ANOMALOUS FLUCTUATIONS FOR A PERTURBED HAMILTONIAN SYSTEM WITH EXPONENTIAL INTERACTIONS

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ABSTRACT. A one-dimensional Hamiltonian system with exponential interactions perturbed by a conservative noise is considered. It is proved that energy superdiffuses and upper and lower bounds describing this anomalous diffusion are obtained.

1. INTRODUCTION

Over the last decade, transport properties of one-dimensional Hamiltonian systems consisting of coupled oscillators on a lattice have been the subject of many theoretical and numerical studies, see the review papers [9, 11, 23]. Despite many efforts, our knowledge of the fundamental mechanisms necessary and/or sufficient to have a normal diffusion remains very limited. Nevertheless, it has been recognized that conservation of momentum plays a major role and numerical simulations provide a strong evidence of the fact that one dimensional chains of anharmonic oscillators conserving momentum are superdiffusive.

An interesting area of current research consists in studying this problem for hybrid models where a stochastic perturbation is superposed to the deterministic evolution. Even if the problem is considerably simplified, several open challenging questions can be addressed for these systems. In [2] it is proved that the thermal conductivity of an unpinned harmonic chain of oscillators perturbed by an energy-momentum conservative noise is infinite while if a pinning potential (destroying momentum conservation) is added it is finite. In the same paper, diverging upper bounds are provided when some nonlinearities are added. This does not, however, exclude the possibility of having a finite conductivity. Therefore much more interesting would be to obtain lower bounds showing that the conductivity is infinite and that energy superdiffuses, but this problem is left open in [2].

In [7], has been introduced and studied numerically, a class of Hamiltonian models for which anomalous diffusion is observed. There, the investigated systems present strong analogies with standard chains of oscillators. They can be described as follows. Let V and U be two non-negative potentials on $\mathbb{R}$ and consider the Hamiltonian system $(\mathbf{r}(t),\mathbf{p}(t))_{t\geq 0}$ whose equations of motion are given by

$$\frac{dp_x}{dt} = V'(r_{x+1}) - V'(r_x), \qquad \frac{dr_x}{dt} = U'(p_x) - U'(p_{x+1}), \qquad x \in \mathbb{Z}, \tag{1}$$

where p_x is the momentum of the particle x, q_x its position and $r_x = q_x - q_{x-1}$ is the "deformation" of the lattice at x. Standard chains of oscillators are recovered for a quadratic kinetic energy $U(p) = p^2/2$. Now, take $V = U$, and call $\eta_{2x-1} = r_x$ and $\eta_{2x} = p_x$. The dynamics can be rewritten as:

$$d\eta_x(t) = \Big(V'(\eta_{x+1}) - V'(\eta_{x-1})\Big)dt. \tag{2}$$

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Notice that with these new variables the energy of the system is simply given by $\sum_{x\in\mathbb{Z}} V(\eta_x)$. In [7] an anomalous diffusion of energy is numerically observed for a generic potential V. Then, following the spirit of [2], the deterministic evolution is perturbed by adding a noise which consists to exchange η_x with η_{x+1} at random exponential times, independently for each bond $\{x,x+1\}$. The dynamics still conserves the energy $\sum_{x\in\mathbb{Z}} V(\eta_x)$ and the "volume" $\sum_{x\in\mathbb{Z}} \eta_x$ and destroys all other conserved quantities. As argued in [7], the volume conservation law is responsible for the anomalous energy diffusion observed for this class of energy-volume conserving dynamics. This can be shown for quadratic interactions ([7]) with a behavior similar to the one observed in [2]. For nonlinear interactions the problem is much more difficult.

The aim of this paper is to show that if the interacting potential is of exponential type then the energy superdiffuses. Therefore, for this class of related models, in a particular case, we answer to the open question stated in [2]. With some additional technical work we think that our methods could be carried out to the Toda lattice perturbed by an energy-momentum conserving noise (considered e.g. in [19]).

The paper is organized as follows. In Section 2 we define precisely the model. The results are stated in Section 3. To prove the theorems we first perform a microscopic change of variables (Section 4) which permits to use a nice orthogonal decomposition of the generator (Section 5). Roughly speaking the upper bound on the energy superdiffusion is proved in Section 6 and the lower bound in Section 7. Section 8 contains a comment about the possible extensions and comparisons of our model to others. In the Appendix we prove the existence of the infinite dynamics.

Notations: For any $a,b\in\mathbb{R}^2$, $a\cdot b$ stands for the standard scalar product between a and b and $|a|=\sqrt{a\cdot a}$ for the norm of a. The transpose of a matrix M is denoted by M^T. If $u:\mathbf{x}=(x_1,\dots,x_n)\in\mathbb{R}^n\to u(\mathbf{x})=(u_1(\mathbf{x}),\dots,u_d(\mathbf{x}))\in\mathbb{R}^d$ is a differentiable function then $(\partial_{x_i}u_j)(\mathbf{x})$ denotes the partial derivative of u_j with respect to the i-th coordinate at $\mathbf{x}$ and $(\nabla u)(\mathbf{x})$ denotes the differential matrix (the gradient if $d=1$) of u at $\mathbf{x}$, i.e. the $d\times n$ matrix whose (i,j)-th entry is $(\partial_{x_i}u_j)(\mathbf{x})$; if $u:=(u_1,\dots,u_d):\mathbb{Z}\to\mathbb{R}^d$ then we adopt the same notation to denote the discrete gradient of u defined by $(\nabla u):=(\nabla u_1,\dots,\nabla u_d)$ with $(\nabla u_i)(x)=u_i(x+1)-u_i(x)$.

2. THE MODEL

Let $b>0$ and $V_b(q)=e^{-bq}-1+bq$. We consider the system $\eta(t)=\{\eta_x(t):x\in\mathbb{Z}\}$ on $\mathbb{R}^{\mathbb{Z}}$ defined by its generator $L=A+\gamma S$, $\gamma>0$, where for local [1] differentiable functions $f:\mathbb{R}^{\mathbb{Z}}\to\mathbb{R}$ we have that

$$(Af)(\eta)=\sum_{x\in\mathbb{Z}}\Big(V_b'(\eta_{x+1})-V_b'(\eta_{x-1})\Big)(\partial_{\eta_x}f)(\eta)$$

and

$$(Sf)(\eta)=\sum_{x\in\mathbb{Z}}\Big(f(\eta^{x,x+1})-f(\eta)\Big).$$

The deterministic system (2) with potential V_b is well known in the integrable systems literature. It has been introduced in [20] by Kac and van Moerbecke and was shown to be completely integrable. Consequently, the energy transport is ballistic ([7, 30]). As we will

[1] A function f defined on an infinite product space is said to be local if it depends only on its variable through a finite number of coordinates.

see this is different when the noise is added: the energy transport is no more ballistic but superdiffusive.

The existence of the dynamics generated by L is proved in the Appendix for a large set of initial conditions and in particular for a set of full measure w.r.t. any invariant state $\mu_{\bar\beta,\bar\lambda}$ (see bellow for its definition).

The system conserves the energy $\sum_{x\in\mathbb{Z}} V_b(\eta_x)$ and the volume $\sum_{x\in\mathbb{Z}} \eta_x$. In fact, we have

$$L(V_b(\eta_x)) = -\nabla \bar j_{x-1,x}(\eta), \quad L(\eta_x) = -\nabla \vec j_{x-1,x}(\eta),$$

where the microscopic currents are given by

$$\bar j_{x,x+1}(\eta) = -b^2 e^{-b(\eta_x+\eta_{x+1})} + b^2(e^{-b\eta_x} + e^{-b\eta_{x+1}}) - \gamma\nabla(V_b(\eta_x))$$

and

$$\vec j_{x,x+1}(\eta) = be^{-b\eta_x} + be^{-b\eta_{x+1}} - \gamma\nabla\eta_x.$$

Every product probability measure $\mu_{\bar\beta,\bar\lambda}$ on $\mathbb{R}^{\mathbb{Z}}$ in the form

$$\mu_{\bar\beta,\bar\lambda}(d\eta) = \prod_{x\in\mathbb{Z}} \bar Z^{-1}(\bar\beta,\bar\lambda)\exp\{-\bar\beta e^{-b\eta_x} - \bar\lambda\eta_x\}, \quad \bar\beta>0\,,\bar\lambda>0$$

is invariant for the dynamics.

Let $\langle\cdot\rangle_{\mu_{\bar\beta,\bar\lambda}}$ denote the average with respect to $\mu_{\bar\beta,\bar\lambda}$. We define $\bar e := \bar e(\bar\beta,\bar\lambda), \bar v := \bar v(\bar\beta,\bar\lambda)$ as the averages of the conserved quantities $V_b(\eta_x)$, η_x with respect to $\mu_{\bar\beta,\bar\lambda}$, respectively, namely $\bar e = \langle V_b(\eta_x)\rangle_{\mu_{\bar\beta,\bar\lambda}}$ and $\bar v = \langle\eta_x\rangle_{\mu_{\bar\beta,\bar\lambda}}$.

A simple computation shows that

$$\langle \bar j_{x,x+1}\rangle_{\mu_{\bar\beta,\bar\lambda}} = b^2(\bar e - b\bar v)^2 \quad \textit{and} \quad \langle \vec j_{x,x+1}\rangle_{\mu_{\bar\beta,\bar\lambda}} = 2b(\bar e - b\bar v). \tag{3}$$

Hence, in the hyperbolic scaling, the hydrodynamical equations are given by

$$\begin{cases} \partial_t \mathfrak{e} - b^2\,\partial_q((\mathfrak{e}-b\mathfrak{v})^2) = 0 \\ \partial_t \mathfrak{v} + 2b\,\partial_q(\mathfrak{e}-b\mathfrak{v}) = 0 \end{cases} \tag{4}$$

and can be written in the compact form $\partial_t\bar{\mathfrak{X}} + \partial_q\bar{\mathfrak{J}}(\bar{\mathfrak{X}}) = 0$ with

$$\bar{\mathfrak{X}} = \begin{pmatrix} \mathfrak{e} \\ \mathfrak{v} \end{pmatrix}, \quad \bar{\mathfrak{J}}(\bar{\mathfrak{X}}) = \begin{pmatrix} -b^2(\mathfrak{e}-b\mathfrak{v})^2 \\ 2b(\mathfrak{e}-b\mathfrak{v}) \end{pmatrix}. \tag{5}$$

This can be proved before the appearance of the shocks (see [7]). The differential matrix of $\bar{\mathfrak{J}}$ is given by

$$(\nabla\bar{\mathfrak{J}})(\bar{\mathfrak{X}}) = 2b\begin{pmatrix} -b(\mathfrak{e}-b\mathfrak{v}) & b^2(\mathfrak{e}-b\mathfrak{v}) \\ 1 & -b \end{pmatrix}.$$

For given $(\bar e,\bar v)$ we denote by $(\bar T_t^+)_{t\ge0}$ (resp. $(\bar T_t^-)_{t\ge0}$) the semigroup on $S(\mathbb{R})\times S(\mathbb{R})$ generated by the linearized system

$$\partial_t\varepsilon + \bar M^T\,\partial_q\varepsilon = 0, \quad (\text{resp. } \partial_t\varepsilon - \bar M^T\,\partial_q\varepsilon = 0), \tag{6}$$

where

$$\bar M := \bar M(\bar e,\bar v) = [\nabla\bar{\mathfrak{J}}](\bar\omega), \quad \bar\omega = \begin{pmatrix} \bar e \\ \bar v \end{pmatrix}.$$

We omit the dependence of these semigroups on $(\bar e,\bar v)$ for lightness of the notations. Above $S(\mathbb{R})$ denotes the Schwartz space of smooth rapidly decreasing functions.

3. Statement of the results

For each integer $z \geq 0$, let $H_z(x) = (-1)^z e^{x^2} \frac{d^z}{dx^z} e^{-x^2}$ be the Hermite polynomial and $h_z(x) = (z!\sqrt{2\pi})^{-1} H_z(x) e^{-x^2}$ the Hermite function. The set $\{h_z, z \geq 0\}$ is an orthonormal basis of $\mathbb{L}^2(\mathbb{R})$. Consider in $\mathbb{L}^2(\mathbb{R})$ the operator $K_0 = x^2 - \Delta$, Δ being the Laplacian on $\mathbb{R}$. For an integer $k \geq 0$, denote by $\mathbb{H}_k$ the Hilbert space induced by $S(\mathbb{R})$ and the scalar product $\langle \cdot, \cdot \rangle_k$ defined by $\langle f, g \rangle_k = \langle f, K_0^k g \rangle_0$, where $\langle \cdot, \cdot \rangle_0$ denotes the inner product of $\mathbb{L}^2(\mathbb{R})$ and denote by $\mathbb{H}_{-k}$ the dual of $\mathbb{H}_k$, relatively to this inner product. Let $\langle \cdot \rangle$ represent the average with respect to the Lebesgue measure.

We take the infinite system at equilibrium under the Gibbs measure $\mu_{\bar\beta,\bar\lambda}$ corresponding to a mean energy $\bar e$ and a mean volume $\bar v$. Our goal is to study the energy-volume fluctuation field in the time-scale $tn^{1+\alpha}$, $\alpha \geq 0$:

$$\mathscr{Y}_t^{n,\alpha}(\mathbf{G}) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} \mathbf{G}(x/n) \cdot \big(\bar\omega_x(tn^{1+\alpha}) - \bar\omega \big), \tag{7}$$

where for $q \in \mathbb{R}$, $x \in \mathbb{Z}$,

$$\mathbf{G}(q) = \begin{pmatrix} G_1(q) \\ G_2(q) \end{pmatrix}, \quad \bar\omega_x = \begin{pmatrix} V_b(\eta_x) \\ \eta_x \end{pmatrix}$$

and G_1, G_2 are test functions belonging to $S(\mathbb{R})$.

If E is a Polish space then $D(\mathbb{R}^+, E)$ (resp, $C(\mathbb{R}^+, E)$) denotes the space of E-valued functions, right continuous with left limits (resp. continuous), endowed with the Skorohod (resp. uniform) topology. Let $Q^{n,\alpha}$ be the probability measure on $D(\mathbb{R}^+, \mathbb{H}_{-k} \times \mathbb{H}_{-k})$ induced by the fluctuation field $\mathscr{Y}_t^{n,\alpha}$ and $\mu_{\bar\beta,\bar\lambda}$. Let $\mathbb{P}_{\mu_{\bar\beta,\bar\lambda}}$ denote the probability measure on $D(\mathbb{R}^+, \mathbb{R}^{\mathbb{Z}})$ induced by $(\eta(t))_{t \geq 0}$ and $\mu_{\bar\beta,\bar\lambda}$. Let $\mathbb{E}_{\mu_{\bar\beta,\bar\lambda}}$ denote the expectation with respect to $\mathbb{P}_{\mu_{\bar\beta,\bar\lambda}}$.

Theorem 1. *Fix an integer $k > 2$. Denote by Q the probability measure on $C(\mathbb{R}^+, \mathbb{H}_{-k} \times \mathbb{H}_{-k})$ corresponding to a stationary Gaussian process with mean 0 and covariance given by*

$$\mathbb{E}_Q [\mathscr{Y}_t(\mathbf{H}) \mathscr{Y}_s(\mathbf{G})] = \langle \bar T_t^{-} \mathbf{H} \cdot \bar\chi \, \bar T_s^{-} \mathbf{G} \rangle$$

for every $0 \leq s \leq t$ and $\mathbf{H}, \mathbf{G}$ in $\mathbb{H}_k \times \mathbb{H}_k$. Here $\bar\chi := \bar\chi(\bar\beta, \bar\lambda)$ is the equilibrium covariance matrix [2] of $\bar\omega_0$. Then, the sequence $(Q^{n,0})_{n \geq 1}$ converges weakly to the probability measure Q.

As a consequence of Theorem 1 we obtain a Central Limit Theorem for the energy flux and for the volume flux through a fixed bond. For that purpose fix a site $x \in \mathbb{Z}$, let $\mathscr{E}_{x,x+1}^n(t)$ (resp. $\mathscr{V}_{x,x+1}^n(t)$) denote the energy (resp. volume) flux through the bond $\{x, x+1\}$ during the time interval $[0, tn]$, defined as:

$$\mathscr{E}_{x,x+1}^n(t) := \sum_{y \geq x+1} \{V_b(\eta_y(tn)) - V_b(\eta_y(0))\} \quad \Big(\text{resp.} \quad \mathscr{V}_{x,x+1}^n(t) := \sum_{y \geq x+1} \{\eta_y(tn) - \eta_y(0)\} \Big).$$

Then, from the previous result we conclude that

Corollary 1. *Fix $x \in \mathbb{Z}$ and let $Z_t^{n,e} := \frac{1}{\sqrt{n}} \{\mathscr{E}_{x,x+1}^n(t) - \mathbb{E}_{\mu_{\bar\beta,\bar\lambda}}[\mathscr{E}_{x,x+1}^n(t)]\}$. For every $k \geq 1$ and every $0 \leq t_1 < t_2 \cdots < t_k$, $(Z_{t_1}^{n,e}, \cdots, Z_{t_k}^{n,e})$ converges in law to a Gaussian vector*

[2] See (18) for an explicit value.

$(Z^e_{t_1},\cdots,Z^e_{t_k})$ *with mean zero and covariance given by*

$$\mathbb{E}_Q[Z^e_t Z^e_s] = \frac{2}{\bar\beta^3}(\bar\lambda - b\bar\beta)^2 s,$$

for all $s \le t$.

Corollary 2. *Fix* $x \in \mathbb{Z}$, *let* $Z^{n,v}_t := \frac{1}{\sqrt{n}}\{\mathcal{V}^n_{x,x+1}(t) - \mathbb{E}_{\mu_{\bar\beta,\bar\lambda}}[\mathcal{V}^n_{x,x+1}(t)]\}$. *For every* $k \ge 1$ *and every* $0 \le t_1 < t_2 \cdots < t_k$, $(Z^{n,v}_{t_1},\cdots,Z^{n,v}_{t_k})$ *converges in law to a Gaussian vector* $(Z^v_{t_1},\cdots,Z^v_{t_k})$ *with mean zero and covariance given by*

$$\mathbb{E}_Q[Z^v_t Z^v_s] = \frac{2}{\bar\beta} s,$$

for all $s \le t$.

We notice that, according to Corollary 1, the limiting energy flux Z^e_t has a vanishing variance for $\bar\lambda = b\bar\beta$ which is equivalent to $\bar e = b\bar v$.

The theorem above means that in the hyperbolic scaling the fluctuations are trivial: the initial fluctuations are transported by the linearized system of (4). To see a nontrivial behavior we have to study, in the transport frame, the fluctuations at a longer time scale $tn^{1+\alpha}$, with $\alpha > 0$. Thus, we consider the fluctuation field $\widehat{\mathscr{Y}^{n,\alpha}_\cdot}$, $\alpha > 0$, defined, for any $\mathbf{G} \in S(\mathbb{R}) \times S(\mathbb{R})$, by

$$\widehat{\mathscr{Y}^{n,\alpha}_t}(\mathbf{G}) = \mathscr{Y}^{n,\alpha}_t\left(\bar T^+_{tn^\alpha}\mathbf{G}\right). \tag{8}$$

According to the fluctuating hydrodynamics theory ([26], pp. 85-96), in the case of a normal (diffusive) behavior $\alpha = 1$, the field $(\widehat{\mathscr{Y}^{n,\alpha}_t})_{t\ge 0}$ should converge to the stationary field $(\widehat{\mathscr{Y}_t})_{t\ge 0}$ simply related to the solution $(\widehat{\mathscr{Z}_t})_{t\ge 0}$ of the linear two dimensional vector valued (infinite-dimensional) stochastic partial differential equation

$$\partial_t \widehat{\mathscr{Z}_t} = \nabla\cdot\left(\mathscr{D}\,\nabla\widehat{\mathscr{Z}_t}\right) + \sqrt{2\mathscr{D}\bar\chi}\,\nabla\cdot W_t. \tag{9}$$

Here $W(x,t)$ is a standard two-dimensional vector valued space-time white noise and the coefficient $\mathscr{D} := \mathscr{D}(\bar e,\bar v)$ is expressed by a Green-Kubo formula (see (12)).

Our second main theorem shows that the correct scaling exponent α is greater than $1/3$:

Theorem 2. *Fix an integer* $k > 1$ *and* $\alpha < 1/3$. *Denote by* Q *the probability measure on* $C(\mathbb{R}^+, \mathbb{H}_{-k}\times\mathbb{H}_{-k})$ *corresponding to a stationary Gaussian process with mean* 0 *and covariance given by*

$$\mathbb{E}_Q\left[\mathscr{Y}_t(\mathbf{H})\,\mathscr{Y}_s(\mathbf{G})\right] = \langle \mathbf{H}\cdot\bar\chi\,\mathbf{G}\rangle$$

for every $0 \le s \le t$ *and* $\mathbf{H},\mathbf{G}$ *in* $\mathbb{H}_k \times \mathbb{H}_k$. *Then, the sequence* $(Q^{n,\alpha})_{n\ge 1}$ *converges weakly to the probability measure* Q.

As in the hyperbolic time scale from the previous result we obtain limiting results for the energy and volume flux. In this case, since we take the system in a time dependent reference frame, we defined the energy and volume flux through the time dependent bond $\{u^{x,\alpha}_t(n), u^{x,\alpha}_t(n)+1\}$, where $u^{x,\alpha}_t(n) := -2b\bar\lambda/\bar\beta tn^{1+\alpha}$. For that purpose, fix a site $x \in \mathbb{Z}$ and let $\mathscr{E}_{u^{x,\alpha}_t(n)}$ (resp. $\mathcal{V}^n_{u^{x,\alpha}_t(n)}(t)$) denote the energy (resp. volume) flux through the bond $\{u^{x,\alpha}_t(n), u^{x,\alpha}_t(n)+1\}$ during the time interval $[0, tn^{1+\alpha}]$, defined as:

$$\begin{aligned}
&\mathscr{E}^n_{u^{x,\alpha}_t(n)}(t) := \sum_{y\ge u^{x,\alpha}_t(n)} \{V_b(\eta_y(tn^{1+\alpha})) - V_b(\eta_y(0))\},\\
&\Big(\text{resp.}\quad \mathcal{V}^n_{u^{x,\alpha}_t(n)}(t) := \sum_{y\ge u^{x,\alpha}_t(n)} \{\eta_y(tn^{1+\alpha}) - \eta_y(0)\}\Big).
\end{aligned} \tag{10}$$

Then, from the previous result we conclude that

Corollary 3. *Fix $t \geq 0$, $x \in \mathbb{Z}$ and $\alpha < 1/3$. Then*

$$\lim_{n\to\infty} \mathbb{E}_{\mu_{\bar\beta,\bar\lambda}} \left[\left(\frac{1}{\sqrt{n}} \{ \mathscr{E}^n_{u^{x,\alpha}_t(n)}(t) - \mathbb{E}_{\mu_{\bar\beta,\bar\lambda}} [\mathscr{E}^n_{u^{x,\alpha}_t(n)}(t)] \} \right)^2 \right] = 0.$$

and

$$\lim_{n\to\infty} \mathbb{E}_{\mu_{\bar\beta,\bar\lambda}} \left[\left(\frac{1}{\sqrt{n}} \{ \mathscr{V}^n_{u^{x,\alpha}_t(n)}(t) - \mathbb{E}_{\mu_{\bar\beta,\bar\lambda}} [\mathscr{V}^n_{u^{x,\alpha}_t(n)}(t)] \} \right)^2 \right] = 0.$$

Similar results have been obtained in [18] by one of the authors for the asymmetric simple exclusion process. The proof of the three corollaries above follows the same arguments as in [18] once the previous theorems are proved. For that reason we will only give a sketch of their proof. The proof of the theorems is more problematic since the multi-scale analysis performed in [18] relies crucially on the existence of a spectral gap so that we cannot follow [18]. Therefore we propose an alternative approach based on computations of some resolvent norms.

Theorem 2 does not exclude the possibility of normal fluctuations, i.e. $\alpha = 1$. In order to show that the system we consider is really superdiffusive we will show that the transport coefficient $\mathscr{D}$ which appears in (9) diverges so that the correct scaling exponent α is strictly smaller than 1. Our third result, stated bellow, shows it is in fact less than $3/4$.

With the notations introduced in the previous section, the *normalized* currents are defined by

$$\hat f_{x,x+1}(\eta) = \begin{pmatrix} \bar j_{x,x+1}(\eta) \\ \bar j'_{x,x+1}(\eta) \end{pmatrix} - \bar{\mathfrak{J}}(\bar\omega) - (\nabla\bar{\mathfrak{J}})(\bar\omega) \begin{pmatrix} V_b(\eta_x) - \bar e \\ \eta_x - \bar\nu \end{pmatrix}. \tag{11}$$

Up to a constant matrix coming from a martingale term (due to the noise) and thus irrelevant for us (see [2], [7]), the coefficient $\mathscr{D}$ is defined by the Green-Kubo formula

$$\mathscr{D} = \int_0^\infty \mathbb{E}_{\mu_{\bar\beta,\bar\lambda}} \left[\sum_{x\in\mathbb{Z}} \hat f_{x,x+1}(t) \left[\hat f_{0,1}(0) \right]^T \right] dt. \tag{12}$$

The signature of the superdiffusive behavior of the system is seen in the divergence of $\mathscr{D}$, i.e. in a slow decay of the current-current correlation function. To study the latter we introduce its Laplace transform

$$\mathscr{F}(\gamma,z) = \int_0^\infty e^{-zt}\, \mathbb{E}_{\mu_{\bar\beta,\bar\lambda}} \left[\sum_{x\in\mathbb{Z}} \hat f_{x,x+1}(t) \left[\hat f_{0,1}(0) \right]^T \right] dt$$

which is well defined for any $z > 0$.

Our third theorem is the following lower bound on $\mathscr{F}(\gamma,z)$. Observe that $\mathscr{F}(\gamma,z)$ is a square matrix of size 2 whose entry (i,j) is denoted by $\mathscr{F}_{i,j}$.

Theorem 3. *Fix $\gamma > 0$. There exists a positive constant $c := c(\gamma) > 0$ such that*

$$\mathscr{F}_{1,1}(\gamma,z) \geq c z^{-1/4}$$

and

$$\mathscr{F}_{i,j}(\gamma,z) = 0, \quad (i,j) \neq (1,1).$$

Moreover, there exists a positive constant $C := C(\gamma)$ such that for any $z > 0$,

$$C^{-1} \mathscr{F}_{1,1}(1, z/\gamma) \leq \mathscr{F}_{1,1}(\gamma,z) \leq C \mathscr{F}_{1,1}(1, z/\gamma). \tag{13}$$

The last part of the theorem is easy to prove but has an important consequence. In [7] numerical simulations are performed to detect the anomalous diffusion of energy. Since it is difficult to estimate numerically the time autocorrelation functions of the currents because of their expected long-time tails, a more tenable approach consists in studying a non equilibrium system in its steady state, i.e. considering a finite system in contact with two thermostats which fix the value of the energy at the boundaries. Then we estimate the dependence of the energy transport coefficient $\kappa(N)$ with the system size N. The latter is defined as N times the average energy current. It turns out that $\kappa(N) \sim N^{\delta}$ with a parameter $\delta := \delta(\gamma) > 0$ increasing with the noise intensity γ (except for the singular value $\delta = 1$ when $\gamma = 0$ which is a manifestation of the ballistic behavior of the Kac-van Moerbecke system). This result is very surprising since the more stochasticity in the model is introduced, the less the system is diffusive. The same has been observed for other anharmonic potentials in [7] and also for the Toda lattice perturbed by an energy-momentum conservative noise ([19]). It has been argued in [19] that this may be explained by the fact that some diffusive phenomena due to non-linearities, like localized breathers, are destroyed by the noise. In [3] simulations have been performed directly with the Green-Kubo formula for other standard anharmonic chains with the same conclusion: current autocorrelation function decreases slower when the noise intensity increases. If all these numerical simulations reproduce correctly the real behavior of the models investigated, they dismiss the theories which pretend that some universality holds, e.g. [29]. It is therefore very important to decide if the phenomena numerically observed are correct or not. The inequality (13) shows that the time decay of the current autocorrelation function is independent of γ (up to possible slowly varying functions corrections, i.e. in a Tauberian sense). Therefore the numerical simulations do not reflect the correct behavior of the system [3]. Of course we could imagine that the exponent δ appearing in the transport coefficient defined by the non-equilibrium stationary state depends on the intensity of the noise and not on the autocorrelation function but this is very improbable. The last part of our theorem is in fact valid for all the models cited above. It applies in particular to the framework studied in [3] and shows that the numerical observations of that paper are not consistent with the real behavior of the system.

4. A CHANGE OF VARIABLES

To study the energy-volume fluctuation field $\mathscr{Y}^{n,\alpha}$, we introduce the following change of variables $\xi_x = e^{-b\eta_x}$, for each $x \in \mathbb{Z}$. Then, the previous Markovian system $(\eta(t))_{t\geq 0}$ defines a new Markovian system $(\xi(t))_{t\geq 0}$ with state space $(0,+\infty)^{\mathbb{Z}}$ whose generator $\mathscr{L}$ is equal to $b^2\mathscr{A} + \gamma\mathscr{S}$, where for local differentiable functions $f : (0,+\infty)^{\mathbb{Z}} \to \mathbb{R}$ we have that

$$(\mathscr{A}f)(\xi) = \sum_{x\in\mathbb{Z}} \xi_x\Big(\xi_{x+1} - \xi_{x-1}\Big)(\partial_{\xi_x} f)(\xi)$$

and

$$(\mathscr{S}f)(\xi) = \sum_{x\in\mathbb{Z}} \Big(f(\xi^{x,x+1}) - f(\xi)\Big).$$

Observe that the energy and volume conservation laws correspond, for the process $(\xi(t))_{t\geq 0}$, to the conservation of the two following quantities $\sum_{x\in\mathbb{Z}} \xi_x$, $\quad \sum_{x\in\mathbb{Z}} \log \xi_x$. The corresponding microscopic currents are defined by the conservation law equations:

$$\mathscr{L}(\xi_x) = -\nabla j_{x-1,x}(\xi), \quad \mathscr{L}(\log \xi_x) = -\nabla j'_{x-1,x}(\xi),$$

[3] It would be very interesting to understand why the numerical simulations are so sensitive to the noise.

where

$$j_{x,x+1}(\xi) = -b^2\xi_x\xi_{x+1} - \gamma\nabla\xi_x, \quad j'_{x,x+1}(\xi) = -b^2(\xi_x + \xi_{x+1}) - \gamma\nabla(\log\xi_x).$$

We have the following relations between the microscopic currents

$$\bar{j}_{x,x+1}(\eta) = j_{x,x+1}(\xi) - j'_{x,x+1}(\xi), \quad \bar{j}'_{x,x+1}(\eta) = -\frac{1}{b}j'_{x,x+1}(\xi). \tag{14}$$

If η is distributed according to $\mu_{\bar\beta,\bar\lambda}$ then ξ defined by $\xi_x = e^{-b\eta_x}$ is distributed according to the probability measure $\nu_{\beta,\lambda}$ on $(0,+\infty)^{\mathbb{Z}}$ given by

$$\nu_{\beta,\lambda}(d\xi) = \prod_{x\in\mathbb{Z}} Z^{-1}(\beta,\lambda)\mathbf{1}_{\{\xi_x>0\}}\exp\{-\beta\xi_x + \lambda\log(\xi_x)\}$$

with $Z(\beta,\lambda)$ the partition function and

$$\beta = \bar\beta, \quad \lambda = -1 + \bar\lambda/b. \tag{15}$$

Remark that $\nu_{\beta,\lambda}$ is nothing but a product probability measure whose marginal follows a Gamma distribution $\gamma_{\lambda+1,\beta^{-1}}$ with parameter $(\lambda+1,\beta^{-1})$. In particular, we have $Z := Z(\beta,\lambda) = \beta^{-(\lambda+1)}\Gamma(\lambda+1)$, where Γ is the usual Gamma function.

Thus, the process $(\xi(t))_{t\geq 0}$ has a family of translation invariant measures $\nu_{\beta,\lambda}$ parameterized by the chemical potentials $(\beta,\lambda)\in(0,+\infty)\times(-1,+\infty)$.

Let $\mathbb{P}_{\nu_{\beta,\lambda}}$ be the probability measure on $D(\mathbb{R}^+,(0,+\infty)^{\mathbb{Z}})$ induced by $(\xi(t))_{t\geq 0}$ and $\nu_{\beta,\lambda}$ and let $\mathbb{E}_{\nu_{\beta,\lambda}}$ denote the expectation with respect to $\mathbb{P}_{\nu_{\beta,\lambda}}$.

Let $\langle\cdot\rangle_{\nu_{\beta,\lambda}}$ denote the average with respect to $\nu_{\beta,\lambda}$. The averages $\rho := \rho(\beta,\lambda), \theta := \theta(\beta,\lambda)$ of the conserved quantities for $(\xi(t))_{t\geq 0}$ at equilibrium under $\nu_{\beta,\lambda}$ are defined by $\rho = \langle\xi_x\rangle_{\nu_{\beta,\lambda}}$ and $\theta = \langle\log(\xi_x)\rangle_{\nu_{\beta,\lambda}}$. By a direct computation we get

$$\rho = 1 + \bar e - b\bar v = \frac{\lambda+1}{\beta}, \quad \theta = -b\bar v = \psi(\lambda+1) - \log(\beta) \tag{16}$$

where ψ is the usual digamma function, i.e. $\psi = \Gamma'/\Gamma$.

It is understood, here and in the whole paper, that (β,λ) are related to $(\bar\beta,\bar\lambda)$ through (15). We will use the following compact notation, for each $x\in\mathbb{Z}$,

$$\omega_x = \begin{pmatrix} \xi_x \\ \log(\xi_x)\end{pmatrix}, \quad \textit{and} \quad \omega = \begin{pmatrix}\rho\\ \theta\end{pmatrix}.$$

Observe that $\bar\omega_x = \Lambda\omega_x - \begin{pmatrix}1\\0\end{pmatrix}$, where

$$\Lambda = \begin{pmatrix} 1 & -1 \\ 0 & -1/b\end{pmatrix}. \tag{17}$$

The covariance matrix $\chi := \chi(\beta,\lambda)$ of ω_0 under $\nu_{\beta,\lambda}$ is given by

$$\chi = \begin{pmatrix} \partial^2_\beta\log(Z) & -\partial_{\beta,\lambda}\log(Z)\\ -\partial_{\beta,\lambda}\log(Z) & \partial^2_\lambda\log(Z)\end{pmatrix} = \begin{pmatrix} (\lambda+1)\beta^{-2} & -\beta^{-1}\\ -\beta^{-1} & (\log\Gamma)''(\lambda+1)\end{pmatrix}.$$

Thus, the covariance matrix χ of ω_0 under $\nu_{\beta,\lambda}$ is related to the covariance matrix $\bar\chi$ of $\bar\omega_0$ under $\mu_{\bar\beta,\bar\lambda}$, by

$$\bar\chi = \Lambda\chi\Lambda^T. \tag{18}$$

A simple computation shows that $\langle j_{x,x+1}\rangle_{\nu_{\beta,\lambda}} = -b^2\rho^2$ and $\langle j'_{x,x+1}\rangle_{\nu_{\beta,\lambda}} = -2b^2\rho$.

The hydrodynamical equations for the process $(\xi(t))_{t\geq 0}$ are given by

$$\begin{cases} \partial_t \rho - b^2 \partial_q(\rho^2) = 0 \\ \partial_t \theta - 2b^2 \partial_q \rho = 0 \end{cases} \tag{19}$$

and can be written in the compact form $\partial_t \mathfrak{X} + \partial_q \mathfrak{J}(\mathfrak{X}) = 0$ with

$$\mathfrak{X} = \begin{pmatrix} \rho \\ \theta \end{pmatrix}, \quad \mathfrak{J}(\mathfrak{X}) = \begin{pmatrix} -b^2\rho^2 \\ -2b^2\rho \end{pmatrix}.$$

The differential matrix of $\mathfrak{J}$ is given by

$$(\nabla \mathfrak{J})(\mathfrak{X}) = \begin{pmatrix} -2b^2\rho & 0 \\ -2b^2 & 0 \end{pmatrix}.$$

As above, let $(T_t^+)_{t\geq 0}$ (resp. $(T_t^-)_{t\geq 0}$) denote the semigroup on $S(\mathbb{R}) \times S(\mathbb{R})$ generated by the linearized system

$$\partial_t \varepsilon + M^T \partial_q \varepsilon = 0, \quad (\text{resp. } \partial_t \varepsilon - M^T \partial_q \varepsilon = 0). \tag{20}$$

where

$$M := M(\rho, \theta) = (\nabla \mathfrak{J})(\omega),$$

ρ and θ are given by (16). We omit the dependence of these semigroups on (ρ, θ) for lightness of the notations.

We remark that the linearized system of (19) around the constant profiles (ρ, θ) is given by the first equation on the left hand side of (20). It is easy to show that $\bar{M} = \Lambda M \Lambda^{-1}$ and $\Lambda^T \bar{T}_t^- = T_t^- \Lambda^T$.

5. Orthogonal decomposition

Observe that $\nu_{\beta,\lambda}$ is a product of Gamma distributions. Let us recall that the Gamma distribution $\gamma_{\alpha,k}$ with parameter (α, k) is the probability distribution on $(0, +\infty)$ absolutely continuous w.r.t. the Lebesgue measure with density $f_{\alpha,k}$ given by

$$f_{\alpha,k}(q) = \Big(k^\alpha \Gamma(\alpha)\Big)^{-1} q^{\alpha-1} e^{-q/k}, \quad q > 0. \tag{21}$$

Thus, we have $\nu_{\beta,\lambda}(d\xi) = \prod_{x\in\mathbb{Z}} \Big(f_{\lambda+1,\beta^{-1}}(\xi_x) d\xi_x\Big) = \prod_{x\in\mathbb{Z}} \Big(\beta f_{\lambda+1,1}(\beta\xi_x) d\xi_x\Big)$. The generalized Laguerre polynomials $(H_n^{(\lambda)})_{n\geq 0}$ form an orthogonal basis of the space $\mathbb{L}^2(\gamma_{\lambda+1,1})$. They satisfy the following equations:

$$\begin{aligned} &H_0^{(\lambda)} = 1, \\ &q\frac{d}{dq} H_n^{(\lambda)} = n H_n^{(\lambda)} - (n+\lambda) H_{n-1}^{(\lambda)}, \\ &\Big(q\frac{d^2}{dq^2} + (\lambda + 1 - q)\frac{d}{dq} + n\Big) H_n^{(\lambda)} = 0, \\ &(n+1) H_{n+1}^{(\lambda)}(q) = (2n + 1 + \lambda - q) H_n^{(\lambda)}(q) - (n+\lambda) H_{n-1}^{(\lambda)}(q) \end{aligned} \tag{22}$$

and the normalization condition

$$\int_0^{+\infty} \Big(H_n^{(\lambda)}(q)\Big)^2 f_{\lambda+1,1}(q)\, dq = \frac{\Gamma(\lambda + n + 1)}{\Gamma(\lambda + 1)} \frac{1}{n!}.$$

In particular, we have

$$\begin{aligned} H_1^{(\lambda)}(q) &= -q + (\lambda+1), \\ H_2^{(\lambda)}(q) &= \frac{(2+\lambda)(1+\lambda)}{2} - (\lambda+2)q + \frac{q^2}{2}. \end{aligned} \tag{23}$$

Let Σ be the set composed of configurations $\sigma = (\sigma_x)_{x\in\mathbb{Z}} \in \mathbb{N}^{\mathbb{Z}}$ such that $\sigma_x \neq 0$ only for a finite number of x. The number $\sum_{x\in\mathbb{Z}} \sigma_x$ is called the size of σ and is denoted by $|\sigma|$. Let $\Sigma_n = \{\sigma \in \Sigma\,;\, |\sigma| = n\}$. On the set of n-tuples $\mathbf{x} := (x_1, \dots, x_n)$ of $\mathbb{Z}^n$, we introduce the equivalence relation $\mathbf{x} \sim \mathbf{y}$ if there exists a permutation p on $\{1,\dots,n\}$ such that $x_{p(i)} = y_i$ for all $i \in \{1,\dots,n\}$. The class of $\mathbf{x}$ for the relation $\sim$ is denoted by $[\mathbf{x}]$. Then the set of configurations of Σ_n can be identified with the class of n-tuples classes for $\sim$.

To any $\sigma \in \Sigma$, we associate the polynomial function $H_\sigma^{\beta,\lambda}$ given by

$$H_\sigma^{\beta,\lambda}(\xi) = \prod_{x\in\mathbb{Z}} H_{\sigma_x}^{(\lambda)}(\beta\xi_x).$$

Then, the family $\left\{H_\sigma^{\beta,\lambda}\,;\, \sigma\in\Sigma\right\}$ forms an orthogonal basis of $\mathbb{L}^2(\nu_{\beta,\lambda})$ such that

$$\int H_\sigma^{\beta,\lambda}\, H_{\sigma'}^{\beta,\lambda}\, d\nu_{\beta,\lambda} = \delta_{\sigma=\sigma'} \prod_{x\in\mathbb{Z}} \frac{\Gamma(\lambda+\sigma_x+1)}{\Gamma(\lambda+1)} \frac{1}{\sigma_x!} = \delta_{\sigma=\sigma'} \mathscr{W}^{\beta,\lambda}(\sigma), \tag{24}$$

where

$$\mathscr{W}^{\beta,\lambda}(\sigma) := \prod_{x\in\mathbb{Z}} \frac{\Gamma(\lambda+\sigma_x+1)}{\Gamma(\lambda+1)} \frac{1}{\sigma_x!} \tag{25}$$

and δ denotes the Kronecker function, so that $\delta_{\sigma=\sigma'} = 1$ if $\sigma = \sigma'$, otherwise it is equal to zero.

The functions F such that $F(\sigma) = 0$ as soon as $|\sigma| \neq n$ are called degree n functions. A function $F : \Sigma \to \mathbb{R}$ such that $F(\sigma) = 0$ if $\sigma \notin \Sigma_n$ is also called a degree n function. Any function $F : \Sigma_n \to \mathbb{R}$ is nothing but a symmetric function $F : \mathbb{Z}^n \to \mathbb{R}$ through the identification of σ with $[\mathbf{x}]$. We denote by $\langle\cdot,\cdot\rangle$ the scalar product on $\oplus\mathbb{L}^2(\Sigma_n)$, each Σ_n being equipped with the counting measure. Hence, if $F, G : \Sigma \to \mathbb{R}$, we have

$$\langle F, G\rangle = \sum_{n\geq 0} \sum_{\sigma\in\Sigma_n} F_n(\sigma) G_n(\sigma) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\mathbf{x}\in\mathbb{Z}^n} F_n(\mathbf{x}) G_n(\mathbf{x}),$$

with F_n, G_n the restrictions of F, G to Σ_n. Since (β,λ) are fixed through the paper we denote $H_\sigma^{\beta,\lambda}$ by H_σ.

If a local function $f \in \mathbb{L}^2(\nu_{\beta,\lambda})$ is written in the form $f(\xi) = \sum_{\sigma\in\Sigma} F(\sigma) H_\sigma(\xi)$ then we have

$$(\mathscr{A} f)(\xi) = \sum_{\sigma\in\Sigma} (\mathfrak{A}F)(\sigma) H_\sigma(\xi), \quad (\mathscr{S} f)(\xi) = \sum_{\sigma\in\Sigma} (\mathfrak{S}F)(\sigma) H_\sigma(\xi)$$

with

$$(\mathfrak{S}F)(\sigma) = \sum_{x\in\mathbb{Z}} (F(\sigma^{x,x+1}) - F(\sigma)),$$

where $\sigma^{x,x+1}$ is obtained from σ by exchanging the occupation numbers σ_x and σ_{x+1}.

Let us now compute the operator $\mathfrak{A}$. We have

$$(\mathscr{A} H_\sigma)(\xi) = \beta \sum_{x\in\mathbb{Z}} (\xi_{x+1} - \xi_{x-1}) \xi(x) \partial_{\xi_x} H_\sigma(\xi).$$

By the definition of H_σ and by the second equality in (22), it follows that

$$(\mathscr{A}H_\sigma)(\xi) = \beta \sum_{x\in\mathbb{Z}} (\xi_{x+1} - \xi_{x-1})\Big(\sigma_x H_\sigma(\xi) - (\sigma_x + \lambda) H_{\sigma-\delta_x}(\xi)\Big),$$

where $\sigma - \delta_x$ is the configuration where a particle has been deleted at site x (if there was no particle on site x, then $\sigma - \delta_x = \sigma$).

Now, noticing that the fourth equality in (22) can be written as

$$\beta q H_n^{(\lambda)}(\beta q) = (2n+1+\lambda) H_n^{(\lambda)}(\beta q) - (n+\lambda) H_{n-1}^{(\lambda)}(\beta q) - (n+1) H_{n+1}^{(\lambda)}(\beta q)$$

and performing some change of variables, we have that

$$\begin{aligned}(\mathscr{A}H_\sigma)(\xi) = &\sum_{\substack{x,y\in\mathbb{Z}\\ |x-y|=1}} a(y-x)(\sigma_x+\lambda)(\sigma_y+1) H_{\sigma+\delta_y-\delta_x}(\xi)\\ &- \sum_{x\in\mathbb{Z}} (\sigma_x+\lambda)(\sigma_{x+1}-\sigma_{x-1}) H_{\sigma-\delta_x}(\xi)\\ &+ \sum_{x\in\mathbb{Z}} (\sigma_x+1)(\sigma_{x+1}-\sigma_{x-1}) H_{\sigma+\delta_x}(\xi).\end{aligned}$$

Here, $a(z) = -1$ if $z = -1$, $a(z) = 1$ if $z = 1$ and 0 otherwise. It follows that

$$\mathfrak{A} = \mathfrak{A}_0 + \mathfrak{A}_- + \mathfrak{A}_+$$

with

$$\begin{aligned}(\mathfrak{A}_0 F)(\sigma) &= -\sum_{\substack{x,y\in\mathbb{Z}\\ |x-y|=1}} a(y-x)\sigma_x(\sigma_y+1+\lambda) F(\sigma+\delta_y-\delta_x),\\ (\mathfrak{A}_+ F)(\sigma) &= -\sum_{x\in\mathbb{Z}} \sigma_x(\sigma_{x+1}-\sigma_{x-1}) F(\sigma-\delta_x),\\ (\mathfrak{A}_- F)(\sigma) &= \sum_{x\in\mathbb{Z}} (\sigma_x - 1 + \lambda)(\sigma_{x+1}-\sigma_{x-1}) F(\sigma+\delta_x).\end{aligned}$$

Observe that if F vanishes outside of Σ_n then $\mathfrak{A}_\pm F$ vanishes outside of $\Sigma_{n\mp1}$ and $\mathfrak{A}_0$ vanishes outside of Σ_n.

The Dirichlet form $\mathscr{D}(f)$ of a local function $f \in \mathbb{L}^2(\nu_{\beta,\lambda})$ is defined by

$$\mathscr{D}(f) = \langle f, (-\mathscr{S}f)\rangle_{\nu_{\beta,\lambda}} = \frac{1}{2}\sum_{x\in\mathbb{Z}} \int \big(f(\xi^{x,x+1}) - f(\xi)\big)^2 \nu_{\beta,\lambda}(d\xi).$$

Here and below $\langle\cdot,\cdot\rangle_{\nu_{\beta,\lambda}}$ denotes the inner product of $\mathbb{L}^2(\nu_{\beta,\lambda})$.

If f has the decomposition $f = \sum_{\sigma\in\Sigma} F(\sigma) H_\sigma$ then

$$\mathscr{D}(f) = \frac{1}{2}\sum_{x\in\mathbb{Z}} \sum_{\sigma\in\Sigma} \mathscr{W}(\sigma)\big(F(\sigma^{x,x+1}) - F(\sigma)\big)^2. \tag{26}$$

Let $\Delta_+ = \big\{(x,y)\in\mathbb{Z}^2\,;\, y \geq x+1\big\}$, $\Delta_- = \big\{(x,y)\in\mathbb{Z}^2\,;\, y\leq x-1\big\}$ and $\Delta_0 = \{(x,x)\,;\, x\in\mathbb{Z}\}$. We denote by $\mathbb{D}_1$ the Dirichlet form of a symmetric simple one dimensional random walk, i.e.

$$\mathbb{D}_1(F) = \frac{1}{2}\sum_{x\in\mathbb{Z}} (F(x+1) - F(x))^2,$$

where $F : \mathbb{Z} \to \mathbb{R}$ is such that $\sum_{x\in\mathbb{Z}} F^2(x) < +\infty$.

We denote by $\mathbb{D}_2$ the Dirichlet form of a symmetric simple random walk on $\mathbb{Z}^2$ where jumps from $\Delta_\pm$ to Δ_0 and from Δ_0 to $\Delta_\pm$ have been suppressed and jumps from $(x,x) \in \Delta_0$ to $(x\pm 1, x\pm 1) \in \Delta_0$ have been added, i.e.

$$\mathbb{D}_2(F) = \frac{1}{2} \sum_{|\mathbf{e}|=1} \sum_{\mathbf{x}\in\Delta_\pm, \mathbf{x}+\mathbf{e}\in\Delta_\pm} (F(\mathbf{x}+\mathbf{e}) - F(\mathbf{x}))^2 + \frac{1}{2} \sum_{\mathbf{x}\in\Delta_0} (F(\mathbf{x}\pm(1,1)) - F(\mathbf{x}))^2,$$

where $F : \mathbb{Z}^2 \to \mathbb{R}$ is a symmetric function such that $\sum_{\mathbf{x}\in\mathbb{Z}^2} F^2(\mathbf{x}) < +\infty$.

Lemma 1. *Let $f = \sum_{n=1}^2 \sum_{\sigma\in\Sigma_n} F_n(\sigma) H_\sigma$ be a local centered function of degree less or equal to 2. There exists a positive constant $C := C(\lambda)$, independent of f, such that*

$$C^{-1} \left[\mathbb{D}_1(F_1) + \mathbb{D}_2(F_2)\right] \le \mathscr{D}(f) \le C \left[\mathbb{D}_1(F_1) + \mathbb{D}_2(F_2)\right].$$

Proof. Observe that

- If $\sigma \in \Sigma_1$, then $\mathscr{W}(\sigma) = (\lambda+1)$.
- If $\sigma \in \Sigma_2$, $\sigma = \delta_x + \delta_y$, $x \neq y$, then $\mathscr{W}(\sigma) = (\lambda+1)^2$; if $\sigma \in \Sigma_2$, $\sigma = 2\delta_x$, then $\mathscr{W}(\sigma) = [(\lambda+2)(\lambda+1)]/2$.

This follows from the relation $\Gamma(z+1) = z\Gamma(z)$. Then, by using (26) and the identification of functions $F : \Sigma_n \to \mathbb{R}$ of degree n with their representations as symmetric functions on $\mathbb{Z}^n$, the claim follows. □

6. Triviality of the fluctuations

In this section we prove Theorems 1 and 2 and the three Corollaries above. Since $V_b(\eta_x) - 1 = \xi_x - \log(\xi_x)$, $\eta_x = -b^{-1}\log(\xi_x)$, the problem is reduced to study the fluctuation field of the conserved quantities for the process $(\xi(t))_{t\ge 0}$ at equilibrium under the probability measure $\nu_{\beta,\lambda}$. The latter is defined by

$$\mathscr{Z}_t^{n,\alpha}(\mathbf{G}) = \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} \mathbf{G}(x/n) \cdot \left(\omega_x(tn^{1+\alpha}) - \omega\right), \tag{27}$$

where $\mathbf{G}$ is a test function belonging to $S(\mathbb{R}) \times S(\mathbb{R})$. Recalling (17) we have

$$\mathscr{Y}_t^{n,\alpha}(\mathbf{G}) = \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} (\Lambda^T \mathbf{G})(x/n) \cdot (\omega_x(tn^{1+\alpha}) - \omega) = \mathscr{Z}_t^{n,\alpha}(\Lambda^T \mathbf{G}). \tag{28}$$

By the relation $\bar{M} = \Lambda M \Lambda^{-1}$, we are able to translate any result about the convergence of $\mathscr{Z}_\cdot^{n,\alpha}$ into a corresponding result for $\mathscr{Y}_\cdot^{n,\alpha}$.

6.1. **The hyperbolic scaling.** For any local function $g := g(\xi)$ we define the projection $\mathscr{P}_{\rho,\theta}\, g$ of g on the fields of the conserved quantities by

$$(\mathscr{P}_{\rho,\theta} g)(\xi) = g(\xi) - \tilde{g}(\rho,\theta) - (\nabla \tilde{g})(\rho,\theta) \cdot (\omega_0 - \omega)$$

where $\tilde{g}(\rho,\theta) = \langle g \rangle_{\nu_{\beta,\lambda}}$ and $\nabla \tilde{g}$ is the gradient of the function $\tilde{g}$.

We have that

Proposition 1 (Boltzmann-Gibbs principle I)**.** *For every local function $g \in \mathbb{L}^2(\nu_{\beta,\lambda})$, for every $\mathbf{H} \in S(\mathbb{R}) \times S(\mathbb{R})$ and every $t > 0$,*

$$\lim_{n\to\infty} \mathbb{E}_{\nu_{\beta,\lambda}} \left[\left(\int_0^t \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} \mathbf{H}(x/n) \cdot \left[\tau_x \mathscr{P}_{\rho,\theta}\, g\left(\xi(sn)\right)\right] ds \right)^2 \right] = 0,$$

where for a local function g and $\xi \in (0,+\infty)^{\mathbb{Z}}$, $\tau_x g(\xi) := g(\tau_x \xi)$ and $\tau_x \xi(y) := \xi(x+y)$.

Since we prove a refined version of this proposition we omit its proof. As a consequence of last result, we get that the fluctuation field $(\mathscr{Z}_\cdot^{n,0})_{n\geq 1}$ converges in law (in the sense of Theorem 1) to the solution of the equation at the right hand side of (20). Theorem 1 is a simple consequence of this fact.

In order to prove Corollary 1 and Corollary 2 we follow the same argument as in the proof of Theorem 4.2 of [18]. In order to compute the limiting variance for the energy flux, we do the following:

$$\begin{aligned}\mathbb{E}_Q[Z_t^e Z_s^e] &= \mathbb{E}_Q[\{\mathscr{Y}_t(H_0^1)-\mathscr{Y}_0(H_0^1)\}\{\mathscr{Y}_s(H_0^1)-\mathscr{Y}_0(H_0^1)\}]\\ &= \lim_{\ell\to+\infty}\mathbb{E}_Q[\mathscr{Y}_t(G_\ell^1)\mathscr{Y}_s(G_\ell^1)-\mathscr{Y}_t(G_\ell^1)\mathscr{Y}_0(G_\ell^1)-\mathscr{Y}_s(G_\ell^1)\mathscr{Y}_0(G_\ell^1)+\mathscr{Y}_0(G_\ell^1)\mathscr{Y}_0(G_\ell^1)]\end{aligned}$$

where

$$H_0^1(x)=\begin{pmatrix} H_0(x)\\ 0\end{pmatrix},\quad G_\ell^1(x)=\begin{pmatrix} G_\ell(x)\\ 0\end{pmatrix},$$

H_0 is the Heaviside function, namely $H_0(x):=\mathbf{1}_{\{\mathbf{x}\geq\mathbf{0}\}}$ and $G_\ell(x):=(1-x/\ell)\mathbf{1}_{\{\mathbf{0}\leq\mathbf{x}\leq\ell\}}$. By (28), $\mathbb{E}_Q[\mathscr{Y}_t(H)\mathscr{Y}_0(G)]:=\langle T_t^-(\Lambda^T H)\cdot\chi\Lambda^T G\rangle$. By the definition of $(T_t^-)_{t\geq 0}$ we have for G_1,G_2 test functions in $S(\mathbb{R})$:

$$T_t^-\begin{pmatrix} G_1(x)\\ G_2(x)\end{pmatrix}=\begin{pmatrix} \frac{1}{\rho}\Big(G_2(x-2b^2\rho t)-G_2(x)\Big)+G_1(x-2b^2\rho t)\\ G_2(x)\end{pmatrix}.$$

As a consequence we obtain that

$$\begin{aligned}\mathbb{E}_Q[Z_t^e Z_s^e] &= \Big(1-\frac{1}{\rho}\Big)^2\Big(\frac{\lambda+1}{\beta^2}\Big)\\ &\quad\times\lim_{\ell\to+\infty}\int_{\mathbb{R}}\Big(G_\ell^t(x)G_\ell^s(x)-G_\ell^t(x)G_\ell(x)-G_\ell^s(x)G_\ell(x)+G_\ell(x)G_\ell(x)\Big)dx,\end{aligned}$$

where for $t\geq 0$, $G_\ell^t(x):=G_\ell(x-2b^2\rho t)$. Now, using (15) and (16) the proof ends. Analogously, by taking an approximation vector function as $(0,H_0)$ we get the result for the volume flux.

6.2. **The longer time scale.** Since in the hyperbolic time scale the initial fluctuations for the field $\mathscr{Z}_\cdot^{n,\alpha}$ are transported by the linearized system on the right hand side of (20), we redefine the fluctuation field $\widehat{\mathscr{Z}_\cdot^{n,\alpha}}$, $\alpha>0$, on $\mathbf{G}\in S(\mathbb{R})\times S(\mathbb{R})$, by

$$\widehat{\mathscr{Z}_t^{n,\alpha}}(\mathbf{G})=\mathscr{Z}_t^{n,\alpha}\left(T_{tn^\alpha}^+\mathbf{G}\right).$$

For $\xi\in(0,+\infty)^{\mathbb{Z}}$, let $J_{x,x+1}(\xi)$ be the microscopic current vector $\begin{pmatrix} j_{x,x+1}(\xi)\\ j'_{x,x+1}(\xi)\end{pmatrix}$.

By Dynkin's formula,

$$\mathscr{M}_t^{n,\alpha}(\mathbf{G})=\widehat{\mathscr{Z}_t^{n,\alpha}}(\mathbf{G})-\widehat{\mathscr{Z}_0^{n,\alpha}}(\mathbf{G})-\int_0^t\Big\{n^{1+\alpha}\mathscr{L}\left(\widehat{\mathscr{Z}_s^{n,\alpha}}(\mathbf{G})\right)+\partial_s\widehat{\mathscr{Z}_s^{n,\alpha}}(\mathbf{G})\Big\}\,ds$$

is a martingale with quadratic variation given by

$$\langle\mathscr{M}^{n,\alpha}\rangle_t=\int_0^t n^{1+\alpha}\mathscr{L}\left(\widehat{\mathscr{Z}_s^{n,\alpha}}(\mathbf{G})\right)^2-2n^{1+\alpha}\left(\widehat{\mathscr{Z}_s^{n,\alpha}}(\mathbf{G})\right)\mathscr{L}\left(\widehat{\mathscr{Z}_s^{n,\alpha}}(\mathbf{G})\right)ds.$$

A simple computation shows that $\mathbb{E}_{\nu_{\beta,\lambda}}[\langle\mathscr{M}^{n,\alpha}\rangle_t]$ vanishes as n goes to $+\infty$ for $\alpha<1$. This is equivalent to saying that the martingale $\mathscr{M}_t^{n,\alpha}$ vanishes as n goes to $+\infty$ in $\mathbb{L}^2(\mathbb{P}_{\nu_{\beta,\lambda}})$,

for $\alpha < 1$. Observe that, by definition of $(T_t^+)_{t\geq 0}$, we have

$$\begin{aligned}\partial_s \widehat{\mathscr{Z}}_s^{n,\alpha}(\mathbf{G}) &= -\frac{n^\alpha}{\sqrt{n}} \sum_{x\in\mathbb{Z}} M^T \left[\partial_q \left(T^+_{sn^\alpha}\mathbf{G}\right)(x/n)\right]\cdot(\omega_x(sn^{1+\alpha})-\omega)\\ &= -\frac{n^\alpha}{\sqrt{n}} \sum_{x\in\mathbb{Z}} \left[\partial_q \left(T^+_{sn^\alpha}\mathbf{G}\right)(x/n)\right]\cdot M(\omega_x(sn^{1+\alpha})-\omega).\end{aligned}$$

On the other hand, the first term in the integral part of the martingale $\mathscr{M}_t^{n,\alpha}(\mathbf{G})$ is equal to

$$\frac{n^\alpha}{\sqrt{n}} \sum_{x\in\mathbb{Z}} n\Big((T^+_{sn^\alpha}\mathbf{G})\left(\frac{x+1}{n}\right) - (T^+_{sn^\alpha}\mathbf{G})\left(\frac{x}{n}\right)\Big)\cdot\Big(J_{x,x+1}(\xi(sn^{1+\alpha})) - \mathbb{E}_{\nu_{\beta,\lambda}}[J_{x,x+1}]\Big).$$

Performing a Taylor expansion, we can replace this term, up to a term vanishing as n goes to $+\infty$ in $\mathbb{L}^2(\mathbb{P}_{\nu_{\beta,\lambda}})$, by

$$\frac{n^\alpha}{\sqrt{n}} \sum_{x\in\mathbb{Z}} \Big((\partial_q T^+_{sn^\alpha}\mathbf{G})(x/n)\Big)\cdot\Big(J_{x,x+1}(sn^{1+\alpha}) - \langle J_{x,x+1}\rangle \nu_{\beta,\lambda}\Big).$$

Thus, in order to show that

$$\lim_{n\to\infty} \mathbb{E}_{\nu_{\beta,\lambda}}\left[\Big(\widehat{\mathscr{Z}}_t^{n,\alpha}(\mathbf{G}) - \widehat{\mathscr{Z}}_0^{n,\alpha}(\mathbf{G})\Big)^2\right] = 0, \tag{29}$$

it remains to show that

$$\lim_{n\to\infty} \mathbb{E}_{\nu_{\beta,\lambda}}\left[\left(\frac{n^\alpha}{\sqrt{n}}\int_0^t ds \sum_{x\in\mathbb{Z}} (\partial_q T^+_{sn^\alpha}\mathbf{G})(x/n)\cdot\Theta_x(\xi(sn^{1+\alpha}))\right)^2\right] = 0$$

where for $\xi \in (0,+\infty)^{\mathbb{Z}}$

$$\Theta_x(\xi) = J_{x,x+1}(\xi) - \langle J_{x,x+1}\rangle_{\nu_{\beta,\lambda}} - M(\omega_x - \omega).$$

Observe that in this formula, $M := M(\rho,\theta)$ is the differential w.r.t. (ρ,θ) of the function $\langle J_{x,x+1}\rangle_{\nu_{\beta,\lambda}}$ as computed below (20). A simple computation shows that for $\xi \in (0,+\infty)^{\mathbb{Z}}$

$$\Theta_x(\xi) = \begin{pmatrix} -b^2(\xi_{x+1}-\rho)(\xi_x-\rho) - (\gamma + b^2\rho)\nabla\xi_x \\ -\nabla(b^2\xi_x + \gamma\log(\xi_x))\end{pmatrix}.$$

The discrete gradient terms appearing in the previous expression, permit to perform another discrete integration by parts and the resulting terms vanish in $\mathbb{L}^2(\mathbb{P}_{\nu_{\beta,\lambda}})$ as n goes to $+\infty$, for $\alpha < 1$. Using the smoothness of the function $\mathbf{G}$, we see that it only remains to show the following

Theorem 4 (Boltzmann-Gibbs principle II)**.** *Fix $\alpha < 1/3$ and let $\varphi : \mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ be such that $\varphi(\cdot,x)\in S(\mathbb{R})$. For every $t>0$*

$$\lim_{n\to\infty} \mathbb{E}_{\nu_{\beta,\lambda}}\left[\left(\int_0^t \frac{n^\alpha}{\sqrt{n}} \sum_{x\in\mathbb{Z}} \varphi(sn^\alpha, x/n)(\xi_x(sn^{1+\alpha})-\rho)(\xi_{x+1}(sn^{1+\alpha})-\rho)\,ds\right)^2\right] = 0$$

Proof. In the following, $C, C_0, C_1, \ldots$ denote constants independent of n whose values can change from line to line. Let $f_s(\xi)$ be the function $\sum_{x\in\mathbb{Z}} \varphi(s,x/n) H_{\delta_x+\delta_{x+1}}(\xi)$. We have

the following upper bound

$$\mathbb{E}_{\nu_{\beta,\lambda}}\left[\left(\int_0^t f_{sn^\alpha}(\xi(sn^{1+\alpha}))\right)^2\right] \le C\int_0^t \langle f_{sn^\alpha}, (s^{-1}-n^{1+\alpha}\mathscr{L})^{-1} f_{sn^\alpha}\rangle_{\nu_{\beta,\lambda}} ds$$
$$= \frac{C}{n^{1+\alpha}}\int_0^t \left\langle f_{sn^\alpha}, \left(\frac{1}{sn^{1+\alpha}}-\mathscr{L}\right)^{-1}_{sn^\alpha}\right\rangle_{\nu_{\beta,\lambda}} ds$$
$$\le \frac{C}{n^{1+\alpha}}\int_0^t \left\langle f_{sn^\alpha}, \left(\frac{1}{sn^{1+\alpha}}-\gamma\mathscr{S}\right)^{-1} f_{sn^\alpha}\right\rangle_{\nu_{\beta,\lambda}} ds.$$

In the first inequality above we used the result of [25] applied to this setting. We notice that since our test functions depend on time, the result of [25] has to be modified as written here. To prove the last result one can simply adapt the proof of Lemma 4.3 of [10] to this case. We also notice that for our purposes, we need to prove the result for φ equal to the first component of the column vector $\partial_q T^+_{sn^\alpha}\mathbf{G}$.

In order to simplify notations, let us define $\varepsilon = 1/sn^{1+\alpha}$.

We denote by Σ_2^0 the set of configurations σ of Σ_2 such that $\sigma = 2\delta_x$, $x\in\mathbb{Z}$, and $\Sigma_2^\pm$ the complementary set of Σ_2^0 in Σ_2, i.e. the set of configurations $\sigma\in\Sigma_2$ such that $\sigma=\delta_x+\delta_y$, $y\neq x\in\mathbb{Z}$. Observe that f_{sn^α} is a function of degree 2 with a decomposition in the form $f_{sn^\alpha} = \sum_{\sigma\in\Sigma_2}\Phi_{sn^\alpha}(\sigma)H_\sigma$ which satisfies $\Phi_{sn^\alpha}(\sigma)=0$ if $\sigma\in\Sigma_2^0$. We have that (see e.g. [25])

$$\left\langle f_{sn^\alpha}, \left(\frac{1}{sn^{1+\alpha}}-\gamma\mathscr{S}\right)^{-1} f_{sn^\alpha}\right\rangle_{\nu_{\beta,\lambda}} = \sup_g\left\{2\langle f_{sn^\alpha}, g\rangle_{\nu_{\beta,\lambda}} - \varepsilon\langle g, g\rangle_{\nu_{\beta,\lambda}} - \gamma\mathscr{D}(g)\right\}$$

where the supremum is taken over local functions $g\in\mathbb{L}^2(\nu_{\beta,\lambda})$. Decompose g appearing in this variational formula as $g=\sum_\sigma G(\sigma)H_\sigma$. Recall that $\{H_\sigma;\sigma\in\Sigma\}$ are orthogonal, that the function f_{sn^α} is a degree 2 function such that $\Phi_{sn^\alpha}(\sigma)=0$ for any $\sigma\notin\Sigma_2^\pm$ and formula (26) for the Dirichlet form $\mathscr{D}(g)$. Thus, we can restrict this supremum over degree 2 functions g such that $G(\sigma)=0$ if $\sigma\in\Sigma_2^0$. Then, by Lemma 1, we have

$$\left\langle f_{sn^\alpha}, \left(\frac{1}{sn^{1+\alpha}}-\gamma\mathscr{S}\right)^{-1} f_{sn^\alpha}\right\rangle_{\nu_{\beta,\lambda}}$$
$$\le \sup_G\left\{\sum_{x\neq y}\Phi_{sn^\alpha}(x,y)G(x,y) - \varepsilon\sum_{\substack{(x,y)\in\mathbb{Z}^2\\ x\neq y}} G^2(x,y) - C\sum_{|\mathbf{e}|=1}\sum_{\substack{(x,y)\in\Delta^\pm\\ (x,y)+\mathbf{e}\in\Delta^\pm}}\Big(G((x,y)+\mathbf{e})-G(x,y)\Big)^2\right\}$$

where $C:=C(\lambda,\gamma)$, $\Delta_\pm=\{(x,y)\in\mathbb{Z}^2; x\neq y\}$ and as usual we identify the functions defined on Σ_n with symmetric functions defined on $\mathbb{Z}^n$.

To any symmetric function G defined on the set $\Delta_\pm$, we denote by $\tilde{G}$ its extension to $\mathbb{Z}^2$ defined by

$$\tilde{G}(x,y)=G(x,y)\text{ if }x\neq y,\quad \tilde{G}(x,x)=\frac{1}{4}\sum_{|\mathbf{e}|=1}G((x,x)+\mathbf{e}).$$

It is trivial that

$$\sum_{(x,y)\in\mathbb{Z}^2}\tilde{G}^2(x,y)\le C\sum_{\substack{(x,y)\in\mathbb{Z}^2\\ x\neq y}}G^2(x,y),$$

$$\sum_{|\mathbf{e}|=1}\sum_{(x,y)\in\mathbb{Z}^2}\Big(\tilde{G}((x,y)+\mathbf{e})-\tilde{G}(x,y)\Big)^2\le C\sum_{|\mathbf{e}|=1}\sum_{\substack{(x,y)\in\Delta^{\pm}\\ (x,y)+\mathbf{e}\in\Delta^{\pm}}}\Big(G((x,y)+\mathbf{e})-G(x,y)\Big)^2.$$

Thus, we have

$$\begin{aligned}&\Big\langle f_{sn^\alpha},\Big(\frac{1}{sn^{1+\alpha}}-\gamma\mathscr{S}\Big)^{-1}f_{sn^\alpha}\Big\rangle_{\nu_{\beta,\lambda}}\\ &\le C_0\sup_G\Big\{\sum_{(x,y)\in\mathbb{Z}^2}\Phi_{sn^\alpha}(x,y)G(x,y)-C_1\varepsilon\sum_{(x,y)\in\mathbb{Z}^2}G^2(x,y)\\ &\qquad -C_2\sum_{|\mathbf{e}|=1}\sum_{(x,y)\in\mathbb{Z}^2}\Big(G((x,y)+\mathbf{e})-G(x,y)\Big)^2\Big\}\end{aligned}$$

where the supremum is now taken over all symmetric local functions $G:\mathbb{Z}^2\to\mathbb{R}$. By using Fourier transforms, it is easy to show that this supremum is equal to

$$\frac{C_0}{4}\int_{[0,1]^2}\frac{|\hat{\Phi}_{sn^\alpha}(\mathbf{k})|^2}{C_1\varepsilon+4C_2\sum_{i=1}^2\sin^2(\pi k_i)}d\mathbf{k}$$

where the Fourier transform $\hat{\Phi}_{sn^\alpha}$ of Φ_{sn^α} is given by

$$\hat{\Phi}_{sn^\alpha}(\mathbf{k})=\sum_{(x,y)\in\mathbb{Z}^2}\Phi_{sn^\alpha}(x,y)e^{2i\pi(k_1x+k_2y)},\quad \mathbf{k}=(k_1,k_2)\in[0,1]^2.$$

By definition of f_{sn^α}, we have $\Phi_{sn^\alpha}(x,y)=\frac{1}{2}\Big(\varphi(sn^\alpha,x/n)+\varphi(sn^\alpha,y/n)\Big)$ if $|x-y|=1$ and 0 otherwise. Consequently, we have

$$\begin{aligned}&\Big\langle f_{sn^\alpha},\Big(\frac{1}{sn^{1+\alpha}}-\gamma\mathscr{S}\Big)^{-1}f_{sn^\alpha}\Big\rangle_{\nu_{\beta,\lambda}}\le\frac{C_0}{16}\int_{[0,1]^2}\frac{\Big|\sum_{x\in\mathbb{Z}}\varphi(sn^\alpha,x/n)e^{2i\pi x(k_1+k_2)}\Big|^2}{C_1\varepsilon+4C_2\sum_{i=1}^2\sin^2(\pi k_i)}d\mathbf{k}\\ &=\frac{C_0}{16}\int_{[0,1]}dp\left|\sum_{x\in\mathbb{Z}}\varphi(sn^\alpha,x/n)e^{2i\pi xp}\right|^2\int_{[0,1]}\frac{dk_1}{C_1\varepsilon+4C_2\sin^2(\pi k_1)+4C_2\sin^2(\pi(p-k_1))}\\ &\le\frac{C_0}{16}\int_{[0,1]}dp\left|\sum_{x\in\mathbb{Z}}\varphi(sn^\alpha,x/n)e^{2i\pi xp}\right|^2\int_{[0,1]}\frac{dk_1}{C_1\varepsilon+4C_2\sin^2(\pi k_1)}\\ &\le\frac{C}{\sqrt{\varepsilon}}\int_{[0,1]}\left|\sum_{x\in\mathbb{Z}}\varphi(sn^\alpha,x/n)e^{2i\pi xp}\right|^2dp.\end{aligned}$$

Observe now that

$$\int_{[0,1]}\left|\sum_{x\in\mathbb{Z}}\varphi(x/n)e^{2i\pi xp}\right|^2=\sum_{(x,y)\in\mathbb{Z}^2}\varphi(x/n)\varphi(y/n)\int_{[0,1]}e^{2i\pi(x-y)p}dp=\sum_{x\in\mathbb{Z}}\varphi^2(x/n)\le Cn.$$

Putting everything together, we get that

$$\mathbb{E}_{\nu_{\beta,\lambda}}\Big[\Big(\int_0^t \frac{n^\alpha}{\sqrt{n}}\sum_{x\in\mathbb{Z}}\varphi(sn^\alpha,x/n)(\xi_x(sn^{1+\alpha})-\rho)(\xi_{x+1}(sn^{1+\alpha})-\rho)\,ds\Big)^2\Big] \\ \leq \frac{Ctn^{2\alpha-1}}{n^{1+\alpha}}\int_0^t \frac{n}{\sqrt{\varepsilon}}\,ds.$$

Since $\varepsilon := 1/sn^{1+\alpha}$ last expression vanishes as n goes to $+\infty$, if $\alpha < 1/3$. □

Now, in order to prove Corollary 3 we follow the same arguments as in the proof of Proposition 9.3 of [18] and we proceed as follows. We use the change of variables to defined the energy (resp. volume) flux through the time dependent bond $u_t^{x,\alpha}(n)$ during the time interval $[0,tn^{1+\alpha}]$ as:

$$\tilde{\mathscr{E}}^n_{u_t^{x,\alpha}(n)}(t) := \sum_{y\geq u_t^{x,\alpha}(n)}\{\xi_y(tn^{1+\alpha})-\xi_y(0)\}$$

$$\Big(\text{resp.}\quad \tilde{\mathscr{V}}^n_{u_t^{x,\alpha}(n)}(t) := \sum_{y\geq u_t^{x,\alpha}(n)}\{\log(\xi_y(tn^{1+\alpha}))-\log(\xi_y(0))\}\Big).$$

By (14) we have that

$$\mathscr{E}^n_{u_t^{x,\alpha}(n)}(t) := \tilde{\mathscr{E}}^n_{u_t^{x,\alpha}(n)}(t) - \tilde{\mathscr{V}}^n_{u_t^{x,\alpha}(n)}(t), \qquad \mathscr{V}^n_{u_t^{x,\alpha}(n)}(t) := -\frac{1}{b}\tilde{\mathscr{V}}^n_{u_t^{x,\alpha}(n)}(t).$$

Then, from (29) applied to the vector function G_ℓ^1 we obtain that

$$\lim_{n\to\infty}\mathbb{E}_{\nu_{\beta,\lambda}}\left[\left(\frac{1}{\sqrt{n}}\{\tilde{\mathscr{E}}^n_{u_t^{x,\alpha}(n)}(t)-\mathbb{E}_{\nu_{\beta,\lambda}}[\tilde{\mathscr{E}}^n_{u_t^{x,\alpha}(n)}(t)]\}\right)^2\right]=0.$$

On the other hand, applying (29) to the vector function

$$\begin{pmatrix} \frac{1}{\rho}G_\ell(x-u_t^{x,\alpha}(n)) \\ -G_\ell(x-u_t^{x,\alpha}(n)) \end{pmatrix}$$

we obtain that

$$\lim_{n\to\infty}\mathbb{E}_{\nu_{\beta,\lambda}}\left[\left(\frac{1}{\rho}\frac{1}{\sqrt{n}}\{\tilde{\mathscr{E}}^n_{u_t^{x,\alpha}(n)}(t)-\mathbb{E}_{\nu_{\beta,\lambda}}[\tilde{\mathscr{E}}^n_{u_t^{x,\alpha}(n)}(t)]\}-\frac{1}{\sqrt{n}}\{\tilde{\mathscr{V}}^n_{u_t^{x,\alpha}(n)}(t)-\mathbb{E}_{\nu_{\beta,\lambda}}[\tilde{\mathscr{V}}^n_{u_t^{x,\alpha}(n)}(t)]\}\right)^2\right]=0.$$

Now, Corollary 3 follows easily from the previous results.

7. Diffusivity

In this section we prove Theorem 3. Our proof is based on the resolvent methods introduced in [4, 22] and developed in few other contexts (e.g. [5, 24, 28]). Some differences with these previous works is the presence of two and not only one conserved quantity and the degeneracy of the symmetric part of the generator.

We fix $\rho>0,\theta\in\mathbb{R}$ and denote by β,λ the chemical potentials given by (16). Let also $(\bar\beta,\bar\lambda)$ be given in terms of (β,λ) by (15).

Recall the definition of $\hat{J}_{x,x+1}$ given in (11). We introduce the normalized currents $\mathbf{j}_{x,x+1}$, $\mathbf{j}'_{x,x+1}$ and $\mathbf{J}_{x,x+1}$ corresponding to the process $(\xi(t))_{t\geq 0}$, which are defined by

$$\mathbf{j}_{x,x+1}(\xi) = j_{x,x+1}(\xi) - \langle j_{x,x+1}\rangle_{\nu_{\beta,\lambda}} - \partial_\rho\langle j_{x,x+1}\rangle_{\nu_{\beta,\lambda}}(\xi_x-\rho) - \partial_\theta\langle j_{x,x+1}\rangle_{\nu_{\beta,\lambda}}(\log(\xi_x)-\theta),$$
$$\mathbf{j}'_{x,x+1}(\xi) = j'_{x,x+1}(\xi) - \langle j'_{x,x+1}\rangle_{\nu_{\beta,\lambda}} - \partial_\rho\langle j'_{x,x+1}\rangle_{\nu_{\beta,\lambda}}(\xi_x-\rho) - \partial_\theta\langle j'_{x,x+1}\rangle_{\nu_{\beta,\lambda}}(\log(\xi_x)-\theta),$$
$$\mathbf{J}_{x,x+1}(\xi) = \mathbf{j}_{x,x+1}(\xi) - \mathbf{j}'_{x,x+1}(\xi).$$

Since $\langle j_{x,x+1}\rangle_{\nu_{\beta,\lambda}} = -b^2\rho^2$ and $\langle j'_{x,x+1}\rangle_{\nu_{\beta,\lambda}} = -2b^2\rho$, we get

$$\begin{aligned}\mathbf{j}_{x,x+1}(\xi) &= -b^2(\xi_x-\rho)(\xi_{x+1}-\rho) - (\gamma+b^2\rho)\nabla(\xi_x)\\ \mathbf{j}'_{x,x+1}(\xi) &= -\nabla(b^2\xi_x+\gamma\log(\xi_x)).\end{aligned} \tag{30}$$

For any local compactly supported functions $f,g:(0,+\infty)^{\mathbb{Z}}\to\mathbb{R}$ we define the inner product $\ll f,g\gg:=\ll f,g\gg_{\beta,\lambda}$ of f and g by

$$\begin{aligned}\ll f,g\gg &= \sum_{x\in\mathbb{Z}}\Big(\langle\tau_x fg\rangle_{\nu_{\beta,\lambda}} - \langle f\rangle_{\nu_{\beta,\lambda}}\langle g\rangle_{\nu_{\beta,\lambda}}\Big)\\ &= \lim_{k\to+\infty}\sum_{|x|\le k}\Big(\langle\tau_x fg\rangle_{\nu_{\beta,\lambda}} - \langle f\rangle_{\nu_{\beta,\lambda}}\langle g\rangle_{\nu_{\beta,\lambda}}\Big)\\ &= \lim_{k\to+\infty}\frac{1}{2k+1}\sum_{|x|\le k}\left\{\sum_{|y-x|\le k}\Big(\langle\tau_{x+y}f\,\tau_y g\rangle_{\nu_{\beta,\lambda}} - \langle f\rangle_{\nu_{\beta,\lambda}}\langle g\rangle_{\nu_{\beta,\lambda}}\Big)\right\}\\ &= \lim_{k\to+\infty}\left\langle\left(\frac{1}{\sqrt{2k+1}}\sum_{|x|\le k}\tau_x f\right)\left(\frac{1}{\sqrt{2k+1}}\sum_{|x|\le k}\tau_x g\right)\right\rangle_{\nu_{\beta,\lambda}}\end{aligned}$$

where the third equality follows from the invariance of $\nu_{\beta,\lambda}$ by the shift. Observe also that the first sum on $\mathbb{Z}$ is in fact a finite sum since f and g are assumed to be local functions. We denote by $\mathscr{H}_0$ the completed space. Observe that any constant and gradient functions are equal to 0 in $\mathscr{H}_0$.

By (30), the normalized current associated to the volume is a gradient and this shows that $\mathscr{F}_{i,j}(\gamma,z)=0$ if $(i,j)\neq(1,1)$. By the definition of $\hat{J}_{x,x+1}$ and by (14), we are only interested in the behavior, as $z\to 0$, of

$$\mathfrak{L}(z) = \ll \mathbf{J}_{0,1},(z-\mathscr{L})^{-1}\mathbf{J}_{0,1}\gg = \int_0^\infty e^{-zt}\ll \mathbf{J}_{0,1}(t)\,,\mathbf{J}_{0,1}(0)\gg dt.$$

Since gradient functions are equal to 0 in $\mathscr{H}_0$, this is equivalent to estimate

$$\mathfrak{L}(z) = b^4\ll W_{0,1},(z-\mathscr{L})^{-1}W_{0,1}\gg$$

where $W_{x,y}$ is the local function $W_{x,y}=(\xi_x-\rho)(\xi_y-\rho)$.

In this section we prove there exists a constant $C>0$ such that

$$\ll W_{0,1},(z-\mathscr{L})^{-1}W_{0,1}\gg \ge Cz^{-1/4}. \tag{31}$$

But before proving (31) let us show (13) which is a direct consequence of the following lemma.

Lemma 2. *For any $\gamma>0$, there exists a constant $C:=C(\gamma)$ such that*

$$\ll W_{0,1}\,,(z/\gamma-b^2\mathscr{A}-\mathscr{S})^{-1}W_{0,1}\gg \le C\ll W_{0,1},(z-b^2\mathscr{A}-\gamma\mathscr{S})^{-1}W_{0,1}\gg$$

and

$$\ll W_{0,1},(z-b^2\mathscr{A}-\gamma\mathscr{S})^{-1}W_{0,1}\gg \le C\ll W_{0,1}\,,(z/\gamma-b^2\mathscr{A}-\mathscr{S})^{-1}W_{0,1}\gg.$$

Proof. Assume $\gamma>1$ the case $\gamma<1$ being similar. By Lemma 2.1 of [4] we have the variational formula for $\ll W_{0,1},(z-\mathscr{L})^{-1}W_{0,1}\gg$, where $\mathscr{L}=b^2\mathscr{A}-\gamma\mathscr{S}$, given by

$$\sup_f\big\{2\ll W_{0,1},f\gg - \ll f,(z-\gamma\mathscr{S})f\gg - b^4\ll\mathscr{A}f,(z-\gamma\mathscr{S})^{-1}\mathscr{A}f\gg\big\},$$

where the supremum is carried over functions f belonging to the domain of the generator $\mathscr{L}$ or equivalently to a dense subspace included in this domain, say the space of smooth local compactly supported functions. We have that

$$\begin{aligned}
&\sup_f \left\{ 2 \ll W_{0,1}, f \gg - \ll f, (z-\gamma\mathscr{S}) f \gg -b^4 \ll \mathscr{A} f, (z-\gamma\mathscr{S})^{-1} \mathscr{A} f \gg \right\} \\
=&\sup_f \left\{ 2 \ll W_{0,1}, f \gg -\gamma \ll f, (z/\gamma-\mathscr{S}) f \gg -b^4\gamma^{-1} \ll \mathscr{A} f, (z/\gamma-\mathscr{S})^{-1} \mathscr{A} f \gg \right\} \\
\geq&\sup_f \left\{ 2 \ll W_{0,1}, f \gg -\gamma \ll f, (z/\gamma-\mathscr{S}) f \gg -b^4\gamma \ll \mathscr{A} f, (z/\gamma-\mathscr{S})^{-1} \mathscr{A} f \gg \right\} \\
=&\sup_f \left\{ 2\gamma^{-1/2} \ll W_{0,1}, f \gg - \ll f, (z/\gamma-\mathscr{S}) f \gg -b^4 \ll \mathscr{A} f, (z/\gamma-\mathscr{S})^{-1} \mathscr{A} f \gg \right\}
\end{aligned}$$

where the penultimate inequality comes from $\gamma > 1$ and the ultimate is obtained by the change of f into $\gamma^{-1/2} f$. The last term is equal to

$$\gamma^{-1} \ll W_{0,1}\,, (z/\gamma - b^2\mathscr{A} - \mathscr{S})^{-1} W_{0,1} \gg$$

and this proves the first inequality of the lemma.

For the second one we proceed similarly:

$$\begin{aligned}
&\sup_f \left\{ 2 \ll W_{0,1}, f \gg - \ll f, (z-\gamma\mathscr{S}) f \gg -b^4 \ll \mathscr{A} f, (z-\gamma\mathscr{S})^{-1} \mathscr{A} f \gg \right\} \\
=&\sup_f \left\{ 2 \ll W_{0,1}, f \gg -\gamma \ll f, (z/\gamma-\mathscr{S}) f \gg -b^4\gamma^{-1} \ll \mathscr{A} f, (z/\gamma-\mathscr{S})^{-1} \mathscr{A} f \gg \right\} \\
\leq&\sup_f \left\{ 2 \ll W_{0,1}, f \gg -\gamma^{-1} \ll f, (z/\gamma-\mathscr{S}) f \gg -b^4\gamma^{-1} \ll \mathscr{A} f, (z/\gamma-\mathscr{S})^{-1} \mathscr{A} f \gg \right\} \\
=&\sup_f \left\{ 2\gamma^{1/2} \ll W_{0,1}, f \gg - \ll f, (z/\gamma-\mathscr{S}) f \gg -b^4 \ll \mathscr{A} f, (z/\gamma-\mathscr{S})^{-1} \mathscr{A} f \gg \right\} \\
=&\gamma \ll W_{0,1}\,, (z/\gamma - b^2\mathscr{A} - \mathscr{S})^{-1} W_{0,1} \gg .
\end{aligned}$$

□

Recall the orthogonal decomposition obtained in Section 5. Let $f = \sum_\sigma F(\sigma) H_\sigma$ and $g = \sum_\sigma G(\sigma) H_\sigma$ be two centered local functions (i.e. $F(0) = G(0) = 0$, where 0 denotes the empty configuration of Σ). The shifted configuration σ by $z \in \mathbb{Z}$ is denoted by $\tau_z\sigma$. We identify F_n, G_n, the restrictions of F, G to Σ_n, to symmetric functions on $\mathbb{Z}^n$. By (24) we have that

$$\ll f, g \gg = \sum_{z\in\mathbb{Z}} \sum_{\sigma\in\Sigma} F(\tau_z\sigma) G(\sigma) \mathscr{W}(\sigma),$$

where $\mathscr{W}$ was defined in (25).

With some abuse of notations, we denote by $\ll F, G \gg$ the scalar product defined by

$$\ll F, G \gg = \sum_{z\in\mathbb{Z}} \sum_{\sigma\in\Sigma} F(\tau_z\sigma) G(\sigma) \mathscr{W}(\sigma).$$

We also introduce the inner product $\ll \cdot, \cdot \gg_{\text{free}}$ defined by

$$\ll F, G \gg_{\text{free}} = \sum_{y\in\mathbb{Z}} \sum_{\sigma\in\Sigma} F(\tau_y\sigma) G(\sigma).$$

Since the function $\mathscr{W}$ is invariant by the shift, we have a very simple relation between these two inner products:

$$\ll F, G \gg = \ll \mathscr{W}^{1/2} F, \mathscr{W}^{1/2} G \gg_{\text{free}} . \tag{32}$$

On the set Σ_n we introduce the equivalence relation $\star$ defined by $\sigma \star \sigma'$ if and only if there exists $u \in \mathbb{Z}$ such that $\tau_u \sigma = \sigma'$. Let $\Sigma_n^\star = \Sigma_n/\star$ be the set of classes for this relation and $\Sigma^\star = \cup_{n\geq 1}\Sigma_n^\star$. We can rewrite the scalar product $\ll \cdot,\cdot \gg_{\text{free}}$ as

$$\ll F, G \gg_{\text{free}} = \sum_{\bar\sigma \in \Sigma^\star} \bar F(\bar\sigma)\bar G(\bar\sigma).$$

Here $\bar F$ is defined by $\bar F(\bar\sigma) = \left(\sum_{y\in\mathbb{Z}} \tau_y F\right)(\sigma)$ where σ is any element of $\bar\sigma$. The function $\mathscr{W}$ being invariant by the shift, it makes sense to define $\mathscr{W}(\bar\sigma)$ by $\mathscr{W}(\sigma)$, $\sigma \in \bar\sigma$, $\bar\sigma \in \Sigma^\star$. Then, we have

$$\ll F, G \gg = \sum_{\bar\sigma \in \Sigma^\star} \mathscr{W}(\bar\sigma)\bar F(\bar\sigma)\bar G(\bar\sigma).$$

Lemma 3. *There exist constants $C := C(n,\lambda), c := c(n,\lambda)$ such that for any local function $F : \Sigma_n \to \mathbb{R}$ of degree n and any positive real $z > 0$*

(1)

$$C^{-1} \ll F, F \gg_{\text{free}} \leq \ll F, F \gg \leq\ C \ll F, F \gg_{\text{free}}.$$

(2)

$$C^{-1} \ll F, -\mathfrak{S}F \gg_{\text{free}} \leq \ll F, -\mathfrak{S}F \gg \leq\ C \ll F, -\mathfrak{S}F \gg_{\text{free}}.$$

(3)

$$\ll F, (z-\gamma\mathfrak{S})^{-1} F \gg = \ll \mathscr{W}^{1/2}F\,,\, (z-\gamma\mathfrak{S})^{-1}\mathscr{W}^{1/2}F \gg_{\text{free}}.$$

Proof. Recall the definition of $\mathscr{W}$ from (25). Thus, $\mathscr{W}$ is bounded from above (resp. from bellow) by a constant $C(n,\lambda)$ (resp. $C^{-1}(n,\lambda)$) independent of $\sigma \in \Sigma_n$. This is enough to conclude *(1)*. In order to prove *(2)*, it is enough to use (32) and the fact that for any local function $F : \Sigma \to \mathbb{R}$ we have that $\mathfrak{S}(\mathscr{W}^{1/2}F) = \mathscr{W}^{1/2}\mathfrak{S}F$. Finally, for a local function F of degree n, we have by (32) and the fact that

$$\ll F, (z-\gamma\mathfrak{S})^{-1}F \gg = \sup_{G \text{ of degree } n} \{2 \ll F, G \gg - \ll G, (z-\gamma\mathfrak{S})G \gg\},$$

the following equality

$$\ll F, (z-\gamma\mathfrak{S})^{-1}F \gg = \ll \mathscr{W}^{1/2}F, (z-\gamma\mathfrak{S})^{-1}\mathscr{W}^{1/2}F \gg_{\text{free}},$$

which proves *(3)*.

□

Our goal is to get a lower bound for $\ll W_{0,1}, (z-\mathscr{L})^{-1}W_{0,1} \gg$ which by Lemma 2.1 of [4] can be rewritten in the variational form

$$\sup_f \left\{2 \ll W_{0,1}, f \gg - \ll f, (z-\gamma\mathscr{S})f \gg - b^4 \ll \mathscr{A}f, (z-\gamma\mathscr{S})^{-1}\mathscr{A}f \gg\right\}.$$

Any element $\bar\sigma \in \Sigma_n^\star$ can be identified with an element of $\mathbb{N}^{n-1}$ through the application which associates to $(\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{N}^{n-1}$ the class of the configuration $\sigma = \delta_0 + \delta_{\alpha_1} + \ldots + \delta_{\alpha_1+\ldots+\alpha_{n-1}}$.

Observe also that $\mathfrak{S}$ is a self-adjoint operator with respect to $\ll \cdot,\cdot \gg$ and with respect to $\ll \cdot,\cdot \gg_{\text{free}}$.

We restrict the previous supremum over degree 2 functions $f = \sum_{(x,y)\in\mathbb{Z}^2} F([x,y])H_{[x,y]}$. In order to keep notation simple, whenever we identify a configuration $\sigma \in \Sigma_n$ with $[\mathbf{x}] \in \mathbb{Z}^n$ we will simply write $F(x)$, instead of $F([\mathbf{x}])$.

Up to some irrelevant multiplicative constant, a lower bound is given by

$$\sup_{F\text{of degree 2}} \left\{2F(0,1) - \|F\|^2_{1,z} - b^4\|\mathfrak{A}_- F\|^2_{-1,z} - b^4\|\mathfrak{A}_+ F\|^2_{-1,z} - b^4\|\mathfrak{A}_0 F\|^2_{-1,z}\right\}$$

where $\|F\|^2_{\pm 1,z} = \ll F, (z-\gamma\mathfrak{S})^{\pm 1} F \gg$. We also introduce the corresponding $H_{\pm 1,z}$-norms associated to $\ll \cdot,\cdot \gg_{\text{free}}$: $\|F\|^2_{\pm 1,z,\text{free}} = \ll F, (z-\gamma\mathfrak{S})^{\pm 1} F \gg_{\text{free}}$, for $F : \Sigma \to \mathbb{R}$.

By Lemma 3, there exists a constant C such that this lower bound is bounded from bellow by

$$\sup_{F\text{of degree 2}} \Big\{2F(0,1) - C\|F\|^2_{+1,z,\text{free}} \\ -b^4\|\mathscr{W}^{1/2}\mathfrak{A}_- F\|^2_{-1,z,\text{free}} - b^4\|\mathscr{W}^{1/2}\mathfrak{A}_+ F\|^2_{-1,z,\text{free}} - b^4\|\mathscr{W}^{1/2}\mathfrak{A}_0 F\|^2_{-1,z,\text{free}}\Big\}.$$

Let us first show that if F is of degree 2 then the contributions given by $\|\mathscr{W}^{1/2}\mathfrak{A}_- F\|^2_{-1,z,\text{free}}$ and $\|\mathscr{W}^{1/2}\mathfrak{A}_0 F\|^2_{-1,z,\text{free}}$ are equal to zero.

The function $\mathscr{W}$ is constant equal to $(\lambda+1)$ on Σ_1 so that $\mathscr{W}^{1/2}\mathfrak{A}_- F = \sqrt{\lambda+1}\mathfrak{A}_- F$. It is easy to check that the degree one function $\mathfrak{A}_- F$ satisfies

$$(\mathfrak{A}_- F)(u) = (\lambda-1)\Big(F(u-1,u) - F(u,u+1)\Big).$$

For any degree one function G, we have

$$\ll \mathfrak{A}_- F, G \gg_{\text{free}} = \sum_{u,y\in\mathbb{Z}} G(u+y)(\lambda-1)\Big(F(u-1,u) - F(u,u+1)\Big) = 0$$

by a telescopic sum argument. This shows that $\mathfrak{A}_- F$ is equal to zero in the Hilbert space generated by $\ll \cdot,\cdot \gg_{\text{free}}$.

Recall that if F is a degree 2 function, i.e. a symmetric function on $\mathbb{Z}^2$, then F is identified with a function $\bar{F}$ defined on $\mathbb{N}$ by

$$\bar{F}(\alpha) = \sum_{u\in\mathbb{Z}} F(u,u+\alpha)$$

and as a consequence, for F and G degree 2 functions it holds that

$$\ll F, G \gg_{\text{free}} = \sum_{\alpha\in\mathbb{N}} \bar{F}(\alpha)\bar{G}(\alpha). \tag{33}$$

Observe that $(\mathfrak{A}_0 F)(u,v)$ is equal to

$$= \begin{cases} 2(1+\lambda)\Big(F(u-1,u) - F(u,u+1)\Big), & \text{if } u = v, \\ (1+\lambda)\Big(F(u-1,u+1) - F(u,u+2)\Big) + (2+\lambda)\Big(F(u,u) - F(u+1,u+1)\Big), \\ \qquad \text{if } (u,v) = (u,u+1), \\ (1+\lambda)\Big(F(u-1,v) - F(u+1,v) + F(u,v-1) - F(u,v+1)\Big), & \text{if } |u-v| \geq 2 \end{cases}$$

and

$$\mathscr{W}(u,u) = \frac{(\lambda+1)(\lambda+2)}{2}, \quad \mathscr{W}(u,v) = (\lambda+1)^2 \quad \text{for } u \neq v. \tag{34}$$

It is then easy to show that

$$\overline{\mathscr{W}^{1/2}(\mathfrak{A}_0 F)}(\alpha) = 0$$

for any $\alpha \in \mathbb{N}$. Putting together the previous result and (33) it follows that:

$$\|\mathscr{W}^{1/2}\mathfrak{A}_0F\|^2_{-1,z,\text{free}} = \ll \mathscr{W}^{1/2}\mathfrak{A}_0F\ ,\ (z-\gamma\mathfrak{S})^{-1}(\mathscr{W}^{1/2}\mathfrak{A}_0F)\gg_{\text{free}}$$
$$= \sum_{\alpha\in\mathbb{N}} \overline{\mathscr{W}^{1/2}(\mathfrak{A}_0F)}(\alpha)\ \overline{(\lambda-\gamma\mathfrak{S})^{-1}[\mathscr{W}^{1/2}(\mathfrak{A}_0F)]}(\alpha) = 0.$$

Lemma 4. *There exists a positive constant C such that for every symmetric function F of degree 2, if $\bar F(\alpha) = \sum_{z\in\mathbb{Z}} F(z,z+\alpha)$, then*

$$C^{-1}\sum_{\substack{x,y\neq 0,\\ |x-y|=1}} \Big(\bar F(y)-\bar F(x)\Big)^2 \le \ll F,-\mathfrak{S}F\gg_{\text{free}} \le\ C\sum_{\substack{x,y\neq 0,\\ |x-y|=1}} \Big(\bar F(y)-\bar F(x)\Big)^2.$$

Proof. This follows easily from the following equalities together with (33):

$$\overline{\mathfrak{S}F}(0) = \sum_{y\in\mathbb{Z}} (\mathfrak{S}F)(y,y)$$
$$= \sum_{y\in\mathbb{Z}} \Big(F(y+1,y+1)-F(y,y)\Big) + \Big(F(y-1,y-1)-F(y,y)\Big) = 0,$$
$$\overline{\mathfrak{S}F}(1) = \sum_{y\in\mathbb{Z}} (\mathfrak{S}F)(y,y+1)$$
$$= \sum_{y\in\mathbb{Z}} \Big(F(y-1,y+1)-F(y,y+1)\Big) + \sum_{y\in\mathbb{Z}} \Big(F(y,y+2)-F(y,y+1)\Big)$$
$$= 2\Big(\bar F(2)-\bar F(1)\Big),$$
$$\overline{\mathfrak{S}F}(\alpha) = \sum_{y} (\mathfrak{S}F)(y,y+\alpha)$$
$$= \sum_{y\in\mathbb{Z}} \Big(F(y-1,y+\alpha)-F(y,y+\alpha)\Big) + \sum_{y\in\mathbb{Z}} \Big(F(y+1,y+\alpha)-F(y,y+\alpha)\Big)$$
$$+ \sum_{y\in\mathbb{Z}} \Big(F(y,y+\alpha+1)-F(y,y+\alpha)\Big) + \sum_{y\in\mathbb{Z}} \Big(F(y,y+\alpha-1)-F(y,y+\alpha)\Big)$$
$$= 2\Big(\bar F(\alpha+1)-\bar F(\alpha)\Big) + 2\Big(\bar F(\alpha-1)-\bar F(\alpha)\Big), \quad \alpha\ge 2.$$

□

To any degree three function G, i.e. a symmetric function G on $\mathbb{Z}^3$, the function $\bar G$ is identified with a function on $\mathbb{N}^2$:

$$\bar G(u,v) = \sum_{y\in\mathbb{Z}} G(y,u+y,u+v+y).$$

Since G is symmetric on $\mathbb{Z}^3$, then $\bar G$ is symmetric on $\mathbb{Z}^2$. As above, for F and G degree 3 functions it holds that

$$\ll F,G\gg_{\text{free}} = \sum_{(\alpha,\beta)\in\mathbb{N}^2} \bar F(\alpha,\beta)\bar G(\alpha,\beta). \tag{35}$$

Let $\mathbb{D}_3$, acting on the local functions on $\mathbb{N}^2$, be defined by

$$\mathbb{D}_3(\bar G) = \sum_{u\ge 1} \Big(\bar G(u+1,0)-\bar G(u,0)\Big)^2 + \sum_{v\ge 1} \Big(\bar G(0,v+1)-\bar G(0,v)\Big)^2 + \Big(\bar G(1,0)-\bar G(0,1)\Big)^2$$
$$+ \sum_{u,v\ge 1} \Big(\bar G(u+1,v)-\bar G(u,v)\Big)^2 + \Big(\bar G(u,v+1)-\bar G(u,v)\Big)^2. \tag{36}$$

This is the Dirichlet form of a symmetric nearest neighbors random walk on $\mathbb{N}^2$ where all the jumps between $\{0\}\times\mathbb{N}$ and $\mathbb{N}^*\times\mathbb{N}^*$[4], all the jumps from $\mathbb{N}\times\{0\}$ and $\mathbb{N}^*\times\mathbb{N}^*$ and all the jumps from 0 have been suppressed, and a jump between $(0,1)$ and $(1,0)$ has been added.

Lemma 5. *There exists a constant $C>0$ such that for any symmetric function G on $\mathbb{Z}^3$*

$$C^{-1}\mathbb{D}_3(\bar G)\ \le \ll G,-\mathfrak{S}G\gg_{\text{free}}\ \le\ C\mathbb{D}_3(\bar G).$$

Proof. We have the following equalities

$$\begin{aligned}
&\overline{\mathfrak{S}G}(0,0)=0,\\
&\overline{\mathfrak{S}G}(0,1)=2\Big(\bar G(0,2)-\bar G(0,1)\Big)+\Big(\bar G(1,0)-\bar G(0,1)\Big),\\
&\overline{\mathfrak{S}G}(0,\beta)=2\Big(\bar G(0,\beta+1)-\bar G(0,\beta)\Big)+2\Big(\bar G(0,\beta-1)-\bar G(0,\beta)\Big),\quad \beta\ge 2,\\
&\overline{\mathfrak{S}G}(1,0)=2\Big(\bar G(2,0)-\bar G(1,0)\Big)+\Big(\bar G(0,1)-\bar G(1,0)\Big)\\
&\overline{\mathfrak{S}G}(\alpha,0)=2\Big(\bar G(\alpha+1)-\bar G(\alpha,0)\Big)+2\Big(\bar G(\alpha-1)-\bar G(\alpha,0)\Big),\quad \alpha\ge 2,\\
&\overline{\mathfrak{S}G}(\alpha,\beta)=\Big(\bar G(\alpha+1,\beta)-\bar G(\alpha,\beta)\Big)+\Big(\bar G(\alpha,\beta+1)-\bar G(\alpha,\beta)\Big)\\
&+\mathbf{1}_{\{\alpha\ge 2\}}\Big(\bar G(\alpha-1,\beta+1)-\bar G(\alpha,\beta)\Big)+\mathbf{1}_{\{\alpha\ge 2\}}\Big(\bar G(\alpha-1,\beta)-\bar G(\alpha,\beta)\Big)\\
&+\mathbf{1}_{\{\beta\ge 2\}}\Big(\bar G(\alpha+1,\beta-1)-\bar G(\alpha,\beta)\Big)+\mathbf{1}_{\{\beta\ge 2\}}\Big(\bar G(\alpha,\beta-1)-\bar G(\alpha,\beta)\Big),\quad \alpha,\beta\ge 1.
\end{aligned}$$

We recognize in these expressions the generator of a symmetric nearest neighbors random walk on $\mathbb{N}^2$ where

- all the jumps between $\{0\}\times\mathbb{N}$ and $\mathbb{N}^*\times\mathbb{N}^*$, all the jumps between $\mathbb{N}\times\{0\}$ and $\mathbb{N}^*\times\mathbb{N}^*$, and all the jumps from 0 have been suppressed;
- a jump between $(0,1)$ and $(1,0)$ with rate 1 has been added;
- jumps between (α,β) and $(\alpha\pm1,\beta\mp1)$ for $(\alpha,\beta)\in\mathbb{N}^*\times\mathbb{N}^*$ with rate 1 have been added.
- the non vanishing jumps on $\mathbb{N}\times\{0\}$ and on $\{0\}\times\mathbb{N}$ have been multiplied by 2.

This together with (35), imply the lemma. □

We choose as a test function F the degree 2 symmetric function F such that

$$\begin{aligned}
&\overline{F}(\alpha)=z^{-1/4}e^{-z^{3/4}(\alpha-1)},\quad \alpha\ge 1,\\
&\overline{F}(0)=\bar F(1).
\end{aligned}\tag{37}$$

This function exists since given a function G defined on $\mathbb{N}$ we can find a symmetric function F defined in $\mathbb{Z}^2$ such that $\bar F=G$. For that purpose, take $F(x,y)=G(|y-x|)[\phi(x)+\phi(y)]$ where the function ϕ is defined on $\mathbb{Z}$ and is such that $\sum_{x\in\mathbb{Z}}\phi(x)=1/2$. Then for any $\alpha\in\mathbb{N}$, $\bar F(\alpha)=\sum_{u\in\mathbb{Z}}F(u,u+\alpha)=G(\alpha)\sum_{u\in\mathbb{Z}}[\phi(u)+\phi(u+\alpha)]=G(\alpha)$.

Observe that with this choice, by Lemma 4,

$$\ll F,-\mathfrak{S}F\gg_{\text{free}}\sim z^{1/4},\quad \bar F(1)=z^{-1/4},\quad z\sum_{\alpha\in\mathbb{N}}\bar F^2(\alpha)\sim z^{-1/4}.\tag{38}$$

It remains to estimate the last contribution given by $\|\mathscr{W}^{1/2}G\|^2_{-1,z,\text{free}}$ where $G=\mathfrak{A}_+F$ is a degree 3 function.

[4]Here and in the sequel $\mathbb{N}^*:=\mathbb{N}\backslash\{0\}$

Lemma 6. *Let $G = \mathfrak{A}_+ F$ where F is defined by (37). There exists a constant $C > 0$ such that*

$$\|\mathscr{W}^{1/2} G\|^2_{-1,z,\mathrm{free}} \geq C z^{-1/4}.$$

Proof. For any $u, v, w \in \mathbb{Z}$, we have

$$\begin{aligned}
&G(u,u+1,u+2) = F(u,u+1) - F(u+1,u+2),\\
&G(u,u+1,v) = F(u,v) - F(u+1,v), \quad v > u+1,\\
&G(v,u,u+1) = F(v,u) - F(v,u+1), \quad v < u,\\
&G(u,u,u+1) = 2\Big(F(u,u) - F(u,u+1)\Big),\\
&G(u,u,u-1) = 2\Big(F(u-1,u) - F(u,u)\Big),\\
&G(u,v,w) = 0 \quad \text{otherwise.}
\end{aligned}$$

Let us now compute $\bar{G}(u,v)$, $u, v \in \mathbb{N}$. We get

$$\begin{aligned}
&\bar{G}(0,1) = -\bar{G}(1,0) = 2\bar{F}(0) - 2\bar{F}(1),\\
&\bar{G}(1,v) = \bar{F}(v+1) - \bar{F}(v), \quad v \geq 2,\\
&\bar{G}(u,1) = \bar{F}(u) - \bar{F}(u+1), \quad u \geq 2,\\
&\bar{G}(u,v) = 0 \quad \text{otherwise.}
\end{aligned} \tag{39}$$

By (37) we have that $\bar{G}(0,1) = \bar{G}(1,0) = 0$. Also notice that $\bar{G}(u,u) = 0$ and by (34) we have that $\mathscr{W}^{1/2}(u,v) = (1+\lambda)$ for $u \neq v$.

It follows, by Lemma 5, that $\|\mathscr{W}^{1/2} G\|^2_{-1,z,\mathrm{free}}$ is upper bounded by the variational formula:

$$\begin{aligned}
\|\mathscr{W}^{1/2} G\|^2_{-1,z,\mathrm{free}} &= \sup_R \left\{ 2 \sum_{(u,v)\in\mathbb{N}^2} R(u,v) \mathscr{W}^{1/2}(u,v) \bar{G}(u,v) - C_0 \mathbb{D}_3(R) \right\}\\
&= \sup_R \left\{ 2(1+\lambda) \sum_{(u,v)\in\mathbb{N}^2} R(u,v) \bar{G}(u,v) - C_0 \mathbb{D}_3(R) \right\}
\end{aligned}$$

where the supremum is taken over local functions on $\mathbb{N}^2$. By (39), we have that

$$\begin{aligned}
\sum_{(u,v)\in\mathbb{N}^2} R(u,v)\bar{G}(u,v) &= \sum_{v\geq 2} R(1,v)\Big(\bar{F}(v+1) - \bar{F}(v)\Big) - \sum_{u\geq 2} R(u,1)\Big(\bar{F}(u+1) - \bar{F}(u)\Big)\\
&= \sum_{v\geq 3} \bar{F}(v)\Big(R(1,v-1) - R(1,v)\Big) - \sum_{u\geq 3} \bar{F}(u)\Big(R(u-1,1) - R(u,1)\Big)\\
&\quad + \bar{F}(2)\Big(R(2,1) - R(1,2)\Big)\\
&= \sum_{v\geq 2} \bar{F}(v)\Big(R(1,v-1) - R(1,v)\Big) - \sum_{u\geq 2} \bar{F}(u)\Big(R(u-1,1) - R(u,1)\Big).
\end{aligned} \tag{40}$$

We use now the following parametrization of R. For $k \geq 1$, $v \in \mathbb{Z}$, let us define

$$R(k,v) = \phi(k-1, v-k), \quad v \geq k, \quad R(u,k) = \phi(k-1, -u+k), u \geq k,$$

where $\{\phi(k,\cdot)\,;\,k\geq 0\}$ are functions from $\mathbb{Z}\to\mathbb{R}$. We have the following lower bound for $\mathbb{D}_3(R)$:

$$\mathbb{D}_3(R)\geq \sum_{u,v\geq 1}\Big(R(u+1,v)-R(u,v)\Big)^2+\Big(R(u,v+1)-R(u,v)\Big)^2$$

which is nothing but the Dirichlet form of a random walk where only jumps connecting sites of $\mathbb{N}^*\times\mathbb{N}^*$ have been conserved. With the choice of the parametrization for R and this lower bound, it is not difficult to show there exists a constant $C>0$ such that

$$\mathbb{D}_3(R)\geq C\sum_{k\geq 0}\sum_{v\in\mathbb{Z}}\Big(\phi(k,v+1)-\phi(k,v)\Big)^2+\Big(\phi(k+1,v)-\phi(k,v)\Big)^2.$$

The right hand side of the previous inequality is the Dirichlet form of a symmetric simple random walk on $\mathbb{N}\times\mathbb{Z}$.

By (40), we get

$$\sum_{(u,v)\in\mathbb{N}^2}R(u,v)\bar{G}(u,v)=\sum_{u\in\mathbb{Z}}\phi(0,u)\Big(\tilde{F}(u-1)-\tilde{F}(u)\Big)$$

where $\tilde{F}:\mathbb{Z}\to\mathbb{R}$ is defined by $\tilde{F}(u)=-\bar{F}(u+2)\mathbf{1}_{\{u\geq 0\}}-\bar{F}(1-u)\mathbf{1}_{\{u\leq -1\}}$. We extend the function ϕ defined on $\mathbb{N}\times\mathbb{Z}$ to $\mathbb{Z}^2$ by defining $\phi(-k,u)=\phi(k,u)$, $k\geq 1, u\in\mathbb{Z}$. Observe then that

$$\begin{aligned}\mathbb{D}_3(R)&\geq C\sum_{k\geq 0}\sum_{v\in\mathbb{Z}}\Big(\phi(k,v+1)-\phi(k,v)\Big)^2+\Big(\phi(k+1,v)-\phi(k,v)\Big)^2\\&=\frac{C}{2}\sum_{k\in\mathbb{Z}}\sum_{v\in\mathbb{Z}}\Big(\phi(k,v+1)-\phi(k,v)\Big)^2+\Big(\phi(k+1,v)-\phi(k,v)\Big)^2.\end{aligned}$$

Consequently we have, for suitable positive constants C_1, C_2:

$$\|\mathscr{W}^{1/2}G\|^2_{-1,z,\text{free}}\leq C_1\sup_{\phi}\Big\{2\sum_{u\in\mathbb{Z}}\phi(0,u)\Big(\tilde{F}(u-1)-\tilde{F}(u)\Big)-C_2\sum_{\substack{(\mathbf{u},\mathbf{v})\in\mathbb{Z}^2\\|\mathbf{u}-\mathbf{v}|=1}}\Big(\phi(\mathbf{u})-\phi(\mathbf{v})\Big)^2\Big\}. \tag{41}$$

A standard Fourier computation shows this supremum is of order $z^{-1/4}$. Indeed, let $\widehat{u}$ be the Fourier transform of the function $u:\mathbb{Z}^n\to\mathbb{R}$, defined by

$$\widehat{u}(\mathbf{k})=\sum_{\mathbf{x}\in\mathbb{Z}^n}e^{2i\pi\mathbf{x}\cdot\mathbf{k}}u(\mathbf{x}),\quad \mathbf{k}=(k_1,\dots,k_n),$$

and denote by $\widehat{u}^*(\mathbf{k})$ the complex conjugate of $\widehat{u}(\mathbf{k})$. Using the expression of the sum of a convergent geometric series, we obtain the following expression for the Fourier transform $\Psi(k_1,k_2)$ of the function $(x,y)\in\mathbb{Z}^2\to\delta_0(y)\tilde{F}(x)$:

$$\Psi(k_1,k_2)=-z^{-1/4}e^{-z^{3/4}}\left\{\frac{1}{1-e^{2i\pi k_1}e^{-z^{3/4}}}-\frac{e^{-2i\pi k_1}}{1-e^{-2i\pi k_1}e^{-z^{3/4}}}\right\}$$

which satisfies

$$|\Psi(k_1,k_2)|\leq\frac{C_3\sqrt{z}}{z^{3/2}+C_4\sin^2(\pi k_1)}$$

for some positive constants C_3, C_4. The supremum appearing in (41) is then given by

$$C_2^{-1}\int_{[0,1]^2}\frac{|\Psi(k_1,k_2)|^2}{z+4\sin^2(\pi k_1)+4\sin^2(\pi k_2)}dk_1dk_2.$$

Then the result follows by a standard study of this integral. □

To obtain (31), by (38) and Lemma 6, it suffices to take a test function in the form aF with F given by (37) and a sufficiently small.

8. Stochastic perturbations of Hamiltonian systems

In this section we discuss some other possible stochastic perturbations and make some connections with the recent models considered in [8]. Let us start with the Hamiltonian system (2) with potential V and generator A given by

$$A = \sum_{x\in\mathbb{Z}} \Big(V'(\eta_{x+1}) - V'(\eta_{x-1}) \Big) \partial_{\eta_x}.$$

The energy $\sum_{x\in\mathbb{Z}} V(\eta_x)$ and the volume $\sum_{x\in\mathbb{Z}} \eta_x$ are conserved by these dynamics. Remark that in fact $\sum_{x\in\mathbb{Z}} \eta_{2x}$ and $\sum_{x\in\mathbb{Z}} \eta_{2x+1}$ are also conserved and that we cannot exclude the case that still many others exist. This is the case for example for the exponential interaction for which an infinite number of conserved quantities can be explicitly identified. Anyway, we are only interested in these two first quantities. The product probability measures $\mu_{\beta,\lambda}$ defined by

$$d\mu_{\beta,\lambda}(\eta) = \prod_{x\in\mathbb{Z}} Z(\beta,\lambda)^{-1} \exp\{-\beta V(\eta_x) - \lambda\eta_x\}\, d\eta_x,$$

where

$$Z(\beta,\lambda) = \int_{-\infty}^{\infty} \exp\left(-\beta V(r) - \lambda r\right) dr.$$

are invariant for the infinite dynamics.

In [7] we proposed to perturb this deterministic dynamics by the Poissonian noise considered in this paper and conserving both the energy and the volume. One could also consider the " Brownian" noise whose generator S is given by $S = \sum_{x\in\mathbb{Z}} Y_x^2$ where

$$Y_x = (V'(\eta_{x+1}) - V'(\eta_{x-1}))\partial_{\eta_x} + (V'(\eta_{x-1}) - V'(\eta_x))\partial_{\eta_{x+1}} + (V'(\eta_x) - V'(\eta_{x+1}))\partial_{\eta_{x-1}},$$

is the vector field tangent to the curve

$$\{(\eta_{x-1},\eta_x,\eta_{x+1}) \in \mathbb{R}^3;\ \sum_{y=x-1}^{x+1} \eta_y = 0,\ \sum_{y=x-1}^{x+1} V(\eta_y) = 1\}.$$

It is easy to see that the process with generator $L = A + S$ conserves the energy and the volume and has $\mu_{\beta,\lambda}$ as invariant measures. A priori, it should be possible to extend our result to this system for V of exponential type but the noise S seems to have a quite complicated expression in the orthogonal basis we used in this paper. The advantage of the Poissonian noise is its very simple form. Notice also that the Poissonian noise is a less strong perturbation of the Hamiltonian dynamics than the Brownian noise. In this sense we are closer to the Hamiltonain dynamics with the Poissonian noise.

We could also decide to conserve energy and not the volume by adding a suitable perturbation. The invariant states are then given by $\mu_{\beta,0}$, $\beta > 0$. If V is even, a simple Poissonian noise consists to change the sign of η_x independently on each site x at random exponential times. In this case one can prove, as in [6], that the energy diffuses in the sense that the Green-Kubo formula converges to a well defined finite value. For a generic V a Brownian noise with generator S given by $S = \sum_{x\in\mathbb{Z}} K_x^2$ with $K_x = V'(\eta_{x+1})\partial_{\eta_x} - V'(\eta_x)\partial_{\eta_{x+1}}$ makes the job.

Consider now the case where we want to add a stochastic perturbation conserving only the volume. It does not seem to be easy to define a simple Poissonian noise with such a property. A Brownian noise is obtained by the following scheme. Fix $\beta > 0$, consider the

vector field $X_x = \partial_{\eta_{x+1}} - \partial_{\eta_x}$ which is tangent to the hyperplane $\{(\eta_x, \eta_{x+1}) \in \mathbb{R}^2 \,;\, \eta_x + \eta_{x+1} = 1\}$ and define the Langevin operator S_β by

$$\begin{aligned} S_\beta &= \frac{1}{2}\sum_{x\in\mathbb{Z}} e^{-\mathscr{H}_{\beta,\lambda}} X_x (e^{\mathscr{H}_{\beta,\lambda}} X_x) \\ &= \frac{1}{2}\sum_{x\in\mathbb{Z}} X_x^2 + \frac{\beta}{2}\sum_{x\in\mathbb{Z}} \Big(V'(\eta_{x+1}) - V'(\eta_x)\Big) X_x \end{aligned}$$

where $\mathscr{H}_{\beta,\lambda} = \beta \sum_{x\in\mathbb{Z}} V(\eta_x) + \lambda \sum_{x\in\mathbb{Z}} \eta_x$. Observe that S_β depends on β but is independent of λ. The operator S_β is a nonpositive self-adjoint operator in $\mathbb{L}^2(\mu_{\beta,\lambda})$ for any λ and $S_\beta(\sum_{x\in\mathbb{Z}} \eta_x) = 0$. Then, the perturbed volume-conserving model has a generator L_β^V given by

$$L_\beta^V = A + \gamma S_\beta \tag{42}$$

where $\gamma > 0$ is a parameter fixing the strength of the noise. By construction, the Markov process generated by L_β^V has $\mu_{\beta,\lambda}$ as a set of invariant probability measures. In fact, using the same methods as in [7, 17] one can prove that the only space-time invariant probability measures with finite local entropy density are mixtures of the $(\mu_{\beta,\lambda})_\lambda$. We can also rewrite L_β^V as

$$\begin{aligned} L_\beta^V = \sum_{x\in\mathbb{Z}} \left\{ \left(1 - \frac{\gamma\beta}{2}\right) V'(\eta_{x+1}) + \gamma\beta V'(\eta_x) - \left(1 + \frac{\gamma\beta}{2}\right) V'(\eta_{x-1}) \right\} \partial_{\eta_x} \\ + \gamma \sum_{x\in\mathbb{Z}} (\partial^2_{\eta_x} - \partial^2_{\eta_x,\eta_{x+1}}). \end{aligned}$$

The microscopic flux $j_{x,x+1}$ associated to the volume conservation law is defined by

$$L_\beta^V(\eta_x) = -\nabla j_{x-1,x}, \quad j_{x-1,x} = -\left(1 + \frac{\gamma\beta}{2}\right) V'(\eta_{x-1}) - \left(1 - \frac{\gamma\beta}{2}\right) V'(\eta_x).$$

The semi-discrete directed polymer model considered in [8] is, up to an irrelevant scaling factor 2, recovered by taking $V(\eta) = e^{-\eta}$, $\beta = 1$ and $\gamma = 2$ (see (3.7) in [27]). In [8] the authors show that for a particular non stationary initial condition ("wedge"), by developing a very nice theory of Macdonald processes, the system belongs to the Kardar-Parisi-Zhang universality class ([27]). Unfortunately one can not use their results or their methods to derive a more precise picture for the model with exponential interactions considered in this paper. For other potentials V the theory developed by Borodin and Corwin in [8] can not be adapted but it would be very interesting to see if one can relate the models generated by L_β^V to the semi-discrete directed polymer and deduce some qualitative information from the latter. The use of the variational formulas considered in this paper could be the way.

Appendix A. Existence of the infinite dynamics

In this section we prove existence of the infinite volume dynamics $(\xi(t))_{t\geq 0}$. We focus here on the process ξ but the same proof can be carried for the process η (or just define η in terms of ξ by $\eta_x(t) = -b^{-1}\log\xi_x(t)$, $x \in \mathbb{Z}$. To simplify notations we will assume $b = 1$.

Since the interaction coming from the deterministic part is quadratic, proving the existence of the infinite dynamics is a non trivial task. Nevertheless nice sophisticated techniques have been introduced by Dobrushin and Fritz in [12]. Here, we follow closely the approach of [13] (see also [14, 15]) adapted to our case. By itself, the strategy of the proof of existence of solutions is standard: we consider finite subsystems and prove compactness

of this family by means of an a priori bound for a quantity $\bar{E}$ which plays the role of an energy density. The obtention of this a priori bound is however non trivial and is the main step to get the existence of the dynamics. The aim of this appendix is to show how to get such an a priori bound. The a priori bound we derive here for the infinite dynamics is also valid for finite subsystems corresponding to a finite set $\Lambda \subset \mathbb{Z}$ with a bound which is independent of the size of Λ. This proves then that the finite subsystems form a compact family from which one can extract a subsequence converging to the infinite dynamics.

We have first to specify the space of allowed configurations $\Omega \subset (0,+\infty)^{\mathbb{Z}}$. For $x \in \mathbb{Z}$, let $g(x) = 1 + \log(1+|x|)$ and denote by $E(\xi,\mu,\sigma)$, $\xi \in (0,+\infty)^{\mathbb{Z}}$, $\mu \in \mathbb{Z}$, $\sigma > 0$, the quantities

$$E(\xi,\mu,\sigma) = \sum_{|x-\mu|\le\sigma} (1 + 2\xi_x - \log \xi_x),$$
$$\bar{E}(\xi) = \sup_{\mu\in\mathbb{Z}} \sup_{\sigma\ge g(\mu)} \sigma^{-1} E(\xi,\mu,\sigma).$$

The quantity $\bar{E}$ is called the logarithmic fluctuation of energy and the set Ω is defined as

$$\Omega := \{\xi \in (0,+\infty)^{\mathbb{Z}} : \bar{E}(\xi) < +\infty\}.$$

The configuration space Ω is equipped with the product topology and with the associated Borel structure. It is easy to see that $\nu_{\beta,\lambda}(\Omega) = 1$ for any $\beta > 0$ and $\lambda > -1$.

Let $\mathbf{N}(t) = \{N_{x,x+1}(t)\,; x \in \mathbb{Z}\}$ be a collection of independent Poison processes of intensity $\gamma > 0$. The equations of motion corresponding to the generator $\mathscr{L}$ read as

$$d\xi_x = \xi_x(\xi_{x+1} - \xi_{x-1})dt + \nabla\left((\xi_x - \xi_{x-1})dN_{x-1,x}(t)\right), \quad x \in \mathbb{Z}. \tag{43}$$

Let $D(\mathbb{R}_+,\mathbb{R})$ denote the space of càdlàg functions of $\mathbb{R}_+$ into $\mathbb{R}$ with the Skorohod topology and let $\mathbb{D} = [D(\mathbb{R}_+,\mathbb{R})]^{\mathbb{Z}}$ equipped with the product topology and the associated Borel field $\mathscr{B}$. The smallest σ-algebra on which all projection restricted to the time interval $[0,t]$ are measurable will be denoted by $\mathscr{B}_t$. Finally, suppose that we are given a probability measure $\mathbf{P}$ on $\mathscr{B}$ such that our Poisson processes $N_{x,x+1}$ are realized as components of the random element of $\mathbb{D}$.

Definition 1. *A $\mathscr{B}_t$-adapted mapping $\xi(t) := \xi(t,\mathbf{N})$ of $\mathbb{D}$ into itself is called a tempered solution of (43) with initial configuration $\xi^0 \in \Omega$ if $\xi(0) = \xi^0$, almost each trajectory $\xi(\cdot,\mathbf{N})$ satisfies the integral form of (43), and the logarithmic energy fluctuation $\bar{E}(\xi(t))$ is bounded on finite intervals of time with probability one.*

Theorem 5. *For any $\xi^0 \in \Omega$, there exists a unique tempered solution of (43) with initial configuration $\xi^0 \in \Omega$.*

As explained above, the main step to prove this theorem is to obtain an a priori bound that we prove in Proposition 3. For a complete proof, we refer to [13] (or [14, 15]).

Now we notice that the Gibbs state $\nu_{\beta,\lambda}$, $(\beta,\lambda) \in (0,+\infty)\times(-1,+\infty)$ is formally invariant for the infinite dynamics generated by $(\xi(t))_{t\ge0}$. This can be seen by observing that $\int(\mathscr{L}f)(\xi)d\nu_{\beta,\lambda}(d\xi) = 0$ for nice functions $f : \Omega \to \mathbb{R}$. Nevertheless, some care has to be taken to prove this. Indeed, we do not know that $\mathscr{L}$ is really the generator of the semigroup generated by $(\xi(t))_{t\ge0}$ on the space of bounded measurable functions on Ω in the usual Hille-Yosida theory. This can be a very difficult question that we prefer to avoid (see [15]). Instead we use the fact that the infinite dynamics can be approximated by finite subsystems.

Proposition 2. *For any $\beta > 0, \lambda > -1$, the probability measure $\nu_{\beta,\lambda}$ is invariant for the process $(\xi(t))_{t\geq 0}$.*

Proof. Let $n \geq 2$ and consider the local dynamics generated by the generator $\mathscr{L}_n = \mathscr{A}_n + \gamma\mathscr{S}_n$ where

$$(\mathscr{A}_n f)(\xi) = \sum_{x=-n}^{n} \xi_x(\xi_{x+1} - \xi_{x-1})(\partial_{\xi_x} f)(\xi) - \xi_{n+1}\left(\xi_n + \frac{\lambda+1}{\beta}\right)(\partial_{\xi_n} f)(\xi) + \xi_{-n-1}\left(\xi_{-n} + \frac{\lambda+1}{\beta}\right)(\partial_{\xi_{-n-1}} f)(\xi),$$

$$(\mathscr{S}_n f)(\xi) = \sum_{x=-n}^{n} \Big(f(\xi^{x,x+1}) - f(\xi)\Big)$$

where $f : \Omega \to \mathbb{R}$ is a compactly supported continuously differentiable function. The dynamics is essentially finite-dimensional since the particles outside the box $\{-n-1, \ldots, n+1\}$ are frozen. Thus, the classical Hille-Yosida theory can be applied. The boundary conditions have been chosen to have

$$\int (\mathscr{L}_n f)(\xi) d\nu_{\beta,\lambda}(\xi) = 0$$

for any compactly supported continuously differentiable function f which shows that $\nu_{\beta,\lambda}$ is invariant for the local dynamics. Since, as a consequence of the a priori bound, the infinite dynamics is obtained as a limit of finite local dynamics, this implies that $\nu_{\beta,\lambda}$ is invariant for the infinite dynamics. □

Then this defines a strongly continuous semigroup of contractions $(P_t)_{t\geq 0}$ on the Hilbert space $\mathbb{L}^2(\Omega, \mathscr{B}, \nu_{\beta,\lambda})$. Moreover, Itô's formula shows that its generator is a closable extension of $\mathscr{L}$ given by $\mathscr{A} + \gamma\mathscr{S}$ since for any local compactly supported continuously differentiable function f, we have

$$(P_t f)(\xi) = f(\xi) + \int_0^t (P_s \mathscr{L} f)(\xi) ds, \quad \xi \in \Omega, \quad t \geq 0.$$

A.1. **Logarithmic energy fluctuation.** We have first to consider a clever smooth modification of $\bar{E}$. Let $0 < \lambda < 1$ and consider a twice continuously differentiable nonincreasing function $\varphi : \mathbb{R} \to (0,1)$ such that $\varphi(u) = e^{\lambda(1-u)}$ if $u \geq 2$, $\varphi(u) = (1 + \lambda + \lambda^2/2)e^{-\lambda}$ if $u \leq 1$, and φ is concave for $u \leq 3/2$, convex if $u \geq 3/2$. Finally, $0 \leq -\varphi'(u) \leq \lambda\varphi(u) \leq e^{\lambda(1-u)}$, $\varphi(u) \geq e^{-\lambda(1+u)}$ and $|\varphi''(u)| \leq \varphi(u)$ for all $u > 0$.

For $x \in \mathbb{Z}$ and $\sigma \geq 1$ we define the function f as

$$f(x,\sigma) = \int_{\mathbb{R}} \varphi(|x-y|/\sigma) e^{-2\lambda|y|} dy.$$

In [15] are proved the following properties on f:

$$c_1 \exp(-\lambda|x|/\sigma) \leq f(x,\sigma) \leq c_2 \exp(-\lambda|x|/\sigma), \tag{44}$$

$$f(x,\sigma) \leq f(y,\sigma) e^{2\lambda|x-y|}, \quad \partial_\sigma f(x,\sigma) \leq e^{2\lambda|x-y|} \partial_\sigma f(y,\sigma). \tag{45}$$

$$|\partial_x f(x,\sigma)| \leq \min\{\partial_\sigma f(x,\sigma), \sigma^{-1} f(x,\sigma)\}, \tag{46}$$

$$g(x)|\partial_x f(x-\mu,\sigma)| \leq 4g(|\mu|+\sigma)(\partial_\sigma f)(x-\mu,\sigma). \tag{47}$$

Here the constants depend only on λ.

For $\xi \in (0,+\infty)^{\mathbb{Z}}$, $\mu \in \mathbb{Z}$ and $\sigma > 0$, consider the function

$$W(\xi,\mu,\sigma) = \sum_{x\in\mathbb{Z}} f(x-\mu,\sigma)(1 + 2\xi_x - \log\xi_x) \tag{48}$$

and let

$$\bar{W}(\xi) = \sup_{\mu\in\mathbb{Z}} \sup_{\sigma\geq g(\mu)} \sigma^{-1} W(\xi,\mu,\sigma). \tag{49}$$

Observe that by (44),

$$W(\xi,\mu,\sigma) \geq c_1 e^{-\lambda} E(\xi,\mu,\sigma), \tag{50}$$

for all $\xi \in (0,+\infty)^{\mathbb{Z}}$, $\mu \in \mathbb{Z}$ and $\sigma > 0$.

For $\xi \in (0,+\infty)^{\mathbb{Z}}$, we also consider the function

$$\widehat{W}(\xi) = \sup_{\mu\in\mathbb{Z}} \frac{W(\xi,\mu,g(\mu))}{g(\mu)}. \tag{51}$$

The following lemma shows that these two modifications of the logarithmic energy fluctuation are equivalent to $\bar{E}$.

Lemma 7. *There exists a constant C such that for all $\xi \in (0,+\infty)^{\mathbb{Z}}$:*

$$C^{-1}\widehat{W}(\xi) \leq \bar{W}(\xi) \leq C\widehat{W}(\xi), \quad C^{-1}\bar{E}(\xi) \leq \bar{W}(\xi) \leq C\bar{E}(\xi).$$

Proof. The inequality $\widehat{W}(\xi) \leq \bar{W}(\xi)$ for all $\xi \in (0,+\infty)^{\mathbb{Z}}$, is trivial. Let us prove the second one by taking $\sigma \geq g(\mu)$, $\mu \in \mathbb{Z}$ and denoting $1+2\xi_x - \log\xi_x$ by H_x. By (44), we have

$$W(\xi,\mu,\sigma) \leq c_2 \sum_{x\in\mathbb{Z}} \exp(-\lambda|x-\mu|/\sigma) H_x = c_2 \sum_{n=0}^{\infty} e^{-\lambda n/\sigma} \sum_{|x-\mu|=n} H_x$$

$$= c_2(1-e^{-\lambda/\sigma}) \sum_{n=0}^{\infty} e^{-\lambda n/\sigma} \sum_{|x-\mu|\leq n} H_x,$$

where the last equality follows from $\sum_{|x-\mu|=n} H_x = \sum_{|x-\mu|\leq n} H_x - \sum_{|x-\mu|\leq n-1} H_x$ and a discrete integration by parts. Let $r \geq 1$ the integer such that $r-1 < g(\mu) \leq r$ and decompose the set $\{x \in \mathbb{Z}; |x-\mu| \leq n\}$ as $\cup_{j=1}^{K+1}\Lambda_j$ where the Λ_j are non intersecting intervals of length r for $j = 1,\dots,K$ and Λ_{K+1} is of length at most $r-1$. Observe that $K+1$ is of order $n/g(\mu)$. By using (50), we have easily that

$$\sum_{x\in\Lambda_j} H_x \leq C\, g(\mu)\, \widehat{W}(\xi)$$

where C depends only on λ. Thus we get

$$W(\xi,\mu,\sigma) \leq C(1-e^{-\lambda/\sigma}) \sum_{n=0}^{\infty} e^{-\lambda n/\sigma} n\widehat{W}(\xi) \leq C'\sigma\widehat{W}(\xi).$$

which concludes the proof of the second inequality.

The proof of $C^{-1}\bar{E}(\xi) \leq \bar{W}(\xi) \leq C\bar{E}(\xi)$ for all $\xi \in (0,+\infty)^{\mathbb{Z}}$, is the same. The first inequality follows from (50) and the constant can be taken equal to $c_1 e^{-\lambda}$. The second inequality follows from a similar argument to the one used above. □

A.2. **The *a priori* bound.**

Proposition 3 (A priori bound)**.** *For each $w \geq 1$ there exists a continuous function $q_w(t)$, $t \geq 0$, such that*

$$\mathbf{P}\left\{ \sup_{0\leq s\leq t} \bar{W}(\xi(s)) > \exp(q_w(t)g(u)) \right\} \leq e^{-u}$$

for each $u \geq 1$, $t \geq 0$, whenever $\bar{W}(\xi^0) \leq w$ and $(\xi(t))_{t\geq 0}$ is a tempered solution of (43) with initial condition ξ^0.

Proof. We consider a tempered solution $(\xi(t))_{t\geq 0}$ of (43) with initial configuration $\xi^0 \in \Omega$.

For each $k \geq 1$, $\mu \in \mathbb{Z}$ and $t \geq 0$ we define the stochastic process ρ_k by

$$\rho_k(t) = kg(\mu) - C_0 \int_0^t g(|\mu| + |\rho_k(s)|)Z'(s)ds \tag{52}$$

where $C_0 := C_0(\gamma, \lambda)$ is a positive constant that will be chosen later and

$$Z(t) = \int_0^t \bar{W}(\xi(s))ds.$$

Since the function $f(\cdot)$ is positive, $\bar{W}(\cdot)$ is also positive and this turns $Z(\cdot)$ positive. The trajectories of ρ_k are differentiable, decreasing and satisfy $\rho_{k+1}(t) - \rho_k(t) \leq g(\mu)$ a.s. for each $t \geq 0$. We consider also the sequence of stopping times $\tau_k = \inf\{t \geq 0; \rho_k(t) \leq g(\mu)\}$ which satisfy $\tau_k < \tau_{k+1} < +\infty$ and $\lim_{k\to+\infty} \tau_k = +\infty$ a.s. We evaluate now the stochastic differential of $t \to W(\xi(t), \mu, \rho_k(t))$ for $t \leq \tau_k$ (so that $\rho_k(t) \geq 1$). This is given by

$$\begin{aligned} d\left[W(\xi(t), \mu, \rho_k(t))\right] &= I_0^{(k)}(t)dt \; - \; C_0 \left(\partial_\sigma W\right)(\xi(t), \mu, \rho_k(t))\, g(|\mu| + \rho_k(t))\bar{W}(\xi(t))dt \\ &\quad + \; dI_1^{(k)}(t) \end{aligned}$$

where

$$\begin{aligned} I_0^{(k)}(t) =& 2\sum_{x\in\mathbb{Z}} \Big(f(x-\mu, \rho_k(t)) - f(x+1-\mu, \rho_k(t))\Big)\xi_x(t)\xi_{x+1}(t) \\ &+ \sum_{x\in\mathbb{Z}} \Big(f(x+1-\mu, \rho_k(t)) - f(x-1-\mu, \rho_k(t))\Big)\xi_x(t) \end{aligned} \tag{53}$$

and

$$dI_1^{(k)} = \sum_{x\in\mathbb{Z}} f(x-\mu, \rho_k)\Big\{2\nabla\Big((\xi_x - \xi_{x-1})dN_{x-1,x}\Big) - \nabla\Big((\log\xi_x - \log\xi_{x-1})dN_{x-1,x}\Big)\Big\}.$$

We first estimate the term $I_0^{(k)}(t)$ and we show that if C_0 is taken sufficiently large then, for $t \leq \tau_k$ we have that

$$I_0^{(k)}(t) \; - \; C_0 \left(\partial_\sigma W\right)(\xi(t), \mu, \rho_k(t))\, g(|\mu| + \rho_k(t))\bar{W}(\xi(t)) \leq 0. \tag{54}$$

The second term on the right hand side of (53) can be estimated, by using (45) and (46), to get to

$$\begin{aligned} \Big|f(x+1-\mu, \rho_k(t)) - f(x-1-\mu, \rho_k(t))\Big| &= \Big|\int_{-1}^1 d\alpha\, (\partial_x f)(x-\mu+\alpha, \rho_k(t))\Big| \\ &\leq \int_{-1}^1 d\alpha\, |(\partial_x f)(x-\mu+\alpha, \rho_k(t))| \\ &\leq \int_{-1}^1 d\alpha\, (\partial_\sigma f)(x-\mu+\alpha, \rho_k(t)) \\ &\leq 2 \sup_{[x-\mu-1, x-\mu+1]} \partial_\sigma f(\cdot, \rho_k(t)) \\ &\leq 2e^{2\lambda}\, \partial_\sigma f(x-\mu, \rho_k(t)) \end{aligned} \tag{55}$$

which gives us that

$$\begin{aligned} \sum_{x\in\mathbb{Z}} \Big(f(x+1-\mu, \rho_k(t)) - f(x-1-\mu, \rho_k(t))\Big)\xi_x(t) &\leq C \sum_{x\in\mathbb{Z}} \partial_\sigma f(x-\mu, \rho_k(t))\xi_x(t) \\ &\leq C(\partial_\sigma W)(\xi(t), \mu, \rho_k(t)). \end{aligned}$$

Now, notice that for any $x \in \mathbb{Z}$ and for all $\xi \in (0,\infty)^{\mathbb{Z}}$ we have that

$$\bar{W}(\xi) \geq \hat{W}(\xi) \geq \frac{W(\xi,x,g(x))}{g(x)}.$$

On the other hand, by (50) and since for all $x > 0$ it holds that $\log(x) \leq 1 + x$, then we have that $W(\xi,x,g(x)) \geq c_1 e^{\lambda} E(\xi,x,g(x)) \geq c_1 e^{-\lambda} \xi_{x+1}$. Then, we conclude that there exists a constant C such that for all $x \in \mathbb{Z}$ and $\xi \in (0,\infty)^{\mathbb{Z}}$,

$$\xi_{x+1} \leq C g(x) \bar{W}(\xi). \tag{56}$$

To estimate the first term on the right hand side of (53) we use the previous estimate, (47) and a similar argument as done in (55). It follows that

$$\begin{aligned}
&\Big| \sum_{x\in\mathbb{Z}} \left(f(x-\mu,\rho_k(t)) - f(x+1-\mu,\rho_k(t))\right) \xi_x(t)\xi_{x+1}(t) \Big| \\
&\leq C\bar{W}(\xi(t))\, g(|\mu|+\rho_k(t)) \sum_{x\in\mathbb{Z}} \partial_\sigma f(x-\mu,\rho_k(t))\xi_x(t) \\
&\leq C\bar{W}(\xi(t))\, g(|\mu|+\rho_k(t))\, (\partial_\sigma W)(\xi(t),\mu,\rho_k(t)).
\end{aligned}$$

Then, (54) follows.

The term $dI_1^{(k)}$ can be written as

$$\begin{aligned}
dI_1^{(k)} &= \sum_{x\in\mathbb{Z}} f(x-\mu,\rho_k) \left\{2\nabla\left((\xi_x - \xi_{x-1})dN_{x-1,x}\right) - \nabla\left((\log\xi_x - \log\xi_{x-1})dN_{x-1,x}\right)\right\} \\
&= -\sum_{x\in\mathbb{Z}} \left(f(x+1-\mu,\rho_k) - f(x-\mu,\rho_k)\right) \left\{2\nabla\xi_x - \nabla\log\xi_x\right\} dN_{x,x+1} \\
&= -\sum_{x\in\mathbb{Z}} \left(f(x+1-\mu,\rho_k) - f(x-\mu,\rho_k)\right) \left\{2\nabla\xi_x - \nabla\log\xi_x\right\} (dN_{x,x+1} - \gamma dt) \\
&\quad - \gamma \sum_{x\in\mathbb{Z}} \left(f(x+1-\mu,\rho_k) - f(x-\mu,\rho_k)\right) \left\{2\nabla\xi_x - \nabla\log\xi_x\right\} dt.
\end{aligned}$$

Since the compensated Poisson processes $N_{x,x+1}(t) - \gamma t$ are orthogonal martingales with quadratic variation $\gamma^2 t$, then

$$dM_\mu^{(k)} = -\sum_{x\in\mathbb{Z}} \left(f(x+1-\mu,\rho_k) - f(x-\mu,\rho_k)\right) \left\{2\nabla\xi_x - \nabla\log\xi_x\right\} (dN_{x,x+1} - \gamma dt)$$

defines a martingale with a quadratic variation equal to

$$d\langle M_\mu^{(k)}\rangle_t = \gamma^2 \sum_{x\in\mathbb{Z}} \left(f(x+1-\mu,\rho_k) - f(x-\mu,\rho_k)\right)^2 \left\{2\nabla\xi_x - \nabla\log\xi_x\right\}^2 dt.$$

Using a similar argument to the one in (55), together with the fact that for all $x,y \in \mathbb{Z}$ such that $|x|,|y| \leq C$ it holds that $|x-y|^2 \leq 2C|x-y|$, the boundedness of the function f, (45), (46), (47) and (56), one has that there exists a constant C such that

$$d\langle M_\mu^{(k)}\rangle_t \leq C\, g(|\mu|+\rho_k(t))\, \bar{W}(\xi(t))\, \partial_\sigma W(\xi(t),\mu,\rho_k(t))\, dt.$$

Similarly we obtain that

$$\left| \sum_{x\in\mathbb{Z}} \left[f(x+1-\mu,\rho_k(t)) - f(x-\mu,\rho_k(t))\right] \left\{2\nabla\xi_x(t) - \nabla\log\xi_x(t)\right\} \right| \leq C\, \partial_\sigma W(\xi(t),\mu,\rho_k(t)).$$

Thus, if the constant C_0 is chosen sufficiently large, we have

$$\sup_{t\geq 0} W(\xi(t\wedge\tau_k),\mu,\rho_k(t\wedge\tau_k)) \leq W(\xi(0),\mu,kg(\mu)) + \sup_{t\geq 0} N(\mu,k,t)$$

where $N(\mu,k,t) = M_\mu^{(k)}(t\wedge\tau_k) - \frac{1}{2}\langle M_\mu^{(k)}\rangle_{t\wedge\tau_k}$. Observe that $\exp(M_\mu^{(k)}(t\wedge\tau_k) - \frac{1}{2}\langle M_\mu^{(k)}\rangle_{t\wedge\tau_k})$ is a martingale with expectation equal to 1. By the exponential supermartingale inequality, we have that

$$\mathbf{P}(\sup_{t\geq 0} N(\mu,k,t) > u) \leq e^{-u}.$$

Thus we proved that for each $k \geq 1$, $\mu \in \mathbb{Z}$ and $u > 0$,

$$\sup_{t\geq 0} W(\xi(t\wedge\tau_k),\mu,\rho_k(t\wedge\tau_k)) \leq W(\xi(0),\mu,kg(\mu)) + u \tag{57}$$

with a probability greater than $1-e^{-u}$. Applying (57) for each $\mu \in \mathbb{Z}$ and $k \geq 1$ with u replaced by $u + Akg(\mu)$ where $A \geq 1$ is sufficiently large to have $\sum_{k\geq 1}\sum_{\mu\in\mathbb{Z}} e^{-Akg(\mu)} \leq 1$, we obtain

$$\begin{aligned}\sup_{t\geq 0} W(\xi(t\wedge\tau_k),\mu,\rho_k(t\wedge\tau_k)) &\leq W(\xi(0),\mu,kg(\mu)) + Akg(\mu) + u \\ &\leq kg(\mu)\bar{W}(\xi(0)) + Akg(\mu) + u\end{aligned} \tag{58}$$

with a probability greater than $1-e^{-u}$ uniformly in k and μ.

Define now $k := k_t$, $t \geq 0$ as the smallest integer $k \geq 1$ for which $\rho_k(t) > g(\mu)$; then $\tau_k > t$ and $\rho_k(t) \leq 2g(\mu)$ as $\rho_{k-1}(t) \leq g(\mu)$; thus choosing $k = k_t$ in (58) and using that $W(\xi,\mu,\sigma)$ is increasing in σ (since $\partial_\sigma f \geq 0$ by the conditions imposed on φ), we get

$$\frac{W(\xi(t),\mu,g(\mu))}{g(\mu)} \leq \frac{W(\xi(t),\mu,\rho_k(t))}{g(\mu)} \leq k\bar{W}(\xi(0)) + Ak + \frac{u}{g(\mu)} \leq k\bar{W}(\xi(0)) + Ak + u,$$

where in the last inequality we used the fact that $g(x) \geq 1$ for all $x \in \mathbb{R}$. Taking the supremum over μ and using Lemma 7, we obtain

$$\bar{W}(\xi(t)) \leq Ck_t\bar{W}(\xi(0)) + u$$

for each $t \geq 0$ with probability at least $1-e^{-u}$. On the other hand,

$$2g(\mu) \geq k_t g(\mu) - C_0\int_0^t g(|\mu| + |\rho_k(s)|)Z'(s)ds$$

whence

$$k_t \leq 2 + C_0\int_0^t \frac{g(|\mu| + |\rho_k(s)|)}{g(\mu)}Z'(s)ds.$$

Since $\rho_k(s) \leq k_t g(\mu)$ for any $s \in [0,t]$ and g is increasing, we have that $g(|\mu| + |\rho_k(s)|) \leq g(\mu + k_t g(\mu))$. On the other hand for $x \geq 2$, $g(x) \leq x$ together with the fact that for $x, y \in \mathbb{R}$ $g(|x||y|) \leq g(|x|)g(|y|)$ and since $g(1+x) \leq 1 + g(x)$ for $x \geq 1$, we obtain that $g(\mu + k_t g(\mu)) \leq g(\mu)(1+g(k_t))$. As a consequence we obtain that

$$k_t \leq 2 + C_0 Z(t)(1 + g(k_t)). \tag{59}$$

Since for all $x \geq 1$ we have that $g(x) \leq 1 + 2\sqrt{|x|}$, then

$$k_t \leq 2 + C_0 Z(t)(2 + 2\sqrt{k_t}).$$

Finally, it follows that $\sqrt{k_t} \leq 2 + 4C_0 Z(t)$. Then, since g is increasing and by plugging the previous inequality in (59), we obtain that

$$k_t \leq 2 + C_0 Z(t)(1 + g((2 + 4C_0 Z(t))^2)).$$

Recalling that $Z'(t) = \bar{W}(\xi(t))$ we obtained that there exists a constant $M > 0$ depending only on λ such that for any $w \geq 1$ and any initial condition $\xi(0)$ satisfying $\bar{W}(\xi(0)) \leq w$,

$$\mathbf{P}\left[\sup_{t\geq 0} M^{-1}Z'(t) - w(1 + Z(t)g(Z(t))) \leq u\right] \geq 1 - e^{-u}.$$

The a priori bound follows from this last inequality (see [13], Proposition 1). □

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