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On the critical value function in the divide and color model

András Bálint * Vincent Beffara † Vincent Tassion ‡

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Abstract

The divide and color model on a graph G arises by first deleting each edge of G with probability $1-p$ independently of each other, then coloring the resulting connected components (*i.e.*, every vertex in the component) black or white with respective probabilities r and $1-r$, independently for different components. Viewing it as a (dependent) site percolation model, one can define the critical point $r_c^G(p)$.

In this paper, we first give upper and lower bounds for $r_c^G(p)$ for general G via a stochastic comparison with Bernoulli percolation, and discuss (non-)monotonicity and (non-)continuity properties of $r_c^G(p)$ in p. Then we focus on the case $G = \mathbb{Z}^2$ and prove continuity of $r_c^{\mathbb{Z}^2}(p)$ as a function of p in the interval $[0, 1/2)$, and examine the asymptotic behavior of the critical value function as p tends to its critical value.

Keywords: DaC model, critical value, continuity, monotonicity, stochastic domination

AMS 2000 Subject Classification: 60K35, 82B43, 82B20

Introduction and main results

The divide and color (DaC) model is a natural dependent site percolation model introduced by Häggström in [8]. It has been studied directly in [8, 6, 4, 3], and as a member of a more general family of models in [10, 4, 2, 7].

This model is defined on a multigraph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is a multiset (*i.e.*, it may contain an element more than once), thus allowing parallel edges between pairs of vertices. For simplicity, we will imprecisely call G a *graph* and $\mathcal{E}$ the *edge set*, even if G contains self-loops or multiple edges. The DaC model on a general (finite or infinite) graph G with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, with parameters $p, r \in [0, 1]$, is defined by the following two-step procedure:

- First step: Bernoulli bond percolation. We independently declare each edge in $\mathcal{E}$ to be open with probability p, and closed with probability $1-p$. We can identify a bond percolation configuration with an element $\eta \in \{0,1\}^{\mathcal{E}}$: for all $e \in \mathcal{E}$, we define $\eta(e) = 1$ if e is open, and $\eta(e) = 0$ if e is closed.

- Second step: Bernoulli site percolation on the resulting cluster set. Given $\eta \in \{0,1\}^{\mathcal{E}}$, we call *p-clusters* or *bond clusters* the connected components in the graph with vertex set $\mathcal{V}$ and edge set $\{e \in \mathcal{E} : \eta(e) = 1\}$. Note that the set of p-clusters of η gives a partition of $\mathcal{V}$. For each p-cluster $\mathcal{C}$, we assign the same color to all the vertices in $\mathcal{C}$. The chosen color is

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black with probability r and white with probability $1-r$, and this choice is independent for different p-clusters.

These two steps yield a site percolation configuration $\xi \in \{0,1\}^{\mathcal{V}}$ by defining, for each $v \in \mathcal{V}$, $\xi(v) = 1$ if v is black, and $\xi(v) = 0$ if v is white. The connected components (via the edge set $\mathcal{E}$) in ξ of the same color are called (black or white) *r-clusters*. The resulting measure on $\{0,1\}^{\mathcal{V}}$ is denoted by $\mu^G_{p,r}$.

Let $E^b_\infty \subset \{0,1\}^{\mathcal{V}}$ denote the event that there exists an infinite black r-cluster. By standard arguments (see Proposition 2.5 in [8]), for each $p \in [0,1]$, there exists a *critical coloring value* $r^G_c(p) \in [0,1]$ such that

$$\mu^G_{p,r}(E^b_\infty) \begin{cases} = 0 & \text{if } r < r^G_c(p), \\ > 0 & \text{if } r > r^G_c(p). \end{cases}$$

This value should not be confused with the *critical edge parameter* $p^G_c \in [0,1]$, which is defined as follows: the probability that there exists an infinite bond cluster is 0 for all $p < p^G_c$, and positive for all $p > p^G_c$. It immediately follows from ergodicity that the latter probability is, in fact, 1 for all $p > p^G_c$, whence $r^G_c(p) = 0$ for all such p. Note, however, that although $\mu^G_{p,r}(E^b_\infty) \in \{0,1\}$ for all $p < p^G_c$, there exist graphs (such as the square lattice, defined below) with $\mu^G_{p,r}(E^b_\infty) \in (0,1)$ for some $r > r^G_c(p)$.

Our main goal in this paper is to understand how the critical coloring parameter r^G_c depends on the edge parameter p. Since the addition or removal of self-loops obviously does not affect the value of $r^G_c(p)$, we will assume that all the graphs G that we consider in this paper are without self-loops. On the other hand, G is allowed to contain multiple edges.

Our first result gives bounds on $r^G_c(p)$ in terms of $r^G_c(0)$, which is simply the critical value for Bernoulli site percolation on G. By the *degree* of a vertex v, we mean the number of edges incident on v (counted with multiplicity).

Proposition 1. *For any graph G with maximal degree Δ, it holds for all $p \in [0,1)$ that*

$$1 - \frac{1 - r^G_c(0)}{(1-p)^\Delta} \le r^G_c(p) \le \frac{r^G_c(0)}{(1-p)^\Delta}.$$

Proposition 1 will be a simple corollary of the much more general Theorem 11, to which we will return in Section 2.

An important special case arises when we take G to be the *square lattice*, whose vertex set is $\mathbb{Z}^2$, and edge set $\mathcal{E}^2$ is given by the edges between elements of $\mathbb{Z}^2$ at Euclidean distance 1 from each other. With an abuse of notation, we denote this graph by $\mathbb{Z}^2$. It has been shown in [4, 8] that for all $p > 1/2$, we have $r^{\mathbb{Z}^2}_c(p) = 0$, that $r^{\mathbb{Z}^2}_c(1/2) = 1$, and that for $p < 1/2$, $r^{\mathbb{Z}^2}_c(p) \in [1/2, 1)$. In the interval $p \in [0, 1/2)$, the exact value of $r^{\mathbb{Z}^2}_c(p)$ is unknown.

Recall that $r^{\mathbb{Z}^2}_c(0)$ is the critical value for Bernoulli site percolation on the square lattice. It is generally believed that $r^{\mathbb{Z}^2}_c(0) \approx 0.593$, but the best rigorous lower bound known to date, given in [5], is 0.556, and the best rigorous upper bound is 0.679, see [17]. The previous proposition, combined with these bounds on $r^{\mathbb{Z}^2}_c(0)$, gives rigorous bounds on $r^{\mathbb{Z}^2}_c(p)$.

Corollary 2. *For all $p \in [0,1)$,*

$$1 - \frac{0.444}{(1-p)^4} \le r^{\mathbb{Z}^2}_c(p) \le \frac{0.679}{(1-p)^4}.$$

Note that these bounds are new only for small values of p. In particular, the upper bound is nontrivial only for $p \leq 1-(0.679)^{1/4} \approx 0.092$. Since $r_c^{\mathbb{Z}^2}(p) \geq 1/2$ is known, the lower bound is interesting only where it implies $r_c^{\mathbb{Z}^2}(p) > 1/2$. This is the case for $p < 1-(0.888)^{1/4} \approx 0.029$.

Our other main results are concerned with the monotonicity and continuity properties of the critical value function. Proposition 2.7 in [8] (or the more general Remark 16 in this paper) implies that when G is a tree, $r_c^G(p)$ is non-increasing on $[0,1]$, and continuous on $[0,1)$. Continuity in 1 fails, for example, for $G = \mathbb{Z}$ (*i.e.*, the infinite regular tree of degree 2), for which $r_c^G(p) = 1$ for all $p < 1$, but $r_c^G(1) = 0$. A less trivial example for discontinuous r_c^G is $G = \mathbb{Z}^2$, where the above mentioned results in [4, 8] imply that the function $r_c^{\mathbb{Z}^2}(p)$ has a discontinuity at $p = p_c^{\mathbb{Z}^2} = 1/2$, and also that monotonicity fails since $r_c^{\mathbb{Z}^2}(1/2) > r_c^{\mathbb{Z}^2}(p)$ for all $p \neq 1/2$. Taking $p = 1/2$ is "cheating" in that $1/2$ is precisely the critical value for Bernoulli bond percolation on $\mathbb{Z}^2$, and the structure of bond clusters at $1/2$ is fundamentally different from those for $p < 1/2$. The relevant questions, therefore, are whether monotonicity and continuity hold for $p < p_c^G$.

The problem of continuity is solved at $p = 0$ when the graph G has bounded degree:

Proposition 3. *For any graph G with bounded degree, $r_c^G(p)$ is continuous in p at* 0.

The above result is a simple consequence of Proposition 1, which in turn relies on a stochastic domination argument. Since bond clusters can be arbitrarily large for $p > 0$, we cannot extend this argument to the interval $[0, p_c^G)$. In fact, we will show that Proposition 3 is sharp both in that there exists a graph of unbounded degree whose critical value function is discontinuous at $0 < p_c^G$, and also that continuity may fail for bounded degree graphs at any other place than 0 (but, with the exception of 1, still in the subcritical regime).

Proposition 4. *There exists a graph G with $p_c^G > 0$ such that r_c^G is discontinuous at* 0.

Theorem 5. *For all $p_0 \in (0,1)$, there exists a graph G of bounded degree such that $r_c^G(p)$ is discontinuous at $p_0 < p_c^G$.*

The proofs of these and all further results in this section are given in Section 4.

The construction that we will use to prove Theorem 5 is admittedly somewhat artificial, and the question remains whether there exist more regular, such as transitive or quasi-transitive, graphs with a discontinuity of r_c^G below p_c^G. For the definition of (quasi-)transitivity, see, for example, Definition 1.2 in [9].

Open question 6. *Is $r_c^G(p)$ continuous in p on $[0, p_c^G)$ for every quasi-transitive graph G?*

Our next result shows that not only near-trivial reasons (such as the essentially different structure of bond clusters in different phases) can cause non-monotonicity of the critical value function, moreover, that such a phenomenon may occur on a "nice" graph as well.

Proposition 7. *There exists a quasi-transitive graph G such that r_c^G is not monotone on the interval $[0, p_c^G)$.*

The graph that we define in the proof of Proposition 7 is quasi-transitive, but not transitive. In fact, all transitive graphs whose critical value functions were studied so far have been proved or conjectured to possess the monotonicity property.

Open question 8. *Is $r_c^G(p)$ a monotone function of p for $p \in [0, p_c^G)$ for all transitive graphs G?*

Note that strict monotonicity cannot be expected for all transitive graphs as it fails for the triangular lattice $\mathbb{T}$, where $r_c^{\mathbb{T}}(p) = 1/2$ for all $p < p_c^{\mathbb{T}}$ (see [4]). Our feeling is, however, that Open question 8 can be answered in the affirmative.

Now we turn our attention from general graphs to the square lattice, whose critical value function, as mentioned above, is not monotone on $[0,1]$ and has a discontinuity at $1/2$. Our next result, Theorem 9 below, implies that this is the only discontinuity point of $r_c^{\mathbb{Z}^2}(p)$. The proof of this theorem will be given in Section 4.3.

Theorem 9. *The critical coloring value $r_c^{\mathbb{Z}^2}(p)$ is a continuous function of p on the interval $p \in [0, 1/2)$.*

Unfortunately, we have not yet been able to prove (or disprove) monotonicity of $r_c^{\mathbb{Z}^2}(p)$ for subcritical p. However, our numerical experiments suggest (see Conjecture 22) that $r_c^{\mathbb{Z}^2}(p)$ is strictly decreasing on the interval $[0, 1/2)$.

A brief outline of the paper is as follows. We set the notation and collect a few results from the literature in Section 1. In Section 2, we stochastically compare $\mu_{p,r}^G$ with Bernoulli site percolation in Theorem 11, and show how this result implies Proposition 1. We determine the critical value function for a class of tree-like graphs in Section 3, and use these results for proving several of our main results in Sections 4.1 and 4.2. We then prove Theorem 9 in Section 4.3 by a finite-size argument. Finally, Section 5 deals with the asymptotic behavior of $r_c^G(p)$ as p tends to p_c^G.

1 Definitions and notation

We consider a graph $G = (\mathcal{V}, \mathcal{E})$ where the set of vertices $\mathcal{V} = \{v_0, v_1, v_2, \ldots\}$ is countable. We define a total order "$<$" on $\mathcal{V}$ by saying that $v_i < v_j$ if and only if $i < j$. In this way, for any subset $V \subset \mathcal{V}$, we can uniquely define $\min(V) \in V$ as the minimal vertex in V with respect to the relation "$<$". For a set S, we denote $\{0,1\}^S$ by Ω_S. We call the elements of $\Omega_{\mathcal{E}}$ *bond configurations*, and the elements of $\Omega_{\mathcal{V}}$ *site configurations*. As defined in the Introduction, in a bond configuration η, an edge $e \in \mathcal{E}$ is called *open* if $\eta(e) = 1$, and *closed* otherwise; in a site configuration ξ, a vertex $v \in \mathcal{V}$ is called *black* if $\xi(e) = 1$, and *white* otherwise. Finally, for $\eta \in \Omega_{\mathcal{E}}$ and $v \in \mathcal{V}$, we define the *bond cluster* $\mathcal{C}_v(\eta)$ of v as the maximal connected induced subgraph containing v of the graph with vertex set $\mathcal{V}$ and edge set $\{e \in \mathcal{E} : \eta(e) = 1\}$, and denote the vertex set of $\mathcal{C}_v(\eta)$ by $C_v(\eta)$.

For $a \in [0,1]$ and a set S, we define ν_a^S as the probability measure on Ω_S that assigns to each $s \in S$ value 1 with probability a and 0 with probability $1-a$, independently for different elements of S. We define a function

$$\begin{array}{rccc} \Phi : & \Omega_{\mathcal{E}} \times \Omega_{\mathcal{V}} & \to & \Omega_{\mathcal{E}} \times \Omega_{\mathcal{V}}, \\ & (\eta, \kappa) & \mapsto & (\eta, \xi), \end{array}$$

where $\xi(v) = \kappa(\min(C_v(\eta)))$. For $p, r \in [0,1]$, we define $\mathbb{P}_{p,r}^G$ to be the image measure of $\nu_p^{\mathcal{E}} \otimes \nu_r^{\mathcal{V}}$ by the function Φ, and denote by $\mu_{p,r}^G$ the marginal of $\mathbb{P}_{p,r}^G$ on $\Omega_{\mathcal{V}}$. Note that this definition of $\mu_{p,r}^G$ is consistent with the one in the Introduction.

Finally, we give a few definitions and results that are necessary for the analysis of the DaC model on the graph $\mathbb{Z}^2$. The *matching graph* $\mathbb{Z}_*^2$ of the square lattice is the graph with vertex set $\mathbb{Z}^2$ and edge set $\mathcal{E}_*^2 = \{\langle v, w\rangle : v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{Z}^2,\ \max(|v_1 - w_1|, |v_2 - w_2|) = 1\}$. In the same manner as in the Introduction, we define, for a color configuration $\xi \in \{0,1\}^{\mathbb{Z}^2}$, (black or white) $*$-*clusters* as connected components (via the edge set $\mathcal{E}_*^2$) in ξ of the same color. We denote by $\Theta^*(p,r)$ the $\mathbb{P}_{p,r}^{\mathbb{Z}^2}$-probability that the origin is contained in an infinite black $*$-cluster, and define

$$r_c^*(p) = \sup\{r : \Theta^*(p,r) = 0\}$$

for all $p \in [0,1]$. (Note that this value may differ from $r_{c^*}^{\mathbb{Z}^2}(p)$!) The main result in [4] is that for all $p \in [0,1/2)$, the critical values $r_c^{\mathbb{Z}^2}(p)$ and $r_c^*(p)$ satisfy the duality relation

$$r_c^{\mathbb{Z}^2}(p) + r_c^*(p) = 1. \tag{1}$$

We will also use the famous exponential decay result for subcritical Bernoulli bond percolation on $\mathbb{Z}^2$. Let $\mathbf{0}$ denote the origin in $\mathbb{Z}^2$, and for each $n \in \mathbb{N} = \{1, 2, \ldots\}$, let us define $S_n = \{v \in \mathbb{Z}^2 : dist(v, \mathbf{0}) = n\}$ (where *dist* denotes graph distance), and the event $M_n = \{\eta \in \Omega_{\mathcal{E}^2} :$ there is a path of open edges in η from $\mathbf{0}$ to $S_n\}$. Then we have the following result:

Theorem 10 ([13, 1])**.** *For $p < 1/2$, there exists $\psi(p) > 0$ such that for all $n \in \mathbb{N}$, we have that*

$$\nu_p^{\mathcal{E}^2}(M_n) < e^{-n\psi(p)}.$$

2 Stochastic domination

In this section, we will prove Proposition 1 via a stochastic comparison between the DaC measure and Bernoulli site percolation. Before stating the corresponding result, however, let us recall the concept of stochastic domination.

We define a natural partial order on $\Omega_{\mathcal{V}}$ by saying that $\xi' \geq \xi$ for $\xi, \xi' \in \Omega_{\mathcal{V}}$ if, for all $v \in \mathcal{V}$, $\xi'(v) \geq \xi(v)$. A random variable $f : \Omega_{\mathcal{V}} \to \mathbb{R}$ is called *increasing* if $\xi' \geq \xi$ implies that $f(\xi') \geq f(\xi)$, and an event $E \subset \Omega_{\mathcal{V}}$ is increasing if its indicator random variable is increasing. For probability measures μ, μ' on $\Omega_{\mathcal{V}}$, we say that μ' is *stochastically larger* than μ (or, equivalently, that μ is *stochastically smaller* than μ', denoted by $\mu \leq_{\text{st}} \mu'$) if, for all bounded increasing random variables $f : \Omega_{\mathcal{V}} \to \mathbb{R}$, we have that

$$\int_{\Omega_{\mathcal{V}}} f(\xi)\ d\mu'(\xi) \geq \int_{\Omega_{\mathcal{V}}} f(\xi)\ d\mu(\xi).$$

By Strassen's theorem [16], this is equivalent to the existence of an appropriate coupling of the measures μ' and μ; that is, the existence of a probability measure $\mathbb{Q}$ on $\Omega_{\mathcal{V}} \times \Omega_{\mathcal{V}}$ such that the marginals of $\mathbb{Q}$ on the first and second coordinates are μ' and μ respectively, and $\mathbb{Q}(\{(\xi', \xi) \in \Omega_{\mathcal{V}} \times \Omega_{\mathcal{V}} : \xi' \geq \xi\}) = 1$.

Theorem 11. *For any graph $G = (\mathcal{V}, \mathcal{E})$ whose maximal degree is Δ, at arbitrary values of the parameters $p, r \in [0,1]$,*

$$\nu^{\mathcal{V}}_{r(1-p)^\Delta} \leq_{st} \mu^G_{p,r} \leq_{st} \nu^{\mathcal{V}}_{1-(1-r)(1-p)^\Delta}.$$

Before turning to the proof, we show how Theorem 11 implies Proposition 1.

Proof of Proposition 1. It follows from Theorem 11 and the definition of stochastic domination that for the increasing event E^b_∞ (which was defined in the Introduction), we have $\mu^G_{p,r}(E^b_\infty) > 0$ whenever $r(1-p)^\Delta > r_c^G(0)$, which implies that $r_c^G(p) \leq r_c^G(0)/(1-p)^\Delta$. The derivation of the lower bound for $r_c^G(p)$ is analogous. □

Now we give the proof of Theorem 11, which bears some resemblance with the proof of Theorem 2.3 in [8].

Proof of Theorem 11. Fix $G = (\mathcal{V}, \mathcal{E})$ with maximal degree Δ, and parameter values $p, r \in [0,1]$. We will use the relation "$<$" and the minimum of a vertex set with respect to this relation as defined in Section 1. In what follows, we will define several random variables; we will denote the joint distribution of all these variables by $\mathbb{P}$.

First, we define a collection $(\eta^e_{x,y} : x, y \in \mathcal{V}, e = \langle x, y\rangle \in \mathcal{E})$ of i.i.d. Bernoulli(p) random variables (*i.e.*, they take value 1 with probability p, and 0 otherwise); one may imagine having each edge $e \in \mathcal{E}$ replaced by two directed edges, and the random variables represent which of these edges are open. We define also a set $(\kappa_x : x \in \mathcal{V})$ of Bernoulli(r) random variables. Given a realization of $(\eta^e_{x,y} : x, y \in \mathcal{V}, e = \langle x, y\rangle \in \mathcal{E})$ and $(\kappa_x : x \in \mathcal{V})$, we will define an $\Omega_\mathcal{V} \times \Omega_\mathcal{E}$-valued random configuration (η, ξ) with distribution $\mathbb{P}^G_{p,r}$, by the following algorithm.

1. Let $v = \min\{x \in \mathcal{V} : \text{no } \xi\text{-value has been assigned yet to } x \text{ by this algorithm}\}$. (Note that v and V, v_i, H_i ($i \in \mathbb{N}$), defined below, are running variables, *i.e.*, their values will be redefined in the course of the algorithm.)

2. We explore the "directed open cluster" V of v iteratively, as follows. Define $v_0 = v$. Given $v_0, v_1, \ldots, v_i$ for some integer $i \geq 0$, set $\eta(e) = \eta^e_{v_i,w}$ for every edge $e = \langle v_i, w\rangle \in \mathcal{E}$ incident to v_i such that no η-value has been assigned yet to e by the algorithm, and write $H_{i+1} = \{w \in \mathcal{V} \setminus \{v_0, v_1, \ldots, v_i\} : w \text{ can be reached from any of } v_0, v_1, \ldots, v_i \text{ by using only those edges } e \in \mathcal{E} \text{ such that } \eta(e) = 1 \text{ has been assigned to } e \text{ by this algorithm}\}$. If $H_{i+1} \neq \emptyset$, then we define $v_{i+1} = \min(H_{i+1})$, and continue exploring the directed open cluster of v; otherwise, we define $V = \{v_0, v_1, \ldots, v_i\}$, and move to step 3.

3. Define $\xi(w) = \kappa_v$ for all $w \in V$, and return to step 1.

It is immediately clear that the above algorithm eventually assigns a ξ-value to each vertex. Note also that a vertex v can receive a ξ-value only after all edges incident to v have already been assigned an η-value, which shows that the algorithm eventually determines the full edge configuration as well. It is easy to convince oneself that (η, ξ) obtained this way indeed has the desired distribution.

Now, for each $v \in \mathcal{V}$, we define $Z(v) = 1$ if $\kappa_v = 1$ and $\eta^e_{w,v} = 0$ for all edges $e = \langle v, w\rangle \in \mathcal{E}$ incident on v (*i.e.*, all directed edges towards v are closed), and $Z(v) = 0$ otherwise. Note that every vertex with $Z(v) = 1$ has $\xi(v) = 1$ as well, whence the distribution of ξ (*i.e.*, $\mu^G_{p,r}$) stochastically dominates the distribution of Z (as witnessed by the coupling $\mathbb{P}$).

Since the distribution of Z is a product measure on $\Omega_\mathcal{V}$ with parameter $r(1-p)^{d(v)}$ at v, where $d(v) \leq \Delta$ is the degree of v, we conclude that $\mu^G_{p,r}$ stochastically dominates the product measure on $\Omega_\mathcal{V}$ with parameter $r(1-p)^\Delta$, which gives the desired stochastic lower bound. The upper bound can be proved analogously; alternatively, it follows from the lower bound by exchanging the roles of black and white. $\square$

Remark 12. *The same stochastic bounds hold for any DaC(q) measure $\mu^G_{p,q,r}$ (which is obtained by replacing Bernoulli bond percolation in the definition of the DaC model with a so-called random-cluster measure $\Phi^G_{p,q}$ with parameters p and q) with $q \geq 1$, while one needs to replace p in both bounds by $p/(p + (1-p)q)$ in case of $q < 1$ (assuming that the DaC(q) model on G exists in the latter case). Of course, these stochastic bounds also imply deterministic bounds for the critical coloring value in terms of $r^G_c(0)$.*

Sketch of proof. This is just a straightforward generalization of the above proof. We define an $\Omega_\mathcal{V} \times \Omega_\mathcal{E}$-valued random configuration (η, ξ) by an algorithm as above, with two differences. The first one is that here we define $(\eta^e_{x,y} : x, y \in \mathcal{V}, e = \langle x, y\rangle \in \mathcal{E})$ to be i.i.d. random variables

with uniform distribution on the interval $[0,1]$. Secondly, in step 2, we fix η for those edges e incident to v_i that are without an η-value one at a time, taking $\eta(e) = 0$ for $e = \langle v_i, w\rangle \in \mathcal{E}$ if and only if $\eta^e_{v_i,w} < \phi$, where ϕ is the conditional probability of e being closed under the conditional distribution $\Phi^G_{p,q}$ given the state of those edges whose η-value has been determined before; otherwise, we take $\eta(e) = 1$. This indeed gives the desired distribution: since the algorithm assigns ξ-values only after the whole bond cluster of the vertices in question has been determined, the ξ-values do not affect the further edge distribution.

Since the $\Phi^G_{p,q}$-probability of an edge e being closed given the state of every other edge is either $1-p$ or $1-p/(p+(1-p)q)$ (see, e.g., [2]), it follows that $\phi \geq b$ with $b = 1-p$ if $q \geq 1$ and $b = 1-p/(p+(1-p)q)$ if $q < 1$. Now define, for each vertex v, $Z(v) = 1$ if $\kappa_v = 1$ and $\eta^e_{w,v} < b$ for all edges $e = \langle v, w\rangle \in \mathcal{E}$ incident on v, and $Z(v) = 0$ otherwise. Upon noting that the distribution of Z is a product measure on $\Omega_{\mathcal{V}}$ with parameter $rb^{d(v)}$ at v and that $\eta^e_{w,v} < b$ implies that the corresponding directed edge from w to v is closed, the rest of the proof goes as above. $\square$

3 The critical value functions of tree-like graphs

In this section, we will study the critical value functions of graphs that are constructed by replacing edges of an infinite tree by a sequence of finite graphs. We will then use several such constructions in the proofs of our main results in Section 4.

Let us fix an arbitrary sequence $D_n = (\mathcal{V}_n, \mathcal{E}_n)$ of finite connected graphs and, for every $n \in \mathbb{N}$, two distinct vertices $a_n, b_n \in \mathcal{V}_n$. Let $\Gamma_3 = (V_3, E_3)$ denote the (infinite) regular tree of degree 3, and fix an arbitrary vertex $\rho \in V_3$. Then, for each edge $e \in E_3$, we denote the end-vertex of e which is closer to ρ by $f(e)$, and the other end-vertex by $s(e)$. Let $\Gamma_D = (\tilde{V}, \tilde{E})$ be the graph obtained by replacing every edge e of Γ_3 between levels $n-1$ and n (*i.e.*, such that $dist(s(e), \rho) = n$) by a copy D_e of D_n, with a_n and b_n replacing respectively $f(e)$ and $s(e)$. Each vertex $v \in V_3$ is replaced by a new vertex in $\tilde{V}$, which we denote by $\tilde{v}$. It is well known that $p_c^{\Gamma_3} = r_c^{\Gamma_3}(0) = 1/2$. Using this fact and the tree-like structure of Γ_D, we will be able to determine bounds for $p_c^{\Gamma_D}$ and $r_c^{\Gamma_D}(p)$.

First, we define $h^{D_n}(p) = \nu_p^{\mathcal{E}_n}(a_n$ and b_n are in the same bond cluster), and prove the following, intuitively clear, lemma.

Lemma 13. *For any $p \in [0,1]$, the following implications hold:*

a) if $\limsup_{n\to\infty} h^{D_n}(p) < 1/2$, then $p \leq p_c^{\Gamma_D}$;

b) if $\liminf_{n\to\infty} h^{D_n}(p) > 1/2$, then $p \geq p_c^{\Gamma_D}$.

Proof. We couple Bernoulli bond percolation with parameter p on Γ_D with inhomogeneous Bernoulli bond percolation with parameters $h^{D_n}(p)$ on Γ_3, as follows. Let η be a random variable with law $\nu_p^{\tilde{E}}$, and define, for each edge $e \in E_3$, $W(e) = 1$ if $\tilde{f(e)}$ and $\tilde{s(e)}$ are connected by a path consisting of edges that are open in η, and $W(e) = 0$ otherwise. The tree-like structure of Γ_D implies that $W(e)$ depends only on the state of the edges in D_e, and it is clear that if $dist(s(e), \rho) = n$, then $W(e) = 1$ with probability $h^{D_n}(p)$.

It is easy to verify that there exists an infinite open self-avoiding path on Γ_D from $\tilde{\rho}$ in the configuration η if and only if there exists an infinite open self-avoiding path on Γ_3 from ρ in the configuration W. Now, if we assume $\limsup_{n\to\infty} h^{D_n}(p) < 1/2$, then there exists $t < 1/2$ and $N \in \mathbb{N}$ such that for all $n \geq N$, $h^{D_n}(p) \leq t$. Therefore, the distribution of the restriction of W on $L = \{e \in E_3 : dist(s(e), \rho) \geq N\}$ is stochastically dominated by the projection of $\nu_t^{E_3}$ on L.

This implies that, a.s., there exists no infinite self-avoiding path in W, whence $p \le p_c^{\Gamma_D}$ by the observation at the beginning of this paragraph. The proof of b) is analogous. $\square$

We now turn to the DaC model on Γ_D. Recall that for a vertex v, C_v denotes the vertex set of the bond cluster of v. Let $E_{a_n,b_n} \subset \Omega_{\mathcal{E}_n} \times \Omega_{\mathcal{V}_n}$ denote the event that a_n and b_n are in the same bond cluster, or a_n and b_n lie in two different bond clusters, but there exists a vertex v at distance 1 from C_{a_n} which is connected to b_n by a black path (which also includes that $\xi(v) = \xi(b_n) = 1$). We note that this is the same as saying that C_{a_n} is *pivotal* for the event that there is a black path between a_n and b_n, *i.e.*, that such a path exists if and only if C_{a_n} is black. It is important to note that E_{a_n,b_n} is independent of the color of a_n. Define $f^{D_n}(p,r) = \mathbb{P}_{p,r}^{D_n}(E_{a_n,b_n})$, and note also that, for $r > 0$, $f^{D_n}(p,r) = \mathbb{P}_{p,r}^{D_n}(\text{there is a black path from } a_n \text{ to } b_n \mid \xi(a_n) = 1)$.

Lemma 14. *For any $p, r \in [0,1]$, we have the following:*

a) if $\limsup_{n\to\infty} f^{D_n}(p,r) < 1/2$, then $r \le r_c^{\Gamma_D}(p)$;

b) if $\liminf_{n\to\infty} f^{D_n}(p,r) > 1/2$, then $r \ge r_c^{\Gamma_D}(p)$.

Proof. We couple here the DaC model on Γ_D with inhomogeneous Bernoulli *site* percolation on Γ_3. For each $v \in V_3 \setminus \{\rho\}$, there is a unique edge $e \in E_3$ such that $v = s(e)$. Here we denote D_e (*i.e.*, the subgraph of Γ_D replacing the edge e) by $D_{\tilde{v}}$, and the analogous event of E_{a_n,b_n} for the graph $D_{\tilde{v}}$ by $E_{\tilde{v}}$. Let (η, ξ) with values in $\Omega_{\tilde{E}} \times \Omega_{\tilde{V}}$ be a random variable with law $\mathbb{P}_{p,r}^{\Gamma_D}$. We define a random variable X with values in Ω_{V_3}, as follows:

$$X(v) = \begin{cases} \xi(\tilde{\rho}) & \text{if } v = \rho, \\ 1 & \text{if the event } E_{\tilde{v}} \text{ is realized by the restriction of } (\eta,\xi) \text{ to } D_{\tilde{v}}, \\ 0 & \text{otherwise.} \end{cases}$$

As noted after the proof of Lemma 13, if $u = f(\langle u, v\rangle)$, the event $E_{\tilde{v}}$ is independent of the color of $\tilde{u}$, whence $(E_{\tilde{v}})_{v \in V_3 \setminus \{\rho\}}$ are independent. Therefore, as $X(\rho) = 1$ with probability r, and $X(v) = 1$ is realized with probability $f^{D_n}(p,r)$ for $v \in V_3$ with $dist(v,\rho) = n$ for some $n \in \mathbb{N}$, X is inhomogeneous Bernoulli site percolation on Γ_3.

Our reason for defining X is the following property: it holds for all $v \in V_3 \setminus \{\rho\}$ that

$$\tilde{\rho} \overset{\xi}{\leftrightarrow} \tilde{v} \quad \text{if and only if} \quad \rho \overset{X}{\leftrightarrow} v, \tag{2}$$

where $x \overset{Z}{\leftrightarrow} y$ denotes that x and y are in the same *black* cluster in the configuration Z. Indeed, assuming $\tilde{\rho} \overset{\xi}{\leftrightarrow} \tilde{v}$, there exists a path $\rho = x_0, x_1, \cdots, x_k = v$ in Γ_3 such that, for all $0 \le i < k$, $\tilde{x_i} \overset{\xi}{\leftrightarrow} \tilde{x_{i+1}}$ holds. This implies that $\xi(\tilde{\rho}) = 1$ and that all the events $(E_{\tilde{x_i}})_{0<i\le k}$ occur, whence $X(x_i) = 1$ for $i = 0, \ldots, k$, so $\rho \overset{X}{\leftrightarrow} v$ is realized. The proof of the other implication is similar. It follows in particular from (2) that $\tilde{\rho}$ lies in an infinite black cluster in the configuration ξ if and only if ρ lies in an infinite black cluster in the configuration X.

Lemma 14 presents two scenarios when it is easy to determine (via a stochastic comparison) whether the latter event has positive probability. Assume, for example, that $\liminf_{n\to\infty} f^{D_n}(p,r) > 1/2$, whence there exists $t > 1/2$ and $N \in \mathbb{N}$ such that for all $n \ge N$, $f^{D_n}(p,r) \ge t$. In this case, the distribution of the restriction of X on $K = \{v \in V_3 : dist(v,\rho) \ge N\}$ is stochastically larger than the projection of $\nu_t^{E_3}$ on K. Let us further assume that $r > 0$. In that case, $X(\rho) = 1$ with positive probability, and $f^{D_n}(p,r) > 0$ for every $n \in \mathbb{N}$. Therefore, under the assumptions $\liminf_{n\to\infty} f^{D_n}(p,r) > 1/2$ and $r > 0$, ρ is in an infinite black cluster in X (and, hence, $\tilde{\rho}$ is in

an infinite black cluster in ξ) with positive probability, which can only happen if $r \geq r_c^{\Gamma_D}(p)$. On the other hand, if $\liminf_{n\to\infty} f^{D_n}(p,0) > 1/2$, then it is clear that $\liminf_{n\to\infty} f^{D_n}(p,r) > 1/2$ (whence $r \geq r_c^{\Gamma_D}(p)$) for all $r > 0$, which implies that $r_c^{\Gamma_D}(p) = 0$. The proof of part a) is similar. □

When $D_n \equiv D$ for some finite connected graph $D = (V, E)$, the previous lemmas determine the critical p- and r-values, as follows. Since D is finite and connected, $h^D(p)$ is a polynomial in p with $h^D(0) = 0$ and $h^D(1) = 1$, and it is strictly increasing on $[0, 1]$. We can therefore define $p_D \in (0, 1)$ as the unique value such that $h^D(p_D) = 1/2$. Also, for fixed $p \in [0, 1]$, $f^D(p, 0) = h^D(p)$ and $f^D(p, 1) = 1$, and it is also clear that, for any fixed $p < 1$, $f^D(p, .)$ is strictly increasing and continuous on $[0, 1]$. Hence, for $p < p_D$, we can define $r_D(p) \in (0, 1)$ to be the unique value such that $f^D(p, r_D(p)) = 1/2$. For $p \geq p_D$, we define $r_D(p) = 0$, and note that $f^D(p, r) > 1/2$ for all $r > 0$.

Lemma 15. *If $D_n \equiv D$, then $p_c^{\Gamma_D} = p_D$, and for all $p \in [0, 1]$, $r_c^{\Gamma_D}(p) = r_D(p)$.*

Proof. These statements immediately follow from Lemmas 13 and 14 upon noting that $\lim_{n\to\infty} h^{D_n}(p) = h^D(p)$, and similarly with f^{D_n}. □

Remark 16. *A trivial extension of the above proofs gives that if Γ_3 is replaced in the definition of Γ_D by an arbitrary tree Γ, then analogous statements to those in Lemmas 13–15 hold with $1/2$ replaced everywhere by $1/br(\Gamma)$, where $br(\Gamma)$ is the* branching number *of Γ (see [12] for the definition). Furthermore, if Γ_D is constructed by taking an arbitrary enumeration $\{e_1, e_2, \ldots\}$ of the edge set of Γ and replacing e_i by D_i for every $i \in \mathbb{N}$, the same implications still hold.*

4 Proofs of the main results

We will first apply the machinery developed in Section 3 for proving some of our main results in Sections 4.1 and 4.2. In these sections, we will freely use the notations introduced in Section 3). We then turn to the proof of Theorem 9 in Section 4.3.

4.1 Non-monotonicity

The results in Section 3 enable us to prove that (a small modification of) the construction considered by Häggström in the proof of Theorem 2.9 in [8] is a graph whose critical coloring value is non-monotone in the subcritical phase.

Proof of Proposition 7. We will apply Lemma 15 with $D_n \equiv D = D^k$, which is defined, for $k \in \mathbb{N}$, to be the complete bipartite graph with the vertex set partitioned into $\{z_1, z_2\}$ and $\{a, b, v_1, v_2, \ldots, v_k\}$ (see Figure 1). We call e_1, e_1' and e_2, e_2' the edges incident to a and b respectively, and for $i = 1, \ldots, k$, f_i, f_i' the edges incident to v_i. For each $k \in \mathbb{N}$, we denote the graph constructed with D^k by Γ_{D^k}. The resulting graphs, as is always the case when D_n is the same for all n, are clearly quasi-transitive.

We will show below that it holds for all $k \in \mathbb{N}$ that

$$p_{D^k} > 1/3, \tag{3}$$

$$r_{D^k}(0) < 2/3, \quad \text{and} \tag{4}$$

$$r_{D^k}(1/3) < 2/3. \tag{5}$$

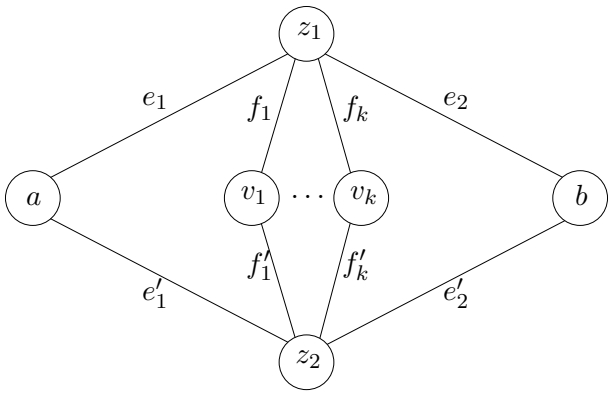

Figure 1: The graph D^k.

Furthermore, there exists $k \in \mathbb{N}$ and $p_0 \in (0, 1/3)$ such that

$$r_{D^k}(p_0) > 2/3. \tag{6}$$

Proving (3)–(6) will finish the proof of Proposition 7 since these inequalities imply by Lemma 15 that, with $k \in \mathbb{N}$ as in (6), the quasi-transitive $G = \Gamma_{D^k}$ is a graph with a non-monotone critical value function in the subcritical regime.

Throughout this proof, we will omit superscripts in the notation when no confusion is possible. For the proof of (3), recall that h^{D^k} is strictly increasing in p, and $h^{D^k}(p_{D^k}) = 1/2$. Since $1 - h^{D^k}(p)$ is the ν_p-probability of a and b being in two different bond clusters, we have that

$$1 - h^{D^k}(1/3) \geq \nu_{1/3}(\{e_1 \text{ and } e_1' \text{ are closed}\} \cup \{e_2 \text{ and } e_2' \text{ are closed}\}).$$

From this, we get that $h^{D^k}(1/3) \leq 25/81$, which proves (3).

To get (4), we need to remember that for fixed $p < p_{D^k}$, $f^{D^k}(p, r)$ is strictly increasing in r, and $f^{D^k}(p, r_{D^k}(p)) = 1/2$. One then easily computes that $f(0, 2/3) = 16/27$, whence (4) follows.

Now, define A to be the event that at least one edge out of e_1, e_1', e_2 and e_2' is open. Then

$$\begin{aligned} f^{D^k}(1/3, 2/3) &\geq \mathbb{P}_{1/3,2/3}(E_{a,b} \mid A)\mathbb{P}_{1/3,2/3}(A) \\ &\geq \mathbb{P}_{1/3,2/3}(C_b \text{ black} \mid A) \cdot 65/81, \end{aligned}$$

which gives that $f^{D^k}(1/3, 2/3) \geq 130/243$, and implies (5).

To prove (6), we consider B_k to be the event that e_1, e_1', e_2 and e_2' are all closed and that there exists i such that f_i and f_i' are both open. One can easily compute that

$$\mathbb{P}_{p,r}(B_k) = (1-p)^4 \left(1 - (1-p^2)^k\right),$$

which implies that we can choose $p_0 \in (0, 1/3)$ (small) and $k \in \mathbb{N}$ (large) such that $\mathbb{P}_{p_0,r}(B_k) > 17/18$. Then,

$$\begin{aligned} f^{D^k}(p_0, 2/3) &= \mathbb{P}_{p_0,r}(E_{a,b} \mid B_k)\mathbb{P}_{p_0,r}(B_k) + \mathbb{P}_{p_0,r}(E_{a,b} \mid B_k^c)(1 - \mathbb{P}_{p_0,r}(B_k)) \\ &< (2/3)^2 \cdot 1 + 1 \cdot 1/18, \end{aligned}$$

whence inequality (6) follows with these choices, completing the proof. □

4.2 Graphs with discontinuous critical value functions

Proof of Proposition 4. For $n \in \mathbb{N}$, let D_n be the graph depicted in Figure 2, and let G be Γ_D constructed with this sequence of graphs as described at the beginning of Section 3.

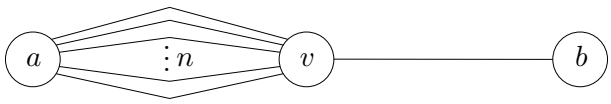

Figure 2: The graph D_n.

It is elementary that $\lim_{n\to\infty} h^{D_n}(p) = p$, whence $p_c^G = 1/2$ follows from Lemma 13, thus $p = 0$ is subcritical. Since $\lim_{n\to\infty} f^{D_n}(0, r) = r^2$, Lemma 14 gives that $r_c^G(0) = 1/\sqrt{2}$. On the other hand, $\lim_{n\to\infty} f^{D_n}(p, r) = p + (1-p)r$ for all $p > 0$, which implies by Lemma 14 that for $p \le 1/2$,

$$r_c^G(p) = \frac{1/2 - p}{1-p} \to 1/2$$

as $p \to 0$, so r_c^G is indeed discontinuous at $0 < p_c^G$. □

In the rest of this section, for vertices v and w, we will write $v \leftrightarrow w$ to denote that there exists a path of open edges between v and w. Our proof of Theorem 5 will be based on the following modification of Lemma 2.1 in [14]:

Lemma 17. *For all $p_0, p_1 \in (0,1)$, there exists a sequence $G_n = (V^n, E^n)$ of graphs and $x_n, y_n \in V^n$ of vertices ($n \in \mathbb{N}$) such that*

1. *$\nu_{p_0}^{E^n}(x_n \leftrightarrow y_n) > p_1$ for all n;*
2. *$\lim_{n\to\infty} \nu_p^{E^n}(x_n \leftrightarrow y_n) = 0$ for all $p < p_0$, and*
3. *there exists $\Delta = \Delta(p_0, p_1) < \infty$ such that, for all n, G_n has maximal degree Δ.*

Proof. There are several ways of constructing such a sequence. Since the concrete choice of G_n will be irrelevant for the rest of this paper, we will just show how to modify the proof of Lemma 2.1 in [14], using the terminology of that proof.

Choose $\varepsilon_2 > 0$ and $c > 0$ such that $(1-\varepsilon_2)^2(1 - e^{-c/3+1}) > p_1$, fix $\varepsilon_1 = \varepsilon_2/3$, and let m be so large that $\nu_{p_0}^H(\mathbf{0}$ is in an infinite bond cluster$) > \sqrt{1-\varepsilon_1}$. Fixing $A_n = \{v_i : 1 \le i \le c/p_0^n\}$, it is easy to see that $\forall n\ \exists H_n : \forall v \in A_n\ \nu_{p_0}^{H_n}(\mathbf{0} \leftrightarrow v) \ge 1 - \varepsilon_1$. Defining G_n with these H_n as in [14] and taking x_n and y_n as $\mathbf{0}$ in H_n and the copy of H_n respectively, the computations in the proof of [14] give the statement of Lemma 17. □

Lemma 17 provides a sequence of bounded degree graphs that exhibit sharp threshold-type behavior at p_0. We will use such a sequence as a building block to obtain discontinuity at p_0 in the critical value function in the DaC model. The basic idea behind the following construction was suggested to us by Jeff Steif.

Proof of Theorem 5. Fix $p_0 \in (0,1)$ and the smallest $k \in \mathbb{N}$ such that

$$\left(\frac{1+2p_0}{3}\right)^k < 1/2.$$

As $((1+2p_0)/3)^{k-1} \geq 1/2$, we may fix

$$p_1 \in \left(\left(\frac{1+p_0}{2} \right)^{1-k} /2, 1 \right),$$

and also $G_n = (V^n, E^n), x_n, y_n$ $(n \in \mathbb{N})$ for p_0 and p_1 as in Lemma 17. Let P be a graph with vertex set $\{v_0, v_1, \ldots, v_{k+1}\}$ and edge set $\{\langle v_i, v_j \rangle : |i-j| = 1\}$ (*i.e.*, P is a path with k edges). We define, for each $n \in \mathbb{N}$, D_n by replacing the edge between v_k and v_{k+1} by G_n, with x_n replacing v_k and y_n replacing v_{k+1}. Let a_n and b_n be the two vertices of D_n that correspond to the former v_0 and v_{k+1}, respectively, and let G be the graph Γ_D with the sequence D_n as defined in Section 3.

Since $h^{D_n}(p) = p^k \nu_p^{E^n}(x_n \leftrightarrow y_n) \leq p^k$ for all n, we have $\limsup h^{D_n}((1+2p_0)/3) < 1/2$, whence Lemma 13 implies that $p_0 < (1+2p_0)/3 \leq p_c^G$. Define

$$c = \begin{cases} 0 & \text{if } k = 1; \\ \frac{(2p_1)^{-1/(k-1)} - p_0}{1-p_0} & \text{if } k > 1, \end{cases}$$

and note that $c < 1/2$ by the choice of p_1. We will show below that $r_c^G(p_0) \leq c$ while $r_c^G(p) \geq 1/2$ for all $p < p_0$, which implies discontinuity of r_c^G at $p_0 < p_c^G$, finishing the proof.

Fix an arbitrary $r \in (c, 1/2)$, and note that for all $p \in [0,1]$,

$$f^{D_n}(p,r) = (p + (1-p)r)^{k-1}(\nu_p^{E^n}(x_n \leftrightarrow y_n) + (1 - \nu_p^{E^n}(x_n \leftrightarrow y_n))a),$$

with a being the conditional $\mathbb{P}_{p,r}^{G_n}$-probability of x_n and y_n being connected by a black path, given that they are not in the same bond cluster and x_n is black. It is immediately clear that $a \leq r$, whence we have for any fixed $p < p_0$ that $\limsup_{n\to\infty} f^{D_n}(p,r) \leq \lim_{n\to\infty} \nu_p^{E^n}(x_n \leftrightarrow y_n) + r < 1/2$ by $r < 1/2$, which implies by Lemma 14 that $r_c^G(p) \leq r$. On the other hand, $\liminf_{n\to\infty} f^{D_n}(p_0, r) \geq (p_0 + (1-p_0)r)^{k-1} \liminf_{n\to\infty} \nu_p^{E^n}(x_n \leftrightarrow y_n) > 1/2$ by $r > c$, whence Lemma 14 gives that $r \geq r_c^G(p_0)$. These computations go through for any $r \in (c, 1/2)$, thus we indeed have $r_c^G(p) \geq 1/2$ for all $p < p_0$ and $r_c^G(p_0) \leq c$. □

4.3 Continuity of $r_c^{\mathbb{Z}^2}(p)$ on the interval $[0, 1/2)$

In this section, we will prove Theorem 9. Our first task is to prove a technical result valid on more general graphs stating that the probability of any event A whose occurrence depends on a finite set of ξ-variables is a continuous function of p for $p < p_c^G$. The proof relies on the fact that although the color of a vertex v may be influenced by edges arbitrarily far away, if $p < p_c^G$, the corresponding probability decreases to 0 in the limit as we move away from v. Therefore, the occurrence of the event A depends essentially on a finite number of η- and κ-variables, whence its probability can be approximated up to an arbitrarily small error by a polynomial in p and r.

Once we have proved Proposition 18 below, which is valid on general graphs, we will apply it on $\mathbb{Z}^2$ to certain "box-crossing events," and appeal to results in [4] to deduce the continuity of $r_c^{\mathbb{Z}^2}(p)$.

Proposition 18. *For every site percolation event $A \subset \{0,1\}^{\mathcal{V}}$ depending on the color of finitely many vertices, $\mu_{p,r}^G(A)$ is a continuous function of (p,r) on the set $[0, p_c^G) \times [0,1]$.*

Proof. In this proof, when μ is a measure on a set S, X is a random variable with law μ and $F : S \longrightarrow \mathbb{R}$ is a bounded measurable function, we write abusively $\mu[F(X)]$ for the expectation of $F(X)$. We show a slightly more general result: for any $k \geq 1$, $\boldsymbol{x} = (x_1, \ldots, x_k) \in \mathcal{V}^k$ and $f : \{0,1\}^k \to \mathbb{R}$ bounded and measurable, $\mu^G_{p,r}\left[f(\xi(x_1), \ldots, \xi(x_k))\right]$ is continuous in (p,r) on the product $[0, p^G_c) \times [0,1]$. Proposition 18 will follow by choosing an appropriate family $\{x_1, \ldots, x_k\}$ such that the states of the x_i suffices to determine whether A occurs, and take f to be the indicator function of A.

To show the previous affirmation, we condition on the vector $\boldsymbol{m_x}(\eta) = (\min C_{x_1}(\eta), \ldots, \min C_{x_k}(\eta))$ which takes values in the finite set

$$\boldsymbol{V} = \left\{(v_1, \ldots, v_k) \in \mathcal{V}^k \ : \ \forall i \ v_i \leq \max\{x_1, \ldots, x_k\}\right\},$$

and we use the definition of $\mathbb{P}^G_{p,r}$ as an image measure. By definition,

$$\begin{aligned}
&\mu^G_{p,r}\left[f(\xi(x_1), \ldots, \xi(x_k))\right] \\
&\quad = \sum_{\boldsymbol{v} \in \boldsymbol{V}} \mathbb{P}^G_{p,r}\left[f(\xi(x_1), \ldots, \xi(x_k)) | \{\boldsymbol{m_x} = \boldsymbol{v}\}\right] \mathbb{P}^G_{p,r}\left[\{\boldsymbol{m_x} = \boldsymbol{v}\}\right] \\
&\quad = \sum_{\boldsymbol{v} \in \boldsymbol{V}} \nu^{\mathcal{E}}_p \otimes \nu^{\mathcal{V}}_r\left[f(\kappa(v_1), \ldots, \kappa(v_k)) | \{\boldsymbol{m_x} = \boldsymbol{v}\}\right] \nu^{\mathcal{E}}_p\left[\{\boldsymbol{m_x} = \boldsymbol{v}\}\right] \\
&\quad = \sum_{\boldsymbol{v} \in \boldsymbol{V}} \nu^{\mathcal{V}}_r\left[f(\kappa(v_1), \ldots, \kappa(v_k))\right] \nu^{\mathcal{E}}_p\left[\{\boldsymbol{m_x} = \boldsymbol{v}\}\right].
\end{aligned}$$

Note that $\nu^{\mathcal{V}}_r\left[f(\kappa(v_1), \ldots, \kappa(v_k))\right]$ is a polynomial in r, so to conclude the proof we only need to prove that for any fixed $\boldsymbol{x}$ and $\boldsymbol{v}$, $\nu^{\mathcal{E}}_p\left(\{\, \boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{v} \,\}\right)$ depends continuously on p on the interval $[0, p^G_c)$.

For $n \geq 1$, write $F_n = \{|C_{x_1}| \leq n, \ldots, |C_{x_k}| \leq n\}$. It is easy to verify that the event $\{\boldsymbol{m_x} = \boldsymbol{v}\} \cap F_n$ depends on the state of finitely many edges. Hence, $\nu^{\mathcal{E}}_p\left[\{\boldsymbol{m_x} = \boldsymbol{v}\} \cap F_n\right]$ is a polynomial function of p.

Fix $p_0 < p^G_c$. For all $p \leq p_0$,

$$\begin{aligned}
0 \leq \nu^{\mathcal{E}}_p\left[\{\boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{v}\}\right] - \nu^{\mathcal{E}}_p\left[\{\boldsymbol{m_x} = \boldsymbol{v}\} \cap F_n\right] &\leq \nu^{\mathcal{E}}_p\left[F_n^c\right] \\
&\leq \nu^{\mathcal{E}}_{p_0}\left[F_n^c\right]
\end{aligned}$$

where $\lim_{n \to \infty} \nu^{\mathcal{E}}_{p_0}\left[F_n^c\right] = 0$, since $p_0 < p^G_c$. So, $\nu^{\mathcal{E}}_p\left[\boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{v}\right]$ is a uniform limit of polynomials on any interval $[0, p_0]$, $p_0 < p^G_c$, which implies the desired continuity. □

Remark 19. *In the proof we can see that, for fixed $p < p^G_c$, $\mu^G_{p,r}(A)$ is a polynomial in r.*

Remark 20. *If G is a graph with uniqueness of the infinite bond cluster in the supercritical regime, then it is possible to verify that $\nu^{\mathcal{E}}_p\left[\{\boldsymbol{m}(\boldsymbol{x}) = \boldsymbol{v}\}\right]$ is continuous in p on the whole interval $[0,1]$. In this case, the continuity given by the Proposition 18 can be extended to the whole square $[0,1]^2$.*

Proof of Theorem 9. In order to simplify our notations, we write $\mathbb{P}_{p,r}, \nu_p$, respectively, $r_c(p)$, for $\mathbb{P}^{\mathbb{Z}^2}_{p,r}, \nu^{\mathcal{E}^2}_p$ and $r^{\mathbb{Z}^2}_c(p)$. Fix $p_0 \in (0, 1/2)$ and $\varepsilon > 0$ arbitrarily. We will show that there exists $\delta = \delta(p_0, \varepsilon) > 0$ such that for all $p \in (p_0 - \delta, p_0 + \delta)$,

$$r_c(p) \geq r_c(p_0) - \varepsilon, \tag{7}$$

and

$$r_c(p) \leq r_c(p_0) + \varepsilon. \tag{8}$$

Note that by equation (1), for all small enough choices of $\delta > 0$ (such that $0 \le p_0 \pm \delta < 1/2$), (7) is equivalent to

$$r_c^*(p) \le r_c^*(p_0) + \varepsilon. \tag{9}$$

Below we will show how to find $\delta_1 > 0$ such that we have (8) for all $p \in (p_0 - \delta_1, p_0 + \delta_1)$. One may then completely analogously find $\delta_2 > 0$ such that (9) holds for all $p \in (p_0 - \delta_2, p_0 + \delta_2)$, and take $\delta = \min(\delta_1, \delta_2)$.

Fix $r = r_c(p_0) + \varepsilon$, and define the event $V_n = \{(\xi, \eta) \in \Omega_{\mathbb{Z}^2} \times \Omega_{\mathcal{E}_2}$: there exists a vertical crossing of $[0, n] \times [0, 3n]$ that is black in $\xi\}$. By "vertical crossing," we mean a self-avoiding path of vertices in $[0, n] \times [0, 3n]$ with one endpoint in $[0, n] \times \{0\}$, and one in $[0, n] \times \{3n\}$. Recall also the definition of M_n in Theorem 10. By Lemma 2.10 in [4], there exists a constant $\gamma > 0$ such that the following implication holds for any $p, a \in [0, 1]$ and $L \in \mathbb{N}$: if both $(3L + 1)(L + 1)\nu_p(M_{\lfloor L/3 \rfloor}) \le \gamma$ and $\mathbb{P}_{p,a}(V_L) \ge 1 - \gamma$ are satisfied, then $a \ge r_c(p)$. (As usual, $\lfloor x \rfloor$ for $x > 0$ denotes the largest integer m such that $m \le x$.) Fix such a γ.

By Theorem 10, there exists $N \in \mathbb{N}$ such that

$$(3n + 1)(n + 1)\nu_{p_0}(M_{\lfloor n/3 \rfloor}) < \gamma$$

for all $n \ge N$. On the other hand, since $r > r_c(p_0)$, it follows from Lemma 2.11 in [4] that there exists $L \ge N$ such that

$$\mathbb{P}_{p_0,r}(V_L) > 1 - \gamma.$$

Note that both $(3L + 1)(L + 1)\nu_p(M_{\lfloor L/3 \rfloor})$ and $\mathbb{P}_{p,r}(V_L)$ are continuous in p at p_0. Indeed, the former is simply a polynomial in p, while the continuity of the latter follows from Proposition 18. Therefore, there exists $\delta_1 > 0$ such that for all $p \in (p_0 - \delta_1, p_0 + \delta_1)$,

$$\begin{aligned} (3L + 1)(L + 1)\nu_p(M_{\lfloor L/3 \rfloor}) &\le \gamma, \\ \text{and } \mathbb{P}_{p,r}(V_L) &\ge 1 - \gamma. \end{aligned}$$

By the choice of γ, this implies that $r \ge r_c(p)$ for all such p, which is precisely what we wanted to prove.

Finding $\delta_2 > 0$ such that (9) holds for all $p \in (p_0 - \delta_2, p_0 + \delta_2)$ is analogous: one only needs to substitute $r_c(p_0)$ by $r_c^*(p_0)$ and "crossing" by "$*$-crossing," and the exact same argument as above works. It follows that $\delta = \min(\delta_1, \delta_2) > 0$ is a constant such that both (8) and (9) hold for all $p \in (p_0 - \delta, p_0 + \delta)$, completing the proof of continuity on $(0, 1/2)$. Right-continuity at 0 may be proved analogously; alternatively, it follows from Proposition 3.

Remark 21. *It follows from Theorem 9 and equation (1) that $r_c^*(p)$ is also continuous in p on $[0, 1/2)$.*

5 Asymptotic behavior as p tends to p_c^G

We performed an extensive simulation of the DaC model on the square lattice $\mathbb{Z}^2$ and on the hexagonal lattice $\mathbb{H}$, for various values of p; our results will be published separately in a separate, more numerically oriented paper [3]. In all this section, only these two lattices will be considered, and the letter G will be used to denote either of them. We obtained the confidence intervals represented in Figure 3.

Having looked at Figure 3, we conjecture the following concerning the behavior of $r_c^G(p)$ as a function of p:

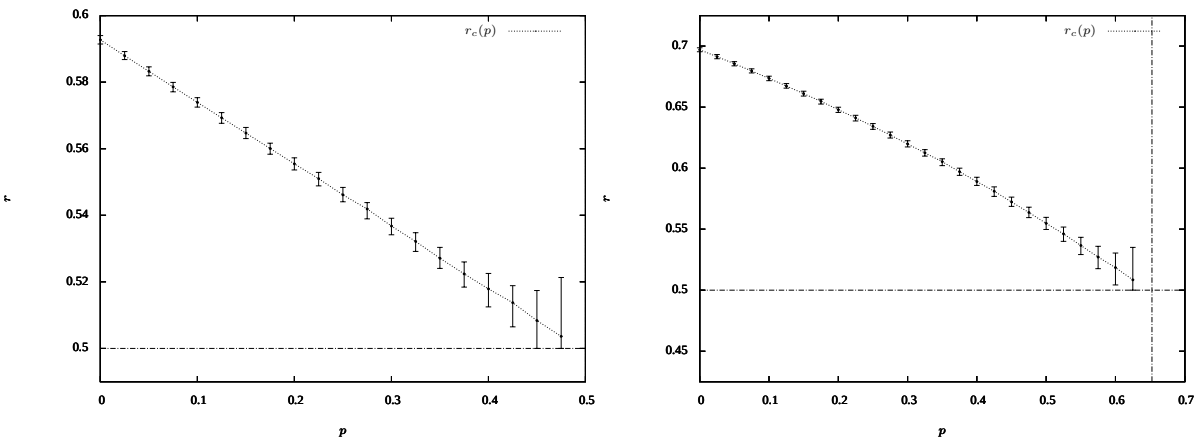

Figure 3: Simulation results for different values of $p < p_c$ (left: on the square lattice; right: on the hexagonal lattice). The dashed line was obtained via a non-rigorous correction method.

Conjecture 22. *For G denoting either $\mathbb{Z}^2$ or $\mathbb{H}$, in the interval $p \in [0, p_c(G))$, $r_c^G(p)$ is a strictly decreasing function of p, and $\lim_{p \to p_c(G)-} r_c(p) = 1/2$.*

Since it is rigorously known that $r_c^G(0) > 1/2$ and $r_c^G(p) \geq 1/2$ for all $p \in [0, p_c(G))$, Conjecture 22 would imply that $r_c^G(p) > 1/2$ for all $p < p_c(G)$. This suggests that the DaC(1) model on $\mathbb{Z}^2$ or $\mathbb{H}$ is qualitatively different from the DaC(1) model on the triangular lattice, where the critical value of r is $1/2$ for all subcritical p (see Theorem 1.6 in [4]). However, $\lim_{p \to 1/2-} r_c(p) = 1/2$ would mean that the difference disappears as p tends to $p_c(G)$.

The fact that the difference should disappear was conjectured by one of the authors (VB) and Federico Camia, based on the following heuristic reasoning. Near $p = p_c(G)$, the structure of the random graph determined by the bond configuration (whose vertices correspond to the bond clusters, and there is an edge between two vertices if the corresponding bond clusters are adjacent in $\mathbb{Z}^2$) is given by the geometry of "near-critical percolation clusters," which is expected to be universal for 2-dimensional planar graphs. This suggests that the critical r for p close to its critical value should not depend much on the original underlying lattice, and we expect the convergence of $r_c(p)$ to $1/2$ to be universal and hold in the case of any 2-dimensional lattice.

There is an additional, strange feature appearing in the case of the square lattice: $r_c(p)$ seems to be close to being an affine function of p on the interval $[0, 1/2)$. This is not at all the same on the hexagonal lattice, and we have not found any interpretation of this observation, or of the special role $\mathbb{Z}^2$ seems to play here.

Open question 23. *Is $r_c^{\mathbb{Z}^2}(p)$ an affine function of p for $p < 1/2$?*

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