equiv \mathcal{V}^{ma} H_{3+}^a + \mathcal{V}^{m\dot{a}} H_{3-}^{\dot{a}}. \quad (6.6)$$

Electric and magnetic three-form charges combine into an  $SO(5, 5)$  vector  $Q_I = (p^m, q_m)$ . The quadratic and cubic U-duality invariants of charges are given by

$$\mathcal{I}_2 = \frac{1}{2} \eta^{IJ} Q_I Q_J, \quad \mathcal{I}_3 = \frac{1}{2\sqrt{2}} (\Gamma^I)_{\Lambda\Sigma} Q_I p^\Lambda p^\Sigma, \quad \mathcal{I}'_3 = \frac{1}{2\sqrt{2}} (\Gamma^I)^{\Lambda\Sigma} Q_I q_\Lambda q_\Sigma \quad (6.7)$$

with the  $SO(5,5)$  Gamma matrices  $(\Gamma^I)^{\Lambda\Sigma}$ ,  $(\Gamma^I)_{\Lambda\Sigma}$ .

The central charges (3.14), (3.20) are defined as

$$\begin{aligned} Z_{\text{mag}}^{AA} &= J_2 \mathcal{V}_\Lambda^{AA} p^\Lambda, & Z_{\text{el}}^{AA} &= J'_2 (\mathcal{V}^{-1})^{AA\Lambda} q_\Lambda, \\ Z_a &= J_3 (\mathcal{V}^{-1})_a^I Q_I, & Z_{\dot{a}} &= J_3 (\mathcal{V}^{-1})_{\dot{a}}^I Q_I. \end{aligned} \quad (6.8)$$

In terms of these central charges one can rewrite the U-duality invariants (6.7) as

$$\begin{aligned} 2 J_3^2 \mathcal{I}_2 &= Z_a^2 - Z_{\dot{a}}^2, \\ 2\sqrt{2} (J_2^2 J_3) \mathcal{I}_3 &= Z_{\text{mag}}^{AA} Z_{\text{mag}}^{BB} (Z_a \gamma_{AB}^a \Omega_{\dot{A}\dot{B}} + Z_{\dot{a}} \Omega_{AB} \gamma_{\dot{A}\dot{B}}^{\dot{a}}), \\ 2\sqrt{2} (J_2'^2 J_3) \mathcal{I}'_3 &= Z_{\text{el},AA} Z_{\text{el},BB} (Z_a \gamma_{AB}^a \Omega_{\dot{A}\dot{B}} - Z_{\dot{a}} \Omega_{AB} \gamma_{\dot{A}\dot{B}}^{\dot{a}}). \end{aligned} \quad (6.9)$$

The intersecting-branes effective potential  $V_{\text{eff}}$  for the considered theory is defined as

$$V_{\text{eff}} = \frac{1}{2} Z_a^2 + \frac{1}{2} Z_{\dot{a}}^2 + \frac{1}{2} Z_{\text{el},AA} Z_{\text{el}}^{AA} + \frac{1}{2} Z_{\text{mag},AA} Z_{\text{mag}}^{AA}. \quad (6.10)$$

The Maurer-Cartan equations (6.5) imply

$$\begin{aligned} \nabla Z_a &= P_{a\dot{a}} Z_{\dot{a}}, & \nabla Z_{\dot{a}} &= P_{a\dot{a}} Z_a \\ \nabla Z_{\text{el}}^{AA} &= \frac{1}{2} P_{a\dot{a}} \gamma_a^{AB} \gamma_{\dot{a}}^{\dot{A}\dot{B}} Z_{\text{el},B\dot{B}}, & \nabla Z_{\text{mag}}^{AA} &= -\frac{1}{2} P_{a\dot{a}} \gamma_a^{AB} \gamma_{\dot{a}}^{\dot{A}\dot{B}} Z_{\text{mag},B\dot{B}}. \end{aligned} \quad (6.11)$$

Thus the extremization equations take the form

$$\begin{aligned} \nabla V_{\text{eff}} &= Z_a \nabla Z_a + Z_{\dot{a}} \nabla Z_{\dot{a}} + Z_{\text{el},AA} \nabla Z_{\text{el}}^{AA} + Z_{\text{mag},AA} \nabla Z_{\text{mag}}^{AA} \\ &= \left( 2Z_a Z_{\dot{a}} + \frac{1}{2} \gamma_{AB}^a \gamma_{\dot{A}\dot{B}}^{\dot{a}} Z_{\text{el}}^{AA} Z_{\text{el}}^{BB} - \frac{1}{2} \gamma_{AB}^a \gamma_{\dot{A}\dot{B}}^{\dot{a}} Z_{\text{mag}}^{AA} Z_{\text{mag}}^{BB} \right) P_{a\dot{a}} \stackrel{!}{=} 0. \end{aligned} \quad (6.12)$$

The Hessian matrix at the horizon can be written as

$$\begin{aligned} \nabla \nabla V_{\text{eff}} &= 2P_{a\dot{a}} P_{b\dot{b}} \left\{ \delta_{ab} Z_{\dot{a}} Z_{\dot{b}} + \delta_{\dot{a}\dot{b}} Z_a Z_b \right. \\ &\quad \left. + \frac{1}{4} (\gamma^a \gamma^b)_A{}^B (\gamma^{\dot{a}} \gamma^{\dot{b}})_{\dot{A}}{}^{\dot{B}} \left( Z_{\text{el}}^{AA} Z_{\text{el},B\dot{B}} + Z_{\text{mag}}^{AA} Z_{\text{mag},B\dot{B}} \right) \right\} \quad (6.13) \end{aligned}$$

### 6.3 $AdS_3 \times S^3$

The analysis of  $D = 6$  maximal supersymmetric supergravity solutions follows the same steps as in the half-maximal case with minor modifications. We start from the ansatz

$$\begin{aligned} ds^2 &= r_{\text{AdS}}^2 ds_{\text{AdS}_3}^2 + r_S^2 ds_{S^3}^2, \\ H_3^m &= p^m \alpha_{S^3} + e^m \beta_{\text{AdS}_3}, \end{aligned} \quad (6.14)$$

for the  $AdS_3 \times S^3$  near-horizon geometry. The fixed scalar equation (6.12) admits the two solutions

$$\begin{aligned} Z_{\text{mag},A\dot{A}} &= Z_{\text{el},A\dot{A}} = Z_a = 0, \\ Z_{\text{mag},A\dot{A}} &= Z_{\text{el},A\dot{A}} = Z_{\dot{a}} = 0, \end{aligned} \tag{6.15}$$

which are both supersymmetric. Combining this with (6.9) one can write the effective potential at the horizon in the scalar independent form

$$V_{\text{eff}} = \frac{1}{2}(Z_a^2 + Z_{\dot{a}}^2) = \frac{1}{2}|Z_a^2 - Z_{\dot{a}}^2| = J_3^2 |\mathcal{I}_2|. \tag{6.16}$$

Again the effective potential is given by the general formula (4.1) but now  $\mathbf{I}_2 = \mathcal{I}_2$  is given by the quadratic invariant (6.7) of  $SO(5, 5)$ . Similarly,  $\vec{r}$ -extremization of the entropy function shows that the sphere and AdS radii and the black string central charges are given by (4.1) in terms of the  $SO(5, 5)$  invariant  $\mathcal{I}_2$ .

Let us consider the moduli space of these solutions. The two solutions are equivalent and we can focus on the  $Z_a = 0$  case. Using an  $SO(5)$  rotation this solution can be recast in the form  $Z_{\dot{a}} = z\delta_{\dot{a},1}$ . The symmetry group leaving this solution invariant is  $SO(5, 4)$ . The moduli space is hence given by the quotient of this group by its maximal compact subgroup  $SO(5) \times SO(4)$ , i.e. ([27, 28, 29])

$$\mathcal{M}_{\text{BPS}} = \frac{SO(5, 4)}{SO(5) \times SO(4)}. \tag{6.17}$$

Alternatively, the same conclusion can be reached by evaluating the Hessian (6.13) at the solution

$$\nabla\nabla V_{\text{eff}} = 2z^2 P_{a\dot{i}} P_{a\dot{i}}, \tag{6.18}$$

one finds 5 strictly positive eigenvalues. More precisely, the  $(\mathbf{5}, \mathbf{5})$  scalars decompose in terms of  $SO(5) \times SO(4)$  as

$$(\mathbf{5}, \mathbf{5}) \longrightarrow \underbrace{(\mathbf{5}, \mathbf{4})}_{m^2=0} \oplus \underbrace{(\mathbf{5}, \mathbf{1})}_{m^2>0}, \tag{6.19}$$

with the  $(\mathbf{5}, \mathbf{4})$  components along  $P_{a,b>i}$  spanning the moduli space of the solution.

The story goes the same way for the solution with  $Z_{\dot{a}} = 0$  which has moduli space  $\mathcal{M}_{\text{BPS}} = \frac{SO(4,5)}{SO(4) \times SO(5)}$ . The two solutions are equivalent and they both preserve the same amount of supersymmetry (namely the minimal one:  $\frac{1}{8}$ -BPS). Actually, they can be interpreted as the *supersymmetry uplift* of the two distinct  $\frac{1}{4}$ -BPS solutions (given by Eqs. (5.19) and (5.20)) of the half-maximal  $D = 6$  supergravity coupled to  $n_V = 4$  vector multiplets.

## 6.4 $AdS_3 \times S^2 \times S^1$

The ansatz for this near-horizon geometry is

$$\begin{aligned} ds^2 &= r_{\text{AdS}}^2 ds_{\text{AdS}_3}^2 + r_S^2 ds_{S^2}^2 + r_1^2 d\theta^2, \\ F_2^A &= p^A \alpha_{S^2} \quad H_3^m = p^m \alpha_{S^2 \times S^1}. \end{aligned} \quad (6.20)$$

Notice that a magnetic string corresponds to the  $SO(5, 5)$  invariant constructed with the 10-dimensional vector  $(p^m, 0)$  having vanishing norm. This is the  $\frac{1}{2}$ -BPS constraint for a  $D = 6$  string configuration, derived in [27].

The solutions of the fixed-scalar equations (6.12) on this background can be written up to an  $SO(5) \times SO(5)$  rotation as<sup>1</sup>

$$\begin{aligned} Z_a &= z\delta_{a1}, & Z_{\dot{a}} &= z\delta_{\dot{a}1}, \\ Z_{\text{el}, A\dot{A}} &= 0, & Z_{\text{mag}}^{A\dot{A}} &= \sqrt{2} \text{diag}(z, z, 0, 0). \end{aligned} \quad (6.21)$$

Using (6.9) one can express  $z$  in terms of the cubic U-invariant (6.7)

$$(J_2^2 J_3) \mathcal{I}_3 = 2\sqrt{2} z^3. \quad (6.22)$$

Combining (6.10), (6.21), (6.22), one finally writes the effective potential in the scalar independent form

$$V_{\text{eff}} = 3z^2 = \frac{3}{2} |J_2^2 J_3 \mathcal{I}_3|^{2/3} = \frac{3}{2} \frac{v_{\text{AdS}_3} v_T^{1/3}}{v_S} |\mathcal{I}_3|^{2/3}. \quad (6.23)$$

Like in the half-maximal case, the effective potential, the black string central charge and the near-horizon geometry are given by the general formulas (4.2) but now in terms of the  $SO(5, 5)$  cubic invariant  $\mathbf{I}_3 = \mathcal{I}_3$ . Again, the radius  $r_1$  of the extra  $S^1$  is not fixed by the extremization equations.

Let us consider the moduli space of this attractor. The symmetry of the solution is  $SO(4, 3)$ , which is the subgroup of  $SO(5, 5)$  leaving invariant (6.21). To see this we notice that  $SO(4, 3)$  is the maximal subgroup of  $SO(5, 5)$  under which the decompositions of both the vector and the spinor representations of  $SO(5, 5)$  contain a singlet

$$\begin{aligned} \mathbf{10} &= \mathbf{7} \oplus \mathbf{3} \cdot \mathbf{1}, \\ \mathbf{16} &= \mathbf{8} \oplus \mathbf{7} \oplus \mathbf{1}. \end{aligned} \quad (6.24)$$

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<sup>1</sup> The explicit form of the solution clearly depends on the particular form of  $SO(5)$  gamma-matrices considered. In our conventions, this choice of  $Z_{\text{mag}}^{A\dot{A}}$  induces a matrix  $\gamma_{AB}^a \gamma_{\dot{A}\dot{B}}^{\dot{a}} Z_{\text{mag}}^{A\dot{A}} Z_{\text{mag}}^{B\dot{B}}$  which has only one non-vanishing entry.

The moduli space is then given by the quotient of the symmetry group by its maximal compact subgroup

$$\mathcal{M}_{\text{BPS}} = \frac{SO(4, 3)}{SO(4) \times SO(3)}. \quad (6.25)$$

More precisely, decomposing the scalar components  $P_{a\dot{a}}$  under  $SO(4) \times SO(3)$  one finds

$$(\mathbf{5}, \mathbf{5}) \longrightarrow \underbrace{(\mathbf{4}, \mathbf{3})}_{m^2=0} \oplus \underbrace{(2 \cdot (\mathbf{4}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus 2 \cdot (\mathbf{1}, \mathbf{1}))}_{m^2>0}. \quad (6.26)$$

This can be confirmed by explicitly evaluating the Hessian (6.13) at this extremum. As a result one finds 12 vanishing and 13 strictly positive eigenvalues.

The moduli space in (6.25) can be understood in terms of orbits of  $\frac{1}{2}$ -BPS strings and  $\frac{1}{4}$ -BPS black holes [27, 28, 29]. Indeed, the  $U$ -invariant  $\mathcal{I}_3$  can be considered as an intersection of a  $\frac{1}{2}$ -BPS string with supporting charge orbit  $\frac{SO(5,5)}{SO(4,4) \times_s \mathbb{R}^8}$  and of a  $\frac{1}{4}$ -BPS black hole with supporting charge orbit  $\frac{SO(5,5)}{SO(4,3) \times_s \mathbb{R}^8}$  [28]. The common stabilizer of the charge vectors  $\mathbf{10}$  and  $\mathbf{16}$  of the  $D = 6$   $U$ -duality  $SO(5, 5)$  is  $SO(4, 3)$ . Indeed, we find that the resulting moduli space of the considered intersecting configuration is given by Eq. (6.25). This is also what expected by the supersymmetry uplift of the BPS moduli space of the half-maximal (1, 1) theory to maximal (2, 2) supergravity.

## 6.5 $AdS_2 \times S^3 \times S^1$

The near-horizon geometry ansatz is

$$\begin{aligned} ds^2 &= r_{\text{AdS}}^2 ds_{\text{AdS}_2}^2 + r_S^2 ds_{S^3}^2 + r_1^2 d\theta^2, \\ F_2^A &= e^A \beta_{\text{AdS}_2}, \quad H_3^m = e^m \beta_{\text{AdS}_2 \times S^1}. \end{aligned} \quad (6.27)$$

The computation of the effective potential proceeds as for the  $AdS_3 \times S^2 \times S^1$  case replacing magnetic by electric charges. The final result read

$$V_{\text{eff}} = \frac{3}{2} \left| J_2'^2 J_3 \mathcal{I}_3' \right|^{\frac{2}{3}} = \frac{3}{2} \frac{v_{\text{AdS}_2}}{v_{S^3} v_T^{1/3}} |\mathcal{I}_3'|. \quad (6.28)$$

Extremizing the entropy function  $F$  in the radii  $\vec{r}$  one confirms that the AdS, sphere radii and the black hole entropy are given again by the general formulae (4.3) with  $\mathbf{I}_3 = \mathcal{I}_3'$  the magnetic  $SO(5, 5)$  cubic invariant. The analysis of the moduli space is identical to that of  $AdS_3 \times S^2 \times S^1$  case and the result is again given by (6.25).

## 7 Maximal $D = 7$

### 7.1 $\mathcal{N} = 2, D = 7$ Supersymmetry Algebra

The maximal  $\mathcal{N} = 2, D = 7$  Poincaré supersymmetry algebra has pseudo-Majorana supercharges and  $\mathcal{R}$ -symmetry  $USp(4)$ . Its central extension reads as follows (see *e.g.* [19, 20, 21])

$$\{\mathcal{Q}_\gamma^A, \mathcal{Q}_\delta^B\} = C_{\gamma\delta} Z^{(AB)} + \gamma_{\gamma\delta}^\mu Z_\mu^{[AB]} + \gamma_{\gamma\delta}^{\mu\nu} Z_{\mu\nu}^{[AB]} + \gamma_{\gamma\delta}^{\mu\nu\rho} Z_{\mu\nu\rho}^{(AB)} \quad (7.1)$$

where  $A = 1, \dots, 4$ , so that the supercharges are  $SO(5)$ -spinors. The "trace" part of  $Z_\mu^{[AB]}$  is the momentum  $P_\mu \Omega^{AB}$ , where  $\Omega^{AB}$  is the  $4 \times 4$  symplectic metric.

Black holes and their magnetic dual (black 3-brane) central extensions  $Z^{(AB)}, Z_{\mu\nu\rho}^{(AB)}$  sit in the  $\mathbf{10}$  of the  $\mathcal{R}$ -symmetry group, and they are embedded in the  $\mathbf{10}$  (and  $\mathbf{10}'$ ) of the  $U$ -duality group  $SL(5, \mathbb{R})$ . Thus, they correspond to the decomposition  $\mathbf{10}^{(l)} \rightarrow \mathbf{10}$  of  $SL(5, \mathbb{R})$  into  $SO(5)$ .

On the other hand, black strings and their magnetic dual (black 2-brane) central extensions  $Z_\mu^{[AB]}, Z_{\mu\nu}^{[AB]}$  sit in the  $\mathbf{5}$  of  $USp(4)$ , and they are embedded in the  $\mathbf{5}'$  (and  $\mathbf{5}$ ) of the  $U$ -duality group. Thus, they correspond to the decomposition  $\mathbf{5}^{(l)} \rightarrow \mathbf{5}$  of  $SL(5, \mathbb{R})$  into  $SO(5)$ .

In our analysis, the corresponding central charges are denoted by  $Z_{\text{el}}^{mn}$  and  $Z_{\text{mag}}^{mn}$  ( $m, n = 1, \dots, 5$  are  $SO(5)$  indices) for black holes and their magnetic duals, and by  $Z_{\text{el}}^m$  and  $Z_{\text{mag}}^m$  for black strings and their magnetic duals.

### 7.2 $\mathcal{N} = 2, D = 7$ Supergravity

The global symmetry group of maximally supersymmetric  $D = 7$  supergravity [30] is  $SL(5, \mathbb{R})$ . The bosonic field content comprises the graviton, 14 scalars, 10 vectors and 5 two-form fields. Under the  $U$ -duality group  $SL(5, \mathbb{R})$  these organize as

$$\begin{aligned} \mathcal{V}_{m_i}^i &: \frac{SL(5, \mathbb{R})}{SO(5)} & i, m = 1, \dots, 5, \\ F_2^{[ij]} &: \mathbf{10}', \\ H_{3i} &: \mathbf{5}. \end{aligned} \quad (7.2)$$

The corresponding charges will be denoted by  $p^{ij}, q_{ij}, p_i, q^i$ . For near-horizon geometries  $AdS_2 \times S^3 \times T^2$  and  $AdS_3 \times S^2 \times T^2$ , there are two independent electric and magnetic three-cycles, respectively, depending on which of the two circles of  $T^2 = S_1^1 \times S_2^1$  is part of the cycle. The corresponding three-charges will be denoted by  $p_{ia}, q_r^i$  with  $a, r = 1, 2$ .

### 7.3 $AdS_3 \times S^3 \times S^1$

We start from the ansatz

$$\begin{aligned} ds^2 &= r_{\text{AdS}}^2 ds_{\text{AdS}_3}^2 + r_S^2 ds_{S^3}^2 + r_1^2 d\theta_1^2 , \\ H_{3i} &= e_i \beta_{\text{AdS}_3} + p_i \alpha_{S^3} , \end{aligned} \tag{7.3}$$

for the near-horizon geometry. The central charges and the relevant quadratic U-duality invariant are given by

$$\begin{aligned} Z_{\text{mag},m} &= J_3 \mathcal{V}^j{}_m p_j , & Z_{\text{el},m} &= J'_3 (\mathcal{V}^{-1})^m{}_i q^i , \\ \mathcal{I}_2 &= q^i p_i = Z_{\text{mag},i} Z_{\text{el}}^i (J_3 J'_3)^{-1} , \end{aligned} \tag{7.4}$$

with  $J_3 = v_T J'_3 = \left( \frac{v_{\text{AdS}_3} v_T}{v_{S^3}} \right)^{\frac{1}{2}}$ . The effective potential can be written as

$$V_{\text{eff}} = \frac{1}{2} Z_{\text{mag}}^m Z_{\text{mag},m} + \frac{1}{2} Z_{\text{el},m} Z_{\text{el},m} . \tag{7.5}$$

Using the Maurer-Cartan equations, we obtain

$$\nabla Z_{\text{mag}}^m = Z_{\text{mag}}^n P_{mn} , \quad \nabla Z_{\text{el},m} = -Z_{\text{el},n} P_{mn} , \tag{7.6}$$

with  $P_{mn}$  a symmetric and traceless matrix ( $P_{mm} = 0$ ). Here, indices  $m, n$  are raised and lowered with  $\delta_{mn}$ . For the variation of the effective potential we thus obtain

$$\nabla V_{\text{eff}} = \left( Z_{\text{mag}}^m Z_{\text{mag}}^n - Z_{\text{el},m} Z_{\text{el},n} \right) P_{mn} \stackrel{!}{=} 0 , \tag{7.7}$$

Equation (7.7) is solved by

$$Z_{\text{mag}}^m = \pm Z_{\text{el},m} . \tag{7.8}$$

In this case we find

$$V_{\text{eff}} \Big|_{\nabla V_{\text{eff}}=0} = J_3 J'_3 |\mathcal{I}_2| = \frac{v_{\text{AdS}_3}}{v_{S^3}} |\mathcal{I}_2| , \tag{7.9}$$

in agreement with (4.1). Extremization of  $F$  w.r.t. the radii yields the black string central charge and near-horizon geometry (4.1) with  $\mathbf{I}_2 = \mathcal{I}_2$  the  $SL(5, \mathbb{R})$  quadratic invariant. Notice that this solution can be thought of as the  $D = 7$  lift of the  $AdS_3 \times S^3$  solution studied in the last section. The radius  $r_1$  of the additional  $S^1$  is not fixed by the attractor equations.

Finally let us consider the moduli space of this black string solution. For this purpose we notice that upon  $SO(5)$  rotation the solution can be written in the form

$$Z_{\text{mag}}^m = \pm Z_{\text{el},m} = z\delta_{m1} . \quad (7.10)$$

This form is clearly invariant under  $SL(4, \mathbb{R})$  rotations. The moduli space can then be written as

$$\mathcal{M}_{\text{BPS}} = \frac{SL(4, \mathbb{R})}{SO(4)} . \quad (7.11)$$

Alternatively, evaluating the Hessian at the solution one finds

$$\nabla\nabla V_{\text{eff}} = 4z^2 P_{1n} P_{1n} \quad (7.12)$$

a matrix with 5 strictly positive eigenvalues and 9 zeros. More precisely the 14 scalars in the symmetric traceless **14** of  $SL(5, \mathbb{R})$  decompose under  $SO(4) \sim SU(2) \times SU(2)$  into the following representations

$$\mathbf{14} \longrightarrow \underbrace{(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1})}_{m^2 > 0} \oplus \underbrace{(\mathbf{3}, \mathbf{3})}_{m^2 = 0} . \quad (7.13)$$

Notice that these 9 moduli, together with the free radius  $r_1$  and the 10 degrees of freedom associated to Wilson lines of the 10 vector fields along  $S^1$  sum up to 20 free parameters characterizing the solution. This precisely matches the dimension of the moduli space of the six-dimensional solution of which the present solution is a lift. In other words, in going from six to seven dimensions, an 11-dimensional part of the moduli space translates into “geometrical moduli” describing the circle radius and Wilson lines. This will be always the case for all  $D > 6$  solutions under consideration here.

It is worth noticing that the solution with  $\mathcal{I}_2 \neq 0$  can be considered as an intersection of one  $\frac{1}{2}$ -BPS electric string and one  $\frac{1}{2}$ -BPS magnetic black-3-brane, respectively in the **5'** and **5** of the  $D = 7$   $U$ -duality group  $SL(5, \mathbb{R})$  [27]. The corresponding supporting charge orbit is  $\frac{SL(5, \mathbb{R})}{SL(4, \mathbb{R}) \times_s \mathbb{R}^4}$  [28], but the common stabilizer of the two charge vectors is  $SL(4, \mathbb{R})$  only, with resulting moduli space of the considered intersecting configuration given by Eq. (7.11).

## 7.4 $AdS_3 \times S^2 \times T^2$

We start from the near-horizon ansatz:

$$\begin{aligned} ds^2 &= r_{\text{AdS}}^2 ds_{\text{AdS}_3}^2 + r_S^2 ds_{S^2}^2 + r_1^2 d\theta_1^2 + r_2^2 d\theta_2^2 , \\ F_2^{ij} &= p^{ij} \alpha_{S^2} , \quad H_{3i} = e_i \beta_{\text{AdS}_3} + \sum_{a=1,2} p_{ia} \alpha_{S^2 \times S_a^1} , \end{aligned} \quad (7.14)$$

where  $T^2 = S^1 \times S^1$ . In particular, in this case there are two magnetic three cycles  $S^2 \times S_a^1$  which we label by  $a = 1, 2$ . The corresponding central charges are given by

$$\begin{aligned} Z_{\text{mag}, mn} &= J_2 (\mathcal{V}^{-1})^m{}_i (\mathcal{V}^{-1})^n{}_j p^{ij} , \\ Z_{\text{mag}, m}^a &= J_{3b} \delta^{ab} \mathcal{V}^j{}_m p_{jb} \quad Z_{\text{el}, m} = J'_3 (\mathcal{V}^{-1})^m{}_i q^i , \end{aligned} \quad (7.15)$$

with  $J_2 = \left( \frac{v_{\text{AdS}_3} v_{T^2}}{v_{S^2}} \right)^{\frac{1}{2}}$ ,  $J'_3 = \left( \frac{v_{\text{AdS}_3}}{v_{S^2} v_{T^2}} \right)^{\frac{1}{2}}$ ,  $J_{3a} = \left( \frac{v_{\text{AdS}_3} v_{T^2}}{v_{S^2} (v_{S_a^1})^2} \right)^{\frac{1}{2}}$ .

In terms of these charges one can built two U-duality invariants

$$\mathcal{I}_3 = \frac{1}{8} \epsilon_{ijklm} p^{ij} p^{kl} q^m , \quad \tilde{\mathcal{I}}_3 = \frac{1}{2} p_{ia} p_{jb} p^{ij} \epsilon^{ab} . \quad (7.16)$$

Note that the existence of  $\tilde{\mathcal{I}}_3$  hinges on the fact that there are two inequivalent magnetic three-cycles. In terms of the central charges (7.15) the invariants can be written as

$$\frac{1}{8} \epsilon_{ijklm} Z_{\text{mag}}^{ij} Z_{\text{mag}}^{kl} Z_{\text{el}}^m = J_2^2 J'_3 \mathcal{I}_3 , \quad Z_{\text{mag}, ia} Z_{\text{mag}, jb} Z_{\text{mag}}^{ij} \epsilon^{ab} = J_2 J_{3a} J_{3b} \epsilon^{ab} \tilde{\mathcal{I}}_3 . \quad (7.17)$$

The effective potential is now given by

$$V_{\text{eff}} = \frac{1}{4} Z_{\text{mag}, mn} Z_{\text{mag}, mn} + \frac{1}{2} Z_{\text{mag}}^{ma} Z_{\text{mag}}^{ma} + \frac{1}{2} Z_{\text{el}, m} Z_{\text{el}, m} . \quad (7.18)$$

The Maurer-Cartan equations give  $\nabla \mathcal{V}^i{}_m = \mathcal{V}^i{}_n P_{mn}$  with  $P_{mn}$  symmetric and traceless. The variation of the central charges is thus given by

$$\nabla Z_{\text{mag}, mn} = 2 Z_{\text{mag}, k[m} P_{n]k} , \quad \nabla Z_{\text{mag}}^{ma} = Z_{\text{mag}}^{na} P_{mn} , \quad \nabla Z_{\text{el}, m} = - Z_{\text{el}, n} P_{mn} .$$

Hence, we obtain

$$\nabla V_{\text{eff}} = \left( - Z_{\text{mag}, mk} Z_{\text{mag}, nk} + Z_{\text{mag}}^{ma} Z_{\text{mag}}^{na} - Z_{\text{el}, m} Z_{\text{el}, n} \right) P_{mn} \stackrel{!}{=} 0 , \quad (7.19)$$

By  $SO(5)$  rotation, the antisymmetric matrix  $Z_{\text{mag}, mn}$  can be brought into skew-diagonal form

$$Z_{\text{mag}, mn} = 2z_1 \delta_{1[m} \delta_{n]2} + 2z_2 \delta_{3[m} \delta_{n]4} . \quad (7.20)$$

Plugging this into the attractor equation (7.19) one finds the following solutions

$$\mathbf{A)} \quad Z_{\text{mag}, mn} = 2z (\delta_{1[m} \delta_{n]2} + \delta_{3[m} \delta_{n]4}) , \quad Z_{\text{el}, m} = z \delta_{m5} , \quad Z_{\text{mag}}^{ma} = 0 . \quad (7.21)$$

$$\mathbf{B)} \quad Z_{\text{mag}, mn} = 2z \delta_{1[m} \delta_{n]2} , \quad Z_{\text{el}, m} = 0 , \quad Z_{\text{mag}, m}^a = z \delta_m^a . \quad (7.22)$$

The corresponding effective potentials are given by

$$\begin{aligned}
V_{\text{eff},A} &= \frac{3}{2}z^2 = \frac{3}{2}(J_2^2 J_3 |\mathcal{I}_3|)^{2/3} = \frac{3}{2} \frac{v_{\text{AdS}_3}^3 v_{T^2}^{1/3}}{v_{S^2}} |\mathcal{I}_3|^{2/3}, \\
V_{\text{eff},B} &= \frac{3}{2}z^2 = \frac{3}{2}(J_2 J_{3,1} J_{3,2} |\tilde{\mathcal{I}}_3|)^{2/3} = \frac{3}{2} \frac{v_{\text{AdS}_3}^3 v_{T^2}^{1/3}}{v_{S^2}} |\tilde{\mathcal{I}}_3|^{2/3},
\end{aligned} \tag{7.23}$$

in agreement with (4.2) with  $\mathbf{I}_3 = \mathcal{I}_3$  and  $\mathbf{I}_3 = \tilde{\mathcal{I}}_3$  for the solutions **A** and **B**, respectively. After the  $\vec{r}$ -extremization one finds again that the AdS and sphere radii and the entropy function are given by the general formula (4.2). Again, the radii  $r_a$  of the two circles  $S_a^1$  are not fixed by the extremization equations.

Let us finally consider the associated moduli spaces. We start with solution **A**. The symmetry of (7.21) is  $Sp(4, \mathbb{R}) \sim SO(3, 2)$ . The moduli space is the quotient of this group by its maximal compact subgroup

$$\mathcal{M}_{\text{BPS},A} = \frac{SO(3, 2)}{SO(3) \times SO(2)}. \tag{7.24}$$

More precisely, in terms of  $SO(3) \times SO(2)$  representations one finds that the 14 scalar components decompose according to

$$P_{mn} : \quad \mathbf{14} \longrightarrow \underbrace{\mathbf{3}_+ \oplus \mathbf{3}_-}_{m^2=0} \oplus \underbrace{\mathbf{1}_{+2} \oplus \mathbf{1}_{-2} \oplus \mathbf{1}_0 \oplus \mathbf{5}_0}_{m^2>0}, \tag{7.25}$$

subscripts referring to  $SO(2)$  charges. Indeed, evaluating the Hessian

$$\begin{aligned}
\nabla\nabla V_{\text{eff}} &= 2 P_{mp} P_{nq} (Z_{\text{mag},mp} Z_{\text{mag},nq} + Z_{\text{mag},mk} Z_{\text{mag},nk} \delta_{pq} + Z_{\text{mag}}^{ma} Z_{\text{mag}}^{na} \delta_{pq} \\
&\quad + Z_{\text{el},m} Z_{\text{el},n} \delta_{pq}),
\end{aligned}$$

at the solution (7.21) one finds a matrix with 6 vanishing and 8 strictly positive eigenvalues.

For solution **B** one finds as a symmetry  $SL(3, \mathbb{R})$  and thus the moduli space

$$\mathcal{M}_{\text{BPS},B} = \frac{SL(3, \mathbb{R})}{SO(3)}. \tag{7.26}$$

The scalars decompose into  $SO(3)$  representations according to

$$P_{mn} : \quad \mathbf{14} \longrightarrow \underbrace{\mathbf{5}}_{m^2=0} \oplus \underbrace{\mathbf{2} \cdot \mathbf{3} \oplus \mathbf{3} \cdot \mathbf{1}}_{m^2>0}. \tag{7.27}$$

Indeed from the Hessian

$$\nabla\nabla V_{\text{eff}}|_{\mathbf{B}} = 6z^2 (P_{1p}P_{1p} + P_{2p}P_{2p}), \quad (7.28)$$

one finds a matrix with 9 strictly positive and 5 vanishing eigenvalues.

As mentioned above, at  $D = 7$  the charge orbit supporting one  $\frac{1}{2}$ -BPS black string (or black-2-brane) is given by  $\frac{SL(5, \mathbb{R})}{SL(4, \mathbb{R}) \times_s \mathbb{R}^4}$  [28]. On the other hand, the charge orbit supporting one black hole (or black-3-brane) is  $\frac{SL(5, \mathbb{R})}{SL(3, \mathbb{R}) \times SL(2, \mathbb{R}) \times_s \mathbb{R}^6}$  and  $\frac{SL(5, \mathbb{R})}{SO(3, 2) \times_s \mathbb{R}^4}$  in the  $\frac{1}{2}$ -BPS and  $\frac{1}{4}$ -BPS cases, respectively [28].

Solution A corresponds to an intersection of one  $\frac{1}{4}$ -BPS black-3-brane (with charges in the  $\mathbf{10}'$  of  $SL(5, \mathbb{R})$ ) with one  $\frac{1}{2}$ -BPS black string (with charges on the  $\mathbf{5}'$  of  $SL(5, \mathbb{R})$ ). The stabilizer of both charge vectors is  $SO(3, 2)$  only, and thus the resulting moduli space of the considered intersecting configuration is given by Eq. (7.24).

Solution B corresponds to an intersection of one  $\frac{1}{2}$ -BPS black-3-brane with two parallel  $\frac{1}{2}$ -BPS black 2-branes (with charges on two different  $\mathbf{5}$ s of  $SL(5, \mathbb{R})$ ). Accordingly, the stabilizer of the three charge vectors is  $SL(3, \mathbb{R})$  only, and thus the resulting moduli space of the considered intersecting configuration is given by Eq. (7.26).

## 7.5 $AdS_2 \times S^3 \times T^2$

The analysis of the black hole solutions is very similar to the previous one of the black strings replacing electric with magnetic charges and vice versa. Now we start from the near-horizon ansatz:

$$\begin{aligned} ds^2 &= r_{\text{AdS}}^2 ds_{\text{AdS}_2}^2 + r_S^2 ds_{S^3}^2 + r_1^2 d\theta_1^2 + r_2^2 d\theta_2^2, \\ F_2^{ij} &= e^{ij} \beta_{\text{AdS}_2}, \quad H_{3i} = p_i \alpha_{S^3} + \sum_{r=1,2} e_i^r \beta_{\text{AdS}_2 \times S^3_r}, \end{aligned} \quad (7.29)$$

where  $r = 1, 2$  labels the two inequivalent electric three-cycles and  $T^2 = S^1_1 \times S^1_2$ . The central charges and U-duality invariants are given by

$$\begin{aligned} Z_{\text{el}, mn} &= J'_2 \mathcal{V}^i_m \mathcal{V}^j_n q_{ij}, \\ Z_{\text{el}, m}^r &= J_{3s}^r \delta^{rs} (\mathcal{V}^{-1})^m_j q_s^j, \quad Z_{\text{mag}, m} = J_3 \mathcal{V}^i_m p_i, \\ \mathcal{I}'_3 &= \frac{1}{8} \epsilon^{ijklm} q_{ij} q_{kl} p_m = \frac{1}{8} \epsilon^{ijklm} Z_{\text{el}, ij} Z_{\text{el}, kl} Z_{\text{mag}, m} (J_2'^2 J_3)^{-1}, \\ \tilde{\mathcal{I}}'_3 &= \frac{1}{2} q_r^i q_s^j q_{ij} \epsilon^{rs} = \frac{1}{2} Z_{\text{el}, r}^i Z_{\text{el}, s}^j Z_{\text{el}, ij} \epsilon^{rs} (J_2' J_{3,1}' J_{3,2}')^{-1}, \end{aligned} \quad (7.30)$$

with  $J'_2 = \left( \frac{v_{\text{AdS}_2}}{v_T v_{S^3}} \right)^{\frac{1}{2}}$ ,  $J_3 = \left( \frac{v_T v_{\text{AdS}_2}}{v_{S^3}} \right)^{\frac{1}{2}}$ ,  $J_{3s} = \left( \frac{v_{\text{AdS}_2} (v_{S^3}^1)^2}{v_{S^3} v_{T^2}} \right)^{\frac{1}{2}}$ .

The two solutions of the associated attractor equations are given by

$$\mathbf{A)} \quad Z_{\text{el},mn} = 2z(\delta_{1[m}\delta_{n]2} + \delta_{3[m}\delta_{n]4}) , \quad Z_{\text{mag},m} = z\delta_{m5} , \quad Z_{\text{el},m}^a = 0 . \quad (7.31)$$

$$\mathbf{B)} \quad Z_{\text{el},mn} = 2z\delta_{1[m}\delta_{n]2} , \quad Z_{\text{mag},m} = 0 , \quad Z_{\text{el},m}^a = z\delta_m^a . \quad (7.32)$$

The effective potentials, entropy function and near-horizon geometry are given by the general formula (4.3) with  $\mathbf{I}_3$  given by  $\mathcal{I}'_3$  and  $\tilde{\mathcal{I}}'_3$  for the cases  $\mathbf{A}$  and  $\mathbf{B}$ , respectively. The analysis of the moduli spaces is identical to that of the  $AdS_3$  cases and the results are again given by (7.24) and (7.26), respectively.

Solution A has  $\tilde{\mathcal{I}}'_3 = 0$ , which comes from  $q_{ia}q_{jb}\epsilon^{ab} = 0$ , meaning that the two  $\mathbf{5}$ 's are reciprocally parallel. On the other hand, solution B has  $\mathcal{I}'_3 = 0$ ; this derives from the condition  $\epsilon^{ijklm}q_{ij}q_{kl} = 0$  for a  $D = 7$  black hole to be  $\frac{1}{2}$ -BPS [27].

## 8 Maximal $D = 8$

### 8.1 $\mathcal{N} = 2, D = 8$ Supersymmetry Algebra

The maximal  $\mathcal{N} = 2, D = 8$  Poincaré supersymmetry algebra has complex chiral supercharges (as in  $D = 4$ ) and  $\mathcal{R}$ -symmetry  $SU(2) \times U(1) = Spin(3) \times Spin(2)$ . Its central extension reads as follows (see *e.g.* [19, 20, 21])

$$\{\mathcal{Q}_\gamma^A, \mathcal{Q}_\delta^B\} = C_{\gamma\delta}Z^{(AB)} + \gamma_{\gamma\delta}^{\mu\nu}Z_{\mu\nu}^{[AB]} + \gamma_{\gamma\delta}^{\mu\nu\rho\lambda}Z_{\mu\nu\rho\lambda}^{(AB)} \quad (\text{and h.c.}) \quad (8.1)$$

$$\{\mathcal{Q}_\gamma^A, \mathcal{Q}_{\delta|B}\} = \gamma_{\gamma\delta}^\mu Z_{\mu|B}^A + \gamma_{\gamma\delta}^{\mu\nu\rho} Z_{\mu\nu\rho|B}^A, \quad (8.2)$$

where  $A, B = 1, 2$ , so that the supercharges are  $SU(2)$ -doublets. The trace part of  $Z_{\mu|B}^A$  is the momentum  $P_\mu\delta_B^A$ .

Black holes and their magnetic dual (black 4-brane) central extensions  $Z^{(AB)}, Z_{\mu\nu\rho\lambda}^{(AB)}$  sit in the  $(\mathbf{3}, \mathbf{2})$  (and  $(\mathbf{3}', \mathbf{2})$ ) of  $SU(2) \times U(1)$ , and they are embedded in the  $(\mathbf{3}, \mathbf{2})$  of the  $U$ -duality group  $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$ .

On the other hand, dyonic black membrane central extensions  $Z_{\mu\nu}^{[AB]}$  are in the  $(\mathbf{1}, \mathbf{2})$  of  $SU(2) \times U(1)$ , and they are embedded in the  $(\mathbf{1}, \mathbf{2})$  of  $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$ .

Black strings and their magnetic dual (black 3-brane) central extensions  $Z_{\mu|B}^A, Z_{\mu\nu\rho|B}^A$  sit in the  $(\mathbf{3}', \mathbf{1})$  (and  $(\mathbf{3}, \mathbf{1})$ ) of  $SU(2) \times U(1)$  (namely in the adjoint of  $SU(2)$ , and they do not carry  $U(1)$  charge, because they are real), and they are embedded in the  $(\mathbf{3}, \mathbf{1})$  of the  $U$ -duality group  $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$ .

In our analysis, the corresponding central charges are denoted by  $Z_{\text{el},iA}$  and  $Z_{\text{mag},iA}$  ( $i = 1, 2, 3$  and  $A = 1, 2$ ) for black holes and their magnetic duals, by  $Z_{\text{el},i}$  and  $Z_{\text{mag},i}$  for black strings and their magnetic duals, and by  $Z_A$  for dyonic black 2-branes.

## 8.2 $\mathcal{N} = 2, D = 8$ Supergravity

The bosonic field content of  $D = 8$  supergravity [31] with maximal supersymmetry includes beside the graviton, scalars in the symmetric manifold

$$\mathcal{V}_i^m, \mathcal{V}_A^B : \quad \frac{SL(3, \mathbb{R})}{SO(3)} \times \frac{SL(2, \mathbb{R})}{SO(2)}, \quad (8.3)$$

with  $i, m = 1, \dots, 3, A = 1, 2$ , and forms in the following representations of the  $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$  U-duality group:

$$\begin{aligned} F_2^{iA} &: \quad (\mathbf{3}, \mathbf{2}), \\ H_{3i} &: \quad (\mathbf{3}', \mathbf{1}), \\ F_4^A &: \quad (\mathbf{1}, \mathbf{2}). \end{aligned} \quad (8.4)$$

The Lagrangian carries only one three-form potential  $C$ , whose field strength  $F_4$  together with its magnetic dual spans the  $SL(2, \mathbb{R})$  doublet  $F_4^A$ .

The scalar vielbeins  $\mathcal{V}_i^m, \mathcal{V}_A^B$ , corresponding to the two factors, vary as

$$\nabla \mathcal{V}_i^m = \mathcal{V}_i^n P_{mn}, \quad \nabla \mathcal{V}_A^B = \mathcal{V}_A^C P_{BC}, \quad (8.5)$$

with  $P_{mn}$  and  $P_{AB}$ , symmetric and traceless. Here we raise and lower indices  $m$  and  $A$  with  $\delta_{mn}$  and  $\delta_{BA}$ , respectively.

## 8.3 $AdS_3 \times S^3 \times T^2$

We start with the  $AdS_3 \times S^3 \times T^2$  near-horizon ansatz:

$$\begin{aligned} ds^2 &= r_{\text{AdS}}^2 ds_{\text{AdS}_3}^2 + r_S^2 ds_{S^3}^2 + \sum_{s=1,2} r_s^2 d\theta_s^2, \quad F_{2,iA} = q_{iA} \alpha_{T^2}, \\ H_{3i} &= p_i \alpha_{S^3} + e_i \beta_{\text{AdS}_3}, \quad F_4 = \sum_{r=1,2} e_r \beta_{\text{AdS}_3 \times S^3_r} + \sum_{a=1,2} p_a \alpha_{S^3 \times S^3_a}. \end{aligned} \quad (8.6)$$

Depending on the choice of the circle within  $T^2 = S_1^1 \times S_2^1$ , there are two inequivalent electric and magnetic four-cycles. The four-form charges combine into the  $SL(2, \mathbb{R})$  doublet  $Q_{Ar} = (q_r, p_r)$ .

The central charges are given by

$$Z_{\text{mag},m} = J_3 (\mathcal{V}^{-1})_m^j p_j, \quad Z_{\text{el}}^m = J'_3 \mathcal{V}_i^m q^i, \quad Z_A^r = J_{4s} \delta^{sr} (\mathcal{V}^{-1})_A^B Q_{Bs}, \quad (8.7)$$

with  $J_3 = \left( \frac{v_{\text{AdS}_3} v_{T^2}}{v_{S^3}^2} \right)^{\frac{1}{2}}$ ,  $J'_3 = \left( \frac{v_{\text{AdS}_3}}{v_{S^3} v_{T^2}} \right)^{\frac{1}{2}}$ ,  $J_{4s} = \left( \frac{v_{\text{AdS}_3} v_{T^2}}{v_{S^3} (v_{S^1})^2} \right)^{\frac{1}{2}}$ .

The U-duality invariants that can be built with these set of charges are

$$\begin{aligned} \mathcal{I}_2 &= p_i q^i = Z_{\text{el},i} Z_{\text{mag},i} (J_3 J'_3)^{-1}, \\ \tilde{\mathcal{I}}_2 &= q_{Ar} q_{Bs} \epsilon^{AB} \epsilon^{rs} = Z_{\text{el},Ar} Z_{\text{el},Bs} \epsilon^{AB} \epsilon^{rs} (J_{4,1} J_{4,2})^{-1}. \end{aligned} \quad (8.8)$$

Note that the existence of two inequivalent electric four-cycles is crucial for the existence of  $\tilde{\mathcal{I}}_2$ . The effective potential can be written as

$$V_{\text{eff}} = \frac{1}{2} Z_{\text{mag},m} Z_{\text{mag},m} + \frac{1}{2} Z_{\text{el}}^m Z_{\text{el}}^m + \frac{1}{2} Z_{Ar} Z_{Ar}, \quad (8.9)$$

and the attractor equations take the form

$$\left( Z_{\text{mag},m} Z_{\text{mag},n} - Z_{\text{el}}^m Z_{\text{el}}^n \right) P_{mn} + \left( Z_{Ar} Z_{Br} \right) P_{AB} \stackrel{!}{=} 0. \quad (8.10)$$

We will consider the following two solutions to these equations

$$\mathbf{A)} \quad Z_{\text{mag},m} = \pm Z_{\text{el}}^m = z \delta_{m1}, \quad Z_{Ar} = 0. \quad (8.11)$$

$$\mathbf{B)} \quad Z_{\text{mag},m} = Z_{\text{el}}^m = 0, \quad Z_{Ar} = z \delta_{Ar}. \quad (8.12)$$

The effective potentials at the horizon become

$$\begin{aligned} V_{\text{eff}}|_{\mathbf{A}} &= z^2 = J_3 J'_3 |\mathcal{I}_2| = \frac{v_{\text{AdS}_3}}{v_{S^3}} |\mathcal{I}_2|, \\ V_{\text{eff}}|_{\mathbf{B}} &= z^2 = J_{4,1} J_{4,2} |\tilde{\mathcal{I}}_2| = \frac{v_{\text{AdS}_3}}{v_{S^3}} |\tilde{\mathcal{I}}_2|, \end{aligned} \quad (8.13)$$

respectively. Plugging this into the entropy function and extremizing with respect to the radii one recovers the near-horizon geometry central charge (4.1) with  $\mathbf{I}_2$  taken as  $\mathcal{I}_2$  or  $\tilde{\mathcal{I}}_2$  for the solution  $\mathbf{A}$  and  $\mathbf{B}$ , respectively.

Let us consider the associated moduli spaces. The symmetry groups leaving (8.11) and (8.12) invariant are  $SL(2, \mathbb{R})^2$  and  $SL(3, \mathbb{R})$ , respectively. The moduli spaces are thus

$$\mathcal{M}_{\text{BPS},A} = \left( \frac{SL(2, \mathbb{R})}{SO(2)} \right)^2, \quad \mathcal{M}_{\text{BPS},B} = \frac{SL(3, \mathbb{R})}{SO(3)}. \quad (8.14)$$

The same results follow from evaluating the Hessians at the solutions

$$\nabla\nabla V_{\text{eff},A} = 2z^2 P_{1m}^2, \quad \nabla\nabla V_{\text{eff},B} = 2z^2 P_{AA}^2, \quad (8.15)$$

which shows that one has 3(2) strictly positive and 4(5) vanishing eigenvalues for the solution  $\mathbf{A}(\mathbf{B})$ , in agreement with the dimensions of the moduli spaces (8.14).

At  $D = 8$  there are two dyonic  $\frac{1}{2}$ -BPS black-2-brane, whose charge orbits are  $\frac{SL(3, \mathbb{R}) \times SL(2, \mathbb{R})}{SL(3, \mathbb{R}) \times \mathbb{R}}$ . The  $\frac{1}{2}$ -BPS black strings (and their dual black 3-branes) are in the  $(\mathbf{3}', \mathbf{1})$  and  $(\mathbf{3}, \mathbf{1})$  of the  $D = 8$   $U$ -duality group  $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$ , and their individual charge orbit is  $\frac{SL(3, \mathbb{R}) \times SL(2, \mathbb{R})}{(SL(2, \mathbb{R}) \times_s \mathbb{R}^2) \times SL(2, \mathbb{R})}$  [28]. The black holes (and their dual black 4-branes) are in the  $(\mathbf{3}, \mathbf{2})$  and  $(\mathbf{3}', \mathbf{2})$  of  $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$ , and their individual charge orbit is  $\frac{SL(3, \mathbb{R}) \times SL(2, \mathbb{R})}{SL(2, \mathbb{R}) \times_s \mathbb{R}^2}$  and  $\frac{SL(3, \mathbb{R}) \times SL(2, \mathbb{R})}{GL(2, \mathbb{R}) \times_s \mathbb{R}^3}$  for the  $\frac{1}{4}$ -BPS and  $\frac{1}{2}$ -BPS cases, respectively [28].

In the considered  $AdS_3 \times S^3 \times T^2$  near-horizon geometry, solution A corresponds to the intersection of one  $\frac{1}{2}$ -BPS black string and one  $\frac{1}{2}$ -BPS black 3-brane. The stabilizer of both charge vectors is  $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$  only, and thus the resulting moduli space of the considered intersecting configuration is  $\left( \frac{SL(2, \mathbb{R})}{SO(2)} \right)^2$ , given in the left hand side of Eq. (8.14).

On the other hand, solution B corresponds to the intersection of two dyonic  $\frac{1}{2}$ -BPS black 2-branes. Thus the stabilizer of both charge vectors is  $SL(3, \mathbb{R})$  only, and the resulting moduli space of the considered intersecting configuration is  $\frac{SL(3, \mathbb{R})}{SO(3)}$ , given in the right-hand side of Eq. (8.14).

## 8.4 $AdS_3 \times S^2 \times T^3$

The near-horizon ansatz is given by

$$\begin{aligned} ds^2 &= r_{\text{AdS}}^2 ds_{\text{AdS}_3}^2 + r_S^2 ds_{S^2}^2 + \sum_{s=1,2,3} r_s^2 d\theta_s^2, \\ F_2^{iA} &= p^{iA} \alpha_{S^2}, \quad H_{3i} = \sum_{a=1,2,3} p_i^a \alpha_{S^2 \times S_a^1}, \\ F_4^A &= \sum_{a=1,2,3} e_a^A \beta_{\text{AdS}_3 \times S_a^1} + \sum_{a=1,2,3} \frac{1}{2} |\epsilon_{abc}| p_a^A \alpha_{(S^2 \times S_b^1 \times S_c^1)}, \end{aligned} \quad (8.16)$$

with  $T^3 = S_1^1 \times S_2^1 \times S_3^1$ . In this near-horizon geometry there are thus three inequivalent electric and magnetic four-cycles and three magnetic three-cycles. The four-form charges again combine into the  $SL(2, \mathbb{R})$  doublet  $Q_{Aa} = (q_a, p_a)$ .

The central charges take the form

$$\begin{aligned} Z_{\text{mag}}^{mA} &= J_2 \mathcal{V}_B{}^A \mathcal{V}_k{}^m p^{kB}, & Z_{\text{mag},m}^a &= J_{3b} \delta^{ba} (\mathcal{V}^{-1})_m{}^j p_j^b, \\ Z_{Aa} &= J_{4b} \delta_{ab} (\mathcal{V}^{-1})_A{}^B Q_{Bb}, \end{aligned} \quad (8.17)$$

$$\text{with } J_2 = \left( \frac{v_{\text{AdS}_3} v_{T^3}}{v_{S^2}} \right)^{\frac{1}{2}}, \quad J_{3a} = \left( \frac{v_{\text{AdS}_3} v_{T^3}}{v_{S^2} (v_{S_a^1})^2} \right)^{\frac{1}{2}}, \quad J_{4a} = \left( \frac{v_{\text{AdS}_3} (v_{S_a^1})^2}{v_{S^2} v_{T^3}} \right)^{\frac{1}{2}}.$$

The U-duality invariants are given by

$$\begin{aligned} \mathcal{I}_3 &= p^{iA} p_i^a Q_{Aa} = Z_{\text{mag}}^{iA} Z_{\text{mag},ia} Z_{Aa} (J_2 J_{3a} J_{4a})^{-1}, \\ \tilde{\mathcal{I}}_3 &= \frac{1}{3} p_i^a p_j^b p_k^c \epsilon^{ijk} \epsilon_{abc} = \frac{1}{3} Z_{\text{mag},i}^a Z_{\text{mag},j}^b Z_{\text{mag},k}^c \epsilon^{ijk} \epsilon_{abc} (J_{3,1} J_{3,2} J_{3,3})^{-1}. \end{aligned} \quad (8.18)$$

From variation of the effective potential

$$V_{\text{eff}} = \frac{1}{2} Z_{\text{mag}}^{mA} Z_{\text{mag}}^{mA} + \frac{1}{2} Z_{\text{mag},m}^a Z_{\text{mag},m}^a + \frac{1}{2} Z_{Ar} Z_{Ar}, \quad (8.19)$$

we thus obtain the attractor equations

$$\begin{aligned} \left( Z_{\text{mag}}^{mA} Z_{\text{mag}}^{nA} - Z_{\text{mag},m}^a Z_{\text{mag},n}^a \right) P_{mn} &\stackrel{!}{=} 0, \\ \left( Z_{\text{mag}}^{mA} Z_{\text{mag}}^{mB} - Z_{Ar} Z_{Br} \right) P_{AB} &\stackrel{!}{=} 0, \end{aligned} \quad (8.20)$$

with  $P_{mn}$  and  $P_{AB}$  symmetric and traceless.

We will consider the solutions

$$\mathbf{A)} \quad Z_{\text{mag}}^{mA} = 0 = Z_{Aa}, \quad Z_{\text{mag},m}^a = z \delta_m^a. \quad (8.21)$$

$$\mathbf{B)} \quad Z_{\text{mag}}^{mA} = z \delta_{m1} \delta_{A1}, \quad Z_{\text{mag},m}^a = \delta^{a1} \delta_{m1} z, \quad Z_{Ar} = \delta_{r1} \delta_{A1} z. \quad (8.22)$$

The effective potentials become

$$\begin{aligned} V_{\text{eff},A} &= \frac{3}{2} z^2 = \frac{3}{2} (J_2 J_{3a} J_{4a} |\tilde{\mathcal{I}}_3|)^{2/3} = \frac{3 v_{\text{AdS}_3} v_{T^3}^{\frac{1}{3}}}{2 v_{S^2}} |\tilde{\mathcal{I}}_3|^{2/3}, \\ V_{\text{eff},B} &= \frac{3}{2} z^2 = \frac{3}{2} (J_{3,1} J_{3,2} J_{3,3} |\mathcal{I}_3|)^{2/3} = \frac{3 v_{\text{AdS}_3} v_{T^3}^{\frac{1}{3}}}{2 v_{S^2}} |\mathcal{I}_3|^{2/3}, \end{aligned} \quad (8.23)$$

respectively, and  $\vec{r}$ -extremization leads to the near-horizon geometry and central charge (4.2) with  $\mathbf{I}_3 = \tilde{\mathcal{I}}_3$  and  $\mathbf{I}_3 = \mathcal{I}_3$  for the cases **A** and **B**, respectively.

The symmetry group leaving (8.21) and (8.22) invariant, is  $SL(2, \mathbb{R})$  and the moduli space thus given by

$$\mathcal{M}_{\text{BPS},A/B} = \frac{SL(2, \mathbb{R})}{SO(2)}. \quad (8.24)$$

Alternatively the moduli space can be determined from the vanishing eigenvalues of the Hessians

$$\begin{aligned} \nabla\nabla V_{\text{eff},A} &= 2z^2(P_{1m}^2 + P_{2m}^2), \\ \nabla\nabla V_{\text{eff},B} &= 4z^2P_{1n}^2 + 6z^2P_{1A}^2, \end{aligned} \quad (8.25)$$

respectively, showing 5 strictly positive and 2 vanishing eigenvalues in each case.

We now derive the nature of the moduli spaces of solutions A and B from the charge orbits discussed in [28].

Solution A corresponds to an intersection of three black 3-branes, with  $\tilde{\mathcal{I}}_3 = \det(p_i^a) \neq 0$  but  $\mathcal{I}_3 = p^{iA}p_i^a Q_{Aa} = 0$ . The charge orbit for each of them is  $\frac{SL(3, \mathbb{R}) \times SL(2, \mathbb{R})}{(SL(2, \mathbb{R}) \times \mathbb{R}^2) \times SL(2, \mathbb{R})}$ , and the common stabilizer is the  $SL(2, \mathbb{R})$  commuting with  $SL(3, \mathbb{R})$ . This agrees with the moduli space  $\left. \frac{SL(2, \mathbb{R})}{SO(2)} \right|_A$  of solution A (see Eq. (8.24)).

Solution B corresponds to the intersection of three parallel black 2-branes, three parallel black 3-branes and  $\frac{1}{2}$ -BPS black 4-branes, respectively characterized by the constraints

$$Q_{Aa}Q_{Bb}\epsilon^{AB} = 0, \quad p_i^a p_j^b \epsilon^{ijk} = 0, \quad p^{iA} p^{jB} \epsilon_{AB} = 0, \quad (8.26)$$

with  $\mathcal{I}_3 = p^{iA} p_i^a Q_{Aa} \neq 0$  and  $\tilde{\mathcal{I}}_3 = 0$ .

The three parallel 3-branes have a common charge orbit  $\frac{SL(3, \mathbb{R}) \times SL(2, \mathbb{R})}{(SL(2, \mathbb{R}) \times_s \mathbb{R}^2) \times SL(2, \mathbb{R})}$ , whereas the parallel 2-branes have a common charge orbit  $\frac{SL(3, \mathbb{R}) \times SL(2, \mathbb{R})}{SL(3, \mathbb{R}) \times \mathbb{R}^1}$ , and the  $\frac{1}{2}$ -BPS 4-brane has charge orbit  $\frac{SL(3, \mathbb{R}) \times SL(2, \mathbb{R})}{(GL(3, \mathbb{R}) \times_s \mathbb{R}^2) \times \mathbb{R}^1}$  [28].

Since the coset is factorized, the common stabilizer of the three parallel 3-branes and of  $\frac{1}{2}$ -BPS 4-brane is  $SL(2, \mathbb{R})$  inside  $SL(3, \mathbb{R})$ , and this agrees with the moduli space  $\left. \frac{SL(2, \mathbb{R})}{SO(2)} \right|_B$  of solution B (see Eq. (8.24)).

## 8.5 $AdS_2 \times S^3 \times T^3$

This case is very similar to the previous discussion. We start with the near-horizon ansatz

$$\begin{aligned}
ds^2 &= r_{\text{AdS}}^2 ds_{\text{AdS}_2}^2 + r_S^2 ds_{S^3}^2 + \sum_{s=1,2,3} r_s^2 d\theta_s^2, \\
F_2^{iA} &= e^{iA} \beta_{\text{AdS}_2}, \quad H_{3i} = \sum_{a=1,2,3} e_i^r \beta_{\text{AdS}_2 \times S_r^1}, \\
F_4 &= \sum_{a,b,c=1,2,3} \frac{1}{2} |\epsilon_{abc}| e_c^A \beta_{\text{AdS}_2 \times S_a^1 \times S_b^1} + \sum_{a=1,2,3} p_a^A \alpha_{S_3 \times S_a^1}, \tag{8.27}
\end{aligned}$$

with  $T^3 = S_1^1 \times S_2^1 \times S_3^1$ . Again, there are thus three inequivalent electric and magnetic four-cycles. In addition, there are three inequivalent electric three-cycles.

The associated central charges and U-duality invariants are given by

$$\begin{aligned}
Z_{\text{el},mA} &= J'_2 (\mathcal{V}^{-1})_A{}^B (\mathcal{V}^{-1})_m{}^k q_{kB}, \quad Z_{\text{el}}^{mr} = J'_{3s} \delta_{sr} \mathcal{V}_j{}^m q_s^j, \\
Z_{Ar} &= J_{4s} \delta_{rs} (\mathcal{V}^{-1})_A{}^B Q_{Bs}, \\
\mathcal{I}'_3 &= q_{iA} q_a^i Q^{Aa} = Z_{\text{el},iA} Z_{\text{el}}^{ia} Z_a^A (J'_2 J'_{3a} J'_{4a})^{-1}, \\
\tilde{\mathcal{I}}'_3 &= \frac{1}{3} q_a^i q_b^j q_c^k \epsilon_{ijk} \epsilon^{abc} = \frac{1}{3} Z_{\text{el},a}^i Z_{\text{el},b}^j Z_{\text{el},c}^k \epsilon_{ijk} \epsilon^{abc} (J_{3,1} J_{3,2} J_{3,3})^{-1}, \tag{8.28}
\end{aligned}$$

with  $J'_2 = \left( \frac{v_{\text{AdS}_2}}{v_S^2 v_{T^3}} \right)^{\frac{1}{2}}$ ,  $J'_{3r} = \left( \frac{v_{\text{AdS}_2} (v_{S_r^1})^2}{v_{S^3} v_{T^3}} \right)^{\frac{1}{2}}$ ,  $J_{4r} = \left( \frac{v_{\text{AdS}_2} v_{T^3}}{v_{S^3} (v_{S_r^1})^2} \right)^{\frac{1}{2}}$ .

The possible solutions of the attractor equations are

$$\begin{aligned}
\mathbf{A)} \quad & Z_{\text{el},mA} = 0 = Z_{As}, \quad Z_{\text{el}}^{mr} = z \delta^{mr}, \\
\mathbf{B)} \quad & Z_{\text{el},mA} = z \delta_{m1} \delta_{A1} \quad Z_{\text{el}}^{mr} = \delta^{r1} \delta_{m1} z, \quad Z_{Ar} = \delta_{r1} \delta_{A2} z. \tag{8.29}
\end{aligned}$$

The effective potentials, entropy function and near-horizon geometry are given by the general formula (4.3) with  $\mathbf{I}_3 = \mathcal{I}_3$  and  $\mathbf{I}_3 = \tilde{\mathcal{I}}_3$  for the cases **A** and **B** respectively. The analysis of the moduli spaces is identical to that on  $AdS_3 \times S^2$  case and the results are given again by (8.24).

## 9 The Lift to Eleven Dimensions

The attractor solutions we have discussed throughout this paper have a simple lift to eleven-dimensional supergravity. The black string solutions with  $AdS_3 \times S^3 \times T^{D-6}$  near-horizon geometry follow from dimensional reduction of M2M5 branes intersecting on a

string. The supersymmetric solutions with  $AdS_3 \times S^2 \times T^{D-5}$  follow from reductions of triple M2-intersection on a string. Finally  $AdS_2 \times S^3 \times T^{D-5}$  near-horizon geometries correspond to triple M5 intersections on a time-like line. The orientations of the M2,M5 branes in the three cases are summarized in table I.

	0	1	2	3	4	5	6	7	8	9	10	near-horizon
M2	—	—	•	•	•	•	•	•	•	•	—	$AdS_3 \times S^3 \times T^5$
M5	—	—	•	•	•	•	—	—	—	—	•	
M2	—	•	•	•	•	•	•	•	•	—	—	$AdS_2 \times S^3 \times T^6$
M2	—	•	•	•	•	•	•	—	—	•	•	
M2	—	•	•	•	•	—	—	•	•	•	•	
M5	—	—	•	•	•	•	•	—	—	—	—	$AdS_3 \times S^2 \times T^6$
M5	—	—	•	•	•	—	—	•	•	—	—	
M5	—	—	•	•	•	—	—	—	—	•	•	

Table I: Supersymmetric M-intersections

After dimensional reductions down to  $D = 6, 7, 8$ -dimensions the solutions expose a variety of charges with respect to forms of various rank. Indeed, a single brane intersection in  $D = 11$  leads to different solutions after reduction to  $D$ -dimensions depending on the orientation of the M-branes along the internal space. Different solutions carry charges with respect to a different set of forms in the  $D$ -dimensional supergravity. They can be fully characterized by U-duality invariants built out of the brane charges. The list of U-duality invariants leading to extremal black  $p$ -brane solutions in  $D = 6, 7, 8$  dimensions are listed in table II. The reader can easily check that there is a one-to-one correspondence between the entries in this table and the solutions found in the previous sections.

dim	$p = 0$	$p = 1$
5	$q_2 q_2 q_2$	$p_2 p_2 p_2$
6	$q_2 q_2 q_3$	$p_3 q_3, p_2 p_2 p_3$
7	$q_2 q_2 p_3, q_3 q_3 q_2$	$p_3 q_3, p_2 p_2 q_3, p_3 p_3 p_2$
8	$q_2 q_3 p_4, q_3 q_3 q_3$	$p_3 q_3, p_4 q_4, p_2 p_3 q_4, p_3 p_3 p_3$

Table II: Electric and magnetic charges for M-brane intersections.  $p = 0, 1$  corresponds to intersections on a black hole and a black string respectively.  $q_n(p_n)$  denotes the electric(magnetic) charge of the brane solution and  $n$  specifies the rank of the form.

## 10 Final Remarks

In the present paper we analyzed the attractor nature of solutions of some supergravity theories in  $D = 6, 7, 8$ , with static, asymptotically flat, spherically symmetric extremal black- $p$  brane backgrounds and scalar fields turned on.

We have found that for such theories, with the near-horizon geometry containing a factor  $AdS_{p+2}$  ( $p = 0, 1$ ), a generalization of the “entropy function” [10] and “effective potential” [4, 5, 6, 7, 8] formalisms occurs, which allows one to determine the scalar flow and the related moduli space near the horizon. The value of the entropy function at its minimum is given in terms of U-duality invariants built out of the brane charges and it measures the central charges of the dual CFT living on the AdS boundary. The resulting central charges were shown to satisfy a Bekenstein-Hawking like area law generalizing the familiar results of black hole physics.

In order to make further contact with previous work on  $p$ -brane intersections and their supersymmetry-preserving features [3], we have found that for maximal supergravities in  $D$  space-time dimensions, the moduli spaces of attractors with  $AdS_3 \times S^3 \times T^{D-6}$  near-horizon geometries have rank  $10 - D$ . Actually, this holds also for the  $D = 4$  case (with near-horizon geometry  $AdS_2 \times S^2$ ) in the non-BPS configuration, with the related moduli space given by the rank-6 symmetric space  $\frac{E_{6(6)}}{USp(8)}$  [22].

Furthermore, for  $D$ -dimensional maximal supergravities, the moduli spaces of attractors with  $AdS_3 \times S^2 \times T^{D-5}$  (or  $AdS_2 \times S^3 \times T^{D-5}$ ) near-horizon geometries have rank  $9 - D$ . This holds also for the  $D = 5$  case (with near-horizon geometry  $AdS_3 \times S^2$  or  $AdS_2 \times S^3$ ) in the  $\frac{1}{8}$ -BPS configuration, with the related moduli space given by the rank-4 symmetric space  $\frac{F_{4(4)}}{USp(2) \times USp(6)}$  [32, 22].

These results imply that the dilatons of the  $p$ -brane intersections in  $D = 11$  described in [3] are not all on equal footing, because only one or two (combinations) of them get(s) fixed at the horizon, while the other ones have asymptotical values which enter the flow, although the function  $F$  does not depend on such values.

Finally, we would like to comment on the fact that the half-maximal non-chiral  $(1, 1)$ ,  $D = 6$  theory analyzed in Sect. 5 may be considered as Type IIA compactified on  $K3$  [33]. The result obtained in the present paper for the  $AdS_3 \times S^3$  near-horizon geometry supports the conjecture of [34]. On the other hand, we do not find an agreement with the other *Ansätze* for the near-horizon geometry, because we only find solutions where the charges of strings (or that of their magnetic duals) are turned on.

We note that the techniques we have developed here apply to any supergravity flow ending on an AdS horizon even in presence of higher derivative terms and gaugings. It would be interesting to apply this formalism to the study of higher derivative corrections

to central charges in ungauged and gauged supergravities extending the black holes results found in [35] and [36]. The study of non-BPS black p-brane flows along the lines of [37] deserves also further investigations.

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