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# On Ziv's rounding test* 

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#### Abstract

A very simple test, introduced by Ziv, allows one to determine if an approximation to the value $f(x)$ of an elementary function at a given point $x$ suffices to return the floating-point number nearest $f(x)$. The same test may be used when implementing floating-point operations with input and output operands of different formats, using arithmetic operators tailored for manipulating operands of the same format. That test depends on a "magic constant" $e$. We show how to choose that constant $e$ to make the test reliable and efficient. Various cases are considered, depending on the availability of an fma instruction, and on the range of $f(x)$.


keywords: Floating-Point arithmetic, correct rounding, elementary functions

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## 1 Introduction

The result of an accurate numerical calculation is frequently available as the unevaluated sum of two floating-point numbers: a "main term" $y_{h}$ and a "correcting term" $y_{\ell}$, such that if $y$ is the exact result for which we are trying to find an approximation, $y_{h}+y_{\ell}$ is very close to $y$ (say, within some relative error bound $\epsilon$ ). Examples occur in: evaluation of elementary function, compensated summation algorithms, and (as we will see below) implementation of "heterogeneous" floating-point operations. The problem we face here is to check

[^0]whether we can easily deduce the floating-point nearest $y$ from $y_{h}, y_{\ell}$ and $\epsilon$. Before going further, let us recall some elementary notions of floating-point arithmetic and define some notation.

### 1.1 Floating-point numbers

A binary, precision- $p$, floating-point number is a number of the form

$$
\begin{equation*}
x=X \cdot 2^{k_{x}-p+1}, \tag{1}
\end{equation*}
$$

where $X$ is an integer, $|X| \leq 2^{p}-1$, and $k_{x}$ is an integer. In practical implementations of floating-point arithmetic, it is required that $k_{x}$ should be between two extremal exponents, $k_{\text {min }}$ and $k_{\text {max }}$.

The floating-point representation (1) of $x$ is said normal if $|X|$ is the largest possible that still satisfies $|X| \leq 2^{p}-1$. When (1) is the normal representation of $X, X$ is called the integral significand of $x, X \cdot 2^{-p+1}$ is called the significand of $x$, and $k_{x}$ is called the exponent of $x$. A normal number has absolute value larger than or equal to $2^{k_{\text {min }}}$. Its integral significand is larger than or equal to $2^{p-1}$. A subnormal number has absolute value less than $2^{k_{\text {min }}}$.

Assuming a given precision $p$, we call midpoint a number that is exactly halfway between two consecutive floating-point numbers, and we define a function ulp (which is an acronym for unit in the last place [7]) as follows:

- if $|x| \in\left[2^{k}, 2^{k+1}\right)$, where $k$ is an integer larger than or equal to $k_{\min }$, then $\operatorname{ulp}(x)=2^{k-p+1}$;
- if $x$ is subnormal then $\operatorname{ulp}(x)=2^{k_{\text {min }}-p+1}$.

Roughly speaking, $\operatorname{ulp}(x)$ is the distance between two consecutive floatingpoint numbers in the neighborhood of $x$.

We define $\mathrm{RN}_{p}(t)$ as the precision- $p$ floating-point number that is closest to $t$ (when $t$ falls exactly halfway between two consecutive floating-point numbers, $\mathrm{RN}_{p}(t)$ will be the one of these two numbers whose integral significand is even: this is the so-called round to nearest even rule of the IEEE 754 standard), and $\mathrm{RU}_{p}(t)$ as the smallest floating-point number that is larger than or equal to $t$. When there is no ambiguity on the value of $p$, we just write " $\mathrm{RN}(t)$ " or " $\mathrm{RU}(t)$ ". With a correctly rounded arithmetic, assuming that round to nearest even is the chosen rounding mode, the number x obtained after executing the C-instruction $x=y+z$ is $R N(y+z)$.

We will also need the following algorithm, due to Dekker:

```
ALGORITHM 1: \(\operatorname{Fast2Sum}(a, b)\)
    \(s \leftarrow \operatorname{RN}(a+b)\)
    \(z \leftarrow \operatorname{RN}(s-a)\)
    \(t \leftarrow \operatorname{RN}(b-z)\)
```

    It satisfies:
    Theorem 1 (Dekker's Fast2Sum algorithm). (for a proof, see [3, 8, 11])
Assume a binary floating-point system.
Let $a$ and $b$ be floating-point numbers, and assume that the exponent of a is larger than or equal to that of $b$. Algorithm 1 computes two floating-point numbers $s$ and $t$ that satisfy:

- $s+t=a+b$ exactly;
- $s$ is the floating-point number that is closest to $a+b$.

In the following, we assume we use a binary floating-point arithmetic that is compliant with the IEEE 754 standard [4], which implies that the arithmetic operations are correctly rounded, and we also assume that the "round-to-nearest ties-to-even" rounding direction is selected (or that none is selected, since that one is the default).

### 1.2 Instances of the "main+correcting terms" problem

### 1.2.1 Evaluation of functions

Assume one wishes to write a program that evaluates a real-valued function $f$ with correct rounding, that is, for any input floating-pointnumber $x$, the program must return $\mathrm{RN}(f(x)) .^{1}$ A strategy for doing this, first suggested and implemented by Ziv [15] in his library of correctly-rounded transcendentals is the following one:

- First, compute an approximation, say $y_{1}$, to $f(x)=y$ with relative error less than $\epsilon_{1}$. From that approximation and the error bound, one can deduce an interval $I_{1}$ such that $y \in I_{1}$.
- If all elements of $I_{1}$ round to the same floating-point number $\hat{y}$ (in other words, if $I_{1}$ does not contain a midpoint), then, necessarily, $\mathrm{RN}(f(x))=\hat{y}$.
- Otherwise, we need to perform another step, i.e., to compute a more accurate approximation $y_{2}$, with relative error bound $\epsilon_{2}<\epsilon_{1}$, and so on: we may need an arbitrarily long sequence of approximations $y_{1}, y_{2}, y_{3}$, etc., that become more and more costly to compute.
This is illustrated by Fig. 1. This process can be improved if we know in advance what is the smallest nonzero relative distance between $f(z)$, where $z$ is a floating-point number, and a midpoint, provided that this smallest distance is not too small. When we know that smallest distance, we can implement an efficient program with a small (in practice, 2 or 3 ) number of steps. Knowing that smallest distance requires solving a problem called the Table Maker's Dilemma [7, 10].

[^1]The important fact is the following: since the cost of computing an approximation increases as the error bound $\epsilon_{i}$ decreases, it is very important to make sure that we almost always never use the second, third, etc. steps. Hence, the test that decides whether we need to perform a second step must satisfy the following properties:

- of course, it must be reliable: if $\left(y_{1}, \epsilon_{1}\right)$ does not suffice to deduce the value of $\mathrm{RN}(y)$, the test must say so;
- the number of "wrong alarms" (i.e., of cases for which $\left(y_{1}, \epsilon_{1}\right)$ would have sufficed to deduce the value of $\mathrm{RN}(y)$, and yet step 2 is performed) must be as small as possible;
- the test itself must be very "cheap", i.e., cost a very few arithmetic operations and comparisons only.


Figure 1: Given an approximation $y_{1}$ to $y=f(x)$ and a bound $\epsilon_{1}$ on the approximation error, we know that $y$ lies in an interval $I_{1}$. In the first case, all elements of $I_{1}$ round to the same floating-point number, so that we can return $\mathrm{RN}(y)$. In the second case, a new, more accurate, approximation $y_{2}$ must be computed.

In the following, we assume the availability of an algorithm that makes it possible to approximate $y=f(x)$ by the unevaluated sum of two floatingpoint numbers: a "main term" $y_{h}$ and a "correcting term" $y_{\ell}$, with relative error bounded by some $\epsilon$, that is,

$$
\begin{equation*}
y_{h}+y_{\ell}=y \cdot(1+\alpha), \quad \text { with } \quad|\alpha| \leq \epsilon . \tag{2}
\end{equation*}
$$

Such algorithms do exist for evaluating the usual transcendental functions (in general, through some judicious combination of table look-up and polynomial approximation). We can even assume that $y_{\ell}$ is small enough in front of $y_{h}$, so that the floating-point number nearest $y_{h}+y_{\ell}$ is $y_{h}$ (otherwise, it suffices to use the Fast2Sum algorithm, given above, to get into this situation).

We wish to find some way of very quickly checking whether $y_{h}=\mathrm{RN}(y)$ or not. We will call that problem the "main+correcting terms" problem (see below for a more accurate definition).

### 1.2.2 Implementation of heterogeneous operations

A similar situation also occurs when we want to implement "heterogeneous" floating-point operations (that is, operations whose operands have different floating-point formats), assuming we only have hardware operators for "homogeneous" operations. Notice that such operations are required by the IEEE 754-2008 Standard for Floating-Point Arithmetic [4]. Suppose we have two different formats available on the same system: a format of precision $p$, and another of precision $p^{\prime}$, where $p<p^{\prime}$ (typical real-life examples are with $p=53$ and $p^{\prime}=64$, which correspond to the binary $64 /$ double precision and the "Intel double-extended" formats, and with $p=24$ and $p^{\prime}=53$, which correspond to the binary 32 /single precision and binary64/double precision formats). Also suppose we only have addition/subtraction operators for adding two numbers of precision $p$ and returning a result in precision $p$; and for adding two numbers of precision $p^{\prime}$ and returning a result in precision $p^{\prime}$ (we consider here the case of addition, but the very same applies to multiplication, division, and square root). We want to add two numbers of precision $p^{\prime}$, and get the result in precision $p$, rounded to nearest. That is, given $a^{\left(p^{\prime}\right)}$ and $b^{\left(p^{\prime}\right)}$ of precision $p^{\prime}$, we wish to compute

$$
y^{(p)}=\mathrm{RN}_{p}\left(a^{\left(p^{\prime}\right)}+b^{\left(p^{\prime}\right)}\right)
$$

To do so, the first idea that springs in mind is to first compute, using the precision- $p^{\prime}$ floating-point addition instruction:

$$
y^{\left(p^{\prime}\right)}=\mathrm{RN}_{p^{\prime}}\left(a^{\left(p^{\prime}\right)}+b^{\left(p^{\prime}\right)}\right)
$$

which satisfies ${ }^{2} y^{\left(p^{\prime}\right)}=\left(a^{\left(p^{\prime}\right)}+b^{\left(p^{\prime}\right)}\right)(1+\mu)$, with $|\mu| \leq 2^{-p^{\prime}}$, and then to round $y^{\left(p^{\prime}\right)}$ to the nearest precision- $p$ floating-point number, that is to compute

$$
y_{\text {temp }}^{(p)}=\operatorname{RN}_{p}\left(y^{\left(p^{\prime}\right)}\right)
$$

Unfortunately, although $y_{\text {temp }}^{(p)}$ will frequently be equal to $y^{(p)}=\mathrm{RN}_{p}\left(a^{\left(p^{\prime}\right)}+\right.$ $\left.b^{\left(p^{\prime}\right)}\right)$, this will not always be the case: this is a typical issue of the double rounding problem $[9,1]$. An example with $p=53$ and $p^{\prime}=64$ is $a^{\left(p^{\prime}\right)}=2^{63}+2^{10}$ and $b^{\left(p^{\prime}\right)}=2^{-2}$, for which one easily finds that $y_{\text {temp }}^{(p)}=2^{63}$ whereas $y^{(p)}=2^{63}+2^{11}$.

[^2]Once $y_{\text {temp }}^{(p)}$ is computed, one may compute

$$
y_{\ell}^{*}=y^{\left(p^{\prime}\right)}-y_{\text {temp }}^{(p)} .
$$

Since $y_{\text {temp }}^{(p)}$ is a precision- $p$ floating-point number and $p<p^{\prime}, y_{\text {temp }}^{(p)}$ is also a precision- $p^{\prime}$ floating-point number. Therefore, $y_{\ell}^{*}$ can be computed using the precision $-p^{\prime}$ floating-point addition instruction.

One easily shows that $y_{\ell}^{*}$ is exactly representable in precision $p^{\prime}$. Hence, $y_{\ell}^{*}$ is exactly computed. We now consider two cases:

- if $p^{\prime} \leq 2 p+1$ then $y_{\ell}^{*}$ is exactly representable in precision $p$. Hence, after having (exactly) computed $y_{\ell}^{*}$ using precision- $p^{\prime}$ addition, one can convert it to the precision- $p$ format without any loss of accuracy. By choosing $y_{h}=y_{\text {temp }}^{(p)}, y_{\ell}=y_{\ell}^{*}$, and $\alpha=\mu$, we find

$$
y_{h}+y_{\ell}=\left(a^{\left(p^{\prime}\right)}+b^{\left(p^{\prime}\right)}\right)(1+\alpha),
$$

with $|\alpha| \leq 2^{-p^{\prime}}$ and $\operatorname{RN}\left(y_{h}+y_{\ell}\right)=y_{h}$, which is just another instance of (2);

- if $p^{\prime} \geq 2 p+1$, then define $y_{\ell}=\operatorname{RN}_{p}\left(y_{\ell}^{*}\right)$. We have

$$
\left|y_{\ell}-y_{\ell}^{*}\right| \leq 2^{-p}\left|y_{\ell}^{*}\right| \leq 2^{-2 p}\left|y^{\left(p^{\prime}\right)}\right| \leq 2^{-2 p}(1+\mu)\left|a^{\left(p^{\prime}\right)}+b^{\left(p^{\prime}\right)}\right|,
$$

Hence, by choosing $y_{h}=y_{\text {temp }}^{(p)}$, we find

$$
y_{h}+y_{\ell}=\left(y_{\mathrm{temp}}^{(p)}+y_{\ell}^{*}\right)+\left(y_{\ell}-y_{\ell}^{*}\right)=\left(a^{\left(p^{\prime}\right)}+b^{\left(p^{\prime}\right)}\right) \cdot(1+\alpha),
$$

with $|\alpha| \leq 2^{-p^{\prime}}+2^{-2 p}+2^{-2 p-p^{\prime}}$ and $\mathrm{RN}\left(y_{h}+y_{\ell}\right)=y_{h}{ }^{3}$ which is just another instance of (2);

### 1.2.3 Summation algorithms

There is a large literature of summation algorithms that first approximate a sum $x_{1}+x_{2}+\cdot+x_{m}$ by a main term and a correcting term and then return the floating-point sum of these two terms (see for instance [14, 12, 13]). Being able to quickly check whether we finally obtain the floating-point number closest to the exact sum is, of course, of interest.

### 1.3 Statement of the "main+correcting terms" problem

Let us now state the "main+correcting terms" problem more formally.

[^3]Problem 1 (main+correcting terms problem). Assume that an exact, real number $y$, is approximated by two precision-p binary floating-point numbers $y_{h}$ and $y_{\ell}$, that satisfy

$$
\begin{equation*}
\left|\left(y_{h}+y_{\ell}\right)-y\right|<\epsilon \cdot|y|, \tag{3}
\end{equation*}
$$

with $\epsilon<2^{-p-1}$, and

$$
\begin{equation*}
y_{h}=\operatorname{RN}\left(y_{h}+y_{\ell}\right) . \tag{4}
\end{equation*}
$$

We assume that $y_{h}$ is a normal number (otherwise, the only floating-point value $y_{\ell}$ that satisfies (4) is $y_{\ell}=0$ ). The problem we address here is the following one: find some simple test that makes it possible to quickly determine whether $y_{h}$ is the floating-point number closest to $y$ or not.

### 1.4 Ziv's rounding technique

In Ziv's Accurate Portable Mathlib (or libultim) library of elementary functions [15], the test being performed to solve Problem 1 is the following one:

$$
\text { Is } y_{h} \text { equal to } \mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right) \text { ? }
$$

where $e$ is some "magic constant" that must be adequately chosen (this is the main purpose of this paper). A similar test has also been used, later on, in the CRLibm library [2]. Notice that RN $\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right)$ is what we get ${ }^{4}$ by evaluating, in floating-point arithmetic, the $C$ expression:

$$
y h+(y l * e) .
$$

We wish to find that "magic constant" $e$. It will be a nonnegative floatingpoint number, as small as possible (due to Observation 1, below), such that

$$
\begin{equation*}
y_{h}=\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right) \Rightarrow y_{h}=\mathrm{RN}(y) . \tag{5}
\end{equation*}
$$

Notice that (3) implies that $y_{h}$ and $y$ have the same sign. Without loss of generality, we assume in the following that they are positive.

In the following, we will say that e allows for a correct Ziv rounding test if (5) holds as soon as (3) and (4) hold. We will say that $\left(y, y_{h}, y_{\ell}\right)$ is a false negative if (3) and (4) hold, $y_{h} \neq \mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right)$ and yet $y_{h}=\mathrm{RN}(y)$. Our goal is to make sure that $e$ should allow for a correct Ziv rounding test, and that false negatives should be as infrequent as possible.

Let us now raise two observations:
Observation 1. If $\left(y, y_{h}, y_{\ell}\right)$ is a false negative for the Ziv rounding test with constant $e$, then it will be a false negative with any constant $e^{\prime}>e$. In other words, the number of false negatives will increase with $e$.

Observation 2. If constant e does not allow for a correct Ziv rounding test, then no constant $e^{\prime}<e$ will allow for a correct Ziv rounding test.

[^4]These two observations are an almost immediate consequence of the fact that the function $t \mapsto \mathrm{RN}(t)$ is monotonic. From these observations, we conclude that we should try, given $\epsilon$ and $p$, to find the a value of $e$ as small as possible that allows for a correct Ziv rounding test. This is the purpose of the next section.

## 2 Main result

### 2.1 Preliminary properties

Property 1. If $\left(y-y_{h}\right)$ and $y_{\ell}$ have different signs then $y_{h}=\mathrm{RN}(y)$.
Proof. If $\left(y-y_{h}\right)$ and $y_{\ell}$ have different signs then either $y \leq y_{h} \leq y_{h}+y_{\ell}$ or $y_{h}+y_{\ell} \leq y_{h} \leq y$. In any case, $y$ is closer to $y_{h}$ than to $y_{h}+y_{\ell}$, so that (3) implies $\left|y_{h}-y\right|<\epsilon \cdot|y|$. Since $|\epsilon|<2^{-p-1}$, this implies $y_{h}=\mathrm{RN}(y)$. This is illustrated by Figure 2.


Figure 2: Illustration of Property 1: when $\left(y-y_{h}\right)$ and $y_{\ell}$ have different signs, $y$ is closer to $y_{h}$ than to $y_{h}+y_{\ell}$, which implies $y_{h}=\mathrm{RN}(y)$.

Now, notice that (3) implies

$$
\begin{equation*}
\left|y_{h}-y\right|<\epsilon \cdot|y|+\left|y_{\ell}\right|, \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
y<\frac{y_{h}+y_{\ell}}{1-\epsilon} \tag{7}
\end{equation*}
$$

from which we deduce

$$
\begin{equation*}
\left|y_{h}-y\right|<\frac{\epsilon}{1-\epsilon} \cdot\left(y_{h}+y_{\ell}\right)+\left|y_{\ell}\right| . \tag{8}
\end{equation*}
$$

### 2.1.1 If $y_{h}$ is not a power of two

Property 2. Assume that $y_{h}$ is not a power of two. If

$$
\left|y_{\ell}\right| \leq \frac{1}{2} \operatorname{ulp}\left(y_{h}\right) \cdot(1-\epsilon)-y_{h} \epsilon
$$

then $y_{h}=\mathrm{RN}(y)$.

Proof. If $\left|y_{\ell}\right| \leq \frac{1}{2} \operatorname{ulp}\left(y_{h}\right) \cdot(1-\epsilon)-y_{h} \epsilon$, then

$$
\frac{1}{2} \operatorname{ulp}\left(y_{h}\right)-y_{h} \cdot \frac{\epsilon}{1-\epsilon} \geq\left|y_{\ell}\right|\left(1+\frac{\epsilon}{1-\epsilon}\right)
$$

which implies

$$
\frac{\epsilon}{1-\epsilon} \cdot\left(y_{h}+y_{\ell}\right)+\left|y_{\ell}\right| \leq \frac{1}{2} \operatorname{ulp}\left(y_{h}\right)
$$

Therefore, using (8), we find

$$
\left|y_{h}-y\right|<\frac{1}{2} \operatorname{ulp}\left(y_{h}\right)
$$

which implies $y_{h}=\mathrm{RN}(y)$.

Property 3. Assume that $y_{h}$ is not a power of two. Let e be a nonnegative floatingpoint number. If $\frac{1}{2} \mathrm{ulp}\left(y_{h}\right)$ is in the normal range, then if

$$
y_{h}=\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right)
$$

then

$$
\left|y_{\ell}\right| \leq \frac{\left(1+2^{-p}\right) \cdot \operatorname{ulp}\left(y_{h}\right)}{2 e}
$$

Proof. From $y_{h}=\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right)$, and since $y_{h}$ is not a power of 2 , we deduce

$$
y_{h}-\frac{1}{2} \operatorname{ulp}\left(y_{h}\right) \leq y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right) \leq y_{h}+\frac{1}{2} \operatorname{ulp}\left(y_{h}\right)
$$

which implies

$$
\begin{equation*}
\left|\operatorname{RN}\left(y_{\ell} \cdot e\right)\right| \leq \frac{1}{2} \operatorname{ulp}\left(y_{h}\right) \tag{9}
\end{equation*}
$$

Since $\frac{1}{2} \operatorname{ulp}\left(y_{h}\right)$ is in the normal range, (9) implies

$$
\left|y_{\ell} \cdot e\right| \leq\left(1+2^{-p}\right) \cdot \frac{1}{2} \mathbf{u l p}\left(y_{h}\right)
$$

q.e.d.

Combining Properties 2 and 3, we find that if

$$
\begin{equation*}
\frac{\left(1+2^{-p}\right) \cdot \operatorname{ulp}\left(y_{h}\right)}{2 e} \leq \frac{1}{2} \operatorname{ulp}\left(y_{h}\right) \cdot(1-\epsilon)-y_{h} \epsilon \tag{10}
\end{equation*}
$$

then $y_{h}=\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right)$ implies $y_{h}=\mathrm{RN}(y)$.
Now, let us find a value of $e$, as small as possible, such that (10) holds for all floating-point numbers $y_{h}$ that are not in the subnormal range. Condition (10) is equivalent to

$$
e \geq \frac{\left(1+2^{-p}\right) \cdot \operatorname{ulp}\left(y_{h}\right)}{(1-\epsilon) \cdot \operatorname{ulp}\left(y_{h}\right)-2 y_{h} \epsilon}
$$

If $y_{h}$ is in the normal range, then one can write $\operatorname{ulp}\left(y_{h}\right)=\xi \cdot y_{h} \cdot 2^{-p}$, with $1<\xi \leq 2$. We therefore have

$$
\frac{\left(1+2^{-p}\right) \cdot \operatorname{ulp}\left(y_{h}\right)}{(1-\epsilon) \cdot \operatorname{ulp}\left(y_{h}\right)-2 y_{h} \epsilon}=\frac{\left(1+2^{-p}\right) \cdot \xi}{(1-\epsilon) \cdot \xi-2^{p+1} \epsilon}
$$

Elementary calculus shows that if $\epsilon<1 /\left(2^{p+1}+1\right)$ then the largest value of

$$
\frac{\left(1+2^{-p}\right) \cdot \xi}{(1-\epsilon) \cdot \xi-2^{p+1} \epsilon}
$$

for $1 \leq \xi \leq 2$ is attained for $\xi=1$. From this, we immediately deduce
Property 4. Assume that $y_{h}$ is not a power of 2 , that $\frac{1}{2} \mathrm{ulp}\left(y_{h}\right)$ is in the normal range and that $\epsilon<1 /\left(2^{p+1}+1\right)$. If

$$
\begin{equation*}
e \geq \frac{1+2^{-p}}{1-\epsilon-2^{p+1} \epsilon} \tag{11}
\end{equation*}
$$

then $y_{h}=\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right)$ implies $y_{h}=\mathrm{RN}(y)$.

### 2.1.2 If $y_{h}$ is a power of 2

Reminder: we still assume that $y$ and $y_{h}$ are positive.

1. first, if $y \geq y_{h}$ and $y_{\ell} \geq 0$ (i.e., $y_{h}+y_{\ell} \geq y_{h}$ ), then we have

$$
y-y_{h}<\frac{1}{2} \operatorname{ulp}\left(y_{h}\right) \Rightarrow y=\operatorname{RN}\left(y_{h}\right)
$$

and

$$
\left(y_{h}=\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right) \Rightarrow y_{h} \leq y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right) \leq y_{h}+\frac{1}{2} \operatorname{ulp}\left(y_{h}\right)\right.
$$

so that the proofs of properties 2,3 , and 4 remain valid: Property 4 still holds;
2. second, if $\left(y \leq y_{h}\right.$ and $\left.y_{\ell} \geq 0\right)$ or ( $y \geq y_{h}$ and $\left.y_{\ell} \leq 0\right)$ then, from Property $1, y=\operatorname{RN}\left(y_{h}\right)$;
3. the remaining case is $y \leq y_{h}$ and $y_{\ell} \leq 0$. In that case, since $y_{h}$ is a power of $2, \operatorname{ulp}\left(y_{h}\right)=2^{-p+1} y_{h}$, and we have the following properties.

Property 5. Assume that $y_{h}$ is a power of $2, y \leq y_{h}$ and $y_{\ell} \leq 0$. If

$$
\begin{equation*}
\left|y_{\ell}\right| \leq y_{h} \cdot \frac{2^{-p-1}-\frac{\epsilon}{1-\epsilon}}{1-\frac{\epsilon}{1-\epsilon}} \tag{12}
\end{equation*}
$$

then $y_{h}=\mathrm{RN}(y)$.

Notice that Condition (12) cannot be satisfied if $2^{-p-1}-\epsilon /(1-\epsilon) \leq 0$, that is, if $\epsilon \geq 1 /\left(1+2^{p+1}\right)$.

Proof or Property 5. If (12) is satisfied, then

$$
\frac{\epsilon}{1-\epsilon}\left(y_{h}+y_{\ell}\right)+\left|y_{\ell}\right| \leq 2^{-p-1} y_{h},
$$

which implies, using (8), that

$$
0 \leq y_{h}-y \leq 2^{-p-1} y_{h}=\frac{1}{4} \operatorname{ulp}\left(y_{h}\right)
$$

from which we deduce $y_{h}=\mathrm{RN}(y h)$.
Notice that, since $y_{h}$ is a power of 2 , the usual condition $\left|y_{h}-y\right|<$ $\frac{1}{2} \mathrm{ulp}\left(y_{h}\right)$ would not have sufficed to show that $y_{h}=\mathrm{RN}(y)$. This is illustrated by Fig. 3


Figure 3: When $y_{h}$ is a power of $2,\left|y_{h}-y\right|<\frac{1}{2} \mathrm{ulp}\left(y_{h}\right)$ does not necessarily imply that $y_{h}=\mathrm{RN}(y)$.

Property 6. Assume that $y_{h}$ is a power of $2, y \leq y_{h}$, that $y_{\ell} \leq 0$, and that $2^{-p-1} y_{h}=\frac{1}{4} \operatorname{ulp}\left(y_{h}\right)$ is a normal number. If

$$
\begin{equation*}
y_{h}=\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right), \tag{13}
\end{equation*}
$$

then

$$
\left|y_{\ell}\right| \leq \frac{\left(1+2^{-p}\right) \cdot 2^{-p-1} \cdot y_{h}}{e}
$$

Proof. Since $y_{h}$ is a power of $2,(13)$ implies

$$
y_{y}-\frac{1}{4} \operatorname{ulp}\left(y_{h}\right) \leq y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right) \leq y_{y}
$$

which implies (since ulp $\left.\left(y_{h}\right)=2^{-p+1} y_{h}\right)$,

$$
-2^{-p-1} y_{h} \leq \mathrm{RN}\left(y_{\ell} \cdot e\right) \leq 0
$$

which implies (since $2^{-p-1} y_{h}$ is a normal number)

$$
\left|y_{\ell} \cdot e\right| \leq\left(1+2^{-p}\right) \cdot 2^{-p-1} y_{h}
$$

q.e.d

Finally, from Properties 5, 6, and the previous observations, we deduce
Property 7. Assume that $y_{h}$ is a power of 2 , that $\frac{1}{4} \mathrm{ulp}\left(y_{h}\right)$ is a normal number, and that $\epsilon<1 /\left(1+2^{p+1}\right)$. If

$$
\begin{equation*}
e \geq \frac{\left(1+2^{-p}\right) \cdot(1-2 \epsilon)}{1-\epsilon-2^{p+1} \epsilon} \tag{14}
\end{equation*}
$$

then $y_{h}=\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right)$ implies $y_{h}=\mathrm{RN}(y)$.
Proof. As discussed above, we only have to consider the case where $y \leq y_{h}$ and $y_{\ell} \leq 0$. From properties 5 and 6 we immediately see that if

$$
\frac{\left(1+2^{-p}\right) \cdot 2^{-p-1} \cdot y_{h}}{e} \leq y_{h} \cdot \frac{2^{-p-1}-\frac{\epsilon}{1-\epsilon}}{1-\frac{\epsilon}{1-\epsilon}}
$$

then $y_{h}=\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right)$ will imply $y_{h}=\mathrm{RN}(y)$. This immediately gives Condition (14).

### 2.2 On the values of $e$ that allow for a correct Ziv rounding test

From Properties 4 and 7, we easily deduce
Theorem 2. Assume that $y_{h}$ is a floating-point number such that $\frac{1}{4} \mathrm{ulp}\left(y_{h}\right)$ is in the normal range, and that $\epsilon$ is less than $1 /\left(2^{p+1}+1\right)$. Also assume that $y_{h}=\operatorname{RN}\left(y_{h}+y_{\ell}\right)$ and $\left|\left(y_{h}+y_{\ell}\right)-y\right|<\epsilon \cdot|y|$. If

$$
e \geq \frac{1+2^{-p}}{1-\epsilon-2^{p+1} \epsilon}
$$

then $y_{h}=\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right)$ implies $y_{h}=\mathrm{RN}(y)$.
This means that in practice, we will perform the rounding test using

$$
e=\operatorname{RU}\left(\frac{1+2^{-p}}{1-\epsilon-2^{p+1} \epsilon}\right)
$$

where $\mathrm{RU}(t)$ is the smallest floating-point number larger than or equal to $t$.
Notice that if a fused multiply-add (fma) instruction is available, one may compute $\mathrm{RN}\left(y_{h}+y_{\ell} \cdot e\right)$ instead of $\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right)$, and get a very slightly better value of $e$. This will be dealt with in Section 3.2.

### 2.3 A result on near-optimality

Let us show that the value $e$ given by Theorem 2 is close to the best possible one, at least for some range of values of $\epsilon$. To do so, consider $\epsilon$ of the form

$$
\epsilon=\frac{2^{-p-k+1}+2^{-3 p}}{2-2^{-p}+2^{-3 p}}
$$

for $k>1$ and $k<p / 2$. For that $\epsilon$, the minimum value of $e$ given by Theorem 2 is
$\frac{1+2^{-p}}{1-\epsilon-2^{p+1} \epsilon}=\left(\frac{2^{k}}{2^{k}-2}\right)+\left(\frac{4^{k}}{\left(2^{k}-2\right)^{2}}\right) \cdot 2^{-p}-\left(\frac{-4^{k}+2^{1+k}-8^{k}}{\left(2^{k}-2\right)^{3}}\right) \cdot 2^{-2 p}+\ldots$
which is within around $2^{2-2 k}$ from $1+2^{1-k}$, and above it. One can easily check that for

$$
\left\{\begin{array}{l}
y_{h}=2-2^{-p+1} \\
y_{\ell}=2^{-p}-2^{-p-k+1} \\
y=2-2^{-p}+2^{-3 p} \\
e=1+2^{1-k}
\end{array}\right.
$$

we have $\left|y_{h}+y_{\ell}-y\right|<\epsilon \cdot|y|, y_{h}=\mathrm{RN}\left(y_{h}+y_{\ell}\right)$, and $y_{h}=\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right)$, and yet $y_{h} \neq \mathrm{RN}(y)$. This shows that for that value of $e$, that is slightly below the minimum value provided by Theorem 2, we do not always have a correct Ziv rounding.

This was a kind of "generic" example (for arbitrary values of $p$ ). For particular values of $p$ we have even more convincing examples

Assume the floating-point format is the binary64 (formerly called "double precision") of the IEEE 754 Standard. We have $p=53$. Consider

$$
y=\frac{1461983273612937874357096965722}{776934764230052409376713600323}
$$

and assume that we have

$$
\begin{aligned}
\epsilon & =2^{-80} \\
y_{h} & =\frac{2118642268759237}{1125899906842624} \\
y_{\ell} & =\frac{9007199188662643}{81129638414606681695789005144064} .
\end{aligned}
$$

One easily finds that

$$
\left|\left(y_{h}+y_{\ell}\right)-y\right|<\epsilon \cdot|y|
$$

more precisely, we have

$$
\frac{\left|\left(y_{h}+y_{\ell}\right)-y\right|}{\epsilon \cdot|y|} \approx 1-2.54811 \cdot 10^{-37}
$$

The bound on $e$ given by Theorem 2 is

$$
e^{*}=\frac{1+2^{-p}}{1-\epsilon-2^{p+1} \epsilon}=\frac{1208925819614629308923904}{1208925801600230665224191}
$$

One may easily check that $\operatorname{RN}\left(y_{h}+\operatorname{RN}\left(e^{*} \cdot y_{\ell}\right)\right) \neq y_{h}$. Now, if we choose

$$
e=\frac{4503599649443365}{4503599627370496}
$$

then we have

- $\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(e^{*} \cdot y_{\ell}\right)\right)=y_{h}$ whereas

$$
\mathrm{RN}(y)=\frac{8474569075036949}{4503599627370496} \neq y_{h}
$$

which shows that the chosen value of $e$ does not allow for a valid Ziv rounding test;

- and yet, $e / e^{*}=0.9999999900000000987640 \ldots$. the chosen value of $e$ is therefore extremely close to the bound given by Theorem 2 .

This example shows that, at least in some cases, Theorem 2 gives a very sharp bound.

### 2.4 A simpler bound for $e$

In Section 2.2, we give a bound on the value of $e$, such that if $e$ is larger than or equal to that bound, we have a correct Ziv rounding test. Let us call $e^{*}$ that bound, namely,

$$
e^{*}=\frac{1+2^{-p}}{1-\epsilon-2^{p+1} \epsilon} .
$$

Computing that bound $e^{*}$ (or, more precisely, computing $\mathrm{RU}\left(e^{*}\right)$, which is the smallest value of $e$ that is a floating-point number and that satisfies the condition given by Theorem 2) cannot be easily done in precision- $p$ floating-point arithmetic. In most cases this is not a problem: an elementary function library is designed once for all, and will be used many times so that it is worth spending time to develop it, possibly using a multiple-precision environment. This might be different if we consider the automatic design of ad-hoc programs for specific functions. Let us now suggest two simpler bounds that are easily computable. These simpler bounds will always be very close to $e^{*}$. Indeed, it will turn out that in all practical cases one of them will be smaller than the floatingpoint number larger than or equal to $e^{*}$, which implies that it will be the best possible one that one can deduce from Theorem 2. Assume that $1-2^{p+1} \epsilon$ is exactly representable in precision- $p$ floating-point arithmetic (this is not too much
a constraint in practice: $\epsilon$ will frequently be a power of 2 , and larger than $2^{-2 p}$ ). The two simpler bounds will be floating-point approximations to

$$
\frac{1+2^{-p+1}}{1-2^{p+1} \epsilon}
$$

more precisely, we will consider

$$
e_{\text {simple }}^{\mathrm{up}}=\operatorname{RU}\left(\frac{1+2^{-p+1}}{1-2^{p+1} \epsilon}\right),
$$

and

$$
e_{\text {simple }}^{\text {nearest }}=\operatorname{RN}\left(\frac{1+2^{-p+1}}{1-2^{p+1} \epsilon}\right),
$$

Notice that since $1+2^{-p+1}$ is a floating-point number, and since we have assumed that $1-2^{p+1} \epsilon$ is a floating-point number too, $e_{\text {simple }}^{\text {up }}$ and $e_{\text {simple }}^{\text {nearest }}$ are straightforwardly computed. Notice that $e_{\text {simple }}^{\text {up }}$ is always larger than or equal to $e$ and $e_{\text {simple }}^{\text {nearest }}$. We could not show that $e_{\text {simple }}^{\text {nearest }}$ is always larger than $e^{*}$, yet this is true in some important practical cases (See Property 9). However, using $e_{\text {simple }}^{\mathrm{up}}$, one is able to show the following result.
Property 8. If $\epsilon \leq 2^{-p-3}$ and $1-2^{p+1} \epsilon$ is a floating-point number (for instance, this will be satisfied if $\epsilon$ is a power of two larger than or equal to $2^{-2 p-1}$ ) then $e_{\mathrm{simple}}^{\mathrm{up}} \geq e^{*}$, and $e_{\text {simple }}^{\mathrm{up}} / e^{*}$ is less than $1+4 \cdot 2^{-p}$.
Proof. Define $\rho=e_{\text {simple }}^{\mathrm{up}} / e^{*}$. We easily find

$$
\frac{\frac{1+2^{-p+1}}{1-2^{p+1} \epsilon}}{\frac{1+2^{-p}}{1-\epsilon-2^{p+1} \epsilon}} \leq \rho \leq \frac{\frac{\left(1+2^{-p+1}\right) \cdot\left(1+2^{-p+1}\right)}{1-2^{p+1} \epsilon}}{\frac{1+2^{-p}}{1-\epsilon-2^{p+1} \epsilon}} .
$$

Let us denote $\omega=1-2^{p+1} \epsilon$. We have $1-\epsilon-2^{p+1} \epsilon=\omega\left(1-\frac{\epsilon}{\omega}\right)$, so that

$$
\frac{\frac{1+2^{-p+1}}{\omega}}{\frac{1+2^{-p}}{\omega\left(1-\frac{\epsilon}{\omega}\right)}} \leq \rho \leq \frac{\frac{\left(1+2^{-p+1}\right)^{2}}{\omega}}{\frac{1+2^{-p}}{\omega\left(1-\frac{\epsilon}{\omega}\right)}},
$$

which implies

$$
\frac{1+2^{-p+1}}{1+2^{-p}}\left(1-\frac{\epsilon}{\omega}\right) \leq \rho \leq \frac{\left(1+2^{-p+1}\right)^{2}}{1+2^{-p}}\left(1-\frac{\epsilon}{\omega}\right) \leq \frac{\left(1+2^{-p+1}\right)^{2}}{1+2^{-p}} .
$$

Elementary manipulation shows that for all $p \geq 1$, we have

$$
\frac{1+2^{-p+1}}{1+2^{-p}} \geq 1+2^{-p}-2^{-2 p}
$$

and

$$
\frac{\left(1+2^{-p+1}\right)^{2}}{1+2^{-p}} \leq 1+4 \cdot 2^{-p}
$$

Furthermore, $\epsilon \leq 2^{-p-3}$ implies $\omega \geq 3 / 4$ and $1-\epsilon / \omega \leq 1-2^{-p-1} 3$, from which we get, since $\left(1+2^{-p}-2^{-2 p}\right) \cdot\left(1-2^{-p-1} 3\right)=1+\frac{5}{6} 2^{-p}-\frac{7}{6} 2^{-2 p}+\frac{1}{6} 2^{-3 p}$ is larger than 1 as soon as $p \geq 1$,

$$
1 \leq \rho<1+4 \cdot 2^{-p}
$$

Now, for special values of $p$ (namely, those corresponding to the binary interchange formats of the IEEE Standard for Floating-Point Arithmetic and to the INTEL "double-extended precision" format), and special values of $\epsilon$ (those of the form $\gamma \cdot 2^{-p-k}$, with $\gamma=1,3 / 4,5 / 8$, or $7 / 8$, and $3 \leq k \leq p+1$ ), one can get an even more interesting result:

Property 9. If $p \in\{11,24,53,64,113\}$, and $\epsilon=\gamma \cdot 2^{-p-k}$, with $\gamma=1,3 / 4,5 / 8$, or $7 / 8$, and $3 \leq k \leq p+1$, then

$$
e^{*} \leq e_{\text {simple }}^{\text {nearest }}<e^{*}+\operatorname{ulp}\left(e^{*}\right)
$$

To prove that property, we just enumerated all possible cases. When the conditions of Property 9 hold, $e_{\text {simple }}^{\text {nearest }}$ is equal to the smallest floating-point number larger than or equal to $e^{*}$ : it is therefore not possible to deduce a better constant $e$ from Theorem 2.

## 3 Additional results

### 3.1 When the test fails, it gives us information anyway

Let us now see that, under mild conditions on $\epsilon$ and $e$, when we perform a correct Ziv rounding test and the test fails (i.e., we cannot be sure that $y_{h}=$ $\mathrm{RN}(y)$ ), we get some information anyway. More precisely, we have the following property:

Theorem 3. If

- $p \geq 2$;
- $\epsilon<1 /\left(2^{p+3}+9\right)$;
- the chosen value of $e$ is

$$
\operatorname{RU}\left(\frac{1+2^{-p}}{1-\epsilon-2^{p+1} \epsilon}\right)
$$

(i.e., the smallest one for which Theorem 2 guarantees that we have a correct Ziv rounding test),
then when the test fails, i.e., when we have

$$
\begin{gather*}
y_{h}+y_{\ell}=y \cdot(1+\alpha), \quad \text { with } \quad|\alpha| \leq \epsilon,  \tag{15}\\
y_{h}=\operatorname{RN}\left(y_{h}+y_{\ell}\right),
\end{gather*}
$$

and

$$
y_{h} \neq \mathrm{RN}\left(y_{h}+\mathrm{RN}\left(e \cdot y_{\ell}\right)\right)
$$

the two numbers $y_{h}$ and $y_{c}=\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(e \cdot y_{\ell}\right)\right)$ are consecutive floating-point numbers, and $y$ is between them, which implies that $\mathrm{RN}(y)$ is either $y_{h}$ or $y_{c}$.

Theorem 3 is especially useful when implementing a locally monotonic elementary function $f$ at point $x$. It essentially means that it suffices to approximate

$$
f^{-1}\left(\frac{y_{h}+y_{c}}{2}\right)
$$

with large enough accuracy to determine if the correct value to be returned is $y_{h}$ or $y_{c}$. This is of interest when $f^{-1}$ is simpler to approximate than $f$ (e.g., when $f$ is the arcsin function). Theorem 3 is also useful when implementing heterogeneous operations. Consider the problem presented in Section 1.2.2. When the test fails, assuming $p^{\prime} \geq p+1,\left(y_{h}+y_{c}\right) / 2$ is a precision- $p^{\prime}$ floatingpoint number, so that it can easily be compared to $a^{\left(p^{\prime}\right)}+b^{\left(p^{\prime}\right)}$.

Proof of Theorem 3. Without loss of generality, we assume that $y_{h}$ and $y$ are positive. Assume $p \geq 2$ and $\epsilon<1 /\left(2^{p+3}+9\right)$. First, elementary manipulation shows that

$$
\frac{1+2^{-p}}{1-\epsilon-2^{p+1} \epsilon}<2
$$

which implies $e \leq 2$.
Now, from $y_{\ell}=\mathrm{RN}\left(y_{h}+y_{\ell}\right)$ and $e \leq 2$, we easily deduce that

- $\left|\mathrm{RN}\left(e y_{\ell}\right)\right| \leq \operatorname{ulp}\left(y_{h}\right)$ if $y_{h}$ is not a power of 2 or $y_{\ell}$ has the same sign as $y_{h}$; and
- $\left|\mathrm{RN}\left(e y_{\ell}\right)\right| \leq \frac{1}{2} \mathrm{ulp}\left(y_{h}\right)$ if $y_{h}$ is a power of 2 and $y_{\ell}$ has not the same sign as $y_{h}$.

A consequence of this is that $y_{c}=\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(e \cdot y_{\ell}\right)\right)$ is either $y_{h}$ or the floatingpoint predecessor or successor of $y_{h}$.

Since $e \leq 2$, one easily deduces that if $\left|y_{\ell}\right| \leq \frac{1}{8} \operatorname{ulp}\left(y_{h}\right)$ then $\left|\mathrm{RN}\left(e y_{\ell}\right)\right| \leq$ $\frac{1}{4} \operatorname{ulp}\left(y_{h}\right)$, so that $y_{c}=\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(e \cdot y_{\ell}\right)\right)=y_{h}$ (this is straightforward if $y_{h}$ is not a power of two, and this is a consequence of the round to nearest even rule if $y_{h}$ is a power of two). As a consequence, if $y_{c} \neq y_{h}$ then, necessarily, $\left|y_{\ell}\right|>\frac{1}{8} \operatorname{ulp}\left(y_{h}\right)>2^{-p-3} y_{h}$. Now, from (15) we have

$$
\begin{equation*}
y-y_{h}=y_{\ell}\left(1-\alpha \frac{y}{y_{\ell}}\right) \tag{16}
\end{equation*}
$$

with $|\alpha| \leq \epsilon$. We have just shown above that $y_{c} \neq y_{h}$ implies

$$
\begin{equation*}
\left|\frac{y_{h}}{y_{\ell}}\right|<2^{p+3} \tag{17}
\end{equation*}
$$

Also, $\mathrm{RN}\left(y_{h}+y_{\ell}\right)=y_{h}$ implies

$$
\begin{equation*}
\left|\frac{y_{h}}{y_{\ell}}\right| \geq 2^{p} \tag{18}
\end{equation*}
$$

We have

$$
y=\frac{y_{h}+y_{\ell}}{1+\alpha}
$$

therefore, using (18),

$$
\left|\frac{y}{y_{\ell}}\right| \leq \frac{y_{h}+\left|y_{\ell}\right|}{\left|y_{\ell}\right|(1+\alpha)} \leq \frac{y_{h}\left(1+2^{-p}\right)}{\left|y_{\ell}\right| \cdot(1-\epsilon)},
$$

therefore, using (17),

$$
\left|\frac{y}{y_{\ell}}\right|<2^{p+3} \cdot \frac{1+2^{-p}}{1-\epsilon}
$$

therefore

$$
\begin{equation*}
|\alpha| \cdot\left|\frac{y}{y_{\ell}}\right|<2^{p+3} \cdot\left(1+2^{-p}\right) \cdot \frac{\alpha}{1-\epsilon} \leq 2^{p+3} \cdot\left(1+2^{-p}\right) \cdot \frac{\epsilon}{1-\epsilon} . \tag{19}
\end{equation*}
$$

Elementary manipulation shows that the right-hand side of (19) will be less than 1 as soon as $\epsilon<1 /\left(2^{p+3}+9\right)$. Hence, using (16), as soon as $\epsilon<1 /\left(2^{p+3}+9\right)$, if $y_{c} \neq y_{h}$ then i) $y-y_{h}$ and $y_{\ell}$ have the same sign, and ii) $y$ is less than or equal to the floating-point successor of $y_{h}$, and larger than or equal to its floatingpoint predecessor. Therefore:

- if $y_{\ell}>0$ then $y>y_{h}$ and (straightforwardly), $y_{c}>y_{h}$;
- if $y_{\ell}<0$ then $y<y_{h}$ and (straightforwardly), $y_{c}<y_{h}$.

Hence, when the test fails, $y$ is between the two consecutive floating-point numbers $y_{c}$ and $y_{h}$, which implies that $\mathrm{RN}(y)$ is either $y_{h}$ or $y_{c}$.

### 3.2 When an fma instruction is available

If an fma instruction is available, we can use it to change the test (5) into another one that is very slightly better in the general case and, more important, that works even if $\frac{1}{4} \mathrm{ulp}\left(y_{h}\right)$ is subnormal. The fma instruction evaluates expressions of the form

$$
\mathrm{RN}(a+b c)
$$

It is available on processors such at the Intel Itanium, IBM PowerPC, AMD Bulldozer, and will be available on the Intel Haswell. It allows for faster and,
in general, more accurate dot products, matrix multiplications, and polynomial evaluations. The fma instruction is included in the new IEEE 754-2008 standard for floating-point arithmetic, so that within a few years, it will probably be available on most general-purpose platforms.

If an fma instruction is available, we may want to replace the test

$$
\text { is } y_{h} \text { equal to } \mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \cdot e\right)\right) \text { ? }
$$

by the test
is $y_{h}$ equal to $\mathrm{RN}\left(y_{h}+y_{\ell} \cdot e\right)$ ?
Analyzing that new test, that is, trying, given $\epsilon$, to find a bound $e^{*}$ on $e$ such that if $e \geq e^{*}$ then

$$
\begin{equation*}
y_{h}=\mathrm{RN}\left(y_{h}+y_{\ell} \cdot e\right) \Rightarrow y_{h}=\mathrm{RN}(y), \tag{20}
\end{equation*}
$$

is done in a way very similar to what we did in Section 2. Assuming $y_{h}$ is in the normal range, Property 3 becomes:

Assume $y_{h}$ is not a power of 2 . If $y_{h}=\mathrm{RN}\left(y_{h}+y_{\ell} \cdot e\right)$ then $\left|y_{\ell}\right| \leq \operatorname{ulp}\left(y_{h}\right) / 2 e$.
That is, compared to Property 3, we get rid of the term $\left(1+2^{-p}\right)$ and of the requirement that $\frac{1}{2} \operatorname{ulp}\left(y_{h}\right)$ should be in the normal range. A consequence is that we have a property analogous to Property 4, without that term and that requirement. Properties 6 and 7 are transformed in a similar way, and we deduce:

Theorem 4. Assume that $y_{h}$ is a normal floating-point number and that $\epsilon$ is less than $1 /\left(2^{p+1}+1\right)$. Also assume that $y_{h}=\mathrm{RN}\left(y_{h}+y_{\ell}\right)$ and $\left|\left(y_{h}+y_{\ell}\right)-y\right|<\epsilon \cdot|y|$. If

$$
e \geq \frac{1}{1-\epsilon-2^{p+1} \epsilon}
$$

then $y_{h}=\mathrm{RN}\left(y_{h}+y_{\ell} \cdot e\right)$ implies $y_{h}=\mathrm{RN}(y)$.

### 3.3 If $\frac{1}{4} \operatorname{ulp}\left(y_{h}\right)$ is subnormal and there is no fma instruction

Now, if $\frac{1}{4} \mathfrak{u l p}\left(y_{h}\right)$ is subnormal, then the relative error committed when rounding $y_{\ell} \cdot e$ may be very large. If we try to adapt the proof of Proposition 3, we easily find that from

$$
\left|\mathrm{RN}\left(y_{\ell} \cdot e\right)\right| \leq \frac{1}{2} \mathrm{ulp}\left(y_{h}\right)
$$

the only thing we can deduce is

$$
\left|y_{\ell} \cdot e\right| \leq \frac{3}{4} \operatorname{ulp}\left(y_{h}\right)
$$

Therefore, the bound of Theorem 2 becomes

$$
\begin{equation*}
e \geq \frac{3 / 2}{1-\epsilon-2^{p+1} \epsilon} \tag{21}
\end{equation*}
$$

which is quite bad. Unfortunately, at least in some cases, values of e slightly below that bound will fail. Consider for instance:

$$
\left\{\begin{array}{l}
y=2^{k_{\min }+1}+2^{k_{\min }-p+1}+2^{k_{\min }-2 p} \\
y_{h}=2^{k_{\min }+1} \\
y_{\ell}=2^{k_{\min }-p+1} \\
\epsilon=2^{-2 p-1} .
\end{array}\right.
$$

For which we have

$$
y_{h}=\operatorname{RN}\left(y_{h}+y_{\ell}\right),
$$

and

$$
\left|\left(y_{h}+y_{\ell}\right)-y\right| \leq \epsilon \cdot|y| .
$$

The bound on $e$ provided by (21) is

$$
e^{*}=\frac{3 / 2}{1-2^{-p}-2^{-2 p-1}} \approx \frac{3}{2} \cdot\left(1+2^{-p}\right) .
$$

If we choose the following value of $e$, which is only very slightly below $e^{*}$ :

$$
e=\frac{3}{2}-2^{-p+1}=\underbrace{1.0111 \cdots 1}_{p \text { bits }},
$$

then we have

$$
\mathrm{RN}\left(y_{\ell} \cdot e\right)=2^{k_{\min }-p+1},
$$

so that

$$
\operatorname{RN}\left(y_{h}+\operatorname{RN}\left(y_{\ell} \cdot e\right)\right)=y_{h},
$$

and yet,

$$
\mathrm{RN}(y)=2^{k_{\min }+1}+2^{k_{\min }-p+2} \neq y_{h} .
$$

This example shows that if $/ u l p\left(y_{h}\right) / 4$ can be subnormal, it is much preferable to perform Ziv's rounding test using an fma instruction.

It is a real problem for the implementation of the heterogeneous operations: small values must be handled separately. However, if we consider the problem of evaluating usual transcendental functions with correct rounding, this problem will almost never need to be considered. Assume we wish to compute $f(x)$, for an input floating-point number $x$ such that $f(x)$ is very near zero:

- if $x$ is very near zero too, for most usual functions, we can directly deduce the correctly rounded value of $f(x)$ from the Taylor series expansion of $f$ at 0 . Consider for instance the case of the sine function. Assume we wish to evaluate $\sin (x)$, where $x>0$ is a tiny floating-point number. Let $x=m_{x} \cdot 2^{e_{x}}$, with $1 \leq m_{x}<2$. Assume $x$ is not a power of 2 (that case is easily handled separately) If

$$
x<\sqrt{3} \cdot 2^{-p / 2},
$$

then

$$
m_{x} \cdot x^{2}<6 \cdot 2^{-p}
$$

so that

$$
m_{x}^{3} \cdot 2^{3 e_{x}}<6 \cdot 2^{e_{x}-p}
$$

which implies

$$
\frac{x^{3}}{6}<2^{e_{x}-p}=\frac{1}{2} \operatorname{ulp}(x)
$$

As a consequence, all values between $x-x^{3} / 6$ and $x$ will round to $x$, which implies $\mathrm{RN}(\sin (x))=x$. Similar reasonings can be done for various functions and, possibly, better bounds can be obtained for particular values of $p$. Table 1 shows how to handle many usual functions in the double-precision/binary64 format of IEEE 754.

Table 1: Some results for small values in the double-precision/binary64 format, assuming rounding to nearest (that table is extracted from [11]). These results make evaluating these functions at tiny arguments straightforward. They can be generalized to other precisions. The number $\alpha=\mathrm{RN}\left(3^{1 / 3}\right) \times 2^{-26}$ is approximately equal to $1.4422 \cdots \times 2^{-26}$, and $\eta \approx 1.1447 \times 2^{-26}$. If $x$ is a real number, we let $x^{-}$denote the largest floating-point number strictly less than $x$, and $x^{+}$ denote the smallest floating-point number strictly larger than $x$.

| This function | can be <br> replaced by | when |
| :--- | :---: | :--- |
| $\exp (\epsilon)-1$ | $\epsilon$ | $\|\epsilon\|<\mathrm{RN}(\sqrt{2}) \times 2^{-53}$ |
| $\exp (\epsilon)-1, \epsilon \geq 0$ | $\epsilon^{+}$ | $\mathrm{RN}(\sqrt{2}) \times 2^{-53}$ <br> $\leq \epsilon<\mathrm{RN}(\sqrt{3}) \times 2^{-52}$ |
| $\ln (1+\epsilon)$ | $\epsilon$ | $\|\epsilon\|<\mathrm{RN}(\sqrt{2}) \times 2^{-53}$ |
| $\sin (\epsilon), \sinh (\epsilon), \sinh ^{-1}(\epsilon)$ | $\epsilon$ | $\|\epsilon\| \leq \alpha=\mathrm{RN}\left(3^{1 / 3}\right) \times 2^{-26}$ |
| $\arcsin (\epsilon)$ | $\epsilon$ | $\|\epsilon\|<\alpha=\mathrm{RN}\left(3^{1 / 3}\right) \times 2^{-26}$ |
| $\sin (\epsilon), \sinh ^{-1}(\epsilon)$ | $\epsilon^{-}=\epsilon-2^{-78}$ | $\alpha<\epsilon \leq 2^{-25}$ |
| $\sinh (\epsilon)$ | $\epsilon^{+}=\epsilon+2^{-78}$ | $\alpha<\epsilon<2^{-25}$ |
| $\arcsin (\epsilon)$ | $\epsilon^{+}=\epsilon+2^{-78}$ | $\alpha \leq \epsilon<2^{-25}$ |
| $\tan (\epsilon), \tanh ^{-1}(\epsilon)$ | $\epsilon$ | $\|\epsilon\|<\eta=\mathrm{RN}\left(12^{1 / 3}\right) \times 2^{-27}$ |
| $\tanh (\epsilon), \arctan (\epsilon)^{\tan (\epsilon), \tanh ^{-1}(\epsilon)} \quad \epsilon^{+}=\epsilon+2^{-78}$ | $\eta \leq \epsilon \leq 1.650 \times 2^{-26}$ |  |
| $\arctan (\epsilon), \tanh ^{\prime}(\epsilon)$ | $\epsilon^{-}=\epsilon-2^{-78}$ | $\eta<\epsilon \leq 1.650 \times 2^{-26}$ |

- in practice, for larger arguments, when $x$ is a floating-point number, $f(x)$ is always far from the subnormal range, even if, in theory, there is a
root of $f$ near $x$ : for instance, for the trigonometric functions, using a continued-fraction argument first presented by Kahan [6], one can show that the double-precision/binary64 number larger than 1 which is nearest an integer multiple of $\pi / 2$ is

$$
6381956970095103 \times 2^{+797}
$$

whose cosine is around $-4.69 \times x^{-19}: 1 / 4$ ulp of that value is far above the subnormal threshold.

## 4 Experiments

### 4.1 General remarks

We have limited our tests to IEEE binary64 floating-point numbers ( $p=53$ ): both $y_{h}$ and $y_{\ell}$ are of the type "double" of the C language. We (allegedly) tried to approximate an arbitrary precision number $y$, with relative error bounded by $\epsilon$. We also have limited ourselves to peculiar values of $y_{\ell}$, where it is positive (the negative case is symmetric), and close to ( $1 / 2$ ) ulp $\left(y_{h}\right)$ since, considering the small values of $e$ that we obtain for reasonable values of $\epsilon$, nothing special happens for values of $y_{\ell}$ notably smaller than $(1 / 2) \operatorname{ulp}\left(y_{h}\right)$. The value of $y$ is chosen extremely close to the limit imposed by $\epsilon$. When $y_{\ell}$ is very close to $(1 / 2) \operatorname{ulp}\left(y_{h}\right), y$ is almost guaranteed to be larger than or equal to $y_{h}+(1 / 2) \operatorname{ulp}\left(y_{h}\right)$ unless $\epsilon$ is very small. Therefore our experiments can be viewed as mimicking a worst case situation, where $y$ is always cornered close to the upper bound rather than having its value scattered all over the interval $\left(\left(y_{h}+y_{\ell}\right) \times(1-\epsilon),\left(y_{h}+y_{\ell}\right) \times(1+\epsilon)\right)$.

Every result of our tests falls in one of the following categories:

- positive: $y_{h}=\mathrm{RN}\left(y_{h}+R N\left(y_{\ell} \times e\right)\right)$ and $y_{h}=\mathrm{RN}(y)$. In such a case, we are done: Ziv 's rounding test indicates that $y_{h}$ is the correct rounding of $y$;
- false positive: $y_{h}=\mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \times e\right)\right)$ and $y_{h} \neq \mathrm{RN}(y)$. That case must never occur, since it means that the rounding test is not correct;
- negative: $y_{h} \neq \mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \times e\right)\right)$ and $y_{h} \neq \mathrm{RN}(y)$. In such a case, a different strategy (e.g. a tighter approximation) should be used;
- false negative: $y_{h} \neq \mathrm{RN}\left(y_{h}+\mathrm{RN}\left(y_{\ell} \times e\right)\right)$ and $y_{h}=\mathrm{RN}(y)$, which causes us to unduly switch to a more expensive strategy. This case must occur as scarcely as possible.

More precisely, the various parameters involved in our experiments satisfy:

- $y_{h} \in[1,2)$;
- $y_{\ell} \in\left[15 / 32 \mathrm{ulp}\left(y_{h}\right), 16 / 32 \mathrm{ulp}\left(y_{h}\right)\right)$;
- $\epsilon \in\left(2^{-81}, 2^{-55}\right]$;
- as a consequence, $e \in[1.0000000074505808,2.0000000000000004]$.


## 4.2 $\mathbf{Z i v}$ rounding test and $\epsilon$ (for $p=53$ )

The results of our experiments (100, 000, 000 values $\epsilon$ chosen randomly for each interval $\left[2^{-k}, 2^{-k+1}\right.$ ), for $k$ from -81 to -56 , one random $y_{h}$ in $[1,2)$ and one random $y_{\ell}$ in $\left[15 / 32 \mathrm{ulp}\left(y_{h}\right), 16 / 32 \mathrm{ulp}\left(y_{h}\right)\right)$ for each $\epsilon$ ) are presented in Table 2. For the largest values of $\epsilon$, the values of $e$ are so large that test fails system-

Table 2: Results of our experiments, for $\epsilon \in\left(2^{-81}, 2^{-55}\right]$

| $\epsilon$ | Positives | False Positives | Negatives | False Negatives |
| :---: | ---: | ---: | ---: | ---: |
| $\left[2^{-55}, 2^{-56}[ \right.$ | 0 | 0 | 100000000 | 0 |
| $\left[2^{-56}, 2^{-57}[ \right.$ | 0 | 0 | 99997603 | 2397 |
| $\left[2^{-57}, 2^{-58}[ \right.$ | 23284922 | 0 | 76116019 | 599059 |
| $\left[2^{-58}, 2^{-59}[ \right.$ | 61962104 | 0 | 37740947 | 296949 |
| $\left[2^{-59}, 2^{-60}[ \right.$ | 80926366 | 0 | 18924027 | 149607 |
| $\left[2^{-60}, 2^{-61}[ \right.$ | 90593819 | 0 | 9332714 | 73467 |
| $\left[2^{-61}, 2^{-62}[ \right.$ | 95167993 | 0 | 4794367 | 37640 |
| $\left[2^{-62}, 2^{-63}[ \right.$ | 97625196 | 0 | 2356226 | 18578 |
| $\left[2^{-63}, 2^{-64}[ \right.$ | 98776533 | 0 | 1213736 | 9731 |
| $\left[2^{-64}, 2^{-65}[ \right.$ | 99399403 | 0 | 596044 | 4553 |
| $\left[2^{-65}, 2^{-66}[ \right.$ | 99703144 | 0 | 294513 | 2343 |
| $\left[2^{-66}, 2^{-67}[ \right.$ | 99853799 | 0 | 145113 | 1088 |
| $\left[2^{-67}, 2^{-68}[ \right.$ | 99924441 | 0 | 74914 | 645 |
| $\left[2^{-68}, 2^{-69}[ \right.$ | 99962425 | 0 | 37261 | 314 |
| $\left[2^{-69}, 2^{-70}[ \right.$ | 99981576 | 0 | 18286 | 138 |
| $\left[2^{-70}, 2^{-71}[ \right.$ | 99990768 | 0 | 9154 | 78 |
| $\left[2^{-71}, 2^{-72}[ \right.$ | 99995282 | 0 | 4677 | 41 |
| $\left[2^{-72}, 2^{-73}[ \right.$ | 99997740 | 0 | 2239 | 21 |
| $\left[2^{-73}, 2^{-74}[ \right.$ | 99998842 | 0 | 1148 | 10 |
| $\left[2^{-74}, 2^{-75}[ \right.$ | 99999413 | 0 | 584 | 3 |
| $\left[2^{-75}, 2^{-76}[ \right.$ | 99999715 | 0 | 284 | 1 |
| $\left[2^{-76}, 2^{-77}[ \right.$ | 99999853 | 0 | 147 | 0 |
| $\left[2^{-77}, 2^{-78}[ \right.$ | 99999928 | 0 | 70 | 0 |
| $\left[2^{-78}, 2^{-79}[ \right.$ | 99999955 | 0 | 44 | 0 |
| $\left[2^{-79}, 2^{-80}[ \right.$ | 99999981 | 99999991 | 0 | 19 |
| $\left[2^{-80}, 2^{-81}[ \right.$ | 9 | 0 | 0 |  |

atically (yielding a "negative" result) because $y$ is certain to be larger than or equal to $y_{h}+\operatorname{ulp}\left(y_{h}\right)$ and $R N\left(y_{\ell} \times e\right) \geq(1 / 2) \operatorname{ulp}\left(y_{h}\right)$. As $\epsilon$ decreases, so does $e$ and $y$ has less latitude to wander beyond $y_{h}+\operatorname{ulp}\left(y_{h}\right)$.

As a consequence, the number of "negatives" decreases and so does the number of "false negatives". At some point, the "false negatives" seem to totally vanish: the probability of a false negative is too low for there to be one among the tested cases, and all we have left is a small and decreasing (by a factor 2 each time $\epsilon$ itself decreases by a factor 2 ) number of "negatives".

The comforting result is that, at no point, do we have a "false positive".

### 4.3 Near optimality of the value of $e$ given by Theorem 2

To minimize the number of "false negatives" $e$ must be as small as possible (and yet large enough to allow for a correct Ziv testing). Hence, it is important to know how optimal is the value of $e$ given by Theorem 2. Let us call $e^{*}$ that value. We want to know how much we can subtract from $e^{*}$ before we start getting "false positive" values. In several experiments, we got our first false positive value at $e^{*}-2^{21} \times$ ulp $(e)$. "False positive" obtained in a specific run, with value $e^{*}-2^{21} \times \operatorname{ulp}(e)$, are given in Table 3.

Table 3: False positives for $e=e^{*}-2^{21} \times \operatorname{ulp}(e)$

| Example 1 |  |  |
| :---: | :---: | :---: |
| $y_{h}=1.9947587312896187$ | $y_{\ell}=1.1102229216237452 e-16$ | $\epsilon=5.1757461373254229 e-24$ |
| $\mathrm{RN}(y)=1.9947587312896189$ | $\begin{aligned} & y=1.99475873128961878055 \\ & 77597160974603068928590135 \\ & 23728434792028406527975734 \\ & 416324 \end{aligned}$ | $e=1.000000092772301$ |
| Example 2 |  |  |
| $y_{h}=1.9883894880686108$ | $y_{\ell}=1.110222993284877 e-16$ | $\epsilon=1.5927638864921235 e-24$ |
| $\mathrm{RN}(y)=1.988389488068611$ | $\begin{aligned} & y=1.98838948806861093299 \\ & 14757256952656859493280554 \\ & 75352754840860468630843325 \\ & 121040 \end{aligned}$ | $e=1.0000000282270229$ |
| Example 3 |  |  |
| $y_{h}=1.9802155978993843$ | $y_{\ell}=1.1102230085276806 e-16$ | $\epsilon=8.2224179116152375 e-25$ |
| $\mathrm{RN}(y)=1.9802155978993845$ | $\begin{aligned} & \hline y=1.98021559789938439433 \\ & 14345963756318263894453847 \\ & 32167637700306671773559863 \\ & 566352 \end{aligned}$ | $e=1.0000000143465304$ |

### 4.4 Simplified and accurate formulas

As noticed in Section 2.4, two simpler bounds, $e_{\text {simple }}^{\text {nearest }}$ and $e_{\text {simple }}^{\text {up }}$, can be computed in precision- $p$ floating-point arithmetic. Since $e_{\text {simple }}^{\text {nearest }}$ is always less than or equal to $e_{\text {simple }}^{\text {up }}$, it is preferable to choose $e=e_{\text {simple }}^{\text {nearest }}$, whenever we are sure that this yields to a correct Ziv rounding test. Unfortunately, we could not prove that $e_{\text {simple }}^{\text {nearest }}$ is always larger than or equal to the bound $e^{*}$ of Theorem 2 (we could only show that for some specific values of $\epsilon$, for the binary formats specified by the IEEE 754-2008 Standard, see Property 9). Below are the results of some tests done in the binary64 format, for approximation accuracy bounds $\epsilon$ varying from $2^{-56}$ to $2^{-90}$. Notice that in all the previous developments, we have considered $\epsilon \leq 2^{-p-2}$. In our case that should give $\epsilon<2^{-55}$. But when $\epsilon \in\left(2^{-56}, 2^{-55}\right]$ we have found many cases where $e_{\text {simple }}^{\text {nearest }}<e^{*}$. This is a limited annoyance since, as can be found up from section 4.2, in this approximation accuracy interval the test is far from pertinent (all our corner cases are "negative", but that could also be true for many of then in more "conventional" situations).

| max relative error $\frac{e_{\text {simple }}^{\text {nearest }}-e^{*}}{e^{*}}$ | $2.2204460492339477 \mathrm{e}-16$ |
| :--- | :--- |
| $e_{\text {simple }}^{\text {nearest }}=e^{*}$ cases | 1514615533 |
| Total number of cases | 3500000000 |

## Conclusion

We have given and proven conditions on the constant $e$ of Ziv's rounding test that allow one to make sure that the test is always reliable (i.e., we never have "false positives"). Furthermore, these conditions are very tight, and for values of the approximation error $\epsilon$ that make sense in practice, the probability of observing a "false negative" (i.e., the test makes us believe that a more accurate computation is required, whereas $y_{h}$ was already the right answer) is very low. This makes Ziv's rounding test an excellent solution for checking whether we can return a correctly-rounded result or not. There are several already-existing applications of that test for implementing correctly-rounded elementary functions, and there are interesting potential applications for implementing "heterogeneous" arithmetic operations.

## References

[1] S. Boldo and G. Melquiond. Emulation of FMA and correctly rounded sums: proved algorithms using rounding to odd. IEEE Transactions on Computers, 57(4):462-471, April 2008.
[2] Florent de Dinechin, Alexey V. Ershov, and Nicolas Gast. Towards the post-ultimate libm. In Proceedings of the 17th IEEE Symposium on Computer Arithmetic, ARITH '05, pages 288-295, Washington, DC, USA, 2005. IEEE Computer Society.
[3] T. J. Dekker. A floating-point technique for extending the available precision. Numerische Mathematik, 18(3):224-242, 1971.
[4] IEEE Computer Society. IEEE Standard for Floating-Point Arithmetic. IEEE Standard 754-2008, August 2008. available at http://ieeexplore. ieee.org/servlet/opac?punumber=4610933.
[5] C.-P. Jeannerod, N. Louvet, J.-M. Muller, and A. Panhaleux. Midpoints and exact points of some algebraic functions in floating-point arithmetic. IEEE Transactions on Computers, 60(2), February 2011.
[6] W. Kahan. Minimizing q*m-n. Text accessible electronically at http: //http.cs.berkeley.edu/~wkahan/. At the beginning of the file "nearpi.c", 1983.
[7] W. Kahan. A logarithm too clever by half. Available at http://http. cs.berkeley.edu/~wkahan/LOG10HAF.TXT, 2004.
[8] D. Knuth. The Art of Computer Programming, volume 2. Addison-Wesley, Reading, MA, 3rd edition, 1998.
[9] D. Monniaux. The pitfalls of verifying floating-point computations. ACM TOPLAS, 30(3):1-41, 2008. A preliminary version is available at http: //hal.archives-ouvertes.fr/hal-00128124.
[10] J.-M. Muller. Elementary Functions, Algorithms and Implementation. Birkhäuser Boston, MA, 2nd edition, 2006.
[11] Jean-Michel Muller, Nicolas Brisebarre, Florent de Dinechin, ClaudePierre Jeannerod, Vincent Lefèvre, Guillaume Melquiond, Nathalie Revol, Damien Stehlé, and Serge Torres. Handbook of Floating-Point Arithmetic. Birkhäuser Boston, 2010. ACM G.1.0; G.1.2; G.4; B.2.0; B.2.4; F.2.1., ISBN 978-0-8176-4704-9.
[12] A. Neumaier. Rundungsfehleranalyse einiger Verfahren zur Summation endlicher Summen. ZAMM, 54:39-51, 1974. In German.
[13] T. Ogita, S. M. Rump, and S. Oishi. Accurate sum and dot product. SIAM Journal on Scientific Computing, 26(6):1955-1988, 2005.
[14] M. Pichat. Correction d'une somme en arithmétique à virgule flottante. Numerische Mathematik, 19:400-406, 1972. In French.
[15] A. Ziv. Fast evaluation of elementary mathematical functions with correctly rounded last bit. ACM Transactions on Mathematical Software, 17(3):410-423, September 1991.


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[^1]:    ${ }^{1}$ We assume that $f(x)$ is not exactly halfway between two consecutive floating-point numbers. This case can be shown to never happen in nontrivial cases with the most usual transcendental functions such as $\sin , \exp , \ln , \ldots$, and concerning the most frequent algebraic functions, this problem is partly dealt with in [5].

[^2]:    ${ }^{2}$ Unless underflow occurs, but for the usual floating-point formats, an underflow in precision $p^{\prime}$ will correspond to a number of absolute value under the smallest nonzero floating-point number in precision $p$.

[^3]:    ${ }^{3}$ Unless $y_{\ell}=(1 / 2) \operatorname{ulp}\left(y_{h}\right)$ exactly, but this case corresponds to the case when $a^{\left(p^{\prime}\right)}+b^{\left(p^{\prime}\right)}$ is extremely close to a precision- $p$ midpoint: in such a case, the test will correctly reply that we are unable to conclude whether $y_{h}$ equals $\mathrm{RN}\left(a^{\left(p^{\prime}\right)}+b^{\left(p^{\prime}\right)}\right)$ or not.

[^4]:    ${ }^{4}$ Assuming round-to-nearest, ties-to-even rounding direction attribute, which is the default.

