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Comparison between binary64 and decimal64 floating-point numbers

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Abstract—We introduce an algorithm that allows one to quickly compare a binary64 floating-point (FP) number and a decimal64 FP number, assuming the “binary encoding” of the decimal formats specified by the IEEE 754-2008 standard for FP arithmetic is used. It is a two-step algorithm: a first pass, based on the exponents only, makes it possible to quickly eliminate most cases, then when the first pass does not suffice, a more accurate second pass is required. We provide an implementation of several variants of our algorithm, and compare them.

I. INTRODUCTION

The IEEE 754-2008 Standard for Floating-Point Arithmetic [3] specifies several binary and decimal formats. The “basic interchange formats” of the standard are presented in Table I.

Although the standard does not require that a binary and a decimal number can be compared (it says *floating-point data represented in different formats shall be comparable as long as the operands’ formats have the same radix*), such comparisons may nevertheless offer several advantages: it is not infrequent to read decimal data from some database and to have to compare it to some binary floating-point number. That comparison may be slightly inaccurate if the decimal number is preliminary converted to binary, or, respectively, if the binary number is first converted to decimal.

While it does not require comparisons between floating-point numbers of different radices, the IEEE 754-2008 standard does not forbid them for languages or systems. As the technical report on decimal floating-point arithmetic in C [4] is currently a mere draft, compilers supporting decimal floating-point arithmetic handle code sequences such as the following one at their discretion and often in an unsatisfactory way:

```
double x = ...;
_decimal64 y = ...;
if (x <= y) {
    ...
}
```

As it occurs, this sequence is translated, for example by Intel’s icc 12.1.3, into a conversion from binary to decimal followed by a decimal comparison. The compiler emits no warning that the boolean result might not be the expected one because

Name	binary32	binary64	binary128
precision (bits)	24	53	113
e_{\max}	+127	+1023	+16383
e_{\min}	−126	−1022	−16382

Name	decimal64	decimal128
precision (digits)	16	34
e_{\max}	+384	+6144
e_{\min}	−383	−6143

TABLE I
THE BASIC BINARY AND DECIMAL INTERCHANGE FORMATS SPECIFIED BY THE 754-2008 STANDARD.

of rounding in the comparison. A direct mixed-radix binary-decimal-comparison might be an answer.

Also, in the long run, allowing “exact” comparison of decimal and binary floating-point numbers will certainly be the only rigorous way of executing program instructions of the form

if $x > 0.1$ then ...

unless compilers learn how to round (binary or decimal) constants figuring in comparisons in a way that does not affect the comparison.

As an example illustrating the possible floating-point pitfalls with such comparisons, assume that x is the binary64 number

$$\frac{3602879701896397}{2^{55}}.$$

One easily sees that $x > 0.1$. And yet, x is equal to the binary64 floating-point number nearest 0.1, and less than the binary32 floating-point number nearest 0.1. Hence, although the test $x > 0.1$ should return `true` and the test $x < 0.1$ should return `false`, the first test will return `false` if 0.1 is replaced by the binary64 number nearest 0.1, and the second test will return `true` if 0.1 is replaced by the binary32 number nearest 0.1.

In the following, we aim at introducing an algorithm for comparing a binary64 floating-point number

$$x_2 = M_2 \cdot 2^{e_2-52},$$

and a decimal64 floating-point number

$$x_{10} = M_{10} \cdot 10^{e_{10}-15}.$$

Here M_2 and M_{10} are integers, with $|M_2| \leq 2^{53} - 1$ and $|M_{10}| \leq 10^{16} - 1$, and e_2 and e_{10} are the floating-point exponents of x_2 and x_{10} .

Although most of what will be presented in this paper is generalizable to other formats, we restrict ourselves to these two formats, for the sake of simplicity. We will also assume that the so-called *binary encoding* [3], [6] of IEEE 754-2008 is used for the decimal64 format, so that the integer M_{10} is easily accessible in binary. Since comparisons are straightforward if x_2 and x_{10} have different signs or are zero, we assume that $M_2 > 0$ and $M_{10} > 0$. Also, to let the IEEE 754 flags untouched (unless when required), we will use integer arithmetic rather than floating-point arithmetic to perform our tests.

When x_2 and x_{10} have significantly different orders of magnitude, examining their exponents will suffice to compare them. Hence, we first address the problem of performing a first, exponent-based, test. We then show how to perform the comparison when the first test does not suffice.

II. FIRST STEP: ELIMINATING THE “SIMPLE CASES” BY EXAMINING THE EXPONENTS

Through a possible preliminary binary shift of M_2 (when x_2 is subnormal), we may assume that $2^{52} \leq M_2$. This gives the following constraints on e_2 (taking into account that possible shift):

$$-1074 \leq e_2 \leq 1023. \quad (1)$$

The constraints on e_{10} , read from Table I, are

$$-383 \leq e_{10} \leq 384. \quad (2)$$

Also, from

$$1 \leq M_{10} \leq 10^{16} - 1,$$

we easily deduce that there exists a unique $\nu \in \{0, 1, 2, \dots, 53\}$ such that

$$2^{53} \leq 2^\nu M_{10} \leq 2^{54} - 1.$$

Hence our initial problem of comparing x_2 and x_{10} reduces to comparing $M_2 \cdot 2^{e_2-52+\nu}$ and $(2^\nu M_{10}) \cdot 10^{e_{10}-15}$.

The fact that we “normalize” the decimal significand M_{10} by a binary shift between two consecutive powers of two is of course questionable; M_{10} could also be normalized into the range $10^{15} \leq 10^t \cdot M_{10} \leq 10^{16} - 1$. However, as hardware support for decimal floating-point arithmetic is not widespread, a decimal normalization would require a loop multiplying by 10 and testing, whereas the proposed binary normalization can exploit an existing hardware leading-zero counter with straight-line code. The pipeline stalls in the loop are hence avoided.

We hence define

$$\begin{cases} m = M_2, \\ h = \nu + e_2 - e_{10} - 37, \\ n = M_{10} \cdot 2^\nu, \\ g = e_{10} - 15, \end{cases} \quad (3)$$

so that

$$\begin{cases} x_2 = m \cdot 2^h \cdot 2^{g-\nu}, \\ x_{10} = n \cdot 5^g \cdot 2^{g-\nu}. \end{cases} \quad (4)$$

Our comparison problem becomes:

$$\text{Compare } m \cdot 2^h \text{ with } n \cdot 5^g.$$

We have

$$\begin{aligned} 2^{52} &\leq m \leq 2^{53} - 1, \\ 2^{53} &\leq n \leq 2^{54} - 1. \end{aligned} \quad (5)$$

From the ranges of e_2 , e_{10} and ν , one easily deduces

$$\begin{aligned} -398 &\leq g \leq 369, \\ -1495 &\leq h \leq 1422. \end{aligned} \quad (6)$$

Note that once the first step, eliminating comparison cases purely based on the exponents, is complete, we will be able to reduce these domains (see Lemma 1). It makes no sense, for instance, to assume that e_2 is tiny and e_{10} is huge: in such a case, the comparison is straightforward.

Now define two functions φ and ψ by

$$\varphi(h) = \lfloor h \cdot \log_5 2 \rfloor, \quad \psi(g) = \lfloor g \cdot \log_2 5 \rfloor. \quad (7)$$

The function φ is appropriate to perform the first comparison step, as stated in Property 1 below. The other function will be useful in the sequel. We will propose an efficient and easy-to-implement way to compute these functions φ and ψ .

Property 1. *We have*

$$\begin{aligned} g < \varphi(h) &\Rightarrow x_2 > x_{10}, \\ g > \varphi(h) &\Rightarrow x_2 < x_{10}. \end{aligned}$$

Proof: If $g < \varphi(h)$ then $g \leq \varphi(h) - 1$, hence $g \leq h \log_5 2 - 1$. This implies that $5^g \leq (1/5) \cdot 2^h$, therefore $2^{54} \cdot 5^g \leq (4/5) \cdot 2^{52} \cdot 2^h$. As a consequence, $n \cdot 5^g < m \cdot 2^h$. If $g > \varphi(h)$ then $g \geq \varphi(h) + 1$, hence $g > h \log_5 2$, so that $5^g > 2^h$. This implies $2^{53} \cdot 5^g > (2^{53} - 1) \cdot 2^h$, and hence $n \cdot 5^g > m \cdot 2^h$. ■

Property 2. *Denoting by $\lfloor \cdot \rfloor$ the nearest integer function, let*

$$\begin{aligned} L_\varphi &= \lfloor 2^{19} \log_5 2 \rfloor = 225799, \\ L_\psi &= \lfloor 2^{12} \log_2 5 \rfloor = 9511. \end{aligned}$$

For all h in the range (6), we have

$$\varphi(h) = \lfloor h \cdot L_\varphi \cdot 2^{-19} \rfloor.$$

Similarly, we have

$$\psi(g) = \lfloor L_\psi \cdot g \cdot 2^{-12} \rfloor \quad \text{for } |g| \leq 204,$$

and in the special case $g = 16q$,

$$\psi(16q) = \lfloor L_\psi \cdot q \cdot 2^{-8} \rfloor \quad \text{for } |q| \leq 32.$$

The products $L_\varphi \cdot h$ and $L_\psi \cdot g$, for g, h in the indicated ranges, can all be computed exactly in (signed or unsigned) 32-bit integer arithmetic. Computing $\lfloor \xi \cdot 2^{-\beta} \rfloor$ of course reduces to a right-shift by β bits.

Proof: Using Maple, we exhaustively check that $\varphi(h) = \lfloor h \cdot L \cdot 2^{-19} \rfloor$ for $|h| \leq 1831$, and that $\psi(g) = \lfloor L_\psi \cdot g \cdot 2^{-12} \rfloor$ when either $|g| \leq 204$ or $g = 16q$ with $|q| \leq 32$. The corresponding quantities $L_\varphi \cdot h$ and $L_\psi \cdot g$ all fit on at most 29 bits in magnitude, plus one sign bit. ■

From Properties 1 and 2, we easily derive the following algorithm, that performs the first step, based on the exponents of x_2 and x_{10} .

Algorithm 1. First step

- compute $h = \nu + e_2 - e_{10} - 37$ and $g = e_{10} - 15$;
- compute $\varphi(h) = \lfloor L_\varphi \cdot h \cdot 2^{-19} \rfloor$ using 32-bit integer arithmetic;
- if $g < \varphi(h)$ then return “ $x_2 > x_{10}$ ”, else if $g > \varphi(h)$ then return “ $x_2 < x_{10}$ ”, else perform the second step.

Note that, when x_{10} admits multiple distinct representations in binary64 format (i.e. when its cohort is non-trivial [3]), the success of the first step may depend on the specific representation passed as input. For instance, both $A = \{M_{10} = 10^{15}, e_{10} = 0\}$ and $B = \{M_{10} = 1, e_{10} = 15\}$ are valid representations of the integer 1. Assume we are trying to compare $x_{10} = 1$ to $x_2 = 2$. Using representation A , we have $\nu = 4$, $h = -32$, and $\varphi(h) = -14 > g = -15$, hence the test from Algorithm 1 shows that $x_{10} < x_2$. In contrast, if x_{10} is given in the form B , we get $\nu = 53$, $\varphi(h) = \varphi(2) = 0 = g$, and the test is inconclusive.

We may quantify the quality of this first filter as follows. We say that Algorithm 1 *succeeds* if it answers “ $x_2 > x_{10}$ ” or “ $x_2 < x_{10}$ ” without proceeding to the second step, and *fails* otherwise. Let X_2 and X_{10} denote the sets of *representations* of positive, finite numbers, respectively in binary64 and in decimal64 format. (In the case of X_2 , each number has a single representation.) Assuming zeros, infinities, and NaNs have been handled before, the input of Algorithm 1 may be thought of as a pair $(\xi_2, \xi_{10}) \in X_2 \times X_{10}$.

Proposition 1. *Algorithm 1 succeeds for more than 99.9% of the input pairs $(\xi_2, \xi_{10}) \in X_2 \times X_{10}$.*

Proof: The first step succeeds if and only if $\varphi(h) \neq g$, that is, iff $\varphi(\nu + e_2 - e_{10} - 37) \neq e_{10} - 15$. The value of ν depends only on M_{10} . Knowing ν , the number of possible values of M_{10} is

$$n_\nu = \begin{cases} 2^{53-\nu}, & \nu > 0, \\ 10^{16} - 2^{53}, & \nu = 0. \end{cases}$$

In addition, for each fixed ν , we can easily compute the number k_ν of pairs (e_2, e_{10}) such that $-1022 \leq e_2 \leq 1023$, $-383 \leq e_{10} \leq 384$, and $\varphi(h) = g$.

Let X_2^{norm} be the subset of X_2 consisting of normal numbers. The number of pairs (e_2, ξ_{10}) such that $\varphi(h) = g$ and $\xi_2 \in X_2^{\text{norm}}$ is

$$N_1 = \sum_{\nu=0}^{53} n_\nu k_\nu = 14\,292\,575\,372\,220\,927\,484 \leq 1.55 \cdot 2^{63}.$$

If now x_2 is subnormal, then $h < -623$ and $\varphi(h) < -268$. A rough bound on the number of $(\xi_2, \xi_{10}) \in (X_2 \setminus X_2^{\text{norm}}) \times X_{10}$ such that $g = \varphi(h)$ is hence

$$N_2 = 2^{52} \cdot 10^{16} \cdot (398 - 268) \leq 1.12 \cdot 2^{112}.$$

Thus, the number of elements of $X_2 \times X_{10}$ for which the first step fails is bounded by

$$N = 2^{52} N_1 + N_2 \leq 1.691 \cdot 2^{115}.$$

This is to be compared with

$$|X_2 \times X_{10}| = (2^{52} - 1) \cdot 2047 \cdot (10^{16} - 1) \cdot 768 \geq 1.66 \cdot 2^{125}.$$

We obtain $N/|X_2 \times X_{10}| \leq 10^{-3}$. ■

As a matter of course, pairs (ξ_2, ξ_{10}) will almost never be equidistributed in practice. Hence the previous estimate should not be interpreted as a *probability* of success of Step 1. It seems more realistic to assume that a well-written numerical algorithm will mostly perform comparisons between numbers which are suspected to be close to each other. For instance, in an iterative algorithm where comparisons are used as part of a convergence test, it is to be expected that most comparisons need to proceed to the second step. Conversely, there are scenarios, e.g., checking for out-of-range data, where the first step should almost always be enough.

III. SECOND STEP: A CLOSER LOOK AT THE SIGNIFICANDS

In the following we assume that $g = \varphi(h)$ (otherwise, the first step already allowed us to compare x_2 and x_{10}).

Define a function

$$f(h) = 5^{\varphi(h)} \cdot 2^{-h} = 2^{\lfloor h \log_5 2 \rfloor \cdot \log_2 5 - h}.$$

We have

$$\begin{cases} f(h) \cdot n > m \Rightarrow x_{10} > x_2, \\ f(h) \cdot n < m \Rightarrow x_{10} < x_2, \\ f(h) \cdot n = m \Rightarrow x_{10} = x_2. \end{cases} \quad (8)$$

The second test consists in performing this comparison, with $f(h) \cdot n$ replaced by an approximation that is accurate enough not to introduce false results.

Ensuring that an approximate test is indeed equivalent to (8) requires to find a lower bound η on the minimum nonzero value of $|m/n - 5^{\varphi(h)}/2^h|$ that may appear at this stage. We want η to be as tight as possible in order to avoid unduly costly computations when approximating $f(h) \cdot n$. The search space is constrained by the following observations.

Lemma 1. *The equality $g = \varphi(h)$ where g and h are defined by (3) can hold only if $-787 \leq h \leq 716$. Additionally,*

- if $n \geq 10^{16}$, then n is necessarily even;

- if $h \geq 680$, then $\nu' = h + \varphi(h) - 971 > 0$ and n is divisible by $2^{\nu'}$.

Proof: Substituting $e_2 = h + g - \nu + 52$ and $g = \varphi(h)$ into inequality (1) yields

$$-1126 \leq e_2 - 52 = h + \varphi(h) - \nu \leq 971. \quad (9)$$

Since we have $0 \leq \nu \leq 53$ and $h \cdot \log_5 2 - 1 \leq \varphi(h) \leq h \cdot \log_5 2$, this implies

$$-1126 \leq h \cdot (1 + \log_5 2) \leq 1025.$$

The bound on h follows.

Both conditions on n come from the fact that 2^ν divides n (which we denote $2^\nu \mid n$ in the sequel of the paper) by definition of n . If $n \geq 10^{16}$, then $\nu \geq 1$ and hence n is even. In general, the last inequality from (9) implies $\nu \geq \nu'$. This last bound is nontrivial when $h \geq 680$. ■

We now deal with the problem of finding η . This problem is very close to that, considered by Cornea et al. in [1], of finding the required accuracy for performing correctly rounded conversions between the binary and decimal formats of the IEEE 754-2008 Standard. The constraints on m, n and h follow from inequalities (5) and Lemma 1.

Problem 1. Find the smallest nonzero value of

$$d_h(m, n) = \left| \frac{5^{\varphi(h)}}{2^h} - \frac{m}{n} \right|$$

under the following constraints

$$\begin{cases} 2^{52} \leq m \leq 2^{53} - 1, \\ 2^{53} \leq n \leq 2^{54} - 1, \\ -787 \leq h \leq 716, \\ n \text{ is even if } n \geq 10^{16}, \\ \text{if } h \geq 680, \text{ then } \nu' = h + \varphi(h) - 971 > 0 \\ \text{and } 2^{\nu'} \mid n. \end{cases}$$

For each integer $h \in [0, 716]$, a numerator of $d_h(m, n)$ is $|5^{\varphi(h)}n - m2^h|$ and a denominator is $2^h n$.

If $0 \leq h \leq 53$ and $d_h(m, n) \neq 0$, we have $|5^{\varphi(h)}n - m2^h| \geq 1$, hence $d_h(m, n) \geq 1/(2^h n) \geq 2^{-107}$. This easy argument actually means that for the range of h where equality cases might occur, the non-zero minimum is no less than 2^{-107} .

If $h \geq 54$, since $5^{\varphi(h)}$ and 2^h are coprime and $n \leq 2^{54} - 1$, we have $d_h(m, n) \neq 0$. Minimizing $d_h(m, n)$ is a classical question related to the theory of continued fractions [2], [5], [7]. We now give a short reminder on the results we need in our setting.

Let $\alpha \in \mathbb{Q}$. We build two finite sequences $(a_i)_{0 \leq i \leq n}$ and $(r_i)_{0 \leq i \leq n}$ defined by:

$$\begin{cases} r_0 = \alpha, \\ a_i = \lfloor r_i \rfloor, \\ r_{i+1} = 1/(r_i - a_i) \text{ if } a_i \neq r_i. \end{cases}$$

Note that this process is actually Euclid's algorithm. For all $0 \leq i \leq n$, the rational number

$$\frac{p_i}{q_i} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_i}}}}$$

is the i th convergent of the continued fraction expansion of α . We write $\alpha = [a_0; a_1, \dots, a_n]$. The a_i are the partial quotients of the expansion. Note that, if we assume $p_{-1} = 1, q_{-1} = 0, p_0 = a_0, q_0 = 1$, we have $p_{i+1} = a_{i+1}p_i + p_{i-1}, q_{i+1} = a_{i+1}q_i + q_{i-1}$, and $\alpha = p_n/q_n$.

First assume that the minimum nonzero value will be less than or equal to 2^{-109} . Then, the integers $2^{52} \leq m \leq 2^{53} - 1$ and $2^{53} \leq n \leq 2^{54} - 1$ that we want to determine necessarily satisfy $|5^{\varphi(h)}/2^h - m/n| \leq 2^{-109} < 1/(2n^2)$: a classical result (see Theorem 19 of [5] for instance) yields that m/n is a convergent of $5^{\varphi(h)}/2^h$. Hence, we only have to compute, for each $h \in [54, 716]$, the convergents of $5^{\varphi(h)}/2^h$ the denominators of which are smaller than 2^{54} . Moreover, we only consider the convergents p_i/q_i which improve the minimum nonzero value and for which there exists $k \in \mathbb{N}$ such that $m = kp_i$ and $n = kq_i$ satisfy the constraints of Problem 1.

We stress that this pruning trick is validated if and only if one eventually obtains a minimum nonzero value lesser than or equal to 2^{-109} . If not, one can use the techniques presented in the Appendix of [1] to get worst and bad cases for the approximation problem we consider. We have favored a different approach, here, because it leads to a less computationally intensive and mathematically simpler method, which is easier to check either manually or with a proof-assistant. However, if one wishes to generate many cases for which x_2 and x_{10} are very close (typically, for the purpose of testing comparison algorithms), the techniques of [1] become of huge interest. We will take advantage of them in Section VI.

The computation shows that the smallest admissible value of $d_h(m, n)$ for $h \geq 0$ is $6.3566 \dots \times 10^{-35} < 2^{-113.59}$, attained for

$$h = 612, \quad \frac{m}{n} = \frac{3521275406171921}{8870461176410409}.$$

We address the case $h \in [-787, 0]$ in a similar way. In this case, a numerator of $d_h(m, n)$ is $|2^{-h}n - 5^{-\varphi(h)}m|$ and a denominator is $5^{-\varphi(h)}n$. We notice that $5^{-\varphi(h)} \geq 2^{54}$ if and only if $h \leq -54$. If $-53 \leq h \leq 0$ and $d_h(m, n) \neq 0$, we have $|2^{-h}n - 5^{-\varphi(h)}m| \geq 1$, hence $d_h(m, n) \geq 1/(5^{-\varphi(h)}n) \geq 2^{-108}$. For $-787 \leq h \leq -54$, we use the same continued fraction tools as above, leading to the following result.

Theorem 1. The minimum nonzero value of $d_h(m, n)$ under the constraints from Problem 1 is

$$6.0485 \dots \times 10^{-35} < 2^{-113.67},$$

attained for

$$h = -275, \quad \frac{m}{n} = \frac{4988915232824583}{12364820988483254}.$$

We may thus take $\eta = 2^{-113.7}$.

IV. INEQUALITY TESTING

As already mentioned, the first step of the full comparison algorithm is straightforwardly implementable in 32-bit integer arithmetic. The second one reduces to comparing m and $f(h) \cdot n$, and we have seen in Section III that it is enough to know $f(h)$ with relative accuracy $2^{-113.67}$. We now discuss several ways to perform this comparison. The main difficulty is to efficiently evaluate $f(h) \cdot n$ with just enough accuracy.

A. Direct method

Perhaps the first thing that comes to mind is to implement the criterion from Equation (8) directly, only with $f(h)$ replaced by a sufficiently accurate approximation, read in a table that is indexed with h . Theorem 2 below implies that it is more than enough to compute the product $f(h) \cdot n$ in 128-bit arithmetic, using a 128-bit approximation of $f(h)$.

By Lemma 1, there are 1504 values of h to consider, which corresponds to a 23.5 KB table with 128-bit precision entries for $f(h)$. The resulting table is quite large but could possibly be reused, for instance, in conversions between binary and decimal formats. One advantage of this method is that the table value $f(h)$ only depends on h , which is available early in the algorithm flow. Thus, the memory latency for the table access can be hidden behind the first step.

Theorem 2. Assume that μ approximates $f(h) \cdot n$ with relative accuracy $\epsilon < \eta/4$ or better, that is,

$$|f(h) \cdot n - \mu| \leq \epsilon f(h) \cdot n < \frac{\eta}{4} f(h) \cdot n. \quad (10)$$

The following implications hold:

$$\begin{cases} \mu > m + \epsilon \cdot 2^{54} \implies x_{10} > x_2, \\ \mu < m - \epsilon \cdot 2^{54} \implies x_{10} < x_2, \\ |m - \mu| \leq \epsilon \cdot 2^{54} \implies x_{10} = x_2. \end{cases} \quad (11)$$

Proof: First notice that $f(h) \leq 1$ for all h . Since $n < 2^{54}$, condition (10) implies $|\mu - f(h) \cdot n| < 2^{54}\epsilon$, and hence $|f(h) \cdot n - m| > |\mu - m| - 2^{54}\epsilon$.

Now consider each of the possible cases that appear in (11). If $|\mu - m| > 2^{54}\epsilon$, then $f(h) \cdot n - m$ and $\mu - m$ have the same sign. The first two implications from (8) then translate into the corresponding cases from (11).

If finally $|\mu - m| \leq 2^{54}\epsilon$, then the triangle inequality yields $|f(h) \cdot n - m| \leq 2^{55}\epsilon$, which implies $|f(h) \cdot n - m|/n < 2^{55}\epsilon/n < 2^{53}\eta/n \leq \eta$. But by definition of η , this cannot happen unless $m/n = f(h)$. This accounts for the last case. ■

Even though a 23.5 KB table for the direct table method may still be acceptable in size, let us mention a way this table could be reduced. The function $\varphi(h)$ is a staircase-function that grows slower than identity. This means several consecutive h map to a same value $g = \varphi(h)$. The corresponding values of $f(h) = 2^{\varphi(h) \cdot \log_2 5 - h}$ are binary shifts of each other. Hence, we may reduce the table size (at the price of a few additional operations) by storing the most significant bits of $f(h)$ as a function of g . We then shift the value read off the table by an appropriate amount to recover $f(h)$. The number of elements

to tabulate goes down from 1504 to 647, that is, by a factor of about 2.3. This would correspond to a 10.4 KB table (for a memory alignment on 8-byte boundaries).

More precisely, define $F(g) = 5^g 2^{-\psi(g)}$ (cf. Sec. II). We then have

$$f(h) = F(\varphi(h)) \cdot 2^{-\rho(h)}, \quad (12)$$

where

$$\rho(h) = h - \psi(\varphi(h)).$$

The shift $\rho(h)$ is easily computed based on Property 2. In addition, we may check that

$$1/2 < F(\varphi(h)) \leq 1 \quad \text{and} \quad 0 \leq \rho(h) \leq 2$$

for all h . Since $\rho(h)$ is nonnegative, multiplication by $2^{\rho(h)}$ can be implemented with a bitshift, not requiring branches. Additionally, since there is enough headroom in the 64-bit variable holding m (which fits on 54 bits), we can use a left shift on m rather than a two-word right shift on $F(g)$.

Note that we did not implement this size reduction and hence could not check its suitability in practice. Instead, we concentrated on another size reduction opportunity, that we shall describe now.

B. Bipartite table

The table size can still be reduced more, using an alternative method, based on a bipartite table. The technique takes advantage of the fact that the exact value of 5^g fits on 64 bits for $g \leq 27$. Assuming that the entries are 8-byte aligned, the table uses only 800 B of storage. Compared to the straightforward method from Section IV-A, this second method requires quite a few more arithmetic operations for the sake of a considerably smaller table. Most of the additional overhead in arithmetic operations can however be hidden on current processors through increased instruction-level parallelism. The reduction in table size also helps decreasing the probability of cache misses, as it holds on only a few cache lines.

Recall that we suppose $g = \varphi(h)$ and hence $-787 \leq h \leq 716$. Write

$$g = \varphi(h) = 16q - r, \quad q = \left\lfloor \frac{g}{16} + 1 \right\rfloor, \quad (13)$$

so that $f(h) = 5^{16q} \cdot 5^{-r} \cdot 2^{-h}$. We thus have

$$-20 \leq q \leq 21, \quad 1 \leq r \leq 16. \quad (14)$$

In particular, 5^r is an integer and fits on 38 bits.

Instead of tabulating $f(h)$ directly, we will use two tables: one containing the (exact) value 5^r for $1 \leq r \leq 16$, the other, the most significant bits of 5^{16q} to a precision of roughly 128 bits. It will prove convenient to store these values with their leading non-zero bit left-aligned, in a fashion similar to floating-point significands. They will be stored respectively on one and two 64-bit words each. Thus, we set

$$\begin{aligned} \theta_1(q) &= 5^{16q} \cdot 2^{-\psi(16q)+127}, \\ \theta_2(r) &= 5^r \cdot 2^{-\psi(r)+63}, \end{aligned}$$

where the power-of-two factors provide for the desired left-alignment. We can check that

$$2^{127} \leq \theta_1(q) < 2^{128} - 1, \quad 2^{63} < \theta_2(r) < 2^{64} \quad (15)$$

for all h .

The value $f(h)$ now decomposes as

$$f(h) = \frac{\theta_1(q)}{\theta_2(r)} 2^{-64-\sigma(h)}$$

where

$$\sigma(h) = \psi(r) - \psi(16q) + h. \quad (16)$$

Equations (7) and (16) imply that

$$h - g \log_2 5 - 1 < \sigma(h) < h - g \log_2 5 + 1.$$

As we also have $h \log_5 2 - 1 \leq g = \varphi(h) < h \log_5 2$ by definition of φ , it follows that

$$0 \leq \sigma(h) \leq 3. \quad (17)$$

In particular, the quantity $m \cdot 2^{\sigma(h)}$ fits on 56 bits, leaving headroom for additional 8 bits on a 64-bit variable.

Now let

$$\Delta = \theta_1(q) \cdot n \cdot 2^{-64+8} - \theta_2(r) \cdot m \cdot 2^{8+\sigma(h)}.$$

Obviously, we have $|x_2| > |x_{10}|$, $|x_2| = |x_{10}|$ or $|x_2| < |x_{10}|$ depending respectively if $\Delta < 0$, $\Delta = 0$ or $\Delta > 0$. The case $|x_2| = |x_{10}|$ implies $x_2 = x_{10}$ because this second stage of the algorithm is used only when x_2 and x_{10} have the same sign.

Lemma 2. *Unless $x_2 = x_{10}$, we have $|\Delta| \geq 2^{124}\eta$.*

Proof: The definition of Δ rewrites as

$$\Delta = 2^{8+\sigma(h)} \theta_2(r) n \left(f(h) - \frac{m}{n} \right).$$

The bounds (5), (15), (17) together imply that $2^{8+\sigma(h)} \theta_2(r) n \geq 2^{124}$. We know from Theorem 1 that either $f(h) = m/n$, which is equivalent to $x_2 = x_{10}$, or $|f(h) - m/n| \geq \eta$. ■

As already explained, the values of $\theta_2(r)$ are all integers and can exactly be tabulated as-is. In contrast, only an approximation can be tabulated for $\theta_1(q)$. We chose to represent these values as $\lceil \theta_1(q) \rceil$, hence replacing Δ by the easy-to-compute

$$\tilde{\Delta} = \lceil \theta_1(q) \rceil \cdot n \cdot 2^{8-64} - \theta_2(r) \cdot m \cdot 2^{8+\sigma(h)}. \quad (18)$$

Here n , $\theta_2(r)$ and $m \cdot 2^{8+\sigma(h)}$ are all nonnegative integers of at most 64 bits, and $\lceil \theta_1(q) \rceil$ is a positive integer of at most 128 bits.

Computing the floor function in the first term of $\tilde{\Delta}$ comes down to dropping the low-order 64-bit word in a three-word integer. This is also the reason we chose to take the ceiling $\lceil \theta_1(q) \rceil$ when representing $\theta_1(q)$ in the table: by dropping words, we can only compute under-approximations. By choosing over-approximations for the table value, we can manage for the two approximations to cancel out, at least partially. As we will see in the sequel, in particular with

Theorem 3, this effect allows equality cases (cases where $x_2 = x_{10}$) to be handled extremely easily.

The multiplications by 2^8 in (18) are not essential. They serve to shift the significant bits of m and n as much as possible to the left, so that the two-word comparison that follows finishes after only one branching in as many cases as possible.

Lemma 2 is in principle enough to compare x_2 to x_{10} , using a criterion similar to that from Theorem 2. But we can actually prove finer properties that allow for a more efficient final decision step. Indeed, the use of Theorem 2 would require adding and subtracting a particular constant ϵ . Here, we show that the sign of $\tilde{\Delta}$ already provides the answer to the comparison.

Theorem 3. *Assume $g = \varphi(h)$, and let $\tilde{\Delta}$ be the 128-bit signed integer defined by (18). Then following equivalences hold:*

$$\begin{aligned} \tilde{\Delta} < 0 &\iff |x_{10}| < |x_2|, \\ \tilde{\Delta} = 0 &\iff x_{10} = x_2, \\ \tilde{\Delta} > 0 &\iff |x_{10}| > |x_2|. \end{aligned}$$

Proof: Write

$$\lceil \theta_1(q) \rceil = \theta_1(q) + \delta_{\text{tbl}}$$

$$\lfloor \lceil \theta_1(q) \rceil \cdot n \cdot 2^{-56} \rfloor = \lceil \theta_1(q) \rceil \cdot n \cdot 2^{-56} + \delta_{\text{rnd}}$$

for some

$$0 \leq \delta_{\text{tbl}} < 1, \quad -1 < \delta_{\text{rnd}} \leq 0.$$

Setting $\delta = \delta_{\text{rnd}} + n 2^{-56} \delta_{\text{tbl}}$, we have $\tilde{\Delta} = \Delta + \delta$, and, by (5),

$$-1 < \delta < 1/4.$$

If $\Delta \neq 0$, it follows that

$$|\tilde{\Delta}| \geq 2^{124}\eta - |\delta| > 2^{10}$$

by Lemma 2. In particular, Δ and $\tilde{\Delta}$ have the same sign, and $\tilde{\Delta} < 0 \iff |x_{10}| < |x_2|$. If $\Delta = 0$, we get $|\tilde{\Delta}| = \delta$, and we can conclude that $\tilde{\Delta} = 0$ because $\tilde{\Delta}$ is an integer. ■

The conclusion $\tilde{\Delta} = 0 = \Delta$ when $x_2 = x_{10}$ means that in this case, there is no error in the approximate computation of Δ . Specifically, the tabulation error δ_{tbl} and the rounding error δ_{rnd} cancel out, thanks to our choice to tabulate the ceiling of $\theta_1(q)$.

Theorem 3 implies that the following algorithm correctly computes the sign of $x_2 - x_{10}$.

Algorithm 2. Step 2 (second method).

- Compute q and r as defined in (13). Compute $\sigma(h)$ using Property 2.
- Read the 128 bits of $\lceil \theta_1(q) \rceil$ as two 64 bit words.
- Compute the 128 high-order bits α of

$$\lceil \theta_1(q) \rceil \cdot (n 2^8)$$

with one 128×64 -bit multiplication, dropping the 64 low order bits.

- Compute

$$\beta = \theta_2(r) \cdot (m 2^{8+\sigma(h)})$$

with one full 64×64 -bit multiplication, keeping all 128 bits.

- Compute the signed difference $\tilde{\Delta} = \alpha - \beta$.
- Return

$$\begin{cases} "x_2 > x_{10}" & \text{if } \tilde{\Delta} < 0, \\ "x_2 = x_{10}" & \text{if } \tilde{\Delta} = 0, \\ "x_2 < x_{10}" & \text{if } \tilde{\Delta} > 0. \end{cases}$$

V. EQUALITY CASES

As seen in the last Sections, the direct and bipartite methods are able to precisely determine cases when x_2 is *equal* to x_{10} , besides deciding the two possible inequalities. However, when it comes to solely determine such equality cases additional properties may simplify and accelerate the previous tests. The condition $x_2 = x_{10}$ is equivalent to

$$m \cdot 2^h = n \cdot 5^g.$$

Lemma 3. Assume $x_2 = x_{10}$. Then,

- either $0 \leq g \leq 22$, $0 \leq h \leq 53$, and then m is divisible by 5^g and n is divisible by 2^h ;
- or $-22 \leq g \leq 0$, $-51 \leq h \leq 0$, in which case we have $2^{-h} \mid m$ and $5^{-g} \mid M_{10}$.

Proof: Notice that h and $g = \varphi(h)$ always have the same sign. When they are nonnegative, n must be a multiple of 2^h and m must be a multiple of 5^g . It follows that $5^g \leq m < 2^{53}$, hence $g \leq \log_5 2^{53} < 22.9$, which in turn implies $h \leq 53$. When g and h are negative, m is a multiple of 2^{-h} while $n = M_{10} \cdot 2^\nu$ and hence M_{10} are multiples of 5^{-g} . We get $-g < \log_5 10^{16} < 22.9$, whence $-h \leq 51$. ■

We deduce the following algorithm, which decides whether $x_2 = x_{10}$ using only 64-bit integer operations. Note that here we do not need to compute $\varphi(h)$.

Algorithm 3. Equality test.

- if $0 \leq h \leq 53$ and $0 \leq g \leq 22$ and $2^h \mid n$ then
 - read 5^g in a table or compute it on the fly
 - compute $m' = 5^g \cdot (n 2^{-h})$ with a 64-bit integer multiplication, yielding 64 output bits
 - return ($m' = m$)
- else if $h \geq -51$ and $-22 \leq g \leq 0$ and $2^{-h} \mid m$ then
 - read 5^{-g} in a table or compute it on the fly
 - compute $n' = 5^{-g} \cdot (m 2^h)$ with a 64-bit integer multiplication
 - return ($n' = n$)
- else return *false*

The constants 5^g , $0 \leq g \leq 22$ used in the algorithm all fit on 51 bits. They can for instance be read in a small table or computed on the fly (which requires at most 5 squarings and multiplications by 5 using binary powering). The tests on g in Algorithm 3 are there only to make sure that the values of $5^{\pm g}$ used later are in the allowable range.

VI. EXPERIMENTAL RESULTS

Both algorithms, the direct method presented in Section IV-A and the bipartite table method presented in Section IV-B, have been implemented and thoroughly tested. Our implementation is written in plain C, using the decimal floating-point type support offered by the compiler we are using, Intel `icc` 12.1.3. To the extent possible when using decimal floating-point numbers, we gave priority to portability over performance. No particular effort at optimization was made. For instance, multiple-word operations are implemented in portable C, with no use of hardware support for operations such as $64 \times 64 \rightarrow 128$ bit multiplication except for that automatically inserted by the compiler.

Testing was done using test vectors that extensively cover all floating-point input classes, such as normal numbers, subnormals, Not-A-Numbers, zeros, infinities, for both the binary input x_2 and the decimal input x_{10} .

The test vectors also exercise the worst-cases (m, n) for the critical value $d_h(m, n) = \lceil 5^{\varphi(h)} / 2^h - m/n \rceil$ for each admissible value of h , as explained in Section III. Additionally, the 50-but-worst cases (m, n) for each value h were computed using the method described in Appendix of [1] and added to the test vectors. The rationale behind exercising not only the worst-case for each h but also some bad cases is that an implementation might work for the worst-case input just by chance, whereas chances decrease rapidly for a larger number of difficult test cases.

The test vectors were finally completed with a reasonably large set of random inputs that fully exercise all possible values of the normalization $2^\nu M_{10}$ for the decimal mantissa (cf. Section II) as well as both the first step succeeding or the second step being necessary.

Both implemented methods have been compared for performance to the naïve comparison method where the binary or decimal input is converted to decimal resp. to binary before getting compared. For performance testing, the code is executed on a system equipped with a quad-core Intel Core i7 M 620 processor clocked at 2.67GHz running Linux 3.2.0-2 in Intel64 mode. The comparison functions are compiled at optimization level 3, with the `-xSSE4.2` flag set.

The timing measurements were done using the Read-Time-Step-Counter instruction after pipeline serialization. The serialization and function call overhead was subtracted off after timing an empty function. Measurements were taken once the caches were preheated by previous comparison function calls, the results of which were discarded. The indicated numbers are in cycles and given for the minimum/ average/ maximum value that was observed. The average values depend on the nature of the comparison inputs and should not be used in a general context. No average values are given for case involving special inputs, such as zeros, Not-A-Numbers (NaNs) or infinities.

As can be seen from Table II, our unoptimized implementations of both the Direct Method presented in Section IV-A and the Bipartite Table Method described in Section IV-B outperform, for the important case when x_2 is normal, the

	Naïve method converting x_2 to decimal64 (<i>incorrect</i>)	Direct method described in Section IV-A	Bipartite table method described in Section IV-B
Input type	min/avg/max	min/avg/max	min/avg/max
Special cases (± 0 , NaNs, Inf)	13/-/215	6/-/118	6/-/115
x_2 subnormal x_{10} of same sign	80/129/229	189/206/232	197/211/229
x_2 finite x_{10} of opposite sign	48/128/228	5/46/117	6/51/119
x_2 normal x_{10} of same sign	29/105/228	19/60/181	16/62/214

TABLE II
TIMINGS FOR BOTH PRESENTED METHODS AND FOR A NAÏVE METHOD

naïve method based on converting x_2 into the decimal64 format. Let us mention again that this naïve comparison is what a C compiler would most likely do and what the one we used does (cf. Section I). Besides being faster, the methods presented in this paper have the advantage of always providing correct boolean answers.

The Direct Method, based on direct tabulation of $f(h)$ (cf. Section IV-A), is slightly faster than the Bipartite Table Method in average and maximum timing. This is mainly due to the more computationally expensive second step of the Bipartite Table Method. The difference in performance, 181 cycles with respect to 214 for the bipartite method, might however be too small to justify the usage of an about 30 times larger table for the direct method. We do not have experimental data for the Direct Method using the table size reduction technique through tabulation of $F(g)$ (cf. Section IV-A).

VII. CONCLUSION

Though not foreseen in the IEEE 754-2008 standard, exact comparisons between floating-point formats of different radices would enrich the current floating-point environment and enhance the safety and provability of numerical software.

This paper has investigated the feasibility of such comparison at the instance of the binary64 and decimal64 formats. A very efficient test has been presented that eliminates most of the comparison inputs. For the remaining cases, two algorithms were proposed, a more direct method and a technique based on a bipartite table. The bipartite table uses only 800 bytes of table space.

Both methods have been proven, implemented and thoroughly tested. They outperform the naïve comparison technique consisting in conversion of one of the inputs to the respectively other format. Furthermore, they always return a correct answer, which is not the case of the naïve technique.

Future work will have to address the other common IEEE 754-2008 binary and decimal formats, such as binary32 and decimal128. Since the algorithmic problems of exact binary to decimal comparison and correct rounding of binary to decimal conversions (and vice versa) are similar, future investigations should also consider the possible reuse of

tables for both problems. Finally, one should mention that considerable additional work is required in order to enable mixed-radix comparisons in the case when the decimal floating-point number is stored in dense-packed-decimal representation.

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