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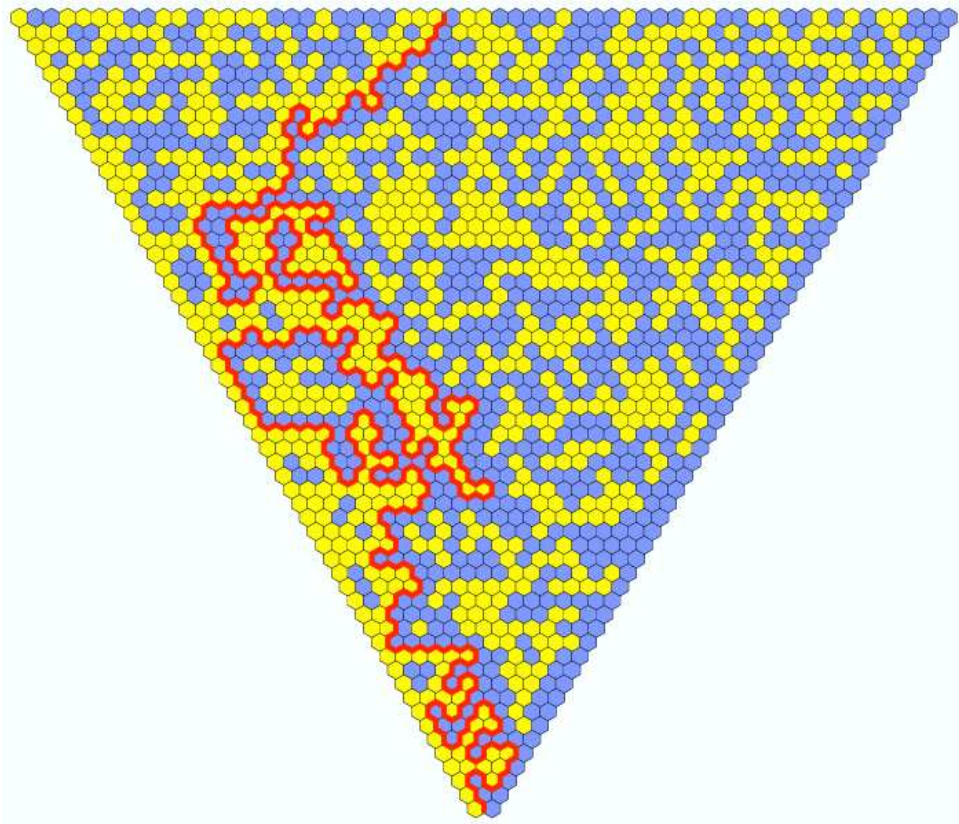
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# Planar percolation with a glimpse of Schramm-Loewner Evolution

Vincent BEFFARA and Hugo DUMINIL-COPIN

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## Abstract

In recent years, important progress has been made in the field of two-dimensional statistical physics. One of the most striking achievement is probably the proof of conformal invariance of percolation. This theorem, together with the introduction of the so-called Schramm-Loewner Evolution and techniques developed over the years in percolation, allow to describe the critical and near-critical regime of percolation very precisely.

**Note to the reader.** Many lecture notes [39] and books [5, 13, 14] already exist on this subject. The present notes are not intended to provide a general exposition but to gather what was explained during a course given by the two authors at the 2011 Florence Summer School, and to provide a few additional details. Most proofs are omitted, since they are given in a very satisfying fashion in other places (we refer to such places where needed). Still, we present a few proofs, which are either quite short and naturally included in the text, or done in a non-standard manner. We include an extended bibliography to which the avid reader is advised to refer.

# 1 Introduction

The model of percolation was introduced by Broadbent and Hammersley in 1950 [7]. For  $p \in (0, 1)$ , *(site) percolation* on the triangular lattice  $\mathbb{T}$  is a random coloring of sites, each site being open with probability  $p$  and closed otherwise, independently of the other sites. Site percolation on the triangular lattice can be seen as a model on faces of the hexagonal lattice (via duality): each face being open with probability  $p$  and closed otherwise, independently of the other faces. In these notes, we will work only with this model and will denote the measure by  $\mathbb{P}_p$ . For general background on percolation, we refer the reader to the books of Grimmett [13] and Kesten [18].

We will be interested in the connectivity properties of the model. Two faces  $a$  and  $b$  of the hexagonal lattice are *connected* (denoted by  $a \leftrightarrow b$ ) if there exists a path of adjacent open faces starting at  $a$  and ending at  $b$ . If there exists a path of adjacent closed faces starting at  $a$  and ending at  $b$ , we write  $a \overset{\star}{\leftrightarrow} b$ . A *cluster* is a maximal connected component of open faces.

It is classical that there exists  $p_c \in (0, 1)$  –called the *critical point*– such that for  $p < p_c$ , there exists almost surely no infinite cluster, while for  $p > p_c$ , there exists almost surely a unique one. Since the model is invariant under translations, the definition of  $p_c$  can be expressed in terms of the probability  $\theta(p) := \mathbb{P}_p(0 \leftrightarrow \infty)$  that the origin is in an infinite cluster: when  $p < p_c$ ,  $\theta(p) = 0$  while for  $p > p_c$ ,  $\theta(p) > 0$ . The aim of these lectures is the following result, that we attribute to several contributors (in the chronological order):

**Theorem 1.1** (Kesten, Schramm, Lawler, Werner, Smirnov). *For the (face) percolation on the hexagonal lattice,  $p_c = 1/2$  and for  $p \searrow 1/2$*

$$\theta(p) = (p - 1/2)^{5/36+o(1)}.$$

The two first lectures will study crossing probabilities (the probability to have an open path connecting two boundary arcs of a domain) when the parameter  $p$  equals  $1/2$ : we will present the Russo-Seymour-Welsh theory and the Cardy-Smirnov formula. The latter, sometimes referred to as conformal invariance of critical percolation, will allow us to describe the scaling limit of interfaces. This fundamental fact is the crucial step in the computation of so-called critical exponents. We will not give any details of the computation of the critical exponents, yet we will discuss some of their applications. In the fourth lecture, we will leave percolation with parameter  $1/2$  to study its dependence in  $p$ . We will prove that  $p = 1/2$  is the critical parameter of site percolation on the triangular lattice. The techniques that we use will allow us to study near-critical percolation, and to derive Kesten scaling relations. This last fact, which is the last step towards Theorem 1.1, will take us the last lecture. We will finish by giving several open questions and conjectures related to percolation.

**Notation and basic properties.** Except otherwise stated,  $\mathbb{H}$  is the hexagonal lattice with mesh-size 1, centered such that the origin 0 is the center of one of the faces, and rotated so that  $i\mathbb{R}$  is an axis of symmetry for  $\mathbb{H}$ . We will often use complex coordinates to specify the position of a point. In particular, a face will be indexed by its center (which corresponds to the vertex in the triangular dual lattice). Rectangles  $[a, b] \times [c, d]$  are the set of faces such that  $e^{i\pi/3}n + m$  with  $n \in [a, b]$  and  $m \in [c, d]$ .

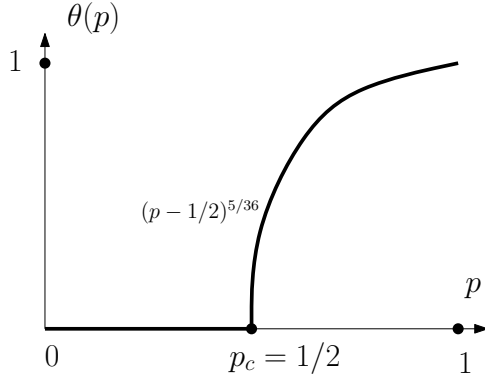


Figure 1: Cluster density with respect to  $p$ . Non-trivial facts in this picture include  $p_c = 1/2$ ,  $\theta(p_c) = 0$  and the behavior near the critical point. We do not investigate properties such as continuity of the cluster density away from  $p_c$ , concavity above  $p_c$  and other known facts.

We will denote by  $u_p \asymp v_p$  if there exist two constants  $A, B$  not depending on  $p$  such that  $Au_p \leq v_p \leq Bu_p$  for all  $p$ .

We will harness the FKG inequality and the monotonicity of percolation a few times. We recall these two facts now. An event is called *increasing* if it is preserved by addition of open faces, see Section 2.2 of [13] (a typical example is the existence of an open path from one set to another). Recall that  $p < p'$  implies that  $\mathbb{P}_p(A) \leq \mathbb{P}_{p'}(A)$ . Moreover, for every  $p \in [0, 1]$  and  $A, B$  two increasing events,

$$\mathbb{P}_p(A)\mathbb{P}_p(B) \leq \mathbb{P}_p(A \cap B) \quad (\text{FKG inequality}).$$

## 2 Crossing probabilities at $p = 1/2$

In this section, fix  $p = 1/2$ .

### 2.1 Russo-Seymour-Welsh theory

Define the ball  $B_n := \{x \in \mathbb{H} : d_{\mathbb{H}}(x, 0) \leq n\}$  ( $d_{\mathbb{H}}(\cdot, \cdot)$  is the graph distance, so that balls have hexagonal shapes) and the annulus  $A_n = B_{3n} \setminus B_n$ . Let  $\mathcal{E}_n$  be the event that there exists an open circuit of adjacent faces in  $A_n$  that surrounds the origin.

**Theorem 2.1** (Russo [30], Seymour-Welsh [31]). *There exists  $C$  such that for every  $n > 0$ ,*

$$\mathbb{P}_{1/2}(\mathcal{E}_n) \geq C > 0.$$

Russo-Seymour-Welsh Theorem (RSW for short) had a great impact on two-dimensional percolation and more generally statistical physics. We will use it extensively in the notes. Though, this property is typical of  $p = 1/2$ : it is natural to expect that the probability of this event goes to 0 (resp. 1) below  $p_c$  (resp. above) since percolation looks like a big sea of closed faces with small islands of open faces in the middle (resp. the opposite). Making this vague statement rigorous is not completely elementary and is the object of Theorem 5.1.

We present one of the many proofs of RSW (this one is inspired by a proof due to Smirnov, and available in french in [40]).

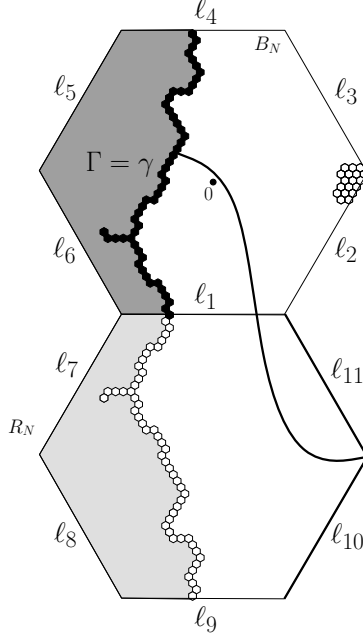


Figure 2: The dark grey area is the set of faces which are discovered after conditioning on  $\{\Gamma = \gamma\}$ . The white area is  $\Omega_\gamma$ .

*Proof.* STEP 1: Let  $n > 0$  and index the edges of  $B_n$  as in Fig. 2. Consider the event that  $\ell_1$  is connected by an open path to  $\ell_3 \cup \ell_4$ . The complement of this event is that  $\ell_2$  is connected by a closed path to  $\ell_5 \cup \ell_6$ . Using the symmetry between closed and open faces and the invariance under the rotation of angle  $\pi/3$ , we find that  $\mathbb{P}_{1/2}(\ell_1 \leftrightarrow \ell_3 \cup \ell_4) = 1/2$ . In fact, we also have that  $\mathbb{P}_{1/2}(\ell_1 \leftrightarrow \ell_4) \geq 1/8$ . Indeed, either this is true or  $\mathbb{P}_{1/2}(\ell_1 \leftrightarrow \ell_3) \geq 1/2 - 1/8$ . But in this case, using FKG,

$$\mathbb{P}_{1/2}(\ell_1 \leftrightarrow \ell_4) \geq \mathbb{P}_{1/2}(\ell_1 \leftrightarrow \ell_3)\mathbb{P}_{1/2}(\ell_2 \leftrightarrow \ell_4) \geq (3/8)^2 \geq 1/8. \quad (2.1)$$

This shows that crossing a rotationally symmetric shape is not hard. The difficult step is to prove that one can cross a thin shape in the long direction.

STEP 2: Now consider the “rectangle”  $R_n = B_n \cup (B_n - 2ni)$  and index its edges as in Fig. 2. For a path  $\gamma$  from  $\ell_1$  to  $\ell_4$ , define the domain  $\Omega_\gamma$  to be the faces of  $R_n$  strictly on the right of  $\gamma \cup \sigma(\gamma)$ , where  $\sigma$  is the symmetry with respect to  $\ell_1$ . Once again, the complement of  $\{\ell_4 \cup \gamma \leftrightarrow \ell_{10} \cup \ell_{11} \text{ in } \Omega_\gamma\}$  is  $\{\ell_9 \cup \sigma(\gamma) \overset{\star}{\leftrightarrow} \ell_2 \cup \ell_3 \text{ in } \Omega_\gamma\}$ . Using the switching of colors and the symmetry with respect to  $\ell_1$ , we deduce that the probability of the former is  $1/2^1$ .

If  $E := \{\ell_1 \leftrightarrow \ell_4\}$  occurs, set  $\Gamma$  to be the left-most crossing between  $\ell_1$  and  $\ell_4$ . For a given path  $\gamma$  from  $\ell_1$  to  $\ell_4$ , the event  $\{\Gamma = \gamma\}$  is measurable only in terms of faces to the left of  $\gamma$ . In particular, conditionally to  $\{\Gamma = \gamma\}$ , the configuration in  $\Omega_\gamma$  is a percolation configuration, so that

$$\mathbb{P}_{1/2}((\ell_4 \cup \gamma) \leftrightarrow (\ell_{10} \cup \ell_{11}) \text{ in } \Omega_\gamma \mid \Gamma = \gamma) = 1/2.$$

<sup>1</sup>Actually it is slightly larger since the face of  $\gamma \cap \ell_1$  is open.

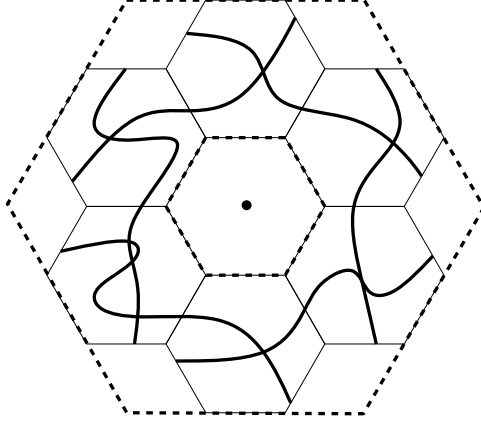


Figure 3: Six “rectangles” which, when crossed, ensure the existence of a circuit in the annulus.

Therefore, we find

$$\begin{aligned}
\mathbb{P}_{1/2}(\ell_4 \leftrightarrow (\ell_{10} \cup \ell_{11})) &= \mathbb{P}_{1/2}(\ell_4 \leftrightarrow (\ell_{10} \cup \ell_{11}), E) \\
&= \sum_{\gamma} \mathbb{P}_{1/2}(\ell_4 \leftrightarrow (\ell_{10} \cup \ell_{11}), \Gamma = \gamma) \\
&\geq \sum_{\gamma} \mathbb{P}_{1/2}((\ell_4 \cup \gamma) \leftrightarrow (\ell_{10} \cup \ell_{11}) \text{ in } \Omega_{\gamma}, \Gamma = \gamma) \\
&\geq \sum_{\gamma} \frac{1}{2} \mathbb{P}_{1/2}(\Gamma = \gamma) = \frac{1}{2} \mathbb{P}_{1/2}(E) = \frac{1}{16}.
\end{aligned}$$

STEP 3: Invoking the FKG inequality, we obtain

$$\mathbb{P}_{1/2}(\ell_4 \leftrightarrow \ell_9) \geq \mathbb{P}_{1/2}(\ell_4 \leftrightarrow (\ell_{10} \cup \ell_{11})) \mathbb{P}_{1/2}((\ell_2 \cup \ell_3) \leftrightarrow \ell_9) \geq \frac{1}{16^2}.$$

Assuming that the six rectangles described in Fig. 3 are crossed (in the sense that there are paths between opposite short edges), we obtain the result using FKG a last time.  $\square$

The first corollary of RSW is the following lower bound on  $p_c$  (historically, this result was first proved this way; a modern and robust argument due to Zhang implies this result very easily, see Section 11 of [13]).

**Corollary 2.2** (Russo [30]). *We have  $\theta(1/2) = 0$ ; in particular,  $p_c \geq 1/2$ .*

*Proof.* We prove that 0 is almost surely not connected by a closed path to infinity (it is the same probability for an open path). Let  $N > 0$ . The origin being connected to  $\partial B_{3N}$  by a closed path implies that for every  $n < N$ ,  $\mathcal{E}_{3n}^c$  occurs. Therefore,

$$\mathbb{P}_{1/2}(0 \overset{*}{\leftrightarrow} \partial B_{3N}) \leq \mathbb{P}_{1/2} \left( \bigcap_{n < N} \mathcal{E}_{3n}^c \right) \leq \prod_{n < N} \mathbb{P}_{1/2}(\mathcal{E}_{3n}^c) \leq (1 - C)^N. \quad (2.2)$$

In the second inequality, we have used independence between percolation in different annuli. In particular, it converges to 0 when  $N$  goes to infinity, implying  $\theta(1/2) = 0$ . The definition of  $p_c$  implies  $p_c \geq 1/2$ .  $\square$

In general, we are interested in crossing probabilities for more general shapes. More precisely, we wish to let the size of the graph go to infinity, but keeping the same global shape. A natural way to do this is to shrink the lattice instead of looking at bigger and bigger scales. This is called taking *scaling limits*.

Consider a topological rectangle  $(\Omega, A, B, C, D)$ , *i.e.* a smooth, bounded simply connected domain  $\Omega \neq \mathbb{C}$  with four distinct points  $A, B, C$  and  $D$  on the boundary, indexed in counter-clockwise order. For  $\delta > 0$ , we will be interested in percolation on  $\Omega_\delta := \Omega \cap \delta\mathbb{H}$ . The graph  $\Omega_\delta$  should be seen as a discretization of  $\Omega$  at scale  $\delta$ . Let  $A_\delta, B_\delta, C_\delta$  and  $D_\delta$  be the vertices of  $\partial\Omega_\delta$  that are closest to  $A, B, C$  and  $D$  respectively. Let  $C_\delta(\Omega, A, B, C, D)$  be the event that there is an open path in  $\Omega_\delta$ , between the intervals  $A_\delta B_\delta$  and  $C_\delta D_\delta$  of its boundary. We call such a path a *crossing*, and the event a *crossing event*. Sometimes, we will say that the rectangle is *crossed* if there exists a crossing.

With a slight abuse of notation, we will denote the percolation of parameter  $1/2$  on  $\delta\mathbb{H}$  by  $\mathbb{P}_{1/2}$  (even though the measure is the push-forward of  $\mathbb{P}_{1/2}$  by the homothety  $x \mapsto \delta x$ ).

**Corollary 2.3.** *Let  $(\Omega, a, b, c, d)$  be a topological rectangle. There exist  $C_1, C_2 > 0$  such that for every  $\delta > 0$ ,*

$$0 < C_1 \leq \mathbb{P}_{1/2}(C_\delta(\Omega, A, B, C, D)) \leq C_2 < 1.$$

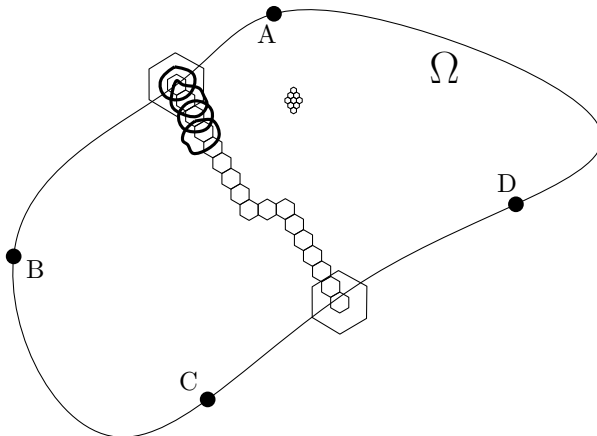


Figure 4: Paths of annuli linking two edges of a topological rectangle. If each of these annuli contains a circuit disconnecting the interior from the exterior boundary, we obtain an open path connecting the two edges.

*Proof.* It is sufficient to prove the lower bound. Indeed, the upper bound is a consequence of the following fact: since the complement of  $C_\delta(\Omega, A, B, C, D)$  is the existence of a closed circuit from  $B_\delta C_\delta$  to  $D_\delta A_\delta$ , it has same probability as  $C_\delta(\Omega, B, C, D, A)$ . Therefore, if this probability is bounded from below, the probability of  $C_\delta(\Omega, A, B, C, D)$  will be bounded from above.

Fix  $\varepsilon > 0$  independent of  $\delta > 0$ . For a hexagon  $h$  of radius  $\varepsilon > 0$ , we set  $\tilde{h}$  to be the hexagon with the same center and radius  $3\varepsilon$ . Now, consider a collection  $h_1, \dots, h_k$  of hexagons of radius  $\varepsilon$  satisfying the following conditions:

- $h_1$  intersects  $AB$  and  $h_k$  intersects  $CD$ ,

- $\tilde{h}_1, \dots, \tilde{h}_k$  are not intersecting  $BC$  nor  $DA$ ,
- $h_i$  are adjacent.

Let  $E_i^\delta$  the event that there is an open circuit in  $\Omega_\delta \cap (\tilde{h}_i \setminus h_i)$  surrounding  $\Omega_\delta \cap h_i$ . By construction, if each  $E_i^\delta$  occurs, there is a path from  $AB$  to  $CD$ , see Fig. 4. Using Theorem 2.1, the probability is bounded from below by  $C^k$  uniformly in  $\delta$ .  $\square$

In particular, we deduce that long rectangles are crossed in the long direction with probability bounded away from 0 as  $\delta \rightarrow 0$ . This result is the classical formulation of RSW.

We finish with a property of percolation with parameter  $1/2$ : connectivity properties decay as power laws:

**Corollary 2.4.** *There exist  $\alpha, \beta > 0$  such that for every  $n > 0$ ,*

$$n^{-\alpha} \leq \mathbb{P}_{1/2}(0 \leftrightarrow \partial B_n) \leq n^{-\beta}.$$

*Proof.* The existence of  $\beta$  is given by (2.2). For the lower bound, use the following construction: define  $R_n := [0, 2^n] \times [0, 2^{n+1}]$  if  $n$  is odd, and  $R_n := [0, 2^{n+1}] \times [0, 2^n]$  if it is even. Set  $F_n$  to be the event that  $R_n$  is crossed in the “long” direction. Corollary 2.3 implies the existence of  $C > 0$  such that  $\mathbb{P}_{1/2}(F_n) \geq C$  for every  $n > 0$ . We have

$$\mathbb{P}_{1/2}(0 \leftrightarrow \partial B_{3N}) \geq \mathbb{P}_{1/2}\left(\bigcap_{n < N} F_n\right) \geq \prod_{n < N} \mathbb{P}_{1/2}(F_n) \geq C^N.$$

This yields the existence of  $\alpha$ .  $\square$

The event  $\{0 \leftrightarrow \partial B_n\}$  is called the one-arm event, meaning that there exists an open path (an arm) from 0 to the boundary of the box. More generally, for  $j \geq 1$ , fix a sequence  $\sigma$  of colors (“open”  $O$  or “closed”  $C$ ) of length  $j$ . For  $n < N$ , define  $A_{j,\sigma}(n, N)$  to be the event that there are  $j$  *disjoint* paths from  $\partial B_n$  to  $\partial B_N$  with colors  $\sigma_1, \dots, \sigma_j$  when the paths are indexed in counter-clockwise order. For instance,  $A_{1,O}(n, N)$  is the one-arm event corresponding to the existence of a crossing from the inner to the outer boundary of  $B_N \setminus B_n$ .

Let us define  $\pi_{j,\sigma}(n, N) = \mathbb{P}_{1/2}(A_{j,\sigma}(n, N))$  and  $\pi_{j,\sigma}(n) = \mathbb{P}_{1/2}(A_{j,\sigma}(j, n))$ . It can be shown following the previous proof that  $\pi_{j,\sigma}(n, N)$  is between two powers of  $n/N$  uniformly in  $n$  and  $N$ . It is therefore natural to predict that there exists a *critical exponent*  $\alpha_{j,\sigma} \in (0, \infty)$  such that

$$\pi_{j,\sigma}(n, N) = (n/N)^{-\alpha_{j,\sigma} + o_{n/N}(1)}.$$

We mention that when  $j = 5$  and  $\sigma = OCOOC$ , it is possible to compute the critical exponent explicitly using **only RSW**. This is an example of *universal exponent*. We state it as a proposition, and refer to [21, 28] for a proof.

**Proposition 2.5** (Kesten-Sidoravicius-Zhang). *Fix  $j = 5$  and  $\sigma = OCOOC$ , for every  $n > 0$ ,  $\pi_{5,OCOOOC}(k, n) \asymp (k/n)^2$ .*

Note that the result is in fact stronger. Assume that one has RSW below scale  $n$  with a uniform lower bound  $\varepsilon > 0$  on the probability to see a circuit in an annulus, then there exists  $A_\varepsilon$  and  $B_\varepsilon$  such that

$$A_\varepsilon(k/n)^2 \leq \pi_{5,OCOOOC}(k, n) \leq B_\varepsilon(k/n)^2.$$



**Morality.** *RSW allows to prove that connectivity properties decay polynomially fast at criticality. In general, RSW is not sufficient to compute  $\alpha_{j,\sigma}$  or even to guarantee its existence. One needs to understand deeply the behavior of the scaling limit (the limit when the meshsize  $\delta$  goes to 0) of percolation. The main step towards this understanding is the proof that crossing probabilities actually converge. This is the subject of the next section.*

The fact that critical exponents are not implied by RSW is not surprising. Actually, RSW is a very general feature of two-dimensional statistical models with continuous phase transition (even though we know how to prove it in very few cases). Somehow, it is equivalent to the fact that there are scaling limits along subsequences. Note that for different models, critical exponents can be completely different, which reminds us that RSW is not sufficient (except for the universal exponents, which are universal by definition). On the other hand, it can seem surprising that proving the convergence of crossing probabilities is sufficient...

### 3 The Cardy-Smirnov formula

In 1994, Langlands, Poulliot and Saint-Aubin [23] published a number of numerics in favor of conformal invariance of crossing probabilities in the scaling limit (the idea of looking at interfaces and crossings is apparently due to Aizenman). They checked that taking different rectangles, the probability of  $C_\delta(\Omega, A, B, C, D)$  converges when  $\delta$  goes to 0 towards a limit which is the same if  $(\Omega, A, B, C, D)$  and  $(\Omega', A', B', C', D')$  are image of each other by a conformal map. The paper [23], while only numerical, attracted many mathematicians to the domain. The same year, Cardy [9] proposed an explicit formula for the limit of crossing probabilities. Finally, in 2001, Smirnov proved Cardy's formula rigorously.

**Theorem 3.1** (Smirnov [33]). *The probability of the event  $C_\delta(\Omega, A, B, C, D)$  has a limit  $f(\Omega, A, B, C, D)$  as  $\delta$  goes to 0. Moreover, the limit is conformally invariant, in the following sense: If  $\Phi$  is a conformal map from  $\Omega$  to another simply connected domain  $\Omega' = \Phi(\Omega)$ , and extends continuously to  $\partial\Omega$ , then*

$$f(\Omega, A, B, C, D) = f(\Phi(\Omega), \Phi(A), \Phi(B), \Phi(C), \Phi(D)).$$

The proof of this theorem is very well (and very shortly) exposed in the original paper [33]. It has been rewritten in a number of places including [5, 14, 40]. We provide the proof that we used in the lecture (in particular with the same notations), which is mainly inspired by [33] and [3].

As will appear naturally in the proof and was first pointed to by Carleson,  $f$  has a most simple expression in the case where  $\Omega$  itself is an equilateral triangle with vertices  $A$ ,  $B$  and  $C$ , and the fourth point  $D$  is on the interval  $(CA)$ . In this case,  $f(\Omega, A, B, C, D)$  equals  $|CD|/|CA|$ . By conformal invariance, this gives the value of  $f$  for every conformal rectangle, since Riemann mapping Theorem yields that any topological rectangle is conformal to one of these topological rectangles.

*Proof.* For every vertex  $z$  in  $\Omega_\delta \setminus \partial\Omega_\delta$ , we define  $E_{A,\delta}(z)$  to be the event that there exists a simple path of open faces in  $\Omega_\delta$ , separating  $A_\delta$  and  $z$  from  $B_\delta$  and  $C_\delta$  — and  $E_{B,\delta}(z)$ ,  $E_{C,\delta}(z)$  similarly, with obvious circular permutations of the letters. Let  $H_{A,\delta}(z)$  (resp.  $H_{B,\delta}(z)$ ,  $H_{C,\delta}(z)$ ) be the probability of  $E_{A,\delta}(z)$  (resp.  $E_{B,\delta}(z)$ ,  $E_{C,\delta}(z)$ ).

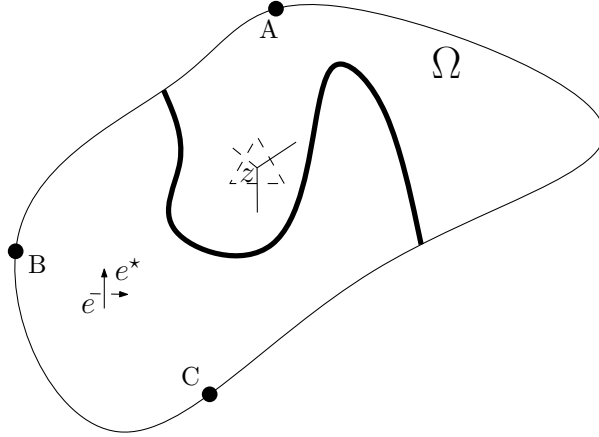


Figure 5: Picture of the event  $E_{A,\delta}(z)$ . We also depicted one oriented edge  $e$  with its associated edge  $e^*$ . The graph  $\mathbb{H}_\delta$  is drawn with solid lines while its dual  $\mathbb{T}_\delta$  is drawn with dotted lines.

The proof runs into three steps, the second one being the most important:

- First, we prove that  $(H_{A,\delta}, H_{B,\delta}, H_{C,\delta})_{\delta>0}$  is a precompact family of functions (with variable  $z$ ).

- Second, introducing the following two sequences of functions defined by

$$H_\delta(z) := H_{A,\delta}(z) + \tau H_{B,\delta}(z) + \tau^2 H_{C,\delta}(z) \quad S_\delta(z) = H_{A,\delta}(z) + H_{B,\delta}(z) + H_{C,\delta}(z),$$

we show that any subsequential limits  $h$  and  $s$  of these sequences are holomorphic. Here,  $\tau = e^{i2\pi/3}$ . In order to prove this statement, we use Morera's theorem and we study discrete integrals.

- Third, we use boundary conditions to identify the possible  $h$  and  $s$ , and thus guarantee the possible subsequential limit of  $(H_{A,\delta}, H_{B,\delta}, H_{C,\delta})_{\delta>0}$  to be unique. A byproduct of the proof is the exact computation of the limit of  $h$  and  $s$ , and thus of the limits of  $(H_{A,\delta})$ ,  $(H_{B,\delta})$  and  $(H_{C,\delta})$ .

Then, making the additional remark that  $E_{C,\delta}(D_\delta)$  is the event  $C_\delta(\Omega, A, B, C, D)$ , this will conclude the proof of Theorem 3.1.

**Precompactness.** Using RSW (Theorem 2.1) in concentric annuli<sup>2</sup>, we obtain the following: There are two positive constants  $K$  and  $\varepsilon$  such that, for every  $\delta > 0$  and any two points  $z$  and  $z'$  in  $\Omega_\delta$  up to the boundary of the domain, yet at distance  $\varepsilon$  away from  $A$ ,  $B$  and  $C$ ,

$$|H_{A,\delta}(z') - H_{A,\delta}(z)| \leq K |z' - z|^\varepsilon \quad (3.1)$$

and a similar bound for  $H_{B,\delta}$  and  $H_{C,\delta}$ . Hence, if we suitably extend these functions continuously to  $\Omega$ , we obtain a family of uniformly Hölder maps from  $\Omega$  to  $[0, 1]$ . The family is then relatively compact with respect to uniform convergence, and it is hence possible to extract a subsequence  $(H_{A,\delta_n}, H_{B,\delta_n}, H_{C,\delta_n})_{n>0}$ , with  $\delta_n \rightarrow 0$ , which converges uniformly to a triple of Hölder maps  $(h_A, h_B, h_C)$  from  $\overline{\Omega}$  to  $[0, 1]$ . From now on, we set  $h = h_A + \tau h_B + \tau^2 h_C$  and  $s = h_A + h_B + h_C$  (they are the limits of  $(H_{\delta_n})_{n>0}$  and  $(S_{\delta_n})_{n>0}$  respectively).

<sup>2</sup>Note that if two points  $z, z'$  are surrounded by an open (or a closed) circuit, then  $H_{A,\delta}(z') = H_{A,\delta}(z)$ .

**Holomorphicity of  $h$  and  $s$ .** To prove that  $h$  is holomorphic, one can try to prove that  $H_{\delta_n}$  is a sequence of (almost) *discrete holomorphic* functions, where one needs to specify what is meant by discrete holomorphic. In our case, it will be that discrete contour integrals vanish<sup>3</sup>. We refer to [36] for more details on discrete holomorphicity and its connections to statistical physics.

Consider a simple, closed, smooth curve  $\gamma$  contained in  $\Omega$ . For every  $\delta > 0$ , let  $\gamma_\delta$  be a discretization<sup>4</sup> of  $\gamma$  contained in  $\Omega_\delta$ , *i.e.* a finite chain  $(\gamma_\delta(k))_{0 \leq k \leq N_\delta}$  of pairwise distinct vertices of  $\mathbb{T}_\delta = \delta\mathbb{H}^*$ , ordered in the positive direction, such that for every index  $k$ ,  $\gamma_\delta(k)$  and  $\gamma_\delta(k+1)$  are nearest neighbors, and chosen in such a way that the Hausdorff distance between  $\gamma_\delta$  and  $\delta$  goes to 0 with  $\delta$ . Notice that  $N_\delta$  can be taken of order  $\delta^{-1}$ , which we shall assume from now on.

The discrete curve  $\gamma_\delta$  surrounds a finite family of triangular faces of  $\mathbb{T}_\delta$ , which we shall denote by  $Int(\gamma_\delta)$ . An oriented edge  $e \in \mathbb{T}_\delta$  *belongs* to  $\gamma_\delta$  if it is of the form  $\gamma_\delta(k)\gamma_\delta(k+1)$  (we set  $e \in \gamma_\delta$ ). It *belongs* to  $Int(\gamma_\delta)$  if it is inside  $\gamma_\delta$  (we write  $e \in Int(\gamma_\delta)$ ).

Define the discrete integral  $I_\gamma^\delta(H)$  of  $H_\delta$  (and similarly  $I_\gamma^\delta(S)$  for  $S_\delta$ ) along  $\gamma_\delta$  by

$$I_\gamma^\delta(H) := \sum_{k=0}^{N_\delta} \frac{H_\delta(\gamma_\delta(k)) + H_\delta(\gamma_\delta(k+1))}{2} (\gamma_\delta(k+1) - \gamma_\delta(k)).$$

Our goal is to prove that  $I_\gamma^\delta(H)$  and  $I_\gamma^\delta(S)$  converge to 0 when  $\delta$  goes to 0. Since along the sequence  $(\delta_n)$ , they also converge to  $\oint_\gamma h(z)dz$  and  $\oint_\gamma s(z)dz$ , it will imply that  $h$  and  $s$  are holomorphic via Morera Theorem (recall that  $\gamma$  is arbitrary and  $h$  and  $s$  are continuous as uniform limits of continuous functions).

For an edge  $e \in \mathbb{T}_\delta$ , define  $e^*$  to be the rotation by  $\pi/2$  of  $e$  (it is an edge of  $\Omega_\delta$ ). For every oriented edge  $e = xy \in \mathbb{T}_\delta$ , set

$$P_{A,\delta}(e) = \mathbb{P}_{1/2}(E_{A,\delta}(y) \setminus E_{A,\delta}(x)),$$

and similarly  $P_B$  and  $P_C$ .

**Lemma 3.2.** *For any smooth  $\gamma$ , we have when  $\delta$  goes to 0,*

$$I_\gamma^\delta(H) = \sum_{e \in Int(\gamma_\delta)} e^* [P_A(e) + \tau P_B(e) + \tau^2 P_C(e)] + o(1) \quad (3.2)$$

$$I_\gamma^\delta(S) = \sum_{e \in Int(\gamma_\delta)} e^* [P_A(e) + \tau^2 P_B(e) + \tau^4 P_C(e)] + o(1) \quad (3.3)$$

*Proof.* We treat the case of  $H_\delta$ . For every oriented edge  $e = xy$  in  $\mathbb{T}_\delta$ , define the following notations:

$$H_\delta(e) := \frac{H_\delta(x) + H_\delta(y)}{2}, \quad \partial_e H_\delta := H_\delta(y) - H_\delta(x).$$

---

<sup>3</sup>Recall that Morera's theorem (see *e.g.* [22]) yields that for any a simply connected domain  $\Omega$  of the complex plane, and any  $f : \Omega \rightarrow \mathbb{C}$  continuous,  $f$  is holomorphic if, and only if, for every simple, closed, smooth curve  $\gamma$  contained in  $\Omega$ , the integral of  $f$  along  $\gamma$  vanishes. Therefore, the previous definition is a relevant discretization of this definition.

<sup>4</sup>discrete contours are often defined on the dual graph. In our case, it is a triangular lattice.

If  $f$  is a (triangular) face of  $\mathbb{T}_\delta$ , let  $\partial f$  be its oriented boundary, seen as a set of oriented edges. With these notations, we get the following identity:

$$I_\gamma^\delta(H) = \sum_{e \in \gamma_\delta} eH_\delta(e) = \sum_{f \in \text{Int}(\gamma_\delta)} \sum_{e \in \partial f} eH_\delta(e). \quad (3.4)$$

Indeed, in the last equality, each boundary term is obtained exactly once with the correct sign, and each interior term appears twice with opposite signs. The sum of  $eH_\delta(e)$  around  $f$  can be rewritten in the following fashion:

$$\sum_{e \in \partial f} eH_\delta(e) = \sum_{e=xy \in \partial f} \left( \frac{x+y}{2} - f \right) \partial_e H_\delta.$$

Putting this quantity in the sum (3.4), the term  $\partial_e H_\delta = H_\delta(y) - H_\delta(x)$  appears twice notice for  $x, y$  nearest neighbors bordered by two triangles in  $\gamma_\delta$ , and the factors  $(x+y)/2$  cancel between the two occurrences, leaving only the difference between the centers of the faces, *i.e.* the dual edge of  $xy$ . Therefore,

$$I_\gamma^\delta(H) = \frac{1}{2} \sum_{e \subset \text{Int}(\gamma_\delta)} e^* \partial_e H_\delta + o(1). \quad (3.5)$$

In the previous equality, we used the fact that the total contribution of the boundary goes to 0 with  $\delta$ . Indeed,  $e^*$  is of order  $\delta$ , and

$$\partial_e H_\delta = P_{A,\delta}(e) - P_{A,\delta}(-e) + \tau(P_{B,\delta}(e) - P_{B,\delta}(-e)) + \tau^2(P_{C,\delta}(e) - P_{C,\delta}(-e)) \quad (3.6)$$

so that RSW gives a bound of  $\delta^{1+\varepsilon}$  for  $e^* \partial_e H_\delta$ . Since there are roughly  $\delta^{-1}$  boundary terms, we obtain that the boundary contributes for at most  $\delta^\varepsilon$ .

Replacing in (3.5)  $\partial H_\delta$  by its expression (3.6), and re-indexing the sum to obtain each oriented edge in exactly one term, we get the required equality.  $\square$

**Lemma 3.3** (Smirnov [33]). *For every edge  $e$  of  $\Omega_\delta$ , we have the following identities:*

$$P_{A,\delta}(e_1) = P_{B,\delta}(e_2) = P_{C,\delta}(e_3),$$

where  $e_1, e_2, e_3$  are three edges emanating from a vertex  $x$ .

Even though we include the proof for completeness, we refer the reader to [33] for the (elementary, but very clever) proof of this result. The proof extends to site-percolation with parameter  $1/2$  on any planar triangulation.

*Proof.* Index the three faces around  $x$  by  $a, b$  and  $c$ , and the vertices by  $y, z$  and  $t$  as depicted in Fig. 6. We see events as subsets of  $\{\text{Open}, \text{Closed}\}^{|\Omega_\delta|}$ .

Let us prove that  $P_{A,\delta}(e_1) = P_{B,\delta}(e_2)$ . The event  $E_{A,\delta}(y) \setminus E_{A,\delta}(x)$  occurs if and only if there are open paths from  $AB$  to  $a$  and from  $AC$  to  $c$ , and a closed path from  $BC$  to  $b$ .

Consider the interface  $\Gamma$  between the open cluster connected to  $AC$  and the closed cluster connected to  $BC$ , starting at  $C$  up to the first time it hits  $x$  (it will do it if and only if there exist an open path from  $AB$  to  $a$  and a closed path from  $AC$  to  $c$ ). Fix a deterministic path from  $C$  to  $x$ , the event  $\{\Gamma = \gamma\}$  depends only on faces adjacent to  $\gamma$  (we denote the space of such faces  $\bar{\gamma}$ ). Now, on  $\{\Gamma = \gamma\}$ , there exists a bijection between configurations with an open path from  $a$  to  $AB$  and configurations with a closed path

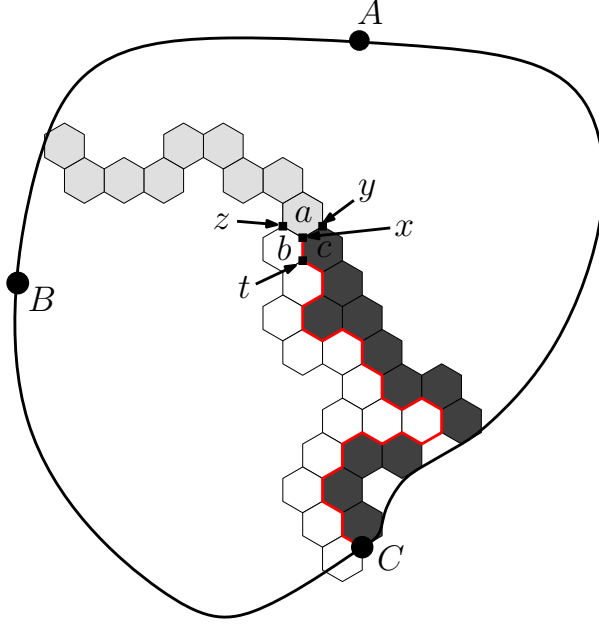


Figure 6: The dark grey and the white hexagons are the hexagons on  $\bar{\Gamma}$ ,  $\Gamma$  being in black.

from  $a$  to  $AB$  (by symmetry between open and closed edges in the domain  $\Omega_\delta \setminus \bar{\gamma}$ ). This is true for any  $\gamma$ , hence there is a bijection between the event

$$E_{A,\delta}(y) \setminus E_{A,\delta}(x) = \bigcup_{\gamma} \{\Gamma = \gamma\} \cap \{a \leftrightarrow AB \text{ in } \Omega_\delta \setminus \bar{\gamma}\}$$

and

$$E := \bigcup_{\gamma} \{\Gamma = \gamma\} \cap \{a \overset{*}{\leftrightarrow} AB \text{ in } \Omega_\delta \setminus \bar{\gamma}\}.$$

Note that  $E_{B,\delta}(z) \setminus E_{B,\delta}(x)$  is the image of  $E$  after switching the colors, so that it is in bijection with it. This part is the key step of the lemma, and is sometimes called **color-switching trick**. Since  $\mathbb{P}_{1/2}$  is simply the uniform measure on configurations, we obtain  $P_{A,\delta}(e_1) = P(E) = P_{B,\delta}(e_2)$ .  $\square$

We are now in a position to prove that  $I_\gamma^\delta(H)$  and  $I_\gamma^\delta(S)$  converge to 0. From Lemmata 3.2 and 3.3, we obtain by reindexing the sum

$$I_\gamma^\delta(H) = \sum_{e \in \mathcal{C}_{\gamma_\delta}} (e^* + \tau(\tau.e)^* + \tau^2(\tau^2.e)^*) P_A(e) + o(1) = o(1)$$

using that

$$e^* + \tau(\tau.e)^* + \tau^2(\tau^2.e)^* = 0. \quad (3.7)$$

Similarly, for  $s$ :

$$I_\gamma^\delta(S) = \sum_{e \in \mathcal{C}_{\gamma_\delta}} (e^* + (\tau.e)^* + (\tau^2.e)^*) P_A(e) + o(1) = o(1).$$

Here, we have used

$$e^* + (\tau.e)^* + (\tau^2.e)^* = 0. \quad (3.8)$$

This concludes the proof of the holomorphicity of  $h$  and  $s$ .

**Identification of  $s$  and  $h$  via boundary conditions.** Let us start with  $s$ . Since  $s$  is holomorphic and real-valued, it must be constant. It is easy to see from the boundary conditions (near a corner for instance) that it is equal to 1. Now, consider  $h$ . Since  $h$  is holomorphic, it is sufficient to identify enough boundary conditions to specify it uniquely.

Let  $z \in \Omega$ . Since  $h_A(z) + h_B(z) + h_C(z) = 1$ ,  $h(z)$  is a barycenter of 1,  $\tau$  and  $\tau^2$  and it belongs to the triangle with vertices 1,  $\tau$  and  $\tau^2$ . Furthermore, if  $z$  is on the boundary of  $\Omega_\delta$  lying between  $B$  and  $C$ ,  $h_A(z) = 0$  (using RSW), and thus  $h_B(z) + h_C(z) = 1$  (since  $s = 1$ ). Hence,  $h(z)$  lies on the interval  $[\tau, \tau^2]$  of the complex plane. Besides,  $h(B) = \tau$  and  $h(C) = \tau^2$ , so  $h$  induces a continuous map from the boundary interval  $[BC]$  of  $\Omega$  onto  $[\tau, \tau^2]$ . By RSW yet again,  $h$  is one-to-one on this boundary interval. Similarly,  $h$  induces a bijection between the boundary interval  $[AB]$  (resp.  $[CA]$ ) of  $\Omega$  and the complex interval  $[1, \tau]$  (resp.  $[\tau^2, 1]$ ), so putting the pieces together we see that  $h$  is a holomorphic map from  $\Omega$  to the triangle with vertices at 1,  $\tau$  and  $\tau^2$  which extends continuously to  $\bar{\Omega}$  and induces a continuous bijection between  $\partial\Omega$  and the boundary of the triangle.

From standard results of complex analysis (“principle of corresponding boundaries”, cf. for instance Theorem 4.3 in [22]), this implies that  $h$  is actually a conformal map from  $\Omega$  to the interior of the same triangle. But we know that  $h$  maps  $A$  (resp.  $B$ ,  $C$ ) to 1 (resp.  $\tau$ ,  $\tau^2$ ), and this determines it uniquely. In other words, there is only one possible limit for the triple  $(H_A, H_B, H_C)$  as  $\delta$  goes to 0, which gives conformal invariance for free and concludes the proof of Theorem 3.1.  $\square$

As a corollary of the proof, we get a nice expression for  $h_A$ : If  $\Phi_{\Omega, A, B, C}$  is the conformal map from  $\Omega$  to the triangle mapping  $A$ ,  $B$  and  $C$  as previously (which means of course that  $\Phi_{\Omega, A, B, C} = h$ ) then

$$H_{A, \delta}(z) \rightarrow \frac{2\Re(\Phi_{\Omega, A, B, C}(z)) + 1}{3}.$$

If  $\Omega$  is the equilateral triangle itself, then  $h$  is the identity map and we obtain Cardy’s formula in Carleson’s form: if  $D \in [CA]$  then

$$f(\Omega, A, B, C, D) = \frac{|CD|}{|AB|}.$$

It is also to be noted that (3.7) actually characterizes the triangular lattice (and therefore its dual the hexagonal one). So, it seems that the triangular lattice is the only one (apart from trivial modifications of it) in which a fully combinatorial proof of the holomorphicity of  $h$  is possible. On the other hand, the holomorphicity of  $s$  and therefore the fact that it equals 1 relies only on (3.8), which is true for any triangulation where RSW holds. This seems to be a fundamental property of critical two-dimensional percolation (and *might* be the key to understanding universality in this particular, limited case, though this is hardly even speculative). As of this time, no direct, combinatorial proof of this fact seems to be known.

**Morality.** *The Cardy-Smirnov formula provides a precise understanding of crossing probabilities for critical percolation. In fact, these crossing probabilities allow to describe the interface of the model, as well as the critical exponents. We do so in the next section.*

## 4 Scaling limit and arm exponents

A natural question at this point is the exact amount of information contained in Theorem 3.1; is it enough to derive precise results about the geometry of critical percolation clusters, for instance? It turns out that it is indeed the case, and in fact the full structure of the percolation scaling limit can be recovered from it through Schramm-Loewner Evolution (SLE for short); but this is quite indirect, and one of the aims of these lectures is to investigate what can be obtained *without* SLE.

**Scaling limit of the hull.** Consider once again a bounded, simply connected domain  $\Omega$  in the plane, with three marked points  $A$ ,  $B$  and  $C$ , and discretized as above at mesh size  $\delta$  (which we will omit from the notation when it doesn't lead to confusion); let  $\Gamma_\delta$  be the *exploration path* defined as follows, see the picture on the cover. Assume all the hexagons on the arc  $AC$  are open, while all the hexagons on the arc  $BC$  are closed. Then,  $\Gamma_\delta$  is the unique interface lying on  $\mathbb{H}$  separating the open cluster of  $AC$  of the closed cluster of  $BC$ . The complement of  $\Gamma_\delta$  in  $\Omega$  is composed of finitely many connected components. Set  $U_A$  and  $U_B$  to be the connected components containing  $A$  and  $B$  respectively. Let  $K_\delta := \Gamma_\delta \setminus (K_A \cup K_B)$ .  $K_\delta$  is a (relatively) compact, simply connected random set separating  $A$  from  $B$ , and whose complement has exactly two connected components — we call such a set a *hull*<sup>5</sup>. The Cardy-Smirnov formula allows us to prove the following:

**Theorem 4.1.** *As  $\delta \rightarrow 0$ ,  $K_\delta$  converges in distribution to a random hull  $\tilde{K}$  in  $\Omega$ .*

*Proof.* Let  $E_A, E_B$  be two disjoint connected compact sets containing  $A$  and  $B$  respectively such that  $\Omega \setminus (E_A \cup E_B)$  is still simply connected and contains  $C$ . The probability that  $K_\delta$  is disjoint from  $E_A \cup E_B$  can be written in terms of Cardy's formula. Indeed, the latter gives the law of the hitting point in a the topological triangle  $ABC$  (in the equilateral triangle of side length 1, it is uniform since the probability to hit on the left of  $D$  is equal to  $AD$ , and one can use conformal invariance to deduce the law in any domain). Now, the probability that  $K_\delta$  is disjoint from  $E_A \cup E_B$  is the probability to hit  $AB$  in the domain  $\Omega \setminus (E_A \cup E_B)$  before hitting  $\partial E_A \cup \partial E_B$ . But since we know the law of the hitting point on  $AB \cup \partial E_A \cup \partial E_B$ , the probability is determined.

Now, RSW allows to show that the boundary of  $K_\delta$  is fairly regular (in particular Hölder continuous uniformly in  $\delta$ ). An inclusion-exclusion argument and a continuity argument then show that the probability that  $K_\delta$  avoids a union of compact sets  $E_1, \dots, E_k$  also converge. The data of all such probabilities characterizes the distribution of a random continuous hull.  $\square$

While this proof is quite clever, it is not very constructive and says very little about the geometry of  $\tilde{K}$ . Nevertheless, the limit is determined by “crossing probabilities”. If one can exhibit a continuous hull with same crossing probabilities, the result will follow. This hull has to be conformally invariant, therefore reflected Brownian motion comes naturally to mind, since it is well known that it is conformally invariant. Giving a precise meaning to the construction requires some background in stochastic analysis, so we keep it deliberately informal.

**Proposition 4.2.** *Assume that the boundary of  $\Omega$  is smooth. Let  $(X_t)$  be a Brownian motion in  $\bar{\Omega}$ , started from  $C$  and reflected on the boundary of  $\Omega$ , with a reflexion angle*

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<sup>5</sup>there is no fixed convention on what is exactly meant by that word in the literature, but *something with nice topology that touches the boundary of a domain* is often a good enough approximation.

of  $\pi/3$  pointing towards  $C$  along the boundary arcs  $CA$  and  $BC$ ; let  $\tau$  be the first hitting time of the arc  $AB$  by  $X$ , and let  $\tilde{X}$  be the hull determined by  $X_{[0,\tau]}$ . Then, the random hulls  $\tilde{K}$  and  $\tilde{X}$  have the same distribution.

**Scaling limit of interfaces.** A question that is perhaps even more natural is whether the exploration path has a scaling limit. Convergence of the exploration path would give a very precise description of cluster boundaries, while the boundary of  $K$  is harder to describe from the microscopic configuration. There *is* such a limit:

**Theorem 4.3** (Smirnov). *As  $\delta \rightarrow 0$ ,  $\Gamma$  converges in distribution to the trace of an  $SLE_6$  process in  $\Omega$ , started from  $C$  and stopped when first hitting the arc  $AB$ .*

A natural approach to proving this theorem is the following. Draw a disk of radius  $\varepsilon$  centered at  $C$ , and follow the exploration until it exits the disk. From the previous theorem, if  $\delta$  goes to 0 while keeping  $\varepsilon$  fixed, the outer shape of this curve converges in distribution. Let  $z_1$  be the point at which it exits the disk; draw a second disk of radius  $\varepsilon$ , centered at  $z_1$ , and continue the exploration until it exits the second disk. Again, its outer shape converges in distribution. Iterating the construction, one gets an approximation of  $\Gamma$  by a chain of balls of radius  $\varepsilon$ , with explicit distribution, and finally letting  $\varepsilon$  go to 0 gives the wanted result.

Also note that the hull defined by SLE(6) is the same that the hull defined by a Brownian motion reflected with an angle  $\pi/3$  (thank to Proposition 4.2). One nice consequence of this is the fact that all three have the same boundary geometry; for instance, their boundary is almost surely of Hausdorff dimension  $4/3$  — that is actually one of the shortest ways to the determination of the dimension of the Brownian frontier.

**Arm exponents.** It is easy to show, using a color-switching argument very similar to the one harnessed in Lemma 3.3, that  $\alpha_{j,\sigma}$  depends only on the length of the sequence, as long as we consider polychromatic sequences. From now on, we set  $\alpha_j$  to be the exponent for polychromatic sequences of length  $j$ . By extension, we set  $\alpha_1$  to be the exponent of the one-arm event.

**Theorem 4.4** (Werner-Smirnov [37]).  $\alpha_1 = \frac{5}{48}$  and  $\alpha_j = \frac{j^2 - 1}{12}$  for  $j > 1$ .

The proof of this is heavily based on the use of Schramm-Loewner Evolution [25, 26, 27]; we refer the reader to existing literature on the topic (for instance the lecture notes [38, 39]), and rather focus on relations between these exponents and geometric features of critical percoation clusters.

**Fractal properties of critical percolation.** These arm exponents can be used to measure the size (Hausdorff dimension) of various sets describing percolation clusters. A set  $S$  is said to be fractal of dimension  $d_S$  if the density of points in  $S$  within a box of size  $n$  decays as  $n^{-x_S}$ , with  $x_S = 2 - d_S$  in two dimensions. The codimension  $x_S$  is related to arm exponents in many cases:

- The 1-arm exponent is related to the existence of long connections, from the center of a box to its boundary. It will thus measure the size of big clusters, like the incipient infinite cluster (IIC) as defined by Kesten [19], which scales as  $n^{2-5/48} = n^{91/48}$ .



- The monochromatic 2-arm exponent describes the size of the backbone of a cluster. The fact that this *backbone* is much thinner than the cluster itself was used by Kesten [19] to prove that the random walk on the IIC is subdiffusive (while it has been proved to converge toward a Brownian Motion on a supercritical infinite cluster).
- The polychromatic 2-arm exponent is related to the boundaries (hulls) of big clusters, which are thus of fractal dimension  $2 - \alpha_2 = 7/4$ .
- The 3-arm exponent concerns the external (accessible) perimeter of a cluster, which is the accessible part of the boundary: one excludes fjords which are connected to the exterior only by 1-site wide passages. The dimension of this frontier is  $2 - \alpha_3 = 4/3$ . These last two exponents can be observed on random interfaces, numerically and in real-life experiments as well (see [10, 32] for instance).
- The 4-arm exponent with alternating colors counts the pivotal sites (see the next section for more information). Its dimension is  $2 - \alpha_4 = 3/4$ . This exponent is crucial in the study of noise-sensitivity of percolation.

**Morality.** *The understanding of the critical phase is now very sharp. But what about the off-critical phase? More precisely, what happens if  $p$  goes to  $1/2$  when the size of the graph goes to infinity? We first deal with the case of fixed  $p$  in the next section, and then with the case of  $p \rightarrow 1/2$  in Section 6.*

## 5 The critical point of percolation

We now arrive to a milestone of modern probability, Kesten's ' $p_c = 1/2$ '-Theorem. It was proved in the case of bond-percolation on the square lattice, but the same argument applies to face-percolation on the hexagonal lattice.

**Theorem 5.1** (Kesten [18]). *The critical value of percolation on the hexagonal lattice is  $1/2$ .*

The rough philosophy of the proof is the following:

- First, exhibit a property at  $p = 1/2$ , which should be witness of the critical phase.
- Second, prove that the property holds only at  $p = 1/2$ , identifying  $1/2$  to be the only possible value for the critical point.

The property “identifying” the critical value is RSW. To prove that it holds only for  $p = 1/2$ , we show that the crossing probabilities go to 0 as  $\delta \rightarrow 0$  whenever  $p < 1/2$ , while they go to 1 whenever  $p > 1/2$ .

In order to prove this fact, we consider a more general question. We aim to understand the behaviour of the function  $p \mapsto \mathbb{P}_p(A)$  for a non-trivial increasing event  $A$  depending on faces of a subgraph of the hexagonal lattice (think of this event as being a crossing event). This increasing function is equal to 0 at  $p = 0$  and to 1 at  $p = 1$ , and we are interested in the range of  $p$  for which its value is between  $\varepsilon$  and  $1 - \varepsilon$  for some positive  $\varepsilon$  (this range is usually referred to as a *window*). Under mild conditions on  $A$ , the window will be narrow for large graphs, and its width can be bounded above in terms of the size of the underlying graph, which is known as a *sharp threshold* behaviour.

In lattice models such as percolation, the study of  $p \mapsto \mathbb{P}_p(A)$  harnesses a differential equality known as Russo's formula:

**Proposition 5.2** (Russo [30], Section 2.3 of [13]). *Let  $p \in (0, 1)$  and  $A$  an increasing event depending on a finite set of faces  $F$ , then*

$$\frac{d}{dp} \mathbb{P}_p(A) = \sum_{f \in F} \mathbb{P}_p(f \text{ pivotal for } A),$$

where  $f$  is pivotal for  $A$  if  $A$  occurs when  $f$  is open, and does not if  $f$  is closed.

If the typical number of pivotal faces is sufficiently large, for instance when the probability of  $A$  is not close to 0 nor 1, the window is necessarily narrow. There has been an extensive study of the largest probability to be pivotal. We present one of the most striking result on the subject:

**Theorem 5.3** (Kahn, Kalai, Linial [16], see also [11, 12, 17]). *Let  $\varepsilon > 0$ ; there exists a constant  $c = c(\varepsilon) \in (0, \infty)$  such that the following holds. Consider a percolation model on a graph  $G$  with  $|F|$  denoting the number of faces of  $G$ . For every  $p \in [\varepsilon, 1 - \varepsilon]$  and every increasing event  $A$ , there exists  $f \in F$  such that*

$$\mathbb{P}_p(f \text{ pivotal for } A) \geq c \mathbb{P}_p(A)(1 - \mathbb{P}_p(A)) \frac{\log |F|}{|F|}.$$

This theorem does not imply that there are always many pivotal points since it deals only with the maximal probability over all points. It could be that this maximum would be attained only at one point (for instance for the event that the origin is open). There is a particularly efficient way (first appeared in [5, 6]) to avoid this problem. In the case of a **translation-invariant event**  $A$  on a torus with  $n$  faces, faces play a symmetric role, so that the probability to be pivotal is the same for all of them. Proposition 5.2 together with Theorem 5.3 thus imply for  $p \in (\varepsilon, 1 - \varepsilon)$ ,

$$\frac{d}{dp} \mathbb{P}_p(A) \geq c(\mathbb{P}_p(A)(1 - \mathbb{P}_p(A)) \log n.$$

Integrating the previous inequality between two parameters  $\varepsilon < p_1 < p_2 < 1 - \varepsilon$ , we obtain

$$\frac{\mathbb{P}_{p_2}(A)}{1 - \mathbb{P}_{p_2}(A)} \geq \frac{\mathbb{P}_{p_1}(A)}{1 - \mathbb{P}_{p_1}(A)} n^{c(p_2 - p_1)}.$$

If we further assume that  $\mathbb{P}_{p_1}(A)$  stays bounded away from 0 uniformly in  $n \geq 1$ , we can find  $c, C > 0$  such that

$$\mathbb{P}_{p_2}(A) \geq 1 - Cn^{-c(p_2 - p_1)}. \quad (5.1)$$

Now that the theory is settled, we can prove the fundamental lemma which shows that RSW fails when  $p \neq 1/2$  (in the sense that crossing probabilities go to 0 when  $p < 1/2$ , and to 1 when  $p > 1/2$ ). We prove it only for one particular topological rectangle (a rectangle twice longer than large), and for  $p > 1/2$ . The other shapes work the same way, and the case  $p < 1/2$  follows from duality between open faces for percolation of parameter  $p$  and closed faces for percolation of parameter  $1 - p$ .

**Lemma 5.4.** *Let  $p > 1/2$ , there exist  $\varepsilon = \varepsilon(p) > 0$  and  $c = c(p) > 0$  such that for every  $n \geq 1$ ,*

$$\mathbb{P}_p([0, n] \times [0, 2n] \text{ is crossed vertically}) \geq 1 - cn^{-\varepsilon}. \quad (5.2)$$

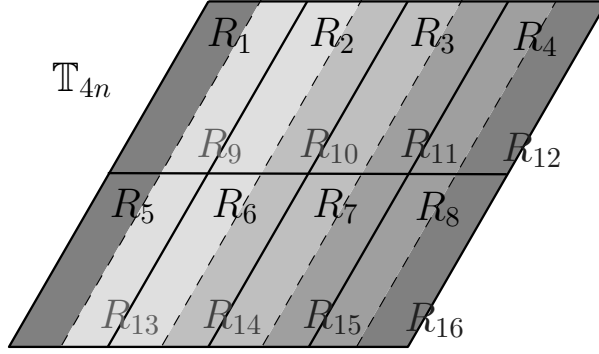


Figure 7: rectangles  $R_1, \dots, R_{16}$ . They are all translates of  $R_1$ .

*Proof.* Consider the torus  $\mathbb{T}_{4n}$  of size  $4n$ . Let  $B$  be the event that there exists a vertical crossing of a rectangle with dimensions  $(n/2, 4n)$  in the torus of size  $4n$ . This event is invariant under translations and satisfies

$$\mathbb{P}_{1/2}(B) \geq \mathbb{P}_{1/2}([0, n/2] \times [0, 4n] \text{ is crossed vertically}) \geq c > 0$$

uniformly in  $n$ . Since  $B$  is increasing, we can apply (5.1) to deduce that for  $p > 1/2$ , there exist  $\varepsilon$  and  $c$  such that

$$\mathbb{P}_p(B) \geq 1 - cn^{-\varepsilon}. \quad (5.3)$$

If  $B$  holds, one of the 16 rectangles  $R_1, \dots, R_{16}$  drawn in Fig. 7 must be crossed from top to bottom. We denote these events by  $F_1, \dots, F_{16}$  — they are translates of the event that  $[0, n] \times [0, 2n]$  is crossed horizontally. Using the FKG inequality in the second line, we find

$$\begin{aligned} \mathbb{P}_p(B) &= 1 - \mathbb{P}_p(B^c) = 1 - \mathbb{P}_p\left(\bigcap_{i=1}^{16} A_i^c\right) \\ &\leq 1 - \prod_{i=1}^{16} \mathbb{P}_p(A_i^c) = 1 - [1 - \mathbb{P}_p([0, n] \times [0, 2n] \text{ is crossed vertically})]^{16}. \end{aligned}$$

Plugging (4.2) into the previous inequality, we deduce

$$\mathbb{P}_p([0, n] \times [0, 2n] \text{ is crossed vertically}) \geq 1 - (cn^{-\varepsilon})^{1/16}.$$

□

*Proof of Theorem 5.1.* The inequality  $p_c \geq 1/2$  was proved in Corollary 2.2. The other inequality follows from the following reasoning: define rectangles  $R_n$  and events  $F_n$  as in the proof of Corollary 2.4. Then, as before,

$$\mathbb{P}_{1/2}(0 \leftrightarrow \infty) \geq \mathbb{P}_{1/2}\left(\bigcap_{n \geq 0} F_n\right) \geq \prod_{n \geq 0} \mathbb{P}_{1/2}(F_n) \geq \prod_{n \geq 0} (1 - c2^{-\varepsilon n}) > 0,$$

which implies the claim. □

It is actually possible to prove that there are many pivotal points without the help of the power-hammer Theorem 5.3. Indeed, a face is pivotal for the event that  $[0, n]^2$  is crossed if and only if there exist four arms of alternating colors emanating from  $f$  and going to the boundary of the box. Therefore, the number of pivotal faces is related to the number of faces where the four arm event  $A_{4,OCOC}$  occurs. Using Theorem 4.4, there exists of order  $n^2 n^{-5/4} = n^{3/4}$  such points. Of course, this reasoning also harnesses a powerful theorem. It is also possible to prove directly that there are many points with the four arm events occurring.

**Proposition 5.5.** *There exists  $\alpha > 0$  such that for any  $k < n$ ,*

$$\mathbb{P}_{1/2}(A_{4,OCOC}(k, n)) \geq (k/n)^{2-\alpha}.$$

*Proof.* We know from Proposition 2.5 that

$$\mathbb{P}_{1/2}(A_{5,OCOO}(k, n)) \asymp (k/n)^{-2}$$

when  $k < n$ . Now, Reimer's inequality (see [29]) implies

$$\mathbb{P}_{1/2}(A_{4,OCOC}(k, n)) \mathbb{P}_{1/2}(A_{1,O}(k, n)) \geq \mathbb{P}_{1/2}(A_{5,OCOO}(k, n)) \asymp (k/n)^2.$$

Moreover,  $\pi_1(k, n) \geq (k/n)^\alpha$  for every  $k < n$ . We deduce

$$\mathbb{P}_{1/2}(A_{4,OCOC}(k, n)) \geq (k/n)^{2-\alpha}. \tag{5.4}$$

□

This shows that there are of order  $n^\alpha$  pivotal points at  $p = 1/2$ . Note that this result is not sufficient to deduce the theorem, since it holds only at  $p = 1/2$ . In fact, it is possible to prove that it holds as long as crossing probabilities are not too small. Recently, Smirnov gave an elementary argument showing that there are many pivotal points (see [40] in french or Section 5.6 of [14] in english).

**Morality.** *We have studied how probabilities of increasing events evolve as functions of  $p$ . If  $p$  is fixed and we consider bigger and bigger rectangles (of size  $n$ ), crossing probabilities go to 0 whenever  $p < 1/2$ . But what happens if  $(p, n) \rightarrow (1/2, \infty)$  (this regime is called the near-critical regime)?*

## 6 The near-critical regime

We now investigate the near-critical regime, when  $p$  is close to  $p_c$ . We develop the notion of correlation length and use it to deduce exponents for the near-critical regime.

### 6.1 Correlation length

For  $\varepsilon > 0$ , define the *correlation length* for  $p < 1/2$  by

$$L_p(\varepsilon) := \inf \{ n > 0 : \mathbb{P}_p([0, n] \times [0, n] \text{ is crossed}) \leq \varepsilon \}$$

and  $L_p(\varepsilon) = L_{1-p}(\varepsilon)$  for  $p > 1/2$ . We will always be considering an arbitrary  $\varepsilon$  small enough. For this reason, we fix  $\varepsilon \ll 1$  and drop it from the notation. Note that the

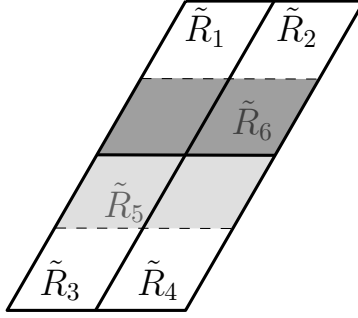


Figure 8: Rectangles  $\tilde{R}_1, \dots, \tilde{R}_6$ .

definition itself of  $L_p(\varepsilon)$  uses the fact that crossing probabilities converge to 0 when  $p < 1/2$  to guarantee that the infimum is well-defined.

We mention that when  $p < 1/2$ ,

$$\mathbb{P}_p([0, n] \times [0, 2n] \text{ is crossed vertically}) \geq \varepsilon^6 \quad \text{for all } n < L_p, \quad (6.1)$$

$$\mathbb{P}_p([0, L_p] \times [0, 2L_p] \text{ is crossed horizontally}) \leq \varepsilon^{1/6}. \quad (6.2)$$

This fact is a consequence of RSW theory (the proof that we presented applies only at criticality, but others proofs are valid outside of the critical point and relate the probability to cross rectangles to the probability to cross squares, see *e.g.* [13, 18]). Formulæ(6.1) and (6.2) thus imply that the shape in the definition of  $L_p(\varepsilon)$  does not really matter.

The correlation length should be understood as the scale at which one starts to see that  $p$  is not critical. Indeed, if one looks at two percolation pictures, one at  $p = 0.5$ , and one at  $p = 0.47$ , it would not be necessarily possible to distinguish between them if the size is not large enough. Yet, when the size of the picture gets bigger and bigger, connectivity properties start to differ drastically. Another way to formulate the previous principle is the following: when studying the super or sub-critical percolation, *coarse-graining arguments* allow to relate properties of a percolation with parameter  $p$  to a percolation with new parameter  $p'$  much closer to 0 or 1. Usually, by taking the grain  $N$  to be large enough, it is even possible to get  $p'$  in the *Peierls regime*, in which counting arguments are sufficient to estimate relevant quantities. Typically, the grain that one needs to consider at parameter  $p$  is of order  $L_p$ .

The fact that  $L_p$  is well-defined implies exponential decay of correlation in the sub-critical phase. This important fact distinguishes between the critical phase and the sub-critical phase: at criticality, probabilities decay according to power laws, while they decay exponentially fast in the sub-critical phase (see Section 5 of [13] for a classical exposition). In our case, there is a very quick proof of this statement:

**Proposition 6.1.** *For any  $p < 1/2$ , we have*

$$\mathbb{P}_p(0 \leftrightarrow \partial B_n) \leq e^{-n/L_p} \quad \text{for all } n \geq L_p.$$

*Proof.* Let  $n > 0$  and consider the rectangles  $\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_6$  defined as in Fig. 8. These rectangles have the property that whenever  $[0, 2n] \times [0, 4n]$  is crossed horizontally, at least

two of the rectangles  $\tilde{R}_i$  are crossed in the easy direction by *disjoint* paths. We deduce, using the BK inequality (see Section 2.3 of [13]), that

$$\mathbb{P}_p([0, 2n] \times [0, 4n] \text{ is cross. hor.}) \leq C_8^2 \mathbb{P}_p([0, n] \times [0, 2n] \text{ is cross. hor.})^2.$$

We easily obtain that for every  $k \geq 0$ ,

$$C_8^2 \mathbb{P}_p([0, 2^k n] \times [0, 2^{k+1} n] \text{ is cross. hor.}) \leq (C_8^2 \mathbb{P}_p([0, n] \times [0, 2n] \text{ is cross. hor.}))^{2^k}.$$

In particular, if  $p < 1/2$ , (6.1) implies that fixing  $n = L_p$ ,

$$C_8^2 \mathbb{P}_p([0, n] \times [0, 2n] \text{ is cross. hor.}) < C_8^2 \varepsilon^{1/6} < 1/e$$

and the claim follows.  $\square$

We conclude this section by mentioning that most often, the correlation length is defined as the ‘‘inverse rate’’ of exponential decay of the connectivity function. More precisely, since the quantity  $\mathbb{P}_p(0 \leftrightarrow nx)$  is super-multiplicative, the quantity  $\xi_p$  can be defined by the formula

$$\frac{1}{\xi_p} = \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}_p(0 \leftrightarrow nx).$$

Then, it is possible to prove that  $L_p \asymp \xi_p$  when  $p < 1/2$  (note that Proposition 6.1 gives one inequality, see Theorem 3.1 of [28] for the other bound).

## 6.2 Exponents for the correlation length and the cluster density

Define  $\pi_{j,\sigma}^p(k, n) := \mathbb{P}_p(A_{j,\sigma}(k, n))$  and  $\pi_{j,\sigma}(n) := \pi_{j,\sigma}(j, n)$  for  $p \neq 1/2$ . Moreover,  $\pi_{1,O}(\cdot)$  and  $\pi_{4,OCCO}(\cdot)$  are denoted by  $\pi_1(\cdot)$  and  $\pi_4(\cdot)$ .

The goal of this section is to prove the following *scaling relations*:

**Theorem 6.2** (Kesten [20]). *For every  $p > 1/2$ , we have*

$$(p - 1/2)L_p^2 \pi_4(L_p) \asymp 1 \quad \text{and} \quad \theta(p) \asymp \pi_1(L_p).$$

Before giving the proof of Theorem 6.2, we explain how it implies Theorem 1.1. Note that if  $\pi_4(n) = n^{-\alpha_4 + o(1)}$  and  $\pi_1(n) = n^{-\alpha_1 + o(1)}$ , it implies the existence of  $\nu$  and  $\beta$  such that  $L_p = (p - 1/2)^{-\nu + o(1)}$  and  $\theta(p) = (p - 1/2)^{\beta + o(1)}$ . Moreover, the relations become

$$(2 - \alpha_4) \nu = 1 \quad \text{and} \quad \beta = \alpha_1 \nu.$$

Therefore,  $\alpha_1 = 5/48$  and  $\alpha_4 = 5/4$  imply  $\nu = 4/3$  and  $\beta = 5/36$ . The latter is exactly the claim of Theorem 1.1.

*Proof.* We first deal with the first equality. We aim to apply Russo’s formula to the event  $A$  that  $[0, L_p]^2$  is crossed. On the one hand, with the definition of  $L_p$ ,  $\mathbb{P}_p(A)$  equals  $1 - \varepsilon$ . On the other hand, one can check that  $\mathbb{P}_{1/2}(A) = 1/2$ . Moreover, a face is pivotal for  $A$  if and only if there are four alternating arms starting from it and going to the boundary of  $[0, L_p]^2$ . Except for points near the boundary, this occurs with  $\mathbb{P}_{p'}$ -probability of order

$\pi_4^{p'}(L_p)$  for every  $p' \in (1/2, p)$ . Therefore, if we neglect the effect of the boundary<sup>6</sup>, we obtain

$$1 \asymp \mathbb{P}_p(A) - \mathbb{P}_{1/2}(A) \asymp \int_{1/2}^p L_p^2 \pi_4^{p'}(L_p) dp'.$$

Assume for a moment that  $\pi_4^{p'}(L_p) \asymp \pi_4(L_p)$  for every  $p' \in (1/2, p)$ , we obtain

$$1 \asymp (p - 1/2) L_p^2 \pi_4(L_p).$$

We now turn to the second relation. On the one hand, it is straightforward that  $\theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty) \leq \mathbb{P}_p(0 \leftrightarrow \partial[0, L_p]^2) = \pi_1^p(L_p)$ . On the other hand, Proposition 6.1 together with a construction similar to the lower bound in Proposition 2.4 implies that there exists  $c > 0$ , not depending on  $p$ , such that

$$\mathbb{P}_p(\exists \text{ a circuit in } [0, L_p]^2 \setminus [0, L_p/2]^2 \text{ surrounding the origin and connected to } \infty) \geq c.$$

Using FKG, we can deduce  $\theta(p) \geq c \mathbb{P}_p(0 \leftrightarrow \partial[0, L_p]^2) \geq c \pi_1^p(L_p)$ . Once again, if one can prove that  $\pi_1^p(L_p) \asymp \pi_1(L_p)$ , then we will have proved the theorem.  $\square$

The previous proof relies on an important assumption ( $\pi_1^p(L_p) \asymp \pi_1(L_p)$  and  $\pi_4^{p'}(L_p) \asymp \pi_4(L_p)$  for  $p' \in (1/2, p)$ ) that we justify now. As described earlier, below the correlation length, the picture should be very close to the critical one. In particular, arm-event probabilities should not vary too much with respect to  $p$ . This is the main step in Kesten's scaling relations.

**Theorem 6.3** (Kesten [20]). *For  $j \geq 1$  and a polychromatic sequence  $\sigma$ , we have  $\pi_{j,\sigma}^p(n) \asymp \pi_{j,\sigma}(n)$  for every  $p$  and  $n \leq L_p$ .*

The idea of the proof is to express the logarithmic derivative of  $p \mapsto \pi_{j,\sigma}^p$  in terms of four-arm events. In order to perform this, we will invoke the following technical lemma:

**Lemma 6.4** (Quasi-multiplicativity). *For  $j \geq 1$  and a polychromatic sequence  $\sigma$ , we have*

$$\pi_{j,\sigma}^p(n_1, n_3) \asymp \pi_{j,\sigma}^p(n_1, n_2) \pi_{j,\sigma}^p(n_2, n_3)$$

for every  $p$  and every  $n_1 < n_2 < n_3 < L_p$ .

It is of note that  $\pi_{j,\sigma}^p(n_1, n_3) \leq \pi_{j,\sigma}^p(n_1, n_2) \pi_{j,\sigma}^p(n_2, n_3)$  is an obvious inequality. When  $j = 1$ , the other inequality follows from the fact that one can “glue” a path in  $B_{n_2} \setminus B_{n_1}$  with a path in  $B_{n_3} \setminus B_{n_2}$  as depicted in Fig. 9, and that this gluing costs only a constant factor thanks to RSW. This kind of reasoning can be generalized when  $j > 1$  and  $\sigma$  is a polychromatic sequence, see Proposition 16 of [28].

*Theorem 6.3.* We treat the case of  $\pi_1^p(n)$  when  $p > 1/2$ . Recall that  $n$  is assumed to be smaller than  $L_p$ , so that RSW holds at every scale smaller than  $n$ . We will be using RSW extensively. We cannot stress enough the fact that it holds as long as (and roughly speaking if and only if)  $n < L_p$ .

---

<sup>6</sup>There are several ways to deal with the boundary effect. One can control the probability to be pivotal for boundary points separately, or one can do the following: for the lower bound, it is sufficient to count points far from the boundary, for the upper bound, one can work with the event that the torus of size  $n$  contains a circuit with non-trivial homotopy (there, the probability to be pivotal is the same for every face, and is smaller than  $\pi_4^p(n)$ ).

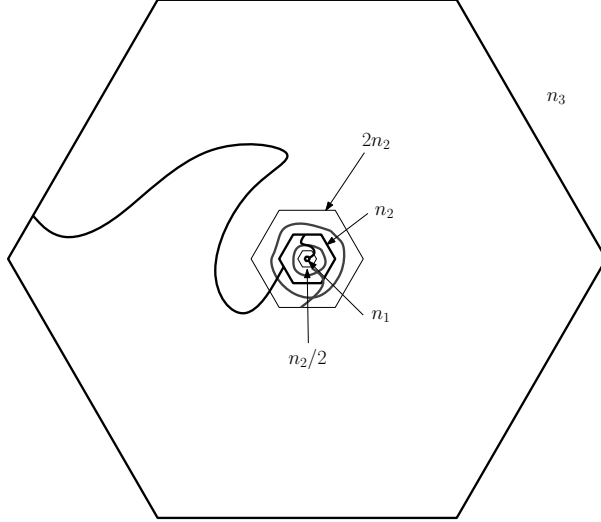


Figure 9: The paths in the annuli  $B_{n_3} \setminus B_{n_2}$  and  $B_{n_2} \setminus B_{n_1}$  are in black. A combination of two circuits connected by a path (in grey) connects the paths together. This figure occurs with probability bounded away from 0 thanks to RSW.

Russo's formula implies

$$\frac{d\pi_1^p(n)}{dp} = \sum_{f \in B_n} \mathbb{P}_p(f \text{ pivotal for } A_{1,O}(n)). \quad (6.3)$$

The face  $f$  is pivotal if and only if there are four arms of alternating colors emanating from it, one of the open arm going to the origin, the other to the boundary of the box, and the two closed arms forming a circuit around the origin. The event that a face  $f$  (at distance  $|f|$  of the origin) is pivotal is thus included in the intersection of events  $A_{1,O}(|f|/2)$ ,  $A_{1,O}(2|f|, n)$  and a translate of  $A_{4,OCOC}(|f|/2)$  (see Fig 10). We deduce, using independence, that

$$\begin{aligned} \mathbb{P}_p(f \text{ pivotal for } A_{1,O}(n)) &\leq \pi_1^p(|f|/2) \pi_1^p(2|f|, n) \pi_4^p(|f|/2) \\ &\leq C \pi_1^p(n) \pi_4^p(|f|/2) \end{aligned}$$

where in the second line we have used quasi-multiplicativity and RSW. Plugging this inequality into (6.3), we find

$$\frac{d\pi_1^p}{dp} \leq C \pi_1^p(n) \cdot \sum_{f \in B_n} \pi_4^p(|f|/2) \quad (6.4)$$

which integrates into

$$\log \pi_1^p(n) - \log \pi_1(n) \leq C \int_{1/2}^p \sum_{f \in B_n} \pi_4^{p'}(|f|/2) dp'. \quad (6.5)$$

It remains to prove that the right-hand side is of order 1.

Fix  $p' \in (1/2, p)$  and note that  $n < L_p < L_{p'}$ . We know that  $\alpha_{4,OCOC} < 2$  thank to Proposition 5.5. In fact, the proof of Proposition 5.5 applies whenever RSW is available, so

$$\pi_4^{p'}(k, n) \geq (n/k)^{2-\alpha}.$$



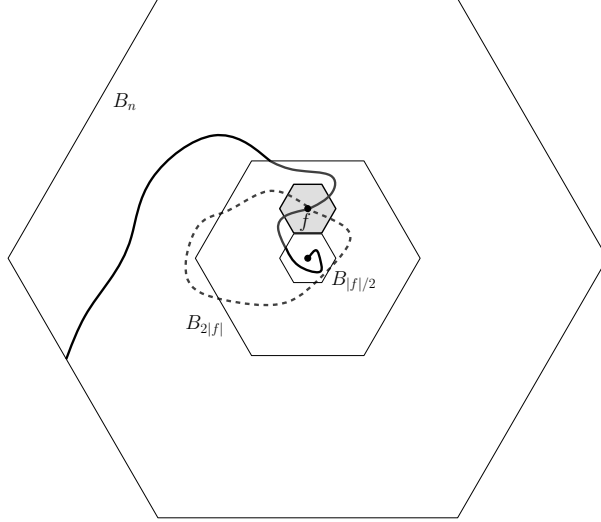


Figure 10: The event that  $f$  is pivotal for  $A_{1,O}(n)$ . The dotted line corresponds to a closed circuit.

From quasi-multiplicativity again, we find

$$\pi_4^{p'}(k) \leq C_1(n/k)^{2-\alpha} \pi_4^{p'}(n),$$

which can be put into (6.5) to give

$$\begin{aligned} \log \pi_1^p(n) - \log \pi_1(n) &\leq C_1 \int_{1/2}^p \left( \sum_{f \in B_n} (2n/|f|)^{2-\alpha} \pi_4^{p'}(n) \right) dp' \\ &\leq C_2 \int_{1/2}^p n^2 \pi_4^{p'}(n) dp'. \end{aligned}$$

To conclude, Russo's formula implies

$$\begin{aligned} 1 &\geq \mathbb{P}_p([0, n/2]^2 \text{ is crossed}) - \mathbb{P}_{1/2}([0, n/2]^2 \text{ is crossed}) = \int_{1/2}^p \sum_{x \in B_{n/2}} \mathbb{P}_{p'}(f \text{ is pivotal}) dp' \\ &\geq \int_{1/2}^p \frac{3n^2}{4} \pi_4^{p'}(n) dp', \end{aligned}$$

where we have used the fact that  $f$  is pivotal for the event  $\{[0, n/2]^2 \text{ is crossed}\}$  if there are four arms of alternating colors going to the boundary of  $f + [0, n]^2$ . In particular, we find the required bound

$$\log \pi_1^p(n) - \log \pi_1(n) \leq 4C_2/3.$$

□

The same reasoning can be applied for  $\pi_{j,\sigma}$ . The main step is to get (6.4) with  $\pi_1$  replaced by  $\pi_{j,\sigma}$ , the end of the proof being the same. In order to obtain this inequality, one harnesses a generalization of Russo's formula; we refer to Theorem 26 of [28] for a complete exposition.

## 7 A few open questions

**Percolation on the hexagonal lattice** Percolation on the hexagonal lattice is now very well understood. Nevertheless, several questions remain open. We selected three of them.

We know the behavior of most thermodynamical quantities (the cluster density  $\theta$ , the truncated mean-cluster size  $\xi(p) = (p - 1/2)^{-\nu+o_p(1)}$ , the two-point functions  $\mathbb{P}_{1/2}(0 \leftrightarrow x) = |x|^{2-d-\eta+o_{|x|}(1)}$  and many others). Nevertheless, the behavior of the following fundamental quantity remains unproved:

**Question 1.** *Prove that the mean number of clusters per vertex  $\kappa(p) = \mathbb{E}_p(|C|^{-1})$  behaves like  $|1/2 - p|^{2+\alpha+o(1)}$ , where  $C$  is the cluster at the origin and  $\alpha = -2/3$ .*

Interestingly, the critical exponent for  $j \neq 1$  disjoint arms of the same color is not equal to the polychromatic arms exponent [2]. A natural open question would be to compute these exponents:

**Question 2.** *Compute the monochromatic exponents.*

**Percolation on other graphs** Conformal invariance of percolation has been proved only on the hexagonal lattice. In physics, it is conjectured that the scaling limit of percolation should be universal, meaning that it should not depend on the lattice (this hypothesis is verified numerically in [23]). For instance, interfaces of bond-percolation on the square lattice at criticality (when the bond-parameter is  $1/2$ ) should also converge to SLE(6).

**Question 3.** *Prove conformal invariance of percolation on another planar lattice.*

For general graphs, the question of embedding the graph becomes crucial. Indeed, if one embeds the square lattice by gluing long rectangles, then the model will not be rotationally invariant. We refer to [4] for further details on the subject.

**Question 4.** *For a general lattice, how to construct a natural embedding on which percolation is conformally invariant?*

In order to understand universality, a natural class of lattices to start with is the class of lattices where RSW estimates can be proved. Note that proofs of RSW always invoke some symmetry (rotational invariance for instance). A proof valid for lattices without any symmetry would be of great importance:

**Question 5.** *Prove RSW for critical percolation on all planar lattices.*

Percolation in high dimension is well understood (see *e.g.* [15]), thanks to the so-called triangular condition and lace-expansion techniques associated to it. In intermediate dimensions, the critical phase is not understood. Of course, one of the main conjectures in probability is to prove that  $\theta(p_c) = 0$  for bond-percolation on  $\mathbb{Z}^3$ . Even weakening of this conjecture seems to be very hard. For instance, the same question on the “sandwich”  $\mathbb{Z}^2 \times \{0, 1\}$  is still open:

**Question 6.** *Prove that  $\theta(p_c) = 0$  on  $\mathbb{Z}^2 \times \{0, 1\}$ .*

**Other two-dimensional models of statistical physics** Conformal invariance is not restricted to percolation (see [35, 36] and references therein). It should hold for a wide class of two-dimensional lattice models at criticality. Among natural generalizations of percolation, we mention the class of random-cluster models and of loop  $O(n)$ -models (including the Ising model and the self-avoiding walk). The only three models in this family for which conformal invariance has been proved are the Ising model, the  $q = 2$ -random cluster model (which is a geometric representation of the Ising model), and the uniform spanning tree (the ' $q = 0$ '-random cluster model).

**Question 7.** *Prove conformal invariance of your favorite model on your favorite lattice (this is assuming that your favorite model is not site-percolation on the triangular lattice).*

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## References

- [1] M. Aizenman and A. Burchard, *Hölder regularity and dimension bounds for random curves*, Duke Math. J. **99** (1999) 419–453.
- [2] V. Beffara and P. Nolin, *On monochromatic arm exponents for 2D critical percolation*, Annals of Probability, to appear (2010).
- [3] V. Beffara, *Cardy's formula on the triangular lattice, the easy way*. Universality and Renormalization, vol. 50 of the Fields Institute Communications (2007) 39–45.
- [4] V. Beffara, *Is critical 2D percolation universal?* In and Out of Equilibrium 2 (2008), vol. 60 of Progress in Probability, Birkhäuser, 31–58.
- [5] B. Bollobás and O. Riordan, *Percolation*, Cambridge Univ. Press (2006).
- [6] B. Bollobás and O. Riordan, *A short proof of the Harris-Kesten Theorem*, Bulletin of the London Mathematical Society **38** (2006), 470–484.
- [7] S.R. Broadbent and J.M. Hammersley, *Percolation processes I. Crystals and mazes*, Proceedings of the Cambridge Philosophical Society **53** (1957) 629–641.
- [8] F. Camia and C. M. Newman, *Two-dimensional critical percolation: the full scaling limit*, **268** (2005), 1–38.
- [9] J. Cardy, *Critical percolation in finite geometries*, J. Phys. A **25** (1992) L201–L206.
- [10] A. Desolneux, B. Sapoval, A. Baldassarri, *Self-organized percolation power laws with and without fractal geometry in the etching of random solids*, in Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot, Proc. Symp. Pure Math. **72** Part 2, (2004) 485–505.
- [11] E. Friedgut, *Influence in product spaces: KKL and BKKKL revisited*. Combinatorics, Probability, Computing, **13** (2004) 17–29.

- [12] E. Friedgut and G. Kalai. *Every monotone graph has a sharp threshold*. Proceedings of the American Mathematical Society, **124** (1996) 2993–3002.
- [13] G. R. Grimmett, *Percolation*, vol. 321 of Grundlehren Math. Wiss., Springer-Verlag, Berlin, second ed., 1999.
- [14] G. R. Grimmett, *Probability on graphs*, Cambridge Univ. Press, Vol. 1 of the IMS textbook series (2010).
- [15] T. Hara and G. Slade *Mean-field behaviour and the lace expansion*, Probability and Phase Transition (G. R. Grimmett, ed.), Kluwer, Dordrecht, (1994) 87–122.
- [16] J. Kahn, G. Kalai and N. Linial. *The influence of variables on Boolean functions*. Proceedings of 29th Symposium on the Foundations of Computer Science, Computer Science Press (1988) 66–80.
- [17] G. Kalai and S. Safra. *Threshold phenomena and influence*. Computational Complexity and Statistical Physics, Oxford University Press, New-York, 2006.
- [18] H. Kesten, *Percolation theory for mathematicians*, vol. 2 of Progress in Probability and Statistics, Birkhäuser, Boston, Mass., 1982.
- [19] H. Kesten, *Subdiffusive behavior of random walk on a random cluster*, Ann. I.H.P. Sec. B **22** (1986) 425–487.
- [20] H. Kesten, *Scaling relations for 2D-percolation*, Comm. Math. Phys. **109** (1987) 109–156.
- [21] H. Kesten, V. Sidoravicius and Y. Zhang, *Almost all words are seen in critical site percolation on the triangular lattice*, Elec. J. Probab. **3**, paper no 10 (1998).
- [22] S. Lang, *Complex Analysis*, vol. 103 of Graduate Texts in Mathematics, Springer-Verlag, New York, fourth ed., 1999.
- [23] R. Langlands, Ph. Pouliot, Y. Saint-Aubin, *Conformal invariance in two-dimensional percolation*, Bull. Amer. Math. Soc. N.S. **30** (1994) 1–61.
- [24] G. F. Lawler, *Conformally Invariant Processes in the Plane*. Math. Surveys Monogr. **114**, Amer. Math. Soc., Providence, RI, 2005.
- [25] G. F. Lawler, O. Schramm, W. Werner, *Values of Brownian intersection exponents I: Half-plane exponents*, Acta Mathematica **187** (2001) 237–273.
- [26] G. F. Lawler, O. Schramm, W. Werner, *Values of Brownian intersection exponents II: Plane exponents*, Acta Mathematica **187** (2001) 275–308.
- [27] G. F. Lawler, O. Schramm, W. Werner, *One-arm exponent for 2D critical percolation*, Electron. J. Probab. **7** (2001) 13 pages.
- [28] P. Nolin, *Near-critical percolation in two dimensions*, Electronic Journal of Probability **13** (2008) 1562–1623.
- [29] D. Reimer *Proof of the van den Berg-Kesten conjecture*. Combin. Probab. Comput. **9** (2000) 27–32.

- [30] L. Russo, *A note on percolation*. Z. Wahrscheinlichkeitstheorie Verw. Gebiete **43** (1978) 39–48.
- [31] P.D. Seymour, D.J.A. Welsh, *Percolation probabilities on the square lattice*, Ann. Discrete Math. **3** (1978) 227–245.
- [32] B. Sapoval, M. Rosso, J.F. Gouyet, *The fractal nature of a diffusion front and the relation to percolation*, J. Phys. Lett. 46 (1985) 146–156.
- [33] S. Smirnov, *Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), 239–244.
- [34] S. Smirnov, *Critical percolation and conformal invariance*, in Zambrini, Jean-Claude (ed.), XIVth International Congress on Mathematical Physics, Lisbon, Portugal, July 28 – August 2, 2003. 99–112, World Sci. Publ., Hackensack, NJ.
- [35] S. Smirnov, *Towards conformal invariance of 2D lattice models*, in Sanz-Solé, Marta (ed.) et al., Proceedings of the International Congress of Mathematicians (ICM), Madrid, Spain, August 22-30, 2006. Volume II, 1421–1451. Zürich: European Mathematical Society, 2006.
- [36] S. Smirnov, *Discrete complex analysis and probability*, Proceedings of the International Congress of Mathematicians (ICM), Hyderabad, India, 2010.
- [37] S. Smirnov and W. Werner, *Critical exponents for two-dimensional percolation*, Mathematical Research Letters, **8** (2001) 729–744.
- [38] W. Werner, *Random planar curves and Schramm-Löwner evolutions*. In Lectures on probability theory and statistics, Lecture Notes in Math. 1840, Springer-Verlag, Berlin (2004) 107–195.
- [39] W. Werner, *Lectures on 2D critical percolation*, Park City (2007).
- [40] W. Werner, *Percolation et modèle d'Ising*, (in french) published by the French Math. Society (SMF) in the Series "Cours spécialisés" (2009).

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