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SCALING LIMIT OF THE PRUDENT WALK

VINCENT BEFFARA
UMPA – ENS Lyon, 46 Allée d'Italie, 69364 Lyon Cedex 07, France
email: vbeffara@ens-lyon.fr

SACHA FRIEDLI
Departamento de Matemática, UFMG-ICEX C.P. 702, Belo Horizonte, 30123-970 MG, Brasil
email: sacha@mat.ufmg.br

YVAN VELENIK
Section de Mathématiques, Université de Genève, 2-4 rue du Lièvre, 1211 Genève 4, Suisse
email: Yvan.Velenik@unige.ch

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Abstract
We describe the scaling limit of the nearest neighbour prudent walk on $\mathbb{Z}^2$, which performs steps uniformly in directions in which it does not see sites already visited. We show that the scaling limit is given by the process $Z_u = \int_0^{3u/7} \big(\sigma_1 \mathbf{1}_{\{W(s)\geq 0\}} \vec{e}_1 + \sigma_2 \mathbf{1}_{\{W(s)<0\}} \vec{e}_2\big) \mathrm{d}s$, $u \in [0,1]$, where W is the one-dimensional Brownian motion and σ_1, σ_2 two random signs. In particular, the asymptotic speed of the walk is well-defined in the L^1-norm and equals $\frac{3}{7}$.

1 Introduction and description of results

The prudent walk was introduced more than 20 years ago, under the name *self-directed walk* in [11, 12] and *outwardly directed self-avoiding walk* in [9], as a particularly tractable variant of the self-avoiding walk. Interest in this model has known a vigorous renewal in recent years, mostly in the combinatorics community [5, 6, 4, 3, 10, 2]. The latter works are concerned with a variant of the original model, more natural from the combinatorial point of view, obtained by considering the uniform probability measure on all allowed paths of given length; we'll refer to it as the uniform prudent walk. In the present work, we consider the original (kinetic) model and describe its scaling limit in details.

Let $\vec{e}_1 = (1,0)$, $\vec{e}_2 = (0,1)$ denote the canonical basis of $\mathbb{Z}^2$. Let us describe the construction of the process associated to the prudent random walk $\gamma_\cdot$. We first set $\gamma_0 := (0,0)$. Assume that we have already constructed[1] $\gamma_{[0,t]}$, then the distribution of γ_{t+1} is obtained as follows. We say that the direction $\vec{e} \in \{\pm\vec{e}_1, \pm\vec{e}_2\}$ is allowed if and only if $\{\gamma_t + k\vec{e}, k > 0\} \cap \gamma_{[0,t]} = \varnothing$; in other words,

[1] For any discrete- or continuous-time process Y_t, we denote by $Y_{[0,t]} := \{Y_s\}_{0\leq s\leq t}$.

allowed directions are those towards which there are no sites that have already been visited. γ_{t+1} is chosen uniformly among the neighbours of γ_t lying in an allowed direction. Observe that there are always at least two allowed directions.

There has been considerable interest in such non-Markovian processes recently. Among the motivations are the necessity of developing specific methods to analyse these processes, and the fact that they sometimes present rather unusual properties. From this point of view, the prudent walk is quite interesting. On the one hand, it is sufficiently simple that a lot of information can be extracted, on the other its scaling limit possesses some remarkable features.

Our main result is the following, which describes the scaling limit (in a topology stronger than weak convergence) of the prudent walk.

Theorem 1.1. *On a suitably enlarged probability space, we can construct the prudent walk $\gamma_\cdot$, a Brownian motion $W_\cdot$ and a pair of ± 1-valued random variables σ_1, σ_2 such that*

$$\lim_{t\to\infty} P\big(\sup_{0\le s\le t} \big\|\tfrac{1}{t}\gamma_s - Z^{\sigma_1,\sigma_2}_{s/t}\big\|_2 > \epsilon\big) = 0,$$

where, for $u \in [0,1]$,

$$Z^{\sigma_1,\sigma_2}_u := \int_0^{3u/7} \big(\sigma_1 \mathbf{1}_{\{W(s)\ge 0\}}\vec{e}_1 + \sigma_2 \mathbf{1}_{\{W(s)<0\}}\vec{e}_2\big)\mathrm{d}s. \tag{1}$$

The Brownian motion $W_\cdot$ and the two random signs σ_1, σ_2 are independent of each other, and $P(\sigma_1 = s, \sigma_2 = s') = 1/4$, for $s, s' \in \{-1, 1\}$.

A typical trajectory of $\gamma_\cdot$ is depicted in Fig. 1. Let us briefly discuss some of the consequences of Theorem 1.1. To simplify the exposition, we assume that $\sigma_1 = \sigma_2 = 1$ (with no loss of generality, given the obvious symmetries of the prudent walk).

- Since $\|Z^{1,1}_u\|_1 = 3u/7$, the asymptotic speed of the prudent walk converges to $3/7$ in L^1-norm in probability. Note that there is no asymptotic speed in the L^2-norm. Actually, using arguments very similar to those of Section 4, one can prove that the speed converges almost surely to $3/7$.

- The angle α_u between $Z^{1,1}_u$ and $\vec{e}_1$ is random, which means that the prudent walk undergoes macroscopic fluctuations in direction, even though it is ballistic. The distribution of α_u can easily be determined, using the arcsine law for Brownian motion:

$$\begin{aligned} P(\alpha_u \le x) &= P(\tan\alpha_u \le \tan x) \\ &= P\big(\frac{\theta^+(W_{[0,u]})}{u} \ge \frac{1}{1+\tan x}\big) = \frac{2}{\pi}\arctan\sqrt{\tan x}, \end{aligned}$$

  where $\theta^+(W_{[0,u]})$ is the time spent by $W_{[0,u]}$ above 0.

- The presence of heavy-tailed random variables (the length of the Brownian excursions) in the limiting process $Z^{1,1}_\cdot$ allows the construction of various natural observables displaying ageing. For example, its first component.

We emphasise that the scaling limit of the kinetic prudent walk seems to be different from the scaling limit of the uniform prudent walk studied in Combinatorics. Indeed, it is shown in [2], for

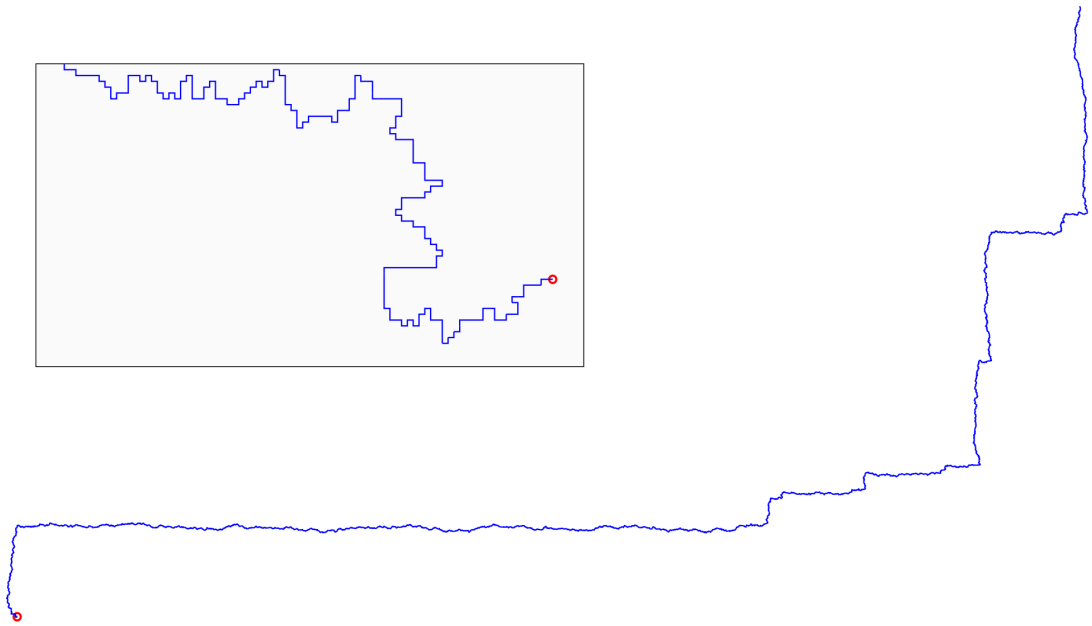

Figure 1: The first 50000 steps of a prudent walk. The inset is a blow-up of the first few hundred steps near the origin.

a simpler variant of the latter (similar to our corner model, see below) that the scaling limit is a straight line along the diagonal, with a speed in L^1-norm approximately given by 0.63. We expect the same to be true for the real uniform prudent walk.

We also observe that our scaling limit is radically different from what is observed for similar continuous random walks that avoid their convex hull, which again are ballistic but, at the macroscopic scale, move along a (random) straight line ([13], [1]).

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2 Excursions

We split the trajectory of the prudent walk into a sequence of excursions. To a path $\gamma_{[0,t]}$, we associate the bounding rectangle $\mathscr{R}_t \subset \mathbb{Z}^2$ with lower left corner $(x_{\min}, y_{\min})$, and upper right corner $(x_{\max}, y_{\max})$. Here, $x_{\min} = \min \mathscr{A}_t$, $y_{\min} = \min \mathscr{B}_t$, $x_{\max} = \max \mathscr{A}_t$, $y_{\max} = \max \mathscr{B}_t$, where

$$\mathscr{A}_t := \left\{ x \in \mathbb{Z} : \exists y \in \mathbb{Z}, (x,y) \in \gamma_{[0,t]} \right\}, \ \mathscr{B}_t := \left\{ y \in \mathbb{Z} : \exists x \in \mathbb{Z}, (x,y) \in \gamma_{[0,t]} \right\}.$$

Notice that γ_t always lies on the boundary of its bounding rectangle. The height and width of $\mathscr{R}_t$ (in units of lattice sites) are defined by $\mathscr{H}_t := |\mathscr{A}_t|$ and $\mathscr{W}_t := |\mathscr{B}_t|$; in particular $\mathscr{H}_0 = \mathscr{W}_0 = 1$. We

say that the path visits a corner at time $s \in \{0, \dots, t\}$ if γ_s coincides with one of the four corners of $\mathscr{R}_s$.

Without loss of generality, we assume that the first step of the path is in the direction $\vec{e}_1$, i.e., $\gamma_1 = \vec{e}_1$, and define the following sequence of random times:

$$T_0 := 0\,,$$
$$U_0 := \inf\{t > 0 : \mathscr{H}_t > 1\} - 1\,,$$

and for $k \geq 0$,

$$T_{k+1} = \inf\{t > U_k : \mathscr{W}_t > \mathscr{W}_{t-1}\} - 1\,,$$
$$U_{k+1} = \inf\{t > T_{k+1} : \mathscr{H}_t > \mathscr{H}_{t-1}\} - 1\,.$$

During the time interval $(T_k, U_k]$, the walk makes an excursion, denoted $\mathscr{E}_k^{\mathrm{v}}$, along one of the vertical sides of $\mathscr{R}_t$. During the time interval $(U_k, T_{k+1}]$, the walk makes an excursion, denoted $\mathscr{E}_k^{\mathrm{h}}$, along one of the horizontal sides of $\mathscr{R}_t$. Two relevant quantities are the horizontal displacement of $\mathscr{E}_k^{\mathrm{v}}$: $X_k := \mathscr{W}_{T_{k+1}} - \mathscr{W}_{T_k}$, and the vertical displacement of $\mathscr{E}_k^{\mathrm{h}}$: $Y_k := \mathscr{H}_{T_{k+1}} - \mathscr{H}_{T_k}$. Observe that by construction, $X_k \geq 1$ and $Y_k \geq 1$.

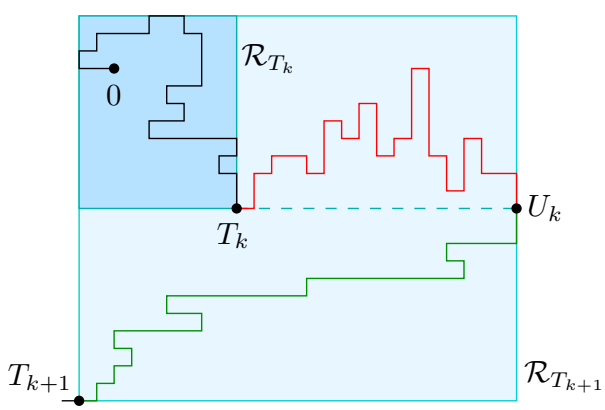

Figure 2: The excursions and their associated times.

2.1 The effective random walk for the excursions

As Figure 2 suggests, the study of excursions along the sides of the rectangle can be reduced to that of the excursions of an effective one-dimensional random walk, with geometric increments.

Let $\xi_1, \xi_2, \dots$ be an i.i.d. sequence, $\xi_i \in \mathbb{Z}$, with $P(\xi_i = k) = \frac{1}{3}(\frac{1}{2})^{|k|}$. Let $S_0 = 0$, $S_n = \xi_1 + \cdots + \xi_n$. We call S_n the effective random walk. Let η_L be the first exit time of S_n from the interval $[0, L-1]$: $\eta_L := \inf\{n > 0 : S_n \notin [0, L-1]\}$. The following lemma will be used repeatedly in the paper.

Lemma 2.1. *For all $k \geq 0$, $m \geq 1$,*

$$P(X_k = m | \gamma_{[0,T_k]}) = P(\eta_{\mathscr{H}_{T_k}} = m)\,, \tag{2}$$
$$P(Y_k = m | \gamma_{[0,U_k]}) = P(\eta_{\mathscr{W}_{U_k}} = m)\,. \tag{3}$$

Proof. We show (2). Assume that the walk is at the lower right corner of the rectangle at time T_k. On $\{X_k = m\} \cap \{\mathscr{H}_{T_k} = h\}$, the excursion $\mathscr{E}_k^{\mathrm{v}}$ can be decomposed into $(\alpha_1, \dots, \alpha_{m-1}, \beta)$, where each α_i is an elementary increment made of a vertical segment of length $l_i \in [-h+1, h-1] \cap \mathbb{Z}$ and of a horizontal segment of length 1 pointing to the right, and β is a purely vertical segment of length $l \in [-h, +h] \cap \mathbb{Z}$, which ensures that the excursion reaches either the top or the bottom of the rectangle by time m (see Figure 3). We thus have

$$P(X_k = m | \gamma_{[0,T_k]}) = \sum_{(\alpha_1,\dots,\alpha_{m-1},\beta)} p(\alpha_1) \dots p(\alpha_{m-1}) p(\beta),$$

where the sum is over all possible sets of such elementary increments, and $p(\alpha_i) = \frac{1}{3}(\frac{1}{2})^{|l_i|}$, $p(\beta) = \frac{1}{3}(\frac{1}{2})^{|l|-1}$. Therefore, in terms of the effective random walk with increments ξ_i,

$$p(\alpha_1) \dots p(\alpha_{m-1}) = P(\xi_1 = l_1, \dots, \xi_{m-1} = l_{m-1}).$$

Moreover, if $l < 0$ (as on Figure 3), then

$$p(\beta) = \tfrac{1}{3}(\tfrac{1}{2})^{|l|} = \tfrac{1}{3} \sum_{j \geq |l|+1} (\tfrac{1}{2})^j \equiv P(S_m < 0 \,|\, S_{m-1} = |l|).$$

Similarly, if $l \geq 0$

$$p(\beta) = P(S_m \geq h \,|\, S_{m-1} = h - l - 1).$$

This shows that

$$P(X_k = m | \gamma_{[0,T_k]}) = P(S_1 \in [0, h-1], \dots, S_{m-1} \in [0, h-1], S_m \notin [0, h-1]),$$

which is (2). □

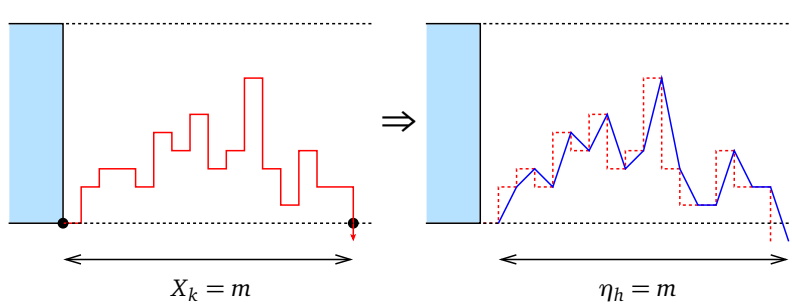

Figure 3: The decomposition of the excursion $\mathscr{E}_k^{\mathrm{v}}$. The full segments on the right image represent the increments of the effective random walk S_n. Observe that S_n starts at γ_{T_k+1}, and that it exits $[0, h-1]$ at time m by jumping to any point on the negative axis.

3 Crossings and visits to the corners

In this section, we show that, almost surely, the prudent walk eventually visits exactly one of the corners of its bounding rectangle an infinite number of times. This excludes, in particular, the

possibility of winding around the origin an infinite number of times, and will allow in the sequel to restrict the study of the prudent walker to a single corner of the rectangle.

Let A_k denote the event in which the kth excursion crosses at least one side of $\mathscr{R}_{T_k}$, that is if the walk is at a bottom (resp. top) corner of the box at time T_k and at a top (resp. bottom) corner of the box at time U_k, or if it is at a right (resp. left) corner of the box at time U_k and at a left (resp. right) corner of the box at time T_{k+1}.

Proposition 3.1. *There exists a constant $c_1 > 0$ such that*

$$P(A_k) \leq \frac{c_1}{k^{4/3}} \tag{4}$$

for all large enough k. In particular, $P(\limsup_k A_k) = 0$, and a.s. exactly one of the corners is visited infinitely many times.

For k large, the bounding rectangle has long sides, which makes crossing unlikely. As is well known from the gambler's ruin estimate for the simple symmetric random walk, the probability of first leaving an interval of length L at the opposite end is of order L^{-1}. To obtain (4), it will therefore be sufficient to show that the sides of the rectangle at the kth visit to a corner grow superlinearly in k.

Before this, we give a preliminary result for the effective random walk S_n starting at 0. Let as before η_L denote the first exit time of S_n from the interval $[0, L-1]$. We will also use $\eta_\infty := \inf\{n > 0 : S_n < 0\}$.

Lemma 3.1. *There exists a constant $c_2 > 0$ such that for all $L \geq 1$,*

$$P(\eta_L \geq n) \geq \frac{c_2}{\sqrt{n}} \quad \textit{for all integer } n \leq c_2 L^{4/3}. \tag{5}$$

Proof. Define $\eta_L^{\rightarrow} := \inf\{n > 0 : S_n \geq L-1\}$, so that

$$\begin{aligned} P(\eta_L \geq n) &= P(\eta_\infty \geq n, \eta_L^{\rightarrow} \geq n) \\ &= P(\eta_\infty \geq n) - P(\eta_L^{\rightarrow} < n, \eta_\infty \geq n) \\ &\geq P(\eta_\infty \geq n) - P(\eta_L^{\rightarrow} < n). \end{aligned}$$

By the gambler's ruin estimate of Theorem 5.1.7. in [8],

$$P(\eta_\infty \geq n) \geq \frac{c_3}{\sqrt{n}} \tag{6}$$

for some constant $c_3 > 0$. On the other hand,

$$P(\eta_L^{\rightarrow} < n) \leq P(\max_{1 \leq j \leq n} |S_j| \geq L) \leq \frac{E[|S_n|^2]}{L^2} = \frac{2n}{L^2}$$

The second inequality follows from the Doob-Kolmogorov Inequality. □

As a consequence, we can show that by the kth visit to the corner, the sides of the rectangle have grown by at least $k^{4/3}$. A refinement of the method below actually shows that $\mathscr{W}_{T_k}$ and $\mathscr{H}_{T_k}$ grow like k^2.

Lemma 3.2. *There exist positive constants c_4 and c_5 such that*

$$P(\mathscr{W}_{T_k} < c_5 k^{4/3}) \le \exp\big(-c_4 k^{1/3}\big)\,, \tag{7}$$

$$P(\mathscr{H}_{T_k} < c_5 k^{4/3}) \le \exp\big(-c_4 k^{1/3}\big)\,. \tag{8}$$

Proof. Let $m = \lfloor k/2 \rfloor$. Since $X_i \ge 1$ and $Y_i \ge 1$ for all i, we have that $\mathscr{H}_{T_j} \ge m$, $\mathscr{W}_{T_j} \ge m$, for all $j \ge m$. We consider the width of the rectangle at time T_k. Let I_j denote the indicator of the event $\{X_j \ge c_2 m^{4/3}\}$. We have, for all $j \ge m$,

$$\begin{aligned} P(I_j = 1 \,|\, \gamma_{[0,T_j]}) &= P(I_j = 1 \,|\, \mathscr{H}_{T_j}) \\ &= P(\eta_{\mathscr{H}_{T_j}} \ge c_2 m^{4/3}) \\ &\ge P(\eta_m \ge c_2 m^{4/3}) \\ \text{by (5)} &\ge c_2^{-3/2} m^{-2/3} \equiv p\,. \end{aligned}$$

Therefore, the I_js can be coupled to Bernoulli variables of parameter p. Since $\mathscr{W}_{T_k} = \sum_{j=0}^{k-1} X_j \ge c_2 m^{4/3} \sum_{j=m}^{k-1} I_j$, we get

$$\begin{aligned} P(\mathscr{W}_{T_k} < 2^{-4/3} c_2 k^{4/3}) &\le P(I_j = 0\,, \forall j = m, \dots, k-1) \\ &\le (1-p)^m \\ &\le e^{-c_4 k^{1/3}}\,. \end{aligned}$$

□

Proof of Proposition 3.1. Consider the event A_k^{v} in which $\mathscr{E}_k^{\mathrm{v}}$ crosses the (vertical) side of the rectangle. We have

$$P(A_k^{\mathrm{v}}) \le P\big(\mathscr{H}_{T_k} < c_5 k^{4/3}\big) + P\big(A_k^{\mathrm{v}} \,\big|\, \mathscr{H}_{T_k} \ge c_5 k^{4/3}\big)$$

The first term is treated with Lemma 3.2. Proceeding as in the proof of Lemma 2.1, we see that, in terms of the effective random walk S_n, the second term is the probability that S_n (started at zero) reaches $[c_5 k^{4/3}, +\infty)$ before becoming negative. Therefore, again by Theorem 5.1.7. in [8], there exists a constant $c_6 > 0$ such that

$$P\big(A_k^{\mathrm{v}} \,\big|\, \mathscr{H}_{T_k} \ge c_5 k^{4/3}\big) \le \frac{c_6}{k^{4/3}}\,.$$

This shows (4). The second claim follows by the Borel-Cantelli Lemma and by recurrence of the effective walk. □

Proposition 3.1 implies that the prudent walker behaves qualitatively, asymptotically, in the same way as a simplified model in which the evolution is along the corner of the infinite rectangle $\mathscr{R}_{SW} := \{(x,y) \in \mathbb{Z}^2 : x \le 0, y \le 0\}$. Namely, redefine a direction $\vec{e} \in \{\pm\vec{e}_1, \pm\vec{e}_2\}$ to be allowed if and only if $\{\gamma_t + k\vec{e}, k > 0\} \cap (\gamma_{[0,t]} \cup \mathscr{R}_{SW}) = \varnothing$. The paths of the corner process $\widehat{\gamma}_\cdot$ are obtained by choosing at each step a nearest neighbour, uniformly in the allowed directions. A trajectory of the corner model is depicted on Figure 4. As before, $\widehat{\gamma}_\cdot$ can be decomposed into a concatenation of excursions along the vertical and horizontal sides of $\mathscr{R}_{SW}$, denoted respectively $\widehat{\mathscr{E}}_k^{\mathrm{v}}$, $\widehat{\mathscr{E}}_k^{\mathrm{h}}$. Since there is no crossing possible here, these are independent.

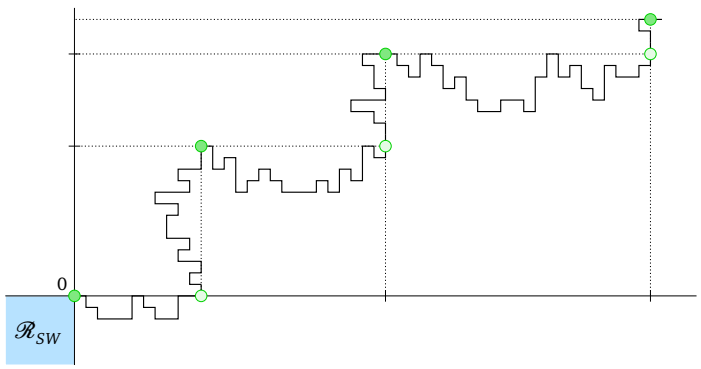

Figure 4: A trajectory of the corner process $\widehat{\gamma}_{\cdot}$.

We relate the true excursions of the prudent walk to those of the corner model, by means of the following coupling. As before, we assume that the first step is horizontal: $\widehat{\gamma}_1 = (1,0)$. Consider a given realisation of $(\widehat{\mathscr{E}}^{\mathrm{v}}_k, \widehat{\mathscr{E}}^{\mathrm{h}}_k)_{k\geq 1}$, from which we construct $(\mathscr{E}^{\mathrm{v}}_k, \mathscr{E}^{\mathrm{h}}_k)_{k\geq 1}$, with the same distribution as the excursions of the prudent walk. Let (see Fig. 5)

$$m^{\mathrm{v}}_k := \widehat{\gamma}_{T_{k-1}}(2) - \min_{T_{k-1}\leq t\leq U_{k-1}} \widehat{\gamma}_t(2), \quad m^{\mathrm{h}}_k := \widehat{\gamma}_{U_{k-1}}(1) - \min_{U_{k-1}\leq t\leq T_k} \widehat{\gamma}_t(1),$$

where we have denoted by $x(1)$ and $x(2)$ the coordinates of $x \in \mathbb{Z}^2$. If $m^{\mathrm{v}}_k > \ell$, let t_ℓ denote the first time $T_{k-1} \leq t \leq U_{k-1}$ such that $\widehat{\gamma}_{T_{k-1}}(2) - \widehat{\gamma}_t(2) = \ell + 1$, and let $\mathrm{Trunc}_\ell(\widehat{\mathscr{E}}^{\mathrm{v}}_k)$ denote the restriction of $\widehat{\mathscr{E}}^{\mathrm{v}}_k$ up to time t_ℓ. Similarly, one defines the corresponding objects in the horizontal case. Let $H_0 := 0$ and set

$$\mathscr{E}^{\mathrm{v}}_1 := \begin{cases} \widehat{\mathscr{E}}^{\mathrm{v}}_1 & \text{if } m^{\mathrm{v}}_1 \leq H_0\,, \\ \mathrm{Trunc}_{H_0}(\widehat{\mathscr{E}}^{\mathrm{v}}_1) & \text{if } m^{\mathrm{v}}_1 > H_0\,. \end{cases}$$

Let X_0 be the horizontal displacement associated to $\mathscr{E}^{\mathrm{v}}_1$ as in Figure 3, and set $W_0 = X_0$. We then set

$$\mathscr{E}^{\mathrm{h}}_1 := \begin{cases} \widehat{\mathscr{E}}^{\mathrm{h}}_1 & \text{if } m^{\mathrm{h}}_1 \leq W_0\,, \\ \mathrm{Trunc}_{W_0}(\widehat{\mathscr{E}}^{\mathrm{h}}_1) & \text{if } m^{\mathrm{h}}_1 > W_0\,. \end{cases}$$

Let $k \geq 1$. Assume $(\mathscr{E}^{\mathrm{v}}_j, \mathscr{E}^{\mathrm{h}}_j)_{j=1,\dots,k}$ have already been defined. To each vertical excursion $\mathscr{E}^{\mathrm{v}}_j$, we associate a horizontal displacement X_j as in Figure 3. Similarly, to each horizontal excursion $\mathscr{E}^{\mathrm{h}}_j$, we associate a vertical displacement Y_j.
Set $H_k := Y_0 + \cdots + Y_{k-1}$, and

$$\mathscr{E}^{\mathrm{v}}_{k+1} := \begin{cases} \widehat{\mathscr{E}}^{\mathrm{v}}_{k+1} & \text{if } m^{\mathrm{v}}_k \leq H_k\,, \\ \mathrm{Trunc}_{H_k}(\widehat{\mathscr{E}}^{\mathrm{v}}_{k+1}) & \text{if } m^{\mathrm{v}}_k > H_k\,. \end{cases}$$

Let now $W_k := X_0 + \cdots + X_k$, where X_k is the horizontal displacement of the vertical excursion $\mathscr{E}^{\mathrm{v}}_{k+1}$, and set

$$\mathscr{E}^{\mathrm{h}}_{k+1} := \begin{cases} \widehat{\mathscr{E}}^{\mathrm{h}}_{k+1} & \text{if } m^{\mathrm{h}}_k \leq W_k\,, \\ \mathrm{Trunc}_{W_k}(\widehat{\mathscr{E}}^{\mathrm{h}}_{k+1}) & \text{if } m^{\mathrm{h}}_k > W_k\,. \end{cases}$$

By construction, vertical and horizontal excursions of the prudent walk alternate. Therefore,

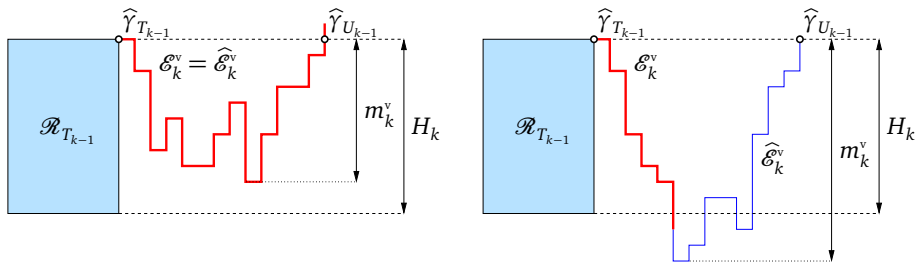

Figure 5: The construction of the vertical excursion $\mathscr{E}_k^{\mathrm{v}}$ of $\gamma_\cdot$ from the vertical excursion $\widehat{\mathscr{E}}_k^{\mathrm{v}}$ of $\widehat{\gamma}_\cdot$. When $\widehat{\mathscr{E}}_k^{\mathrm{v}}$ is compatible with $\mathscr{R}_{T_{k-1}}$, it is kept (left); when it is not compatible, it is truncated (right).

$(\mathscr{E}_k^{\mathrm{v}}, \mathscr{E}_k^{\mathrm{h}})_{k\geq 1}$ is distributed exactly as the excursions of the true prudent walk up to the appropriate reflection. By Proposition 3.1, almost surely, $\mathscr{E}_k^{\mathrm{h}} = \widehat{\mathscr{E}}_k^{\mathrm{h}}$ and $\mathscr{E}_k^{\mathrm{v}} = \widehat{\mathscr{E}}_k^{\mathrm{v}}$ for all but a finite number of ks.

According to Proposition 3.1, the prudent walker $\gamma_\cdot$ eventually fixates in one of the four quadrants and couples with the trajectory of the corner model $\hat{\gamma}_\cdot$. Obvious lattice symmetries present in the model imply that the quadrant is chosen with uniform probability 1/4. In order to lighten notations, we shall assume that the chosen quadrant is the first one.

Theorem 3.1. *Let us denote by $\mathscr{Q}_1$ the event that $\gamma_\cdot$ eventually settles in the first quadrant; notice that $P(\mathscr{Q}_1) = 1/4$. We then have, for any $\epsilon > 0$,*

$$\lim_{t\to\infty} P\big(\sup_{0\leq s\leq t} \big\|\tfrac{1}{t}\widehat{\gamma}_s - \tfrac{1}{t}\gamma_s\big\|_2 \geq \epsilon \;\big|\; \mathscr{Q}_1\big) = 0.$$

Proof. By Proposition 3.1, $\gamma_\cdot$ and $\widehat{\gamma}_\cdot$ couple almost surely in finite time. Since, on $\mathscr{Q}_1$, the distance between the two processes (at the same time) remains constant after coupling, it follows that, almost surely, $\sup_{t\geq 0} \|\widehat{\gamma}_t - \gamma_t\|_1 < \infty$. □

4 The scaling limit: Proof of Theorem 1.1

Theorem 1.1 is a consequence of Theorem 3.1 and the following result.

Theorem 4.1. *On a suitably enlarged probability space, one can construct simultaneously a realization of $\widehat{\gamma}_\cdot$ and of a Brownian motion $W_\cdot$ such that, for any $\epsilon > 0$,*

$$\lim_{t\to\infty} P\big(\sup_{0\leq s\leq t} \big\|\tfrac{1}{t}\widehat{\gamma}_s - Z^{1,1}_{s/t}\big\|_2 \geq \epsilon\big) = 0,$$

where $Z^{1,1}_\cdot$ was defined in (1).

The rest of this section is devoted to the proof of Theorem 4.1. We start by establishing a suitable coupling between the corner process and the effective random walk $S_\cdot$ introduced in Subsection 2.1. To this end, we construct a process $\hat{S}_\cdot$ associated to the effective random walk $S_\cdot$. This construction is illustrated on Figure 6.

Consider the following alternating ladder times for $S_\cdot$: $\tau_0 := 0$ and, for $k \in \mathbb{Z}_{\geq 0}$,

$$\tau_{2k+1} := \inf\big\{n > \tau_{2k} \,:\, S_n < S_{\tau_{2k}}\big\},$$
$$\tau_{2k+2} := \inf\big\{n > \tau_{2k+1} \,:\, S_n > S_{\tau_{2k+1}}\big\}.$$

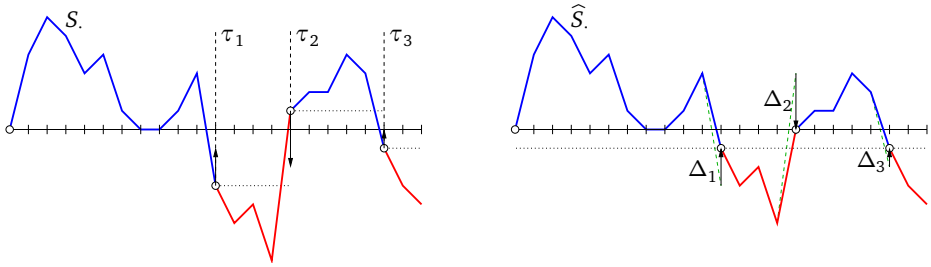

Figure 6: The construction of the process $\hat{S}_\cdot$.

Observe that $\tau_{k+1} - \tau_k$ has the same distribution as η_∞. Define the overshoots as $\Delta_0 := 0$, and for all $k \in \mathbb{Z}_{\geq 0}$,

$$\Delta_{2k+1} := -1 - (S_{\tau_{2k+1}} - S_{\tau_{2k}}),$$
$$\Delta_{2k+2} := +1 - (S_{\tau_{2k+2}} - S_{\tau_{2k+1}}).$$

Thanks to the memoryless property of the geometric distribution, we have that, for all k, $(-1)^{k+1}\Delta_k$ is a non-negative random variable, with geometric distribution of parameter $1/2$. We now define

$$\widehat{S}_n := S_n + \sum_{j \geq 0} \Delta_j \mathbf{1}_{\{\tau_j \leq n\}}. \tag{9}$$

The trajectory of the corner model can be constructed from the process $\widehat{S}_\cdot$, by decomposing the trajectory of the latter into two types of "excursions": those starting at 0 and ending at -1, and those starting at -1 and ending at 0. Applying the inverse of the procedure described in Fig. 3 to an excursion of the first type, we construct a vertical excursion of $\widehat{\gamma}_\cdot$; similarly, applying the same procedure to an excursion of the second type, we construct a horizontal excursion of $\widehat{\gamma}_\cdot$. The fact that these have the proper distribution follows from Lemma 2.1.
The next result shows that the sup-norm between the two processes $S_\cdot$ and $\widehat{S}_\cdot$ up to any fixed time never gets too large.

Lemma 4.1. *For any $\delta > 0$,*

$$\lim_{n\to\infty} P\big(\max_{0\leq k\leq n} |S_k - \widehat{S}_k| \geq n^{1/4+\delta}\big) = 0.$$

Proof. By (9), we have $S_k - \widehat{S}_k = \tilde{S}_{N_o(k)} - \Delta_{k+1}/2$, where $\tilde{S}_N := \sum_{j=0}^N \tilde{\Delta}_i$, $\tilde{\Delta}_i := (\Delta_i + \Delta_{i+1})/2$ and $N_o(k) := \max\big\{j \geq 0 : \tau_j \leq k\big\}$. Observe that the random variables $\tilde{\Delta}_i$ are i.i.d. and symmetric; in particular, $\tilde{S}_N$ is a martingale. If $L := n^{1/4+\delta}$,

$$\begin{aligned}
\lim_{n\to\infty} P\big(\max_{0\leq k\leq n} |S_k - \widehat{S}_k| \geq n^{1/4+\delta}\big) &\leq \lim_{n\to\infty} P\big(\max_{k\leq n} |\tilde{S}_{N_o(k)}| \geq L/2\big) \\
&\leq \lim_{n\to\infty} P\big(\max_{j\leq N_o(n)} |\tilde{S}_j| \geq L/2\big) \\
&\leq \lim_{n\to\infty} P\big(\max_{j\leq L^2} |\tilde{S}_j| \geq L/2\big) + \lim_{n\to\infty} P(N_o(n) \geq L^2).
\end{aligned}$$

The first limit is zero by the Doob-Kolmogorov inequality, and so is the second by a standard renewal argument, since $P(\tau_1 = n) \sim n^{-3/2}$. □

Lemma 4.2. *Fix $1/8 > \delta > 0$. For all $\epsilon > 0$, we have*

$$\lim_{n\to\infty} P\Big(\max_{1\le k\le n}\Big|\tfrac{1}{n}\sum_{i=1}^{k}\mathbf{1}_{\{\widehat{S}_i\ge 0\}} - \tfrac{1}{n}\sum_{i=1}^{k}\mathbf{1}_{\{S_i\ge n^{1/4+\delta}\}}\Big| > \epsilon\Big) = 0,$$
$$\lim_{n\to\infty} P\Big(\max_{1\le k\le n}\Big|\tfrac{1}{n}\sum_{i=1}^{k}\mathbf{1}_{\{\widehat{S}_i< 0\}} - \tfrac{1}{n}\sum_{i=1}^{k}\mathbf{1}_{\{S_i< -n^{1/4+\delta}\}}\Big| > \epsilon\Big) = 0.$$

Proof. Fix $1/8 > \delta > 0$. By the local central limit theorem, $E(\sum_{i=1}^{n}\mathbf{1}_{\{|S_i|<n^{1/4+\delta}\}}) \le c_7 n^{3/4+\delta}$. Therefore

$$\lim_{n\to\infty} P\Big(\sum_{i=1}^{n}\mathbf{1}_{\{|S_i|<n^{1/4+\delta}\}} \ge n^{3/4+2\delta}\Big) = 0. \tag{10}$$

Observe now that the signs of S_k and $\widehat{S}_k$ coincide when $|S_k| \ge n^{1/4+\delta}$, by Lemma 4.1. The conclusion follows, since

$$P\Big(\max_{1\le k\le n}\Big|\tfrac{1}{n}\sum_{i=1}^{k}\mathbf{1}_{\{\widehat{S}_i\ge 0\}} - \tfrac{1}{n}\sum_{i=1}^{k}\mathbf{1}_{\{S_i\ge n^{1/4+\delta}\}}\Big| > \epsilon\Big)$$
$$\le P\Big(\max_{\epsilon n/2\le k\le n}\Big|\tfrac{1}{n}\sum_{i=1}^{k}\mathbf{1}_{\{\widehat{S}_i\ge 0\}} - \tfrac{1}{n}\sum_{i=1}^{k}\mathbf{1}_{\{S_i\ge n^{1/4+\delta}\}}\Big| > \epsilon/2\Big).$$

□

The next ingredient is the existence of a strong coupling between the effective random walk $S_\cdot$ and the Brownian motion $B_\cdot$ [7]: on a suitably enlarged probability space, we can construct both processes in such a way that, for n large enough,

$$P\big(\max_{k\le n}|S_k - \sigma B_k| > n^{1/4}\big) \le e^{-n^{1/4}},$$

where $\sigma^2 = 2$ is the variance of ξ_i. It thus follows from Lemma 4.1 that, for any $\delta > 0$,

$$\lim_{n\to\infty} P\big(\max_{k\le n}|\widehat{S}_k - \sigma B_k| > n^{1/4+\delta}\big) = 0. \tag{11}$$

Lemma 4.3. *Fix $1/8 > \delta > 0$. For all $\epsilon > 0$, we have*

$$\lim_{n\to\infty} P\Big(\max_{k\le n}\Big|\tfrac{1}{n}\int_0^k \mathbf{1}_{\{B_s\ge 0\}}\mathrm{d}s - \tfrac{1}{n}\sum_{i=1}^{k}\mathbf{1}_{\{S_i\ge n^{1/4+\delta}\}}\Big| > \epsilon\Big) = 0,$$
$$\lim_{n\to\infty} P\Big(\max_{k\le n}\Big|\tfrac{1}{n}\int_0^k \mathbf{1}_{\{B_s< 0\}}\mathrm{d}s - \tfrac{1}{n}\sum_{i=1}^{k}\mathbf{1}_{\{S_i< -n^{1/4+\delta}\}}\Big| > \epsilon\Big) = 0.$$

Proof. It follows from (10) that the number of times $k \le n$ for which $|S_{k-1}| \wedge |S_k| \wedge |S_{k+1}| > n^{1/4+\delta}$ is at least $n - 3n^{3/4+2\delta}$, with probability going to 1 as $n \to \infty$. Of course, the same remains true if we impose additionally that S_{k-1}, S_k and S_{k+1} have the same sign. For such times k,

$$P\Big(\mathbf{1}_{\{S_k\ge n^{1/4+\delta}\}} \ne \int_{k-1/2}^{k+1/2}\mathbf{1}_{\{B_s\ge 0\}}\mathrm{d}s\Big) \le e^{-c_8 n^{1/2+2\delta}},$$

and the conclusion follows. □

Let us now introduce

$$\theta^+(\widehat{S}_{[0,n]}) := \sum_{k=0}^{n} \mathbf{1}_{\{\widehat{S}_k \geq 0\}}, \qquad \theta^-(\widehat{S}_{[0,n]}) := n - \theta^+(\widehat{S}_{[0,n]}),$$
$$\theta^+(B_{[0,t]}) := \int_0^t \mathbf{1}_{\{B_s \geq 0\}} \mathrm{d}s, \qquad \theta^-(B_{[0,t]}) := t - \theta^+(B_{[0,t]}).$$

Using these notations, we can deduce from Lemmas 4.2 and 4.3 that

$$\lim_{n\to\infty} P\Big(\max_{k\leq n}\big|\tfrac{1}{n}\theta^+(B_{[0,k]}) - \tfrac{1}{n}\theta^+(\widehat{S}_{[0,k]})\big| > \epsilon\Big) = 0, \tag{12}$$

for all $\epsilon > 0$, and similarly for $\theta^-(B_{[0,k]})$ and $\theta^-(\widehat{S}_{[0,k]})$. Let us introduce, for $m \geq 0$,

$$\Gamma_m := \theta^+(\widehat{S}_{[0,m]})\vec{e}_1 + \theta^-(\widehat{S}_{[0,m]})\vec{e}_2,$$
$$\mathsf{Z}_m := \theta^+(B_{[0,m]})\vec{e}_1 + \theta^-(B_{[0,m]})\vec{e}_2.$$

It follows from (12) that, for all $\epsilon > 0$,

$$\lim_{n\to\infty} P\big(\sup_{0\leq m\leq n}\big\|\tfrac{1}{n}\Gamma_m - \tfrac{1}{n}\mathsf{Z}_m\big\|_2 > \epsilon\big) = 0.$$

Given $n \in \mathbb{N}$, we denote by $t(n) := \sum_{i=1}^{n}\big(1 + |\widehat{S}_i - \widehat{S}_{i-1}|\big)$ the microscopic time such that the point $\widehat{S}_n$ is mapped on the point $\widehat{\gamma}_{t(n)}$ by the transformation described before Lemma 4.1.

Lemma 4.4. *For any $\epsilon > 0$,*

$$\lim_{n\to\infty} P\big(\sup_{0\leq m\leq n}\big\|\tfrac{1}{n}\widehat{\gamma}_{t(m)} - \tfrac{1}{n}\Gamma_m\big\|_2 > \epsilon\big) = 0. \tag{13}$$

Proof. This follows from Lemma 4.1 and the fact that, for any $\delta > 0$, $\max_{0\leq k\leq n}|S_k| \leq n^{1/2+\delta}$, with probability going to 1 as $n \to \infty$. □

It remains to relate more explicitly the real microscopic time $t(n)$ and the time n of the effective random walk.

Lemma 4.5. *For any $\epsilon > 0$,*

$$\lim_{n\to\infty} P\big(\sup_{0\leq m\leq n}|t(m) - \tfrac{7}{3}m| > \epsilon n\big) = 0.$$

Proof. As we have seen $\lim_{n\to\infty} P(N_{\mathrm{o}}(n) \geq n^{1/2+\delta}) = 0$, for all $\delta > 0$. We thus deduce that

$$\lim_{n\to\infty} P\big(\sup_{1\leq m\leq n}\Big|\sum_{i=1}^{m}|\widehat{S}_i - \widehat{S}_{i-1}| - \sum_{i=1}^{m}|S_i - S_{i-1}|\Big| > \epsilon n\big)$$
$$= \lim_{n\to\infty} P\big(\sup_{1\leq m\leq n}\Big|\sum_{i=1}^{m}|\widehat{S}_i - \widehat{S}_{i-1}| - \sum_{i=1}^{m}|S_i - S_{i-1}|\Big| > \epsilon n, N_{\mathrm{o}}(n) < n^{1/2+\delta}\big).$$

Now, observing that

$$\sup_{1\leq m\leq n}\Big|\sum_{i=1}^{m}|\widehat{S}_i - \widehat{S}_{i-1}| - \sum_{i=1}^{m}|S_i - S_{i-1}|\Big| \leq \sum_{i=1}^{N_{\mathrm{o}}(n)}|\Delta_i|,$$

we deduce that

$$\lim_{n\to\infty} P\big(\sup_{1\le m\le n}\Big|\sum_{i=1}^{m}|\widehat{S}_i-\widehat{S}_{i-1}|-\sum_{i=1}^{m}|S_i-S_{i-1}|\Big|>\epsilon n\big)$$
$$\le \lim_{n\to\infty} P\big(\sum_{i=1}^{n^{1/4+\delta}}|\Delta_i|>\epsilon n\big)=0.$$

It then follows from the Doob-Kolmogorov inequality that, since $E(|S_i-S_{i-1}|)=E(|\xi_i|)=4/3$,

$$\lim_{n\to\infty} P\big(\sup_{1\le m\le n}|t(m)-\tfrac{7}{3}m|>\epsilon n\big)$$
$$\le \lim_{n\to\infty} P\big(\sup_{1\le m\le n}\Big|\sum_{i=1}^{m}\big(1+|S_i-S_{i-1}|-\tfrac{7}{3}\big)\Big|>\epsilon n\big)=0.$$

□

Actually, we rather need to express n in terms of the real microscopic time t: $n(t):=\inf\{n\ge 0\,:\,t(n)\ge t\}$. However, since $t(n+1)-t(n)=|\xi_{n+1}|$, it follows from the previous lemma that $\lim_{t\to\infty}P(\sup_{0\le s\le t}|n(s)-\frac{3}{7}s|>\epsilon t)=0$.

To sum up, we have established that, for all $\epsilon>0$,

$$\lim_{t\to\infty} P\big(\sup_{0\le s\le t}\big\|\tfrac{1}{t}\widehat{\gamma}_s-\tfrac{1}{t}\mathrm{Z}_{3s/7}\big\|_2>\epsilon\big)=0.$$

The claim of Theorem 4.1 follows since, for $u\in[0,1]$,

$$\tfrac{1}{t}\mathrm{Z}_{3ut/7}=\int_0^{3u/7}\big(\mathbf{1}_{\{W_s\ge 0\}}\vec{e}_1+\mathbf{1}_{\{W_s<0\}}\vec{e}_2\big)\mathrm{d}s=Z_u^{1,1},$$

where we have set $W_u:=B_{ut}/\sqrt{t}$.

5 Concluding remarks

In the present work, we have focused on some of the most striking features of the scaling limit of the (kinetic) prudent walk. There remain however a number of open problems. We list a few of them here.

- It would be interesting to determine the scaling limit of the prudent walk on other lattices, e.g., triangular. Observe that the scaling limit we obtain reflects strongly the symmetries of the $\mathbb{Z}^2$ lattice, and is thus very likely to be different for other lattices. Nevertheless, numerical simulations indicate that similar scaling limits hold.

- In this work, we relied heavily on properties specific to the 2-dimensional square lattice. There are two natural generalizations of the prudent walk in higher dimensions: (i) the natural extension (forbidding steps in directions where visited sites are present), (ii) forbidding steps such that the corresponding half-line intersect the bounding parallelepiped. Notice that both coincide in dimension 2.

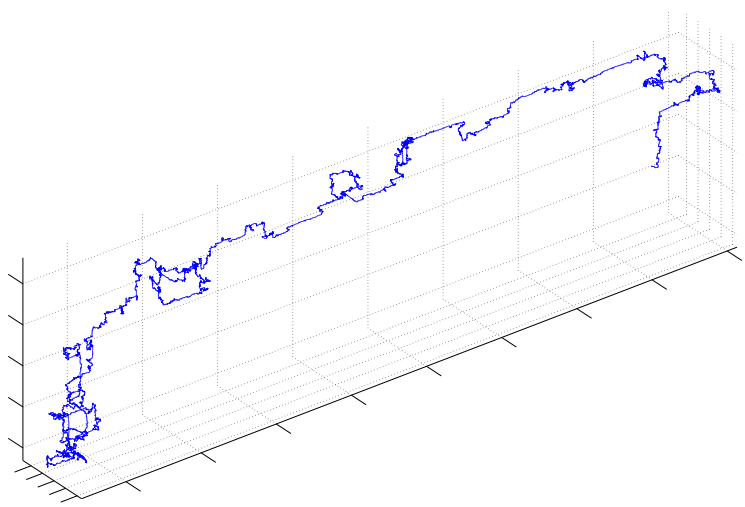

Figure 7: A trajectory of a 3-dimensional generalization of the prudent walk (first variant) with 6 million steps.

The second variant is easier, and it is likely to be in the scope of a suitable extension of our techniques. The first variant, however, is much more subtle (and interesting). Numerical simulations suggest the existence of an anomalous scaling exponent: If γ_t is the location of the walk after t steps, its norm appears to be of order t^α with $\alpha \simeq .75$ (but we see no reason to expect α to be equal to 3/4).

- Derive the scaling limit of the uniform prudent walk. In particular, it was observed numerically [3] that the number of prudent walks has the same growth rate as that of the (explicitly computed) number of so-called 2-sided prudent walks (which are somewhat analogous to our corner model). This should not come as a surprise, since one expects the uniform prudent walk to visit all corners but one only finitely many times. It would be interesting to see whether it is possible to make sense of the fact that the uniform prudent walk is more strongly concentrated along (one of) the diagonal. This would permit to derive a proof of the previous result from the one of the kinetic model.

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