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On counting untyped lambda terms

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Abstract

We present several results on counting untyped lambda terms, i.e., on telling how many terms belong to such or such class, according to the size of the terms and/or to the number of free variables.

1 Introduction

This paper presents several results on counting untyped lambda terms, i.e., on telling how many terms belong to such or such class, according to the size of the terms and/or to the number of free variables. In particular, it gives results on the distribution of terms according to their number of free variables and explores properties of these distributions, especially through generating functions. In addition to the inherent interest of these results from the mathematical point of view, we expect that a knowledge on the distribution of terms will improve the implementation of reduction and that results on asymptotic distributions of terms will give a better insight of the lambda calculus. We show strong evidence for the claim that when the number m of free variables tends to infinity, the ratio “number of normal forms of size n over number of terms of size n ” decreases and tends to 1, meaning that when m grows, more and more terms are normal forms. The results presented here are a milestone in describing probabilistic properties of lambda terms with answers to questions like: How does a random lambda term look like? How does a random normal form look like? How to generate a random lambda term (a random normal form)?

Related works

Previous works on counting lambda terms were by O. Bodini et al. [2], R. David et al. [3] and J. Wang [11]. Related works are on counting types and/or counting tautologies [12, 8, 4, 9].

2 Untyped lambda terms with de Bruijn indices

I am dedicating this book to N. G. “Dick” de Bruijn, because his influence can be felt on every page. Ever since the 1960s he has been my chief mentor, the main person who would answer my question when I was stuck on a problem that I had not been taught how to solve.

Donald Knuth in preface of [10]

The λ -calculus [1] is a logic formalism to describe functions, for instance, the function $f \mapsto (x \mapsto f(f(x)))$, which takes a function f and applies it twice. For historical reason, this function is written $\lambda f.\lambda x.f(fx)$, which contains the two variables f and x , bounded by λ .

In this paper we represent terms by De Bruijn indices [6], this means that variables are represented by numbers $\underline{1}, \underline{2}, \dots, \underline{m}, \dots$, where an index, for instance \underline{k} , is the number of λ , above the location of the index and below the λ that binds the variable, in a representation of λ -terms by trees. For instance, the term with variables $\lambda x.\lambda y.xy$ is represented by the term with de Bruijn indices $\lambda\lambda\underline{2}\underline{1}$. The variable x is bound by the top λ . Above the occurrence of x , there are two λ 's, therefore x is represented by $\underline{2}$ and from the occurrence of y , one counts just the λ that binds y ; so y is represented by $\underline{1}$. In what follows we will call *terms*, the untyped terms with de Bruijn indices. Let us call $\mathcal{T}_{n,m}$, the set of terms of size n , with m de Bruijn indices, i.e., with indices in $\mathcal{I}(m) = \{\underline{1}, \underline{2}, \dots, \underline{m}\}$. A term in $\mathcal{T}_{n,m}$, is either a de Bruijn index or an abstraction on a term with $m+1$ indices i.e., a term in $\mathcal{T}_{n,m+1}$ or an application of a term in $\mathcal{T}_{n,m}$ on a term in $\mathcal{T}_{n,m}$. We can write, using $@$ as the application symbol,

$$\mathcal{T}_{n+1,m} = \mathcal{I}(m) \uplus \lambda\mathcal{T}_{n,m+1} \uplus \bigoplus_{k=0}^n \mathcal{T}_{n-k,m} @ \mathcal{T}_{k,m}.$$

Moreover terms of size 1 are only made of de Bruijn indices, therefore

$$\mathcal{T}_{1,m} = \mathcal{I}(m).$$

From this one gets:

$$\begin{aligned} T_{n+1,m} &= T_{n,m+1} + \sum_{k=0}^n T_{n-k,m} \cdot T_{k,m} \\ T_{1,m} &= m \\ T_{0,m} &= 0 \end{aligned}$$

$T_{n,0}$ is the set of closed terms (terms with no non bound indices). Notice that

$$T_{n+1,m} = T_{n,m+1} + \sum_{k=1}^{n-1} T_{n-k,m} \cdot T_{k,m}$$

Let us illustrate this result by the array of closed terms up to size 5:

| n | terms | $T_{n,0}$ |
|-----|--|-----------|
| 1 | none | 0 |
| 2 | $\lambda\mathbf{1}$ | 1 |
| 3 | $\lambda\lambda\mathbf{1}, \lambda\lambda\mathbf{2}$, | 2 |
| 4 | $\lambda\lambda\lambda\mathbf{1}, \lambda\lambda\lambda\mathbf{2}, \lambda\lambda\lambda\mathbf{3}, \lambda(\mathbf{1}\mathbf{1})$ | 4 |
| 5 | $\lambda\lambda\lambda\lambda\mathbf{1}, \lambda\lambda\lambda\lambda\mathbf{2}, \lambda\lambda\lambda\lambda\mathbf{3}, \lambda\lambda\lambda\lambda\mathbf{4}, \lambda\lambda(\mathbf{1}\mathbf{1}), \lambda\lambda(\mathbf{1}\mathbf{2}), \lambda\lambda(\mathbf{2}\mathbf{1}), \lambda\lambda(\mathbf{2}\mathbf{2}),$ $\lambda(\mathbf{1}\mathbf{1}\mathbf{1}), \lambda(\mathbf{1}\mathbf{1}\mathbf{2}), \lambda((\lambda\mathbf{1})\mathbf{1}), \lambda((\lambda\mathbf{2})\mathbf{1}), \lambda\mathbf{1}\mathbf{1}$ | 13 |

The equation that defines $T_{n,m}$ allows us to compute it, since it relies on entities $T_{i,j}$ where either $i < n$ or $j < m$. Figure 1 is a table of the first values of $T_{n,m}$ up to $T_{18,7}$. We are mostly interested by the sequence of sizes of the closed terms, namely $T_{n,0}$, in other words the first column of the table. The values of $T_{n,0}$ correspond to sequence **A135501** (see <http://www.research.att.com/~nudges/sequences/A135501>) due to Christophe Raffalli, which is the *number of closed lambda-terms of size n*. His formula for those numbers is more complex. He considers the values of the double sequence $f_{n,m}$. $T_{n,m}$ and $f_{n,m}$ coincide for $m = 0$, i.e., $T_{n,0} = f_{n,0}$.

$$\begin{aligned}
f_{1,1} &= 1 \\
f_{0,m} &= 0 \\
f_{n,m} &= 0 \text{ if } m > 2n - 1 \\
f_{n,m} &= f_{n-1,m} + f_{n-1,m+1} + \sum_{p=1}^{n-2} \sum_{c=0}^m \sum_{l=0}^{m-c} \binom{c}{m} \binom{l}{m-c} f_{p,l+c} f_{n-p-1,m-l}.
\end{aligned}$$

He adds

The last term is for the application where c is the number of common variables in both subterms. $f_{n,m}$ can be computed only using $f_{n',m'}$ with $n' < n$ and $m' \leq m + n - n'$.

Notice that he deals only with sequence $T_{n,0}$, whereas we consider the values for any value of m , from which we can expect to extract interesting informations. The main interesting statement we can draw from this is that considering lambda terms with explicit variables or considering lambda terms with de Bruijn indices makes no difference, at least when no β -reduction is taken into account. We feel that considering lambda terms with de Bruijn indices makes the task easier and produces more results.

3 The functions $m \mapsto T_{n,m}$

Due to properties of the generating function (see Section 5) we are not able to give a simple expression for the function $n \mapsto T_{n,m}$, so we focus on the function $m \mapsto T_{n,m}$. These functions are polynomial P_n^T , defined recursively as follows:

$$P_0^T(m) = 0 \tag{1}$$

$$P_1^T(m) = m \tag{2}$$

$$P_{n+1}^T(m) = P_n^T(m+1) + \sum_{k=1}^{n-1} P_k^T(m) P_{n-k}^T(m). \quad (3)$$

See Figure 2 for the first 18 polynomials. The degrees of those polynomials increase two by two and we can describe their leading coefficients.

Proposition 1 $\deg(P_{2p-1}^T) = \deg(P_{2p}^T) = p$.

Proof: This is true for $P_1^T = m$ and $P_2^T = m+1$ which have degree 1. Assume the property true up to p . Note that all the coefficients are positive. In

$$P_n^T(m+1) + \sum_{k=1}^{n-1} P_k^T P_{n-k}^T,$$

the degree of $P_{n+1}^T(m)$ comes from the sum. Consider the degree of $P_k^T P_{n-k}^T$ according to the parity of n and k .

$n = 2p+1$ **and** $k = 2h-1$. In this case, $p \geq h \geq 1$ and the degree of P_{2h-1}^T is h and the degree of $P_{2p+1-2h+1}^T$ is $p-h+1$, hence the degree of $P_{2h-1}^T P_{2p+1-2h+1}^T$ is $p+1$.

$n = 2p+1$ **and** $k = 2h$. In this case, $p \geq h \geq 1$ and the degree of P_{2h}^T is h and the degree of $P_{2p+1-2h}^T$ is $p-h+1$, hence the degree of $P_{2h}^T P_{2p+1-2h}^T$ is $p+1$.

$n = 2p$ **and** $k = 2h-1$. In this case, $p+1 \geq h \geq 1$ and the degree of P_{2h-1}^T is h and the degree of $P_{2p-2h+1}^T$ is $p-h+1$, hence the degree of $P_{2h-1}^T P_{2p-2h+1}^T$ is $p+1$.

$n = 2p$ **and** $k = 2h$. In this case, $p+1 \geq h \geq 1$ and the degree of P_{2h}^T is h and the degree of P_{2p-2h}^T is $p-h$, hence the degree of $P_{2h}^T P_{2p-2h}^T$ is p . These products $P_{2h}^T P_{2p-2h}^T$ do not contribute to the degree of P_{2p+1}^T .

□

In what follows, for short, we write θ_{2q+1} and θ_{2q} the leading coefficients of P_{2q+1}^T and P_{2q}^T .

Proposition 2 *The leading coefficients of P_{2q+1}^T are $\frac{1}{q+1} \binom{2q}{q}$, i.e., the Catalan numbers.*

Proof: From Equation 3 and the last two steps of the proof of Proposition 1, we deduce the following relation:

$$\theta_{2q+1} = \sum_{h=0}^q \theta_{2h-1} \theta_{2q-2h+1}$$

Let us set

$$\mathcal{O}d(z) = \sum_{i=0}^{\infty} \theta_{2i+1} z^i.$$

We get:

$$\mathcal{O}d(z) = 1 + z \mathcal{O}d(z)^2.$$

which shows that

$$\mathcal{O}d(z) = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

and $\mathcal{O}d$ is the generating function of the Catalan numbers. \square

Proposition 3 *The leading coefficients of P_{2q}^T are $\binom{2q-1}{q}$.*

Proof: Without loss of generality, we assume that $\theta_0 = 0$. From Equation 3, we get

$$\begin{aligned} \theta_{2q} &= \theta_{2q-1} + \sum_{k=0}^{2q-1} \theta_k \theta_{2q-1-k} \\ &= \theta_{2q-1} + \sum_{h=1}^q \theta_{2h-1} \theta_{2q-2h} + \sum_{h=0}^{q-1} \theta_{2h} \theta_{2q-1-2h} \\ &= \theta_{2q-1} + 2 \sum_{h=0}^{q-1} \theta_{2h} \theta_{2q-2h-1}. \end{aligned}$$

If one sets

$$\mathcal{E}v(z) = \sum_{i=0}^{\infty} \theta_{2i} z^i$$

one gets:

$$\mathcal{E}v(z) = \mathcal{O}d(z) + 2z\mathcal{E}v(z)\mathcal{O}d(z),$$

hence

$$\mathcal{E}v(z) = \frac{\mathcal{O}d(z)}{1 - 2z\mathcal{O}d(z)} = \frac{(1 - 4z)^{-\frac{1}{2}} - 1}{2z}.$$

which is the generating function of $\binom{2q-1}{q}$. \square

The table below gives the coefficients of the polynomials P_n^T up to 13. From this table, we induce that second leading coefficients of P_{2q+1}^T are $q\binom{2q-1}{q}$. Hence typically

$$P_{2q+1}^T(m) = \frac{1}{q+1} \binom{2q}{q} m^{q+1} + q \binom{2q-1}{q} m^q + \dots + T_{2q+1,0}.$$

| $n \setminus m^i$ | m^7 | m^6 | m^5 | m^4 | m^3 | m^2 | m | 1 |
|-------------------|-------|-------|-------|-------|--------|--------|--------|--------|
| 1 | | | | | | | 1 | 0 |
| 2 | | | | | | | 1 | 1 |
| 3 | | | | | | 1 | 1 | 2 |
| 4 | | | | | | 3 | 5 | 4 |
| 5 | | | | | 2 | 6 | 17 | 13 |
| 6 | | | | | 10 | 26 | 49 | 42 |
| 7 | | | | 5 | 30 | 111 | 179 | 139 |
| 8 | | | | 35 | 134 | 405 | 683 | 506 |
| 9 | | | 14 | 140 | 652 | 1658 | 2629 | 1915 |
| 10 | | | 126 | 676 | 2812 | 7122 | 10725 | 7558 |
| 11 | | 42 | 630 | 3610 | 12760 | 30783 | 45195 | 31092 |
| 12 | | 462 | 3334 | 17670 | 60240 | 138033 | 196355 | 132170 |
| 13 | 132 | 2772 | 19218 | 87850 | 285982 | 635178 | 880379 | 580466 |

4 Normal forms

Lest us call \mathcal{F}_m the set of normal forms with m de Bruijn indices and \mathcal{G}_m the sets of normal forms with no head λ . The combinatorial structure equations are

$$\begin{aligned}\mathcal{G}_m &= \mathcal{I}(m) \uplus \mathcal{G}_m @ \mathcal{F}_m \\ \mathcal{F}_m &= \lambda \mathcal{F}_{m+1} \uplus \mathcal{G}_m\end{aligned}$$

Let $G_{n,m}$ be the number of normal forms of size n with no head λ and with de Bruijn indices in $\mathcal{I}(m)$ and $F_{n,m}$ be the number of normal forms of size n with de Bruijn indices in $\mathcal{I}(m)$. The relations between $G_{n,m}$ and $F_{n,m}$ are

$$\begin{aligned}G_{1,m} &= m \\ G_{n+1,m} &= \sum_{k=0}^n G_{n-k,m} F_{k,m} \\ F_{1,m} &= m = G_{1,m} \\ F_{n+1,m} &= F_{n,m+1} + G_{n+1,m}\end{aligned}$$

whereas the relations between generating functions are

$$\begin{aligned}G_m(z) &= mz + zG_m(z)F_m(z) \\ F_m(z) &= zF_{m+1}(z) + G_m(z).\end{aligned}$$

The coefficients $F_{n,m}$ are given in Figure 3.

The functions $m \mapsto F_{n,m}$

Like for $m \mapsto T_{n,m}$, the functions $m \mapsto F_{n,m}$ are polynomials of degree $n+1 \div 2$, which we write P_n^{NF} and which are given in Figure 4. We can conjecture facts about the coefficients of polynomials P_n^{NF} somewhat similar to those proved for polynomials P_n^T .

- Like for P_{2q+1}^T the coefficients of P_{2q+1}^{NF} are Catalan numbers.
- The coefficients of P_{2q+2}^{NF} are $2\binom{2q-1}{q}$, i.e., $P_{2q+2}^{NF} = 2P_{2q}^T$.

Once proved, this means that the leading coefficients of P_n^T and P_n^{NF} have asymptotically the same behavior, so this shows the following fact: when the number m of free variables grows, $P_{2q+1}^T(m) \sim P_{2q+1}^{NF}(m)$ and since $P_{2q-1}^T < P_{2q}^T < P_{2q+1}^T$ and $P_{2q-1}^{NF} < P_{2q}^{NF} < P_{2q+1}^{NF}$ then when m grows the set of normal forms tends to be the same as the sets of terms and the set of normal forms tends to be no more negligible.

5 Generating functions for terms

We consider the bivariate generating function for terms which coefficients are the $T_{n,m}$ and the vertical generating functions which gives the $T_{n,m}$ for each value of m .

The bivariate generating function for terms

Let $T(z, u)$ be the bivariate generating function of $T_{n,m}$.

$$\begin{aligned}
T(0, u) &= 0 \\
\text{and} \\
T(z, u) &= \sum_{n,m \geq 0} T_{n,m} z^n u^m \\
&= z \sum_{m \geq 0} m u^m + \sum_{n \geq 1, m \geq 0} T_{n-1, m+1} z^n u^m + zT^2(z, u) \\
&= \frac{z u}{(1-u)^2} + \sum_{n \geq 0, m \geq 1} T_{n,m} z^{n+1} u^{m-1} + zT^2(z, u) \\
&= \frac{z u}{(1-u)^2} + \frac{z}{u} (T(z, u) - T(z, 0)) + zT^2(z, u)
\end{aligned}$$

Hence

$$z u T^2(z, u) + (z - u) T(z, u) + \frac{z u^2}{(1-u)^2} - z T(z, 0) = 0$$

and

$$\boxed{T(z, u) = \frac{-(z-u) + \sqrt{(z-u)^2 + 4zu\left(\frac{z u^2}{(1-u)^2} - zT(z, 0)\right)}}{2zu}}.$$

We notice that

$$\lim_{u \rightarrow 0} T(z, u) = T(z, 0).$$

On the other hand we see the presence of $T(z, 0)$ in the lefthand side, which shows that the equation is not a definition of T but rather an equation which

characterizes T as a fixed point. Presented like that, the equation is not amenable to the same treatment as those presented in Flajolet and Sedgewick [7] and specialists consider it as difficult and no result is known about the asymptotic behavior of its coefficient.

Vertical generating functions

We see that

$$T_{n,m+1} = T_{n+1,m} - \sum_{k=0}^n T_{n-k,m} T_{k,m}.$$

Hence

$$T^{(m)}(0) = 0$$

and

$$\begin{aligned} T^{(m+1)}(z) &= \sum_{n=0}^{\infty} T_{n,m+1} z^n \\ &= \sum_{n=0}^{\infty} T_{n+1,m} z^n - \sum_{n=0}^{\infty} \sum_{k=0}^n T_{n-k,m} T_{k,m} z^n \\ &= \frac{T^{(m)}(z)}{z} - (T^{(m)}(z))^2 \end{aligned}$$

In other words

$$z(T^{(m)}(z))^2 - T^{(m)}(z) + zT^{(m+1)}(z) = 0.$$

Hence

$$T^{(m)}(z) = \frac{1 - \sqrt{1 - 4z^2 T^{(m+1)}(z)}}{2z}.$$

One sees that $T^{(m)}$ is defined from $T^{(m+1)}$. Like the bivariate generating function $T(z, u)$, $T^{(m)}(z)$ is also difficult to study, because we have $T^{(m)}$ defined in term of $T^{(m+1)}$.

6 Conclusion

We have given several parameters on numbers of untyped lambda terms and untyped normal forms and proved or conjectured facts about them. Further tracks can lead to compute moments which will allow us to characterize precisely properties of large random lambda terms which hold with high probability. On another direction, it could be worth to study typed lambda terms, whereas we have only analyzed untyped lambda terms in this paper.

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| $n \setminus m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------------|------------|------------|-------------|--------------|--------------|---------------|---------------|----------------|
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 2 | 4 | 8 | 14 | 22 | 32 | 44 | 58 |
| 4 | 4 | 12 | 26 | 46 | 72 | 104 | 142 | 186 |
| 5 | 13 | 38 | 87 | 172 | 305 | 498 | 763 | 1112 |
| 6 | 42 | 127 | 324 | 693 | 1294 | 2187 | 3432 | 5089 |
| 7 | 139 | 464 | 1261 | 2890 | 5831 | 10684 | 18169 | 29126 |
| 8 | 506 | 1763 | 5124 | 12653 | 27254 | 52671 | 93488 | 155129 |
| 9 | 1915 | 7008 | 21709 | 57070 | 130863 | 269260 | 508513 | 896634 |
| 10 | 7558 | 29019 | 94840 | 265129 | 646458 | 1406983 | 2791564 | 5136885 |
| 11 | 31092 | 124112 | 427302 | 1264362 | 3262352 | 7502892 | 15703602 | 30429782 |
| 12 | 132170 | 548264 | 1977908 | 6168242 | 16811366 | 40776020 | 89671904 | 181746638 |
| 13 | 580466 | 2491977 | 9384672 | 30755015 | 88253310 | 225197061 | 520076012 | 1104714147 |
| 14 | 2624545 | 11629836 | 45585471 | 156409882 | 471315501 | 1263116040 | 3058077451 | 6789961206 |
| 15 | 12190623 | 55647539 | 226272369 | 810506769 | 2558249963 | 7184911623 | 18208806189 | 42244969589 |
| 16 | 58083923 | 272486289 | 1146515237 | 4275219191 | 14098296495 | 41417170373 | 109721440529 | 265618096347 |
| 17 | 283346273 | 1363838742 | 5923639803 | 22933607180 | 78832280277 | 241776779298 | 668513708207 | 1686996660888 |
| 18 | 1413449148 | 6968881025 | 31177380822 | 125027527671 | 446961983408 | 1428444131853 | 4116538065930 | 10816530842627 |

Figure 1: Values of $T_{n,m}$ up to (18, 7)

| n | $P_n^T(m)$ |
|-----|--|
| 1 | m |
| 2 | $m + 1$ |
| 3 | $m^2 + m + 2$ |
| 4 | $3m^2 + 5m + 4$ |
| 5 | $2m^3 + 6m^2 + 17m + 13$ |
| 6 | $10m^3 + 26m^2 + 49m + 42$ |
| 7 | $5m^4 + 30m^3 + 111m^2 + 179m + 139$ |
| 8 | $35m^4 + 134m^3 + 405m^2 + 683m + 506$ |
| 9 | $14m^5 + 140m^4 + 652m^3 + 1658m^2 + 2629m + 1915$ |
| 10 | $126m^5 + 676m^4 + 2812m^3 + 7122m^2 + 10725m + 7558$ |
| 11 | $42m^6 + 630m^5 + 3610m^4 + 12760m^3 + 30783m^2 + 45195m + 31092$ |
| 12 | $462m^6 + 3334m^5 + 17670m^4 + 60240m^3 + 138033m^2 + 196355m + 132170$ |
| 13 | $132m^7 + 2772m^6 + 19218m^5 + 87850m^4 + 285982m^3 + 635178m^2 + 880379m + 580466$ |
| 14 | $1716m^7 + 16108m^6 + 104034m^5 + 449290m^4 + 1390246m^3 + 2991438m^2 + 4052459m + 2624545$ |
| 15 | $429x^8 + 12012x^7 + 99386x^6 + 560854x^5 + 2308173x^4 + 6895122x^3 + 14436365x^2 + 19144575x + 12190623$ |
| 16 | $6435x^8 + 76444x^7 + 584878x^6 + 3076878x^5 + 12039895x^4 + 34815210x^3 + 71170791x^2 + 92631835x + 58083923$ |
| 17 | $1430x^9 + 51480x^8 + 502384x^7 + 3389148x^6 + 16925916x^5 + 63753310x^4 + 179178860x^3 + 358339416x^2 + 458350525x + 283346273$ |
| 18 | $24310x^9 + 357256x^8 + 3176112x^7 + 19799164x^6 + 93981244x^5 + 342274990x^4 + 938333964x^3 + 1840448776x^2 + 2317036061x + 1413449148$ |

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Figure 2: The polynomials P_n^T for the function $m \mapsto T_{n,m}$

| $n \setminus m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|-----------|-----------|------------|-------------|-------------|--------------|--------------|---------------|---------------|
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 3 | 2 | 4 | 8 | 14 | 22 | 32 | 44 | 58 | 74 |
| 4 | 4 | 10 | 20 | 34 | 52 | 74 | 100 | 130 | 164 |
| 5 | 10 | 25 | 58 | 121 | 226 | 385 | 610 | 913 | 1306 |
| 6 | 25 | 72 | 185 | 400 | 753 | 1280 | 2017 | 3000 | 4265 |
| 7 | 72 | 223 | 614 | 1497 | 3244 | 6347 | 11418 | 19189 | 30512 |
| 8 | 223 | 728 | 2195 | 5716 | 12863 | 25688 | 46723 | 78980 | 125951 |
| 9 | 728 | 2549 | 8108 | 22745 | 56360 | 125093 | 253004 | 473753 | 832280 |
| 10 | 2549 | 9254 | 31253 | 93734 | 244997 | 564854 | 1173029 | 2237558 | 3983189 |
| 11 | 9254 | 35168 | 124778 | 395720 | 1109222 | 2770904 | 6261818 | 12999728 | 25130630 |
| 12 | 35168 | 138606 | 512898 | 1720040 | 5097660 | 13347978 | 31308206 | 66902388 | 132274680 |
| 13 | 138606 | 563907 | 2174894 | 7645095 | 23948550 | 66818531 | 167837142 | 384821079 | 816168830 |
| 14 | 563907 | 2369982 | 9459993 | 34771380 | 114618495 | 335857722 | 880524117 | 2092596528 | 4571548155 |
| 15 | 2369982 | 10231830 | 42221886 | 161568762 | 558056526 | 1723895502 | 4785906510 | 12073186866 | 28016723742 |
| 16 | 10231830 | 45381558 | 192944940 | 765787548 | 2764390146 | 8947158690 | 25962816408 | 68135021640 | 163627733358 |
| 17 | 45381558 | 206266797 | 901441688 | 3701763855 | 13912595562 | 47127027713 | 143678500332 | 397091138883 | 1005324501470 |
| 18 | 206266797 | 959283300 | 4302919895 | 18223902654 | 71123969121 | 251343711032 | 799893538635 | 2302171013970 | 6046781201429 |

Figure 3: Values of $F_{n,m}$ up to $(18, 8)$

| n | $P_n^{NF}(m)$ |
|-----|---|
| 1 | m |
| 2 | $m + 1$ |
| 3 | $m^2 + m + 2$ |
| 4 | $2m^2 + 4m + 4$ |
| 5 | $2m^3 + 3m^2 + 10m + 10$ |
| 6 | $6m^3 + 15m^2 + 26m + 25$ |
| 7 | $5m^4 + 12m^3 + 49m^2 + 85m + 72$ |
| 8 | $20m^4 + 62m^3 + 155m^2 + 268m + 223$ |
| 9 | $14m^5 + 50m^4 + 240m^3 + 589m^2 + 928m + 728$ |
| 10 | $70m^5 + 263m^4 + 870m^3 + 2146m^2 + 3356m + 2549$ |
| 11 | $42m^6 + 210m^5 + 1153m^4 + 3658m^3 + 8351m^2 + 12500m + 9254$ |
| 12 | $252m^6 + 1128m^5 + 4658m^4 + 14838m^3 + 33575m^2 + 48987m + 35168$ |
| 13 | $132m^7 + 882m^6 + 5446m^5 + 21198m^4 + 63138m^3 + 137695m^2 + 196810m + 138606$ |
| 14 | $924m^7 + 4862m^6 + 24086m^5 + 93748m^4 + 275898m^3 + 587814m^2 + 818743m + 563907$ |
| 15 | $429m^8 + 3696m^7 + 25372m^6 + 117120m^5 + 429435m^4 + 1223102m^3 + 2558090m^2 + 3504604m + 2369982$ |
| 16 | $3432m^8 + 20996m^7 + 121286m^6 + 556920m^5 + 2011411m^4 + 5601948m^3 + 11448828m^2 + 15384907m + 10231830$ |
| 17 | $1430m^9 + 15444m^8 + 116892m^7 + 624768m^6 + 2717670m^5 + 9524196m^4 + 26064412m^3 + 52459126m^2 + 69361301m + 45381558$ |
| 18 | $12870m^9 + 90683m^8 + 598120m^7 + 3162562m^6 + 13513606m^5 + 46329205m^4 + 124109404m^3 + 245453736m^2 + 319746317m + 206266797$ |

Figure 4: The polynomials P_n^{NF} for the function $m \mapsto F_{n,m}$