HAL
open science

A simple particle model for a system of coupled equations with absorbing collision term

Cedric Bernardin, Valeria Ricci

▶ To cite this version:

Cedric Bernardin, Valeria Ricci. A simple particle model for a system of coupled equations with absorbing collision term. 2011. ensl-00610145

HAL Id: ensl-00610145
https://ens-lyon.hal.science/ensl-00610145

Preprint submitted on 21 Jul 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A SIMPLE PARTICLE MODEL FOR A SYSTEM OF COUPLED EQUATIONS WITH ABSORBING COLLISION TERM

Cédric Bernardin

Université de Lyon and CNRS,
UMPA, UMR-CNRS 5669, ENS-Lyon,
46, allée d'Italie, 69364 Lyon Cedex 07 - France

Valeria Ricci

Dipartimento di Metodi e Modelli Matematici
Università di Palermo, Viale delle Scienze Edificio 8
I90128 Palermo - Italy

Abstract. We study a particle model for a simple system of partial differential equations describing, in dimension $d \geq 2$, a two component mixture where light particles move in a medium of absorbing, fixed obstacles; the system consists in a transport and a reaction equation coupled through pure absorption collision terms. We consider a particle system where the obstacles, of radius ε, become inactive at a rate related to the number of light particles travelling in their range of influence at a given time and the light particles are instantaneously absorbed at the first time they meet the physical boundary of an obstacle; elements belonging to the same species do not interact among themselves. We prove the convergence (a.s. w.r.t. the product measure associated to the initial datum for the light particle component) of the densities describing the particle system to the solution of the system of partial differential equations in the asymptotics $a_n^d n^{-\kappa} \to 0$ and $a_n^d \varepsilon^\zeta \to 0$, for $\kappa \in (0, \frac{1}{2})$ and $\zeta \in (0, \frac{1}{2} - \frac{1}{2d})$, where a_n^{-1} is the effective range of the obstacles and n is the total number of light particles.

1. Introduction

In this paper we propose a microscopic model for a system of partial differential equations consisting in a transport equation, having pure loss collision term, and a pure loss reaction equation; the equations are self–consistently coupled and they are meant to describe the evolution in time of the phase space densities in a binary mixture where the two species interact in such a way that each species inhibits the activity of the other, with an interaction proportional to their macroscopic (position space) densities. In particular, we want to rigorously derive the system of equations in a suitable asymptotics for the particle system.

The system of equations we shall deal with is a sort of very simple reactive transport system, though without conservation of masses; more complex reactive transport systems arise in many different contexts, like for instance in the modeling of biological systems, of porous media, of radiative transfer or, more in general, of various systems in the presence of chemical reactions (see e.g.[SDL, MM]). We

1991 *Mathematics Subject Classification.* 82C22, 82C21, 60F05, 60K35.

Key words and phrases. Interacting particle systems, large numbers limit, absorption.

Valeria Ricci acknowledges the support by the GNFM (research project 2008: "Approssimazione di equazioni alle derivate parziali tramite sistemi di particelle").

are interested here in the analysis of the simplest reactive self-consistent coupling (i.e. the absorption coupling), and we shall therefore not include other terms in the equations.

From the point of view of particle modeling, an analysis of the absorption self-consistent coupling for one species nonlinear equations can be found in models for reaction-diffusion equations, such as the ones proposed in [NOR] and [Szn]; in both papers, the particle system evolves according to a Brownian motion (to get the diffusion) and destruction of particles occurs with different mechanism: in the first paper the destruction is stochastic, with a death rate for the Brownian particles which is a function of stochastic exponential times, in the second one it is deterministic, and it happens as soon as collisions between particles occur.

We shall consider a binary semi–deterministic system where both kinds of interactions occur. More precisely, the particle system we shall consider consists in point like (light) particles moving uniformly among fixed, spherical particles (obstacles). Particles not belonging to the same species do not interact among themselves and particles of different species interact in the following way: a light particle becomes inactive (or is absorbed) at the first time it meets an obstacle and an obstacle becomes inactive (or disappears) in a stochastic interval of time whose size is connected, through a local mean-field type interaction, to the number of light particles traveling within the area of detection (range) of the obstacle.

The main difficulty in the derivation of the (otherwise simple) system of partial differential equations originates from the self consistent structure of the problem: in order to overcome the mathematical troubles introduced from the self–consistent terms, we shall adopt a natural scheme for facing self–consistent problems in partial differential equations and particle systems and we shall prove the convergence using two levels of approximation of the original particle system, the first one eliminating the self-consistent structure and the second one reducing the correlations between the two species with respect to the original many-particle system. Our formalism is quite explicit and the strategy we shall adopt is similar to the procedure used in [NOR]. The adjustment of this procedure to our particle system is not trivial, since, at variance with the situation treated in [NOR], we deal with a two species system interacting in a non symmetric way and the instantaneous absorption of the light particles is in some sense a singular interaction with respect to the local mean-field type interaction which govern the obstacles lifetime, i.e. the interaction type considered in [NOR]. The presence of the deterministic component (the light particles) imposes a sufficiently careful analysis of the trajectories, in the style of the analysis performed for particle models of linear equations based on similar deterministic components (see e.g. [BGW], [DR1, DR2]). Nevertheless, the structure of the stochastic component (the obstacles) simplifies by a considerable amount the mathematics with respect to a two component, totally deterministic system.

Our final theorem establishes a weak law of large numbers for the empirical measures (associated to the two species of particles in the mixture) to the solution of the system of partial differential equations, almost surely with respect to the initial distribution of positions in the phase space of the light particles and in probability with respect to the distributions of positions and life time of the obstacles. This weak law of large numbers is valid in an asymptotics where the radius and the effective range of the obstacles vanish, keeping a finite action in the limit, and the number of particles grows up to infinity, these quantities being related in a way

that we shall determine while proving the theorem. As a part of this relationship, we shall prove a simple condition (relating the diminishing rate of the effective range of the obstacles to the number of light particles) which guarantees the weak convergence of the product of an empirical measure times a sufficiently regular approximant of a Dirac delta distribution toward the product of their weak limits (under suitable regularity assumptions on the weak limit of the empirical measure and on the choice of the delta's approximant). This condition entails very useful bounds for our estimates and allows to get easily the required asymptotic result.

2. The equations, the particles model and the main result

Throughout the paper we shall use the following notations: in dimension $d \geq 2$, we denote by $x \in \mathbb{R}^d$ the position of a light particle and by $v \in \mathbb{R}^d$ its velocity; $t \in \mathbb{R}_+$ is the time variable. In general, configurations of M variables in $\mathbb{R}^d$ will be denoted by boldface letters with subscript M ($\mathbf{y}_M = (y_1, \ldots, y_M)$) and sequences of variables by capital letters with the subscript ∞ ($Y_\infty = \{y_i\}_{i=1}^\infty$). When needed (e.g., in the functional spaces) we shall use $\mathbb{R}^d_x$ and $\mathbb{R}^d_v$ resp. for the position and the velocity spaces. For $p \in \mathbb{R}^d$ and $r \in \mathbb{R}_+$ we shall denote by $B_r(p) = \{y \in \mathbb{R}^d : |p-y| \leq r\}$ the ball of radius r centered in p. For a given $z = (x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, the free flow associated to the light particle with initial position (in the phase space) z is

$$T^t(z) = (T^t_1(z), T^t_2(z)) = (x + vt, v), \qquad t \in \mathbb{R}_+,$$

while $x(t) = T^t_1(z)$, $t \in \mathbb{R}_+$, is its trajectory, and for $\varepsilon > 0$

$$\mathcal{T}_\varepsilon(t, z) = \left\{ y \in \mathbb{R}^d \quad ; \exists s \in [0, t), \quad |y - x(s)| \leq \varepsilon \right\} \tag{1}$$

denotes the flow tube of radius ε associated to the trajectory up to the time t.

Unless differently stated, for a given random variable η, we denote by P_η its associated probability distribution, and for a measure π (random variable η), we denote by $\mathbb{E}_\pi$ ($\mathbb{E}_\eta$) the expectation w.r.t. π (η). In order to simplify the notation, in many stochastic variables depending on configurations $\mathbf{y}_M$, we shall label this dependence by a simple subscript M, instead of rewriting each time the whole configuration.

We shall moreover denote $\rightharpoonup$ the weak convergence (convergence in law) in the space of finite measures and $\overset{*}{\rightharpoonup}$ the *-weak (vague) convergence on the space of Radon measures, and by K (with some subscript) any generic constant whose value needs not to be specified. We shall sometimes shorten the notation for sums and difference of functions with the same argument as $(f \pm g)(w) = f(w) \pm g(w)$.

For $T > 0$, we consider the following system of partial differential equations for the two densities $f = f(t, x, v)$, $\sigma = \sigma(t, x)$, $t \in [0, T]$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = -C_d |v| \sigma f \\ \partial_t \sigma = -\Theta(\int_{\mathbb{R}^d} dv f)\sigma \\ f(0, x, v) = f_0(x, v) \\ \sigma(0, x) = \sigma_0(x), \end{cases} \tag{2}$$

where Θ and $C_d = \int_{\{\omega \in \mathbb{R}^d : |\omega| = 1\}} |n \cdot \omega| d\omega$, $|n| = 1$, are positive constants. In order to have a convenient existence and uniqueness theorem for the solution (cf. Appendix 6.1), we assume $f_0 \in L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x))$, $vf_0 \in L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x)) \cap L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v)$, $v^2 f_0 \in L^1(\mathbb{R}^d_v; L^\infty(\mathbb{R}^d_x))$ and $\sigma_0 \in W^{1,\infty}(\mathbb{R}^d_x)$.

We shall show that the system (2) can be derived from a semi-deterministic particle system of the kind we specified in the Introduction. The particle system will be described in next paragraphs.

Obviously, once established the correct asymptotics to get equation (2) from the particle system we have chosen, we shall simply write $\lim_{n\to\infty}$ to denote a limit in this asymptotics (n and ε being related). This peculiar use of the notation will be consequent to the hypothesis in the environments where the notation is used.

2.1. **The particle system: initial data statistics.** We consider configurations of spherical fixed obstacles of radius ε with stochastically distributed positions at time $t = 0$ and we denote by $\mathbf{c}_M = (c_1, \dots, c_M)$ the coordinates of their centers. Obstacles may overlap (i.e., configurations such that for some i, k, $|c_i - c_k| \le 2\varepsilon$ are allowed) and $M = 0$ corresponds to absence of obstacles. In this paper, we shall assume that obstacles positions follow a Poisson distribution with parameter μ_ε, i.e. that the probability distribution of finding M obstacles in a bounded measurable set $\Lambda \subset \mathbb{R}^d$ is given by:

$$P(d\mathbf{c}_M) = e^{-\mu_\varepsilon [\Lambda]_L} \frac{\mu_\varepsilon^M}{M!} dc_1 \dots dc_M, \tag{3}$$

where (here and in what follows) $[\Lambda]_L$ denotes the Lebesgue measure of the set Λ. We shall adapt the initial datum $\sigma_0(x)$ to this choice for the statistics of the obstacles.

We consider then n point-like particles, located initially at positions $x_1, \dots, x_n$ and moving uniformly among the obstacles with velocities $v_1, \dots, v_n$; we shall denote the phase space coordinate of the i-th light particles as $z_i = (x_i, v_i)$. A point in the n-particles phase space is denoted as $\mathbf{z}_n = (x_1, v_1, .., x_n, v_n)$ and a sequence of initial data is denoted as $Z_\infty = \{z_i\}_{i=1}^\infty \in (\mathbb{R}^d \times \mathbb{R}^d)^\infty$.

We describe both species of particles by means of their empirical measure (see e.g. [G1]).

Given a 1-particle probability density f_0, we denote by $\mathscr{P}$ the (infinite product) probability measure defined on the space of infinite sequences $(\mathbb{R}^d \times \mathbb{R}^d)^\infty$ by f_0, i.e. such that, for $n = 1, 2, \dots$ and $A_x^{(i)} \times A_v^{(i)} \subset \mathbb{R}^d \times \mathbb{R}^d$,

$$\mathscr{P}(Z_\infty : z_i \in A_x^{(i)} \times A_v^{(i)}, 1 \le i \le n) = \prod_{k=1}^n \int_{A_x^{(k)} \times A_v^{(k)}} f_0(x, v) dx dv. \tag{4}$$

The sequence of initial empirical measures for the light particles $\{\mu_n^0\}_{n=1}^\infty$ is then such that:

$$\mu_n^0(t, x, v; \mathbf{z}_n) = \frac{1}{n} \sum_{i=1}^n \delta_{z_i}(x, v) \rightharpoonup f_0(x, v) \qquad \mathscr{P} - a.e.\ w.r.t. Z_\infty. \tag{5}$$

We shall require moreover some regularity condition on f_0.

2.2. **The particle system: dynamics.** We define now the dynamics of the system.

We consider a sequence $\tau_\infty = \{\tau_1, \dots, \tau_k, \dots\}$ of independently distributed exponential variables and we define the following stochastic functions:

- The *risk function* at a given position c,

$$V_{n,\varepsilon,M}(t,c) = \frac{1}{n}\sum_{i=1}^{n}\int_0^t ds q_n(x_i(s)-c)\xi_{n,\varepsilon,M}(s,z_i), \tag{6}$$

  where $q_n(x) = a_n^d q(a_n x)$, is an (at least continuous) approximant, up to a multiplicative constant, of the Dirac delta distribution, with q a non negative, radial ($q(x) = q(|x|)$) function such that $\int_{\mathbb{R}^d} q(y)dy = \Theta > 0$ and $\xi_{n,\varepsilon,M}$ is the stochastic function defined below.
- The *life functions* [1], resp. $\xi_{n,\varepsilon,M}(t,z)$ for a light particle with initial position and velocity (x,v) and $\eta_{n,\varepsilon,M}(t,c_k)$ for an obstacle centered in $c_k \in \{c_1,\dots,c_M\}$,

$$\begin{aligned} &\xi_{n,\varepsilon,0}(t,z) = 1 \\ &M \geq 1 \\ &\xi_{n,\varepsilon,M}(t,z) = I_{\{x(s)\notin \bigcup_{h=1}^{M} B_\varepsilon(c_h)\eta_{n,\varepsilon,M}(s,c_h) \quad \forall s\in[0,t)\}} \\ &\eta_{n,\varepsilon,M}(t,c_k) = I_{\{V_{n,\varepsilon,M}(t,c_k)<\tau_k\}}. \end{aligned} \tag{7}$$

- The *maximal collision time* between a light particle with initial phase space position $z = (x,v)$ and an obstacle located in c,

$$T_{z,c} = \inf_{s\in\mathbb{R}_+}\{s:\ |x(s)-c| \leq \varepsilon\}, \tag{8}$$

  with no reference to the activity of both particles. Because no life functions are involved in this definition, $T_{z,c}$ can be infinite.

In the particle system defined through (6) and (7), the obstacles become inactive at stochastically distributed times defined at a given position c_k through the risk function (6) as the time $t_k = t_k(c_k,\tau_k)$ s.t. $V(t_k,c_k) = \tau_k$, while the light particles are absorbed at the first time they meet an active obstacle (i.e. as soon as $|x_i(T_{coll})-c_k| = \varepsilon$ for some k and $T_{coll} < t_k$; notice that T_{coll} is defined w.r.t. active particles, and it is therefore a different variable w.r.t. $T_{z,c}$).

Unless we require q to have compact support, a light particle located at x can affect the life function of a far obstacle (it suffices that $q(x-c) \neq 0$); on the contrary, a given light particle interacts only with obstacles which are met by its trajectory in space. Since, for a given n, the volume including all light particle trajectories up to time T, $\mathcal{V}_n$, is such that

$$\mathcal{V}_n \subset B_{\max_{i=1,\dots,n}|x_i|+\max_{j=1,\dots,n}|v_j|T+1}(0) = B_n, \tag{9}$$

and, in studying the interacting system, we do not need to consider quantities related to obstacles which can not be met from any light particles, we may (for a given n) restrict the expectation value with respect to (3) to the volume $\Lambda = \Lambda_n$, for

[1] We use, here and later, the notation $\bigcup_{k=1}^{M} B_\varepsilon(c_k)\eta_{n,\varepsilon,M}(s,c_k)$ to mean $\bigcup_{\substack{k\in\{1,\dots,M\}:\\ \eta_{n,\varepsilon,M}(s,c_k)=1}} B_\varepsilon(c_k)$. Although not formally correct, this notation allows us to write less cumbersome formulas.

a growing sequence of bounded Lebesgue measurable sets $\{\Lambda_n\}_n$ s.t. $B_n \subset \Lambda_n$ and $\lim_{n\to\infty} \Lambda_n = \mathbb{R}^d$. We denote by $\mathbb{E}_c^n$ the expectation value with respect to the centers distribution (3) with such a choice for Λ. We shall denote by $\mathbb{E}^n = \mathbb{E}_c^n \mathbb{E}_{\mathcal{T}} = \mathbb{E}_{\mathcal{T}} \mathbb{E}_c^n$, where $\mathbb{E}_{\mathcal{T}}$ is the expectation value w.r.t. to the exponential times sequence τ_∞, and by P^n the corresponding probability distributions.

2.3. **Scaling laws: heuristics.** In order to derive the system (2) from the particle system defined above in the macroscopic limit where the radius ε of the obstacles vanishes, we have to choose the scaling for densities of both species of particles in a suitable way.

As a first requirement, we want the mean free path of the light particles to be finite at the scale at which the system is observed, in such a way to keep track, in the chosen asymptotics, of the interaction between the light particles and the obstacles. To this purpose, we fix the rate μ_ε of the Poisson process (3) to be such that:

$$\mu_\varepsilon \varepsilon^{d-1} = \mu > 0 \tag{10}$$

and, since we consider a uniform initial macroscopic distribution of obstacles, in (2) we assume $\sigma_0(x) = \mu$.

When $V_{n,\varepsilon,M} \equiv 0$ (i.e. $q \equiv 0$), formula (10) defines the so called Boltzmann-Grad scaling, which has been analysed for linear particle systems in particular in connection with the asymptotics of the Lorentz gas and its variants (see e.g. [Ga, Sp, BBS], [DR1] for the stochastic case, [BGW, G2, RW] for the periodic case and the beautiful review [G3] -focused on the periodic case-). When $V_{n,\varepsilon,M} \not\equiv 0$, (10) guarantees the finiteness of the mean free path for a given light particle as soon as $V_{n,\varepsilon,M}$, in flow tubes and for a non-negligible set of obstacles configurations, is bounded uniformly in n. This is true, under the conditions on f_0 specified at the beginning of Section 2 and the conditions on q given in Section 2.2 (i.e. $q \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$), whenever $a_n^d n^{-\frac{1}{2}} \leq const$.

A second requirement is to have, in the $\varepsilon \to 0$ asymptotics, vanishing correlations between light particles and obstacles, so as to obtain a Markovian limit (i.e. without memory effects), coherently with the structure of (2). Since the dominant part of correlations is associated to grazing crossings of light particle trajectories, and in particular to the mean volume V_g^ε spanned from grazing trajectories crossings within the effective range of an obstacle, this requirement connects the scaling in the effective range a_n^{-1} to the scaling in the mean density of the obstacles. In order to have negligible correlations, the mean number of multiple collisions per unit volume has to vanish in the limit. This leads to the condition $\mu_\varepsilon a_n^d V_g^\varepsilon \to 0$. Notice that possibly $V_g^\varepsilon = O(\varepsilon^{d-1})$, and having a better asymptotic behavior depends essentially on the regularity of the limit measure of the light particle component.

We have then a third, technical, requirement due to the form of (6): since eventually we want to obtain a mean-field limit, and therefore we want to be able to perform limits of mean values with respect to μ_n^0 of sequences of functions converging to singular limits (such as Dirac delta distributions), we need a condition assuring the convergence (in a suitable sense) of the product of such sequences times the initial empirical measure: this is achieved if the empirical measure converges faster to its (sufficiently regular) limit than the chosen approximant concentrates, in such a way that the empirical measure is in practice equivalent to its limit density well before than the delta's approximant concentrates in its center. We can

guess roughly that this happens if the fraction of particles fluctuating around mean values in a volume corresponding to the effective range of an obstacle vanishes in the chosen asymptotics, and therefore $\frac{1}{\sqrt{n}} = o(a_n^{-d})$. We shall prove a more precise asymptotics in Appendix 6.2

Given (10) and the just described scaling laws, for a given configuration of obstacles, the empirical measure at time t for the light particle component (representing its mesoscopic density in phase space) is:

$$\mu_n(t,x,v;\mathbf{z}_n,\mathbf{c}_M) = \frac{1}{n}\sum_{i=1}^{n} \delta_{T^t(z_i)}(x,v)\xi_{n,\varepsilon,M}(t,z_i) \tag{11}$$

and the (macroscopic) density of the obstacle component (i.e. the number of obstacles per unit volume) is expressed as:

$$\sigma_n(t,x;\mathbf{z}_n,\mathbf{c}_M) = \varepsilon^{d-1}\sum_{k=1}^{M} \delta_{c_k}(x)\eta_{n,\varepsilon,M}(t,c_k). \tag{12}$$

2.4. **The limit process.** Under the conditions we assumed on (f_0,σ_0), as shown in paragraph 6.1 in the Appendix, the problem (2) admits a unique solution $(f,\sigma) \in L^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)\times L^\infty([0,T]\times\mathbb{R}^d)$, with $\int dvf \in L^\infty([0,T]\times\mathbb{R}^d)$ (and actually $\int dvf(t,\cdot,v) \in C_b(\mathbb{R}^d)$). This solution can be expressed in semi-explicit form as:

$$\begin{aligned} f(t,x,v) &= f_0(x-vt,v)e^{-C_d|v|\int_0^t ds\sigma(s,x-v(t-s))} \\ \sigma(t,x) &= \sigma_0(x)e^{-\Theta\int_0^t ds\int_{\mathbb{R}^d} dvf(s,x,v)}. \end{aligned} \tag{13}$$

In order to be able to compare solutions of (2) with the stochastic measures (11) and (12), we define the risk functions and the life functions associated to the limit process described by (2) as:

$$\begin{aligned} V_L^f(t,c) &= \Theta\int_0^t ds\int_{\mathbb{R}^d} dvf(s,c,v) \\ \eta_L^f(t,c) &= I_{\{V_L(t,c)<\tau\}} \end{aligned} \tag{14}$$

$$\begin{aligned} U_L^\sigma(t,z) &= C_d|T_2^t(z)|\int_0^t ds\sigma(s,T_1^s(z)) \\ \xi_L^\sigma(t,z) &= I_{\{U_L(t,z)<\tau_p\}}, \end{aligned} \tag{15}$$

where τ and τ_p are independently distributed exponential variables and $\tau = \tau_k$ when $c = c_k$.

In this way, the semi-explicit form of the solution (13) can be re-expressed as

$$\begin{aligned} f(t,x,v) &= f_0(x-vt,v)\mathbb{E}_{\tau_p}\left[\xi_L^\sigma(t,T^{-t}(x,v))\right] \\ \sigma(t,x) &= \sigma_0(x)\mathbb{E}_\tau\left[\eta_L^f(t,x)\right]. \end{aligned} \tag{16}$$

In order to simplify the notation, we shall omit in what follows the dependence on the density functions ($V_L = V_L^f$, $U_L = U_L^\sigma$).

We want to establish a (weak) law of large numbers for the measures (11) and (12). More precisely, we shall prove the following theorem (we recall that $\mathscr{S}(\mathbb{R}^n)$ denotes the space of $C^\infty(\mathbb{R}^n)$ functions of rapid decay at infinity [Sch, DL]):

Theorem 1. *Consider the non-negative functions f_0 and q and assume*

- $f_0 \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d) \bigcap L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x))$ *is a probability density such that*

$$\int_{\mathbb{R}^d} dv f_0 \in \mathscr{S}(\mathbb{R}^d) \qquad \text{with} \qquad \int_{\mathbb{R}^d} dv f_0(0,v) > 0, \tag{17}$$

  and $vf_0 \in L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x))$, $v^2 f_0 \in L^1(\mathbb{R}^d_v; L^\infty(\mathbb{R}^d_x))$*;*
- q *is a radial function s.t.* $q \in \mathscr{S}(\mathbb{R}^d)$ *and* $\int_{\mathbb{R}^d} dx q(x) = \Theta > 0$*;*
- $\{a_n\}_{n=1}^\infty$ *is such that* $a_n > 0$, $\lim_{n\to\infty} a_n = \infty$ *and there exists some* $\kappa \in (0, \frac{1}{2})$ *such that*

$$\lim_{n\to\infty} \frac{a_n^d}{n^\kappa} = 0; \tag{18}$$

- $\{\varepsilon\}_{n=1}^\infty = \{\varepsilon_n\}_{n=1}^\infty$ *s.t.* $\varepsilon_n > 0$ *and*

$$\lim_{n\to\infty} a_n^d \varepsilon^\zeta = 0 \tag{19}$$

  for some $\zeta \in (0, \frac{1}{2} - \frac{1}{2d})$.

Then, $\mathscr{P}$-almost everywhere w.r.t. sequences of initial data Z_∞ and in probability w.r.t. P and P_τ, when $n \to \infty$

$$\mu_n(t,x,v;\mathbf{z}_n,\boldsymbol{c}_M) = \frac{1}{n}\sum_{i=1}^n \delta_{T^t(z_i)}(x,v)\xi_{n,\varepsilon,M}(t,z_i) \rightharpoonup f(t,x,v) \tag{20}$$

$$\sigma_n(t,x;\mathbf{z}_n,\boldsymbol{c}_M) = \varepsilon^{d-1}\sum_{i=1}^M \delta_{c_i}(x)\eta_{n,\varepsilon,M}(t,c_i) \overset{*}{\rightharpoonup} \sigma(t,x), \tag{21}$$

where $\xi_{n,\varepsilon,M}$, $\eta_{n,\varepsilon,M}$ are defined in paragraph 2.2 and (f,σ) is the unique solution of

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = -C_d|v|\sigma f \\ \partial_t \sigma = -\Theta(\int_{\mathbb{R}^d} dv f)\sigma \\ f(0,x,v) = f_0(x,v) \\ \sigma(0,x) = \mu. \end{cases} \tag{22}$$

We shall see in paragraph 6.2 in the appendix that hypothesis (18), together with suitable regularity assumptions on f_0 (all included in our theorem), guarantees the following $\mathscr{P}$-a.s. convergence

$$\begin{aligned} |v|^j \mu_n^0(x,v)q_n(x) &\rightharpoonup \Theta|v|^j f_0(x,v)\delta_0(x) \\ |v|^j \mu_n^0 &\rightharpoonup |v|^j f_0 \\ \otimes_{i=1}^k |v|^j \mu_n^0 &\rightharpoonup \otimes_{i=1}^k |v|^j f_0 \end{aligned} \tag{23}$$

for $j = 0,1,2,\dots,P$, $k = 1,\dots Q$, with given positive integers P, Q.

3. Definitions of the approximating systems and proof of the main theorem

The main difficulty in studying (2) and its associated particle system (6),(7) is that we have to deal with a self-consistent problem. The first step is therefore to find, both for the limit system (2) and for the particle system defined from (6),(7), suitable approximating systems which do not share this self-consistent structure.

We first recall here without proof, since it will be useful for the sequel and we shall use it largely in various steps of the convergence proof, lemma 3.2 in [NOR], concerning bounds of distances of stochastic variables of the form $\eta(t) = I_{\{S(t)<\tau\}}$ in terms of distances between their associated risk function S.

Lemma 1. *Let $\tau > 0$ be an exponentially distributed real variable and define, for $i = 1, 2$, $t \in [0, T]$, $\eta_i(t) = I_{\{S_i(t)<\tau\}}$, where S_i, S are non negative, non decreasing, right continuous random functions. Then, for any $\delta > 0$*

$$|\eta_1(t) - \eta_2(t)| \leq \frac{1}{\delta}|S_1(t) - S(t)| + \frac{1}{\delta}|S_2(t) - S(t)| + I_{\{|S(t)-\tau|<\delta\}}. \tag{24}$$

Remark 1. Whenever at least one or both risk functions S_i, $i = 1, 2$ are independent of the exponential time τ, the bound simplifies in the first case as $\mathbb{E}_\tau|\eta_1(t) - \eta_2(t)| \leq \frac{1}{\delta}\mathbb{E}_\tau|S_1(t) - S_2(t)| + 2\delta$, and in the second case as $\mathbb{E}_\tau|\eta_1(t) - \eta_2(t)| \leq |S_1(t) - S_2(t)|$.

3.1. **Approximation of the limit system.** As shown in Appendix 6.1, under suitable hypothesis on f_0, the solution of the system of equations (2) can be obtained as the $k \to \infty$ limit of the sequence of solutions of the sequence of linear systems defined as:

$$f^{(0)}(t, x, v) = f_0(x - vt, v), \quad \sigma^{(0)}(t, x) = \mu \tag{25}$$

$$\begin{cases} \partial_t f^{(k)} + v \cdot \nabla_x f^{(k)} = -C_d|v|\sigma^{(k-1)} f^{(k)} \\ \partial_t \sigma^{(k)} = -(\Theta \int_{\mathbb{R}^d} dv f^{(k-1)})\sigma^{(k)} \\ f(0, x, v) = f_0(x, v) \\ \sigma(0, x) = \mu \end{cases} \qquad k = 1 \dots \ .$$

More precisely, the sequence of semiexplicit solutions of (25) is, for $k = 1, 2, \dots$,

$$\begin{aligned} f^{(k)}(t, x, v) &= f_0(x - vt, v)e^{-C_d|v| \int_0^t ds \sigma^{(k-1)}(s, x-v(t-s))} \\ \sigma^{(k)}(t, x) &= \mu e^{-\Theta \int_0^t ds \int_{\mathbb{R}^d} dv f^{(k-1)}(s,x,v)}, \end{aligned} \tag{26}$$

and we have, when $k \to \infty$ (cf. Appendix 6.1)

$$\begin{aligned} &\|f - f^{(k)}\|_{L^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)} \to 0\,, \quad \| \int_{\mathbb{R}^d} dv(f - f^{(k)})\|_{L^\infty([0,T]\times\mathbb{R}^d)} \to 0, \\ &\|\sigma - \sigma^{(k)}\|_{L^\infty([0,T]\times\mathbb{R}^d)} \to 0. \end{aligned} \tag{27}$$

The risk functions associated to the k-th system (25) are defined as:

$$\bar{V}^{(k)}(t, c) = \Theta \int_0^t ds \int_{\mathbb{R}^d} dv f^{(k-1)}(s, c, v) \tag{28}$$

$$\bar{U}^{(k)}(t, z) = C_d|T_2^t(z)| \int_0^t ds \sigma^{(k-1)}(s, T_1^s(z)), \tag{29}$$

and, because we shall need it later (cf. the definition of the system (33)), we define $\bar{V}^{(0)}(t,c)=0$. Their associated life functions are:

$$\begin{aligned}\bar{\eta}^{(k)}(t,c) &= I_{\{\bar{V}^{(k)}(t,c)<\tau\}}\\ \bar{\xi}^{(k)}(t,z) &= I_{\{\bar{U}^{(k)}(t,z)<\tau_p\}},\end{aligned} \tag{30}$$

with τ and τ_p exponentially distributed times. Of course, (26) can be expressed in terms of $\bar{V}^{(k)}$ and $\bar{U}^{(k)}$ in a form analogous to (16).

3.2. **Approximation of the particle system.** In the same spirit, we may also approximate, for each n, the particle system described by (6), (7) by a suitable sequence of systems. For an initial datum $\mathbf{z}_n$ for the n particles phase space position and configuration of obstacles $\mathbf{c}_M$ ($M=0$ in the absence of obstacles), we define this system, for integers $M,k\geq 1$, $j=1,\dots,n$ and $\ell=1,\dots,M$, as:

$$\begin{aligned}&\xi^{(0)}_{n,\varepsilon,0}(t,z_j)=\xi^{(k)}_{n,\varepsilon,0}(t,z_j)=1\\ &V^{(0)}_{n,\varepsilon,M}(t,c_\ell)=0,\quad \eta^{(0)}_{n,\varepsilon,M}(t,c_\ell)=1,\quad \xi^{(0)}_{n,\varepsilon,M}(t,z_j)=1\\ &V^{(k)}_{n,\varepsilon,M}(t,c_\ell)=\frac{1}{n}\sum_{i=1}^{n}\int_0^t ds q_n(x_i(s)-c_\ell)\xi^{(k-1)}_{n,\varepsilon,M}(s,z_i)\\ &\eta^{(k)}_{n,\varepsilon,M}(t,c_\ell)=I_{\{V^{(k)}_{n,\varepsilon,M}(t,c_\ell)<\tau_\ell\}}\\ &\xi^{(k)}_{n,\varepsilon,M}(t,z_j)=I_{\{x_j(s)\notin\bigcup_{h=1}^{M}B_\varepsilon(c_h)\eta^{(k-1)}_{n,\varepsilon,M}(s,c_h)\ \forall\in[0,t)\}}.\end{aligned} \tag{31}$$

This sequence is a linearization of the original particle system (which is its formal limit when $k\to\infty$) in the same way as (25) is a linearization of (2) and it satisfies what is called in [NOR] *sandwiching* property (see (48)). This property implies, in a given asymptotics for n and ε:

$$\begin{aligned}&\mathbb{E}^n|V^{(k)}_{n,\varepsilon,M}-\bar{V}^{(k)}|\to 0 &&\Longrightarrow \mathbb{E}^n|V_{n,\varepsilon,M}-V_L|\to 0\\ &\mathbb{E}^n|\xi^{(k)}_{n,\varepsilon,M}-\bar{\xi}^{(k)}|\to 0 &&\Longrightarrow \mathbb{E}^n\mathbb{E}_{\tau_p}|\xi_{n,\varepsilon,M}-\xi_L|\to 0\\ &\mathbb{E}^n\mathbb{E}_\tau|\eta^{(k)}_{n,\varepsilon,M}-\bar{\eta}^{(k)}|\to 0 &&\Longrightarrow \mathbb{E}^n\mathbb{E}_\tau|\eta_{n,\varepsilon,M}-\eta_L|\to 0\\ &\mathbb{E}^n|\int dz\mu_n\phi(\xi^{(k)}_{n,\varepsilon,M}-\bar{\xi}^{(k)})|\to 0 &&\Longrightarrow \mathbb{E}^n\mathbb{E}_{\tau_p}|\int dz\mu_n\phi(\xi_{n,\varepsilon,M}-\xi_L)|\to 0\end{aligned} \tag{32}$$

and using these implications we shall be able to bypass the direct evaluation of quantities related to the particle system (6),(7).

The advantage in dealing with the two approximating sequences of systems defined by (25) and (31) instead of the original systems is that, for each k, the two components evolve in a given field, associated to the functions defined at the previous step $k-1$ in the sequence, and the original self-consistent structure is lost.

The next step would be then to show that (25) and (31) are, for a given k, asymptotically equivalent. Unfortunately, the system defined from (31), when ε is positive, still keeps strong correlations between all light particles and obstacles positions (in the phase space), correlations which are absent in (25); this makes hard the direct comparison of the two systems. Therefore, we need to define an intermediate system in which correlations among light particles and obstacles are further reduced. We shall prove then that this system is equivalent in the limit $n \to \infty$ both to (25) and to (31).

3.3. **A system asymptotically equivalent to both approximating systems.** For an initial datum $\mathbf{z}_n$ for the n particles phase space position, and for configurations of obstacles $\mathbf{c}_M$ ($M = 0$ in the absence of obstacles), we define, for integers $M, k \geq 1$, $j = 1, \dots, n$ and $\ell = 1, \dots, M$, the intermediate system in the following way

$$
\begin{aligned}
&\hat{\xi}^{(0)}_{n,\varepsilon,0}(t, z_j) = \hat{\xi}^{(k)}_{n,\varepsilon,0}(t, z_j) = 1 \\
&\hat{\xi}^{(0)}_{n,\varepsilon,M}(t, z_j) = 1, \\
&\hat{\xi}^{(k)}_{n,\varepsilon,M}(t, z_j) = I_{\{x_j(s) \notin \bigcup_{h=1}^{M} B_\varepsilon(c_h) I_{\{\bar{V}^{(k-1)}(s,c_h) < \tau_h\}} \quad \forall s \in [0,t)\}} \\
&\hat{A}^{(k)}_{n,\varepsilon,M}(t, c_\ell) = \frac{1}{n} \sum_{i=1}^{n} \int_0^t ds\, q_n(x_i(s) - c_\ell) \hat{\xi}^{(k-1)}_{n,\varepsilon,M}(s, z_i) \\
&\hat{\eta}^{(k)}_{n,\varepsilon,M}(t, c_\ell) = I_{\{\hat{A}^{(k)}_{n,\varepsilon,M}(t,c_\ell) < \tau_\ell\}}
\end{aligned}
\tag{33}
$$

where τ_k, $1 \leq k \leq M$, are independent exponentially distributed times.

In (33), the life function of a light particle at level k is defined through fictitious obstacles life functions ($I_{\{\bar{V}^{(k-1)}(t,c) < \tau_c\}}$), corresponding to the obstacles life functions at level $k-1$ associated to (25); the life function of an obstacle at level k, $I_{\{\hat{A}^{(k)}_{n,\varepsilon,M} < \tau_c\}}$, is defined through the light particles life functions at level $k-1$. In this way, the correlation between light particles and obstacles is weaker, compared to the same correlation in system (31), and this allows us to prove, when n and ε verify conditions (18) and (19), the convergence to both systems (25) and (31) in quadratic mean w.r.t. the expectation value $\mathbb{E}^n$.

3.4. **Proof of the main theorem.** We proceed now with the proof of our main theorem. To this purpose, we assume we have already proved the sandwiching property for the system (31) and the asymptotic equivalence of this last system to (25): we shall postpone to next sections the proof of lemmas and propositions concerning these two facts, since they are the core of the derivation; as pointed out in section 3.2, the convergence of (31) to (25) will be obtained passing through the equivalence in this asymptotics of (33) to both systems (31) and (25).

Proof of Theorem 1. Given the definitions in the previous paragraphs, we write

$$\xi_{n,\varepsilon,M} = (\xi_{n,\varepsilon,M} - \xi_L) + (\xi_L - \bar{\xi}^{(k)}) + \bar{\xi}^{(k)}$$

and

$$f = (f - f^{(k)}) + f^{(k)}.$$

Then, using formula (13) for f and (26) for $f^{(k)}$, we can write, for all $\phi \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$ and for all $k \geq 1$:

$$\mathbb{E}^n \left[\left| \frac{1}{n} \sum_{i=1}^n \phi(T^t(z_i)) \xi_{n,\varepsilon,M}(t, z_i) - \int_{\mathbb{R}^d \times \mathbb{R}^d} dx dv \phi(x,v) f(t,x,v) \right| \right] \leq$$

(34)
$$\mathbb{E}^n \mathbb{E}_{\tau_p} \left[\left| \frac{1}{n} \sum_{i=1}^n \phi(T^t(z_i)) (\xi_{n,\varepsilon,M} - \xi_L)(t, z_i) \right| \right] +$$

(35)
$$\|\phi\|_\infty \left\{ \mathbb{E}^n \mathbb{E}_{\tau_p} \left[\frac{1}{n} \sum_{j=1}^n |\xi_L - \bar{\xi}^{(k)}|(t, z_j) \right] + \right.$$
$$\left. \int_{\mathbb{R}^d \times \mathbb{R}^d} dx dv C_d |v| f_0(x,v) \int_0^t ds |\sigma - \sigma^{(k-1)}|(s, x(s)) \right\} +$$

(36)
$$\mathbb{E}^n \mathbb{E}_{\tau_p} \left[\left| \frac{1}{n} \sum_{i=1}^n \phi(T^t(z_i)) \bar{\xi}^{(k)}(t, z_i) - \int_{\mathbb{R}^d \times \mathbb{R}^d} dx dv f^{(k)}(t,x,v) \phi(x,v) \right| \right].$$

The term (34) vanishes on a set of full measure w.r.t. $\mathscr{P}$ as a consequence of the sandwiching property valid for (31) (cf. Corollary 1, Sec. 4.3) and of the convergence of (31) to (25) in the asymptotics specified in (18), (19) (cf. Proposition 4, Sec. 5).

Thanks to (23), and in particular to the convergence $|v|\mu_n^0 \rightharpoonup |v| f_0$, the term in curly brackets (35) is bounded (excepted on a set of zero measure with respect to $\mathscr{P}$) by $K_a \|\sigma - \sigma^{(k-1)}\|_{L^\infty([0,T] \times \mathbb{R}^d)}$, so that, by choosing a suitable k, it can be made arbitrarily small because of (27).

Using Cauchy-Schwarz's inequality, we get then for (36)

$$\left| \mathbb{E}^n \mathbb{E}_{\tau_p} \left[\left| \frac{1}{n} \sum_{i=1}^n \phi(T^t(z_i)) \bar{\xi}^{(k)}(t, z_i) - \int_{\mathbb{R}^d \times \mathbb{R}^d} dx dv f^{(k)}(t,x,v) \phi(x,v) \right| \right] \right|^2 \leq$$
$$\mathbb{E}^n \left[\left| \frac{1}{n} \sum_{i=1}^n \phi(T^t(z_i)) \mathbb{E}_{\tau_p}[\bar{\xi}^{(k)}(t, z_i)] \right|^2 \right] + \int_{\mathbb{R}^d \times \mathbb{R}^d} dx dv f^{(k)}(t,x,v) \phi(x,v) \times$$
$$\times \left[\int_{\mathbb{R}^d \times \mathbb{R}^d} dx dv f^{(k)}(t,x,v) \phi(x,v) - \frac{2}{n} \sum_{i=1}^n \phi(T^t(z_i)) \mathbb{E}_{\tau_p}[\bar{\xi}^{(k)}(t, z_i)] \right].$$

Because of the hypothesis on f_0, $\phi(T^t(\cdot)) \mathbb{E}_{\tau_p}[\bar{\xi}^{(k)}(t, \cdot)] \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, so that, thanks to (23) (and in particular to $\mu_n^0 \rightharpoonup f_0$), on a set of full measure w.r.t. $\mathscr{P}$ we have

$$\frac{1}{n} \sum_{i=1}^n \phi(T^t(z_i)) \mathbb{E}_{\tau_p}[\bar{\xi}^{(k)}(t, z_i)] \to \int_{\mathbb{R}^d \times \mathbb{R}^d} dx dv f^{(k)}(t,x,v) \phi(x,v),$$

and (36) vanishes on this set.

In the same way, by writing

$$\eta_{n,\varepsilon,M} = (\eta_{n,\varepsilon,M} - \eta_L) + (\eta_L - \bar{\eta}^{(k)}) + \bar{\eta}^{(k)}$$

and

$$\sigma = (\sigma - \sigma^{(k)}) + \sigma^{(k)},$$

we get, for all $\psi \in C_K(\mathbb{R}^d)$ and for all $k \geq 1$,

$$\mathbb{E}^n \left[\left| \varepsilon^{d-1} \sum_{i=1}^{M} \psi(c_i) \eta_{n,\varepsilon,M}(t, c_i) - \int_{\mathbb{R}^d} dx \psi(x) \sigma(t,x) \right| \right] \leq$$

(37)

$$\|\psi\|_\infty \left\{ \mathbb{E}^n \left[\varepsilon^{d-1} \sum_{i=1}^{M} I_{\mathrm{supp}\psi}(c_i) |\eta_{n,\varepsilon,M} - \eta_L|(t, c_i) \right] + \right.$$

(38)

$$\left. \left(\mathbb{E}^n \left[\varepsilon^{d-1} \sum_{i=1}^{M} I_{\mathrm{supp}\psi}(c_i) |\bar{\eta}^{(k)} - \eta_L|(t, c_i) \right] + \int_{\mathrm{supp}\psi} dx |\sigma^{(k)} - \sigma|(t,x) \right) \right\}$$

(39)

$$+ \mathbb{E}^n \left[\left| \varepsilon^{d-1} \sum_{i=1}^{M} \psi(c_i) \bar{\eta}^{(k)}(t, c_i) - \int_{\mathbb{R}^d} dx \psi(x) \sigma^{(k)}(t,x) \right| \right],$$

where the term in round brackets (38) is (everywhere) bounded by $K_b \| \int dv (f - f^{(k-1)}) \|_{L^\infty([0,T] \times \mathbb{R}^d)}$ and can be made arbitrarily small because of (27).

The term (37) vanishes on a set of full measure w.r.t. $\mathscr{P}$ for the same reasons as (34), i.e. as a consequence of the sandwiching property valid for (31) and of the convergence of (31) to (25) in the asymptotics specified in (18), (19).

As for (39), we can write

$$\left| \mathbb{E}^n \left[\left| \varepsilon^{d-1} \sum_{i=1}^{M} \psi(c_i) \bar{\eta}^{(k)}(t, c_i) - \int_{\mathbb{R}^d} dx \psi(x) \sigma^{(k)}(t,x) \right| \right] \right|^2 \leq$$

$$\left| \int_{\Lambda_n} dc \psi(c) \sigma^{(k)}(t,c) - \int_{\mathbb{R}^d} dc \psi(c) \sigma^{(k)}(t,c) \right|^2 + \varepsilon^{d-1} \int_{\Lambda_n} dc \psi^2(c) \sigma^{(k)}(t,c).$$

Since $\psi \in C_K(\mathbb{R}^d)$, we have $\psi^h(\cdot)\sigma^{(k)}(t,\cdot) \in L^1(\mathbb{R}^d))$ for $h = 1, 2$ and when n grows to infinity we have both $\Lambda_n \to \mathbb{R}^d$ and $\varepsilon \to 0$, so that this term vanishes, in the chosen asymptotics.

Collecting all the assertions about the different terms, the theorem is proved. □

4. Asymptotic equivalence of (31) and (25) and convergence of the particle system to the limit system

We collect in the present section the Lemmas and Propositions which will help us to build up the last step of the proof of our main theorem.

4.1. **A few useful bounds.**

Let us list a few bounds and formulas which we shall use often in our calculations.

It will be useful to adopt in next sections, for stochastic variables $\lambda^{\mathcal{F}_{n,M}}$ of the form

$$\lambda^{\mathcal{F}_{n,M}}(t, z_j) = I_{\{x_j(s) \notin \bigcup_{h=1}^{M} B_\varepsilon(c_h) I_{\{\mathcal{F}_{n,M}(s,\, c_h) < \tau_h\}} \quad \forall s \in [0,t)\}},$$

(with $\mathcal{F}_{n,M}(s,c_h) = \mathcal{F}_{\mathbf{c}_M,\tau_M,\mathbf{z}_n}(s,c_h)$ a given random function) the following representation:

$$\lambda^{\mathcal{F}_{n,M}}(t,z_j) = \prod_{h=1}^{M}\Big(1 - I_{\{c_h\in\mathcal{T}_\varepsilon(t,z_j)\}}I_{\{\mathcal{F}_{n,M}(T_{z_j,c_h},\,c_h)<\tau_{c_h}\}}\Big)\,. \tag{40}$$

Given any couple of stochastic variables $\lambda^{\mathcal{F}^{(1)}_{n,M}}$, $\lambda^{\mathcal{F}^{(2)}_{n,M}}$ of the form (40), using the trivial inequality

$$|\prod_h I_{A_h} - \prod_h I_{B_h}| \le \sum_h |I_{A_h} - I_{B_h}|,$$

for $M \ge 1$, we get the bound

$$\left|\lambda_1^{\mathcal{F}^{(1)}_{n,M}} - \lambda_2^{\mathcal{F}^{(2)}_{n,M}}\right|(t,z_j) \le \sum_{h=1}^{M} I_{c_h\in\mathcal{T}_\varepsilon(t,z_j)}|\eta^{\mathcal{F}^{(1)}_{n,M}} - \eta^{\mathcal{F}^{(2)}_{n,M}}|(T_{z_j,c_h},c_h), \tag{41}$$

where

$$\eta^{\mathcal{F}^{(i)}_{n,M}}(T_{z_j,c_h},c_h) = I_{\{\mathcal{F}^{(i)}_{n,M}(T_{z_j,c_h},c_h)<\tau_{c_h}\}}, \qquad i=1,2.$$

The bound (41) will be largely used in the last paragraph of this section.

Moreover, because of the hypothesis on f_0 (assigning initially uniformly bounded number and kinetic energy limit densities for the light particle component, cf. Remark 9 in Appendix 6.2), $\mathscr{P}$-a.e. with respect to Z_∞, we may use the bounds

$$\|\frac{1}{n}\sum_{j=1}^{n} q_n(x_j(s)-c)\|_\infty \quad\le\quad K_1 \tag{42}$$

$$\|\frac{1}{n}\sum_{j=1}^{n} q_n(x_j(s)-c)[\mathcal{T}_\varepsilon(s,z_j)]_L\|_\infty \quad\le\quad \varepsilon^{d-1}C_dTK_2 \tag{43}$$

where $K_1, K_2 > 0$ are constant independent of n.

4.2. **Quantities related to flow tubes.** When bounding correlations in our particle system, we shall need to evaluate expectation values with respect to the Poisson distribution (3) and to μ_n^0 on volumes corresponding to the intersections of flow tubes.

Defining, for two vectors $v, w \in \mathbb{R}^d$, $\cos\alpha(v,w) = \frac{v\cdot w}{|v||w|}$ (and $\alpha_{ij} = \alpha(v_i,v_j)$), and recalling the notation $B_r(p)$ introduced at the beginning of Section 2 to denote the ball of radius r centered in p, we have, for the intersection of two flow tubes associated to the particles with initial phase space positions z_i and z_j, the following trivial bound:

Lemma 2. *For any $\beta \in (0,\frac{1}{d})$:*

$$[\mathcal{T}_\varepsilon(t,z_j)\cap\mathcal{T}_\varepsilon(t,z_i)]_L \le \tag{44}$$
$$[B_2(0)]_L\varepsilon^{d(1-\beta)}I_{|\sin\alpha_{ij}|\ge\varepsilon^\beta} + C_d\varepsilon^{d-1}\min(|v_i|,|v_j|)TI_{|\sin\alpha_{ij}|<\varepsilon^\beta}.$$

Proof. We have always

$$\begin{aligned}[\mathcal{T}_\varepsilon(t,z_j)\cap\mathcal{T}_\varepsilon(t,z_i)]_L &\le \min([\mathcal{T}_\varepsilon(t,z_j)]_L,[\mathcal{T}_\varepsilon(t,z_i)]_L)\\ &= C_d\varepsilon^{d-1}\min(|v_i|,|v_j|)t\end{aligned}$$

and, whenever $\sin\alpha_{ij} \neq 0$, denoting as y the (unique) crossing point for the trajectories $x_i(s)$, $x_j(s)$ (i.e. $y = x_i + v_i s_1 = x_j + v_j s_2$ for some $s_1, s_2 \in \mathbb{R}$),

$$\mathcal{T}_\varepsilon(t, z_j) \cap \mathcal{T}_\varepsilon(t, z_i) \subset B_{\frac{2\varepsilon}{\sin\alpha_{ij}}}(y).$$

We can therefore write:

$$|\mathcal{T}_\varepsilon(t, z_j) \cap \mathcal{T}_\varepsilon(t, z_i)| \leq |B_2(0)| \left(\frac{\varepsilon}{\sin\alpha_{ij}}\right)^d \wedge C_d \varepsilon^{d-1} \min(|v_i|, |v_j|) t \tag{45}$$

and (44) follows.

□

Thanks to the weak convergence of the initial empirical measure toward a regular density, we can estimate the measure (w.r.t. μ_n^0) of the set corresponding, for a given velocity $w \in \mathbb{R}^d$, to the grazing collisions. We can prove in fact the following Lemma:

Lemma 3. *Let $\mu_n^0 \rightharpoonup f_0 \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$, $\beta > 0$ and $w \in \mathbb{R}^d$. Assume (18). Denoting $\cos\alpha_i = \frac{v_i \cdot w}{|v_i||w|}$, the following bound is verified*

$$\frac{1}{n}\sum_{i=1}^n I_{|\sin\alpha_i| < \varepsilon^\beta} \leq K\varepsilon^\zeta + o(a_n^{-d})$$

for any $\zeta \in (0, \frac{d-1}{2}\beta)$.

Proof. We observe first that we have $\lim_{n\to\infty} \frac{\#\{j : |v_j \cdot w| = |w||v_j|\}}{n} = 0$, since $f_0 \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$, so that, uniformly in ε, the contribution of the corresponding term vanishes when $n \to \infty$ and it is actually $o(a_n^{-d})$ in the prescribed asymptotics[2].

We now evaluate the contribution of grazing crossings of particle trajectories. We choose a suitable (standard) C^∞ regularization $\bar{R}^\delta$ of $I_{|v\cdot w| \neq |v||w|}$, for instance $\bar{R}^\delta$ is s.t.:

$$I_{|v\cdot w| \neq |v||w|} - \bar{R}^\delta \leq e^{-\frac{\sin^2\alpha(v,w)}{\delta^2(\delta^2 - \sin^2\alpha(v,w))}} I_{|\sin\alpha(v,w)| \leq \delta}.$$

Here the parameter δ denotes the radius of the set where the regularization differs from the original characteristic functions (i.e. the set in $\mathbb{R}^d \times \mathbb{R}^d$ s.t. $|\sin\alpha(v,w)| \leq \delta$), and, for all $\delta > 0$ and $0 < \iota < \frac{d-1}{2}$, we write:

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n I_{|v_i\cdot w| \neq |v_i||w|} I_{|\sin\alpha_i| < \varepsilon^\beta} &\leq \frac{1}{n}\sum_{i=1}^n \frac{\varepsilon^{\iota\beta}}{(1 - \frac{|v_i\cdot w|^2}{|v_i|^2|w|^2})^{\frac{\iota}{2}}} \bar{R}^\delta(v_i) + o(\delta) \\ &\leq \varepsilon^{\iota\beta} \int_{\mathbb{R}^d\times\mathbb{R}^d} dxdv \frac{f_0(x,v)}{(1 - \frac{|v\cdot w|^2}{|v|^2|w|^2})^{\frac{\iota}{2}}} + K_R \frac{\varepsilon^{\iota\beta}}{\delta^{3+\iota}} + o(\delta).\end{aligned}$$

The error coming from the regularization of the characteristic function $I_{|v\cdot w| \neq |v||w|}$ is $o(\delta)$ because of the weak convergence of μ_n^0 toward $f_0 \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ and, for the same reason, all constants (here and in the following estimates) are uniform in w

[2]In fact, since by Toeplitz's lemma, for all $a \in (0, 1]$, $\frac{1}{n}\sum_{k=1}^n (\frac{\#\{j : |v_j\cdot w| = |w||v_j|\}}{k})^a$ vanishes when $n \to \infty$, we have at least $\frac{\#\{j : |v_j\cdot w| = |w||v_j|\}}{n} = o(\frac{1}{\sqrt{n}})$).

and n and depend only on the dimension d and on few L^p norms of f_0. We obtain then, for suitable choices of δ,

$$\frac{1}{n}\sum_{i=1}^{n} I_{|v_i\cdot w|\neq|v_i||w|} I_{|\sin\alpha_i|<\varepsilon^\beta} \leq K\varepsilon^\zeta,$$

for any $\zeta\in(0,\iota\beta)$ and therefore for any $\zeta\in(0,\frac{d-1}{2}\beta)$. Collecting the two estimates, the lemma is proved. □

We shall use Lemma 3 for free choices of $\beta\in(0,\frac{1}{d})$, so that $\zeta\in(0,\frac{d-1}{2}\beta)\subset(0,\frac{1}{2}-\frac{1}{2d})$.

Remark 2. The estimate obtained in Lemma 3 is valid for any given sequence of empirical measures $\{\mu_n^0\}$ such that $\mu_n^0 \rightharpoonup f_0\in\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)$ and it is actually a stronger assertion with respect to what we need to prove Theorem 1 (which is valid $\mathscr{P}-a.e.$ w.r.t. Z_∞). In order to prove Lemma 3, where, as stated in the introductory sentence to the lemma, we bound the measure of a set with respect to the empirical measure μ_n^0, we need to evaluate the measure of the set $\{(x,v): |v\cdot w|=|w||v|\}$ with respect to the measure μ_n^0 because the measure of this set vanishes only asymptotically (i.e. with respect to the limit measure with density f_0). We choose to evaluate this measure with respect to the parameter a_n^d (i.e. as $o(a_n^{-d})$) for further convenience.

A similar, simpler statement can be proved $\mathscr{P}$-a.e. w.r.t. Z_∞: in this case we obtain the bound $\frac{1}{n}\sum_{i=1}^n I_{|\sin\alpha_i|<\varepsilon^\beta}\leq K\varepsilon^\zeta$ ($\mathscr{P}$-a.e. w.r.t. Z_∞). This alternative statement could be used instead of Lemma 3 to prove our main theorem. We prefer nevertheless to use Lemma 3, getting in this way bounds which are (as much as possible) valid on the whole subset of initial sequences Z_∞ such that the associated sequence of empirical measures $\{\mu_n^0\}_{n=1}^\infty$ converges weakly to $f_0\in\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)$.

Remark 3. Notice that, since $\frac{1}{n^2}\sum_{i,j:|v_i\cdot v_j|=|v_i||v_j|} 1 = o(a_n^{-d})$ and all bounds are uniform in the velocity w, we get also

$$\frac{1}{n^2}\sum_{i,j=1}^{n} I_{|\sin\alpha_{ij}|\leq\varepsilon^\beta} \leq \frac{1}{n}\sum_{j=1}^{n}\Big(\frac{1}{n}\sum_{i=1}^{n} I_{|\sin\alpha_{ij}|\leq\varepsilon^\beta}\Big) \leq K\varepsilon^\zeta + o(a_n^{-d}). \tag{46}$$

A last very useful tool will be the following parametrization of the points $c\in\mathcal{T}_\varepsilon(t,z)$. Let $u=T_{z,c}$ be the maximal collision time defined in (8); then we may define the change of variables

$$c = x(u)+\varepsilon\omega,\quad u\in[0,t),\quad \omega\in\partial B_1(0).$$

Using this parametrization, for each non negative function $V\in L^\infty([0,T];W^{1,\infty}(\mathbb{R}^d))$ we can write $V(T_{z,c},c)=V(u,x(u)+\varepsilon\omega)$, getting then

$$\int_{\mathcal{T}_\varepsilon(t,z)} dc e^{-V(T_{z,c},c)} = \varepsilon^{d-1}C_d|v|\left(\int_0^t du[e^{-V(u,x(u))}+\phi_\varepsilon(u,x(u))]\right) \tag{47}$$

where we may bound the remainder through the trivial inequality $|e^{-y}-e^{-x}|\leq|x-y|$, for $x,y\geq 0$, and the function ϕ_ε is such that $\sup_{[0,T]}\|\phi_\varepsilon\|_{L^\infty(\mathbb{R}^d)}=O(\varepsilon)\to 0$ when $\varepsilon\to 0$.

4.3. **The sandwiching property.** We prove now the sandwiching property for the system defined by (31) and its implication on the convergence of (7).

Lemma 4. *(sandwiching property) Consider $V_{n,\varepsilon,M}$, $\xi_{n,\varepsilon,M}$, $\eta_{n,\varepsilon,M}$ defined by (6), (7) and $\xi^{(k)}_{n,\varepsilon,M}$, $V^{(k)}_{n,\varepsilon,M}$, $\eta^{(k)}_{n,\varepsilon,M}$ defined by (31). Then, for $k = 1, 2 \ldots$,*

$$\xi^{(2k-1)}_{n,\varepsilon,M} \leq \xi^{(2k+1)}_{n,\varepsilon,M} \leq \xi_{n,\varepsilon,M} \leq \xi^{(2k)}_{n,\varepsilon,M} \leq \xi^{(2k-2)}_{n,\varepsilon,M}$$

$$V^{(2k-2)}_{n,\varepsilon,M} \leq V^{(2k)}_{n,\varepsilon,M} \leq V_{n,\varepsilon,M} \leq V^{(2k+1)}_{n,\varepsilon,M} \leq V^{(2k-1)}_{n,\varepsilon,M} \tag{48}$$

$$\eta^{(2k-1)}_{n,\varepsilon,M} \leq \eta^{(2k+1)}_{n,\varepsilon,M} \leq \eta_{n,\varepsilon,M} \leq \eta^{(2k)}_{n,\varepsilon,M} \leq \eta^{(2k-2)}_{n,\varepsilon,M}.$$

Proof. From definitions (6), (7) and (31), we have $0 = V^{(0)}_{n,\varepsilon,M} \leq V_{n,\varepsilon,M} \leq V^{(1)}_{n,\varepsilon,M}$, which implies $\eta^{(0)}_{n,\varepsilon,M} \geq \eta_{n,\varepsilon,M} \geq \eta^{(1)}_{n,\varepsilon,M}$, which in turn implies $\xi^{(1)}_{n,\varepsilon,M} \leq \xi_{n,\varepsilon,M} \leq \xi^{(2)}_{n,\varepsilon,M} \leq \xi^{(0)}_{n,\varepsilon,M} = 1$.

We have therefore, using again these definitions:

$$\xi^{(1)}_{n,\varepsilon,M} \leq \xi^{(3)}_{n,\varepsilon,M} \leq \xi_{n,\varepsilon,M} \leq \xi^{(2)}_{n,\varepsilon,M} \leq \xi^{(0)}_{n,\varepsilon,M}$$

$$V^{(0)}_{n,\varepsilon,M} \leq V^{(2)}_{n,\varepsilon,M} \leq V_{n,\varepsilon,M} \leq V^{(3)}_{n,\varepsilon,M} \leq V^{(1)}_{n,\varepsilon,M}$$

$$\eta^{(1)}_{n,\varepsilon,M} \leq \eta^{(3)}_{n,\varepsilon,M} \leq \eta_{n,\varepsilon,M} \leq \eta^{(2)}_{n,\varepsilon,M} \leq \eta^{(0)}_{n,\varepsilon,M}$$

and (48) is valid for $k = 1$.

Since the following chain is also valid:

$$\begin{aligned}
V^{(2k-2)}_{n,\varepsilon,M} &\leq V^{(2k)}_{n,\varepsilon,M} \leq V_{n,\varepsilon,M} \leq V^{(2k+1)}_{n,\varepsilon,M} \leq V^{(2k-1)}_{n,\varepsilon,M} \implies \\
\eta^{(2k-2)}_{n,\varepsilon,M} &\geq \eta^{(2k)}_{n,\varepsilon,M} \geq \eta_{n,\varepsilon,M} \geq \eta^{(2k+1)}_{n,\varepsilon,M} \geq \eta^{(2k-1)}_{n,\varepsilon,M} \implies \\
\xi^{(2k-1)}_{n,\varepsilon,M} &\leq \xi^{(2k+1)}_{n,\varepsilon,M} \leq \xi_{n,\varepsilon,M} \leq \xi^{(2k+2)}_{n,\varepsilon,M} \leq \xi^{(2k)}_{n,\varepsilon,M} \implies \\
V^{(2k)}_{n,\varepsilon,M} &\leq V^{(2k+2)}_{n,\varepsilon,M} \leq V_{n,\varepsilon,M} \leq V^{(2k+3)}_{n,\varepsilon,M} \leq V^{(2k+1)}_{n,\varepsilon,M}
\end{aligned}$$

the statement of the proposition follows by induction. □

We may then prove the following corollary to Lemma 4:

Corollary 1. *Consider $V_{n,\varepsilon,M}$, $\xi_{n,\varepsilon,M}$, $\eta_{n,\varepsilon,M}$ and $V^{(k)}_{n,\varepsilon,M}$, $\xi^{(k)}_{n,\varepsilon,M}$, $\eta^{(k)}_{n,\varepsilon,M}$ defined as in Lemma 4 and V_L and ξ_L given by (14) and (15), with (f, σ) unique solution of (2). Then, when $n \to \infty, \varepsilon \to 0$, for any sub-linear operators L on $L^\infty([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$*

$$L|V^{(k)}_{n,\varepsilon,M} - \bar{V}^{(k)}| \to 0 \implies L|V_{n,\varepsilon,M} - V_L| \to 0$$

$$L|\xi^{(k)}_{n,\varepsilon,M} - \bar{\xi}^{(k)}| \to 0 \implies L\mathbb{E}_{\tau_p}|\xi_{n,\varepsilon,M} - \xi_L| \to 0$$

$$L\mathbb{E}_{\tau}|\eta^{(k)}_{n,\varepsilon,M} - \bar{\eta}^{(k)}| \to 0 \implies L\mathbb{E}_{\tau}|\eta_{n,\varepsilon,M} - \eta_L| \to 0$$

$$L|\int_{\mathbb{R}^d \times \mathbb{R}^d} dz \mu^0_n \phi(\xi^{(k)}_{n,\varepsilon,M} - \bar{\xi}^{(k)})| \to 0 \implies L\mathbb{E}_{\tau_p}|\int_{\mathbb{R}^d \times \mathbb{R}^d} dz \mu^0_n \phi(\xi_{n,\varepsilon,M} - \xi_L)| \to 0.$$

Proof. We denote here by $\phi_+ = \phi \wedge 0$, $\phi_- = -\phi \vee 0$ resp. the positive and the negative part of ϕ.

Let $\mathcal{F}_{n,\varepsilon,M}$ be one among the non negative functions $V_{n,\varepsilon,M}$, $1-\xi_{n,\varepsilon,M}$, $1-\eta_{n,\varepsilon,M}$, $\int dz\mu_n^0\phi_+\,\xi_{n,\varepsilon,M}$, $\int dz\mu_n^0\phi_-\,\xi_{n,\varepsilon,M}$ and $\mathcal{F}^{(k)}_{n,\varepsilon,M}$ its approximant of order k (defined through (31)). Let $\mathcal{F}_L$ be its associate limit function (defined through (14) and (15)) and $\mathcal{F}^{(k)}$ the approximant of $\mathcal{F}_L$ (defined through (28),(29) and (30)).

Because of (48), we have, for a given $k \geq 1$:

$$\mathcal{F}^{(2k)}_{n,\varepsilon,M} - \mathcal{F}_L \leq \mathcal{F}_{n,\varepsilon,M} - \mathcal{F}_L \leq \mathcal{F}^{(2k+1)}_{n,\varepsilon,M} - \mathcal{F}_L,$$

and therefore,

$$|\mathcal{F}_{n,\varepsilon,M} - \mathcal{F}_L| \leq |\mathcal{F}_L - \mathcal{F}^{(2k)}| + |\mathcal{F}_L - \mathcal{F}^{(2k+1)}| +$$

$$|\mathcal{F}^{(2k+1)} - \mathcal{F}^{(2k+1)}_{n,\varepsilon,M}| + |\mathcal{F}^{(2k)} - \mathcal{F}^{(2k)}_{n,\varepsilon,M}|.$$

Since $\|V_L - \bar{V}^{(k)}\|_{L^\infty([0,T]\times\mathbb{R}^d)}$ and $\|\mathbb{E}_{\tau_p}[\xi_L - \bar{\xi}^{(k)}]\|_{L^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)}$ vanish in the $k\to\infty$ limit (cf. (27)), and since $\mathbb{E}_{\tau_p}[|\eta_L - \bar{\eta}^{(k)}|] \leq |V_L - \bar{V}^{(k)}|$ (cf. Remark 1), the assertion is proved. □

As already pointed out, Corollary 1 is valid in particular when L is an expectation value operator.

4.4. **Equivalence between (33) and (25).**

4.4.1. *Motion of light particles in a decaying medium: equivalence of the light particle components of (33) and (25).* We prove here a lemma which describes the behavior of the light particle component of the system (33) in the $n\to\infty$ limit. In this lemma, we consider a particle system such that the rate of death of the obstacles is given and independent from the light particles component, while the light particles are instantaneously absorbed each time they collide with an obstacle. The correspondent limit system consists of equations which are coupled only in a very weak way and the parameters determining the asymptotics are independent.

Let $g(t,c) \in L^1([0,T]; W^{1,\infty}(\mathbb{R}^d))$ be a non negative function. The risk function $V(t,c)$ defined, for $t \leq T$, by

$$V(t,c) = \int_0^t g(s,c)ds$$

is non decreasing as a function of t.

As in the previous sections, we define:

$$\begin{aligned} &\eta^V(t,c_h) = I_{\{V(t,c_h)<\tau_h\}} \\ &\xi^V_{n,\varepsilon,M}(t,z_i) = I_{\{x_i(s)\notin \bigcup_{k=1}^M B_\varepsilon(c_k)\eta^V(s,c_k)\ \ \forall\in[0,t)\}}. \end{aligned} \tag{49}$$

We have then:

Lemma 5. *Let $g(t,c) \in L^1([0,T]; W^{1,\infty}(\mathbb{R}^d))$ and $\xi^V_{n,\varepsilon,M}$ be defined by (49). Consider the sequence $Z_\infty \in (\mathbb{R}^d\times\mathbb{R}^d)^\infty$ and its associated sequence of empirical measures $\{\mu_n^0\}_{n=1}^\infty$. Assume $\mu_n^0 \rightharpoonup f_0 \in \mathscr{S}$ and $|v|\mu_n^0 \rightharpoonup |v|f_0 \in \mathscr{S}$. Then $\forall\phi \in C_b(\mathbb{R}^d\times\mathbb{R}^d)$,*

$$\lim_{\substack{n\to\infty\\ \varepsilon\to 0}} \mathbb{E}^n\left[\left|\frac{1}{n}\sum_{i=1}^n \xi^V_{n,\varepsilon,M}(t,z_i)\phi(T^t(z_i)) - \int_{\mathbb{R}^d\times\mathbb{R}^d} dxdv f(t,x,v)\phi(x,v)\right|^2\right] = 0$$

(independently of the order of the limits), where (f,σ) is the (unique) solution of the following system of partial differential equations:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = -|v|C_d\sigma f \\ \partial_t\sigma = -g\sigma \\ f(0,x,v) = f_0(x,v) \\ \sigma(0,x) = \mu. \end{cases}$$

Remark 4. Of course, when $V=\bar V^{(k-1)}$, we have $\xi^V_{n,\varepsilon,M} = \hat\xi^{(k)}_{n,\varepsilon,M}$, so that obviously (by the triangular inequality) Lemma 5 implies, when $\varepsilon\to 0$, $n\to\infty$, the limit $\mathbb{E}^n[|\int dxdv\mu^0_n(\hat\xi^{(k)}_{n,\varepsilon,M} - \bar\xi^{(k)})\phi|]\to 0$.

Proof. According to the definition of maximal collision time (8), we have

$$\xi^V_{n,\varepsilon,M}(t,z_j) = 1 \iff \begin{array}{c} \forall h: 1\le h\le M, \\ T_{z_j,c_h}\wedge t \ge \inf_{s\in\mathbb{R}_+}\{\eta^V(s,c_h)=0\}\wedge t. \end{array}$$

The function $F(\cdot,c) = \inf\{z : z = V^{-1}(\cdot,c)\}$ is well defined and

$$\inf\{s\in\mathbb{R}_+; \eta^V(s,c_h)=0\} = F(\tau_{c_h},c_h).$$

Moreover, the maximal collision time $T_{z_j,c}$ is greater than t if and only if c does not belong to the tube $\mathcal{T}_\varepsilon(t,z_j)$. It follows that

$$\begin{aligned} P^n\left[\xi^V_{n,\varepsilon,M}(t,z_j)=1\right] &= P^n\left[\{1\le h\le M : T_{z_j,c_h}\wedge t \ge F(\tau_{c_h},c_h)\wedge t\}\right] \\ &= \mathbb{E}^n\left[\prod_{h=1}^M \left(I_{\{c_h\notin\mathcal{T}_\varepsilon(t,z_j)\}} + I_{\{c_h\in\mathcal{T}_\varepsilon(t,z_j);\ T_{z_j,c_h}\ge V^{-1}(\tau_{c_h},c_h)\wedge t\}}\right)\right] \\ &= \exp\left[-\mu_\varepsilon\int_{\mathcal{T}_\varepsilon(t,z_j)} dce^{-V(T_{z_j,c},c)}\right]. \end{aligned}$$

Using (47), because of the weak convergence of μ^0_n and $|v|\mu^0_n$, we get

$$\mathbb{E}^n\left[\frac{1}{n}\sum_{i=1}^n \xi^V_{n,\varepsilon,M}(t,z_i)\phi(T^t(z_i))\right] \overset{n\to\infty,\varepsilon\to 0}{\to} \int_{\mathbb{R}^d\times\mathbb{R}^d} dxdvf(t,x,v)\phi(x,v), \tag{50}$$

the result being independent of the ordering of the limits.

From (50), we prove Lemma 5 as follows. We call

$$\begin{aligned} \mathcal{E}_{1\,n} &= \mathbb{E}^n\left[\frac{1}{n^2}\sum_{\substack{i,j=1\\ i\neq j}}^n \xi^V_{n,\varepsilon,M}(t,z_i)\xi^V_{n,\varepsilon,M}(t,z_j)\phi(T^t(z_i))\phi(T^t(z_j))\right] \\ \mathcal{E}_{2\,n} &= \int_{\mathbb{R}^d\times\mathbb{R}^d} dxdvf(t,x,v)\phi(x,v) - \mathbb{E}^n\left[\frac{2}{n}\sum_{i=1}^n \xi^V_{n,\varepsilon,M}(t,z_i)\phi(T^t(z_i))\right]. \end{aligned}$$

We write then:

$$\mathbb{E}^n\left[\left|\frac{1}{n}\sum_{i=1}^n \xi^V_{n,\varepsilon,M}(t,z_i)\phi(T^t(z_i)) - \int_{\mathbb{R}^d\times\mathbb{R}^d} dxdvf(t,x,v)\phi(x,v)\right|^2\right] =$$

$$\mathbb{E}^n\left[\frac{1}{n^2}\sum_{i=1}^n \xi^V_{n,\varepsilon,M}(t,z_i)\phi^2(T^t(z_i))\right] + \mathcal{E}_{1\,n} + \mathcal{E}_{2\,n}\int_{\mathbb{R}^d\times\mathbb{R}^d} dxdvf(t,x,v)\phi(x,v)$$

where $\mathbb{E}^n\left[\frac{1}{n^2}\sum_{i=1}^n \xi^V_{n,\varepsilon,M}(t,z_i)\phi^2(T^t(z_i))\right]$, when $n\to\infty$, obviously vanishes and, thanks to (50),

$$\mathcal{E}_{2\,n}\int_{\mathbb{R}^d\times\mathbb{R}^d} dxdv f(t,x,v)\phi(x,v) \to -\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} dxdv f(t,x,v)\phi(x,v)\right)^2.$$

As for $\mathcal{E}_{1\,n}$ we observe that, for $i\neq j$:

$$\mathbb{E}^n\left[\xi^V_{n,\varepsilon,M}(t,z_i)\xi^V_{n,\varepsilon,M}(t,z_j)\right] =$$

$$\exp\left[-\mu_\varepsilon\int_{\mathcal{T}_\varepsilon(t,z_i)} dc\, e^{-V(T_{z_i,c},c)} - \mu_\varepsilon\int_{\mathcal{T}_\varepsilon(t,z_j)} dc\, e^{-V(T_{z_j,c},c)}\right]\times$$

$$\exp\left[\mu_\varepsilon\int_{\mathcal{T}_\varepsilon(t,z_j)\bigcap\mathcal{T}_\varepsilon(t,z_i)} dc\left(I_{\{c:T_{z_i,c}\le T_{z_j,c}\}}e^{-V(T_{z_j,c},c)} + I_{\{c:T_{z_i,c}>T_{z_j,c}\}}e^{-V(T_{z_i,c},c)}\right)\right].$$

We write $1 = I_{|\sin\alpha_{ij}|\ge\varepsilon^\beta} + I_{|\sin\alpha_{ij}|<\varepsilon^\beta}$, for $\beta<\frac{1}{d}$. Then, using (46) and $[\mathcal{T}_\varepsilon(t,z_j)\cap\mathcal{T}_\varepsilon(t,z_i))]_L \le \bar{K}_\mu\varepsilon^{d-1}$ (always valid for intersections of flow tubes, as shown also in the proof of Lemma 2) we have:

$$\frac{1}{n^2}\sum_{\substack{i,j=1\\ i\neq j}}^n I_{|\sin\alpha_{ij}|<\varepsilon^\beta}\mathbb{E}^n\left[\xi^V_{n,\varepsilon,M}(t,z_i)\xi^V_{n,\varepsilon,M}(t,z_j)\right]$$
$$\le \frac{1}{n^2}\sum_{\substack{i,j=1\\ i\neq j}}^n e^{\mu_\varepsilon[\mathcal{T}_\varepsilon(t,z_j)\bigcap\mathcal{T}_\varepsilon(t,z_i)]_L} I_{|\sin\alpha_{ij}|<\varepsilon^\beta} \le K_\mu\varepsilon^\zeta + o(a_n^{-d}).$$

As for the remaining part (since $I_{|\sin\alpha_{ij}|\ge\varepsilon^\beta} = 1 - I_{|\sin\alpha_{ij}|<\varepsilon^\beta}$), from (46) and Lemma 2 we get:

$$\frac{1}{n^2}\sum_{\substack{i,j=1\\ i\neq j}}^n \phi(T^t(z_i))\phi(T^t(z_j))I_{|\sin\alpha_{ij}|>\varepsilon^\beta}\mathbb{E}^n\left[\xi^V_{n,\varepsilon,M}(t,z_i)\xi^V_{n,\varepsilon,M}(t,z_j)\right] =$$

$$o(a_n^{-d}) + O(\varepsilon^\zeta) + \frac{1}{n^2}\sum_{i,j=1}^n \phi(T^t(z_i))\phi(T^t(z_j))\times$$

$$\times\exp\left[-\mu_\varepsilon\left(\int_{\mathcal{T}_\varepsilon(t,z_i)} dce^{-V(T_{z_i,c},c)} + \int_{\mathcal{T}_\varepsilon(t,z_j)} dce^{-V(T_{z_j,c},c)}\right) + K_t\varepsilon^{1-d\beta}\right]$$

since in this case $\mu_\varepsilon[\mathcal{T}_\varepsilon(t,z_j)\bigcap\mathcal{T}_\varepsilon(t,z_i))]_L = O(\varepsilon^{1-d\beta})\to 0$, so that (using (47) once more)

$$\mathcal{E}_{1\,n} \to \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} dxdv f(s,x,v)\phi(x,v)\right)^2$$

and Lemma 5 is proved. □

We emphasize again that all results in this Lemma are independent of the ordering of the limits $n\to\infty$, $\varepsilon\to 0$.

4.4.2. *Equivalence of the obstacle components of (33) and (25).* Lemma 5 describes the asymptotic average behavior of the light particle component of the system (33). The asymptotic average behavior of its risk function, for an obstacle located in c, should be equivalent to the behavior associated to the risk (28). This fact is described by the proposition proved in this section and its corollary (notice that the risk function of a given obstacle with position c_h, differently from the generic risk function associated to a given position, is defined only if there exists an obstacle in c_h). The second proposition will be useful in the proof of equivalence of (33) to (31), in the next section.

It will be convenient to define the following stochastic functions. For $z = (x, v) \in \mathbb{R}^d \times \mathbb{R}^d$:

$$\rho_\varepsilon^{(k)}(t,c,z) = 1 - I_{\{c\in\mathcal{T}_\varepsilon(t,z)\}}\, I_{\{\bar{V}^{(k)}(T_{z,c},c)<\tau_c\}} \tag{51}$$

$$\rho^{(k)}(t,z) = \int_0^t du \int_{\partial B_1(0)} |v\cdot\omega| d\omega \exp\left(-\bar{V}^{(k)}(u,x(u))\right)$$

$$Q_n^{ij}(s,t,c) = q_n(x_i(s)-c)q_n(x_j(t)-c)$$

$$\mathcal{R}_\varepsilon^{ij\,(k)}(s,t,c) = \mathbb{E}_{\tau_c}[\rho_\varepsilon^{(k)}(s,c,z_i)\rho_\varepsilon^{(k)}(t,c,z_j)].$$

Given these definitions, a fundamental tool in all estimates in this section is the following lemma:

Lemma 6. *Take a sequence $Z_\infty \in (\mathbb{R}^d\times\mathbb{R}^d)^\infty$ and its associated sequence of empirical measures $\{\mu_n^0\}_{n=1}^\infty$. Take a sequence $\{\Lambda_n\}_{n=1}^\infty$ of bounded, Lebesgue measurable sets such that $B_n \subset \Lambda_n$ (with B_n defined in (9)) and assume $\mu_n^0 \rightharpoonup f_0 \in \mathscr{S}$ and $|v|\mu_0^n \rightharpoonup |v| f_0 \in \mathscr{S}$. For $0 \le s \le t$, we have*

$$\begin{aligned}\frac{1}{[\Lambda_n]_L}\int_{\Lambda_n} dc\, \mathcal{R}_\varepsilon^{ij\,(k)}(t,s,c) = 1 - \frac{\varepsilon^{d-1}}{[\Lambda_n]_L}[\rho^{(k)}(t,z_i) + \rho^{(k)}(s,z_j)(1-\delta_{ij})] \\ + \frac{R_\varepsilon^{(2)}(z_i,z_j,t,s)}{[\Lambda_n]_L}\end{aligned} \tag{52}$$

where, for each $\zeta \in (0, \frac{1}{2} - \frac{1}{2d})$,

$$\frac{1}{n}\sum_{i=1}^n \|R_\varepsilon^{(2)}(z_i,z_j)\|_{L^\infty([0,T]\times[0,T])} \le K_{r1}\varepsilon^{d-1+\zeta}|v_j| + \varepsilon^{d-1}[o(a_n^{-d})] \tag{53}$$

and

$$\lim_{\varepsilon\to 0}\varepsilon^{1-d}\frac{1}{n^2}\sum_{i,j=1}^n \|R_\varepsilon^{(2)}(z_i,z_j)\|_{L^\infty([0,T]\times[0,T])} = 0. \tag{54}$$

Proof. Formula (52) follows, through simple calculations, from (47), expanding the expectation value in the definition of $\mathcal{R}_\varepsilon^{ij\,(k)}$, given in (51), and writing

$$\int_{\mathcal{T}_\varepsilon(t,z)} dc e^{-\bar{V}^{(k)}(T_{z,c},c)} = \varepsilon^{d-1}\left[\rho^{(k)}(t,z) + C_d|v|\int_0^t du\phi_\varepsilon(u,x(u))\right];$$

doing this, we obtain for the remainder

$$\begin{aligned}|R_\varepsilon^{(2)}(z_i,z_j,t,s)| \le \varepsilon^{d-1}C_d(|v_i|+|v_j|)\|\varphi_\varepsilon\|_{L^\infty([0,T]\times\mathbb{R}^d)} \\ +[\mathcal{T}_\varepsilon(s,z_j)\cap\mathcal{T}_\varepsilon(t,z_i)]_L\end{aligned} \tag{55}$$

where $\|\varphi_\varepsilon\|_{L^\infty([0,T]\times\mathbb{R}^d)} = O(\varepsilon) \to 0$.

We use then the bound (55) to obtain (53) and (54). In the single sum in (53), so as in the double sum in (54), thanks to the weak convergence of $|v|\mu_n^0$, the contribution of the first term on the right-hand side of (55) is $O(\varepsilon^d)$. As for the contribution of $[\mathcal{T}_\varepsilon(s,z_j)\cap\mathcal{T}_\varepsilon(t,z_i)]_L$, we bound the Lebesgue measure of the intersection of the tubes using Lemma 2; then, thanks to the weak convergence of μ_n^0 toward the (sufficiently) regular function f_0, we can use Lemma 3 and (46), with $\beta<\frac{1}{d}$, to estimate the contribution coming from the grazing collisions (being the remaining part in both sums $o(\varepsilon^{d-1})$). So, from Lemma 3 we get straightforwardly (53) and, from (46), we get the bound:

$$\frac{1}{n^2}\sum_{i,j=1}^n \|R_\varepsilon^{(2)}(z_i,z_j)\|_{L^\infty([0,T]\times[0,T])}\le K_R\varepsilon^{d-1}[\varepsilon^\zeta+o(a_n^{-d})], \tag{56}$$

and, when $\varepsilon\to 0$, we obtain (54).

□

We may now prove the main proposition of this section:

Proposition 1. *Consider the stochastic variables defined in (33) and (28) and the sequences of positive real numbers $\{a_n\}_{n=1}^\infty$ and $\{\varepsilon_n\}_{n=1}^\infty$ satisfying conditions (18) and (19), i.e. $\lim_{n\to\infty} a_n^d n^{-\kappa}=0$, for some $\kappa\in(0,\frac12)$, and $\lim_{n\to\infty} a_n^d\varepsilon^\zeta=0$, for some $\zeta\in(0,\frac12-\frac{1}{2d})$.*

Then, $\mathscr{P}$-almost everywhere w.r.t. sequences of initial data Z_∞,

$$\lim_{n\to\infty}[\Lambda_n]_L\sup_{t\in[0,T]}\mathbb{E}^n\left[\left|\hat{A}^{(k)}_{n,\varepsilon,M}(t,c)-\bar{V}^{(k)}(t,c)\right|^2 I_{M\ge1}\right]=0$$

where $\{\Lambda_n\}$ is an increasing sequence of bounded Lebesgue measurable sets such that $B_n\subset\Lambda_n$ (with B_n defined in (9)) and $\lim_{n\to\infty}\Lambda_n=\mathbb{R}^n$.

Proof. We expand the square power in the expectation value and we calculate the value of the three resulting terms.

Because of definition (51), using (40), we can write:

$$\hat{\xi}^{(k-1)}_{n,\varepsilon,M}(s,z_j)=\prod_{h=1}^M\rho_\varepsilon^{(k-2)}(s,c_h,z_j).$$

We substitute this expression in the definition of $\hat{A}^{(k)}_{n,\varepsilon,M}$, in system (33), and we obtain (resumming on the Poissonian variables):

$$[\Lambda_n]_L\mathbb{E}^n\left[\left|\hat{A}^{(k)}_{n,\varepsilon,M}(t,c)\right|^2 I_{\{M\ge1\}}\right]=$$

$$\frac{1}{n^2}\sum_{i,j=1}^n\int_{[0,t]^2}ds_1\,ds_2\left(\exp\left[\mu_\varepsilon[\Lambda_n]_L\Big(\int_{\Lambda_n}dc\,\frac{\mathcal{R}_\varepsilon^{ij\,(k-2)}(s_1,s_2,c)}{[\Lambda_n]_L}-1\Big)\right]-e^{-\mu_\varepsilon[\Lambda_n]_L}\right)\times$$

$$\times\frac{\int_{\Lambda_n}dc\,Q_n^{ij}(s_1,s_2,c)\mathcal{R}_\varepsilon^{ij\,(k-2)}(s_1,s_2,c)}{\int_{\Lambda_n}dc\,\frac{\mathcal{R}_\varepsilon^{ij\,(k-2)}(s_1,s_2,c)}{[\Lambda_n]_L}}.$$

Then, using Lemma 6, we have:

$$[\Lambda_n]_L\mathbb{E}^n\left[\left|\hat{A}^{(k)}_{n,\varepsilon,M}(t,c)\right|^2 I_{\{M\ge1\}}\right]= \tag{57}$$

$$\frac{1}{n^2}\sum_{i,j=1}^n\int_{[0,t]^2}ds_1\,ds_2\,e^{-\mu[\rho^{(k-2)}(s_1,z_i)+\rho^{(k-2)}(s_2,z_j)(1-\delta_{ij})]}$$

$$\times(1+\varepsilon^{1-d}R_\varepsilon^{ij(3)})\left[\int_{\Lambda_n}dc\,Q_n^{ij}(s_1,s_2,c)\mathcal{R}_\varepsilon^{ij\,(k-2)}(s_1,s_2,c)\right]$$

where $R_\varepsilon^{ij(3)} = R_\varepsilon^{(3)}(s_1, s_2, z_i, z_j) \leq K_t R_\varepsilon^{(2)}(s_1, s_2, z_i, z_j)$ is such that

$$\frac{\varepsilon^{1-d}}{n^2} \sum_{i,j=1}^{n} \|R_\varepsilon^{ij(3)}\|_{L^\infty[0,T]\times[0,T]} \leq K_{R_3}(\varepsilon^\zeta + o(a_n^{-d})).$$

We recall that, using Lemma 6 and since $q \in \mathscr{S}(\mathbb{R}^d)$, we have, $\mathscr{P}$-almost everywhere w.r.t. sequences of initial data Z_∞ and for any $D \subset \mathbb{R}^d$, the bounds

$$\frac{\varepsilon^{1-d}}{n^2} \sum_{i,j=1}^{n} \|R_\varepsilon^{(3)}\|_{L^\infty[0,T]\times[0,T]} \int_D dc\, Q_n^{ij}(s,t,c) \leq K_q \Theta a_n^d \|q\|_\infty (\varepsilon^\zeta + o(a_n^{-d})), \tag{58}$$

$$\frac{1}{n^2} \sum_{i,j=1}^{n} \int_{\mathcal{T}_\varepsilon(t,z_i)} dc\, Q_n^{ij}(s,t,c) \leq C_d t \varepsilon^{d-1} (a_n^d \|q\|_\infty) \frac{K_1}{n} \sum_{i=1}^{n} |v_i|.$$

Therefore, using the definition of $\mathcal{R}_\varepsilon^{ij\,(k-2)}$, given in (51), we get the equality

$$[\Lambda_n]_L \mathbb{E}^n \left[\left| \hat{A}_{n,\varepsilon,M}^{(k)}(t,c) \right|^2 I_{\{M\geq 1\}} \right] = R_\varepsilon^{(4)}(t) +$$

$$\frac{1}{n^2} \sum_{i,j=1}^{n} \int_{[0,t]^2} ds_1 ds_2 e^{-\mu[(\rho^{(k-2)}(s_1,z_i)+\rho^{(k-2)}(s_2,z_j))(1-\delta_{ij})]} \int_{\Lambda_n} dc\, Q_n^{ij}(s_1,s_2,c)$$

where

$$\sup_{t\in[0,T]} R_\varepsilon^{(4)}(t) \leq 4K_q \Theta T^2 \|q\|_\infty a_n^d (\varepsilon^\zeta + o(a_n^{-d})) + O(\varepsilon^{d-1} a_n^d) \to 0,$$

since $\lim_{n\to\infty} a_n^d \varepsilon^\zeta = 0$ with $\zeta \in (0, \frac{1}{2} - \frac{1}{2d})$.

Let $z = (x,v)$ and $\tilde{z} = (y,w)$. Define the sequence of measurable sets $\Lambda_n^s[z] = \{a \in \mathbb{R}^d : \exists b \in \Lambda_n \ s.t.\ a = b - x(s)\}$ and the sequence of functions $Q^*_{\Lambda_n^s}(z,x) = (I_{\Lambda_n^s[z]} q_n) * q_n(x)$.

Since q is a radial function, we have the following equality:

$$\frac{1}{n^2} \sum_{i,j=1}^{n} e^{-\mu[(\rho^{(k-2)}(s_1,z_i)+\rho^{(k-2)}(s_2,z_j))(1-\delta_{ij})]} \int_{\Lambda_n} dc\, Q_n^{ij}(s_1,s_2,c) =$$

$$\int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} dz d\tilde{z} [\mu_n^0 \otimes \mu_n^0](z,\tilde{z}) e^{-\mu[\rho^{(k-2)}(s_1,z)+\rho^{(k-2)}(s_2,\tilde{z})]} Q^*_{\Lambda_n^{s_2}}(\tilde{z}, x(s_1) - y(s_2))$$

$$+ \frac{1}{n} \int_{\mathbb{R}^d} dz \mu_n^0(z) e^{-\mu\rho^{(k-2)}(s_1,z)} [1 - e^{-\mu\rho^{(k-2)}(s_2,z)}] Q^*_{\Lambda_n^{s_2}}(\tilde{z}, x(s_1) - y(s_2)),$$

where the last term is bounded by

$$\frac{2}{n} \int dz \mu_n^0 (q_n * q_n)(x(s_1) - y(s_2)) \leq 2 \frac{a_n^d}{n} \|q\|_\infty \Theta.$$

In the first term we can use Fubini's theorem, rewriting

$$\int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} dz d\tilde{z} [\mu_n^0 \otimes \mu_n^0](z,\tilde{z}) e^{-\mu[\rho^{(k-2)}(s_1,z)+\rho^{(k-2)}(s_2,\tilde{z})]} Q^*_{\Lambda_n^{s_2}}(\tilde{z}, x(s_1) - y(s_2)) =$$

$$\int_{\mathbb{R}^d} da \int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} dz d\tilde{z} [\mu_n^0 \otimes \mu_n^0](z,\tilde{z}) \left[e^{-\mu\rho^{(k-2)}(s_1,z)} q_n(a - x(s_1)) \right] \times$$

$$\times \left[e^{-\mu\rho^{(k-2)}(s_2,\tilde{z})} q_n(a - y(s_2)) I_{\Lambda_n^{s_2}[\tilde{z}]}(a - y(s_2)) \right]$$

and, because of (42), we can use Lebesgue's dominated convergence theorem to pass to the limit into the integral.

$\{\Lambda_n\}_{n=1}^{\infty}$ is an increasing sequence of Lebesgue measurable set and $I_{\Lambda_n} \to 1$ in $L^\infty(\mathbb{R}^d)$. From the $\mathscr{P}$-a.e. weak convergence $\otimes_{k=1}^{Q} q_n \mu_n^0 \rightharpoonup \otimes_{k=1}^{Q} \delta_0 f_0$ and the L^∞ convergence of $I_{\Lambda_n^s[y]}$, we have

$$\int_{[0,t]^2\times(\mathbb{R}^d\times\mathbb{R}^d)^2} ds_1 ds_2 dz d\tilde{z} [\mu_n^0 \otimes \mu_n^0](z,\tilde{z}) \times$$

$$\times\, e^{-\mu[\rho^{(k-2)}(s_1,z)+\rho^{(k-2)}(s_2,\tilde{z})]} Q^*_{\Lambda_n^{s_2}}(\tilde{z}, x(s_1)-y(s_2))$$

$$\to \int_{\mathbb{R}^d} dc \left|\bar{V}^{(k)}(t,c)\right|^2,$$

so that we obtain finally

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^{n} \int_{[0,t]^2\times\Lambda_n} ds_1\, ds_2\, dc\, Q_n^{ij} e^{-\mu[\rho^{(k-2)}(s_2,z_i)+\rho^{(k-2)}(s_1,z_j)(1-\delta_{ij})]} =$$

$$\int_{\mathbb{R}^d} dc \left|\bar{V}^{(k)}(t,c)\right|^2.$$

We compute now the double product and, because of the definitions of $\rho_\varepsilon^{(k)}$, Λ_n and of the boundedness of $\bar{V}^{(k)}$, we get :

$$[\Lambda_n]_L \mathbb{E}^n \left[\hat{A}^{(k)}_{n,\varepsilon,M}(t,c)\bar{V}^{(k)}(t,c) I_{\{M\geq 1\}}\right] = O(a_n^d \varepsilon^{d-1}) +$$

$$\tfrac{1}{n} \sum_{j=1}^{n} \int_0^t ds_1 \left(\exp\left[-\mu_\varepsilon \int_{\mathcal{T}_\varepsilon(s_1,z_j)} dc\, e^{-\bar{V}^{(k-2)}(T_{z_j,c,c})}\right]\right.$$

$$\left.\times \int_{\Lambda_n} dc \bar{V}^{(k)}(t,c) q_n(x_j(s_1)-c)\right).$$

Using (47), $\mathscr{P}$-a.e. with respect to Z_∞, we get:

$$[\Lambda_n]_L \mathbb{E}^n \left[\hat{A}^{(k)}_{n,\varepsilon,M}(t,c)\bar{V}^{(k)}(t,c) I_{\{M\geq 1\}}\right] =$$

$$\frac{1}{n} \sum_{i=1}^{n} \int_0^t ds_1 e^{-\mu\rho^{(k-2)}(s_1,z_i)} \int_{\Lambda_n} dc\, \bar{V}^{(k)}(t,c) q_n(x_i(s_1)-c) + o(\varepsilon)$$

so that, because of the $\mathscr{P}$-a.e. weak convergence of $q_n\mu_n^0$ and the hypothesis on f_0, we obtain:

$$-2 \lim_{n\to\infty} [\Lambda_n]_L \mathbb{E}^n \left[\hat{A}^{(k)}_{n,\varepsilon,M}(t,c)\bar{V}^{(k)}(t,c) I_{\{M\geq 1\}}\right] = -2 \int_{\mathbb{R}^d} dc \left|\bar{V}^{(k)}(t,c)\right|^2.$$

For the last term we have:

$$[\Lambda_n]_L \mathbb{E}^n \left[|\bar{V}^{(k)}(t,c)|^2 I_{\{M\geq 1\}}\right] = (1-e^{-\mu_\varepsilon[\Lambda_n]_L}) \int_{\Lambda_n} dc \left|\bar{V}^{(k)}(t,c)\right|^2,$$

whose limit is $\int_{\mathbb{R}^d} dc \left|\bar{V}^{(k)}(t,c)\right|^2$ because of the L^∞ convergence of I_{Λ_n}.

Hence, we have established Proposition 1. □

In addition to Proposition 1, we shall need its corollary

Corollary 2. *Under the same hypothesis as Proposition 1, for the obstacles life functions defined in (30) and in (33) and for all $\phi \in C_K(\mathbb{R}^d)$, $\mathscr{P}$-almost everywhere w.r.t. sequences of initial data Z_∞:*

$$\lim_{n\to\infty} \mathbb{E}^n [\varepsilon^{d-1} \sum_{i=1}^{M} I_{\mathrm{supp}\phi}(c_i) |\hat{\eta}_{n,\varepsilon,M}(t,c_i) - \bar{\eta}(t,c_i)|] = 0.$$

Proof. Since for all $\delta > 0$ (cf. Remark 1)

$$\mathbb{E}^n[\varepsilon^{d-1}\sum_{i=1}^{M} I_{\mathrm{supp}\phi}(c_i)|\hat{\eta}_{n,\varepsilon,M}(t,c_i) - \bar{\eta}(t,c_i)|] \leq$$

$$\frac{1}{\delta}\mathbb{E}^n[\varepsilon^{d-1}MI_{\mathrm{supp}\phi}(c)|\bar{V}^{(k)}(t,c) - \hat{A}^{(k)}_{n,\varepsilon,M}(t,c)|] + 2\delta\mu\,|\mathrm{supp}\phi\cap\Lambda_n|\,,$$

the proof follows easily bounding the first term on the right-hand side by Cauchy-Schwarz's inequality and using the identity

$$\mathbb{E}^n[(\varepsilon^{d-1}M)^2 I_{\mathrm{supp}\phi}(c)] = \mu^2[\Lambda_n]_L[\mathrm{supp}\phi\cap\Lambda_n]_L$$

and Proposition 1. □

We shall need moreover a modified form of the proposition, whose proof will be only sketched, being essentially the same as the one of Proposition 1

Proposition 2. *Under the same hypothesis as Proposition 1,$\mathscr{P}$-almost everywhere w.r.t. sequences of initial data Z_∞:*

$$\lim_{n\to\infty}\sup_{1\leq h\leq n}\left(\frac{1}{|v_h|}\mathbb{E}^n\left[|\hat{A}^{(k)}_{n,\varepsilon,M} - \bar{V}^{(k)}|^2(T_{z_h,c_1},c_1)MI_{c_1\in\mathcal{T}_\varepsilon(T,z_h)}\right]\right) = 0.$$

Proof. The proof is obtained along the same line as the proof of Prop. 1, since T_{z_h,c_1} is independent of the stochastic times $\mathcal{T}_\infty$, is bounded by T and all quantities involved depend on T_{z_h,c_1} in a simple way, so to allow to get estimates uniform in z_h.

Using, instead of (58), the estimates

$$\frac{\varepsilon^{1-d}}{n^2}\sum_{i,j=1}^{n}\|R^{ij(3)}_\varepsilon\|_{L^\infty[0,T]\times[0,T]}Q^{ij}_n(s,t,c) \leq K_r a_n^d(\varepsilon^\zeta + o(a_n^{-d})),$$

$$\frac{1}{n^2}\sum_{i,j=1}^{n}\int_{\mathcal{T}_\varepsilon(t,z_h)\cap\mathcal{T}_\varepsilon(t,z_i)} dc\,Q^{ij}_n(s,t,c) \leq K_Q|v_h|\varepsilon^{d-1}a_n^d(\varepsilon^\zeta + o(a_n^{-d}))$$

which are a consequence of Lemma 6 and of the bounds (42), (43), we perform the change of variables (47) and we get:

$$\frac{1}{|v_h|}\mathbb{E}^n\left[M\left|\hat{A}^{(k)}_{n,\varepsilon,M}(T_{z_h,c_1},c_1)\right|^2 I_{c_1\in\mathcal{T}_\varepsilon(T,z_h)}\right] = O(a_n^d(\varepsilon^\zeta + o(a_n^{-d})))+$$

$$\frac{\mu C_d}{n^2}\sum_{i,j=1}^{n}\int_0^T ds\int_{[0,s]^2} ds_1 ds_2 Q^{ij}_n(s_1,s_2,x_h(s))e^{-\mu[\rho^{(k-2)}(s_1,z_i)+\rho^{(k-2)}(s_2,z_j)]},$$

while for the double product, we have

$$\frac{1}{|v_h|}\mathbb{E}^n\left[MI_{\{c_1\in\mathcal{T}_\varepsilon(T,z_h)\}}\hat{A}^{(k)}_{n,\varepsilon,M}(t,c_1)\bar{V}^{(k)}(t,c_1)\right] = O(a_n^{d+1}\varepsilon)+$$

$$\frac{\mu C_d}{n}\sum_{j=1}^{n}\int_0^T ds\bar{V}^{(k)}(s,x_h(s))\int_0^s ds_1 q_n(x_j(s_1)-x_h(s))\exp[-\mu\rho^{(k-2)}(s_1,z_j)].$$

Of course

$$\frac{1}{|v_h|}\mathbb{E}^n\left[M\left|\bar{V}^{(k)}(t,c_1)\right|^2 I_{c_1\in\mathcal{T}_\varepsilon(T,z_h)}\right] = \mu C_d\int_0^T ds\left|\bar{V}^{(k)}(s,x_h(s))\right|^2 + O(\varepsilon).$$

Then, observing that, thanks to the $\mathscr{P}$-a.e. weak convergence of $q_n\mu_n^0$, the quantity

$$\frac{1}{|v_h|}\mathbb{E}^n\left[|\hat{A}^{(k)}_{n,\varepsilon,M}(T_{z_h,c_1},c_1) - \bar{V}^{(k)}(T_{z_h,c_1},c_1)|^2 MI_{c_1\in\mathcal{T}_\varepsilon(T,z_h)}\right]$$

vanishes for all fixed $h\in\mathbb{N}$, uniformly in z_h because of the choice of the initial data (cf. Remark 9 in Appendix 6.2) and since all error estimates in this section are themselves uniform in z_h, the Proposition is proved. □

4.5. **Equivalence between (33) and (31): the $\mathcal{O},\mathcal{P}$-frozen system.** The last step to complete the proof of our main theorem is to estimate the distance between the particle system (33) and the particle system (31). The difficulty, here, originates from the complicate dependence of the life functions, defining both systems, on the stochastic configuration: in particular, this dependence compels us to use (24) for estimating differences of life functions for both species, preventing the use of any of its simplified forms, mentioned in Remark 1.

Since the system (33) is equivalent in the large particles number limit to the system (25), a mean–field system, what should happen for systems (33) and (31) to be equivalent in the limit is that (31) is somehow stable (in the large particles number limit) w.r.t. deletion (or addition) of a finite number of particles of the two species. This is in fact the case, as we shall show in the first Lemma in this section. Then, deleting light particles and obstacles in a suitable way from system (31), we shall be able to simplify the dependence on the stochastic configuration in the estimates and finally to prove the asymptotic equivalence between (33) and (31).

Let $\mathcal{O}$ and $\mathcal{P}$ be finite subsets of $\mathbb{N}$ and k the level of approximation. In this section, we introduce a sequence of particle systems where the obstacles with labels in $\mathcal{O}$ (resp. light particles with labels in $\mathcal{P}$) have no interaction with the light particles (resp. with the obstacles). These systems will give us a tool to estimate the required distance.

For a given configuration of obstacles, $\mathbf{c}_M = (c_1,\dots,c_M)$ and a particle initial datum $\mathbf{z}_n = (z_1,\dots,z_n)$, fix the sets of integers $\mathcal{O}\subseteq\{1,\dots,M\}$, $\mathcal{P}\subseteq\{1,\dots,n\}$ and define, for integers $M,k\geq 1$:

$$
\begin{aligned}
&\gamma^{(0,\emptyset,\mathcal{P})}_{n,\varepsilon,0}(t,z_j) = \gamma^{(k,\emptyset,\mathcal{P})}_{n,\varepsilon,0}(t,z_j) = I_{\{j\notin\mathcal{P}\}},\\
&\gamma^{(0,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}(t,z_j) = I_{\{j\notin\mathcal{P}\}},\\
&\delta^{(0,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}(t,c_\ell) = I_{\{\ell\notin\mathcal{O}\}},\\
&\gamma^{(k,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}(t,z_j) = I_{\{x_j(s)\notin\bigcup_{h=1}^{M} B_\varepsilon(c_h)\delta^{(k-1,\mathcal{O},\mathcal{P})}(c_h,s)\ \forall s\in[0,t)\}} I_{\{j\notin\mathcal{P}\}},\\
&\delta^{(k,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}(t,c_\ell) = I_{\{A^{(k,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}(t,c_\ell)\leq\tau_{c_\ell}\}} I_{\{\ell\notin\mathcal{O}\}},\\
&A^{(k,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}(t,c_\ell) = \frac{1}{n}\sum_{i=1}^{n}\int_0^t q_n(x_i(s)-c_\ell)\gamma^{(k-1,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}(s,z_i)ds.
\end{aligned}
\tag{59}
$$

Remark that $A^{(k,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}(t,c_\ell)$ does not depend on τ_{c_h}, x_j, v_j for any $h\in\mathcal{O}$ and any $j\in\mathcal{P}$, and it does not depend on c_h, $h\in\mathcal{O}$, whenever $\ell\neq h$. Moreover, for $(\mathcal{O},\mathcal{P})=(\emptyset,\emptyset)$, the system defined above is the k-th level approximation system defined in (31), Section 3.2, i.e.

$$
A^{(k,\emptyset,\emptyset)}_{n,\varepsilon,M} = V^{(k)}_{n,\varepsilon,M},\qquad \gamma^{(k,\emptyset,\emptyset)}_{n,\varepsilon,M} = \xi^{(k)}_{n,\varepsilon,M},\qquad \delta^{(k,\emptyset,\emptyset)}_{n,\varepsilon,M} = \eta^{(k)}_{n,\varepsilon,M}.
$$

4.5.1. *Proof of the equivalence between (33) and (31) using the $\mathcal{O},\mathcal{P}$-frozen system.* Using the bound (41), we can use the same strategy as in [NOR].

To shorten the notation, we define in this paragraph, for sets $\mathcal{O},\mathcal{O}_1\subseteq\{1,\dots,M\}$ and $\mathcal{P},\mathcal{P}_1\subseteq\{1,\dots,n\}$:

$$
\mathcal{A}^{nM(k)}_{\mathcal{O},\mathcal{P}}(\mathcal{O}_1,\mathcal{P}_1) = \left|A^{(k,\mathcal{O}\cup\mathcal{O}_1,\mathcal{P}\cup\mathcal{P}_1)}_{n,\varepsilon,M} - A^{(k,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}\right|,
$$

for $M=1,\dots,$

$$\mathcal{I}^{\mathcal{O},\mathcal{P},0}_{\varepsilon,t,z_i}(c_\ell) = 0$$

$$\mathcal{I}^{\mathcal{O},\mathcal{P},M}_{\varepsilon,t,z_i}(c_\ell) = MI_{\{c_\ell\in\mathcal{T}_\varepsilon(t,z_i)\}}I_{\mathcal{O}\subset\{1,\dots,M\}}I_{\mathcal{P}\subset\{1,\dots,n\}},$$

and $\mathcal{I}^{M}_{\varepsilon,t,z_i} = \mathcal{I}^{\emptyset,\emptyset,M}_{\varepsilon,t,z_i}$.

We prove the following lemmas:

Lemma 7. *Consider the stochastic variables defined in (59). Then, under the same hypothesis as in Proposition 1, for all integers $k \geq 1$, $p \geq 1$ and for all $\mathcal{O}_1, \mathcal{P}_1 \subset \mathbb{N}$ s.t. $0 < \#\mathcal{O}_1, \#\mathcal{P}_1 < \infty$, $\mathscr{P}$-almost everywhere w.r.t. sequences of initial data Z_∞, we have the limits*

(60)
$$\lim_{n\to\infty}\frac{1}{n^p}\sum_{\substack{1\leq j_1\leq n\\ \dots,\\ 1\leq j_p\leq n}}\sup_{z_{j_0}\in\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{|v_{j_0}|}\mathbb{E}^n\Bigg[\int_{[0,T]^p}\prod_{h=1}^{p}ds_{j_h}q_n(x_{j_h}(s_{j_h})-c_{i_h})I_{\{c_{i_h}\in\mathcal{T}_\varepsilon(s_{j_{h-1}},z_{j_{h-1}})\}}$$
$$\times\mathcal{I}^{\mathcal{O}\cup\mathcal{O}_1,\mathcal{P}\cup\mathcal{P}_1,M}_{\varepsilon,s_{j_p},z_{j_p}}(c_\ell)M^{\#\{i_1,\dots,i_p,\ell\}-1}\mathcal{A}^{nM(k)}_{\mathcal{O},\mathcal{P}}(\mathcal{O}_1,\mathcal{P}_1)(T_{z_{j_p},c_\ell},c_\ell)\Bigg]=0$$

$$\lim_{n\to\infty}\sup_{z_{j_0}\in\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{|v_{j_0}|}\mathbb{E}^n\left[\mathcal{I}^{\mathcal{O}\cup\mathcal{O}_1,\mathcal{P}\cup\mathcal{P}_1,M}_{\varepsilon,s_{j_0},z_{j_0}}(c_\ell)\mathcal{A}^{nM(k)}_{\mathcal{O},\mathcal{P}}(\mathcal{O}_1,\mathcal{P}_1)(T_{z_{j_0},c_\ell},c_\ell)\right]=0$$

for all $i_1,\dots,i_p\in\mathcal{O}\cup\mathcal{O}_1$ and $s_{j_0}<T$, $z_{j_0}=(x_{j_0},v_{j_0})$. Limits (60) are valid also in the case where T replaces T_{z_j,c_ℓ}.

Remark 5. This lemma shows that particle systems obtained from (31) by deleting a finite number of light particles and/or obstacles are equivalent, in the prescribed asymptotics.

Proof. The proof is obtained by induction.

For $k=1$ and $p\geq 0$, we have

$$\mathcal{A}^{nM(1)}_{\mathcal{O},\mathcal{P}}(\mathcal{O}_1,\mathcal{P}_1)(T_{z_{j_p},c_\ell},c_\ell)\leq\frac{1}{n}\sum_{h\in\mathcal{P}_1}\int_0^{T_{z_{j_p},c_\ell}}dsq_n(x_h(s)-c_\ell).$$

and the second limit in (60) follows straightforwardly, since

$$\mathcal{I}^{\mathcal{O}\cup\mathcal{O}_1,\mathcal{P}\cup\mathcal{P}_1,M}_{\varepsilon,s_{j_p},z_{j_p}}(c_\ell)\mathcal{A}^{nM(1)}_{\mathcal{O},\mathcal{P}}(\mathcal{O}_1,\mathcal{P}_1)(T_{z_{j_p},c_\ell},c_\ell)\leq$$
$$\mathcal{I}^{\mathcal{O}\cup\mathcal{O}_1,\mathcal{P}\cup\mathcal{P}_1,M}_{\varepsilon,s_{j_p},z_{j_p}}(c_\ell)\frac{a_n^d}{n}\#\mathcal{P}_1\|q\|_\infty T$$

and $\mathbb{E}^n[\mathcal{I}^{\mathcal{O}\cup\mathcal{O}_1,\mathcal{P}\cup\mathcal{P}_1,M}_{\varepsilon,s_{j_p},z_{j_p}}(c_\ell)]\leq\mu C_dT|v_{j_p}|$, for $p\geq 0$.

Denoting $N=\#\{i_1,\dots,i_p,\ell\}$, we build a partition of labels by grouping them in the following way.

We start from the label ℓ and we denote as $i_{k_1^{(1)}},\dots,i_{k_{w_1}^{(1)}}$, with $k_1^{(1)}<k_2^{(1)}<\dots<k_{w_1}^{(1)}$, the w_1 labels among the $p+1$ labels $i_1,\dots,i_p,\ell$ having common value ℓ; notice that $k_{w_1}^{(1)}=p+1$, being $c_{i_{p+1}}=c_\ell$ the obstacle associated to the light particle label j_{p+1}. We then consider $i_{f_1}=\max\{i_s : i_s\neq\ell\}$ and we call $i_{k_1^{(2)}},\dots,i_{k_{w_2}^{(2)}}$ the w_2 labels having common value i_{f_1}, always using the ordering $k_1^{(2)}<k_2^{(2)}<\dots<k_{w_2}^{(2)}$ (i.e., $k_{w_2}^{(2)}=f_1$). We build in this way N groups of labels and we denote $\bar{q}$ the group label such that $k_1^{(\bar{q})}=1$.

We can write then

$$\frac{1}{n^p}\sum_{\substack{1\le j_1\le n\\ \dots,\\ 1\le j_p\le n}}\mathbb{E}^n\Bigg[\int_{[0,T]^p}\prod_{h=1}^p ds_{j_h}q_n(x_{j_h}(s_{j_h})-c_{i_h})I_{\{c_{i_h}\in\mathcal{T}_\varepsilon(s_{j_{h-1}},z_{j_{h-1}})\}} \tag{61}$$

$$\times\mathcal{I}^{\mathcal{O}\cup\mathcal{O}_1,\mathcal{P}\cup\mathcal{P}_1,M}_{\varepsilon,s_{j_p},z_{j_p}}(c_\ell)M^{\#\{i_1,\dots,i_p,\ell\}-1}\frac{1}{n}\sum_{j_{p+1}\in\mathcal{P}_1}\int_0^{T_{z_{j_p},c_\ell}}dsq_n(x_{j_{p+1}}(s)-c_\ell)\Bigg]\le$$

$$\frac{\mathbb{E}^n_c[M^N]}{[\Lambda_n]^N}\frac{1}{n^{p+1}}\sum_{j_{k^{(1)}_{w_1}}\in\mathcal{P}_1}\sum_{\substack{1\le j_{k^{(1)}_h}\le n\\ h=1,\dots,w_1-1}}\sum_{\substack{1\le j_{k^{(2)}_h}\le n\\ h=1,\dots,w_q}}\dots\sum_{\substack{1\le j_{k^{(N)}_h}\le n\\ h=1,\dots,w_N}}\prod_{q=1}^N\int_{\Lambda_n}dc_{i_{k^{(q)}_{w_q}}}\int_{[0,T]^{w_q}}$$

$$\prod_{h=1}^{w_q}\left[ds_{j_{k^{(q)}_h}}q_n(x_{j_{k^{(q)}_h}}(s_{j_{k^{(q)}_h}})-c_{i_{k^{(q)}_{w_q}}})I_{\{c_{i_{k^{(q)}_{w_q}}}\in\bigcap_{h=1}^{w_q}\mathcal{T}_\varepsilon(s_{j_{k^{(q)}_h-1}},z_{j_{k^{(q)}_h-1}})\}}\right]$$

We use then the bound

$$I_{\{c_{i_{k^{(q)}_{w_q}}}\in\bigcap_{h=1}^{w_q}\mathcal{T}_\varepsilon(s_{j_{k^{(q)}_h-1}},z_{j_{k^{(q)}_h-1}})\}}\le I_{\{c_{i_{k^{(q)}_{w_q}}}\in\mathcal{T}_\varepsilon(s_{j_{k^{(q)}_1-1}},z_{j_{k^{(q)}_1-1}})\}}.$$

so that, thanks to (42), we have:

$$(61)\le\frac{\mathbb{E}^n_c[M^N]}{[\Lambda_n]^N}\left(\#\mathcal{P}_1\|q\|_\infty T\frac{a_n^d}{n}\right)\frac{\prod_{q=1}^N(K_1T)^{w_q-1}}{n^{N-1}}\int_{\Lambda_n^N}\prod_{q=1}^N dc_{i_{k^{(q)}_{w_q}}}\Bigg\{I_{\{c_{i_{k^{(\bar q)}_{w_{\bar q}}}}\in\mathcal{T}_\varepsilon(s_{j_0},z_{j_0})\}}$$

$$\prod_{\substack{q=1\\ q\ne\bar q}}^N\left[\sum_{1\le j_{k^{(q)}_1}\le n}\int_{[0,T]}ds_{j_{k^{(q)}_1-1}}q_n(x_{j_{k^{(q)}_1-1}}(s_{j_{k^{(q)}_1-1}})-c_{i_{k^{(q)}_1-1}})I_{\{c_{i_{k^{(q)}_{w_q}}}\in\mathcal{T}_\varepsilon(s_{j_{k^{(q)}_1-1}},z_{j_{k^{(q)}_1-1}})\}}\right]\Bigg\}$$

and, using (43) and $\mathbb{E}^n_c[M^N]\le K(\mu_\varepsilon[\Lambda_n])^N$ we get finally:

$$(61)\le K\left(\#\mathcal{P}_1\|q\|_\infty T\frac{a_n^d}{n}\right)\left(\frac{\mu TC_dK_2}{K_1}\right)^N\frac{(TK_1)^{p+1}}{K_2}|v_{j_0}|$$

so that (60) is valid for $k=1$.

For $k=2$, since

$$\mathcal{I}^{\mathcal{O}\cup\mathcal{O}_1,\mathcal{P}\cup\mathcal{P}_1,M}_{\varepsilon,s_{j_p},z_{j_p}}(c_\ell)\mathcal{A}^{nM(2)}_{\mathcal{O},\mathcal{P}}(\mathcal{O}_1,\mathcal{P}_1)(T_{z_{j_p},c_\ell},c_\ell)\le$$

$$\mathcal{I}^{\mathcal{O}\cup\mathcal{O}_1,\mathcal{P}\cup\mathcal{P}_1,M}_{\varepsilon,s_{j_p},z_{j_p}}(c_\ell)\Bigg[\frac{1}{n}\sum_{h\in\mathcal{P}_1}\int_0^{T_{z_{j_p},c_\ell}}dsq_n(x_h(s)-c_\ell)+$$

$$\frac{1}{n}\sum_{h\notin\mathcal{P}_1}\int_0^{T_{z_{j_p},c_\ell}}dsq_n(x_h(s)-c_\ell)\sum_{m\in\mathcal{O}}I_{\{c_m\in\mathcal{T}_\varepsilon(s_h,z_h)\}}\Bigg],$$

in the same way (using Lemma 2 and Lemma 3) we get the second limit in (60) and

$$(61)\le K_D\left(\#\mathcal{P}_1\frac{a_n^d}{n}+\#\mathcal{O}(a_n^d\varepsilon^\zeta+\varepsilon^{d-1})\right)\left(\frac{\mu TC_dK_2}{K_1}\right)^N\frac{(TK_1)^{p+1}}{K_2}|v_{j_0}|$$

(with K_D depending on $\|q\|_\infty$, T, K_1, K_2, C_d).

Let assume then that (60) is true for $k-1$, with $k>2$. From the definition of $A^{(k,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}(T_{z_{j_p},c_\ell},c_\ell)$ and the bound (41) we have, for $p\geq 0$:

(62)
$$\mathcal{I}^{\mathcal{O}\cup\mathcal{O}_1,\mathcal{P}\cup\mathcal{P}_1,M}_{\varepsilon,s_{j_p},z_{j_p}}(c_\ell)\mathcal{A}^{nM(k)}_{\mathcal{O},\mathcal{P}}(\mathcal{O}_1,\mathcal{P}_1)(T_{z_{j_p},c_\ell},c_\ell)\leq \mathcal{I}^{\mathcal{O}\cup\mathcal{O}_1,\mathcal{P}\cup\mathcal{P}_1,M}_{\varepsilon,s_{j_p},z_{j_p}}(c_\ell)\times$$
$$\left\{\left(\#\mathcal{P}_1\|q\|_\infty T\frac{a_n^d}{n}\right)+\frac{1}{n}\sum_{j_{p+1}\notin\mathcal{P}}\int_0^T dsq_n(x_{j_{p+1}}(s)-c_\ell)\times\right.$$
$$\left.\left[\sum_{m=1}^M I_{\{c_m\in\mathcal{T}_\varepsilon(s,z_{j_{p+1}})\}}|\delta^{(k-2,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}-\delta^{(k-2,\mathcal{O}\cup\mathcal{O}_1,\mathcal{P}\cup\mathcal{P}_1)}_{n,\varepsilon,M}|(T_{z_{j_{p+1}},c_m},c_m)\right]\right\},$$

and from Lemma 1, for $\delta>0$,

$$\mathbb{E}_{\tau_m}\left[\left|\delta^{(k-2,\mathcal{O}\cup\mathcal{O}_1,\mathcal{P}\cup\mathcal{P}_1)}_{n,\varepsilon,M}-\delta^{(k-2,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}\right|(T_{z_{j_{p+1}},c_m},c_m)\right]\leq 2\delta+ \tag{63}$$
$$\frac{1}{\delta}\mathbb{E}_{\tau_m}\left[\mathcal{A}^{nM(k-2)}_{\mathcal{O},\mathcal{P}}(\mathcal{O}_1\cup\{m\},\mathcal{P}_1\cup\{j_{p+1}\})(T_{z_{j_{p+1}},c_m},c_m)\right]$$
$$+\frac{1}{\delta}\mathbb{E}_{\tau_m}\left[\mathcal{A}^{nM(k-2)}_{\mathcal{O}\cup\mathcal{O}_1,\mathcal{P}\cup\mathcal{P}_1}(\{m\},\{j_{p+1}\})(T_{z_{j_{p+1}},c_m},c_m)\right].$$

When substituting (63) through (62) into the expectation values in formula (60), the contribution to the expectation value coming from the first term on the righthand side of (63) is bounded by $2K_I\delta$, where $K_I=K_I(T,C_d,\mu,K_1,K_2,p,\#\{i_1,\dots,i_p,\ell\})$, while the contribution to (60) due to the last terms in (63) vanish asymptotically because of the inductive hypothesis. The remaining term coming from (62) is $O(a_n^d n^{-1})$.

The last part of the Lemma is proved replacing T_{z_j,c_ℓ} by T. □

Lemma 8. *Consider the stochastic variables defined in (33) and (59). Under the same hypothesis as in Proposition 1, $\forall k\geq 1$, $\forall\mathcal{O},\mathcal{P}\subset\mathbb{N}$ such that $0\leq\#\mathcal{O},\#\mathcal{P}<\infty$, $\mathscr{P}$-almost everywhere w.r.t. sequences of initial data Z_∞,*

$$\lim_{n\to\infty}\sup_{1\leq u\leq n}\frac{1}{|v_u|}\mathbb{E}^n\left[\mathcal{I}^{\mathcal{O},\mathcal{P},M}_{\varepsilon,T,z_u}(c_\ell)|A^{(k,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}-\hat{A}^{(k)}_{n,\varepsilon,M}|(T_{z_u,c_\ell},c_\ell)|\right]=0. \tag{64}$$

Proof. We can prove Lemma 8 by induction.

For $k=1$ and $1\leq u\leq n$ we have:

$$\frac{1}{|v_u|}\mathbb{E}^n[\mathcal{I}^{\mathcal{O},\mathcal{P},M}_{\varepsilon,T,z_u}(c_\ell)|A^{(1,\mathcal{O},\mathcal{P})}_n-\hat{A}^{(1)}_n|(T_{z_u,c_\ell},c_\ell)]\leq(\#\mathcal{P})C_dT^2\mu\|q\|_\infty\frac{a_n^d}{n},$$

and for $k=2$ (using Lemma 3)

$$\frac{1}{|v_u|}\mathbb{E}^n[\mathcal{I}^{\mathcal{O},\mathcal{P},M}_{\varepsilon,T,z_u}(c_\ell)|A^{(2,\mathcal{O},\mathcal{P})}_n-\hat{A}^{(2)}_n|(T_{z_u,c_\ell},c_\ell)]\leq K_m(\#\mathcal{P}\frac{a_n^d}{n}+\#\mathcal{O}\varepsilon^{d-1}+a_n^d\varepsilon^\zeta)$$

so that (64), because of (18) and (19), is valid (K_m depends on $\|q\|_\infty$, T, μ, K_2, C_d).

Let assume (64) is valid for a given $k>1$ and $\forall\mathcal{O},\mathcal{P}$.

We shall use the notation $\sum_{j\notin\mathcal{P}}=\sum_{\substack{1\leq j\leq n\\ j\notin\mathcal{P}}}$ and $\sum_{h\notin\mathcal{O}}=\sum_{\substack{1\leq h\leq M\\ h\notin\mathcal{O}}}$.

From the definitions (59) and (33) we obtain

$$\frac{1}{|v_u|}\mathbb{E}^n[\mathcal{I}^{\mathcal{O},\mathcal{P},M}_{\varepsilon,T,z_u}(c_\ell)|A^{(k,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}-\hat{A}^{(k)}_{n,\varepsilon,M}|(T_{z_u,c_\ell},c_\ell)]\leq(\#\mathcal{P})C_dT^2\mu\|q\|_\infty\frac{a_n^d}{n}+$$
$$\mathbb{E}^n\Big[\frac{\mathcal{I}^{\mathcal{O},\mathcal{P},M}_{\varepsilon,T,z_u}(c_\ell)}{|v_u|n}\sum_{j\notin\mathcal{P}}\int_0^T dsq_n(x_j(s)-c_\ell)\left|\hat{\xi}^{(k-1)}-\gamma^{(k-1,\mathcal{O},\mathcal{P})}\right|(s,z_j)\Big].$$

Since both $\hat{\xi}^{(k-1)}(s,x_j,v_j)$ and $\gamma^{(k-1,\mathcal{O},\mathcal{P})}(s,x_j,v_j)$ are of the form (40), using (41) and then Lemma 2 and Lemma 3, $\mathscr{P}$-almost everywhere w.r.t. sequences of initial data Z_∞ we get (we omit the time integral):

$$\frac{1}{n}\sum_{j\notin\mathcal{P}}\mathbb{E}^n\left[\frac{\mathcal{I}^{\mathcal{O},\mathcal{P},M}_{\varepsilon,T,z_u}(c_\ell)}{|v_u|}q_n(x_j(s)-c_\ell)\left|\hat{\xi}^{(k-1)}-\gamma^{(k-1,\mathcal{O},\mathcal{P})}\right|(s,z_j)\right]\leq \tag{65}$$

$$TC_d\mu\|q\|_\infty a_n^d\left[TC_d(\#\mathcal{O})(\frac{1}{n}\sum_{i=1}^n|v_i|)\,\varepsilon^{d-1}+3K_c(\varepsilon^\zeta+o(a_n^{-d}))\right]+$$

$$\sum_{j\notin\mathcal{P}}\mathbb{E}^n\left[\frac{\mathcal{I}^{\mathcal{O},\mathcal{P},M}_{\varepsilon,T,z_u}(c_\ell)}{n|v_u|}q_n(x_j(s)-c_\ell)\sum_{h\notin\mathcal{O}\cup\{\ell\}}I_{c_h\in\mathcal{T}_\varepsilon(s,z_j)}\left|\bar{\eta}^{(k-2)}-\delta^{(k-1,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}\right|(T_{z_j,c_h},c_h)\right].$$

We use then lemma 1 and we write, for $\delta>0$:

$$\mathbb{E}_{\tau_\ell}\left[\left|\bar{\eta}^{(k-2)}-\delta^{(k-1,\mathcal{O},\mathcal{P})}_{n,\varepsilon,M}\right|(T_{z_j,c_h},c_h)\right]\leq 2\delta+ \tag{66}$$

$$\frac{\mathbb{E}_{\tau_\ell}}{\delta}\left[\left\{\left|\bar{V}^{(k-2)}-A^{(k-2,\mathcal{O}\cup\{\ell\},\mathcal{P}\cup\{j\})}_{n,\varepsilon,M}\right|+\mathcal{A}^{nM(k-2)}_{\mathcal{O},\mathcal{P}}(\{\ell\},\{j\})\right\}(T_{z_j,c_h},c_h)\right],$$

since $A^{(k-2,\mathcal{O}\cup\{\ell\},\mathcal{P}\cup\{j\})}_{n,\varepsilon,M}(T_{z_j,c_h},c_h)$ does not depend on τ_ℓ.

The contribution to the right-hand side of (65) coming from the last term in the last expectation value of (66) vanishes in the limit thanks to Lemma 7 and the contribution coming from the first term is bounded by $2\delta(K_a+K_ba_n^d\varepsilon^\zeta+o(a_n^{-d}))$ for each $\delta>0$.

We evaluate now the remaining term, recalling that $h\neq\ell$.

When $h\neq\ell$, since $\left|\bar{V}^{(k-2)}-A^{(k-2,\mathcal{O}\cup\{\ell\},\mathcal{P}\cup\{j\})}_{n,\varepsilon,M}\right|(T_{z_j,c_h},c_h)$ is independent of c_ℓ, we have (again omitting the time integral):

$$\mathbb{E}^n\left[\frac{\mathcal{I}^{\mathcal{O},\mathcal{P},M}_{\varepsilon,T,z_u}(c_\ell)}{|v_u|n}\sum_{\substack{j\notin\mathcal{P}\\ h\notin\mathcal{O}\cup\{\ell\}}}q_n(x_j(s)-c_\ell)I_{c_h\in\mathcal{T}_\varepsilon(s,z_j)}\left|\bar{V}^{(k-2)}-A^{(k-2,\mathcal{O}\cup\{\ell\},\mathcal{P}\cup\{j\})}_{n,\varepsilon,M}\right|(T_{z_j,c_h},c_h)\right]$$

$$\leq\frac{\mu_\varepsilon[\Lambda_n]_L}{n|v_u|}\sum_{j\notin\mathcal{P}}\mathbb{E}^n[I_{c\in\mathcal{T}_\varepsilon(T,z_u)}q_n(x_j(s)-c)]$$

$$\times\,\mathbb{E}^n\left[\mathcal{I}^{\mathcal{O},\mathcal{P},M}_{\varepsilon,s,z_j}(c_1)\left|\bar{V}^{(k-2)}-A^{(k-2,\mathcal{O}\cup\{\ell\},\mathcal{P}\cup\{j\})}_{n,\varepsilon,M}\right|(T_{z_j,c_1},c_1)\right].$$

By the triangular inequality, we may then bound the time integral in $[0,T]$ of this quantity, $\mathscr{P}$-almost everywhere w.r.t. sequences of initial data Z_∞, by the sum of:

$$\frac{\mu_\varepsilon[\Lambda_n]_L}{n|v_u|}\sum_{j\notin\mathcal{P}}\int_0^T ds\,\mathbb{E}^n[I_{\{c\in\mathcal{T}_\varepsilon(T,z_u)\}}q_n(x_j(s)-c)]$$

$$\times\mathbb{E}^n\left[\mathcal{I}^{\mathcal{O},\mathcal{P},M}_{\varepsilon,s,z_j}(c_1)\left|\bar{V}^{(k-2)}-\hat{A}^{(k-2)}_{n,\varepsilon,M}\right|(T_{z_j,c_1},c_1)\right]$$

$$\leq K_V\sqrt{\sup_{1\leq u\leq n}\mathbb{E}^n\left[\tfrac{1}{|v_u|}\mathcal{I}^M_{\varepsilon,T,z_u}(c_1)\left|\bar{V}^{(k-2)}-\hat{A}^{(k-2)}_{n,\varepsilon,M}\right|^2(T_{z_u,c_1},c_1)\right]},$$

vanishing because of Proposition 2, and

$$\frac{\mu_\varepsilon[\Lambda_n]_L}{n|v_u|}\sum_{j\notin\mathcal{P}}\int_0^T ds\,\mathbb{E}^n[I_{\{c\in\mathcal{T}_\varepsilon(T,z_u)\}}q_n(x_j(s)-c)]$$

$$\times\mathbb{E}^n\Big[\mathcal{I}^{\mathcal{O},\mathcal{P},M}_{\varepsilon,s,z_j}(c_1)\left|\hat{A}^{(k-2)}_{n,\varepsilon,M}-A^{(k-2,\mathcal{O}\cup\{\ell\},\mathcal{P}\cup\{j\})}_{n,\varepsilon,M}\right|(T_{z,c_1},c_1)\Big]$$

$$\leq K_A\sup_{1\leq j\leq n}\mathbb{E}^n\left[\frac{\mathcal{I}^{\mathcal{O},\mathcal{P},M}_{\varepsilon,T,z_j}(c_1)}{|v_j|}\left|A^{(k-2,\mathcal{O}\cup\{\ell\},\mathcal{P}\cup\{j\})}_{n,\varepsilon,M}-\hat{A}^{(k-2)}_{n,\varepsilon,M}\right|(T_{z_j,c_1},c_1)\right],$$

vanishing because of the inductive hypothesis (the constants there depend on C_d, μ, T, K_1, K_2). Since, because of condition $a_n^d\varepsilon^\zeta\to 0$, all terms vanish, the Lemma is proved. □

Collecting all results in this section we prove

Proposition 3. *Consider the stochastic variables defined in (31) and (33). Then, under the same hypothesis as in Proposition 1, $\mathscr{P}$-almost everywhere w.r.t. sequences of initial data Z_∞,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n\mathbb{E}^n\Big[\left|\hat{\xi}^{(k)}(t,z_j)-\xi^{(k)}_{n,\varepsilon,M}(t,z_j)\right|\Big]=0 \tag{67}$$

and

$$\lim_{n\to\infty}\mathbb{E}^n\Big[\varepsilon^{d-1}MI_{\mathrm{supp}\phi}(c_\ell)|\eta^{(k)}_{n,\varepsilon,M}(t,c_\ell)-\hat{\eta}^{(k)}_{n,\varepsilon,M}(t,c_\ell)|\Big]=0. \tag{68}$$

Proof. We start from (67), bounding it first as

$$\frac{1}{n}\sum_{j=1}^n\mathbb{E}^n\Big[\left|\hat{\xi}^{(k)}_{n,\varepsilon,M}-\xi^{(k)}_{n,\varepsilon,M}\right|(t,z_j)\Big]\leq\frac{1}{n}\sum_{j=1}^n\mathbb{E}^n\Big[\mathcal{I}^M_{\varepsilon,t,z_j}(c)|\eta^{(k)}_{n,\varepsilon,M}-\bar{\eta}^{(k)}|(T_{z_j,c},c)\Big].$$

Using lemma 1 and the bound

$$|\bar{V}^{(k)}-A^{(k,\{h\},\{j\})}_{n,\varepsilon,M}|\leq|\bar{V}^{(k)}-\hat{A}^{(k)}_{n,\varepsilon,M}|+|\hat{A}^{(k)}_{n,\varepsilon,M}-A^{(k,\{h\},\{j\})}_{n,\varepsilon,M}|, \tag{69}$$

we get, $\forall\delta>0$:

$$\mathbb{E}_{\tau_h}\left[|\eta^{(k)}_{n,\varepsilon,M}-\bar{\eta}^{(k)}|(\cdot,c_h)\right]\leq 2\delta+\frac{1}{\delta}\mathbb{E}_{\tau_h}\left[|\mathcal{A}^{nM(k)}_{\emptyset,\emptyset}(\{h\},\{j\})|(\cdot,c_h)\right]$$

$$+\frac{1}{\delta}\mathbb{E}_{\tau_h}\left[\left\{|\hat{A}^{(k)}_{n,\varepsilon,M}-A^{(k,\{h\},\{j\})}_{n,\varepsilon,M}|+|\hat{A}^{(k)}_{n,\varepsilon,M}-\bar{V}^{(k)}|\right\}(\cdot,c_h)\right].$$

Define

$$\mathscr{E}^{(k)}_{h,l}(t)=\sup_{z\in\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{|v|}\mathbb{E}^n[\mathcal{I}^{\{h\},\{j\},M}_{\varepsilon,t,z}(c_l)\mathcal{A}^{nM(k)}_{\emptyset,\emptyset}(\{h\},\{j\})(T_{z,c_l},c_l)]$$

$$+\sup_{1\leq i\leq n}\frac{1}{|v_i|}\mathbb{E}^n[\mathcal{I}^M_{\varepsilon,t,z_i}(c_l)|\bar{V}^{(k)}-\hat{A}^{(k)}_{n,\varepsilon,M}|^2(T_{z_i,c_l},c_l)]$$

$$+\sup_{1\leq i\leq n}\frac{1}{|v_i|}\mathbb{E}^n[\mathcal{I}^{\{h\},\{j\},M}_{\varepsilon,t,z_i}(c_l)|A^{(k,\{h\},\{j\})}_{n,\varepsilon,M}-\hat{A}^{(k)}_{n,\varepsilon,M}|(T_{z_i,c_l},c_l)].$$

We get, for all $\delta>0$:

$$\frac{1}{n}\sum_{j=1}^n\mathbb{E}^n\Big[\left|\hat{\xi}^{(k)}_{n,\varepsilon,M}-\xi^{(k)}_{n,\varepsilon,M}\right|(t,z_j)\Big]\leq\left(\frac{1}{n}\sum_{i=1}^n|v_i|\right)\left[C_dT\mu\delta+\frac{1}{\delta}\sup_{t\in[0,T]}\mathscr{E}_{h,h}(t)\right]$$

and (67) follows from Lemma 7, Proposition 2 and Lemma 8.

As for (68), we use Lemma 1 to get the bounds

$$\mathbb{E}_{\tau_h}\left[|\eta^{(k)}_{n,\varepsilon,M}-\hat{\eta}^{(k)}_{n,\varepsilon,M}|(t,c_\ell)\right]\leq 2\delta+$$

$$\frac{1}{\delta}\mathbb{E}_{\tau_h}\left[\left\{|\mathcal{A}^{nM(k)}_{\emptyset,\emptyset}(\{h\},\{j\})|+|A^{(k,\{h\},\{j\})}_{n,\varepsilon,M}-\hat{A}^{(k)}_{n,\varepsilon,M}|\right\}(t,c_\ell)\right],$$

and, for $h\neq\ell$,

$$\mathbb{E}_{\tau_h}\left[|\eta^{(k-2)}_{n,\varepsilon,M}-\delta^{k-2,\{\ell\},\{j\}}_{n,\varepsilon,M}|(t,c_h)\right]\leq 2\delta+\frac{1}{\delta}\mathbb{E}_{\tau_h}\left[\mathcal{A}^{nM(k-2)}_{\emptyset,\emptyset}(\{h\},\{j\})|(t,c_h)\right],$$

and we bound the difference of risk functions through (41).

Noticing that the quantity $\mathcal{A}^{nM(k)}_{\emptyset,\emptyset}(\{\ell\},\{j\})(t,c_h)$ does not depend on c_ℓ for $h\neq\ell$ when $c_\ell\notin\bigcup_{i=1}^{n}\mathcal{T}_\varepsilon(t,z_i)$, we get then, through (69), for all $\delta>0$:

$$\mathbb{E}^n\left[\varepsilon^{d-1}MI_{\mathrm{supp}\phi}(c_\ell)|\eta^{(k)}_{n,\varepsilon,M}(t,c_\ell)-\hat{\eta}^{(k)}_{n,\varepsilon,M}(t,c_\ell)|\right]\leq\mu[\mathrm{supp}\phi]_L K_p(\delta+\frac{a_n^d}{\delta n})$$

$$+\frac{K_d}{\delta^3}\left[\sup_{s\in[0,T]}\mathscr{E}^{(k-2)}_{\ell,\ell-1}(s)+a_n^d\varepsilon^{d-1}\sup_{s\in[0,T]}\mathscr{E}^{(k-2)}_{\ell,\ell}(s)\right.$$

$$+\varepsilon^{d-1}\frac{1}{n}\sum_{1\leq i\leq n}\sup_{\substack{s\in[0,T]\\ z\in\mathbb{R}^d\times\mathbb{R}^d}}\frac{1}{|v|}\mathbb{E}^n\left[\int_{[0,T]}ds_i q_n(x_i(s_i)-c_\ell)I_{\{c_\ell\in\mathcal{T}_\varepsilon(s,z)\}}\right.$$

$$\left.\left.\times\mathcal{I}^{\{\ell\},\{j\},M}_{\varepsilon,s_i,z_i}(c_{\ell-1})M\mathcal{A}^{nM(k-2)}_{\emptyset,\emptyset}(\{\ell\},\{j\})(T_{z_i,c_{\ell-1}},c_{\ell-1})\right]\right],$$

where $K_p=K_p(T,\|q\|_\infty,K_1,K_2,\mu,C_d)$ and $K_d=K_d(\mu,\Theta,\||v|f_0\|_{L_1})$. The proof of the proposition follows using Lemmas 7, 8 and Proposition 2. □

5. Final proposition: asymptotic equivalence of (31) and (25)

We may now prove our final proposition, which will allow us to establish the vanishing limits of (34) and (37) :

Proposition 4. *For the life functions defined in (30) and in (31), under the same hypothesis as in Proposition 1 and for all $\phi\in C_b(\mathbb{R}^d\times\mathbb{R}^d)$ and $\psi\in C_K(\mathbb{R}^d)$, the following limits are valid, $\mathscr{P}$-almost everywhere w.r.t. sequences of initial data Z_∞:*

$$\lim_{n\to\infty}\mathbb{E}^n\left[|\frac{1}{n}\sum_{i=1}^{n}\phi(T^t(z_i))(\bar{\xi}^{(k)}(t,z_i)-\xi^{(k)}_{n,\varepsilon,M}(t,z_i))|\right]=0 \tag{70}$$

and

$$\lim_{n\to\infty}\mathbb{E}^n\left[\varepsilon^{d-1}\sum_{i=1}^{M}I_{\mathrm{supp}\psi}(c_i)|\eta^{(k)}_{n,\varepsilon,M}(t,c_i)-\bar{\eta}^{(k)}(t,c_i)|\right]=0. \tag{71}$$

Proof. By the triangular inequality:

$$\mathbb{E}^n\left[|\frac{1}{n}\sum_{i=1}^{n}\phi(T^t(z_i))[\bar{\xi}^{(k)}-\xi^{(k)}_{n,\varepsilon,M}](t,z_i)|\right]\leq$$

$$\mathbb{E}^n\left[|\frac{1}{n}\sum_{i=1}^{n}\phi(T^t(z_i))[\bar{\xi}^{(k)}-\hat{\xi}^{(k)}_{n,\varepsilon,M}](t,z_i)|+\frac{\|\phi\|_\infty}{n}\sum_{i=1}^{n}|\hat{\xi}^{(k)}_{n,\varepsilon,M}-\xi^{(k)}_{n,\varepsilon,M}|(t,z_i)\right]$$

and the right-hand side term vanishes because of Lemma 5 and Proposition 3.

In the same way,

$$\mathbb{E}^n[\varepsilon^{d-1}\sum_{i=1}^{M}I_{\mathrm{supp}\psi}(c_i)|\eta^{(k)}_{n,\varepsilon,M}-\bar{\eta}^{(k)}|(t,c_i)]\leq$$

$$\mathbb{E}^n[\varepsilon^{d-1}MI_{\mathrm{supp}\psi}(c)\{|\eta^{(k)}_{n,\varepsilon,M}-\hat{\eta}^{(k)}_{n,\varepsilon,M}|+|\hat{\eta}^{(k)}_{n,\varepsilon,M}-\bar{\eta}^{(k)}|\}(t,c)]$$

and the right-hand side term vanishes because of Corollary 2 and Proposition 3. □

This proposition completes the proof of theorem 1.

6. Appendix

6.1. **Existence and Uniqueness of solutions for the limit system.** We give here the theorem of existence and uniqueness of solutions of system (2), which we can state as follows:

Theorem 2. *Let $f_0 \geq 0$ and $\sigma_0 \geq 0$ be s.t. $f_0 \in L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x))$, $vf_0 \in L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x)) \cap L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v)$, $v^2 f_0 \in L^1(\mathbb{R}^d_v; L^\infty(\mathbb{R}^d_x))$ and $\sigma_0 \in W^{1,\infty}(\mathbb{R}^d_x)$. Then for each $T > 0$ there exists an unique solution (f, σ) to the initial value problem (2) in the interval $[0, T]$.*

The proof of theorem (2) is the consequence of the following proposition:

Proposition 5. *Consider the space*

$$\mathcal{W} = \{\mathbf{F} = (F_1, F_2, F_3) : F_i \geq 0 \, i = 1, 2, 3, \\ F_1 \in L^\infty([0,T] \times \mathbb{R}^d \times \mathbb{R}^d), F_2, F_3 \in L^\infty([0,T] \times \mathbb{R}^d) \}$$

with norm

$$\|\mathbf{F}\|_{\mathcal{W}} = \|F_1\|_{L^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)} + \|F_2\|_{L^\infty([0,T]\times\mathbb{R}^d)} + \|F_3\|_{L^\infty([0,T]\times\mathbb{R}^d)}$$

and the map $M = (M_1, M_2, M_3) : \mathcal{W} \to \mathcal{W}$ defined, for $f_0, \sigma_0 \geq 0$, as:

$$\begin{aligned} M_1[\mathbf{F}](t,x,v) &= f_0(x - vt, v) e^{-C_d |v| \int_0^t ds F_3(s, x - v(t-s))} \\ M_2[\mathbf{F}](t,x) &= \int_{\mathbb{R}^d} dv f_0(x - vt, v) e^{-C_d |v| \int_0^t ds F_3(s, x - v(t-s))} \\ M_3[\mathbf{F}](t,x) &= \sigma_0(x) e^{-\Theta \int_0^t ds F_2(s,x)}. \end{aligned} \tag{72}$$

Assume $\sigma_0 \in W^{1,\infty}(\mathbb{R}^d)$ and $f_0 \in L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x))$, together with $vf_0 \in L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x)) \cap L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v)$ and $v^2 f_0 \in L^1(\mathbb{R}^d_v; L^\infty(\mathbb{R}^d_x))$.

Then

- *M is a strict contraction on $\mathcal{W}$ when $T < T_0$, where T_0 depends on C_d, Θ, $\|vf_0\|_{L^\infty(\mathbb{R}\times\mathbb{R}^d)}$, $\|vf_0\|_{L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x))}$, $\|\sigma\|_{W^{1,\infty}(\mathbb{R}^d)}$)*
- *Let $M[\mathbf{f}] = \mathbf{f}$ be the (unique) fixed point of M, then for each $t \in [0,T]$, $T < T_0$,*

$$\begin{aligned} f_1(t,\cdot,\cdot) &\in L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x)), \\ vf_1(t,\cdot,\cdot) &\in L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x)) \cap L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_v), \\ v^2 f_1(t,\cdot,\cdot) &\in L^1(\mathbb{R}^d_v; L^\infty(\mathbb{R}^d_x)) \\ f_3(t,\cdot) &\in W^{1,\infty}(\mathbb{R}^d). \end{aligned}$$

Remark 6. The map (72) is defined so to represent the solutions to the linear problem (25) (cf. the proof of Theorem 2 on next page), and therefore it does not depend on F_1 (the sources in the linear problem are indeed F_2 and F_3). Three components are (of course) nevertheless needed to get a map having as unique fixed point the solution to the nonlinear problem (2).

Proof of Proposition 5. Consider $\mathbf{F}, \mathbf{G} \in \mathcal{W}$. Then

$$\|M_1[\mathbf{F}] - M_1[\mathbf{G}]\|_{L^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)} \leq C_d T \|vf_0\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} \|F_3 - G_3\|_{L^\infty([0,T]\times\mathbb{R}^d)}$$

$$\|M_2[\mathbf{F}] - M_2[\mathbf{G}]\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq C_d T \|vf_0\|_{L^1(\mathbb{R}^d; L^\infty(\mathbb{R}^d))} \|F_3 - G_3\|_{L^\infty([0,T]\times\mathbb{R}^d)}$$

$$\|M_3[\mathbf{F}] - M_3[\mathbf{G}]\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq \Theta T \|\sigma_0\|_{L^\infty([0,T]\times\mathbb{R}^d)} \|F_2 - G_2\|_{L^\infty([0,T]\times\mathbb{R}^d)}$$

We can therefore write

$$\|M[\mathbf{F}] - M[\mathbf{G}]\|_{\mathcal{W}} \leq AT\|\mathbf{F} - \mathbf{G}\|_{\mathcal{W}}, \tag{73}$$

where A is a constant depending on C_d, Θ and on the norms $\|vf_0\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}$, $\|\sigma_0\|_{L^\infty([0,T]\times\mathbb{R}^d)}$, $\|vf_0\|_{L^1(\mathbb{R}^d_v;L^\infty(\mathbb{R}^d_x))}$. Whenever $T < T_0$, where $T_0 = \frac{1}{A}$, M is a contraction on the complete space $\mathcal{W}$. Therefore, whenever $T < \frac{1}{A}$, there exists a unique fixed point $(f, g, \sigma) = M(f, g, \sigma)$ (and of course, $g = \int dvf$).

The properties of the fixed point functions listed in the second part of the thesis of the proposition follow trivially from (72) and the assumptions on f_0. □

Proof of theorem 2. The map M maps the point $\mathbf{F}$ in the point $M[\mathbf{F}]$, with $M_2[\mathbf{F}] = \int dvM_1[\mathbf{F}]$ and $(M_1[\mathbf{F}], M_3[\mathbf{F}])$ solution to the linear problem (25), with sources $\int_{\mathbb{R}^d} dvf^{(k-1)} = F_2$ and $\sigma^{(k-1)} = F_3$ and initial data $f(0, x, v) = f_0(x, v)$, $\sigma(0, x) = \sigma_0(x)$. The fixed point $\mathbf{f}$ of M is therefore s.t. $\mathbf{f} = (f, \int dvf, \sigma)$, where (f, σ) is the solution of (2) for $t \in [0, T]$, with $T < \frac{1}{A}$.

Because of the property of the fixed point $(f, \int dvf, \sigma)$, the solution is prolongeable for any value of $T > 0$. □

Remark 7. Since $f_1(t, \cdot, \cdot) \in L^1(\mathbb{R}^d_v; W^{1,\infty}(\mathbb{R}^d_x))$, $\int dvf_1(t, \cdot, v) \in C_b(\mathbb{R}^d)$.

6.2. **Condition for the weak convergence $q_n\mu_0^n \rightharpoonup \delta_0 f_0$.** We state here a simple condition on the growth rate of the generic term of the sequence $\{a_n\}$ such that the product of two weakly convergent measures converges weakly to the product of the two limit measures.

Lemma 9. *Let $\mathscr{P}$ be a probability measure defined as (4), with one particle probability density f_0 s.t.*

$$\int dvf_0(\cdot, v) \in \mathscr{S}(\mathbb{R}^d), \tag{74}$$

and $q \in \mathscr{S}(\mathbb{R}^d)$ a non negative function s.t.

$$\int q(x)dx = \Theta > 0.$$

Take a sequence of positive real numbers $\{a_n\}_{n=1}^\infty$ ($a_n > 0$) such that for some $\kappa \in (0, \frac{1}{2})$

$$\lim_{n\to\infty} \frac{a_n^d}{n^\kappa} = 0.$$

Then, given $\phi \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, $\mathscr{P}$-a.s.,

$$\frac{1}{n}\sum_{h=1}^n a_n^d q(a_n x_h)\phi(x_h, v_h) \to \Theta \int_{\mathbb{R}^d} f_0(0, v)\phi(0, v)dv. \tag{75}$$

Proof. We observe first that, since $q \in \mathscr{S}(\mathbb{R}^d)$, we can use the identity

$$q(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ik\cdot x}\hat{q}(k)dk$$

where $\hat{q} \in \mathscr{S}(\mathbb{R}^d)$ is the Fourier transform of q.

Because of $\hat{q} \in \mathscr{S}(\mathbb{R}^d)$ and (74), in all calculations below we can apply Fubini's theorem.

We can then write:

$$\left|\frac{1}{n}\sum_{h=1}^{n} a_n^d q(a_n x_h)\phi(x_h,v_h) - \Theta\int_{\mathbb{R}^d} f_0(0,v)\phi(0,v)dv\right| \leq$$

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left|\hat{q}(\frac{k}{a_n})\right|\left|\frac{1}{n}\sum_{h=1}^{n} e^{ik\cdot x_h}\phi(x_h,v_h) - \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{ik\cdot x} f_0(x,v)\phi(x,v)dxdv\right|dk \tag{76}$$

$$+\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} a_n^d q(a_n x) f_0(x,v)\phi(x,v)dxdv - \Theta\int_{\mathbb{R}^d} f_0(0,v)\phi(0,v)dv\right|.$$

We consider then, for $h = 1,\ldots$, the sequence of independent stochastic variables

$$\nu_h^\phi(k) = e^{ik\cdot x_h}\phi(x_h,v_h) - \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{ik\cdot x} f_0(x,v)\phi(x,v),$$

s.t. $\mathbb{E}_{\mathscr{P}}[\nu_h^\phi(k)] = 0$ and $\mathbb{E}_{\mathscr{P}}[|\nu_h^\phi|^2(k)] \leq 2\|\phi\|_\infty^2$ and for which we have, for any even $j \in \mathbb{N}$,

$$\mathbb{E}_{\mathscr{P}}\left[|\frac{1}{n}\sum_{h=1}^{n}\nu_h^\phi(k)|^j\right] \leq \frac{K_j}{n^{\frac{j}{2}}}(\mathbb{E}_{\mathscr{P}}[|\nu|^2(k)])^{\frac{j}{2}} \leq \frac{K_j}{n^{\frac{j}{2}}}(\sqrt{2}\|\phi\|_\infty)^j. \tag{77}$$

We define then

$$\hat{\omega}_n^\phi = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} dk\left|\hat{q}(\frac{k}{a_n})\right|\left|\frac{1}{n}\sum_{h=1}^{n}\nu_h^\phi(k)\right| I_{\{k:|\frac{1}{n}\sum_{h=1}^{n}\nu_h^\phi(k)|>\frac{C}{n^\kappa}\}} \tag{78}$$

and we get:

$$\left|\frac{1}{n}\sum_{h=1}^{n} a_n^d q(a_n x_h)\phi(x_h,v_h) - \Theta\int_{\mathbb{R}^d} f_0(0,v)\phi(0,v)dv\right| \leq C\frac{a_n^d}{n^\kappa}\|\hat{q}\|_{L^1} + \hat{\omega}_n^\phi$$

$$+\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} a_n^d q(a_n x) f_0(x,v)\phi(x,v)dxdv - \Theta\int_{\mathbb{R}^d} f_0(0,v)\phi(0,v)dv\right| \tag{79}$$

Since, because of the characteristic function in the definition of $\hat{\omega}_n^\phi$,

$$\mathbb{E}_{\mathscr{P}}(\hat{\omega}_n^\phi) \leq \frac{n^{(j-1)\kappa}}{C(2\pi)^d}\int_{\mathbb{R}^d} dk\left|\hat{q}(\frac{k}{a_n})\right|\mathbb{E}_{\mathscr{P}}\left[|\frac{1}{n}\sum_{h=1}^{n}\nu_h^\phi(k)|^j\right],$$

by Tchebycheff inequality and (77), for $\delta > 0$,

$$\mathscr{P}(|\hat{\omega}_n^\phi| > \delta) \leq \frac{\mathbb{E}_{\mathscr{P}}(\hat{\omega}_n^\phi)}{\delta} \leq \frac{K}{\delta}a_n^d n^{(j-1)\kappa-\frac{j}{2}}$$

(with K depending on $\|\phi\|_\infty$ and $\|\hat{q}\|_{L^1}$); we obtain then, for $\kappa \in (0,\frac{1}{2})$ and for all $\delta > 0$, $\sum_n \mathscr{P}(|\hat{\omega}_n^\phi| > \delta) < \infty$, and therefore $\hat{\omega}_n^\phi \overset{\mathscr{P}a.s.}{\to} 0$.

Since $\lim_{n\to\infty}\frac{a_n^d}{n^\kappa} = 0$ and $q_n(x) = a_n^d q(a_n x)$ is such that $q_n \rightharpoonup \Theta\delta_0$, we get finally

$$\frac{1}{n}\sum_{h=1}^{n} a_n^d q(a_n x_h)\phi(x_h,v_h) \overset{\mathscr{P}a.s.}{\to} \Theta\int_{\mathbb{R}^d} f_0(0,v)\phi(0,v)dv.$$

□

Proposition 6. *In the same hypothesis as in Lemma 9, on a full measure set with respect to* $\mathscr{P}$

$$q_n(x)\mu_0^n(x,v) \rightharpoonup \Theta\delta_0(x)f_0(x,v).$$

Proof. We consider first functions in the separable space $C_0(\mathbb{R}^d\times\mathbb{R}^d)$, the set of continuous functions vanishing at infinity. Let D be a countable dense set in $C_0(\mathbb{R}^d\times\mathbb{R}^d)$ and consider the two sets

$$A=\{Z_\infty:\forall\phi\in D\int_{\mathbb{R}^d\times\mathbb{R}^d}dxdvq_n(x)\mu_0^n(x,v)\phi(x,v)\to\Theta\int_{\mathbb{R}^d}dvf_0(0,v)\phi(0,v)\}$$

and

$$B=\{Z_\infty:\int_{\mathbb{R}^d\times\mathbb{R}^d}dxdvq_n(x)\mu_0^n(x,v)\to\Theta\int_{\mathbb{R}^d}dvf_0(0,v)\}.$$

Because of Lemma 9, both sets have full measure, so as their intersection $A\cap B$.

For $Z_\infty\in A\cap B$ the sequence $\{q_n\mu_n^0\}_{n=1}^\infty$ is a sequence of finite positive measures s.t. for all $\phi\in C_0(\mathbb{R}^d\times\mathbb{R}^d)$

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}dxdvq_n\mu_0^n\phi\to\Theta\int_{\mathbb{R}^d}dvf_0(0,v)\phi(0,v)<\infty. \tag{80}$$

Since the convergence in (80) is valid also for $\phi=1$, on the set $A\cap B$, weakly in the sense of measure,

$$q_n(x)\mu_n(x,v)\rightharpoonup\Theta\delta_0(x)f_0(x,v)$$

(see e.g. [MA], p.90, theorem 6.8), and the proposition is proved. □

Remark 8. Since $f_0\in\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)$ and $\int dvf_0\in\mathscr{S}(\mathbb{R}^d)$, the convergence proved in Proposition 6 is valid also for $|v|^jq_n\mu_0^n$, $j=1,2,\dots$ (i.e. $|v|^jq_n\mu_0^n\rightharpoonup|v|^j\delta_0f_0$). To show it, it suffices to rewrite the proof with ν_h^ϕ replaced with $|v|^j\nu_h^\phi$. Under the same hypothesis, it is also possible to prove, by induction, that $\otimes_{k=1}^Mq_n\mu_n\rightharpoonup\otimes_{k=1}^M\delta_0f_0$, for $M=1,2\dots$.

Remark 9. Whenever, in addition to the hypothesis in Lemma 9, we have $f_0\in L^1(\mathbb{R}^d_v;W^{1,\infty}(\mathbb{R}^d_x))$ and $v^2f_0\in L^1(\mathbb{R}^d_v;L^\infty(\mathbb{R}^d_x))$, we can obtain, following the same steps, the weak convergence of $(\mathcal{T}_a^{v,s}q)_n\mu_n^0$ and $|v|(\mathcal{T}_a^{v,s}q)_n\mu_n^0$, where we define $(\mathcal{T}_a^{v,s}g)(x)=g(x+vt+a)$. The convergence is uniform in $a\in\mathbb{R}^d$, as can be easily checked.

Moreover, a very simplified form of the procedure allows to prove that, given a limit density $f_0\in\mathscr{S}(\mathbb{R}^d\times\mathbb{R}^d)$ and two positive integers P,Q, on a full measure set w.r.t. $\mathscr{P}$, $|v|^j\mu_0^n\rightharpoonup|v|^jf_0$ and $\otimes_{k=1}^i|v_k|^j\mu_n\rightharpoonup\otimes_{k=1}^i|v|^jf_0$, for $j=0,\dots,P$ and $i=1,\dots,Q$.

References

[BBS] (MR0725107) C. Boldrighini, L. A. Bunimovic and Ya. G. Sinai. *On the Boltzmann equation for the Lorentz gas*, J. Stat. Phys. **32** (1983), 477-501.

[BGW] (MR1600299) J. Bourgain, F. Golse and B. Wennberg. *On the distribution of free path lengths for the periodic Lorentz gas*, Commun. Math. Phys. **190** (1998), 491-508.

[DR1] (MR2033100) L. Desvillettes and V. Ricci. *Non-Markovianity of the Boltzmann-Grad limit of a system of random obstacles in a given force field*, Bull. Sci. Math. **128** (2004), 39-46.

[DR2] (MR1859001) L. Desvillettes and V. Ricci. *A rigorous derivation of a linear kinetic of the Fokker-Planck type in the limit of grazing collisions*, J. Stat. Phys. **104** (2001).

[Ga] G. Gallavotti. *Rigorous theory of the Boltzmann equation in the Lorentz gas*, Nota interna n. 358, Istituto di Fisica, Università di Roma (1972).

[G1] (MR2050595) F. Golse. *The mean-field limit for the dynamics of large particle systems*, JEDP Forges les Eaux (2003), Exp. No. IX, 47, Univ. Nantes.

[G2] (MR2499853) F. Golse. *On the periodic Lorentz gas and the Lorentz kinetic equation*, Ann. Fac. Sci. Toulouse, Ser.6 **17** (2008), 735–749.

[G3] F. Golse. *Recent results on the periodic Lorentz gas*, preprint, HAL: hal-00390895, (2009), 1–62.

[DL] (MR969367) P. Dautray and J. L. Lions "Mathematical Analysis and Numerical Methods for Science and Technology", v.2, Springer-Verlag, Berlin, 1988.

[MA] (MR1254211) P. Malliavin and H. Airault. "Integration et Analyse de Fourier, Probabilités et Analyse Gaussienne", 2nd edition, Masson, Paris, 1994.

[MM] (MR781346) D. Mihalas and B. Weibel Mihalas. "Foundations of radiation hydrodynamics", Oxford University Press, 1984, Reprint Dover 1999.

[NOR] (MR1003528) G. Nappo, E. Orlandi and H. Rost. *A reaction-diffusion model for moderately interacting particles*, J. Stat. Phys. **55** (1989), 579–600.

[RW] (MR2056540) V. Ricci and B. Wennberg. *On the derivation of a linear Boltzmann equation from a periodic lattice gas*, Stochastic Process. Appl. **111** (2004), 281–315.

[Sch] (MR0209834) L. Schwartz. "Théorie des Distributions", Hermann, Paris, 1966.

[Sp] (MR0496299) H. Spohn. *The Lorentz process converges to a random flight process*, Commun. Math. Phys. **60** (1978), 277–290.

[SDL] C. I. Steefel, D. J. De Paolo and P. C. Lichtner. *Reactive transport modeling: An essential tool and a new research approach for the Earth sciences*, Earth and Planetary Science Letters **240** (2005), 539–558.

[Szn] (MR910949) A. S. Sznitman. *Propagation of chaos for a system of annihilating Brownian spheres*, CPAM **50** (1987), 663–690.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: `Cedric.Bernardin@umpa.ens-lyon.fr`
E-mail address: `valeria.ricci@unipa.it`