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Midpoints and Exact Points of Some Algebraic Functions in Floating-Point Arithmetic

Claude-Pierre Jeannerod, Member, IEEE, Nicolas Louvet, Jean-Michel Muller, Senior Member, IEEE, and Adrien Panhaleux

Abstract—When implementing a function \( f \) in floating-point arithmetic, if we wish correct rounding and good performance, it is important to know if there are input floating-point values \( x \) such that \( f(x) \) is either the middle of two consecutive floating-point numbers (assuming rounded-to-nearest arithmetic), or a floating-point number (assuming rounded toward \( \pm \infty \) or toward 0 arithmetic). In the first case, we say that \( f(x) \) is a midpoint, and in the second case, we say that \( f(x) \) is an exact point. For some usual algebraic functions and various floating-point formats, we prove whether or not there exist midpoints or exact points. When there exist midpoints or exact points, we characterize them or list all of them (if there are not too many). The results and the techniques presented in this paper can be used in particular to deal with both the binary and the decimal formats defined in the IEEE 754-2008 standard for floating-point arithmetic.

Index Terms—Floating-point arithmetic, correct rounding, algebraic function.

1 INTRODUCTION

In a floating-point system that follows the IEEE 754-1985 standard for radix-2 floating-point arithmetic [1], the user can choose an active rounding mode, also called rounding-direction attribute in the newly revised IEEE 754-2008 standard [5]: rounding toward \(-\infty\), rounding toward \(+\infty\), rounding toward 0, and rounding to nearest, which is the default rounding mode. Given a real number \( x \), we denote, respectively, by RD\( (x) \), RU\( (x) \), RZ\( (x) \), and RN\( (x) \) these rounding modes. Let us also recall that correct rounding is required by the above cited IEEE standards for the four elementary arithmetic operations \( (+, -, \times, \div) \) as well as for the square root: the result of an operation is said to be correctly rounded if for any inputs, its result is the infinitely precise result rounded according to the active rounding mode. We are interested here in facilitating the delivery of correctly rounded results for various simple algebraic functions that are frequently used in numerical analysis or signal processing.

Let us call midpoint for a floating-point format a number that is exactly halfway between two consecutive floating-point numbers of that format. Given a function \( f : \mathbb{R}^{d} \rightarrow \mathbb{R} \) and a floating-point vector \( x \), we say that \( f(x) \) is a midpoint of \( f \) if \( f(x) \) is a midpoint for that format.

Given \( f \) and \( x \), the problem of computing RN\( (f(x)) \) is closely related to the knowledge of the midpoints of the function \( f \). Then, a common strategy (see [15] and [10, chapter 10]) for returning RN\( (f(x)) \) is as follows:

Let us first compute an approximation \( f_{1} \), of accuracy \( \epsilon_{1} \), to \( f(x) \). If there are no midpoints of the considered floating-point format within distance \( \epsilon_{1} \) from \( f_{1} \), then necessarily RN\( (f(x)) = RN(f_{1}) \). If on the contrary, there are such midpoints within distance \( \epsilon_{1} \) from \( f_{1} \), we can progressively increase the quality of the approximations (that is, computing an approximation \( f_{2} \) of accuracy \( \epsilon_{2} < \epsilon_{1} \), and so on) until we are able to provide a correctly rounded result. The point is that this strategy may not terminate if the function \( f \) has midpoints. As a consequence, a correctly rounded implementation of a given function \( f \) can be made more efficient if we know in advance that \( f \) admits no midpoints. If \( f \) admits midpoints, it is also very useful to know how to characterize them.

If now we consider one of the directed rounding modes (RD, RU, or RZ), the strategy that consists in progressively refining the approximations will not terminate if \( f(x) \) is a floating-point number. In this case, we say that \( f(x) \) is an exact point of the function \( f \), and it is also very useful to know a characterization of these exact points when implementing \( f \). Moreover, a characterization of the exact points of \( f \) can be used to set the "inexact" flag required by the IEEE standards [1], [5]. For example, for \( x/\sqrt{x^2+y^2} \) in radix 2, our study shows that this flag must always be raised except when \( x \) or \( y \) is zero, which can be detected easily.

In this paper, we present results on the existence of midpoints and exact points for some algebraic functions: beyond division, inversion, and square root, we study functions like the reciprocal square root \( 1/\sqrt{y} \), the 2D euclidean norm \( \sqrt{x^2+y^2} \) and its reciprocal \( 1/\sqrt{x^2+y^2} \),
and the 2D-normalization function $x/\sqrt{x^2+y^2}$. A part of the results presented on division and square root have been known for some time in binary arithmetic; see, for instance, the pioneering work by Markstein [9], as well as studies by Iordache and Matula [6] and Parks [11]. Let us also recall the work by Lauter and Lefèvre [8] on the function $x^p$, which thus covers integer powers. We present these results for completeness, and extend some of them to other radices, in particular to radix 10.

Before going into further details, we introduce some definitions. A radix-$\beta$, precision-$p$ floating-point number $x$ is either 0 or a rational number of the form

$$x = \pm X \cdot \beta^{e_x} \cdot p^{+1},$$

where $X$ is a positive integer such that $X < \beta^p$. If in addition $\beta^{p-1} \leq X$, then $x = \pm X \cdot \beta^{-p+1}$ is called the normalized representation of $x$, and the integers $X$ and $e_x$ are called, respectively, the integral significand and the exponent of $x$. We can, in fact, speak of the exponent for any nonzero real $x$ in radix $\beta$, it is the unique integer $e_x$ such that $\beta^{e_x} \leq |x| < \beta^{e_x+1}$. On computing systems conforming to the IEEE 754-2008 standard [5], the radix $\beta$ is 2 or 10. Radix 16 is also sometimes used [12]. The exponent $e_x$ is bounded: $e_{\text{min}} \leq e_x \leq e_{\text{max}}$, where $e_{\text{min}}$ and $e_{\text{max}}$ are the extremal exponents of the considered floating-point format. A nonzero number without a normal representation is said subnormal: all subnormal numbers have absolute value less than $\beta^{-\infty}$.

Assuming that we are working with a radix-$\beta$, precision-$p$ floating-point arithmetic, a midpoint is a rational number of the form

$$z = \pm(Z + 1/2) \cdot \beta^{e_x-p+1},$$

where $Z$ is a nonnegative integer such that

$$\begin{cases} \beta^{p-1} \leq Z < \beta^p, & \text{if } e_{\text{min}} < e_x \leq e_{\text{max}}, \\ 0 \leq Z < \beta^p, & \text{if } e_x = e_{\text{min}}. \end{cases}$$

Such a number is exactly halfway between two consecutive floating-point numbers. The midpoints are the values where the function $x \rightarrow \text{RN}(x)$ is discontinuous, as illustrated in Fig. 1 on a toy floating-point format ($\beta = 2$, $p = 3$, $e_{\text{min}} = -1$, and $e_{\text{max}} = 1$).

When using the implementation of a mathematical function in floating-point arithmetic, in most practical cases, the input and output precisions are the same. However, a user may, for example, wish to calculate the single-precision/binary32 number that is closest to the square root of a double-precision/binary64 floating-point number. For the sake of simplicity, we assume, in this paper, that the input and output precisions are the same. Moreover, we give our results assuming an unbounded exponent range, that is, under the hypothesis that no underflow nor overflow occurs. For that purpose, we define $\mathcal{F}_{\beta, p}$ as the set of the radix-$\beta$, precision-$p$ floating-point numbers, with an unbounded exponent range. Similarly, midpoints are restricted to the set

$$\mathcal{M}_{\beta, p} = \{ \pm (Z + 1/2) \cdot \beta^{e_x-p+1} : Z \in \mathbb{Z}, \beta^{p-1} \leq Z < \beta^p, e_x \in \mathbb{Z} \},$$

where $\mathbb{Z}$ denotes the set of integers and $\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$ of nonnegative integers.

The purpose of this paper is, for the floating-point number systems and the algebraic functions mentioned above, to investigate whether these functions admit midpoints or exact points, and to characterize such midpoints and exact points when they exist. The results we obtain are for $\beta = 2^q$ with $q$ a positive integer, and for $\beta = 10$, but in some cases, we managed to weaken these assumptions on $\beta$. Moreover, most of the examples proposed are based on the basic formats defined in the IEEE 754-2008 standard [5] that are briefly recalled below:

<table>
<thead>
<tr>
<th>Binary formats</th>
<th>$p$</th>
<th>Decimal formats</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>binary32</td>
<td>24</td>
<td>decimal32</td>
<td>7</td>
</tr>
<tr>
<td>binary64</td>
<td>53</td>
<td>decimal64</td>
<td>16</td>
</tr>
<tr>
<td>binary128</td>
<td>113</td>
<td>decimal128</td>
<td>34</td>
</tr>
</tbody>
</table>

Table 1 summarizes the results presented in the paper. In this table, “many” indicates that the techniques we used did not allow us to find a simple characterization of the midpoints or of the exact points of the function, an exhaustive enumeration was impractical because of the too large number of cases to consider, and we have experimental evidence that the number of midpoints and/or exact points is
large. Most of the results displayed here for \( \beta = 2 \) are, in fact, obtained in a more general setting, namely, for \( \beta = 2^q \), where \( q \) is a positive integer.

Note that since we considered an unbounded exponent range, subnormal floating-point numbers of the various IEEE 754-2008 formats can be written in normalized form. Hence, subnormal numbers are a subset of floating-point numbers with unbounded exponent range. This implies that the results presented in Table 1 remain unchanged when the inputs are subnormal numbers. If there are no exact points or midpoints for normal floating-point numbers with unbounded exponent range for a given function, then midpoints or exact points cannot occur if the inputs are subnormals. Similarly, if the exact points and midpoints are characterized by one of the theorems, assuming that the subnormals. Similarly, if the exact points and midpoints cannot be in the subnormal range. Although the results are not detailed in the paper, the techniques presented can be used to deal with midpoints in the subnormal range for the other functions listed in Table 1.

Outline. We start with extensions to radices \( 2^q \) and 10 of classical, radix-2 results for square roots (Section 2), reciprocal square roots (Section 3), and positive integer powers (Section 4). In Section 5, we move to the function that maps a real \( x \) and a \( d \)-dimensional real vector \( y = [y_1, y_2, \ldots, y_d] \) to \( x/\|y\|_2 \).

The function \( x/\|y\|_2 \) is interesting for it covers several important special cases, each of them being detailed in a subsequent section: for \( d = 1 \), division and reciprocal (Sections 6 and 7); for \( d = 2 \), reciprocal 2D euclidean norm \( 1/\sqrt{x^2 + y^2} \) and normalization of 2D vectors \( x/\sqrt{x^2 + y^2} \) (Sections 8 and 9). We comment on the 2D euclidean norm in Section 10.

Notation. Throughout the paper, the symbols \( \mathbb{Q} \), \( \mathbb{R} \), and \( \mathbb{N}_{>0} \) denote the rational numbers, the real numbers, and the positive integers, respectively. We write \( i \) for the complex number whose square is \(-1\), and \( \lfloor \cdot \rfloor \) and \( \lceil \cdot \rceil \) for the usual floor and ceiling functions. Also, for \( x, y \in \mathbb{Z} \) such that \( y \neq 0 \), we use the standard notation \( x \mod y = x - y\lfloor x/y \rfloor \) (see, for instance, Graham et al. [4, p. 82]).

## 2 Square Root

### 2.1 Midpoints for Square Root

The following theorem can be viewed as a consequence of a result of Markstein [9, Theorem 9.4]. It says that the square root function has no midpoints, whatever the radix \( \beta \) is. A detailed proof is given here for completeness.

**Theorem 1 (Markstein [9]).** Let \( y \in \mathbb{F}_{\beta,p} \) be positive. Then, \( \sqrt{y} \notin \mathbb{M}_{\beta,p} \).

**Proof.** Let \( z = \sqrt{y} \) and assume that \( z \) is in \( \mathbb{M}_{\beta,p} \). Then, there exist some integers \( Z \) and \( e_z \) such that \( z = (Z + 1/2) \cdot \beta^p/\beta^{p+1} \) and \( \beta^{p-1} \leq Z < \beta^p \). Using \( y = z^2 \) and \( y = Y \cdot \beta^{p-1} \), we deduce that

\[
4Y \cdot \beta^{p-2e_z+p-1} = (2Z + 1)^2.
\]

Now, one may check that \( e_z = \lfloor e_y/2 \rfloor \) so that

\[
e_y - 2e_z = e_y \mod 2,
\]

which is nonnegative. Thus, for \( p \geq 1 \), the left-hand side of (1) is an even integer. This contradicts the fact that the right-hand side is an odd integer.

### 2.2 Exact Points for Square Root

We saw in the previous section that the square root function has no midpoints. The situation for exact points is just opposite: for a given input exponent, the number \( N \) of floating-point numbers having this exponent and whose square root is also a floating-point number grows essentially like \( \beta/2 \). In this section, we make this claim precise for \( \beta = 2^q \) (\( q \in \mathbb{N}_{>0} \) and \( \beta = 10 \) by giving an explicit expression for \( N \) in Theorem 2. To establish this counting formula, we need the following two lemmas:

**Lemma 1.** For \( a, b \in \mathbb{R} \) such that \( 0 \leq a \leq b \), and \( c \in \mathbb{N}_{>0} \), the number of integer multiples of \( c \) that lie in \([a, b)\) is \([b/c] - [a/c]\).

**Proof.** Let us write \( N_{a,b}^{(c)} \) for the number of integer multiples of \( c \) lying in \([a, b)\). Since \( 0 \leq a \leq b \), the set \( [0, b) \) is the union of the disjoint sets \([0, a)\) and \([a, b)\). Hence, \( N_{a,b}^{(c)} = N_{0,b}^{(c)} - N_{0,a}^{(c)} \) and it remains to check that \( N_{0,a}^{(c)} = [a/c] \). If \( a \notin \mathbb{N} \), it follows from \( c \in \mathbb{N}_{>0} \) that \( N_{0,a}^{(c)} = 1 + [a/c] \). If \( a \in \mathbb{N} \), either \( c \) divides \( a \) in which case \( N_{0,a}^{(c)} = a/c \), otherwise, \( N_{0,a}^{(c)} = 1 + [a/c] \).

**Lemma 2.** Let \( y \in \mathbb{F}_{\beta,p} \) be positive. The real number \( \sqrt{y} \) is also in \( \mathbb{F}_{\beta,p} \) if and only if the integral significand \( Y \) of \( y \) satisfies \( \beta^{p-1} \leq Y < \beta^p \) and \( Y = Z^2 \cdot \beta^{p-1}[e_y \mod 2] \) for some integer \( Z \) such that \( \beta^{p-1} \leq Z < \beta^p \).

**Proof.** Let \( z = \sqrt{y} \). Assume first that \( z \in \mathbb{F}_{\beta,p} \). Then, there exists an integer \( Z \) such that \( z = Z \cdot \beta^{p-1} \) and \( \beta^{p-1} \leq Z < \beta^p \). Using \( y = z^2 \) and \( y = Y \cdot \beta^{p-1} \), we deduce that

\[
Y = Z^2 \cdot \beta^{p-1}[e_y \mod 2],
\]

The “only if” statement then follows from (2). Conversely, using \( y = Y \cdot \beta^{p-1} \), we may rewrite the equality \( Y = Z^2 \cdot \beta^{p-1}[e_y \mod 2] \) as

\[
\sqrt{y} = Z \cdot \beta^{p-1}[e_y \mod 2],
\]

where

\[
e_y = (e_y - (e_y \mod 2))/2.
\]

By definition, \( e_y \) is an integer and by assumption, \( Z \) is an integer lying in \([\beta^{p-1}, \beta^p)\). Hence, \( \sqrt{y} \in \mathbb{F}_{\beta,p} \).
Theorem 2. For a given exponent $e_y$, let $N$ denote the number of input floating-point values $y \in \mathbb{F}_{b,p}$ such that $\sqrt{y} \in \mathbb{F}_{b,p}$, and let $e_y = (e_y + p - 1) \mod 2$.

- If $\beta = 2^k$, $q \in \mathbb{N}_{>0}$, then
  \[
  N = \left\lceil 2^{(q(p-1) \mod 2)/2} \right\rceil \cdot \left\lceil 2^{(q(p-1)-e_y(q \mod 2)/2} \right\rceil.
  \]

- If $\beta = 10$, then $N = \left\lceil 10^{(p-1)/2} \right\rceil - \left\lceil 10^{(p-1-e_y)/2} \right\rceil$.

Proof. Let $\gamma = p - 1 + (e_y \mod 2)$. From Lemma 2, $N$ is the number of integers $Y$ in $\left(\mathbb{F}_{b-1}\right)^{\beta-1}$ of the form $Z^2 \cdot Y^{\gamma-1}$ for some integer $Z$ such that $\beta^{-1} \leq Z < \beta$.

Rewriting $Y = Z^2 \cdot Y^{\gamma-1}$ as $Y \cdot \beta^{\gamma-\gamma_0} = Z^2$, we see that $\beta^{\gamma-\gamma_0}$ divides $Z^2$. Since $e_y = \gamma \mod 2$, we know that $\gamma - e_y$ is even and for $p \geq 1$, nonnegative. Using, for instance, the factorizations of $\beta^{\gamma-\gamma_0}$ and $Z$ into primes, we deduce that $\beta^{(\gamma-\gamma_0)/2}$ divides $Z$. Consequently, there exists an integer $X$ such that

\[
Y \cdot \beta^{\gamma} = X^2 \quad \text{and} \quad Z = X \cdot \beta^{(\gamma-\gamma_0)/2}.
\]

Now, the assumption $\beta^{p-1} \leq Y < \beta^p$ is equivalent to

\[
\beta^{p-1+e_y/2} \leq X < \beta^{p+e_y/2},
\]

while the same assumption on $Z$ is equivalent to $\beta^{-1} \leq X < \beta^{p-1+e_y/2}$. The latter interval contains the former because $p - 1 \leq \delta \leq p$. Hence, $N$ is the number of integers $X$ satisfying (4) and whose square is an integer multiple of $\beta^\gamma$. We distinguish between the two cases $e_y = 0$ and $e_y = 1$.

If $e_y = 0$, then $N$ is the number of integers $X$ satisfying (4). Consequently, $N = \left\lceil \beta^{p/2} \right\rceil - \left\lceil \beta^{p+1/2} \right\rceil$ (using either Lemma 1 with $c = 1$ or (4, (3.12))).

If $e_y = 1$, then $X^2$ is a multiple of $\beta$: When $\beta$ has linear factors only (like $\beta = 2$ or $\beta = 10 = 2 \cdot 5$), this implies that $X$ is a multiple of $\beta$. In this case, $N$ is the number of integers $X$ that are multiples of $\beta$ and satisfy $\beta^{p/2} \leq X < \beta^{p+1/2}$. Hence, using Lemma 1, $N = \left\lceil \beta^{p/2} \right\rceil - \left\lceil \beta^{p+1/2} \right\rceil$. Assume now that $\beta = 2^k$ for some positive integer $q$. If $q$ is even, then $2^k$ divides $X^2$ implies $2^k$ divides $X$ so that we take the number of $X$s being an integer multiple of $2^k/\beta$. Lemma 1 thus gives $N = \left\lceil 2^{(q\beta/2)} \right\rceil - \left\lceil 2^{(q\beta+1/2)} \right\rceil$. If $q$ is odd, then $Y \cdot 2 = (X \cdot 2^{-(\delta/2)})^2$, which means that $X \cdot 2^{-(\delta/2)}$ is even. Hence, we keep all the cases $Z$ that are an integer multiple of $2^{-(\delta/2)}$. Using Lemma 1, this gives $N = \left\lceil 2^{(q\beta-1)/2} \right\rceil - \left\lceil 2^{(q\beta+1-1)/2} \right\rceil$.

For a fixed $e_y$, using Theorem 2, one can count the number of input floating-point numbers $y$ whose square root is an exact point. We give below the number $N$ of exact points for the basic formats of the IEEE 754-2008 standard.

<table>
<thead>
<tr>
<th>Format</th>
<th>$p$</th>
<th>decimal32</th>
<th>decimal64</th>
<th>decimal128</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_y = 0$</td>
<td>11</td>
<td>14</td>
<td>1199</td>
<td>$\approx 2.78 \cdot 10^7$</td>
</tr>
<tr>
<td>$e_y = 1$</td>
<td>24</td>
<td>9</td>
<td>849</td>
<td>$\approx 1.97 \cdot 10^7$</td>
</tr>
</tbody>
</table>

Also, for a fixed exponent $e_y$, one can see from Theorem 2 that the number of exact points for the square root function is $\Theta(\beta^{p/2})$, when the radix $\beta$ is either $2^k$ or 10 (it is said that $u(p) = \Theta(v(p))$ if there exist positive constants $c_1$, $c_2$, and $p_0$ such that $0 \leq c_1 v(p) \leq u(p) \leq c_2 v(p)$ for all $p \geq p_0$, see, for instance, Graham et al. [4, p. 448] for more details on the $\Theta$ notation). Except for small precisions, Theorem 2 implies, therefore, that it can be regarded as impractical to enumerate the exact points for the square root. It also shows that when computing the square root of a floating-point number, the probability of that square root being an exact point is very small (it vanishes as $p$ increases). This property may be taken into account when tuning a square root algorithm.

3 Reciprocal Square Root

3.1 Midpoints for Reciprocal Square Root

Theorem 3. Let $y \in \mathbb{F}_{b,p}$ be positive and let $\delta_y$ denote $e_y \mod 2$. If $\beta = 2^k$ ($q \in \mathbb{N}_{>0}$), then $1/\sqrt{y} \in \mathbb{F}_{b,p}$. If $\beta = 10$, one has $1/\sqrt{y} \in \mathbb{F}_{b,p}$ if and only if the integral significand $Y$ of $y$ has the form

\[
Y = 2^{3p-\delta_y+1} \cdot 5^{3p-2e_y-\delta_y-1},
\]

with $\ell \in \mathbb{N}$ such that $\ell \leq (3p - \delta_y - 1)/2$ and

\[
\begin{align*}
2 \cdot 10^{p-1/2} < 5^\ell & < 2 \cdot 10^{p-1/2}, \\
2 \cdot 10^{p-1/2} < 5^\ell & < 2 \cdot 10^p, \\
& \text{if } e_y \text{ is odd}, \\
& \text{if } e_y \text{ is even}.
\end{align*}
\]

Proof. Let $z = 1/\sqrt{y}$ and assume $z \in \mathbb{F}_{b,p}$. Let $y = Y \cdot \beta^{e_y-p+1}$ and $z = (Z + 1/2) \cdot \beta^{e_y-p+1}$ be the normalized representations of $y$ and $z$. From $y^2 = 1$, we deduce that

\[
Y(2Z + 1)^2 = 4 \cdot \beta^{3p-\delta_y-1},
\]

Since $z$ is a midpoint, one has $\beta^{e_y} < z < \beta^{e_y+1}$, and so $\delta_{2e_y} < y < \beta^{-2e_y}$. From this, one may check that

\[
-\delta_y - 2e_y = 2 - \delta_y, \quad \delta_y = e_y \mod 2.
\]

Hence, we obtain from (6) and (7):

\[
Y(2Z + 1)^2 = 4 \cdot \beta^{3p-\delta_y-1}.
\]

When $\beta = 2^k$, (8) has no solution, since the right-hand side of the equality is a power of two, while the left-hand side has an odd factor $(2Z + 1)^2$. Let us now consider the case where $\beta = 10$. Equation (8) then becomes

\[
Y(2Z + 1)^2 = 2^{3p-\delta_y+1} \cdot 5^{3p-2e_y-\delta_y-1},
\]

and it remains to prove the bounds on $\ell$. Since $Y$ is an integer, we have $3p - 2\ell - \delta_y - 1 \geq 0$, and the first bound $\ell \leq (3p - \delta_y - 1)/2$ follows. To prove the bounds in (5), note first that $10^{p-1} < y < 10^{p+1}$ and (7) gives $10^{p+1(1-\delta_y)/2} < z = 1/\sqrt{y} \leq 10^{\ell+1(1-\delta_y)/2}$. Then, using

\[
z = (Z + 1/2) \cdot 10^{(e_y-p+1)}.
\]
Theorem 4. Proof.

Let $y \in \mathbb{R}$, which includes the most frequent case.

The following theorem gives a characterization of the exact points for radix 10, where we do not have general results (which is in contrast with the previous case, we find $y = 2^{(p+1)-e}$, which is an even power of two. Conversely, if $y = 2^{2k}$, then $z = 2^{-k}$. Since any integral power of two is representable in $\mathbb{F}_{2p}$, we conclude that $z$ is an exact point.

All the floating-point numbers $y$ such that $1/\sqrt{y}$ is an exact point can be deduced from the ones lying in the interval $[1, \beta^2]$. In radix 2, Theorem 4 implies that at most two values of $y$ in $[1, 2^{2^{2^{k}}}]$ suffice to characterize the exact points for the reciprocal square root. In radix 10, for instance, the only exact points for input values $y \in [1, 256]$ are:

$$
\begin{array}{|c|c|c|c|}
\hline
y & 1 & 4 & 16 \\
\hline
1/\sqrt{y} & 1/2 & 0.816 & 1/4 = 0.416 \quad 1/8 = 0.216 \\
\hline
\end{array}
$$

Theorem 5. Let $y \in \mathbb{F}_{10p}$ be positive and let $\delta_y$ denote $e_y$ mod 2. One has $1/\sqrt{y} \in \mathbb{F}_{10p}$ if and only if either $y = 10^{-2e_y}$ or the integral significand $Y$ of $y$ differs from $10^{-1}$ and has the form

$$
Y = 2^{3p-1-\delta_y-2k} \cdot 5^{p-1-\delta_y-2l}
$$

with $k, l \in \mathbb{N}$ such that $0 \leq k, l \leq (3p - 1 - \delta_y)/2$.

Proof. Let $z = 1/\sqrt{y}$ and assume $z \in \mathbb{F}_{10p}$. If $z = 10^{e}$, then obviously, $y = 10^{-2e}$. On the other hand, $z$ must differ from the irrational number $10^{-1}/2$. Hence, we now assume that $z \in (10^{-1}/2, 10^{e})$. This implies that $y \in (10^{e-2}, 10^{e+1}/2)$. Therefore, $y$ is not a power of 10 and its normalized representation $y = 10^{e} \cdot 10^{-p+1}$ is such that $y \neq 10^{-p+1}$. Note now that (7) and (10) still hold here so that $yz^2 = 1$ implies that $YZ^2 = 10^{3p-1-\delta_y}$. In particular, $Z$ must have the form $Z = 2^{-k} \cdot 5^{l}$ for some $k, l \in \mathbb{N}$.

$$
Y = 2^{3p-1-\delta_y-2k} \cdot 5^{p-1-\delta_y-2l}
$$

where $Y$ is an integer, $0 \leq k, l \leq (3p - 1 - \delta_y)/2$.

Conversely, the case $y = 10^{-2e}$ being straightforward, let $Y = 2^{3p-1-\delta_y-2k} \cdot 5^{p-1-\delta_y-2l}$ be the integral significand of $y$ such that $10^{-1} < Y < 10^{e}$, and let $z = 1/\sqrt{y}$. Using (7) further leads to $z = 2^{-k} \cdot 5^{l} \cdot 10^{-p+1}$. One has $2^{-k} \cdot 5^{l} \in \mathbb{F}_{10p}$ and $10^{-1} < Y < 10^{e}$, we get $10^{-p+1}/2 < 2^{-k} \cdot 5^{l} < 10^{-p+2}$. Hence, $z \in \mathbb{F}_{10p}$.

Table 2 gives all the integral significands $Y$ of $y$, and the parity of the exponent $e_y$. Therefore, $1/\sqrt{y}$ is a floating-point number too, in the decimal32 format (see also Table 8 in the Appendix for the decimal64 format).

For the basic decimal formats of the IEEE 754-2008, the table below gives the number of significands $Y$ such that $1/\sqrt{y}$ is an exact point, with respect to the parity $\delta_y$ of the exponent of $y$.

<table>
<thead>
<tr>
<th>Format</th>
<th>$e_y$</th>
<th>decimal32</th>
<th>decimal64</th>
<th>decimal128</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = 7$</td>
<td>even</td>
<td>222</td>
<td>4194304</td>
<td></td>
</tr>
<tr>
<td>$p = 16$</td>
<td>odd</td>
<td>243, 512</td>
<td>70368747766400</td>
<td>281479067106560</td>
</tr>
<tr>
<td>$p = 34$</td>
<td>odd</td>
<td>2102, 573</td>
<td>1267650060228229401496703205376000</td>
<td></td>
</tr>
</tbody>
</table>

If $\beta$ is prime, we deduce from (10) that $Z = \beta^p$ for some $\ell \in \mathbb{N}$, Hence, $Y = \beta^{p-1-\delta_y-2l}$ and using (7), $y = \beta^{(p-1-\delta_y-2l)}$ is indeed an even power of $\beta$. Conversely, if $y = \beta^{2k}$, then $z = \beta^{2k}$ is in $\mathbb{F}_{\beta^p}$.

The table below gives the number of significands $Y$ such that $1/\sqrt{y}$ is an exact point, with respect to the parity $\delta_y$ of the exponent of $y$.
4 Positive Integer Powers

We consider here the function \( f(x, k) = x^k \) with \( x \in \mathbb{R} \) and \( k \in \mathbb{N}_{>0} \), assuming that each prime factor appears only once in the prime decomposition of \( k \), which is the case for \( \beta = 2 \) and \( \beta = 10 \). We provide a sufficient condition for the nonexistence of midpoints in such radices. In the particular case \( \beta = 2 \), the results given in this section can be deduced from Lauter and Lefèvre’s study of the power function \( (x, y) \mapsto x^y \) [8], which shows how to check quickly if \( x^y \) is a midpoint or an exact point in double precision (binary64 format).

**Definition 1.** A number fits in \( n \) digits exactly in radix \( \beta \) if it is a precision-\( n \) floating-point number that cannot be exactly represented in precision \( n - 1 \). More precisely, it is a number of the form \( x = \beta^e \cdot \delta, \) where \( e, \delta \in \mathbb{Z}, \delta > 1 \), and \( X > 0 \), and \( X \) is not a multiple of \( \beta \).

**Lemma 3.** Let \( k \in \mathbb{N}_{>0} \) be given. If each factor of \( \beta \) appears only once in its prime number decomposition (which is true for \( \beta = 2 \) or \( \beta = 10 \)), and if \( x \) fits in \( n \) digits exactly, then \( x^k \) fits in \( n \) digits exactly, with \( m \in \mathbb{N} \) such that \( (k+1) < m \leq kn \).

**Proof.** Let \( x = \beta^e \cdot \delta \) be a number that fits in \( n \) digits exactly. From \( \beta^e \cdot \delta > 1 \) and \( \beta^e \cdot \delta > 1 \), it follows that \( \beta^{e(k-1)} < |X| < \beta^e \). Consequently, there exists \( m \in \mathbb{N} \) such that \( (k-1) < m \leq kn \) and \( \beta^{e(k-1)} < |X| < \beta^e \). Moreover, the assumption on the prime factor decomposition of \( X \) and the fact that \( \beta \) does not divide \( X \) imply that \( X^2 \) is not a multiple of \( \beta \). \( \Box \)

An immediate consequence of the previous lemma is the following result:

**Theorem 6.** Assume that the radix \( \beta \) is such that each factor appears only once in its prime number decomposition, and let \( p \) be the precision. If \( x \) fits in \( n \) digits exactly, then \( x^p \) cannot be a midpoint as soon as \( k(n-1) > p \), and it cannot be an exact point as soon as \( k(n-1) + 1 > p \).

Theorem 6 is not helpful when \( k \) is small. For large values of \( k \), however, it allows to quickly determine the possible midpoints and exact points. For instance, in the binary64 format (\( \beta = 2 \) and \( p = 53 \)), the only floating-point numbers \( x \) such that \( x^{10} \) can be an exact point are those that fit in \( n \) bits exactly, where \( n \leq 6 \). For a given value of the exponent, there are at most \( 2^n = 64 \) such points: it, therefore, suffices to check these 64 values to know all the exact points. By accurately computing \( x^{10} \) for these 64 points, we easily find that the exact points for function \( x^{10} \) in the binary64 format correspond to the input values of the form \( x = X \cdot 2^e \), where \( X \) is an integer between 0 and 40.

5 The Function \((x, y) \mapsto x \div y\)

Given \( d \in \mathbb{N}_{>0}, \) the number of exact points of the function that maps \( (x, y) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \) to \( x \div y \) is huge as we will see later in Section 6.2, are exact points for the function \( x \div y \) as well. Therefore, we shall focus here exclusively on midpoints: our aim is to decide whether there exist floating-point inputs \( x, y_1, \ldots, y_d \in \mathbb{R} \) such that \( x \div y_i \) is an exact point for \( i = 1, \ldots, d \).

**Theorem 7.** Let \( x \in \mathbb{R} \) and \( y \neq 0 \). Let be a nonzero, \( d \)-dimensional vector of elements of \( \mathbb{R} \). If \( \beta = 2 \), then \( x \div y \in \mathbb{R} \).

**Proof.** Because of the symmetries of the function that maps \((x, y) \mapsto x \div y\), we can restrict to the case where \( x \) and all the entries of \( y \) are positive. Hence, \( x = X \cdot 2^e \) and \( y = Y_1 \cdot 2^e \) for some integers \( X \) and \( Y_k \), such that \( y_k \leq X, Y_k \leq 2^e \). Let \( z = x \div y \). The above is not a midpoint, that is, \( z \) is an integer in the same range as \( x \) and \( y \). The identity \( x^2 = \|y\|^2 \) holds thus becomes:

\[
4X^2 \cdot \beta^{(e-1)(p-1)} = \sum_k Y_k^2 \cdot \beta^{2e_k} (2Z + 1)^2.
\] (11)

In order to have integers on both sides, it suffices to multiply (11) by \( \beta^{-e} \), where \( e = \min_k e_k \). This gives

\[
4X^2 \cdot \beta^{(e-1)(p-1)} = \sum_k Y_k^2 \cdot \beta^{2(e_k-e)} (2Z + 1)^2.
\] (12)

Now, the power of \( \beta \) involved in the left-hand side of (12) is itself an integer. This is due to the fact that the integer \( e_k - e_k - e \) is nonnegative, which can be seen as follows: Since \( d \geq 1 \) and \( y_k = Y_k \) for \( k = 1, \ldots, d \), one has \( z = x \div y \). Using \( x < y \), and \( z < y \) (in fact, this lower bound is strict, for \( z \) is a midpoint), we deduce that \( e_k \leq y_k - e \). The exponents on both sides of the latter inequality being integers, we conclude that \( e_k \leq e \). When \( \beta = 2 \), (12) becomes

\[
X^2 \cdot 2^{(e-1)(p-1)} = \sum_k Y_k^2 \cdot 2^{2(e_k-e)} (2Z + 1)^2.
\] (13)

The left-hand side of (13) is a multiple of the odd integer \( (2Z + 1)^2 \). Since \( e_k - e_k - e \) is nonnegative, this implies that \( X \) is a multiple of \( 2Z + 1 \), and thus, \( X \geq 2Z + 1 \). However, recalling that \( 2^{p-1} \leq X, Z < 2^e \), we have

\[
X < 2Z + 1.
\] (14)

Hence, a contradiction, which concludes the proof. \( \Box \)
Theorem 7 implies the nonexistence of midpoints in radix \( \beta = 2 \) for a number of important special cases: division \( x/y \) (see Corollary 1), and thus, reciprocal \( 1/y \) as well; reciprocal 2D euclidean norm \( 1/\sqrt{x^2 + y^2} \) and 2D-vector normalization \( x/\sqrt{x^2 + y^2} \).

However, when \( \beta > 2 \), the function \( x/\|y\|_2 \) does have midpoints and some examples will be given in Section 6.1 for \( \beta \in \{3, 4, 10\} \). Thus, rather than trying to characterize all the midpoints of that general function, we focus from Section 6 to Section 9 on the four special cases just mentioned.

6 Division

6.1 Midpoints for Division

Concerning midpoints for division, Theorem 7 gives an answer for the far most frequent case in practice: the radix is two, the input precision equals the output precision, and the results are above the underflow threshold. Indeed, choosing \( d = 1 \) in Theorem 7, we obtain the following corollary:

**Corollary 1.** In binary arithmetic, the quotient of two floating-point numbers cannot be a midpoint in the same precision.

In radix-2 floating-point arithmetic, Corollary 1 can be seen as a consequence of a result presented by Markstein in [9, Theorem 8.4, p. 114]. Note that this result only holds when \( \beta = 2 \) and when the input precision is less than or equal to the output precision. Nevertheless, it is sometimes believed that it holds in prime radices: the first example given below shows that this is not the case. The following examples also illustrate the existence of midpoints when \( \beta > 2 \).

- In radix 3, with precision \( p = 4 \),

\[
\frac{28_{10}}{56_{10}} = \frac{1001_{3}}{2002_{3}} = 0.1111_{3} + \frac{1}{2} \cdot 3^{-4}.
\]

- In radix 4, with \( p = 4 \),

\[
\frac{129_{10}}{128_{10}} = \frac{2001_{4}}{2000_{4}} = 1.0000_{4} + \frac{1}{2} \cdot 4^{-3}.
\]

- In radix 10, midpoint quotients are quite frequent. For instance, when \( p = 2 \), there are 181 midpoints for \( X/Y \) with \( 10 \leq X, Y \leq 99 \) (that is, \( 10/16 = 0.625 \)) and when \( p = 3 \), there are 2,633 cases with \( 100 \leq X, Y \leq 999 \).

We now briefly discuss the case of different input \( (p_i) \) and output \( (p_o) \) precisions. If \( p_i > p_o \), many quotients can be midpoints, even in radix-2 arithmetic: If \( x \) is in precision \( p_i > p_o \), \( x \) can be a midpoint in precision \( p_o \), which is then the case for the quotient \( x/1 \). It is also possible to find less trivial cases. For example, if \( x \) and \( y \) are binary64 numbers \( (p = 53) \) with

\[
x = \frac{1.000000000000000000000000000000}{11111111111111111111111111110000},
\]

\[
y = \frac{1.111111111111111111111111111111}{00000000000000000000000000000000},
\]

then one has

\[
x/y = 0.\underbrace{10000000000000000000000000000000}_{p_o = 24},
\]

which is a midpoint in the binary32 floating-point format \( (p_o = 24) \).

6.2 Exact Points for Division

Let \( x \) and \( y \) be two numbers in IF\( _{\beta,pr} \), and assume that the quotient \( z = x/y \) is also in IF\( _{\beta,pr} \). Using the normalized representations \( x = \beta^{-p+1} \) and \( y = \beta^{-p+1} \), then \( z \) can be written \( z = \beta^{-s-p+\delta} \), with \( \delta \in \{0, 1\} \). Hence, from \( x = yz \), it follows that

\[
\beta^{p-\delta}X = YZ,
\]

with \( \delta \in \{0, 1\} \). In other words, if \( z \) is an exact point, then (15) must be satisfied. For any radix \( \beta \), (15) has many solutions: for each value of \( X \), there is at least the straightforward solution \( (X, Y) = (Z, \beta^{-p}) \), which corresponds to \( x/\beta^s \). As a consequence, the number of exact points of the function \( (x, y) \mapsto x/y \) grows at least like \( \beta^{p-1}(\beta - 1) \) for any given exponents \( e_x, e_y \). This is too large to enumerate all the exact points of division in practice.

7 Reciprocal

As we have seen above, except in radix 2, division admits many midpoints. Moreover, whatever the radix is, division also admits a lot of exact points. Consequently, we now focus on a special case, the reciprocal function \( y \mapsto 1/y \), for which more useful results can be obtained.

7.1 Midpoints for Reciprocal

**Theorem 8.** Let \( y \in \text{IF}_{\beta,pr} \) be nonzero. If \( \beta = 2^q \ (q \in \mathbb{N}_{>0}) \), then \( 1/y \notin \text{ML}_{\beta,pr} \). If \( \beta = 10 \), one has \( 1/y \in \text{ML}_{\beta,pr} \) if and only if the integral significand \( Y \) of \( y \) has the form

\[
Y = 2^{2p} \cdot 5^{2p-1-\ell},
\]

with \( \ell \in \mathbb{N} \) such that \( 2 \cdot 10^{p-1} < 5^\ell < 2 \cdot 10^p \).

**Proof.** Without loss of generality, we assume that \( y > 0 \). Let \( z = 1/y \). First, one may check that

\[
e_z = -e_y - 1.
\]

Now, if \( z \in \text{ML}_{\beta,pr} \), then \( z = (Z + 1/2) \cdot \beta^{s-p+1} \) for some integer \( Z \) such that \( \beta^{p-1} \leq Z < \beta^p \). Using \( yz = 1 \) thus gives

\[
Y(2Z + 1) = 2 \cdot \beta^{2p-1}.
\]

When \( \beta = 2^q \), (18) has no solution, since the right-hand side of the equality is a power of two, while the left-hand side has an odd factor \( 2Z + 1 \). When \( \beta = 10 \), (18) becomes

\[
Y(2Z + 1) = 2^{2p} \cdot 5^{2p-1}.
\]

As \( 2Z + 1 \) is odd, we deduce from (19) that \( 2Z + 1 \) is a power of five. Also, since \( 2 \cdot 10^{p-1} < 2Z + 1 < 2 \cdot 10^p \), there are at most two such powers of 5. Hence, \( y \) is necessarily as in (16). Conversely, if \( y = \beta^{-s-p+1} \) with \( Y \) as in (16), then using (17), \( y = 5^{e_y} \cdot 10^{e_y} \). It follows
that $z$ can be written $z = ((5^i - 1)/2 + 1/2) \cdot 10^{e_2 - p+1}$.

Since $(5^i - 1)/2$ is an integer, and by hypothesis $10^{p-1} \leq (5^i - 1)/2 < 10^p$, we deduce that $z \in \mathbb{M}_{10,p}$, which concludes the proof. \qed

In radix 10, there are at most two values of $\ell \in \mathbb{N}$ such that $2 \cdot 10^{\ell-1} < 5^i < 2 \cdot 10^\ell$. Therefore, to determine all inputs $y$ that give a midpoint $1/y$ for a fixed exponent $e_y$, it suffices to find the at most two $\ell$ such that $2 \cdot 10^{\ell-1} < 5^i < 2 \cdot 10^\ell$. This is easily done, even when the precision $p$ is large. Table 4 gives the integral significands of the floating-point numbers $y$ such that $1/y$ is a midpoint, for the decimal formats of the IEEE 754-2008 standard [5].

### 7.2 Exact Points for Reciprocal

For radices either 10 or a positive power of two, the exact points of the reciprocal function can all be enumerated according to the following theorem:

**Theorem 9.** Let $y \in \mathbb{F}_{10,p}$ be nonzero. One has $1/y \in \mathbb{F}_{10,p}$ if and only if the integral significand $Y$ of $y$ satisfies $\beta^{p-1} \leq Y < \beta^p$ and

$$Y = \begin{cases} 2^k, & 0 \leq k \leq q(2p-1), \text{ if } \beta = 2^k, \quad q \in \mathbb{N}_{>0}, \\ 2^k \cdot 5^\ell, & 0 \leq k, \ell \leq 2p-1, \text{ if } \beta = 10. \end{cases}$$

**Proof.** For the “only if” statement, let $y > 0$ in $\mathbb{F}_{10,p}$ be given, let $z = 1/y$, and assume that $z \in \mathbb{F}_{10,p}$. First, one may check that the exponent of $z$ satisfies $e_z = -e_y - \delta$ with $\delta \in \{0, 1\}$. Then, using the identity $yz = 1$ together with the normalized representations $y = Y \cdot \beta_\circ^{p-1}$ and $z = Z \cdot \beta_\circ^{-p+1}$, we get

$$YZ = \beta_\circ^{2p-2+\delta}, \quad \beta^{p-1} \leq Y, Z < \beta^p. \quad (20)$$

If $\beta = 2^k$ for some integer $q \geq 1$, then (20) implies that $Y = 2^k$ for some integer $k$ such that $0 \leq k \leq q(2p-1)$. If $\beta = 10$, then (20) implies that $2^k \cdot 5^\ell$ for some integers $k$ and $\ell$ such that $0 \leq k, \ell \leq 2p-1$.

Let us now prove the “if” statement. If $Y = \beta^{p-1}$, then $y$ is a power of the radix, and thus, $1/y$ belongs to $\mathbb{F}_{10,p}$. If $\beta^{p-1} < Y < \beta^p$, then defining $Z = Y^{-1} \cdot \beta_\circ^{2p-1}$, we obtain

$$1/y = Z \cdot \beta_\circ^{-p+1}, \quad \beta^{p-1} < Z < \beta^p. \quad (21)$$

To conclude that $1/y$ belongs to $\mathbb{F}_{10,p}$, it remains to show that $Z$ is an integer: If $\beta = 2^k$ and $Y = 2^k$, one has $Z = 2^k(2^{p-1}-1)$, which is an integer for $k \leq q(2p-1)$; If $\beta = 10$ and $Y = 2^k \cdot 5^\ell$, then $Z = 2^{p-1-k} \cdot 5^{\ell-1}$, which is an integer for $k, \ell \leq 2p-1$. Hence, $Z$ is an integer in both cases, showing that $1/y$ is indeed an exact point. This concludes the proof. \qed

In radix 16, for instance, the exact points $1/y$ with $y$ in the interval $[1, 16]$ are listed below.

<table>
<thead>
<tr>
<th>$y$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/y</td>
<td>1/2</td>
<td>0.816</td>
<td>0.416</td>
<td>0.216</td>
</tr>
</tbody>
</table>

In radix 10, all the integers $Y = 2^k \cdot 5^\ell$ with $0 \leq k, \ell \leq 2p-1$ and $10^{p-1} \leq Y < 10^p$ can be enumerated by a simple program, and each one of them gives an exact point. Table 5 gives the 21 integral significands $Y$ such that $1/y$ is an exact point, in the case of the decimal32 format (see also Table 7 in the Appendix for the decimal64 format).

Furthermore, given an input exponent, the result below provides an explicit formula for the number $N$ of floating-point inputs having this exponent and whose reciprocal is a floating-point number.

**Theorem 10.** For a given exponent $e_y$, the number $N$ of positive values $y \in \mathbb{F}_{10,p}$ such that $1/y \in \mathbb{F}_{10,p}$ is

$$N = \left\lfloor \frac{q}{2} \right\rfloor \left[ p \log_2(10) \right] + 1, \quad \text{if } \beta = 2^k, \quad q \in \mathbb{N}_{>0},$$

$$N = 1, \quad \text{if } \beta = 10.$$

**Proof.** When $\beta = 2^k$, Theorem 9 says that each exact point corresponds to an integer $k$ such that $2^{p(\beta-1)} \leq 2^k < 2^{p}$ and $0 \leq k \leq q(2p-1)$. The former condition is equivalent to $q(2p-1) \leq q(2p-1)$, and thus, implies the latter. From this, we deduce that the number of possible values of $k$ is when $\beta = 2^k$.

When $\beta = 10$, Theorem 9 says that each exact point corresponds to a pair of integers $(k, \ell)$ such that

$$10^{p-1} \leq 2^k \cdot 5^\ell < 10^p \quad \text{and} \quad 0 \leq k, \ell \leq 2p-1.$$

The value of $N$ is the number of points $(k, \ell) \in \mathbb{Z}^2$ that satisfy those two sets of constraints. Let $\sigma = \log_2(2) = 0.4306765581 \ldots$ The first set of constraints is equivalent to

$$(p-1)(1 + \sigma) \leq \sigma k + \ell < p(1 + \sigma). \quad (22)$$
It implies in particular that \((p-1)(1+\sigma) \leq \ell < p(1+\sigma)\), which is stronger than \(0 \leq \ell \leq 2p-1\) for \(p \geq 2\), since \(1+\sigma \approx 1.43\). Hence, \(N = \sum_{0 \leq k < 2p} N_k\), where \(N_k\) is the number of integers \(\ell\) satisfying (22) for a given \(k\).

Recalling that half-open real intervals \([a, b)\) such that \(a \leq b\) contain exactly \([b] - [a]\) integers \([4, p. 74]\), we deduce that for \(0 \leq k < 2p\),

\[
N_k = \left[ p(1+\sigma) - \sigma k \right] - \left[ (p-1)(1+\sigma) - \sigma k \right] = \left[ (p-k)\sigma \right] - \left[ (p-1-k)\sigma \right] + 1.
\]

Consequently, the sum \(\sum_{0 \leq k < 2p} N_k\) telescopes to \(2p + [p\sigma] + [p\sigma]\). Since the integer \(p\) is nonzero and \(\sigma\) is irrational, \(p\sigma\) cannot be an integer. Hence, \([p\sigma] = [p\sigma] + 1\), which leads to \(N = 2\left[ p(1+\sigma) \right] + 1\).

According to Theorem 10, when \(\beta = 2^9\), the number \(N\) of different integral significands leading to an exact point is \(g\).

In radix 2, we have \(N = \Theta(\epsilon)\), which confirms the fact that the midpoints for the reciprocal can be easily enumerated, even when the precision \(p\) is large. This is in contrast with the exact points of square root in radix 10 or 2\(^2\), whose number is exponential in \(p\) (see Section 2.2). For the decimal formats of IEEE 754-2008, the corresponding values of \(N\) are listed below.

<table>
<thead>
<tr>
<th>Format</th>
<th>decimal32</th>
<th>decimal64</th>
<th>decimal128</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>7</td>
<td>16</td>
<td>34</td>
</tr>
<tr>
<td>(N)</td>
<td>21</td>
<td>45</td>
<td>97</td>
</tr>
</tbody>
</table>

8 Reciprocal 2D Euclidean Norm

Given a \(d\)-dimensional vector \(y\) with entries in \(\mathbb{F}_{2^p}\), we know from Theorem 7 that \(z = 1/\|y\|_2\) cannot be a midpoint in radix 2. In this section, we focus on the 2D case, studying the midpoints and the exact points of the reciprocal 2D Euclidean norm, in radices \(2^9\) and 10. In radix 10, our study relies on the representation of products of the form \(2^i \cdot 5^j\) as sums of two squares \(a^2 + b^2\), where \(a, b \in \mathbb{N}\). Thus, we first explain in Section 8.1 the method we used for enumerating all the representations of such a product as the sum of two integer squares. Then, midpoints and exact points are studied in Sections 8.2 and 8.3, respectively.

8.1 Decomposing \(2^i \cdot 5^j\) into Sums of Two Squares

Decomposing an integer into sums of two squares is a well-studied problem in the mathematical literature (see, for instance, Wagon [13] and the references therein). In our particular case of interest, we deduce the following theorem that allows us to compute all the decompositions of \(2^i \cdot 5^j\) as sums of two squares. The proof of Theorem 11 relies on the uniqueness of the decomposition of a number into prime factors in the ring of Gaussian integers \(\mathbb{Z}[i]\) (see, for instance, Everest and Ward [3, chap. 2] for more details on this topic).

**Theorem 11.** Let \(r, s \in \mathbb{N}\) be given, and assume \(k \in \mathbb{N}\). All the unordered pairs \((a, b)\) with \(a, b \in \mathbb{N}\) and \(a^2 + b^2 = 2^i \cdot 5^j\) are given by \(a = |R(c)|\) and \(b = |3(c)|\) with \(c = 2^{[r/2]}(1+i)^{mod2}(2+i)^k(2-i)^{-k}\), \(0 \leq k < \lfloor(s+1)/2\rfloor\).

In particular, there are exactly \(\lfloor(s+1)/2\rfloor\) different decompositions of \(2^i \cdot 5^j\) as the sum of two squares.

**Proof.** Let us assume that \(2^i \cdot 5^j = a^2 + b^2\). Since the decomposition of \(2^i \cdot 5^j\) into prime factors in \(\mathbb{Z}[i]\) is unique apart from multiplications by \(\pm 1\) or \(\pm i\), one has \(2^i \cdot 5^j = b_0(1+i)^r(2+i)^j(2-i)^{-j}\), with \(b_0 \in \{1, \pm i\}\). On the other hand, one has \(a^2 + b^2 = (a + ib)(a - ib)\); hence, by uniqueness of the decomposition into prime factors, it follows that \(a + ib = b_0(1+i)^j(1-i)^{r-j}(2+i)^j(2-i)^{-j}\) for some \(k_1, k_2, k_3, k_4 \in \mathbb{N}\) and \(b_0 \in \{1, \pm i\}\). Then, one has \(a^2 + b^2 = 2^{[r/2]}(1+i)^{mod2}(2+i)^k(2-i)^{-k}\), which leads to \(N = 2\left[ p(1+\sigma) \right] + 1\).

A corollary of the factorization into primes in \(\mathbb{Z}[i]\), it can be shown that if we take \(k_1 \neq k_2\) with \(0 \leq k_1, k_2 < \lfloor(s+1)/2\rfloor\), then the corresponding unordered pairs \((a, b)\) and \((a', b')\) are necessarily different. Hence, there are exactly \(\lfloor(s+1)/2\rfloor\) unordered pairs \((a, b)\).

For later use, we also state the following corollary of Theorem 11:

**Corollary 2.** Given \(r \in \mathbb{N}\), the unique decomposition of \(2^i\) as a sum of two integer squares is

\[
2^i = \begin{cases} 
2^{[r/2]}(1+i)^{mod2}(2+i)^k(2-i)^{-k} & \text{if } r \text{ is even,} \\
(2^{[r/2]-1/2})^2 + (2^{(r-1)/2})^2 & \text{if } r \text{ is odd.}
\end{cases}
\]

8.2 Midpoints for Reciprocal 2D Norm

Theorem 12 below can be used to determine all the midpoints of the reciprocal 2D-norm function with exponent \(\epsilon\).

**Theorem 12.** Let \(x, y \in \mathbb{F}_{2^p}\) be such that \((x, y) \neq (0, 0)\), and let \(z = 1/\sqrt{x^2 + y^2}\). If \(\beta = 2^q\) \((q \in \mathbb{N})\), then \(z \notin \mathbb{N}_{2^p}\). If \(\beta = 10\), one has \(z \in \mathbb{N}_{2^p}\) if and only if \(z\) has the form...
Lemma 4. Let \( a \in \mathbb{Q} \). One has \( a^2 \in \mathbb{N} \) if and only if \( a \in \mathbb{Z} \).

As in the proof of Theorem 12, let us assume that \( 1 < z < 10 \) and \( x, y \geq 0 \), which implies that \( e_x, e_y \in \{-1, -2\} \). We denote by \( X \) and \( Y \) the integral significands of \( x \) and \( y \), respectively. From (25), we can deduce that \( X \) and \( Y \) must satisfy

\[
2^{4p+2} \cdot 5^{4p-2\ell} = (X \cdot 10^{e_x+3})^2 + (Y \cdot 10^{e_y+2})^2.
\]

From \( 2 \cdot 10^{p-1} < \ell < 2 \cdot 10^p \), one has \( 5^{4p-2\ell} \in \mathbb{N} \). Since, moreover, \( e_x, e_y \in \{-1, -2\} \), necessarily \( X \cdot 10^{e_x+3} \in \mathbb{N} \), and \( Y^2 \cdot 10^{2(e_y+2)} \) is also in \( \mathbb{N} \). Since \( Y^2 \cdot 10^{2(e_y+2)} \) is a nonnegative rational number whose square is a natural integer, it follows from Lemma 4 that \( Y \cdot 10^{e_y+2} \in \mathbb{N} \). Hence, we know that \( X \cdot 10^{e_x+3} \) and \( Y \cdot 10^{e_y+2} \) both necessarily belong to \( \mathbb{N} \).

As a consequence, to find all inputs \((X, Y)\) that give a midpoint for the function \( 1/\sqrt{x^2 + y^2} \), we know from (26) that we need to find all decompositions of the at most two integers \( 2^{4p+2} \cdot 5^{4p-2\ell} \) as the sum of two squares. We used Theorem 11 to build all values \( x \) and \( y \), \( x \geq y \), such that \( 1/\sqrt{x^2 + y^2} \) is a midpoint, for the decimal formats of the IEEE 754-2008 standard. For the decimal32 format, all the pairs of floating-point numbers \((x, y)\) for which \( 1/\sqrt{x^2 + y^2} \) is a midpoint can be deduced from the pairs listed in Table 6 by either exchanging \( x \) and \( y \) or by multiplying them by the same power of 10 (results for the decimal64 format are listed in Table 9 in the Appendix, and those for the decimal128 format are available at http://prunel.ccsd.cnrs.fr/ensi-00409366/fr/).

The following table gives the number \( N(7) \) of midpoints \( z \) in a decade (i.e., with a fixed exponent \( e_z \)), with respect to the decimal format considered. The table also gives the number \( N(10) \) of pairs of integral significand \((X, Y)\) with \( X \geq Y \) that give these midpoints. In decimal32 arithmetic, for instance, the function \((x, y)\rightarrow 1/\sqrt{x^2 + y^2}\) has two midpoints \( z = 1/\sqrt{x^2 + y^2} \) in a decade \([1, 10)\): the number of pairs \((X, Y)\) that give \( z \) is 10, and nine pairs give \( z \).

<table>
<thead>
<tr>
<th>Format</th>
<th>decimal32</th>
<th>decimal64</th>
<th>decimal128</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>7</td>
<td>16</td>
<td>34</td>
</tr>
<tr>
<td>( N_z )</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( N )</td>
<td>4</td>
<td>10 + 9</td>
<td>20 + 19</td>
</tr>
</tbody>
</table>

### 8.3 Exact Points for Reciprocal 2D Norm

Theorem 13. Let \( x, y \in \mathbb{R}, \) be such that \((x, y) \neq (0, 0)\). Let \( X, Y \) denote the integral significands of \( x, y \), and let also \( z \) denote \( 1/\sqrt{x^2 + y^2} \).

- For \( \beta = 2^q \) (\( q \in \mathbb{Q}\)), the real \( z \) is also in \( \mathbb{F}_{2^p} \) if and only if \((x, y)\in(0, \pm 2^k)\) for some \( k \in \mathbb{Z}\).
- For \( \beta = 10 \), the number \( z \) is in \( \mathbb{F}_{10^p} \) if and only if its integral significand \( Z \) satisfies \( Z = 2^k \cdot 5^f \), with \( 10^{p-1} < 2^k \cdot 5^f < 10^p \) and \( k, f \in \mathbb{Z} \). In this case, one has \( 2^{2k} \cdot 5^{2f} \) in \( \mathbb{N} \) and \((X, Y)\) must satisfy

\[
(X \cdot 10^m)^2 + (Y \cdot 10^n)^2 = 2^{2k+2m} \cdot 5^{2f+2n},
\]

where \( m, n \in \mathbb{Z} \) such that \( X \cdot 10^m \) and \( Y \cdot 10^n \) are in \( \mathbb{N} \).

**Proof.** Without loss of generality, we assume that \( 1 \leq z < \beta \) and \( 0 \leq y < x \). Reasoning as in the proof of Theorem 12, one may check that necessarily \( e_x \in \{-2, -1, 0\} \). Using as usual the normalized representations of \( x, y \), and \( z \), from \((x^2 + y^2)z = 1\), we deduce

\[
Z^2(X^2 \cdot 2^{e_x-2x} + Y^2) = \beta^{e_x-2x}.
\]

If \( \beta = 2^q \) for some \( q \in \mathbb{Q}\), then (27) implies that \( Z = 2^q \) for some \( \ell \in \mathbb{Z} \). From (27), we then deduce
Since $2^{q(p-1)} \leq Z < 2^{qp}$, we deduce that $q(p-1) \leq \ell < qp$; hence, $2^{q(p-2)}$ is in $\mathbb{N}$. Since both $2^{q(p-2)}$ and $X \cdot 2^{(p+2)}$ are in $\mathbb{N}$, it follows that $(Y \cdot 2^{(p+2)})^2$ is also in $\mathbb{N}$, and from Lemma 4, we deduce that $Y \cdot 2^{(p+2)} \in \mathbb{N}$. Then, Corollary 2 implies that the only possible decomposition as the sum of two squares is $2^{q(p-2)} = 0^2 + (2^{q(p-2)})^2$ so that $(X, Y) = \{0, 2^{q(p-2)}\}$. Conversely, if $(x, y) = \{0, \pm 2^k\}$, then $1/\sqrt{x^2 + y^2} = 2^{-k}$ belongs to $\mathbb{F}_{2^p}$. Now, let us assume that $\beta = 10$. Then, (27) becomes

$$Z^2(X^2 \cdot 10^{2^{q-2}c} + Y^2) = 10^{p-2}Z.$$

Since $10^{p-4}Z$ is a multiple of $Z$, necessarily $Z = 2^k \cdot 5^\ell$ with $k, \ell \in \mathbb{N}$ such that $10^{p-1} \leq 2^k \cdot 5^\ell < 10^p$, which implies that $\ell \leq 2p$ and $k \leq 4p$. Moreover, from (29) with $Z = 2^k \cdot 5^\ell$, we have

$$(X \cdot 10^{2p+2} + Y \cdot 10^{2p+2})^2 = 2^{q(p-1)} \cdot 5^{q(p-2)}.$$

Since $(X \cdot 10^{2p+2})^2$ and $2^{q(p-2)} \cdot 5^{q(p-2)}$ are both in $\mathbb{N}$, then $Y \cdot 10^{2p+2}$ also belongs to $\mathbb{N}$, which concludes the proof.

In radix $2^p$, the pairs $(x, y)$ such that $1/\sqrt{x^2 + y^2}$ is a midpoint are characterized by Theorem 13. In radix 10, for each $Z = 2^k \cdot 5^\ell$ with $k, \ell \in \mathbb{N}$ such that $10^{p-1} \leq 2^k \cdot 5^\ell < 10^p$, we are reduced to find all decompositions of $2^{q(p-2)} \cdot 5^{q(p-2)}$ as sums of two squares. This is done as explained in Section 8.1. For each basic decimal format of the IEEE 754-2008 standard, the following table gives the number $N$ of midpoints with a fixed exponent $e_x$, together with the number $N'$ of pairs of significands $(X, Y)$ with $X \geq Y$ such that $1/\sqrt{x^2 + y^2}$ is in $\mathbb{F}_{10^p}$.

<table>
<thead>
<tr>
<th>Format</th>
<th>decimal32</th>
<th>decimal64</th>
<th>decimal128</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>7</td>
<td>16</td>
<td>34</td>
</tr>
<tr>
<td>$N$</td>
<td>42</td>
<td>93</td>
<td>196</td>
</tr>
<tr>
<td>$N'$</td>
<td>160</td>
<td>764</td>
<td>3373</td>
</tr>
</tbody>
</table>

### 9.2 Exact Points for 2D Normalization

The next theorem provides a characterization of the exact points of the 2D-normalization function in radix $2^q$.

**Theorem 15.** Let $q \in \mathbb{N}_{>0}$ and let $x, y \in \mathbb{F}_{2^p}$ be such that $(x, y) \neq (0, 0)$. One has $z = x/\sqrt{x^2 + y^2}$ in $\mathbb{F}_{2^p}$ if and only if $z < 0$ or $y = 0$.

**Proof.** The “if” statement is obvious. Conversely, assume that $z \in \mathbb{F}_{2^p}$ and both $x$ and $y$ are nonzero. We can restrict to $x, y > 0$ with no loss of generality. Let $z = x/\sqrt{x^2 + y^2}$. Since $z \leq 1$, necessarily $e_z \leq 0$. Then, using $x^2(1 - z^2) = y^2z^2$ and the normalized representations of $x$ and $y$,

$$X^2(\beta_{2^p} - Z^2) = Y^2Z^2 \cdot \beta_{2^p}.$$

From $e_z \leq 0$, it follows that the left-hand side of (34) is in $\mathbb{N}$ and, due to Lemma 4, so is $YZ \cdot \beta_{2^p}$. Now, since $Z^2Y^2 \cdot \beta_{2^p}$ is a multiple of $X^2$, we have $YZ \cdot \beta_{2^p} = JX$ for some $J \in \mathbb{N}_{>0}$. Then, we obtain from (34)

$$((\beta_{2^p} + 1)^2) = J^2 + Z^2.$$

When $\beta = 2^q$, Corollary 2 implies that either $J$ or $Z$ is zero, a contradiction.

### 9.3 Normalization of 2D Vectors

Theorem 7 shows that $x/\sqrt{x^2 + y^2}$ cannot be a midpoint in radix 2. Here, we first extend this result to radices 2$^q$ and 10. Then, we characterize the exact points of the 2D-normalization function in radix 2$^q$. In radix 2$^q$, the pairs $(x, y)$ such that $1/\sqrt{x^2 + y^2}$ is a midpoint are characterized by Theorem 13. In radix 10, for each $Z = 2^k \cdot 5^\ell$ with $k, \ell \in \mathbb{N}$ such that $10^{p-1} \leq 2^k \cdot 5^\ell < 10^p$, we are reduced to find all decompositions of $2^{q(p-2)} \cdot 5^{q(p-2)}$ as sums of two squares. This is done as explained in Section 8.1. For each basic decimal format of the IEEE 754-2008 standard, the following table gives the number $N$ of midpoints with a fixed exponent $e_x$, together with the number $N'$ of pairs of significands $(X, Y)$ with $X \geq Y$ such that $1/\sqrt{x^2 + y^2}$ is in $\mathbb{F}_{10^p}$.
This experiment suggests that the number of pairs \((x, y)\)
 such that \(x/\sqrt{x^2 + y^2}\) is an exact point grows very rapidly
with \(p\), and no useful enumeration can be performed.

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Let \(x\) and \(y\) be two numbers in \(F_{2^p}\), and assume that \(z = \sqrt{x^2 + y^2}\) is a midpoint. We use the normalized representations
of \(x\) and \(y\), and write as usual \(z = (Z + 1/2) \cdot \beta^{e_Z+1}\).
Without loss of generality, we assume that \(x \geq y\), which implies that \(e_x \geq e_y\). Then, from \(x^2 + y^2 = z^2\), it follows that

\[
4(y^2 + x^2 \cdot \beta^{2e_x-2e_y}) = (2Z + 1)^2 \cdot \beta^{2e_x-2e_y}. \tag{36}
\]

When \(\beta\) is odd, the right-hand side of (36) is odd, while the left-hand side is always even. Hence, if the radix \(\beta\) is odd, \(\sqrt{x^2 + y^2}\) cannot be a midpoint, and this observation can be
generalized to the euclidean norm in higher dimensions.
Nevertheless, this not a very useful result since it does not
hold for binary, decimal, or hexadecimal arithmetic.

For even radices, we do not have general results. Equation (36) has solutions, and exhaustive enumeration
can be performed at least for small precisions. In radices 2
and 10, and for some small precisions \(p\), the following two
tables display the number \(N\) of input pairs \((x, y)\), with \(x \geq y\), such that \(z = \sqrt{x^2 + y^2}\) is a midpoint in \([1, \beta]\).

<table>
<thead>
<tr>
<th>(p)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>18</td>
<td>30</td>
<td>76</td>
<td>155</td>
<td>348</td>
</tr>
</tbody>
</table>

These experiments suggest that the number of midpoints
for the function \((x, y) \mapsto \sqrt{x^2 + y^2}\) grows very rapidly with \(p\).

On the other hand, in one dimension, the euclidean norm
reduces to the absolute value, which suffices to see that it
admits only exact points.

11 CONCLUSION

We have shown that for several simple algebraic functions
\((y_k, \sqrt{y_k}, x^k\) for \(k \in \mathbb{N}_{\geq 0}, x/\|x\|_2, x/y, 1/y, 1/\sqrt{x^2 + y^2}, x/\sqrt{x^2 + y^2}\), we can obtain useful information on
the existence of midpoints and exact points. This information
can be used for simplifying or improving the performance
of programs that evaluate these functions.

Finding midpoints and exact points would also be of
interest for the most common transcendental functions
(sine, cosine, exponential, logarithm, ...). Providing these
functions with correct rounding is a difficult problem,
known as the Table-Maker’s Dilemma [7], [10]. For the most
simple transcendental functions, these built from the
complex exponential and logarithm, one can deduce the
nonexistence of midpoints from the following corollary of
Lindemann’s theorem (see, for example, [2, p. 6]):

**Theorem 16** (Lindemann). \(e^z\) is transcendental for every
nonzero algebraic complex number \(z\).
In radix-2 arithmetic with integer too (which implies the existence of midpoints in applications (physics, statistics, etc.), almost nothing is known about their midpoints or exact points in floating-point arithmetic. Thus, the only result that can be used to deal with a very few cases is that $\Gamma(x)$ is shown to be irrational if $x \pmod{1}$ belongs to $\{1/6, 1/4, 1/3, 1/2, 2/3, 3/4, 5/6\}$.

**APPENDIX**

**Exact Points for Reciprocal, Square Root Reciprocal, and 2D-Norm Reciprocal**

See Tables 7, 8, and 9.

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**REFERENCES**


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**APPENDIX**

**Exact Points for Reciprocal, Square Root Reciprocal, and 2D-Norm Reciprocal**

See Tables 7, 8, and 9.
Claude-Pierre Jeannerod received the PhD degree in applied mathematics from the Institut National Polytechnique de Grenoble in 2000. After being a postdoctoral fellow in the Symbolic Computation Group at the University of Waterloo, he is now a researcher at INRIA Grenoble—Rhone-Alpes, and a member of the LIP Laboratory (LIP is a joint laboratory of CNRS, Ecole Normale Superieure de Lyon, INRIA, and Universite Claude Bernard Lyon 1). His research interests include computer algebra, linear algebra, and floating-point arithmetic. He is a member of the ACM and the IEEE.

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