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To cite this version:

Anne Vaugon. Reeb periodic orbits after a bypass attachment. 2012. ensl-00768825

HAL Id: ensl-00768825

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Preprint submitted on 24 Dec 2012

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REEB PERIODIC ORBITS AFTER A BYPASS ATTACHMENT

ANNE VAUGON

Abstract. On a 3-dimensional contact manifold with boundary, a bypass attachment is an elementary change of the contact structure consisting in the attachment of a thickened half-disc with a prescribed contact structure along an arc on the boundary. We give a model bypass attachment in which we describe the periodic orbits of the Reeb vector field created by the bypass attachment in terms of Reeb chords of the attachment arc. As an application, we compute the contact homology of a product neighbourhood of a convex surface after a bypass attachment, and the contact homology of some contact structures on solid tori.

1. Introduction

We describe the effect on Reeb dynamics of an elementary change of the contact structure on a 3-manifold with boundary known as a *bypass attachment*. Our study is based on the explicit construction of a bypass. We encode the dynamics of the associated Reeb vector field and give a symbolic representation of the new periodic orbits. As an application, we compute the contact homology of a product neighbourhood of a convex surface after a bypass attachment, and the contact homology of some contact structures on solid tori.

Honda [27] introduced bypass attachments to classify contact structures on solid tori, thickened tori, and lens spaces. Bypasses may be seen as basic building-blocks of contact structures. In particular, cobordisms are constructed out of bypasses as contact structures on a thickened surface are obtained from an invariant contact structure and a finite number of bypass attachments and removals (see [26, Section 11.1]).

Describing new periodic orbits after a bypass attachment is the first step toward computing the *contact homology* of the new contact manifold. Introduced in the vein of Floer homology by Eliashberg, Givental, and Hofer [11] in 2000, contact homology is an invariant of a contact structure on a closed manifold defined through a Reeb vector field. Colin, Ghiggini, Honda, and Hutchings [6] generalised it to an invariant of contact structures on manifolds with boundary called *sutured contact homology*. The simplest associated complex is the $\mathbb{Q}$-vector space generated by Reeb periodic orbits and the differential "counts" pseudo-holomorphic cylinders in the symplectisation of the contact manifold. Gromov [21] introduced pseudo-holomorphic curves in symplectic geometry in 1985. Hofer [22] generalised them to symplectisations in 1993. Our theorem is similar to a theorem of Bourgeois, Ekholm, and Eliashberg [2] describing the new Reeb periodic orbits after a surgery along a Legendrian sphere Λ in terms of Reeb chords of Λ. In addition, the authors deduce exact triangles between contact homology, symplectic homology and Legendrian contact homology. Finding an analogous triangle would be a natural extension to this paper.

The computation of contact homology hinges on finding periodic orbits and solving elliptic partial differential equations and thus is usually out of reach. To our knowledge, Golovko's work [19, 20] contains the only explicit computations in the sutured case. Actual computations are of importance to clarify our intuition

and understand connections between sutured Heegaard-Floer homology and sutured embedded contact homology. Sutured embedded contact homology is a variant of sutured contact homology introduced in [6] in the vein of Hutchings' work [28]. Taubes [37] proved that it is an invariant of the manifold. In the closed case, Kutluhan, Lee, and Taubes [30] and, independently, Colin, Ghiggini, and Honda [5] announced an isomorphism between the two homologies. In addition, computations can be of use to understand the algebraic structures associated to sutured contact homology and obtain a comprehensive gluing theorem from the partial theorem in [6].

Outline. This paper is derived from the PhD thesis of the author [38]. It is organised as follows. In Section 2, we present our main theorems. In Section 3, we recall some usual definitions in contact geometry and contact homology that will be used in the sequel. In Section 4, we apply our main theorem to the simplest manifold with boundary: the product neighbourhood of a convex surface. In Section 5, we compute the contact homology of a product neighbourhood of a convex surface after a bypass attachment, and the contact homology of some contact structures on solid tori. The proof of our main theorem (Theorem 2.1) is technical. We give a sketch of proof in Section 6 and a detailed proof in Section 7. Section 8 is devoted to the proof of Theorem 2.6 which describe the Conley-Zehnder index of the new Reeb periodic orbits.

2. Main results

2.1. **Bypasses.** Let us review some basic definitions (see Section 3 for more details). Let M be a 3-manifold. A 1-form α on M is called a *contact form* if $\alpha \wedge \mathrm{d}\alpha$ is a volume form. A *contact structure* ξ is a plane field locally defined as the kernel of a contact form. To any contact form α, we associate the vector field, called the *Reeb vector field*, such that $\iota_{R_\alpha}\alpha = 1$ and $\iota_{R_\alpha}\mathrm{d}\alpha = 0$. A *Reeb chord* of an arc γ_0 is a Reeb arc with endpoints on γ_0. A curve tangent to ξ is called *Legendrian*. In a contact manifold, "pleasant" surfaces are *convex* surfaces (see Section 3.1 for a precise definition). In a neighbourhood of a convex surface S, the contact structure is encoded by a smooth multi-curve Γ, the *dividing set*, separating S into positive and negative regions (Giroux [16]). In what follows we specify the dividing set associated to a convex surface S by the pair (S, Γ). Convexity is a natural condition to impose to the boundary of a contact manifold. To deal with contact forms as opposed to contact structures, for instance to define sutured contact homology, one usually refines this condition as follows. A contact form α is *adapted* to a convex surface (S, Γ) if Γ is the set of tangency points between R_α and S and, along Γ, the vector field R_α points toward the sub-surface S_+ where R_α is positively transverse to S. On a manifold with convex boundary, the dividing set of the boundary is a *suture* as defined by Gabai [14].

An *attaching arc* of the convex boundary (S, Γ) of (M, ξ) is a Legendrian arc which intersects the dividing set Γ in precisely three points, namely, its two endpoints and one interior point. A *bypass attachment* along an attachment arc γ_0 is the gluing of a half-disc D with a prescribed continuation of ξ along γ_0. We get a new manifold with boundary by thickening (D, ξ).

2.2. **Main theorem.** Let $I_b = \left[-\frac{3\pi}{4}, \frac{11\pi}{4}\right]$. Let (M, α) be a contact manifold with convex boundary (S, Γ) and γ_0 be an attachment arc on S. We assume that

(C1) there exists a neighbourhood Z of γ_0 with coordinates $(x, y, z) \in I_b \times [-y_{\max}, 0] \times I_{\max}$ where $I_{\max} = [-z_{\max}, z_{\max}]$ such that
- $\alpha = \sin(x)\mathrm{d}y + \cos(x)\mathrm{d}z$;
- $\gamma_0 = [0, 2\pi] \times \{0\} \times \{0\}$;

- $S_Z = I_b \times \{1\} \times I_{\max} = S \cap Z$

(C2) α is adapted to $S \setminus S_Z$.

Fix $K > 0$. Let $\delta_K(\gamma_0)$ denote the image of $\gamma_0 \setminus \Gamma$ on S by the Reeb flow for times smaller than K. Additionally, we assume that

(C3) $\delta_K(\gamma_0)$ is transverse to γ_0.

Condition (C3) is generic and ensures that the number of Reeb chord of γ_0 with period smaller than K is finite. We denote by $a_1, \dots a_N$ these chords. Let $l(a_{i_1} \dots a_{i_k}) = T(a_{i_1}) + \dots + T(a_{i_k})$ where $T(a_i)$ is the period of the Reeb chord a_i.

Theorem 2.1. *Under conditions (C1), (C2) and (C3), there exists a contact manifold (M', S', α') obtained from (M, S, α) after a bypass attachment along γ_0, such that*

- *S' is convex;*
- *α' is adapted to S' and arbitrarily close to α in M;*
- *Reeb periodic orbits of period smaller than K intersecting the bypass correspond bijectively to words $\mathbf{a}$ on the letters $a_1, \dots, a_N$ such that $l(\mathbf{a}) < K$ up to cyclic permutation.*

In addition, the periodic orbit $\gamma_{\mathbf{a}}$ associated to $\mathbf{a} = a_{i_1} \dots a_{i_k}$ intersects S_Z in $2k$ points denoted by $p_1^-, p_1^+, \dots, p_k^-, p_k^+$ and is arbitrarily close to the chord a_j between p_j^- and p_j^+.

Therefore, if the contact form α is non-degenerate on M, the Reeb periodic orbits of period smaller than K on M' are exactly the Reeb periodic orbits of M of period smaller than K and the orbits described in Theorem 2.1. This theorem is proved in Sections 6 and 7. We will see in the proof that the condition "α' is adapted to S'" is crucial to obtain this symbolic representation of the new periodic orbits. The following proposition ensures that conditions (C1) and (C2) are satisfied for any contact manifold after an isotopy.

Proposition 2.2. *Let (M, ξ) be a contact manifold with convex boundary (S, Γ) and γ_0 be an attaching arc. There exists a contact structure ξ' isotopic to ξ and a contact form α of ξ' satisfying conditions (C1) and (C2).*

This proposition derives from Giroux theory of convex surfaces (Sections 3 and 4) and the explicit construction of Proposition 5.1.

2.3. **Computations of contact homology.** We now apply Theorem 2.1 to compute some sutured contact homologies. The sutured contact homology is an invariant associated to a contact structure with boundary (M, ξ, Γ) where Γ is the dividing set of the boundary. Though commonly accepted, existence and invariance of contact homology remain unproven. In what follows this assumption will be called Hypothesis H (see Section 3.3.3 for more details).

Let S be a convex surface and $\Gamma = \bigcup_{i=0}^n \Gamma_i$ be a dividing set of S without contractible components. Let $M = S \times [-1, 1]$ be the product neighbourhood of S with invariant contact structure[1].

Proposition 2.3. *There exists a contact form α without contractible Reeb periodic orbits such that the cylindrical sutured contact homology of $(M, \alpha, \Gamma \times \{\pm 1\})$ is the $\mathbb{Q}$-vector space generated by $n+1$ periodic orbits homotopic to $\Gamma_k \times \{0\}, k = 0, \dots, n$ and by their multiples.*

Theorem 2.4. *Let γ_0 be an attachment arc in S intersecting three distinct components of Γ. Let Γ_0 be the component intersecting the interior of γ_0. We denote by (M', ξ') the contact manifold obtained from (M, ξ) after a bypass attachment*

[1]The multi-curve $\Gamma \times \{\pm 1\}$ is a dividing set of the boundary.

along $\gamma_0 \times \{1\}$ and by Γ' a dividing set of $\partial M'$. Then, under Hypothesis H, the cylindrical sutured contact homology of (M', ξ', Γ') is the $\mathbb{Q}$-vector space generated by n periodic orbits homotopic to $\Gamma_k \times \{0\}, k = 1, \dots, n$ and by their multiples.

Thus, a bypass attachment removes Γ_0 and its multiples from the generators of the sutured contact homology.

Contact structures with longitudinal dividing set on the boundary are characterised by the dividing set of any convex meridian disc [27]. Golovko [20] computed the contact homology in the case where the dividing set of a meridian discs consists of segments parallel to the boundary. He also computed contact the homology of solid tori with non-longitudinal boundary dividing set [19]. We extend his computations to contact structures such that (see Figure 1)

(C4) the boundary dividing set Γ has $2n$ longitudinal components;
(C5) if $(D, \Gamma = \bigcup_{i=0}^{n} \Gamma_i)$ is the dividing set of a convex meridian disc D there exists a partition of ∂D in two sub-intervals I_1 and I_2 such that
- ∂I_1 is contained in two bigons (called *extremal bigons*);
- if $I = \{i, \partial\Gamma_i \subset I_1 \text{ or } \partial\Gamma_i \subset I_2\}$ then any connected component of $D \setminus \left(\bigcup_{i \notin I} \Gamma_i\right)$ contains at most one component of Γ.

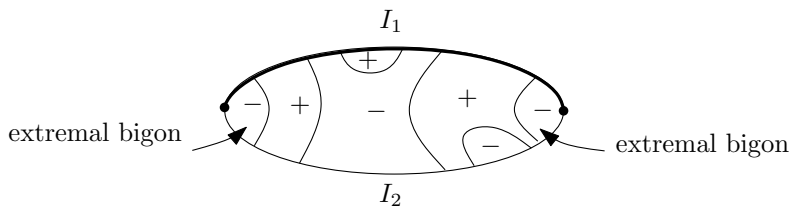

FIGURE 1. A chord diagram satisfying condition (C5)

Theorem 2.5. *Let ξ be a contact structure on $M = D^2 \times S^1$ satisfying conditions (C4) and (C5) above. Under Hypothesis H, the sutured contact homology of (M, ξ, Γ) is the $\mathbb{Q}$-vector space generated by n_+ curves homotopic to $\{*\} \times S^1$, n_- curves homotopic to $\{*\} \times (-S^1)$ and by their multiples where*

$$n_\pm = \chi(S_\pm) + \#\{\mp \textit{non-extremal bigons}\} - \#\{\pm \textit{ bigons}\}.$$

2.4. **Two improvements to the main theorem.** We now describe the Conley-Zehnder index $\mu(\gamma_{\mathbf{a}})$ of the periodic orbit $\gamma_{\mathbf{a}}$ from Theorem 2.1. This index gives the graduation in contact homology and is associated to a trivialisation of the normal bundle of the orbit. We first construct a "nice" trivialisation extending trivialisations along the Reeb chords. In the setting of Theorem 2.1, we denote by a^- and a^+ the inward and outward endpoints of a Reeb chord a, by $[p, p']$ the segment between p and p' in the chart associated to Z and, if p and p' are on $\gamma_{\mathbf{a}}$, by $[p, p']_{\mathbf{a}}$ the arc of $\gamma_{\mathbf{a}}$ between p and p'. For each $i = 1, \dots, N$, choose a collar neighbourhood S_i of $a_i \cup [a_i^+, a_i^-]$. We obtain a collar neighbourhood $S_{\mathbf{a}}$ of the periodic orbit $\gamma_{\mathbf{a}}$ corresponding to $\mathbf{a} = a_{i_1} \dots a_{i_k}$ by gluing together

- collar neighbourhoods of $[p_{i_j}^-, p_{i_j}^+]_{\mathbf{a}} \cup [p_{i_j}^-, p_{i_j}^+]$ given by a small perturbation of S_{i_j},
- and an immersed disc in the bypass with boundary $\bigcup_j [p_{i_j}^+, p_{i_{j+1}}^-]_{\mathbf{a}} \cup [p_{i_j}^+, p_{i_j}^-]$, embedded near its boundary.

For all i, the annulus S_i gives a symplectic trivialisation (e_1, e_2) of ξ along a_i. Let $(R_t)_{t \in [0, T(a_i)]}$ denote the path of symplectic matrices induced by the differential of the Reeb flow along a_i. For all $t \in [0, T(a_i)]$, we denote by θ_t the angle between

e_1 and $R_t(e_1)$. Let $\tilde{\mu}(a_i)$ be the integer such that $\theta_{T(a_i)} \in (\pi\tilde{\mu}(a_i), \pi(\tilde{\mu}(a_i)+1)]$. Then $\tilde{\mu}(a_i)$ depends only on the homology class of S_i.

Theorem 2.6. *If $\mathbf{a} = a_{i_1} \dots a_{i_k}$ is a word such that $l(\mathbf{a}) \leq K$, then $\mu(\gamma_{\mathbf{a}}) = \sum_{j=1}^{k} \tilde{\mu}(a_{i_j})$ in the trivialisation $S_{\mathbf{a}}$.*

In addition, our explicit construction of bypasses allows us to control all the new periodic orbits after a bypass attachment but with less precision. This property is used in our actual computations of contact homology. Let (M, α) be a contact manifold with convex boundary (S, Γ) and γ_0 an attachment arc satisfying conditions (C1) and (C2). We assume that

(C6) there exists $\lambda_0 > 0$ such that for all $\varepsilon > 0$ and for any small enough perturbation of α, the distance between the dividing set and the endpoints of the Reeb chords of $[0, 2\pi] \times \{1\} \times I_{\max}$ is either smaller that ε or greater than λ_0.

Theorem 2.7. *Under conditions (C1), (C2) and (C6), there exists a contact manifold (M', S', α') obtained from (M, S, α) after a bypass attachment along γ_0, such that*

- *S' is convex;*
- *α' is adapted to S' and arbitrarily close to α in M;*
- *if $\psi : [\pi + \lambda_0, 2\pi - \lambda_0] \times I_{max} \to [\lambda_0, \pi - \lambda_0] \times I_{max}$ is the partial function induced by the Reeb flow in M and φ is the map induced by the Reeb flow in the bypass then every periodic orbit intersecting S_Z intersects S_Z^- on a periodic point of $\varphi \circ \psi$.*

If the hypotheses of Theorem 2.1 and Theorem 2.7 are simultaneously satisfied, the associated constructions coincide. In addition, the periodic orbit $\gamma_{\mathbf{a}}$ associated to $\mathbf{a} = a_{i_1} \dots a_{i_k}$ corresponds to the unique fixed point of $\varphi \circ \psi_{i_k} \circ \dots \circ \varphi \circ \psi_{i_1}$ where ψ_{i_j} is the restriction of ψ to the connected component of $\mathrm{dom}(\psi)$ containing $a_{i_j}^-$.

3. Contact geometry

3.1. **Contact geometry and convex surfaces.** A more detailed presentation can be found in [15]. Let $(M, \xi = \ker(\alpha))$ be a contact manifold. A vector field whose flow preserves ξ is said to be *contact*. A fundamental step in the classification of contact structures in dimension 3 was the definition of tight and overtwisted contact structures given by Eliashberg [9] in the line of Bennequin's work [1]. A contact structure ξ is *overtwisted* if there exists an embedded disc tangent to ξ on its boundary. Otherwise ξ is said to be *tight*.

Eliashberg's work [9, 10] initiated the study of surfaces in contact manifolds. The *characteristic foliation* $\mathscr{F}$ of a surface S is the singular 1-dimensional foliation of S such that

- x is a singular point if $\xi_x = T_x S$;
- $\mathscr{F}_x = \xi_x \cap T_x S$ if x is non-singular.

If ω is a volume form on S and $i : S \to M$ is the inclusion, $\mathscr{F}$ is defined by the vector field X satisfying $\iota_X \omega = i^* \alpha$. The characteristic foliation determines the germ of ξ near S [16, Proposition II.1.2].

The development of convexity by Giroux [16] following Eliashberg and Gromov's definition [12] represents a major progress in the study of contact geometry. A surface S is *convex* if there exists a contact vector field transverse to S. If S has a boundary, we require it to be Legendrian. Closed convex surfaces are generic [16, Proposition II.2.6]. The convexity of a surface is equivalent to the existence of a *dividing set* for the characteristic foliation (Giroux, [16, Proposition II.2.1]). A

multi-curve Γ on S is a *dividing set* for a singular 1-dimensional foliation $\mathscr{F}$ of S if there exist two sub-surfaces $S_\pm$ of S, a vector field Y and a volume form ω on S such that

- $\partial S_\pm = \Gamma$;
- $\operatorname{div}_\omega Y > 0$ on S_+ and $\operatorname{div}_\omega Y < 0$ on S_-;
- Y points toward S_+ along Γ.

The dividing set Γ inherits the orientation of ∂S_+. All dividing sets of a given foliation are isotopic. If X is a contact vector field transverse to S, the set of tangency points between X and ξ along S is a dividing set. The dividing set Γ encodes ξ near S as any foliation divided by Γ can be realised as the characteristic foliation of a perturbed surface. This property is due to Giroux [16, Proposition II.3.6] and known as the *realisation lemma*. In favourable situations, the Reeb vector field provides us with a dividing set.

Lemma 3.1. *Let S be a compact surface in (M, α). If R_α is tangent to S along a smooth curve Γ and if, along Γ, the characteristic foliation of S points toward S_+, the sub-surface where R_α is positively transverse to S, then S is convex and Γ is a dividing set.*

Proof. In the definition of dividing set, choose any volume form ω of S and Y such that $\iota_Y \omega = i^*\alpha$ where $i : S \to M$ is the inclusion. □

3.2. **Bypasses.** Let S be a closed convex surface without boundary in a contact manifold. A *bypass* for S is an embedded half-disc D in M such that

- D is transverse to S;
- D has a Legendrian boundary denoted by $\gamma_1 \cup \gamma_2$ and $D \cap S = \gamma_1$;
- the singularities of the characteristic foliation of D are (see Figure 2)
  - a negative elliptic singularity in the interior of γ_1;
  - two positive elliptic singularities at the endpoints of γ_1;
  - positive singularities along γ_2 alternating between elliptic and hyperbolic singularities.

The arc γ_1 is called the *attaching arc* of the bypass.

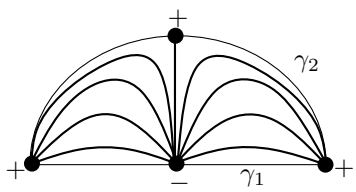

FIGURE 2. The characteristic foliation of a bypass

Proposition 3.2 (Honda [27])**.** *Let D be a bypass for S with attaching arc γ_1. There exists an neighbourhood of $S \cup D$ diffeomorphic to $S \times [0, 1]$ such that*

- *$S \simeq S \times \{\varepsilon\}$;*
- *the contact structure is invariant in $S \times [0, \varepsilon]$;*
- *the surfaces $S \times \{0\}$ and $S \times \{1\}$ are convex with dividing sets Γ and Γ' where Γ and Γ' are identical except in a neighbourhood of γ_1 on which the arrangements are shown in Figure 3.*

Let (M, ξ) be a contact manifold with convex boundary S. Let Γ be a dividing set of S and γ_1 be an attaching arc. A *bypass attachment along* γ_1 is a contact manifold (M', ξ') with convex boundary S' extending (M, ξ) such that there exists a neighbourhood $S \times [0, 1]$ of S' satisfying

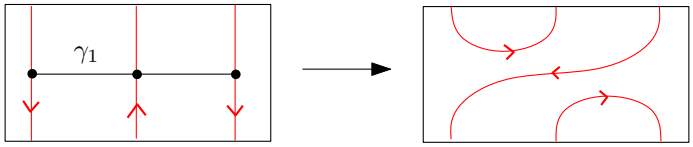

FIGURE 3. Dividing sets Γ and Γ'

- $S' \simeq S \times \{1\}$;
- $S \times \{0\}$ is convex and is the image of S by the flow of an inward contact vector field;
- there exists a contact retraction of $S \times [0,1]$ on an arbitrarily small neighbourhood of $S \times \{0\} \cup D$ where D is a bypass for $S \times \{0\}$ with attaching arc the image of γ_1 on $S \times \{0\}$.

The differences between the dividing sets of S and S' are shown on Figure 3. Honda [27] constructed an explicit bypass attachment on a convex boundary satisfying condition (C1) (see Section 7.1).

There exist two degenerate bypass attachments: the trivial one that does not change the contact structure up to isotopy and the overtwisted one that creates an overtwisted contact structure (see Figure 4).

FIGURE 4. Trivial (left) and overtwisted (right) bypasses

Giroux [17] and Honda [27] independently classified contact structures on solid tori. Honda's proof hinges on bypasses. We follow Mathews' presentation [33]. A *chord diagram* is a finite set of disjoint properly embedded arcs in the disc D^2 up to isotopy relative to the boundary.

Theorem 3.3 (Giroux [17], Honda [27])**.** *Let $F \subset S^1$ be a set with $2n$ elements and $\mathscr{F}$ be a singular foliation on T^2 divided by $\Gamma = F \times S^1$ and containing a meridian leaf which intersects Γ in $2n$ points. Tight contact structures on $D^2 \times S^1$ with characteristic foliation $\mathscr{F}$ on the boundary up to isotopy relative to the boundary correspond bijectively to chord diagrams of n chords with boundary in F. In addition, the associated chord diagram is the dividing set of any convex meridian disc intersecting Γ in $2n$ points.*

By the realisation lemma, we can assume that the characteristic foliation of the boundary satisfies the hypothesis of Theorem 3.3.

Proposition 3.4 (Honda [27])**.** *Let ξ be a contact structure on $D^2 \times S^1$ such that the boundary dividing set Γ has $2n$ longitudinal components. Let D' be a convex meridian disc intersecting Γ in $2n$ points. Fix an attaching arc $\gamma \subset \partial D'$. Then, the contact structure ξ' on $D^2 \times S^1$ obtained after a bypass attachment along γ has a boundary dividing set with $2(n-1)$ longitudinal components. In addition, the chord diagram associated to ξ' is obtained from the diagram associated to ξ by gluing the endpoints of the two chords intersecting[2] γ (see Figure 5).*

[2]This operation corresponds to an annihilation in [33].

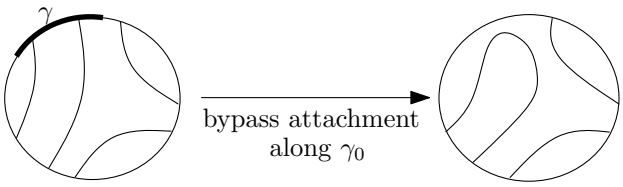

FIGURE 5. Effect of a bypass attachment on solid torus

3.3. **Sutured contact homology.** We consider the generalisation of contact homology to manifolds with boundary called *sutured contact homology* and introduced by Colin, Ghiggini, Honda and Hutchings [6]. Let $(M, \xi = \ker(\alpha))$ be a contact manifold.

3.3.1. *Holomorphic cylinders.* The differential of contact homology "counts" pseudo-holomorphic curves in the symplectisation of the contact manifold. One can refer to [35] for more information. The *symplectisation* of $(M, \xi = \ker(\alpha))$ is the non-compact symplectic manifold $(\mathbb{R} \times M, \mathrm{d}(e^\tau \alpha))$ where τ is the $\mathbb{R}$-coordinate. An *almost complex structure* on a even-dimensional manifold M is a map $J : TM \to TM$ preserving the fibres and such that $J^2 = -\mathrm{Id}$. An almost complex structure J on $\mathbb{R} \times M$ is *adapted* to α if J is τ-invariant, $J\frac{\partial}{\partial \tau} = R_\alpha$, $J\xi = \xi$ and $\omega(\cdot, J\cdot)$ is a Riemannian metric. A map $u : (M_1, J_1) \to (M_2, J_2)$ is *pseudo-holomorphic* if $\mathrm{d}u \circ J_1 = J_2 \circ \mathrm{d}u$. Here we consider pseudo-holomorphic cylinders $u : (\mathbb{R} \times S^1, j) \to \mathbb{R} \times M$. The simplest non-constant pseudo-holomorphic maps are trivial cylinders:

$$\begin{array}{ccc} \mathbb{R} \times S^1 & \longrightarrow & \mathbb{R} \times M \\ (s,t) & \longmapsto & (Ts, \gamma(Tt)). \end{array}$$

where γ is a T-periodic Reeb orbit. Note that there also exist trivial pseudo-holomorphic maps over any Reeb orbit. For every non-constant map

$$u : (\mathbb{R} \times S^1, j) \to (R \times M, J)$$

which is not a trivial cylinder, the points (s,t) such that $\mathrm{d}u = 0$ or $\frac{\partial}{\partial \tau} \in \mathrm{im}(\mathrm{d}u(s,t))$ are isolated (see [35, Lemma 2.4.1]).

The map $u = (a, f) : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ is *positively asymptotic* to a T-periodic orbit γ at $+\infty$ if $\lim_{s \to +\infty} a(s,t) = +\infty$ and $\lim_{s \to +\infty} f(s,t) = \gamma(-Tt)$. It is *negatively asymptotic* to γ at $-\infty$ if $\lim_{s \to -\infty} a(s,t) = -\infty$ and $\lim_{s \to -\infty} f(s,t) = \gamma(+Tt)$. It is a theorem of Hofer [22, Theorem 31] that holomorphic curves $u : (\mathbb{R} \times S^1, j) \to (\mathbb{R} \times M, J)$ with finite Hofer energy are asymptotic to a Reeb periodic orbit at $\pm\infty$ if the contact form α is non-degenerate.

3.3.2. *Conley-Zehnder index.* The Conley-Zehnder index gives the graduation in contact homology. Consider $(M, \xi = \ker(\alpha))$ a contact manifold, γ a T-periodic Reeb orbit and $p \in \gamma$. If φ_t denote the Reeb flow, the map $\mathrm{d}\varphi_T(p) : (\xi_p, \mathrm{d}\alpha) \to (\xi_p, \mathrm{d}\alpha)$ is a symplectomorpism. A non-degenerate periodic orbit γ is called *even* if $\mathrm{d}\varphi_T(p)$ has two real positive eigenvalues and *odd* if $\mathrm{d}\varphi_T(p)$ has two complex conjugate or two real negative eigenvalues. In addition, if $\mathrm{d}\varphi_T(p)$ has real eigenvalues, the orbit is said *hyperbolic*. If it has two complex conjugate eigenvalues, the orbit is called *elliptic*. Let γ_m be the m-th multiple of a simple orbit γ_1. Then γ_m is said to be *good* if γ_1 and γ_m have the same parity, otherwise γ_m is said to be *bad*.

The *Conley-Zehnder index* was introduced in [7] for paths of symplectic matrices. Our short presentation follows [31]. Let $\mathrm{Sp}(2)$ denote the set of symplectic matrices in $\mathcal{M}_2(\mathbb{R})$. The open set $\mathrm{Sp}^* = \{A \in \mathrm{Sp}(2), \det(A - I) \neq 0\}$ has two connected components and they are contractible. Any path $R : [0,1] \to \mathrm{Sp}(2)$ such that

$R_0 = I$ and $R_1 \in \mathrm{Sp}^*(2)$ can be extended by a path $(R_t)_{t\in[1,2]}$ in Sp^* such that

$$R_2 = W_+ = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ or } R_2 = W_- = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Using polar decomposition, we write $R_t = S_t O_t$ where S_t is positive-definite and O_t is a rotation of angle θ_t. The *Conley-Zehnder index* of R is $\mu(R) = \frac{\theta_2-\theta_0}{\pi}$. It is an integer and does not depend on the choice of an extension of R.

As $\mathrm{d}\varphi_t(p) : (\xi_p, \mathrm{d}\alpha) \to (\xi_{\varphi_t(p)}, \mathrm{d}\alpha)$ is a symplectomorpism, a trivialisation of ξ along γ provides us with a path of symplectic matrices. If γ is non degenerate, its Conley-Zehnder index is well defined. It gives a relative (depending on a choice of trivialisation) grading of Reeb periodic orbits. Its parity matches with the above definition.

3.3.3. *Sutured contact homology.* We now assume that (M, ξ) has a convex boundary (S, Γ) and that α is a non-degenerate contact equation adapted to the boundary. We sketch the construction of cylindrical sutured contact homology chain complex $(C_*^{\mathrm{cyl}}(M, \Gamma, \alpha), \partial)$ defined in [6]. The chain complex $C_*^{\mathrm{cyl}}(M, \Gamma, \alpha)$ is the $\mathbb{Q}$-vector space generated by good Reeb periodic orbits (here we consider simple periodic orbits and their good multiples). Choose an almost complex structure J adapted to the symplectisation. To define $\partial\gamma$, consider the set $\mathcal{M}_{[Z]}(J, \gamma, \gamma')$ of equivalence classes (modulo reparametrisation) of solutions of the Cauchy-Riemann equation with finite energy, positively asymptotic to γ, negatively asymptotic to γ' and in the relative homotopy class $[Z]$. The $\mathbb{R}$-translation in $\mathbb{R} \times M$ induces a $\mathbb{R}$-action on $\mathcal{M}_{[Z]}(J, \gamma, \gamma')$. Due to severe transversality issues for multiply-covered curves, there is no complete proof that $\overline{\mathcal{M}}_{[Z]}(J, \gamma, \gamma') = \mathcal{M}_{[Z]}(J, \gamma, \gamma')/_{\mathbb{R}}$ admits a smooth structure. We use Hypothesis H to make this assumption.

Hypothesis H. *There exists an abstract perturbation of the Cauchy-Riemann equation such that $\overline{\mathcal{M}}_{[Z]}(J, \gamma, \gamma')$ is a union of branched labelled manifolds with corners and rational weights whose dimensions are given by $[Z]$ and the Conley-Zehnder indices of the asymptotic periodic orbits.*

There exists several approaches to the perturbation of moduli spaces due to Fukaya and Ono [13], Liu and Tian [32], Hofer, Wysocki and Zehnder [23, 25, 24] or Cieliebak and Oancea in the equivariant contact homology setting [3, 4]. There also exist partial transversality results due to Dragnev [8].

The *differential of a periodic orbit* γ is

$$\partial\gamma = \sum_{\gamma'} \frac{n_{\gamma,\gamma'}}{\kappa(\gamma')}\gamma'$$

where $\kappa(\gamma')$ is the multiplicity of γ' and $n_{\gamma,\gamma'}$ denote the signed weighted counts of points in 0-dimensional components of $\overline{\mathcal{M}}_{[Z]}(J, \gamma, \gamma')$ for all relative homology classes $[Z]$. In particular, the differential of an even (resp. odd) periodic orbit contains only odd (resp. even) periodic orbits.

Under Hypothesis H, it is reasonable to expect the following: if there exists an open set $U \subset \mathbb{R} \times M$ containing all the images of J-holomorphic curves positively asymptotic to γ, negatively asymptotic to γ', then U contains the images of all solutions of perturbed Cauchy-Riemann equations with the same asymptotics for all small enough abstract perturbations.

Hypothesis H is the key ingredient to prove the existence and invariance of contact homology. The condition "α adapted to the boundary" implies that a family of holomorphic cylinders stays in a compact subset in the interior of M.

Theorem 3.5 (Colin-Ghiggini-Honda-Hutchings)**.** *Under Hypothesis H,*

(1) $\partial^2 = 0$*;*

(2) *the associated homology* $HC_*^{cyl}(M,\xi,\Gamma)$ *does not depend on the choice of the contact form, complex structure and abstract perturbation.*

If $\partial^2 = 0$ for some contact form α, we denote $HC_*^{\mathrm{cyl}}(M,\Gamma,\alpha,J)$ the associated homology.

Theorem 3.6 (Golovko [20])**.** *Let ξ be a contact structure on $D^2 \times S^1$ such that the boundary dividing set has $2n$ longitudinal components and the dividing set of a meridian disc has n components parallel to the boundary. Then the sutured cylindrical contact homology is the $\mathbb{Q}$-vector space generated by $n-1$ orbits homotopic to $\{*\} \times S^1$ and by their multiples.*

3.3.4. *Positivity of intersection.* In dimension 4, two distinct pseudo-holomorphic curves C and C' have a finite number of intersection points and that each of these points contributes positively to the algebraic intersection number $C \cdot C'$. This result is known as *positivity of intersection* and was introduced by Gromov [21] and McDuff [34]. In this text we will only consider the simplest form of positivity of intersection: let M be a 4-dimensional manifold, C and C' be two J-pseudo-holomorphic curves and $p \in M$ so that C and C' intersect transversely at p. Consider $v \in T_pC$ and $v' \in T_pC'$ two non-zero tangent vectors. Then (v, Jv, v', Jv') is a direct basis of T_pM (J induces a natural orientation on T_pM). In the symplectisation of a contact manifold, positivity of intersection of a pseudo-holomorphic curve with a trivial holomorphic map results in the following lemma.

Lemma 3.7. *Let (M,ξ) be a contact manifold, α be a contact form and J be an adapted almost complex structure. Consider U an open subset of $\mathbb{C}$, $u = (a,f) : U \to \mathbb{R} \times M$ a J-pseudo holomorphic curve and $p \in U$ such that $\mathrm{d}f_p$ is injective and transverse to $R(f(p))$.Then, $R(f(p))$ is positively transverse to $\mathrm{d}f_p$.*

The hypothesis "$\mathrm{d}f_p$ injective and transverse to $R(f(p))$" is generic. We will use positivity of intersection in the following situation to carry out explicit computations of sutured contact homology in Sections 4 and 5. Let $(M, \xi = \ker(\alpha))$ be a contact manifold with convex boundary and α be a contact form. We assume there exist two sets of Reeb chords of ∂M, denoted by X_+ and X_-, with non-empty interior. Let J be an almost complex structure adapted to α.

Lemma 3.8. *Let $u : (\mathbb{R} \times S^1, j) \to (\mathbb{R} \times M, J)$ be a J-holomorphic cylinder asymptotic to γ_+ and γ_-. Assume that for any Reeb chord $c \in X_\pm$ there exists a path of properly embedded arcs in $M \setminus (\gamma_+ \cup \gamma_-)$ connecting c and a Reeb chord in $\mathrm{int}(X_\mp)$ with reversed orientation. Then $\mathrm{im}(u)$ is disjoint from $\mathrm{int}(X_+) \cup \mathrm{int}(X_-)$.*

Proof. Generically a Reeb chord is transverse to u. Let c_+ be a Reeb chord in X_+ transverse to u. There exists c_- in X_- transverse to u and connected to $-c_+$ by a path of properly embedded arcs in $M \setminus (\gamma_+ \cup \gamma_-)$. By positivity of intersection, $c_\pm \cdot u \geq 0$. Yet $c_+ \cdot u = -c_- \cdot u$ and c_+ does not intersect $\mathrm{im}(u)$. □

4. Thickened convex surfaces

In this section we study the simplest example of contact manifold with boundary. We compute its sutured contact homology and apply our main theorem. Let S be a convex surface and $\Gamma = \bigcup_{i=0}^n \Gamma_i$ be a dividing set of S. Assume Γ has no contractible component. Let $M = S \times [-1,1]$ be the product neighbourhood of S with invariant contact structure. This contact structure is tight [18, Théoreme 4.5a]. Let γ_0 be an attachment arc. The multi-curve $\Gamma \times \{\pm 1\}$ is a dividing set of the boundary. Giroux [16, Proposition 2.1] proved that there exists a contact form α_0 such that

(C7) for all $n = 1, \ldots, n$, there exists a neighbourhood U_i of Γ_i with coordinates

$$(x,y,z) \in [-x_{\max}, x_{\max}] \times [-1,1] \times S^1$$

such that

- $S \cap U_i \simeq [-x_{\max}, x_{\max}] \times \{0\} \times S^1$;
- $\Gamma_i \simeq \{0\} \times \{0\} \times S^1$;
- $\alpha_0 = f(x)\mathrm{d}y + \cos(x)\mathrm{d}z$ where $f : [-x_{\max}, x_{\max}] \to \mathbb{R}$ is non-decreasing, $f = \pm 1$ near $\pm x_{\max}$ and $f = \sin$ near 0;

(C8) $\alpha_0 = \beta_\pm \pm \mathrm{d}y$ on $S_\pm \times [-1,1] \setminus U$ where $\pm \mathrm{d}\beta_\pm > 0$ and $U = \bigcup_{i=0}^n U_i$.

The Reeb periodic orbits are exactly the curves $\Gamma_i \times \{t\}$ for all $t \in [-1,1]$ and $i = 0, \dots, n$ (see Figure 6). The contact form is degenerate and is not adapted to

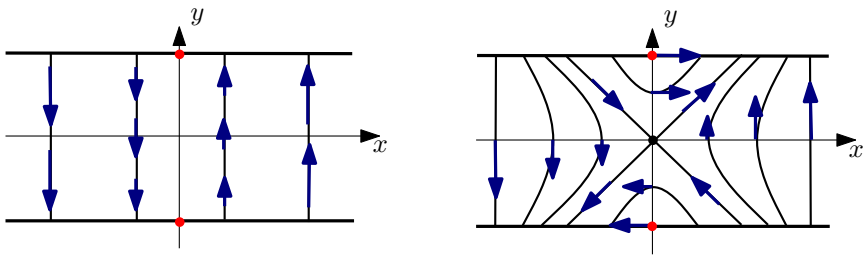

FIGURE 6. The vector fields R_{α_0} and R_{α_p} projected on the (x,y)-plane

the boundary. In U_i, near $x = 0$, we perturb α_0 into

$$\alpha_p = \sin(x)\mathrm{d}y + (1 + k(x)l(y))\cos(x)\mathrm{d}z$$

where

(C9) k is a cut-off function such that $k = 1$ near 0 and $k = 0$ near $\pm x_{\max}$;

(C10) l is a $\mathcal{C}^\infty$-small strictly convex function with minimum 0 at 0.

The associated Reeb vector field is

$$R_{\alpha_p} = \frac{1}{p(x,y)} \begin{pmatrix} l'(y)k(x)\cos(x) \\ p(x,y)\sin(x) - k'(x)l(y)\cos(x) \\ \cos(x) \end{pmatrix}.$$

Proposition 4.1. *The contact form α_p is non-degenerate, adapted to the boundary. Its Reeb periodic orbits are the curves $\Gamma_i \times \{0\}$ for $i = 0, \dots, n$. These orbits are even and hyperbolic.*

Proof. In $M \setminus U$, the Reeb vector field is $R = \pm \frac{\partial}{\partial y}$. Thus the Reeb periodic orbits are contained in U. In addition, if l is small enough, they are contained in the neighbourhood where $k = 1$. In this neighbourhood, the projection of the Reeb vector field in the plane (x,y) is collinear to the Hamiltonian vector field of $(x,y) \mapsto l(y)\cos(x)$. There are no closed levels so the Reeb periodic orbits correspond to critical points of the Hamiltonian (see Figure 6). By linearising the Reeb flow φ_t along $\Gamma_i \times \{0\}$ we get

$$\mathrm{tr}\left(\mathrm{d}\varphi_t(0,0,0)_{|\xi}\right) = 2\cosh\left(\sqrt{tl''(0)}\right) > 2.$$

Thus the Reeb periodic orbits are even and hyperbolic. □

Proof of Proposition 2.3. Proposition 2.3 is a corollary of Proposition 4.1. Indeed $\partial = 0$ in $C_*^{\mathrm{cyl}}(M, \Gamma \times \{\pm 1\}, \alpha_p)$ as all the orbits are even. □

We now apply our main theorem to M. By Proposition 2.2 and [16, Proposition 2.1] there exists an isotopic the contact structure with a contact form satisfying conditions (C1) and (C2) and coordinates in a neighbourhood of Γ, compatible with

the coordinates in Z and satisfying conditions (C7) and (C8). We perturb the contact form into

$$\alpha_b = f(x)\mathrm{d}y + (1 + k(x)l(y)m(z))\cos(x)\mathrm{d}z$$

where k and l satisfy conditions (C9) and (C10) and

(C11) m is a smooth cut-off function, $m = 0$ on $[-z_{\max}, z_{\max}]$ and $m = 1$ outside a small neighbourhood of $[-z_{\max}, z_{\max}]$.

Let $p(x, y, z) = 1 + k(x)l(y)m(z)$. The Reeb vector field is

$$R_{\alpha_b} = \frac{1}{p - k'(x)l(y)m(z)\cos(x)\sin(x)} \begin{pmatrix} l'(y)k(x)m(z)\cos(x) \\ p\sin(x) - k'(x)l(y)m(z)\cos(x) \\ \cos(x) \end{pmatrix}.$$

Proposition 4.2. *The contact form α_b is non-degenerate, adapted to the boundary. Its Reeb periodic orbits are the curves $\Gamma_i \times \{0\}$ for $i = 0, \dots, n$. These orbits are even and hyperbolic.*

Proof. As in the proof of Proposition 4.1, for l small enough, the Reeb periodic orbits are contained in the neighbourhood where $k = 1$. In this neighbourhood, any Reeb orbit intersecting the set $xy \leq 0$ meets the boundary or is equal to $\Gamma_i \times \{0\}$ (see Figure 6). In addition, any Reeb orbit intersecting the set $xy \geq 0$ meets the set $xy \leq 0$. Thus the Reeb periodic orbits are the curves $\Gamma_i \times \{0\}$. By linearising the Reeb flow φ_t along $\Gamma_i \times \{0\}$, we get

$$\mathrm{tr}\left(\mathrm{d}\varphi_t(0,0,0)_{|\xi}\right) > \mathrm{tr}\left(\mathrm{d}\varphi_0(0,0,0)_{|\xi}\right) = 2. \qquad \square$$

We denote by Γ_0 the connected component of Γ which intersects the interior of γ_0. If γ_0 intersects three distinct components of Γ, then γ_0 intersect U_0 along $[-x_{\max}, x_{\max}] \times \{1\} \times \{0\}$. If γ_0 intersects only two distinct components of Γ, there exists $z_1 \in S^1$ such that γ_0 intersects U_0 along $[-x_{\max}, x_{\max}] \times \{1\} \times \{0\}$ and $[0, x_{\max}] \times \{1\} \times \{z_1\}$ (reverse the orientation of S if necessary). If c is a Reeb chord of γ_0, we denote by $\overline{c}$ the union of c and the arc of γ_0 joining c^+ and c^-. Recall that c^+ and c^- are the endpoints of c.

Proposition 4.3. *The contact form α_b satisfies condition (C3) for all $K > 0$. In addition (see Figure 7),*

- *if γ_0 intersects three distinct components of Γ, the set of Reeb chords of γ_0 is $\{c_k, k \in \mathbb{N}^*\}$ and $[\overline{c_k}] = [\Gamma_0]^k$;*
- *if γ_0 corresponds to a trivial bypass attachment, the set of Reeb chords of γ_0 is $\{c_k, d_k, k \in \mathbb{N}^*\}$ where $[\overline{c_k}] = [\overline{d_k}] = [\Gamma_0]^k$ and the z-coordinates of c_k^+ and d_k^+ are respectively 0 and z_1;*
- *if γ_0 corresponds to an overtwisted bypass attachment, the set of Reeb chords of γ_0 is $\{d_0, c_k, d_k, k \in \mathbb{N}^*\}$ where c_k and d_k are as above for $k \in \mathbb{N}^*$, $\overline{d_0}$ is contractible and the z-coordinate of d_0^+ is z_1.*

In a trivialisation that does not intersect small translations of the Reeb chord along $\frac{\partial}{\partial y}$, we have $\tilde{\mu}(c_k) = 1$ and $\tilde{\mu}(d_k) = 0$.

Corollary 4.4. *We assume that γ_0 intersects three distinct components of Γ. Let $L > 0$. There exists a contact manifold (M', α') obtained from (M, α_b)after a bypass attachment along γ_0 such that for all $l < L$ the set of Reeb periodic orbits homotopic to $[\Gamma_0]^l$ is*

$$\left\{\kappa = (k_1, \dots, k_m) \in (\mathbb{N}^*)^l, l = k_1 + \dots + k_m\right\}/_{\{\textit{cyclic permutation}\}}.$$

In addition, $\mu(\gamma_\kappa) = m$ if γ_κ is the orbit associated to $\kappa = (k_1, \dots, k_m)$.

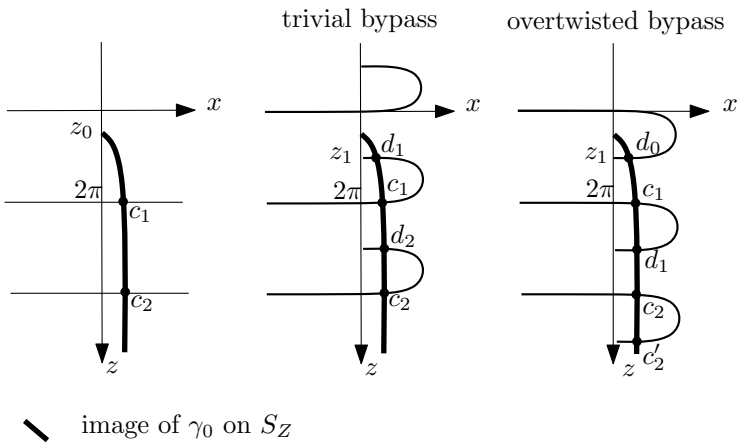

FIGURE 7. Reeb chords

Corollary 4.5. *An overtwisted bypass attachment creates a contractible Reeb periodic orbit.*

Proof of Proposition 4.3. For l small enough the Reeb chords are contained in U_0 and more precisely in the neighbourhood where $k = 1$. Let $z_0 = \sup\{z, m([0, z]) = 0\}$. We lift the S^1-coordinate in U_0 to $\mathbb{R}$ and denote by $\delta = (x_S, z_S)$ the image of γ_0 on $[-x_{\max}, x_{\max}] \times \{1\} \times \mathbb{R}$ (see Figure 7). There exists $s_{\max} < x_{\max}$ such that $\mathrm{dom}(\delta) = (-s_{\max}, 0)$. In addition $\lim_{s\to -s_{\max}} z_S(s) = +\infty$ and $\lim_{s\to 0} z_S(s) = z_0$.

Let (x_{z_1}, y_{z_1}) denote the image of γ_0 by the Reeb flow in the plane $z = z_1$. For all $z \geq z_0$ and for all small enough $\eta > 0$, $\frac{\mathrm{d}}{\mathrm{d}z}\big(x_z(s+\eta) - x_z(s)\big)$ and $\frac{\mathrm{d}}{\mathrm{d}z}\big(y_z(s+\eta) - y_z(s)\big)$ are non-negative. It holds that

$$x_z'(s) \geq 1 \qquad y_z'(s) \geq \frac{1}{\cos^2(x_0)} z_0 \tag{1}$$

(renormalise R_{α_b} and use condition (C10)). Thus, if $(s_0, 1, 0)$ and $(x, 1, z)$ are the endpoints of the lift of a Reeb chord, we have

$$z_S'(s_0) = -\frac{1}{\tan(x_z(s_0))} y_z'(s_0) < 0$$

and δ intersects the segments $[-x_{\max}, x_{\max}] \times \{1\} \times \{2k\pi\}$ exactly for $k \in \mathbb{N}^*$. In addition, there is only one intersection point. This point is the endpoint of a Reeb chord. If the bypass is trivial or overtwisted, let z_1 denote to the smallest positive lift of $z_1 \in S^1$. Then δ also intersects $[0, x_{\max}] \times \{1\} \times \{2k\pi + z_1\}$ if and only if $k \in \mathbb{N}^*$ and there is exactly one intersection point. This concludes the description of the Reeb chords when γ_0 intersects three distinct components of Γ.

We now compute $\tilde{\mu}(a)$ for some Reeb chord a. In the coordinates (R, e_1, e_2) given by the symplectic trivialisation of ξ along a, let v denote the projection of $\frac{\partial}{\partial y}$ on (e_1, e_2). Then $v \neq e_1$ and v is positively collinear to e_2 at $t = 0$. Let R_t denote the symplectic matrix induced by the differential of the Reeb flow on $\xi_{a(t)}$. The vector $\mathrm{d}R_t \cdot e_1$ does not cross $\mathbb{R}_+ v$ as $x_z'(s) \geq 1$. Write $\mathrm{d}R_t \cdot e_1 = r(t)e^{i\theta(t)}$. If $a = c_k$, then v is positively collinear to $-e_2$ at $t = T(a)$. The tangent vector to the image of γ_0 on ξ_{a^+} at the endpoint of a is

$$\begin{pmatrix} x_{z_0}'(s_0) \\ y_{z_0}'(s_0)\cos^2(x_0) \\ -y_{z_0}'(s_0)\cos(x_0)\sin(x_0) \end{pmatrix}$$

where the endpoints of a are $(s_0, 1, 0)$ and $(x_0, 1, z_0)$. Using (1), we get

$$\theta(T(a)) \in (\pi, \frac{3\pi}{2}) + 2\pi\mathbb{Z}.$$

Thus $\theta(T(c_k)) \in [\pi, 2\pi]$ and $\tilde{\mu}(c_k) = 1$. If $a = d_k$, then v is positively collinear to e_2 at $t = T(a)$. In addition, it holds that

$$\theta(T(a)) \in (0, \frac{\pi}{2}) + 2\pi\mathbb{Z}.$$

Thus, we obtain $\theta(T(c_k)) \in [0, \pi]$ and $\tilde{\mu}(d_k) = 0$. □

Proof of Corollary 4.4. There exists $\nu > 0$ such that for any small perturbation of α, the Reeb chords of $[-\nu, \nu] \times \{1\} \times I_{\max}$ are arbitrarily close to $\{0\} \times \{1\} \times I_{\max}$. We apply Theorem 2.1 and Proposition 2.7 for $\lambda = \nu$ and K such that $i_1 + \cdots + i_k \leq L$ implies $\Sigma_{j=1}^{k} T(c_{i_j}) < K$.

The set of Reeb periodic orbits homotopic to $[\Gamma_0]^l$ in Corollary 4.4 corresponds to Reeb periodic orbits with period smaller that K described in Theorem 2.1. It remains to prove that there are no other Reeb periodic orbits homotopic to $[\Gamma_0]^l$. If γ is such a Reeb periodic orbit then γ is associated to a periodic point of $\varphi \circ \psi$ (Proposition 2.7). We decompose ψ into $(\Psi_k)_{k \in \mathbb{N}^*}$ so that, if a is a Reeb chord that contributes to Ψ_k, then $[\overline{a}] = [\Gamma_0]^k$. For $k \leq L$, we have $\Psi_k = \psi_k$ (see Proposition 2.7). Therefore γ corresponds to a fixed point of $\varphi \circ \Psi_{i_k} \circ \cdots \circ \varphi \circ \Psi_{i_1}$ and $i_1 + \cdots + i_k = l$ and we obtain $T(\gamma) < K$. □

5. Applications to sutured contact homology

In this section we apply Theorem 2.1 and Proposition 2.7 to prove Theorem 2.4 and Theorem 2.5. To compute the contact homology of a convex surface after a bypass attachment and of some contact structures on solid tori, we construct suitable contact structures on the associated manifolds. On a thickened convex surface (Theorem 2.4) we start from the contact form described in Section 4. Theorem 2.5 gives the contact homology of a contact structure ξ on $D^2 \times S^1$ such that

- (C4) the boundary dividing set Γ has $2n$ longitudinal components;
- (C5) for a convex meridian disc dividing set $(D, \Gamma = \bigcup_{i=0}^{n} \Gamma_i)$ there exists a partition of ∂D in two sub-intervals I_1 and I_2 such that
  - ∂I_1 is contained in two bigons (called *extremal bigons*);
  - if $I = \{i, \partial\Gamma_i \subset I_1 \text{ or } \partial\Gamma_i \subset I_2\}$ then $D \setminus \left(\bigcup_{i \notin I} \Gamma_i\right)$ contains at most one component of Γ.

All these contact structures are obtained from the contact structure $(D^2 \times S^1, \xi_{/\!/})$ with parallel dividing set on convex meridian discs after a finite number of bypass attachments. In Section 5.1 we compute the sutured contact homology of $(D^2 \times S^1, \xi_{/\!/})$ and apply Theorem 2.1 and Proposition 2.7 to obtain the Reeb periodic orbits after a finite number of bypass attachments. To compute the contact homology (and prove Theorem 2.5), it remains to control the differential. By Corollary 4.4, a bypass attachment creates many homotopic periodic orbits. This complicates the direct study of the differential. We get round this difficulty in Section 5.2 by proving that all the holomorphic cylinders are contained in a standard neighbourhood. We deduce the differential in this neighbourhood by use of computations of contact homology in simple situations on a solid torus.

5.1. Contact forms on solid tori.

For $n \in \mathbb{N}^*$ and $0 < \eta < \frac{\pi}{4}$, let

$$D_{n,\eta} = \{(x, y), x \in [-\pi + \eta - h(y), n\pi - \eta + h(y)], y \in [-1, 1]\}$$

where $h : [-1,1] \to \mathbb{R}$ is a strictly concave function with maximum $h(0) < \eta$, vertical and zero at ± 1 (see Figure 8). On $M_{n,\eta} = D_{n,\eta} \times S^1$, we consider the contact form

$$\alpha = f(x)\mathrm{d}y + \cos(x)\mathrm{d}z + g(x,y)\mathrm{d}x$$

where

- f is 2π-periodic, $f = (-1)^{k+1}$ in a neighbourhood of $\left[k\pi + \frac{\pi}{4}, k\pi + \frac{3\pi}{4}\right]$ and $f = \sin$ near $k\pi$, $k = -1, \dots, n$;
- g does not depend on y for $y \geq \frac{1}{2}$;
- $g = 0$ for $y \leq 0$ and outside a neighbourhood of $x = -\frac{3\pi}{4}$ where $f = -1$;
- the leaves of the characteristic foliation of $\partial M_{n,\eta}$ are closed.

Then $(M_{n,\eta,}, \ker(\alpha))$ satisfies property (C4) and a dividing set $\Gamma_{/\!/}$ of the boundary is given by the curves $x = -\pi + \eta - h(0)$, $x = k\pi + \frac{\pi}{2}, k = 0, \dots, n-1$ and $x = n\pi - \eta + h(0)$. Let D be a disc in $M_{n,\eta}$ transverse to $\frac{\partial}{\partial z}$ with Legendrian boundary. As $\frac{\partial}{\partial z}$ is a contact vector field, D is convex. A dividing set is given by condition $\frac{\partial}{\partial z} \in \xi$ and is thus composed of the curves $x = k\pi + \frac{\pi}{2}, k = -1, \dots, n-1$. Therefore $(M, \ker(\alpha))$ is diffeomorphic to $(D^2 \times S^1, \xi_{/\!/})$ (Theorem 3.3).

Proposition 5.1. *There exists a contact form α_p on $M_{n,\eta}$ without contractible Reeb periodic orbits such that $(M_{n,\eta}, \ker(\alpha_p))$ is diffeomorphic to $(D^2 \times S^1, \xi_{/\!/})$ and the sutured contact homology of $(M_{n,\eta}, \alpha_p, \Gamma_{/\!/})$ is the $\mathbb{Q}$-vector space generated by $n_+ = \left\lceil \frac{n-1}{2} \right\rceil$ curves homotopic to $\{*\} \times S^1$, $n_- = n - 1 - n_+$ curves homotopic to $\{*\} \times (-S^1)$ and by their multiples.*

Proof. As in Section 4, we perturb α into

$$\alpha_p = \sin(x)\mathrm{d}y + (1 + k(x)l(y))\cos(x)\mathrm{d}z$$

in a neighbourhood of $x = k\pi, k = 0, \dots n-1$ such that $x \mapsto k(x - k\pi)$ satisfies condition (C9) and l satisfies condition (C10). By Proposition 4.1, the contact form α_p is non-degenerate, adapted to the boundary. Its Reeb periodic orbits are the curves $\{k\pi\} \times \{0\}\ S^1$ for $k = 0, \dots, n-1$. These orbits are even and hyperbolic. Therefore $\partial = 0$. □

Let ξ be a contact structure on $M = D^2 \times S^1$ satisfying conditions (C4) and (C5) for some $n_0 \in \mathbb{N}^*$. There exist $n, m > 0$, a family of integers $\{k_j\}$ and $\{\varepsilon_j\} \in \{-1, 1\}$ for $j = 1, \dots, m$ such that

- $1 \leq k_j \leq n-2$ and $k_{j+1} - k_j \geq 3$;
- (M, ξ) is diffeomorphic to the contact manifold obtained from $(M_{n,\eta}, \ker(\alpha))$ after bypass attachments along the arcs

$$\gamma_j = [(k_j - 1)\pi, (k_j + 1)\pi] \times \{-\varepsilon_j\} \times \left\{\frac{2\pi k_j}{n}\right\}$$

  for $j = 1, \dots, m$ (see Figure 8).

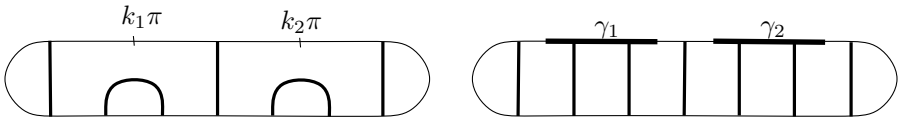

FIGURE 8. Attaching arcs γ_j

Let $\delta_k = \{k\pi\} \times \{0\} \times S^1$. We describe a contact structure on $M_{n,\eta}$ adapted to the bypass attachments along the curves γ_j.

Proposition 5.2. *Let $L > 0$ and $0 < x_{max} \ll 1$. There exists an arbitrarily small perturbation α_b of α in $M_{n,\eta}$ such that*

- *γ_j satisfies conditions (C1) and (C3) for all $K > 0$ and for all $j = 1, \dots, m$;*
- *α_b is adapted to $S \setminus \bigcup S_{Z_j}$;*
- *$\alpha_b = \alpha$ outside $U_k = [-x_{max} + k\pi, x_{max} + k\pi] \times [-1, 1] \times S^1$ for $k = 0, \dots, n-1$;*
- *the Reeb periodic orbits are the curves δ_k for $k = 0, \dots, n-1$, these orbits are even and hyperbolic;*
- *the set of Reeb chords of γ_j is $\{c_{l,j}, l \in \mathbb{N}^*\}$ and $[\overline{c_{l,j}}] = \left[\{*\} \times S^1\right]^{(-1)^{k_j} l}$.*

Proof of Proposition 5.2. In U_k, consider the contact form

$$\alpha_b = f(x)\mathrm{d}y + (1 + k(x)l(y)m(z))\cos(x)\mathrm{d}z$$

such that $x \mapsto k(x - k\pi)$ satisfies condition (C9), l satisfies condition (C10) and $m = 0$ in $Z = \bigcup Z_j$ and $m = 1$ outside a neighbourhood of Z. By Propositions 4.2 and 4.3, we obtain the desired conditions. □

Let $\sigma_+ = \{k_j, k_j \text{ even}\}$ and $\sigma_- = \{k_j, k_j \text{ odd}\}$. As in Corollary 4.4 we apply Theorem 2.1 and Proposition 2.7 to deduce the Reeb periodic orbits after the bypass attachments along the curves $(\gamma_j)_{j=1,\dots,m}$.

Proposition 5.3. *Fix $L > 0$. Let (M', α') be the contact manifold obtained from $(M_{n,\eta}, \alpha_b)$ after bypass attachments along the arcs $(\gamma_j)_{j=1,\dots,m}$ for K large enough.*

- *For all $1 \le l \le L$, the Reeb periodic orbits homotopic to $\left[S^1\right]^l$ are the curves δ^l_{2k} for $1 \le 2k \le n-1$ and the periodic orbits $\gamma_{\mathbf{a}}$ associated to $\mathbf{a} = c_{i_1,j} \dots c_{i_k,j}$ with $i_1 + \dots + i_k = l$ and $k_j \in \sigma_+$.*
- *For all $-L \le l \le -1$, the Reeb periodic orbits homotopic to $\left[S^1\right]^l$ are the curves δ^l_{2k+1} for $1 \le 2k+1 \le n-1$ and the periodic orbits $\gamma_{\mathbf{a}}$ associated to $\mathbf{a} = c_{i_1,j} \dots c_{i_k,j}$ with $i_1 + \dots + i_k = -l$ and $k_j \in \sigma_-$.*

We denote by $\mathcal{B}_j$ the bypass attached to the attaching arc γ_j.

5.2. **Holomorphic cylinders.** It remains to control the holomorphic cylinders in the symplectisation of the contact manifold (M', α') given in Proposition 5.3. Let E^l_j be the $\mathbb{Q}$-vector space generated by the periodic orbit δ_{k_j} and by the periodic orbits obtained after the bypass attachment along γ_j homotopic to $\left[S^1\right]^l$. Let E^l_+ and E^l_- be the $\mathbb{Q}$-vector spaces generated by the periodic orbits δ^l_{2k} for $2k \notin \sigma_+$ and δ^l_{2k+1} for $2k+1 \notin \sigma_-$. The complex of contact homology is written

$$C_*^{[S^1]^l}(M', \alpha') = \bigoplus_{k_j \in \sigma_+} E^l_j \oplus E^l_+ \text{ if } l > 0,$$

$$C_*^{[S^1]^l}(M', \alpha') = \bigoplus_{k_j \in \sigma_-} E^l_j \oplus E^l_- \text{ if } l < 0.$$

Let $I^j_b = \left[k_j\pi - \frac{7\pi}{4}, k_j\pi + \frac{7\pi}{4}\right]$. Consider $I^j_{\max}$ such that

$$S_{Z_j} = [-x_{\max} + k_j\pi, x_{\max} + k_j\pi] \times \{1\} \times I^j_{\max}.$$

Lemma 5.4. *For any adapted almost complex structure on the symplectisation of (M', α')*

- *$\partial_{|E^l_\pm} = 0$;*

- *for all $j = 1, \dots, m$, we have $\partial(E_j^l) \subset E_j^l$ and any holomorphic cylinder with finite energy and asymptotics in E_j is contained in $\mathcal{U}_j \cup \mathcal{B}_j$ (see Figure 9) where*

$$\mathcal{U}_j = U_{k_j-1} \cup U_{k_j} \cup U_{k_j+1} \cup \big(I_b^j \times [-1,1] \times I_{max}^j\big).$$

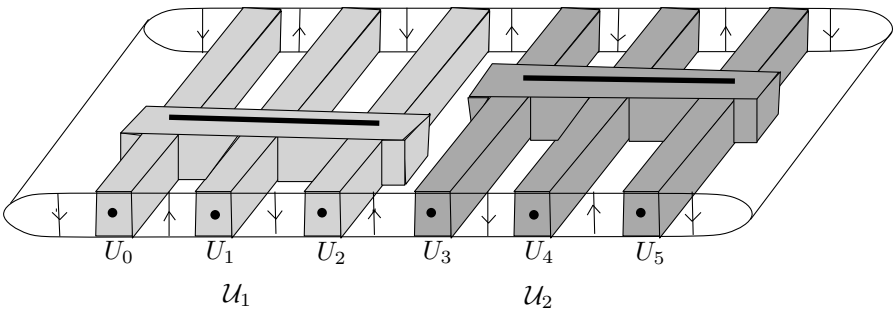

FIGURE 9. Neighbourhoods $\mathcal{U}_j$

Proof. Let

$$\mathcal{U} = \bigcup_{j=1\dots n} U_{k_j} \cup \bigcup_{j\in\sigma_\pm} \big(I_b^j \times [-1,1] \times I_{\max}^j\big) \cup \mathcal{B}_j,$$
$$\mathcal{W} = \big([-\pi+\eta, n\pi-\eta] \times [-1,1] \times S^1\big) \setminus \mathcal{U}.$$

The connected components of $\mathcal{U}$ are the sets U_l for $l \notin \sigma_\pm \cup (\sigma_\pm - 1) \cup (\sigma_\pm + 1)$ and $\mathcal{U}_j$ for $j = 1, \dots, m$. In $\mathcal{W}$, all Reeb orbits are Reeb chords joining the planes $y = 1$ and $y = -1$. Let γ be a Reeb periodic orbit (see Proposition 5.3). If $\gamma = \delta_j^l$ let $U_\gamma = U_j$. If γ is derived from the bypass attachment along γ_j, let

$$U_\gamma = U_j \cup \big(I_b^j \times \{-\varepsilon_j\} \times I_{\max}^j\big) \cup \mathcal{B}_j.$$

In both cases $\gamma \subset U_\gamma$. Let γ_+ and γ_- be two homotopic Reeb periodic orbits and $X = \mathcal{W} \setminus \big(U_{\gamma_+} \cup U_{\gamma_-}\big)$. As γ_+ and γ_- are homotopic and $k_{j+1} - k_j \geq 3$, for any Reeb chord c in X there exists a path of properly embedded arcs in X connecting c and a Reeb chord in $\mathrm{int}(X)$ with reversed orientation. Thus, by positivity of intersection (Proposition 3.8), all holomorphic cylinders are contained in $\mathcal{U}$. □

Let $(M, \xi_0 = \ker(\alpha))$ be the contact structure obtained from $(M_{4,\eta}, \alpha_b)$ after a bypass attachment along $\gamma_1 = [0, 2\pi] \times \{1\} \times \{\pi\}$ (Theorem 2.1). Let Γ an adapted dividing set of the boundary. The dividing set of a convex meridian disc contains exactly three connected components which are parallel to the boundary (Proposition 3.4). By Golovko's result (Theorem 3.6), the contact homology of (M, α, Γ) is generated by two periodic orbits homotopic to S^1 and by their multiples. We now compare this result with our construction and obtain useful properties of $\partial_{E_1^l}$. There are two periodic orbits of $R_{\alpha'}$ homotopic to S^1 (Proposition 5.3). If $l > 0$, we have $C_*^{[S^1]^l}(M', \alpha') = E_+^l$ and $\partial_{E_+^l} = 0$ (Lemma 5.4). If $l < 0$, we have $C_*^{[S^1]^l}(M', \alpha') = E_1^l$. Thus $\ker(\partial_{E_1^l})/\mathrm{im}(\partial_{E_1^l}) = 0$. Let $\overline{\mathcal{U}}$ denote the set $\mathcal{U}_1$ given by Lemma 5.4.

Proof of Theorem 2.5. Let ξ be a contact structure on $M = D^2 \times S^1$ satisfying conditions (C4) and (C5). We choose the contact form given by Proposition 5.2.

The neighbourhoods $\mathcal{U}_j$ described in Lemma 5.4 are contactomorphic to $\overline{\mathcal{U}}$. Thus $\ker(\partial_{E_j^l})/\mathrm{im}(\partial_{E_j^l}) = 0$. Therefore $HC^{[S^1]^{\pm l}}(M, \xi_0, \Gamma) = E_\pm^l$ for $l > 0$. As

$$\dim(E_\pm^l) + \#\{\sigma_\pm\} + \#\{\pm \text{ extremal bigon}\} = \chi(S_\pm) + \#\{\sigma_\mp\},$$

we obtain the desired dimension. □

We now turn to the case of a thickened convex surface. Let S be a convex surface and $\Gamma = \bigcup_{i=0}^n \Gamma_i$ be a dividing set of S without contractible components. Let γ_0 be an attachment arc in S intersecting three distinct components of Γ. Let $M = S \times [-1, 1]$ be the product neighbourhood of S and ξ be the associated invariant contact structure. Choose a contact form α_b of ξ given in Section 4. We denote by (M', ξ') the contact manifold obtained from (M, ξ) after a bypass attachment along $\gamma_0 \times \{1\}$ and by Γ' a dividing set of $\partial M'$. Fix $K > 0$ and apply Theorem 2.1 to obtain a contact form α' of ξ'. Let $\mathcal{B}$ denote the attached bypass. The proof of the following lemma is similar to the proof of Lemma 5.4.

Lemma 5.5. *For any adapted almost complex structure, the J-holomorphic curves in the symplectisation of (M', α') are contained in (see Figure 10)*

$$\mathcal{U} = \bigcup_{i=0}^n U_i \cup (I_b \times [-1, 1] \times I_{max}) \cup \mathcal{B}.$$

In addition, $\mathcal{U}$ is contactomorphic to $\overline{\mathcal{U}}$.

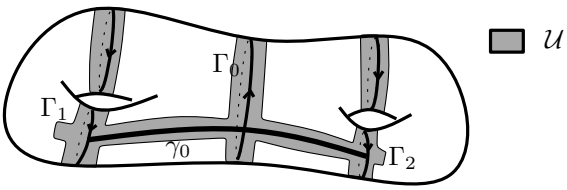

FIGURE 10. The neighbourhood $\mathcal{U}$ projected on S

Proof of Theorem 2.4. Theorem 2.4 is a corollary of Lemma 5.5. The proof is similar to the proof of Theorem 2.5. □

6. Sketch of proof of the bypass attachment theorem

We now sketch the proof of our main theorem (Theorem 2.1). A complete proof is given in Section 7. Fix $K > 0$. Let $(M, \xi = \ker(\alpha))$ be a contact manifold with convex boundary (S, Γ) and γ_0 be an attaching arc satisfying condition (C1), (C2) and (C3). To describe the Reeb periodic orbits after a bypass attachment, we study the maps φ_B and ψ_M induced on S_Z by the Reeb flow in the bypass and in M. Their domains and ranges consist of rectangles and these maps contract or expand the associated fibres. The maximal invariant set of the composite function is hyperbolic and this function is similar to a "generalised horseshoe" (see [29, 36]). The Reeb periodic orbits correspond to the periodic points and are given by the symbolic dynamics.

In Section 6.1 we present our notations. We describe the Reeb dynamics in M in Section 6.2 and in the bypass in Section 6.3. In Section 6.4, we prove that these dynamic properties indeed give the symbolic description of Reeb periodic orbits. Finally, in Section 6.5, we sketch the construction of a *hyperbolic bypass*: a bypass with Reeb dynamics described in Section 6.3.

6.1. **Notations.** We say that a (partial) function $\varphi : X \to Y$ is *decomposed* into $(\varphi_i)_{i\in I}$ if $\mathrm{dom}(\varphi_i)$ is a union of connected components of $\mathrm{dom}(\varphi)$, $(\mathrm{dom}(\varphi_i))_{i\in I}$ is a partition of $\mathrm{dom}(\varphi)$ and $\varphi_i = \varphi_{|\mathrm{dom}(\varphi_i)}$.

In coordinates (x, y, z), let S_{y_0} denote the plane $y = y_0$, $X^{\leq y_0} = X \cap \{(x, y, z), y \leq y_0\}$, $X^{\geq y_0} = X \cap \{(x, y, z), y \geq y_0\}$ and $X^{[y_0,y_1]} = X^{\leq y_1} \cap X^{\geq y_0}$. For all $0 < \lambda < \frac{\pi}{8}$, we consider the following subsets of S_Z (see Figure 11)

$$R_\lambda = \left(\bigcup_{k=-1}^{4} \left[\frac{k\pi}{2} + \lambda, \frac{(k+1)\pi}{2} - \lambda \right] \right) \times I_{\max},$$

$$Q_\lambda = \left(\bigcup_{k=0}^{2} \left[k\pi - \frac{\lambda}{2}, k\pi + \frac{\lambda}{2} \right] \right) \times I_{\max},$$

$$X = \left(\left[0, \frac{\pi}{4}\right] \times [-z_{\max}, 0)) \cup \left(\left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \times I_{\max} \right) \cup \left(\left[\frac{3\pi}{4}, \pi\right] \times (0, z_{\max}] \right),$$

$$Y = \left(\left[\pi, \frac{5\pi}{4}\right] \times [-z_{\max}, 0)) \cup \left(\left[\frac{5\pi}{4}, \frac{7\pi}{4}\right] \times I_{\max} \right) \cup \left(\left[\frac{7\pi}{4}, 2\pi\right] \times (0, z_{\max}] \right).$$

For positive z_{prod} and y_{std}, let $I_{\mathrm{prod}} = [-z_{\mathrm{prod}}, z_{\mathrm{prod}}]$, $S_R = \left[\frac{\pi}{2}, \pi\right] \times \{y_{\mathrm{std}}\} \times I_{\mathrm{prod}}$

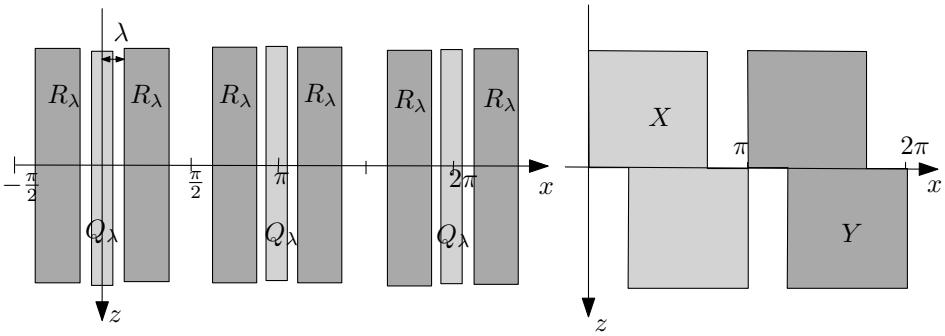

FIGURE 11. The subsets R_λ, Q_λ, X and Y

and $S_{\varepsilon,k} = \left[\frac{k\pi}{2} - \varepsilon, \frac{k\pi}{2} + \varepsilon\right] \times I_{\max}$. A *rectangle* is a closed set diffeomorphic to $[0, 1] \times [0, 1]$. This set inherits horizontal and vertical fibres from $[0, 1] \times [0, 1]$.

In $(\mathbb{R}^2, \langle\cdot,\cdot\rangle)$, let D be a straight line and $\nu > 0$. The *ν-cone centred at D* is the set

$$\mathcal{C}(D, \nu) = \left\{ w \in \mathbb{R}^2, |\langle w, v\rangle| < \nu |\langle w, u\rangle| \right\}$$

where u is tangent to D and (u, v) is an orthonormal basis. We denote by H and V the horizontal and vertical axes. Let U and V be two open sets in $\mathbb{R}^2$ and $f : U \to V$ be a diffeomorphism. The image of a cone field $\mathcal{C}$ on U is the cone field $f_*\mathcal{C}$ on V defined by $(f_*\mathcal{C})_p = \mathrm{d}f_{f^{-1}(p)} \left(\mathcal{C}_{f^{-1}(p)}\right)$. If $\mathcal{C}$ and $\mathcal{C}'$ are two cones fields on U we write $\mathcal{C} \subset \mathcal{C}'$ if $\mathcal{C}_p \subset \mathcal{C}'_p$ for all $p \in U$. If $z \to \gamma(z)$ is a smooth curve in $\mathbb{R}^2$, let $\mathcal{C}_{x,z}(\gamma, \varepsilon) = C(\gamma'(z), \varepsilon)$ and $\mathcal{C}_{x,z}(\gamma^\perp, \varepsilon) = C(\gamma'(z)^\perp, \varepsilon)$.

6.2. **Reeb dynamics in M.** We now study the Reeb dynamics is the manifold M with boundary. To attach an adapted bypass we will perturb the contact form α. We want to control the map ψ_M induced by the Reeb flow in M for times smaller than K and for the contact form α and perturbations of α. The Reeb chords of S_Z that contribute to ψ_M for the contact form α are close to the Reeb chords of γ_0. Nevertheless, as $\Gamma \cap Z$ is contained in Reeb orbits, this decomposition is not stable by perturbation and some Reeb chords may appear near the dividing set.

Let $\lambda > 0$, the pair (S_Z, λ) is said to be *K-hyperbolic* if ψ_M can be decomposed into $(\psi_j)_{j=0,\dots,N}$ and (see Figure 12):

(1) $\mathrm{dom}(\psi_0) \subset Q_\lambda$ and $\mathrm{im}(\psi_0) \subset Q_\lambda$;
(2) if $x \in \left[k\pi - \frac{\lambda}{2}, k\pi + \frac{\lambda}{2}\right]$ and $(x, z) \in \mathrm{dom}(\psi_0)$ then
   - $(\psi_0)_x(x, z) \in \left[k\pi - \frac{\lambda}{2}, k\pi + \frac{\lambda}{2}\right]$;
   - $(\psi_0)_z(x, z) < z$ if k is odd;
   - $(\psi_0)_z(x, z) > z$ if k is even;
(3) for all $j \in [\![1, N]\!]$, $\mathrm{dom}(\psi_j)$ and $\mathrm{im}(\psi_j)$ are rectangles in R_λ with horizontal fibres and the ψ_j reverse the fibres.

Note that one can have $\mathrm{dom}(\psi_0) = \emptyset$ or $N = 0$. Let μ, ν and τ be real positive

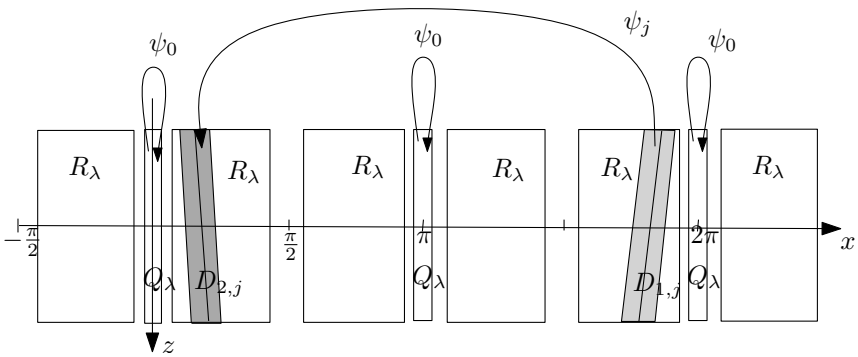

FIGURE 12. A K-hyperbolic surface

numbers. A K-hyperbolic surface is *dominated* by $\omega = (\mu, \nu, \tau)$ if for $j = 1, \dots, N$ there exist segments $D_{1,j} \subset \mathrm{dom}(\psi_j)$ and $D_{2,j} \subset \mathrm{im}(\psi_j)$ with boundary on $z = \pm z_{\max}$ such that (see Figure 12):

(1) the tangents of the vertical fibres of $\mathrm{dom}(\psi_j)$ and $\mathrm{im}(\psi_j)$ are respectively in $\mathcal{C}(D_{1,j}, \nu)$ and $\mathcal{C}(D_{2,j}, \nu)$;
(2) $(\psi_j)_* \mathcal{C}(H, \nu) \subset \mathcal{C}(D_{2,j}, \mu)$ and $\left(\psi_j^{-1}\right)_* \mathcal{C}(H, \nu) \subset \mathcal{C}(D_{1,j}, \mu)$;
(3) the return time of ψ_j is contained in $(T(a_j) - \tau, T(a_j) + \tau)$.

Let $\varepsilon > 0$. A K-hyperbolic (ω-dominated) surface (S_γ, λ) is *ε-stable* if for all ε-perturbation of α preserving γ_0, (S_γ, λ) remains K-hyperbolic (and ω-dominated).

Proposition 6.1. *Let $\tau > 0$ and $\mu > 0$. There exists a contact form arbitrarily close to α, z_{max} small and some real positive numbers ν, λ and ε such that*

- *α satisfies conditions (C1), (C2) and (C3);*
- *(S_Z, λ) is K-hyperbolic (μ, ν, τ)-dominated and ε-stable.*

Proof. After a small perturbation of α, we can assume that the images of $\gamma_0 \setminus \Gamma$ on S_Z by the Reeb flow and the opposite of the Reeb flow for times smaller that K are transverse to γ_0. The intersection points correspond to the endpoints of the Reeb chords $a_1, \dots, a_N$. For $z_{\max}$ small enough, the domain and range of ψ_M are contained in a small neighbourhood of the endpoints of the Reeb chords. We choose $\mathrm{dom}(\psi_0) = \emptyset$. Let $D_{1,j}$ and $D_{2,j}$ be the tangent to the image of $\gamma_0 \setminus \Gamma$ on S_Z at the endpoints of the Reeb chords $a_1, \dots, a_N$ (see Figure 13). We obtain the vertical fibres of $\mathrm{dom}(\psi_M)$ and $\mathrm{im}(\psi_M)$ as the inverse images and images of the horizontal segments in $\mathrm{im}(\psi_M)$ and $\mathrm{dom}(\psi_M)$. For small enough perturbations of α, the structure of ψ_M is preserved outside Q_λ. In $Q_\lambda \times [-y_{\max}, 0]$, the component $|R_z|$ is close to 1 and ψ_0 satisfies the desired conditions. $\square$

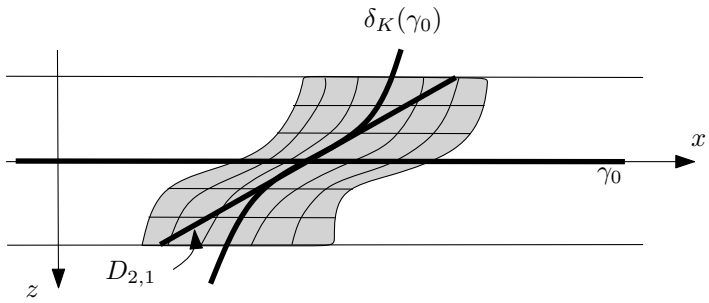

FIGURE 13. The segments $D_{2,1}$ and the rectangle structure of $\mathrm{im}(\psi_1)$

6.3. **Reeb dynamics in the bypass.** We now describe the desired dynamics in the bypass in terms of horizontal rectangles. A *hyperbolic bypass* in Z is a triple $(\mathcal{B}, \alpha_B, \lambda)$ where $(\mathcal{B}, \alpha_B)$ is a contact manifold in Z and λ is a real positive number such that

(1) $\mathcal{B}^{\leq 0} = Z^{\leq 0}$ and α_B is adapted to the boundary in $Z^{\geq 0}$;
(2) the map φ_B induced on S_Z by the Reeb flow in $\mathcal{B}$ can be decomposed into maps φ_0 and φ_1 such that
   - $\mathrm{dom}(\varphi_0) \subset Q_\lambda$ and $\mathrm{im}(\varphi_0) \subset Q_\lambda$;
   - $\mathrm{dom}(\varphi_1) \subset X$ and $\mathrm{im}(\varphi_1) \subset Y$;
   - if $x \in \left[k\pi - \frac{\lambda}{2}, k\pi + \frac{\lambda}{2}\right]$ and $(x, z) \in \mathrm{dom}(\varphi_0)$ then
     - $(\varphi_0)_x(x, z) \in \left[k\pi - \frac{\lambda}{2}, k\pi + \frac{\lambda}{2}\right]$;
     - $(\varphi_0)_z(x, z) < z$ if k is odd;
     - $(\varphi_0)_z(x, z) > z$ if k is even;
(3) the restriction of φ_1 to R_λ can be decomposed into $(\varphi_{i,j})_{i,j\in\{0,1\}}$ (see Figure 14) where $\mathrm{dom}(\varphi_{i,j})$ and $\mathrm{im}(\varphi_{i,j})$ are rectangle as large as R_λ with vertical fibres and the $\varphi_{i,j}$ reverse the fibres.

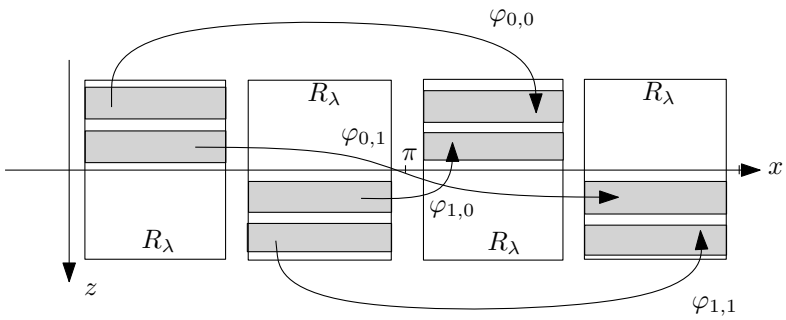

FIGURE 14. The rectangles $\mathrm{dom}(\varphi_{i,j})$ and $\mathrm{im}(\varphi_{i,j})$

As in the previous section we want to control the return time and obtain some cone-preservation properties. A hyperbolic bypass $(\mathcal{B}, \alpha_B, \lambda)$ is *dominated* by $\omega_B = (\nu, \tau, A, \eta)$ if

(1) $(\varphi_{i,j})_* \mathcal{C}(V, A) \subset \mathcal{C}(H, \nu)$ and $\left(\varphi_{i,j}^{-1}\right)_* \mathcal{C}(V, A) \subset \mathcal{C}(H, \nu)$;
(2) $\|\mathrm{d}\varphi_{i,j}(p, v)\| > \frac{1}{\eta}\|v\|$ for all $p \in \mathrm{dom}(\varphi_{i,j})$ and $v \in \mathcal{C}_p(V, A)$;
(3) $\|\mathrm{d}\varphi_{i,j}^{-1}(p, v)\| > \frac{1}{\eta}\|v\|$ for all $p \in \mathrm{im}(\varphi_{i,j})$ and $v \in \mathcal{C}_p(V, A)$;
(4) the return time in R_λ is bounded by 8τ.

The following theorem is the main ingredient in the proof of Theorem 2.1.

Theorem 6.2. *Fix $K > 0$. Let $(M, \xi = \ker(\alpha))$ be a contact manifold with convex boundary (S, Γ) and γ_0 be an attaching arc satisfying conditions (C1), (C2) and (C3). For all real positive numbers ν, τ, A, η, ε and λ and for z_{max} small enough there exists a hyperbolic bypass (V_B, α_B, λ) obtained from (V, ξ_B) after a bypass attachment along γ_0, dominated by (ν, τ, A, η) and such that α_B is ε-close to α.*

If (S_Z, λ) is K-hyperbolic, ε-stable and (μ, ν, τ)-dominated, we call the bypass attachment described in Theorem 6.2 a *K-hyperbolic bypass attachment.*

6.4. **Reeb periodic orbits after a bypass attachment.** Before turning to the proof of Theorem 6.2, we prove that Theorem 6.2 and Proposition 6.1 imply Theorem 2.1 and Proposition 2.7. We prove this result in two steps:

(1) we obtain a symbolic representation of the Reeb flow in restriction to R_λ;
(2) we prove that all new Reeb periodic orbits intersect R_λ.

Fix $K > 0$. Let $(M, \xi = \ker(\alpha))$ be a contact manifold with convex boundary (S, Γ) and γ_0 be an attaching arc satisfying conditions (C1), (C2) and (C3). We assume $K \neq l(\mathbf{a})$ for all words $\mathbf{a}$ on the letters $a_1, \dots, a_N$. Thus, there exists ε_K such that $|K - l(\mathbf{a})| > \varepsilon_K$ for all words $\mathbf{a}$. Let l_0 be such that $l(a_i) > l_0$ for all $i = 1, \dots, N$. There exists ε_0 such that these estimations remain satisfied for all $2\varepsilon_0$-perturbations of α. Without loss of generality $l_0 < 1$ and $\frac{\varepsilon_K}{K} < 1$. Let $\tau < \frac{l_0}{18K}\varepsilon_K$. We apply Proposition 6.1 to obtain a K-hyperbolic surface, (μ, ν, τ)-dominated and ε-stable. Without loss of generality $\varepsilon < \varepsilon_0$, $\nu \leq \mu$ and $\mathcal{C}(D_{i,j}, \mu) \cap \mathcal{C}(H, \mu) = \{0\}$ for $i = 1, 2$ and $j = 1, \dots, N$. Choose $A > 0$ such that $\mathcal{C}(D_{i,j}, \mu) \subset \mathcal{C}(V, A)$ and $\mathcal{C}(V, A) \cap \mathcal{C}(H, \mu) = \{0\}$ for $i = 1, 2$ and $j = 1, \dots, N$ and for all contact forms ε-close to α. Choose $M > 0$ such that $\|\mathrm{d}\psi_j\| < M$ and $\|\mathrm{d}\psi_j^{-1}\| < M$ for $j = 1, \dots, N$ and for all ε-perturbations of α. Let $\eta < \frac{1}{3M}$. Apply Theorem 6.2, to obtain a hyperbolic bypass (M_B, α_B, λ) dominated by (ν, τ, A, η) and such that α_B is ε-close to α.

To obtain a symbolic representation of the new Reeb periodic orbits, we apply a fixed point theorem in hyperbolic situations. The following proposition derives from [36, Theorem 3.2].

Proposition 6.3. *Let R and R' be two rectangles in $[0, 1] \times [0, 1]$ such that the vertical boundaries of R are contained in $\{0, 1\} \times [0, 1]$ and the horizontal boundaries of R' are contained in $[0, 1] \times \{0, 1\}$. Let $F : R \to R'$ be a diffeomorphism such that, for some $A > 0$, $\nu > 0$ and $a > 2$*

- *$F_*\mathcal{C}(V, A) \subset \mathcal{C}(V, A)$ and $F_*^{-1}\mathcal{C}(H, \nu) \subset \mathcal{C}(H, \nu)$;*
- *$\|\mathrm{d}F^{-1}(p, v)\| \geq a\|v\|$ for all $p \in R'$ and $v \in \mathcal{C}_p(H, \nu)$;*
- *$\|\mathrm{d}F(p, v)\| \geq a\|v\|$ for all $p \in R$ and $v \in \mathcal{C}_p(V, A)$.*

Then F has a unique fixed point.

Proposition 6.4. *Let $\mathbf{a} = a_{i_1} \dots a_{i_k}$ and $F_\mathbf{a} = \psi_{i_k} \circ \varphi_B \dots \psi_{i_1} \circ \varphi_B$ in restriction to R_λ. The map $F_\mathbf{a}$ has a unique fixed point. The period $T(\gamma_\mathbf{a})$ of the associated Reeb periodic orbit $\gamma_\mathbf{a}$ satisfies $T(\gamma_\mathbf{a}) \in [l(\mathbf{a}) - 9k\tau, l(\mathbf{a}) + 9k\tau]$.*

Proof. By induction on k, the map $F_\mathbf{a}$ can be decomposed into $F_0^\mathbf{a}$ and $F_1^\mathbf{a}$ such that (see Figure 15) :

- $\mathrm{im}(F_i^\mathbf{a})$ are rectangles as high as R_λ with horizontal fibres;
- $\mathrm{dom}(F_i^\mathbf{a})$ are rectangles with vertical fibres contained in two different components of R_λ and as large as the associated component;
- $F_{\mathbf{a}*}\mathcal{C}(V, A) \subset \mathcal{C}(D_{2,i_k}, \mu)$ and ${F_\mathbf{a}^{-1}}_*\mathcal{C}(H, \nu) \subset \mathcal{C}(H, \nu)$;
- $\|\mathrm{d}F_\mathbf{a}^{-1}(p, v)\| \geq \frac{1}{(\eta M)^k}\|v\|$ for all $p \in \mathrm{im}(F_\mathbf{a})$ and $v \in \mathcal{C}_p(H, \nu)$;
- $\|\mathrm{d}F_\mathbf{a}(p, v)\| \geq \frac{1}{(\eta M)^k}\|v\|$ for all $p \in \mathrm{dom}(F_\mathbf{a})$ and $v \in \mathcal{C}_p(V, A)$.

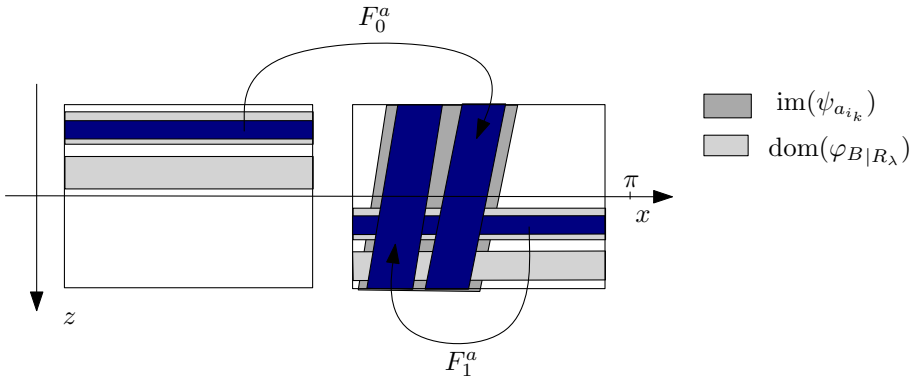

FIGURE 15. The maps $F_0^{\mathbf{a}}$ and $F_1^{\mathbf{a}}$

To obtain a unique fixed point, we apply Proposition 6.3 to the component of $F_{\mathbf{a}}$ such that $\mathrm{dom}(F_i^{\mathbf{a}}) \cap \mathrm{im}(F_i^{\mathbf{a}}) \neq \emptyset$. The estimates on the period of the associated Reeb periodic orbit derives from the estimates on the return time. □

We now turn to the second step of the proof.

Proposition 6.5. *Let γ be a Reeb periodic orbit intersecting S_Z in p_γ and such that $T(\gamma) < K$. Then $p_\gamma \notin Q_\lambda$.*

Proof. We control the Reeb orbits intersecting Q_λ and prove that they are not periodic. Let $X_k = \left[k\pi - \frac{\lambda}{2}, k\pi + \frac{\lambda}{2}\right]$ and $p_0 = q_0 \in \mathrm{im}(\psi_0)$. As long as these expressions are well-defined, let $p_{2l+1} = {\psi_M}^{-1}(p_{2l})$, $p_{2l+2} = {\varphi_B}^{-1}(p_{2l+1})$, $q_{2l+1} = \varphi_B(p_{2l})$ and $q_{2l+2} = \psi_M(q_{2l+1})$. Write $p_l = (x_l, y_l)$ and $q_l = (x'_l, y'_l)$. There exists k such that $x_0 \in X_k$. The following implications hold.

- If k is odd and $z_0 \geq 0$, then $x_l \in X_k$ and $z_{2l} < z_{2l+1} < z_{2l+2}$.
- If k is even and $z_0 \leq 0$, then $x_l \in X_k$ and $z_{2l} > z_{2l+1} > z_{2l+2}$.
- If k is even and $z_0 \geq 0$, then $x'_l \in X_k$ and $z'_{2l} < z'_{2l+1} < z'_{2l+2}$.
- If k is odd and $z_0 \leq 0$, then $x'_l \in X_k$ and $z'_{2l} > z'_{2l+1} > z'_{2l+2}$.

We give a detailed proof in the case k odd and $z_0 \geq 0$. The proof of the other cases is similar. We prove the result by induction. If $x_{2l} \in X_k$, $z_{2l} \geq 0$ and p_{2l+1} is well-defined then $p_{2l} = \psi_M(p_{2l+1})$ and $p_{2l} \in \bigcup_j \mathrm{im}(\psi_i)$. As $p_{2l} \in Q_\lambda$, we have $p_{2l} \in \mathrm{im}(\psi_0)$. Therefore, $p_{2l} = \psi_0(p_{2l+1})$ and $p_{2l+1} \in \mathrm{dom}(\psi_0)$. Thus, we obtain $x_{2l+1} \in X_k$ and $z_{2l+1} > z_{2l}$. If $x_{2l+1} \in X_k$, $z_{2l+1} \geq 0$ and p_{2l+2} is well-defined, then $p_{2l+1} = \varphi_B(p_{2l+2})$ and $p_{2l+1} \in Y \cup \mathrm{im}(\varphi_0)$. As k is odd, we obtain $z_{2l+1} \geq 0$ and $p_{2l+1} \in Q_\lambda$ and therefore $p_{2l+1} \in \mathrm{im}(\varphi_0)$. Thus, we have $p_{2l+1} = \varphi_0(p_{2l+2})$ and $p_{2l+2} \in \mathrm{dom}(\varphi_0)$. Consequently, $x_{2l+2} \in X_k$ and $z_{2l+2} > z_{2l+1}$.

Let γ be a Reeb periodic orbits intersecting S_Z in $p_\gamma \in Q_\lambda$ and such that $T(\gamma) < K$. If $p_\gamma \in S_-$, then $\varphi_B(p_\gamma) \in S_+ \cap Q_\lambda$. Thus, without loss of generality, we can assume $p_\gamma \in S_+$. Therefore $p_\gamma \in \mathrm{dom}(\varphi_B) \cap \mathrm{im}(\psi_0)$ and $x_\gamma \in X_k$. If k is odd and $z_\gamma \geq 0$, then p_l is well defined for all $l \in \mathbb{N}$, z_l in increasing and γ is not periodic. This leads to a contradiction. The proof of the other cases is similar. □

Proof of Theorem 2.1. Let $\mathbf{a} = a_{i_1} \dots a_{i_k}$ be a word such that $l(\mathbf{a}) < K$. By definition of l_0, we have $k \leq \frac{K}{l_0}$. Thus $T(\gamma_{\mathbf{a}}) \in \left[l(\mathbf{a}) - \frac{\varepsilon_K}{2}, l(\mathbf{a}) + \frac{\varepsilon_K}{2}\right]$ and $T(\gamma_{\mathbf{a}}) < K$ (Proposition 6.4).

Conversely, let γ be a Reeb periodic orbit intersecting S_Z and such that $T(\gamma) < K$. Let $p_1, \dots, p_k$ denote its successive intersection points with S_+ and $q_1, \dots, q_k$ its successive intersection points with S_-. By Proposition 6.5, for all $j = 1 \dots k$

there exists i_j such that $q_j \in \mathrm{dom}(\psi_{i_j})$ and $p_{j+1} = \psi_{i_j}(q_j)$. Let $\mathbf{a} = a_{i_1} \dots a_{i_k}$. Then p_0 is the fixed point of $F_{\mathbf{a}}$. Then $l(\mathbf{a}) < K + 9k\tau$ (Proposition 6.4). Thus $k \leq \frac{K+9k\tau}{l_0}$ and $k \leq \frac{2K}{l_0}$. Therefore $l(\mathbf{a}) < K + \varepsilon_K$ and $l(\mathbf{a}) < K$. □

Proof of Proposition 2.7. There exists ε such that for any ε-perturbation of α, the map ψ_M can be decomposed into ψ_0 and ψ_1 such that ψ_0 has properties similar to those described in the definition of K-hyperbolic surface, $\mathrm{dom}(\psi_1) \subset R_{\lambda_0}$ and $\mathrm{im}(\psi_1) \subset R_{\lambda_0}$. Apply Theorem 6.2 for $\lambda = \lambda_0$ and any ν, τ, A and η. As in Proposition 6.5, if γ is a Reeb periodic orbits intersecting S_Z in p_γ, then $p_\gamma \notin Q_\lambda$. Thus any Reeb periodic orbit intersects S_- at a periodic point of $\varphi_B \circ \psi$. □

6.5. **Hyperbolic bypasses.** We now give an overview of the proof of Theorem 6.2. It is the main and last step in the proofs of Theorem 2.1 and Proposition 2.7. The complete proof is technical and is the subject of Section 7.

Honda's construction (see Sections 3.2 and 7.1) provides us with a bypass attachment (M', α') along the attaching arc γ_0 but α' is not adapted to the boundary. This attachment, if properly performed, does not create any Reeb periodic orbit. Indeed, near S_Z, the Reeb vector field is tangent to the planes $x = \mathrm{cst}$ and its slope is $\tan(x)$. Thus, all Reeb orbits intersecting S_Z outside a neighbourhood of $x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$ go out of the bypass (see Figure 16). Therefore the domain and

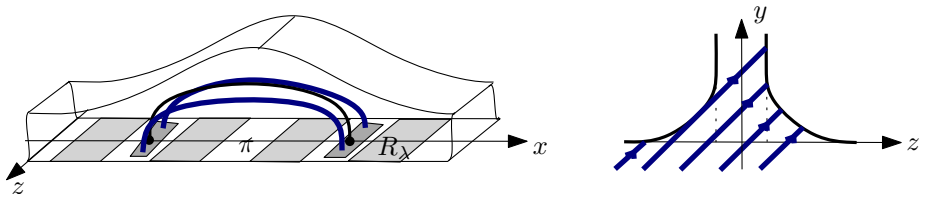

FIGURE 16. A non-convex bypass attachment

range of the map induced on S_Z by the Reeb flow in the bypass are contained in neighbourhoods of $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. By definition of a K-hyperbolic surface, there is no new Reeb periodic orbit.

To obtain a contact form adapted to the boundary, we use the *convexification* process described in [6]. It consists in gluing a small "bump" with prescribed contact form along the non-adapted part of the dividing set (see Figure 17). Inside this

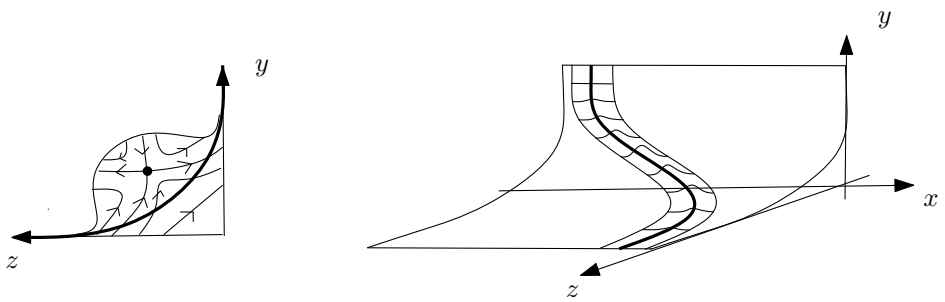

FIGURE 17. The convexification bump

bump the Reeb vector field is nearly tangent to the dividing set. The restriction of φ_B to R_λ is now non-empty. The path of the associated Reeb chords is the following

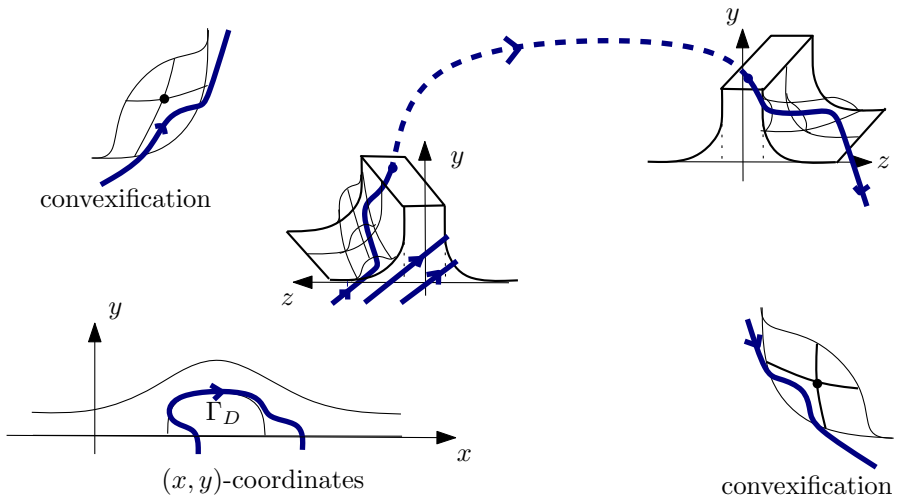

FIGURE 18. Reeb chords of R_λ

- they enter the convexification bump and follow the dividing set (see Figure 18 left);
- then, they reach the area in M' where the Reeb vector field is nearly tangent to the dividing set $\Gamma_D \times \{0\}$ and travel along $\Gamma_D \times \{0\}$ (see Figure 18 centre);
- finally, they go out of the bypass in a similar way and intersect R_λ again (see Figure 18 right).

To understand the Reeb dynamics and obtain cone-preserving properties, we describe the image of vertical curves on intermediate surfaces (see Figure 19). A

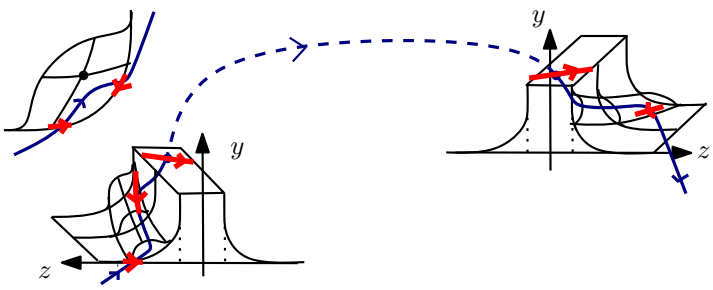

FIGURE 19. Images of vertical curves in the bypass

vertical curve is stretched into the convexification bump, then transported (and slightly stretched) in the upper part of the bypass. After a last visit to the convexification area, the curve becomes nearly horizontal. The effect on the level of rectangles is shown on Figure 20.

We now translate these intuitive pictures into more explicit conditions on the Reeb flow in the bypass. Section 7 is devoted to the construction of a bypass satisfying these conditions. Let $(M, \xi = \ker(\alpha))$ be a contact manifold with convex boundary (S, Γ) and γ_0 be an attaching arc satisfying conditions (C1), (C2) and (C3). We divide the bypass into two regions, the region $y \leq y_{\text{std}}$ where the contact form is standard and the convexification bump is added and the region $y \geq y_{\text{std}}$ where the contact structure corresponds to a thickened half overtwisted disc. We use the notations from Section 7.1. Fix real positive numbers K, ν, τ, A, η, ε

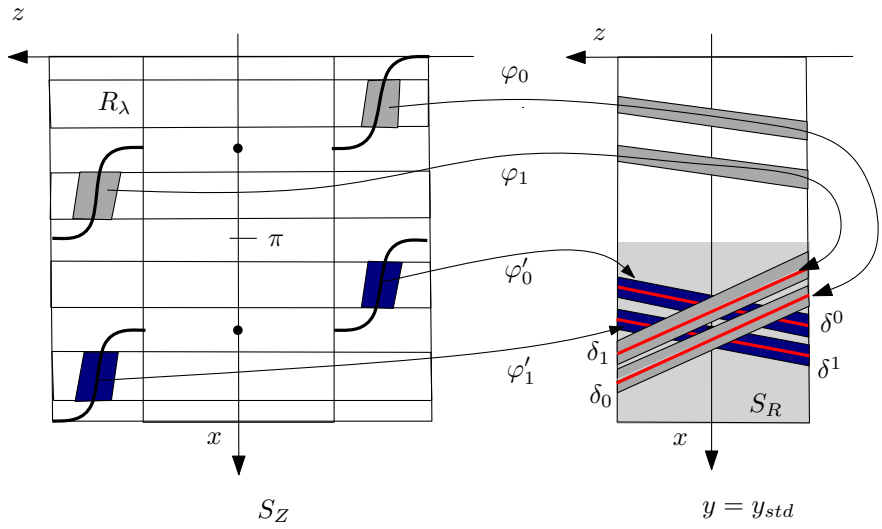

FIGURE 20. Rectangles on intermediate surfaces

and $\lambda < \frac{\pi}{8}$. Let $(\mathcal{B}, \alpha_B)$ be a bypass such that the boundary S_B is convex, α_B is adapted to S_B, α_B is arbitrarily close to α and the Reeb flow satisfies the following properties.

Reeb dynamics restricted to R_λ. (To reduce the number of compositions we consider the map induced by the Reeb flow between S_Z and S_R.)

(B1) There exist real positive numbers δ_1, ε_R and A_R, two graphs in z denoted δ_0 and δ_1 and a decomposition (φ_0, φ_1) of the map induced between S_Z and S_R for positive time such that
- $\mathrm{dom}(\varphi_i)$ are rectangles with vertical fibres and basis $[\frac{\pi}{2} + \lambda, \pi - \lambda]$ and $[\pi + \lambda, \frac{3\pi}{2} - \lambda]$ (see Figure 20);
- $\mathrm{im}(\varphi_i)$ are rectangles in an ε_R-neighbourhood of δ_i with horizontal fibres and basis I_{prod} (see Figure 20);
- φ_i preserves the fibres (see Figure 19);
- $(\varphi_i)_* \mathcal{C}(V, A) \subset \mathcal{C}(\delta_i, \varepsilon_R)$ and $\left(\varphi_i^{-1}\right)_* \mathcal{C}(\delta_i^\perp, A_R) \subset \mathcal{C}(H, \nu)$;
- $\|\mathrm{d}\varphi_i^{-1}(p, v)\| > \frac{1}{\sqrt{\eta}}\|v\|$ for all $p \in \mathrm{im}(\varphi_i)$ and $v \in \mathcal{C}_p(\delta_i^\perp, A_R)$;
- $\|\mathrm{d}\varphi_i(p, v)\| > \frac{1}{\sqrt{\eta}}\|v\|$ for all $p \in \mathrm{dom}(\varphi_i)$ and $v \in \mathcal{C}_p(V, A)$;

– the return time is bounded by 4τ.

(B2) The map induced between S_R and S_Z for positive times can be decomposed into φ'_0 and φ'_1 and there exist two graphs δ^0 and δ^1 satisfying similar properties.

(B3) $\mathcal{C}(\delta_i, \varepsilon_R) \subset \mathcal{C}({\delta^j}^\perp, A_R)$, $\mathcal{C}(\delta^i, \varepsilon_R) \subset \mathcal{C}(\delta_j^\perp, A_R)$ and δ and δ' intersect transversely in one point if $d_{\mathcal{C}^1}(\delta, \delta_i) < \varepsilon_R$ and $d_{\mathcal{C}^1}(\delta', \delta^j) < \varepsilon_R$.

Reeb dynamics in $\mathcal{B}^{\geq y_{std}}$.

(B4) The domain of the map induced by the Reeb flow on $S_{y_{\mathrm{std}}}$ is contained in $\left[\frac{\pi}{2} - \frac{\lambda}{2}, \frac{\pi}{2} + \frac{\lambda}{2}\right] \times I_{\mathrm{prod}}$ and its range in $\left[\frac{3\pi}{2} - \frac{\lambda}{2}, \frac{3\pi}{2} + \frac{\lambda}{2}\right] \times I_{\mathrm{prod}}$.

Reeb dynamics in $\mathcal{B}^{[0, y_{std}]}$.

(B5) There is no return map on $S_{y_{\mathrm{std}}}$.

(B6) The return map on S_Z can be decomposed into θ_0, θ_1 and θ_2 such that
- $\mathrm{dom}(\theta_k) \subset S_{\frac{\lambda}{2}, 2k}$ and $\mathrm{im}(\theta_k) \subset S_{\frac{\lambda}{2}, 2k}$;
- $(\theta_k)_z(x, z) < z$ if k is odd;

- $(\theta_k)_z(x,z) > z$ if k is even.

The Reeb chords which contribute to θ_k do not intersect a neighbourhood of $S_{y_{\mathrm{std}}}$.

(B7) The map induced between S_Z and $S_{y_{\mathrm{std}}}$ for positive times can be decomposed into two maps with domains in X and $X + 2\pi$ and ranges in $S_{\frac{\lambda}{2},1}$ and $S_{\frac{\lambda}{2},5}$.

(B8) The map induced between $S_{y_{\mathrm{std}}}$ and S_Z for positive times can be decomposed into two maps with domains in $S_{\frac{\lambda}{4},3}$ and $S_{\frac{\lambda}{4},-1}$ and ranges in Y and $Y - 2\pi$.

Proposition 6.6. *The bypass $(\mathcal{B}, \alpha_B, \lambda)$ is hyperbolic and (ν, τ, A, η)-dominated.*

Proof. The map φ_B can be decomposed into φ_0 and φ_1 where the Reeb chords which contribute to φ_0 do not intersect $S_{y_{\mathrm{std}}}$ and the Reeb chords which contribute to φ_1 intersect $S_{y_{\mathrm{std}}}$. The properties of φ_0 derive from condition (B6). By condition (B5), φ_1 is the the composite of the maps described in conditions (B7), (B4) and (B8). Thus $\mathrm{dom}(\varphi_1) \subset X$ and $\mathrm{im}(\varphi_1) \subset Y$. By conditions (B4) and (B5), the restriction of φ_1 to R_λ is the composite of the maps described in conditions (B1) and (B2). Conditions (B1) and (B3) ensures that the composite map is a map between rectangles. The hyperbolic properties derive from conditions (B1), (B2) and (B3). □

7. Hyperbolic bypasses

In this section we construct a bypass satisfying conditions (B1) to (B8) and thus end the proofs of Theorem 2.1 and Proposition 2.7. Fix some positive numbers $\lambda < \frac{\pi}{8}$, y_{std}, τ and $z_{\mathrm{prod}} < z_{\max}$. In what follows, we have $z_{\max} \ll 1$.

The construction of a hyperbolic bypass is technical. We start from an explicit contact form on a bypass inspired from Honda [27] (Section 7.1). In Section 7.2, we present some preliminary lemmas ensuring a precise control of the Reeb flow. Section 7.3 presents a preparatory perturbation of the contact structure in the bypass called a *pre-convex bypass*. This pre-convex bypass determines the curves δ_i and δ^j (condition (B1)). The actual construction begins in Section 7.4 with the description of adapted coordinates. In Section 7.5, we present the convexification contact form in the coordinates described in Section 7.4. In Section 7.6 we prove that our construction satisfies the desired conditions.

7.1. Explicit constructions of bypasses.

In this section we present an explicit construction of a bypass attachment. This construction is due to Honda [27] and is the first step of our explicit construction in the proof of Theorem 2.1. We construct a contact structure on the product of a smoothed half overtwisted disc and smoothen the product.

Let (M, α) be a contact manifold with convex boundary (S, Γ) and γ_0 be an attachment arc satisfying condition (C1). In coordinates $(x, y) \in I_b \times \mathbb{R}_+$, let $U_y = \left[-\frac{\pi}{6}, \frac{13\pi}{6}\right] \times [y, +\infty)$ and $\gamma_1 = I_b \times \{0\}$. Consider closed set $\mathcal{A}$ diffeomorphic to a square (see Figure 21) such that $\mathcal{A} = I_b \times [0, y_0]$ outside U_0 and

$$\partial\mathcal{A} = \gamma_1 \cup \left(\left\{-\frac{3\pi}{4}\right\} \times [0, y_0]\right) \cup \left(\left\{\frac{11\pi}{4}\right\} \times [0, y_0]\right) \cup \gamma_2.$$

Choose a 1-form β on $\mathcal{A}$ such that

(1) there exists $y_\beta > 2y_{\mathrm{std}}$ such that $\beta = \sin(x)\mathrm{d}y$ in $\mathcal{A} \setminus U_{2y_{\mathrm{std}}}$ and β is a positive multiple of $\sin(x)\mathrm{d}y$ for $y \leq y_\beta$;

(2) in $U_{y_\beta} \cap \mathcal{A}$ the singularities of β are
   - a half-elliptic negative singularity in (π, y_β);
   - two half-hyperbolic singularities in $(0, y_\beta)$ and $(2\pi, y_\beta)$;

- two positive elliptic singularities on $\partial\mathcal{A}$ for $x = 0$ and $x = 2\pi$;
- positive singularities on $\partial\mathcal{A} \cap U_{y_\beta}$;

(3) there exists a smooth proper multi-curve Γ_A dividing $\mathcal{A}$ into two subsurfaces $\mathcal{A}_\pm$ such that $\pm \mathrm{d}\beta > 0$ on $\mathcal{A}_\pm$, ∂A_+ is oriented as Γ_A and $\beta_{\Gamma_A} > 0$;

(4) if Γ_D is the component of Γ_A joining $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$, there exist coordinates $(r, \theta) \in \left[\frac{3\pi}{2} - \varepsilon, \frac{3\pi}{2} + \varepsilon\right] \times [0, \theta_{\max}] = \mathcal{U}$ near Γ_D such that $\Gamma_D \simeq \left\{\frac{3\pi}{2}\right\} \times [0, \theta_{\max}]$ and $\beta = \sin(r)\mathrm{d}\theta$.

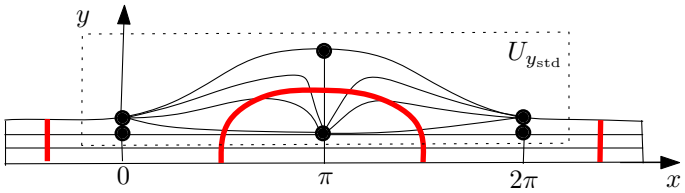

FIGURE 21. A smoothed half overtwisted disc

Remark 7.1. One can assume that $\left|\int_{\Gamma_D} \beta\right| < \tau$ by replacing β by $(1 - b(x)c(y))\beta$ where b and c are suitable cut-off functions.

We now follow [16] to construct an invariant contact structure on $\mathcal{A} \times \mathbb{R}$. Let $\alpha = \beta + f(x, y)\mathrm{d}z$ where

- $f(x, y) = \cos(x)$ for $y \leq \frac{3}{2} y_{\mathrm{std}}$ or $x \notin \left[-\frac{\pi}{3}, \frac{7\pi}{3}\right]$;
- in $\mathcal{U}$, the function f depends only on r and is decreasing, in addition $f(r, \theta) = \cos(r)$ near $r = \frac{3\pi}{2}$;
- in $U_{y_\beta} \setminus \mathcal{U}$, $f = \pm 1$ if $\pm \mathrm{d}\beta > 0$;
- elsewhere $f(\cdot, y)$ has the same variations as cos and
  - if $y \geq 2y_{\mathrm{std}}$, the function f does not depend on y and interpolates between cos and 1
  - if $y < 2y_{\mathrm{std}}$, the function f interpolates between cos and $f(\cdot, 2y_{\mathrm{std}})$.

We now smooth $\mathcal{A} \times I_{\mathrm{prod}}$ to glue it on S_Z. There are three types of corners: the convex corners $\gamma_2 \times \{\pm z_{\mathrm{prod}}\}$, the concave corners $\gamma_1 \times \{\pm z_{\mathrm{prod}}\}$ and the corners associated to $x = -\frac{3\pi}{4}$ and $x = \frac{11\pi}{4}$. We smooth the convex corners using a function

$$l_{\sup} : I_b \times I_{\mathrm{prod}} \to \mathbb{R}_+^*$$

independent of x for $x \notin \left[-\frac{\pi}{6}, \frac{13\pi}{6}\right]$ and such that $l_{\sup}(\cdot, z)$ is strictly concave with maximum $\gamma_2(x)$ at $z = 0$ (see Figure 22). Similarly we smooth the concave corners using an even function

$$l_{\inf} : [-z_{\max}, -z_{\mathrm{prod}}] \cup [z_{\mathrm{prod}}, z_{\max}] \to [0, y_{\mathrm{smooth}}]$$

decreasing and strictly convex on $[z_{\mathrm{prod}}, z_0]$ and zero on $[z_0, z_{\max}]$. In addition, we assume $y_{\mathrm{smooth}} < y_{\mathrm{std}} < \inf(l_{\sup})$ and $\mathrm{im}(l_{\sup}) \cap (\mathcal{U} \times I_{\mathrm{prod}}) = \emptyset$. To smooth the

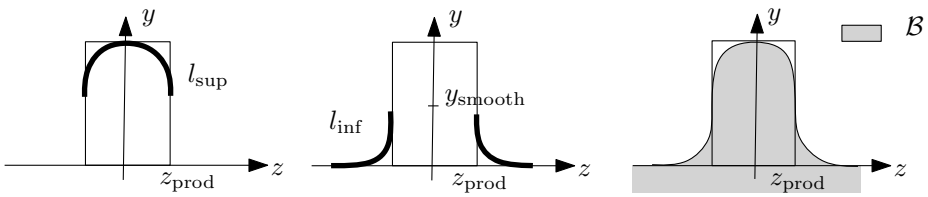

FIGURE 22. Bypass smoothing

remaining corners, we first perturb the previous smoothing for $z < 0$ and x close to $-\frac{\pi}{2}$ so that the perturbed boundary is the graph of a function

$$h : \left[-\frac{\pi}{2} - \varepsilon_s, -\frac{\pi}{2} + \varepsilon_s\right] \times (0, y_0) \to (-z_{\max}, 0)$$

such that $\frac{\partial h}{\partial y} \geq 0$ on $\mathrm{dom}(h)$ and $\frac{\partial h}{\partial y}(x, y) > \eta > 0$ for all $\left|x - \frac{\pi}{2}\right| \leq \frac{\varepsilon_s}{2}$. There exists $\varepsilon < \varepsilon_s$ such that $\mathrm{cotan}\left(-\frac{\pi}{2} - \varepsilon\right) < \eta$. We smooth the boundary of the bypass for $x \in \left[-\frac{\pi}{2} - \varepsilon, -\frac{\pi}{2} - \frac{\varepsilon}{2}\right]$ so that the new boundary is the graph of

$$k : \left[-\frac{\pi}{2} - \varepsilon, -\frac{\pi}{2} - \frac{\varepsilon}{2}\right] \times [-z_{\max}, z_{\max}] \to [0, y_0]$$

and

- $k = 0$ for x close to $-\frac{\pi}{2} - \varepsilon$;
- $0 \leq \frac{\partial k}{\partial z} < \frac{1}{\eta}$ for all $z < 0$;
- $\frac{\partial k}{\partial z} \leq 0$ for all $z \geq 0$.

We smooth the boundary for x close to $\frac{5\pi}{2}$ with a similar construction and denote by $\mathcal{B}$ the smoothed product. Let $M' = M \cup \mathcal{B}$. Then M' is a smooth manifold with boundary S'.

Proposition 7.2. *After an arbitrarily small perturbation of the contact form near $\gamma_2 \times \{0\}$, the boundary S' is convex. A dividing set, denoted by Γ_{smooth}, is given by the tangency points between the Reeb vector field and S' (see Figure 23).*

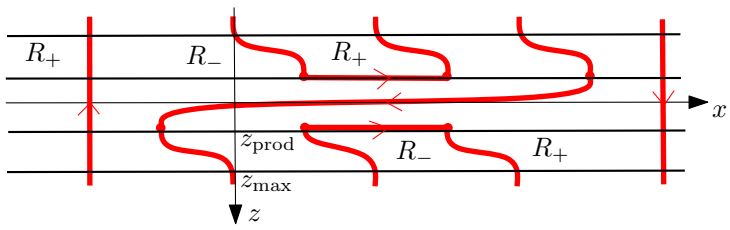

FIGURE 23. The dividing set Γ_{smooth}

Remark 7.3. The dividing set Γ_{smooth} is disjoint from the graphs of h and k. Indeed, the Reeb vector field is tangent to the graphs of h or k if and only if $\frac{\partial h}{\partial y}(x, y) = \mathrm{cotan}(x)$ or $\frac{\partial k}{\partial z}(x, z) = \tan(x)$.

Proof. By use of the implicit function theorem, the set of tangency points between the Reeb vector field and S' is a smooth curve Γ. In addition the characteristic foliation of S' is positively transverse to Γ except along $U_0 \cap (\gamma_2 \times \{0\})$. In a neighbourhood of $U_0 \cap (\gamma_2 \times \{0\})$, we consider the contact form $\alpha + \varepsilon(x, y)\mathrm{d}x$ where ε is a non-positive, small and smooth function that does not depend on y in a neighbourhood of $\left\{-\frac{\pi}{6}, \frac{13\pi}{6}\right\} \times \{y_0\}$. This perturbation does not change Γ and the characteristic foliation of S' is now positively transverse to Γ everywhere. We apply Lemma 3.1 to obtain the convexity of S'. □

Proposition 7.4 (Honda [27])**.** *M' is obtained from M after a bypass attachment along γ_1.*

The dividing set is pictured on Figure 23. Figure 24 shows the associated Reeb vector field. In particular, α is not adapted to S'.

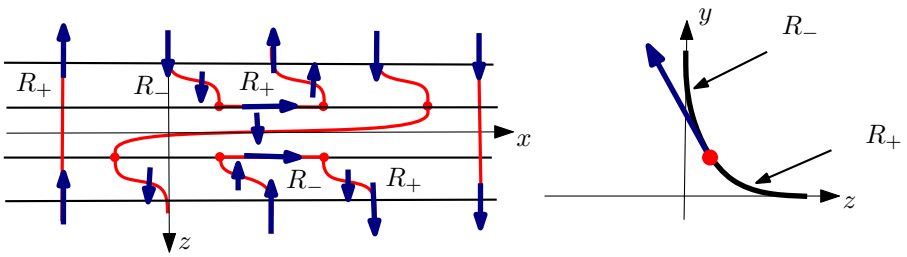

FIGURE 24. The Reeb vector field along Γ_{smooth}

7.2. **Preliminary lemmas.** This section can be skipped on first reading. We present bounds on $z_{\max}$ and on the perturbations of the contact form ensuring a precise control of the Reeb flow. Let $\Omega = I_b \times [0, y_{\text{std}}] \times I_{\max}$ and $\alpha = \sin(x)\mathrm{d}y + \cos(x)\mathrm{d}z$.

Lemma 7.5. *For small enough perturbations of α, the x-coordinate of any Reeb orbit in Ω covers an interval of length at most $\frac{\lambda}{8}$.*

Proof. The x-coordinate of R_α is zero and the amount of time spent in Ω by a Reeb orbit is uniformly bounded. □

Lemma 7.6. *For z_{max} small enough and for any small enough perturbation of α*

- *condition (B6) is satisfied;*
- *for all $0 \leq y \leq \frac{3}{4}y_{std}$, the map induced between S_y and $S_{y_{std}}$ for positive times can be decomposed into two maps with domains in $S_{\frac{\lambda}{4},1}$ and $S_{\frac{\lambda}{4},5}$ and ranges in $S_{\frac{\lambda}{2},1}$ and $S_{\frac{\lambda}{2},5}$;*
- *the map induced between $S_{y_{std}}$ and S_Z for positive times can be decomposed into two maps with domains in $S_{\frac{\lambda}{4},3}$ and $S_{\frac{\lambda}{4},-1}$ and ranges in $S_{\frac{\lambda}{2},3}$ and $S_{\frac{\lambda}{2},-1}$;*
- *the amount of time spent in Ω by a Reeb orbit is bounded by $2(y_{std} + z_{max})$.*

Proof. For $z_{\max}$ small enough, the domain and range of the map from S_y to $S_{y_{\text{std}}}$ are contained in $S_{\frac{\lambda}{4},1}$ and $S_{\frac{\lambda}{4},5}$ for all $0 \leq y \leq \frac{3}{4}y_{\text{std}}$. Similarly, the domain and range of the map from $S_{y_{\text{std}}}$ to S_y are contained in $S_{\frac{\lambda}{4},3}$ and $S_{\frac{\lambda}{4},-1}$. Thus the conditions on the map between S_y and $S_{y_{\text{std}}}$ are satisfied for any small perturbation of α.

We now prove condition (B6) and the return time condition. For any small perturbation of α

- $|R_y| \geq \frac{1}{2}$ outside $\left(S_{\frac{\lambda}{8},0} \cup S_{\frac{\lambda}{8},2} \cup S_{\frac{\lambda}{8},4}\right) \times [0, y_{\text{std}}]$;
- $|R_z| \geq \frac{1}{2}$ into $\left(S_{\frac{\lambda}{2},0} \cup S_{\frac{\lambda}{2},2} \cup S_{\frac{\lambda}{2},4}\right) \times [0, y_{\text{std}}]$;
- the hypotheses of Lemma 7.5 are satisfied.

Thus any Reeb orbit intersecting S_Z outside $S_{\frac{\lambda}{4},0} \cup S_{\frac{\lambda}{4},2} \cup S_{\frac{\lambda}{4},4}$ is not a Reeb chord of S_Z (Lemma 7.5). Any Reeb chord of S_Z stays in $S_{\frac{\lambda}{2},2k}$ for some k. Along any Reeb orbit, $|R_y| \geq \frac{1}{2}$ or $|R_z| \geq \frac{1}{2}$ (Lemma 7.5) and we obtain the desired bound on the amount of time spent in Ω.

Finally, the period of any Reeb chord of S_Z is bounded by $2z_{\max}$. Thus, for small enough perturbations, the y-coordinate covers an interval of length at most $2z_{\max}$ and $2z_{\max} < y_{\text{std}}$ for $z_{\max}$ small enough. Thus the Reeb chords which contribute to θ_k do not intersect a neighbourhood of $S_{y_{\text{std}}}$ and condition (B6) is satisfied. □

Remark 7.7. If the contact form is not perturbed near $x = k\pi$ there is no Reeb chord on S_Z or $S_{y_{\text{std}}}$. More precisely, let $y \in [0, y_{\text{std}})$ and α' be a small perturbation

of α such that $\alpha' = \alpha$ in $\left(S_{\frac{\pi}{4},0} \cup S_{\frac{\pi}{4},2} \cup S_{\frac{\pi}{4},4}\right) \times [y, y_{\text{std}}]$. Then there is no Reeb chord of $S_{y_{\text{std}}}$ contained in $I_b \times [y, y_{\text{std}}] \times I_{\max}$.

We smooth the corners of Ω as described in Section 7.1. The dividing set is given by the smooth curve Γ_{smooth} (see Proposition 7.2). We still denote by Γ_{smooth} its restriction to Ω. The convexification process will take place in a neighbourhood of Γ_{smooth} and will radically change the contact form. To control the new Reeb orbits, we first control the Reeb chord of a neighbourhood of Γ_{smooth}.

Lemma 7.8. *There exists an arbitrarily small neighbourhood V_{smooth} of Γ_{smooth} such that, for any small perturbation of α, the Reeb chords joining two distinct connected components of V_{smooth} are contained in $S_{\frac{\lambda}{4},2k} \times [0, y_{std}]$. In addition, the orientation of these Reeb chords is given by the sign of the z-component of R_α.*

Proof. There exists $\varepsilon < \frac{\lambda}{2}$ such that for any small perturbation of α, any Reeb orbit intersecting $S_{\varepsilon,2k+1} \times [0, y_{\text{std}}] \times [z_{\text{prod}} - \varepsilon, z_{\max}]$ remains in $S_{\varepsilon,2k+1} \times [0, y_{\text{std}}] \times \left[\frac{z_{\text{prod}}}{2}, z_{\max}\right]$ for $k = -1, \dots, 2$. In addition, we ask that the x-coordinate of any Reeb orbit in Ω covers an interval of length at most $\frac{\varepsilon}{2}$.

Choose some neighbourhood V_{smooth} of Γ_{smooth} with radius smaller that $\frac{\varepsilon}{2}$. Any connected component of V_{smooth} is contained in

$$\left[\frac{k\pi}{2} - \frac{\varepsilon}{2}, \frac{(k+1)\pi}{2} + \frac{\varepsilon}{2}\right] \times [0, y_{\text{std}}] \times I_{\max}$$

for $k = -1, \dots, 5$. Consider the connected component contained in

$$\left[l\pi - \frac{\pi}{2} - \frac{\varepsilon}{2}, l\pi + \frac{\varepsilon}{2}\right] \times [0, y_{\text{std}}] \times I_{\max}.$$

Any Reeb chord connecting this component to another is contained in $S_{\varepsilon,2l-1} \times [0, y_{\text{std}}]$ or in $S_{\varepsilon,2l} \times [0, y_{\text{std}}]$. By definition of ε, there is no Reeb orbit in $S_{\varepsilon,2l-1} \times [0, y_{\text{std}}]$. □

Lemma 7.9. *Let $(\mathcal{A}, \beta)$ be a bypass foliation. For z_{max} small enough, for any smoothing as described in Section 7.1 and for any small perturbation α' of α such that $\alpha' = \alpha$ in $\left(S_{\frac{\pi}{4},0} \cup S_{\frac{\pi}{4},2} \cup S_{\frac{\pi}{4},4}\right) \times [y_{std}, 2y_{std}]$ the condition (B4) is satisfied. In addition, the return time is bounded by θ_{max}.*

Proof. For $y \in \left[y_{\text{std}}, \frac{3}{2}y_{\text{std}}\right]$, we have $\alpha = \sin(x)\mathrm{d}y + \cos(x)\mathrm{d}z$. Thus the domain and range of the return map can be made as close to $x = k\pi$ as desired for $z_{\max}$ small enough. In addition, the only R_α-chord is $\Gamma_D \times \{0\}$ and the return time is bounded by $\theta_{\max}$. □

7.3. **Pre-convex bypasses.** Let $(\mathcal{B}, \alpha)$ be a bypass as defined in Section 7.1. In what follows we will always assume the the smoothing map $l_{\inf}$ is invariant by mirror symmetry along the plane $z - z_{\text{prod}} - y = 0$ for $z \geq z_{\text{prod}}$. We assume $y_{\text{smooth}} < \frac{y_{\text{std}}}{2}$. Recall that the Reeb vector field is tangent to the dividing set Γ_{smooth} of $\partial\mathcal{B} = S'$ when S' is vertical and points toward S'_- in the concave part of S' (Proposition 7.2). To apply the convexification process we first "eliminate" the tangency points between R_α and Γ_{smooth} by perturbing α to obtain a *pre-convex bypass*. We use the symmetries of the bypass to extend local constructions. In particular, if $(x, y, z) \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \times [0, y_{\text{std}}] \times [z_{\text{prod}}, z_{\text{prod}} + y_{\text{std}}]$, let

$$\sigma(x, y, z) = \left(-x + \frac{3\pi}{2}, z - z_{\text{prod}}, y + z_{\text{prod}}\right) \tag{2}$$

be the rotation of angle π with axis $x = \frac{3\pi}{4}$, $y = z - z_{\text{prod}}$. For $(x, y, z) \in \left[-\frac{3\pi}{4}, \frac{\pi}{4}\right] \times [0, y_{\text{std}}] \times [0, z_{\max}]$, let

$$\tau(x, y, z) = (-x, y, -z) \tag{3}$$

be the rotation of angle π with axis $x = z = 0$. Let $\Gamma_A^+ = \Gamma_A \times \{z_{\text{prod}}\}$, $\Gamma_A^- = \Gamma_A \times \{-z_{\text{prod}}\}$ and $\Gamma_A^\pm = \Gamma_A^- \cup \Gamma_A^+$. We use similar notations for Γ_D.

A *pre-convex* perturbation $(k_{\text{sup}}, k_{\text{inf}})$ of a bypass $(\mathcal{B}, \alpha)$ is composed of two smooth maps $\mathcal{B} \to \mathbb{R}_+^*$ such that (see Figure 25):

- there exist a neighbourhood V_+ of the restriction of $\left(\Gamma_A^\pm\right)$ to $y \geq \frac{y_{\text{std}}}{2}$ and a neighbourhood V_- of the restriction of $\left(\Gamma_A^\pm\right)$ to $y \leq \frac{y_{\text{std}}}{2}$ such that $k_{\text{sup}} = 1$ outside V_+ and $k_{\text{inf}} = 1$ outside V_-;
- $\frac{\partial k_{\text{sup}}}{\partial z} > 0$ near Γ_A^+ and $\frac{\partial k_{\text{sup}}}{\partial z} < 0$ near Γ_A^-;
- k_{sup} does not depend on x near $\Gamma_A \setminus \Gamma_D$ and on r near Γ_D;
- $k_{\text{inf}}(x, y, z) = (1 - f_{\text{inf}}(y)\rho z)$ near $\Gamma_D^+ \cap \{(x, y, z), x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]\}$ where $\rho > 0$, $f_{\text{inf}} : [0, y_{\text{std}}] \to \mathbb{R}_+$, $f_{\text{inf}} = 0$ near 0 and for $y \geq \frac{y_{\text{std}}}{2}$, f_{inf} is increasing on $[0, y_\rho^-]$, $f_{\text{inf}} = 1$ on $[y_\rho^-, y_\rho^+]$ and is decreasing on $[y_\rho^+, \frac{y_{\text{std}}}{2}]$;
- k_{sup} is τ-invariant and "π-periodic" for $y \in [0, y_{\text{std}}]$ and k_{inf} is τ-invariant and "π-periodic".

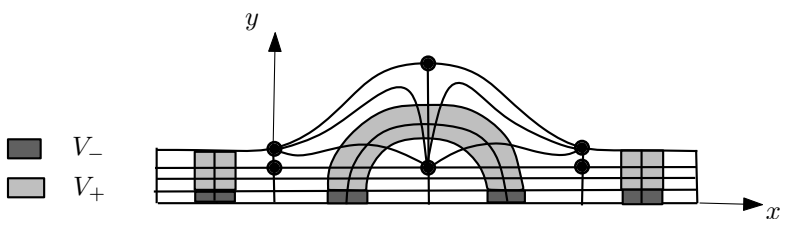

FIGURE 25. The neighbourhoods V_+ and V_-

In what follows, we assume $y_\rho^+ < \frac{2y_{\text{std}}}{5}$, $y_\rho^- > y_{\text{smooth}}$, and that the radius of V_- is smaller than $\frac{\lambda}{2}$. Let $k'_{\text{sup}} = k_{\text{sup}} \circ \sigma$. We extend k'_{sup} in a neighbourhood of $\mathcal{B}$ to obtain a π-periodic, τ-invariant function. We define k'_{inf} similarly. Let

$$\alpha_{\text{prec}} = (k_{\text{sup}} k_{\text{inf}}) \sin(x) \mathrm{d}y + (k'_{\text{sup}} k'_{\text{inf}}) \cos(x) \mathrm{d}z.$$

For k_{sup} and k_{inf} close to 1, α_{prec} is a contact form. Let $\eta(k_{\text{sup}})$ and $\eta(k_{\text{inf}})$ be the radius of the neighbourhoods of $\Gamma_A^\pm$ where k_{sup} and k_{inf} do not depends on x or r.

Proposition 7.10. *Fix $\eta > 0$. Let $(\mathcal{B}, \alpha)$ be a bypass. There exist real positive numbers ε_{inf} and ε_{sup} such that for any pre-convex perturbation (k_{sup}, k_{inf}) satisfying $\eta(k_{sup}) \geq \eta$, $\eta(k_{sup}) \geq \eta$, $\|k_{sup} - 1\| < \varepsilon_{sup}$ and $\|k_{inf} - 1\| < \varepsilon_{inf}$ the following holds*

- *S' is convex with dividing set Γ_{prec}, the set of tangency points between R_{prec} and S';*
- *R_{prec} points toward S_+ for $y > \frac{y_{std}}{2}$ and toward S_- for $y < \frac{y_{std}}{2}$.*

Note that $\varepsilon_{\text{inf}} > 0$ and $\varepsilon_{\text{sup}} > 0$ depend on the smoothing of $\mathcal{A} \times I_{\text{prod}}$.

Proof. The tangency condition is open. By Proposition 7.2 and Lemma 3.1 the first condition is satisfied for small enough perturbations of α.

The Reeb vector field R_α is transverse to Γ_{smooth} for $x \neq \frac{k\pi}{2}$ as the tangents to Γ_{smooth} have a non-vanishing x-component. Thus, for small enough perturbations, the transversality still holds for $\left|x - \frac{k\pi}{2}\right| \geq \eta$.

We study the case $y < \frac{y_{\text{std}}}{2}$. We prove transversality for $\left|x - \frac{\pi}{2}\right| < \eta$ and extend the result by symmetry. The Reeb vector field is

$$R_\alpha = \frac{1}{k_{\text{inf}}(y, z)} \begin{pmatrix} -\frac{\partial k_{\text{inf}}}{\partial z}(y, z) \sin(x) \\ \sin(x) \\ k_{\text{inf}}(y, z) \cos(x) \end{pmatrix} \tag{4}$$

and the tangency condition between $R_{\alpha_{\text{prec}}}$ and S' is

$$k_{\text{inf}}(y,z)\cos(x) - L'_{\text{inf}}(y)\sin(x) = 0$$

where $L_{\text{inf}} = l_{\text{inf}}^{-1}$ (see Section 7.1). Thus a parametrisation of Γ_{prec} is given by

$$y \mapsto \left(\left(\frac{\cos}{\sin}\right)^{-1}\left(\frac{L'_{\text{inf}}(y)}{k_{\text{inf}}(y, L_{\text{inf}}(y))}\right), y, L_{\text{inf}}(y)\right) = (x(y), y, z(y)).$$

The tangency condition between $R_{\alpha_{\text{prec}}}$ and Γ_{prec} implies

$$x'(y) + \frac{\partial k_{\text{inf}}}{\partial z}\big(y, z(y)\big) = 0.$$

Yet $\frac{\partial k_{\text{inf}}}{\partial z}\big(y, z(y)\big) < 0$ and $x'(y)$ has the sign of $-L''_{\text{inf}}(y)$. Thus $x'(y) < 0$ for k_{inf} close to 1 and $R_{\alpha_{\text{prec}}}$ is not tangent to Γ_{prec}.

We now study the case $y > \frac{y_{\text{std}}}{2}$. The tangency condition between $R_{\alpha_{\text{prec}}}$ and S' is

$$k_{\text{sup}}(y,z)\cos(x) - L'_{\text{sup}}(y)\sin(x) = 0.$$

The tangency condition between $R_{\alpha_{\text{prec}}}$ and Γ_{prec} implies

$$x'(y) + \frac{\partial k_{\text{sup}}}{\partial z}\big(y, z(y)\big) = 0.$$

Yet $\frac{\partial k_{\text{sup}}}{\partial z}\big(y, z(y)\big) > 0$ and $x'(y)$ has the sign of $L''_{\text{sup}}(y)$. □

The convexification process takes place near Γ_{prec}. We carefully choose $(\mathcal{B}, \alpha)$ and a pre-convex perturbation to control the image of the dividing set by the Reeb flow. For $s \in [0, \frac{\pi}{2}]$, let

$$\begin{aligned}
\gamma_0(s) &= (s, 0, -z_{\text{prod}})\\
\gamma_1(s) &= (\pi - s, 0, z_{\text{prod}})\\
\gamma^0(s) &= (s + \pi, 0, -z_{\text{prod}})\\
\gamma^1(s) &= (2\pi - s, 0, z_{\text{prod}}).
\end{aligned}$$

We denote by p_V and p_R the images on S_Z and S_R by the Reeb flow.

A quadruple $(\mathcal{B}, \alpha, k_{\text{sup}}, k_{\text{inf}})$ is a *pre-convex bypass* if $(\mathcal{B}, \alpha)$ is a bypass as defined in Section 7.1, $(k_{\text{sup}}, k_{\text{inf}})$ is a pre-convex perturbation and there exist positive real numbers ε_R, A_R and ε_Z (ε_Z is arbitrarily small), four z-graphs in S_R denoted by δ_0, δ_1, δ^0 and δ^1 and product neighbourhoods

$$V_{\Gamma_A} = V_x \times \left[0, \frac{3}{4}y_{\text{std}}\right] \times V_z \tag{5}$$

of the restriction of $\left(\Gamma_A^{\pm}\right)$ to $y \in \left[0, \frac{3y_{\text{std}}}{4}\right]$ with $|V_x| = \lambda_B < \lambda$ such that

(1) there is no Reeb chord between V_{Γ_A} and $S_{y_{\text{std}}}$;
(2) if Γ_B is the restriction of Γ_{prec} for $x \in \left[\frac{\pi}{2} + \frac{\lambda_B}{2}, \pi - \frac{\lambda_B}{2}\right]$ then $\left|\int_{\Gamma_B} \alpha\right| < \tau$ and $d_{\mathcal{C}^1}(p_Z(\Gamma_B), \gamma_1) < \frac{\varepsilon_Z}{2}$;
(3) δ_i is increasing and δ^i decreasing, the graphs intersect the segment $r = \frac{3\pi}{2}$;
(4) $\mathcal{C}(\delta_i, \varepsilon_R) \cap \mathcal{C}(H, \varepsilon_R) = \{0\}$ and $\mathcal{C}(\delta^i, \varepsilon_R) \subset \mathcal{C}(\delta_j^{\perp}, A_R)$ and δ^i satisfies symmetric conditions;
(5) if $d_{\mathcal{C}^1}(\delta, \delta_i) < \varepsilon_R$ and $d_{\mathcal{C}^1}(\delta', \delta^j) < \varepsilon_R$ then δ and δ' intersect transversely in one point ;
(6) if γ is a curve in S_Z such that $d_{\mathcal{C}^1}(\gamma, \gamma_i) < \varepsilon_Z$ or $d_{\mathcal{C}^1}(\gamma, \gamma^i) < \varepsilon_Z$ then either $d_{\mathcal{C}^1}(p_R(\gamma), \delta_i) < \varepsilon_R$ or $d_{\mathcal{C}^1}(p_R(\gamma), \delta^i) < \varepsilon_R$;
(7) the return time between S_Z and S_R is bounded by τ.

Proposition 7.11. *Let $(M, \xi = \ker(\alpha))$ be a contact manifold with convex boundary (S, Γ) and γ_0 be an attaching arc satisfying condition (C1), (C2) and (C3). Let $(\mathcal{A}, \beta)$ be a bypass foliation (see Section 7.1). For z_{max} small enough, there exists a pre-convex bypass $(\mathcal{B}, \alpha, k_{sup}, k_{inf})$ with k_{sup} and k_{inf} arbitrarily close to 1.*

The rest of this section is devoted to the proof of Proposition 7.11. Let $\eta < \frac{\lambda}{4}$. Choose $l_{\sup}$ and $k_{\sup}$ such that $\eta(k_{\sup}) \geq \eta$ and $\|k_{\sup} - 1\| < \varepsilon_{\sup}$.

Lemma 7.12. *If k_{sup} is close to 1 and z_{max} small enough, then $p_R(\gamma_i)$ and $p_R(\gamma^i)$ satisfy condition (3).*

Proof. For $z_{\max}$ small, the Reeb chords of α that contribute to the map between S_Z and S_R are contained in the neighbourhood where $\alpha = \sin(r)\mathrm{d}\theta + \cos(r)\mathrm{d}z$ and $k_{\sup}$ does not depend on r. In coordinates (r, θ, z), the image of γ^1 on S_θ is

$$s \longmapsto \left(2\pi - s, \theta, -\frac{\cos(s)}{\sin(s)}\theta + z_{\text{prod}}\right).$$

For $z_{\max}$ small enough, this curve in a decreasing graph in z and contains $(\frac{3\pi}{2}, \theta, z_{\text{prod}})$. Consider the perturbation $\alpha_{\sup}$ of α associated to $k_{\sup}$. As

$$R_\alpha = \frac{1}{k_{\sup}(y,z)} \begin{pmatrix} -\frac{\partial k_{\sup}}{\partial z}(y,z)\sin(x) \\ \sin(x) \\ k_{\sup}(y,z)\cos(x) \end{pmatrix},$$

we have $\frac{\partial k_{\sup}}{\partial z} = 0$ for $y \leq \frac{y_{\text{std}}}{2}$ and $\frac{\partial k_{\sup}}{\partial z} > 0$ for $z > \frac{y_{\text{std}}}{2}$ near Γ_D^+. Therefore, the curve $p_R(\gamma^1)$ intersects the segment $r = \frac{3\pi}{2}$. The proof is similar in the other cases. □

Let $p_R(\gamma_i) = \delta_i$ and $p_R(\gamma^i) = \delta^i$. The curves δ_i and δ^j intersect transversely in exactly one point and there exist ε_R, A_R and ε_Z satisfying conditions (4) and (5) for any small perturbation of $\alpha_{\sup}$. Additionally, all $R_{\alpha_{\sup}}$-orbits intersecting $\left\{\frac{\pi}{2}\right\} \times \left[0, \frac{3}{4}y_{\text{std}}\right] \times \{z_{\text{prod}}\}$ go out of the bypass. Then, there exists V_{Γ_A} such that the Reeb vector field of all small perturbation of $\alpha_{\sup}$ satisfies condition (1). We now carefully choose $l_{\inf}$. A parametrisation of Γ_B is

$$\begin{array}{ccc} \left[\frac{\pi}{2} + \frac{\lambda_B}{2}, \pi - \frac{\lambda_B}{2}\right] & \longrightarrow & \mathbb{R}^3 \\ x & \longmapsto & \left(x, l_{\inf}\left(\left(l'_{\inf}\right)^{-1}\left(\tan(x)\right)\right), \left(l'_{\inf}\right)^{-1}\left(\tan(x)\right)\right) \end{array}$$

thus $p_Z(\Gamma_B)$ is parametrised by

$$x \longmapsto \left(x, 0, \left(l'_{\inf}\right)^{-1}\left(\tan(x)\right) - \operatorname{cotan}(x) \times l_{\inf}\left(\left(l'_{\inf}\right)^{-1}\left(\tan(x)\right)\right)\right)$$

and its derivative is

$$x \longmapsto \left(1, 0, -\frac{1}{\sin^2(x)} \times l_{\inf}\left(\left(l'_{\inf}\right)^{-1}\left(\tan(x)\right)\right)\right).$$

Thus, the second half of condition (2) is satisfied for $l_{\inf}$ small. In addition, we have

$$\int_\Gamma \alpha = \int_{\frac{\pi}{2}+\frac{\lambda_B}{2}}^{\pi - \frac{\lambda_B}{2}} \frac{1}{\sin^3(x)} \times \frac{1}{l''_{\inf}\left((l'_{\inf})^{-1}(\tan(x))\right)} \mathrm{d}x.$$

Fix $C > 0$ and $c > 0$ we chose $l_{\inf}$ such that $l_{\inf} < c$ and $l''_{\inf}\left((l'_{\inf})^{-1}(y)\right) > C$ for all

$$y \in \left[\tan\left(\frac{\pi}{2} + \frac{\lambda_B}{2}\right), \tan\left(\pi - \frac{\lambda_B}{2}\right)\right].$$

For C big enough and c small enough, condition (2) is satisfied. Additionally, such a function exists: it is the anti-derivative of a function $f : (0,1] \longrightarrow (-\infty, 0]$ such that

- f is increasing and $\left|\int_0^1 f\right| = c$;
- for all $k \in \mathbb{N}$, $f^{(k)}(1) = 0$ and $\lim_{x\to 0} |f^{(k)}(x)| = \infty$;
- $f'\left((f)^{-1}(y)\right) > C$ for all $y \in \left[\tan\left(\frac{\pi}{2} + \frac{\lambda_B}{2}\right), \tan\left(\pi - \frac{\lambda_B}{2}\right)\right]$.

Finally, we choose $k_{\inf}$ small enough. Condition 7 derives from Lemmas 7.6 and 7.9 and Remark 7.1. This concludes the proof of Proposition 7.11.

7.4. **Convexification coordinates.** In the two following sections, we describe the actual construction of an hyperbolic bypass.

Construction 7.13. There exists a pre-convex bypass $(\mathcal{B}, \alpha, k_{\sup}, k_{\inf})$ adapted to τ and λ (Proposition 7.11). For technical reasons, we also consider a second pre-convex bypass $(\mathcal{B}', \alpha, k_{\sup}, k_{\inf})$ extending $\mathcal{B}$ and such that, $d_{\mathcal{C}^0}(\partial\mathcal{B}', \mathcal{B}) > 0$ for all $|z| \leq z_{\text{prod}} + y_{\text{std}}$ and $0 \leq y \leq y_{\text{std}}$. We denote by p'_Z and p'_R the projections in $\mathcal{B}'$. Without loss of generality $\varepsilon_Z < \nu$ and $p_Z(\text{dom}(p_{R|\Gamma_B}) \subset \text{dom}(p'_{R|S_Z})$.

We are now in position to apply the convexification process described in [6]. Recall that S' is the boundary of $M' = M \cup \mathcal{B}$, $k_{\inf} = 1 - \rho z$ on $[y_\rho^-, y_\rho^+]$,

$$y_{\text{smooth}} < y_\rho^- < y_\rho^+ < \frac{2}{5} y_{\text{std}},$$

and the upper boundary of V_{Γ_A} is contained in $y = \frac{3}{4} y_{\text{std}}$. Choose

$$y_\rho^- < y^- < y^+ < y_\rho^+.$$

Our first step is to obtain nice coordinates near $\Gamma_{\text{prec}} \cap \mathcal{B}^{\leq y_{\text{std}}}$. We construct these coordinates near the connected component Γ_0 contained in $\left[\frac{\pi}{2}, \pi\right] \times [0, y_{\text{std}}] \times [0, z_{\text{std}}]$ and extend them using the symmetries of $\mathcal{B}$.

Fact 7.14. *There exists ε_{prec} such that for all ε_{prec}-perturbation of S', the contact form α_{prec} is adapted to the new boundary for $y \geq \frac{2}{3} y_{std}$.*

Proposition 7.15. *There exists a surface Σ with coordinates $(u, v) \in [u_{min}, u_{max}] \times [-v_{max}, v_{max}]$ such that*

1. *Σ is transverse to $R_{\alpha_{prec}}$, $\Sigma \cap S' = \Gamma_0$ and the intersection is transverse;*
2. *$\sigma(i_\Sigma(u, v)) = i_\Sigma(u_{max} + u_{min} - u, v)$ where $i_\Sigma : \Sigma \to \mathbb{R}^3$ is the inclusion and σ is defined by equation (2);*
3. *$\frac{\partial}{\partial u} = \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial v} = \frac{\partial}{\partial z}$ for y close to $[y^-, y^+]$;*
4. *$\alpha_{prec} = \mathrm{d}t + (1 - \rho z_{prod} - \rho v)\mathrm{d}u$ in the flow-box coordinates (t, u, v) associated to Σ.*

We denote by y_Σ the coordinate such that $\left(\frac{\pi}{2}, y_\Sigma\right) = i_\Sigma(u_{\max}, 0)$.

Proof. Choose a surface Σ satisfying conditions (1), (2) and (3) and such that $i_\Sigma^* \alpha_{\text{prec}} = g(u, v)\mathrm{d}v$ and $g(u, 0) = 1 - \rho z_{\text{prod}}$. Then $g(u, v) = 1 - \rho z_{\text{prod}} - \rho v$ for y close to $[y_\rho^-, y_\rho^+]$ and $\alpha_{\text{prec}} = g(u, v)\mathrm{d}u + \mathrm{d}t$. Moser's trick provides us with a diffeomorphism φ_1 such that $\varphi_1 = \text{Id}$ along Γ_{prec} and for y close to $[y^-, y^+]$, φ_1 preserves Σ and $\varphi_1^* \alpha_{\text{prec}} = \mathrm{d}t + (1 - \rho z_{\text{prod}} - \rho v)\mathrm{d}u$ as the two Reeb vector fields coincide. In addition $\varphi_1 \circ \sigma = \sigma \circ \varphi_1$. Thus $\varphi_{1|\Sigma}$ has the desired properties. □

Let $u_\rho^\pm$ and $u^\pm$ be the u-coordinates associated to the intersection points between Γ_0 and $S_{y_\rho^\pm}$ or $S_{y^\pm}$. Let $\psi = (\psi_x, \psi_y, \psi_z)$ be the diffeomorphism associated to the change of coordinates. Let $S_{\mathcal{B}} = \psi^{-1}(S')$. Without loss of generality $\psi : \mathcal{U} \to \mathcal{V}$ and[3]

$$\mathcal{U} = I_t \times I_u \times I_v = [-t_{\max}, t_{\max}] \times [u_{\min}, u_{\max}] \times [-v_{\max}, v_{\max}], \tag{6}$$

$$\mathcal{V} \subset \left(\left[\frac{\pi}{2} - \frac{\lambda}{8}, \pi + \frac{\lambda}{8}\right] \times \left[0, \frac{3}{4} y_{\text{std}}\right] \times I_{\max} \right) \cap V_{\text{smooth}} \cap \mathcal{B}', \tag{7}$$

$$\alpha = (1 - \rho z)\sin(x)\mathrm{d}y + \cos(x)\mathrm{d}z \text{ on } \psi\left(I_t \times [u^-, u^+] \times I_v\right). \tag{8}$$

[3]See Lemma 7.8 for the definition of V_{smooth}.

Fact 7.16. *For all $(t,u,v) \in I_t \times [u^-,u^+] \times I_v$, it holds that*

$$\begin{aligned}
\sigma\big(\psi(t,u,v)\big) &= \psi\big(-t, u_{max}+u_{min}-u, v\big),\\
\psi(0,u,v) &= \left(\frac{\pi}{2}, y_\Sigma - u_{max} + u, z_{prod} + v\right),\\
\psi(t,u,v) &= \psi(t,0,v) + (0, y_\Sigma - u_{max} + u, 0),\\
\psi(-t,u,v) &= \big(\pi - \psi_x(t,u,v), 2\psi_y(0,u,v) - \psi_y(t,u,v), \psi_z(t,u,v)\big).
\end{aligned}$$

We rewrite our objects and conditions in coordinates (t,u,v). We subdivide $[u_{\min}, u_{\max}]$ into $u_{\min} < u_1 < u_2 < u_3 < u_4 < u_5 < u_{\max}$ (see Figure 26) such that

$$\psi(I_t \times [u_5, u_{\max}] \times I_v) \subset V_{\Gamma_A} \tag{9}$$

$$\left[\frac{\pi}{2}, \frac{\pi}{2} + \frac{\lambda}{2}\right] \subset \psi_x(I_t \times [u_4, u_{\max}] \times I_v), \tag{10}$$

$$\left[\frac{\pi}{2} + \frac{3\lambda}{4}, \pi - \frac{3\lambda}{4}\right] \subset \psi_x(I_t \times [u_2, u_3] \times I_v), \tag{11}$$

$$\left[\pi - \frac{\lambda}{2}, \pi\right] \subset \psi_x(I_t \times [u_{\min}, u_1] \times I_v). \tag{12}$$

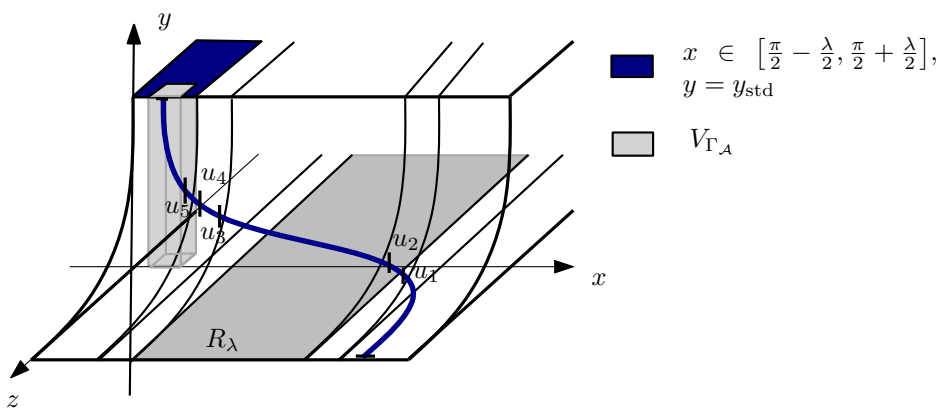

FIGURE 26. The subdivision u_i

Lemma 7.17. *Without loss of generality we may assume that (see Figure 29)*

(1) *$S_{\mathcal{B}}$ is a smooth surface contained in $\mathring{I}_t \times I_u \times [0, v_{max}]$ and*
   (a) *its restriction to the plane $u = cst$ is a smooth curve composed of two graphs containing $(0,0)$, one positive and increasing and the other negative and decreasing on $(0, v_{max}]$;*
   (b) *$S_{\mathcal{B}}$ is u-invariant and invariant by the mirror symmetry along the plane $t = 0$ for $u \in [u^-, u^+]$;*
(2) *there is no Reeb chord of $\partial\mathcal{V}$ outside $\mathcal{V}$;*
(3) *$C_\psi = \|\psi(t,u,v) - \psi(0,u,0)\|_\infty + \| \, d\psi(t,u,v) - \, d\psi(0,u,0)\|_\infty \ll 1$:*

Proof. $S_{\mathcal{B}}$ is a smooth surface containing $\{0\} \times [u_{\min}, u_{\max}] \times \{0\}$, tangent to $\operatorname{Span}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial t}\right)$ along this curve and transverse to $\frac{\partial}{\partial t} = R_{\alpha_{\text{prec}}}$ elsewhere. In addition, $\frac{\partial}{\partial t}$ is positively transverse to $S_{\mathcal{B}}$ for $t < 0$ and negatively transverse for $t > 0$ (see Figure 27).

To prove condition (2), we first note that there is no Reeb chord of Γ_{prec} in $\mathcal{B}^{<\frac{3}{4}y_{\text{std}}}$. Indeed, in the set where $\alpha_{\text{prec}} = \sin(x)\mathrm{d}y + \cos(x)\mathrm{d}z$, we have $R_x = 0$ and Γ_{prec} intersects the planes $x = \text{cst}$ in one point. By symmetry, it remains to prove

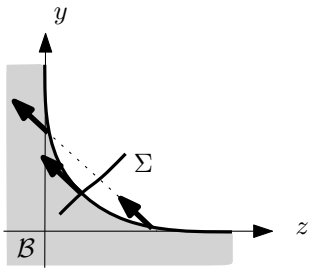

FIGURE 27. The Reeb vector field and $S_{\mathcal{B}}$

the result for x close to $\frac{\pi}{2}$ and $z > 0$. In this set, we have $R_x > 0$, $R_y > 0$. The projection of Γ_{prec} on the plane (y, x) is decreasing and there is no Reeb chord of Γ_{prec} in $\mathcal{B}^{<y_{\text{prec}}}$.

If there exists a sequence $\left(\gamma_n\right)_{n\in\mathbb{N}^*}$ of Reeb chords of $\partial\mathcal{V}_n$ where the radius of $\mathcal{V}_n$ is smaller that $\frac{1}{n}$, then the endpoints of γ_n converge to Γ_{prec}. In addition, the period of these chords is bounded (Lemma 7.6) and bounded below by $t_{\max}$ (associated to the maximal t-coordinate in $\mathcal{V}_1$). Thus γ_n converges to a Reeb chord of Γ_{prec}. This leads to a contradiction and condition 2 is proved. □

Lemma 7.18. *Without loss of generality we may assume that there exist real positive numbers ε_B and B such that for all $t_\Sigma \in I_t$, the maps φ_- and φ_+ induced by the Reeb flow between $\Sigma = \{(t, u, v), t = t_\Sigma\}$ and S_Z and between Σ and S_R satisfy:*

1. *$\mathcal{C}(H, \varepsilon_B) \cap \mathcal{C}(V, B) = \{0\}$;*
2. *$(\varphi_-)_* (\mathcal{C}_p(H, \varepsilon_B)) \subset \mathcal{C}_{\varphi_-(p)}(H, \nu)$ and $(\varphi_-^{-1})_* \left(\mathcal{C}_{\varphi_-(p)}(V, A)\right) \subset \mathcal{C}_p(V, B)$ for all $p \in [u_2, u_3] \times I_v$;*
3. *$(\varphi_+)_* (\mathcal{C}_p(H, \varepsilon_B)) \subset \mathcal{C}_{\varphi_+(p)}(\delta_1, \varepsilon_R)$ and $(\varphi_+^{-1})_* \left(\mathcal{C}_{\varphi_+(p)}(\delta_1^\perp, A_R)\right) \subset \mathcal{C}_p(V, B)$ for all $p \in [u_4, u_5] \times I_v$;*
4. *$\varphi_+([u_4, u_5] \times I_v) \subset \{(x, z), |x - \delta_1(z)| < \varepsilon_R\}$;*
5. *the return time between S_Z and Σ is bounded by τ and by 2τ between S_R and Σ.*

Let L be a bound of $\|\mathrm{d}\varphi_\pm\|$ and $\|\mathrm{d}\varphi_\pm{}^{-1}\|$ where $\|\cdot\|$ is defined in the coordinates (x, y, z) in S_R and S_Z and in the coordinates (t, u, v) in Σ.

Proof. Let $\Gamma_u = \Gamma_{\text{prec}} \cap \psi\left(I_t \times [u_2, u_3] \times I_v\right)$. Then $\Gamma_u \subset \Gamma_B$ and all the Reeb chords between Γ_B and S_Z have an endpoint in Γ_u. There exists $\varepsilon_B > 0$ such that if $d_{\mathcal{C}^1}(\gamma, \Gamma_u) < \varepsilon_B$ then $d_{\mathcal{C}^1}(p_Z(\gamma), p_Z(\Gamma_u)) < \frac{\varepsilon_Z}{2}$. Thus the fist half of condition (2) derive from conditions (2) and (6) in the definition of pre-convex bypasses for $\mathcal{U}$ small enough. As $p_Z(\mathrm{dom}(p_{R|\Gamma_B}) \subset \mathrm{dom}(p'_{R|S_Z})$, condition (4) and the first part of condition (3) derive from conditions (2) and (6) in the definition of pre-convex bypasses. As $(\varphi_-^{-1})_* (\mathcal{C}(V, A)) \cap \mathcal{C}(H, \varepsilon_B) = \emptyset$ there exists B satisfying the second part of condition (2). The proof of the second part of condition (3) is similar. □

We now perturb S' to obtain a u-invariant surface.

Construction 7.19. We perturb S' (see Figure 28) so that there exists $v_0 \in (0, v_{\max})$ satisfying

- $S_{\mathcal{B}|u} = S_{\mathcal{B}|u_{\max}}$ on $[0, v_0]$;
- $S_{\mathcal{B}}$ contains $\{0\} \times [u_{\min}, u_{\max}] \times \{0\}$ and is tangent to $\mathrm{Span}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial t}\right)$ along this curve;
- the Reeb vector field is positively transverse to $S_{\mathcal{B}}$ for $t < 0$, negatively transverse for $t > 0$.

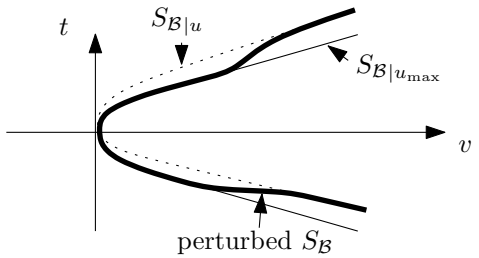

FIGURE 28. The perturbation of S'

These condition are automatically satisfied for $u \in [u_{\min}, u_1]$ and $u \in [u_5, u_{\max}]$. Let $\mathcal{B}$ denote the new bypass. For $\mathcal{U}$ small enough, this perturbation is $\varepsilon_{\text{prec}}$-small. In what follows, let $I_v = [-v_0, v_0]$ and $v_{\max} = v_0$.

7.5. **Convexification.** The convexification process consists in adding a "bump" with prescribed contact structure in a neighbourhood of Γ_{prec}. We first describe the new boundary in Section 7.5.1. In Section 7.5.2, we present the contact structure in the convexification and in Section 7.5.3 we modify this model to obtain the desired cone-preserving properties. Recall that we construct the convexification near the connected component Γ_0 contained in $\left[\frac{\pi}{2}, \pi\right] \times [0, y_{\text{std}}] \times [0, z_{\text{std}}]$.

7.5.1. *Perturbed boundary.* We perturb the boundary for $y \leq \frac{3}{4} y_{\text{std}}$. For $[y^-, \frac{3}{4} y_{\text{std}}]$, the new boundary is the graph of a function. Let $\mathcal{H}$ be the set of smooth functions

$$h : \left[\frac{\pi}{2} - \frac{\lambda}{3}, \frac{\pi}{2} + \frac{\lambda}{3}\right] \times \left[y^-, \frac{3}{4} y_{\text{std}}\right] \to \mathbb{R}$$

such that

(1) $\|h\|_{\mathcal{C}^\infty} \ll 1$ (in particular $\|h\|_{\mathcal{C}^\infty} < \varepsilon_{\text{prec}}$);
(2) $h = 0$ near $x = \frac{\pi}{2} \pm \frac{\lambda}{3}$ and $y = \frac{3}{4} y_{\text{std}}$;
(3) $h(-x, y) = h(x, y)$;
(4) for $y \in \left[y^-, \frac{2}{3} y_{\text{std}}\right]$, the map h does not depend on y and there exists x^h_{flat} such that h is increasing for $x < -x^h_{\text{flat}}$, constant on $\left[\frac{\pi}{2} - x^h_{\text{flat}}, \frac{\pi}{2} + x^h_{\text{flat}}\right]$ and decreasing for $x > x^h_{\text{flat}}$;
(5) $\tilde{S}_h = \left\{(x, y, z_{\text{prod}} + h(x)), x \in \left[\frac{\pi}{2} - \frac{\lambda}{3}, \frac{\pi}{2} + \frac{\lambda}{3}\right], y \in [y^-, y^+]\right\}$ is contained in $\mathcal{V}$.

Let $S_h = \psi^{-1}\left(\tilde{S}_h\right)$. We denote by v_h the maximum of h and $z_{\text{prod}} + v_h$ by z_h. The minimum of S_h on the v-axis corresponds to $t = 0$ and $v = v_h$ (see Figure 29). In addition $\mathcal{H} \neq \emptyset$. Let $v_{\mathcal{H}} = \sup\{\max(h), h \in \mathcal{H}\}$ then $v_{\mathcal{H}} > 0$ and for all $0 < v < v_{\mathcal{H}}$ there exists $h \in \mathcal{H}$ such that $v_h = v$. Given v_h there exists $h \in \mathcal{H}$ with x^h_{flat} arbitrarily small.

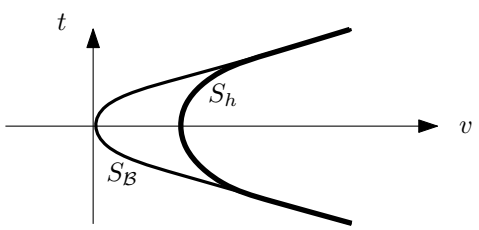

FIGURE 29. The surfaces $S_{\mathcal{B}}$ and S_h

Lemma 7.20. *The surface S_h is u-invariant and its restriction to the plan $u = cst$ is a smooth curve composed of two symmetric graphs containing $(v_h, 0)$, one positive and increasing and the other negative and decreasing on $(v_h, v_{max}]$;*

Proof. For h small enough, the tangency points between the Reeb vector field and $\tilde{S}_h$ are the segment $\left\{\frac{\pi}{2}\right\} \times [y^-, y^+] \times \{z_h\}$. In addition, $\frac{\partial}{\partial t}$ is positively transverse to S_h for $t < 0$ and negatively transverse for $t > 0$. The proof is similar to the proof of Lemma 7.17. The symmetry derives from Fact 7.16. □

We extend the surface S_h by translation in the coordinates (t, u, v) and still denote by S_h the extension. The surface S_h also extends $\sigma\left(\tilde{S}_h\right)$ as S_h is parametrised by $(\pm l(v), u, v)$ and $\sigma(\psi(\pm l(v), u, v) = \psi(\mp l(v), u_{\max} + u_{\min} - u, v)$. The following lemma is a powerful tool to study the Reeb chords in $\mathcal{B}^{\leq y_{\text{std}}}$ (conditions (B6), (B7) and (B8)). We will use its corollary (Corollary 7.37) in Section 7.6.

Lemma 7.21. *There exist positive numbers t'_{max}, v'_{max}, u_λ, Δ and v_Δ such that*

- *$t'_{max} < t_{max}$, $v_\Delta < v'_{max} < v_{max}$ and $v_\Delta < v_{\mathcal{H}}$;*
- *$\psi_x\left(I'_t \times [u_{min}, u_\lambda] \times I'_v\right) \subset \left[\frac{\pi}{2} + \frac{\lambda}{2}, \pi + \lambda\right]$ where $I'_t = [-t'_{max}, t'_{max}]$ and $I'_v = [-v'_{max}, v'_{max}]$;*
- *$\psi_y(p) < \psi_y(p')$ and $\psi_x(p') - \psi_x(p) \leq \frac{\lambda}{12}$ for all $p = (t, u, v)$ and $p' = (t', u', v')$ in $I'_t \times [u_\lambda, u_{max}] \times I'_v$ such that $t' - t > \Delta$ and $u' \geq u$;*
- *the planes $t = \pm\Delta$ intersect $S_{\mathcal{B}}$ for $v \leq v_\Delta$.*

Proof. Without loss of generality, there exists η such that $\frac{1}{\eta} < k_{\inf} < \eta < 2$ and $\eta^2 - \sin\left(\frac{\pi}{2} + \lambda\right) < \frac{\eta}{32}$. We start with $t'_{\max} = t_{\max}$ and $v'_{\max} = v_{\max}$ and progressively reduce them. There exists u_λ such that for $t'_{\max}$ and $v'_{\max}$ small enough

$$\psi_x\left(I'_t \times [u_\lambda, u_{\max}] \times I'_v\right) \subset \left[\frac{\pi}{2} - \lambda, \frac{\pi}{2} + \lambda\right],$$
$$\psi_x\left(I'_t \times [u_{\min}, u_\lambda] \times I'_v\right) \subset \left[\frac{\pi}{2} + \frac{\lambda}{2}, \pi + \lambda\right].$$

Let $M = \left\|\frac{\partial \psi_y}{\partial v}\right\|_\infty + \left\|\frac{\partial \psi_x}{\partial v}\right\|_\infty + \left\|\frac{\partial k_{\inf}}{\partial z}\right\|_\infty$. Choose Δ such that, upon reducing $t'_{\max}$ and $v'_{\max}$

- $t'_{\max} = 4\Delta < \frac{\lambda}{96M}$;
- $v'_{\max} < \min\left(\frac{\Delta}{8M}, \frac{\lambda}{48M}, v_{\mathcal{H}}\right)$;
- the planes $t = \pm\Delta$ intersect $S_{\mathcal{B}}$ for $v \leq v'_{\max}$.

For all $(t, u, v) \in [0, t'_{\max}] \times [u_\lambda, u_{\max}] \times I'_v$ we have[4]

$$\psi_y(0, u, v) + \frac{t}{\eta}\sin\left(\frac{\pi}{2} + \lambda\right) \leq \psi_y(t, u, v) \leq \psi_y(0, u, v) + \eta t,$$
$$\psi_y(0, u, 0) - v'_{\max}\left\|\frac{\partial \psi_y}{\partial v}\right\|_\infty \leq \psi_y(0, u, v) \leq \psi_y(0, u, 0) + v'_{\max}\left\|\frac{\partial \psi_y}{\partial v}\right\|_\infty.$$

Thus $\psi_y(p') - \psi_y(p) \geq \frac{\Delta}{4} > 0$. Similarly, it holds that

$$\psi_x(0, u, 0) - v'_{\max}\left\|\frac{\partial \psi_x}{\partial v}\right\|_\infty - 2t'_{\max}\left\|\frac{\partial k_{\inf}}{\partial z}\right\|_\infty \leq \psi_x(t, u, v),$$
$$\psi_x(t, u, v) \leq \psi_x(0, u, 0) + v'_{\max}\left\|\frac{\partial \psi_x}{\partial v}\right\|_\infty + 2t'_{\max}\left\|\frac{\partial k_{\inf}}{\partial z}\right\|_\infty,$$

and $\psi_x(p') - \psi_x(p) \leq 2v'_{\max}\left\|\frac{\partial \psi_x}{\partial v}\right\|_\infty + 4t'_{\max}\left\|\frac{\partial k_{\inf}}{\partial z}\right\|_\infty$. □

[4]See equation(4) for the explicit form of the Reeb vector field.

Construction 7.22. We apply Lemma 7.21 and choose a map $h \in \mathcal{H}$ such that $v_\Delta < \max(h) < v_{\max}$ and such that the planes $v = v'_{\max}$ intersect S_h outside the flat part of $\tilde{S}_h$.

Fact 7.23. *There exists ε^h_{pert} such that for any ε^h_{pert}-perturbation of α_{prec} in $\mathcal{C}^1$-norm, the Reeb vector field is transverse to $\tilde{S}_h$ for all x satisfying $x^h_{flat} \leq |x - \frac{\pi}{2}| \leq \frac{\lambda}{3}$.*

7.5.2. *Convexification model.* We now construct a convexification model for $y \in \left[y_-, \frac{2}{5}y_{\text{std}}\right]$ and interpolate with the adapted model for $y \in \left[\frac{2}{5}y_{\text{std}}, \frac{2}{3}y_{\text{std}}\right]$. The new boundary is smoothed for $y \in \left[\frac{2}{3}y_{\text{std}}, \frac{3}{4}y_{\text{std}}\right]$. Let $z > 0$ and

$$\begin{cases} \Phi_{z_0}(z) &= \exp(-\frac{1}{z-z_0}), & \text{if } z > z_0, \\ \Phi_{z_0}(z) &= 0, & \text{otherwise.} \end{cases}$$

For $y \in [y^-_\rho, y^+_\rho]$ and near Γ_0,

$$\alpha_{\text{prec}} = k_{\text{inf}}(z)\sin(x)\mathrm{d}y + \cos(x)\mathrm{d}z$$

and $k_{\text{inf}}(z) = 1 - \rho z$. Let $k_{\text{conv}}(z) = k_{\text{inf}}(z) + a\Phi_{z_0}(z)$ and

$$\alpha_{\text{conv}} = k_{\text{conv}}(z)\sin(x)\mathrm{d}y + \cos(x)\mathrm{d}z \tag{13}$$

where $a > 0$.

Fact 7.24. *The contact form α_{conv} is adapted to $\tilde{S}_h$ for $y \in \left[y_-, \frac{2}{3}y_{std}\right]$.*

In the coordinates (t, u, v), the associated contact form is not a convexification model in the sense of [6]. We use a weakened version of convexification. Let J_u be such that $[u_1, u_5] \subset J_u \subset I_u$. Two functions f and g from $I_t \times J_u \times I_v$ to $\mathbb{R}^*_+$ form a *convexification pair* if

(1) $f = 1$ and $g(t, u, v) = 1 - \rho z_{\text{prod}} - \rho v$ near S' and for $v \geq v'_{\max}$;
(2) f and g do not depend on u for $u \in [u_1, u_5]$;
(3) $\frac{\partial f}{\partial v} \geq 0$ and $\frac{\partial f}{\partial v} > 0$ near $(0, v_h)$;
(4) in a neighbourhood of $[u_1, u_5]$, in the planes $u = \text{cst}$, the vector field $X_g = \left(-\frac{\partial g}{\partial v}, \frac{\partial g}{\partial t}\right)$ is negatively transverse to S_h for $t > 0$, positively transverse for $t < 0$ and points toward the half-space $t < 0$ for $t = 0$ (see Figure 30).

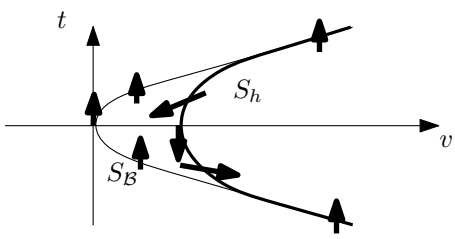

FIGURE 30. The surface S_h and the Reeb vector field

Proposition 7.25. *For ρ and C_ψ small enough, there exists $\varepsilon_{z_0} > 0$ such that for any small ρ_0 and any $z_h - \varepsilon_{z_0} < z_0 < z_h$, there exists a contact form α and a pair of convexification (f, g) with $J_u = [u_{max} + u_{min} - u_+, u_+]$ satisfying*

(1) *$R_u \geq 0$, R is positively transverse to S_h for $t < 0$, negatively transverse for $t > 0$ and points toward the half-space $t < 0$ for $t = 0$;*
(2) *$\sigma^*\alpha = -\alpha$;*
(3) *$\alpha = \alpha_{prec}$ for $v \geq v'_{max}$ and $v \leq v_\Delta$;*
(4) *$\alpha = f(t, u, v)\mathrm{d}t + g(t, u, v)\mathrm{d}u$ on $I_t \times J_u \times I_v$;*

(5) $\psi^{-1*}\alpha = \alpha_{conv}$ *and* $k'_{conv}(z_h) = \rho_0$ *in a neighbourhood of* $\{\frac{\pi}{2}\}\times\left[y^+, \frac{2}{5}y_{std}\right]\times\{z_{prod}\}$.

By definition $\|k_{\text{conv}}\|_{\mathcal{C}^1} \leq \rho_0$. The end of Section 7.5.2 is devoted to the proof of Proposition 7.25. Before extending α_{conv} for all u, we study some properties of α_{conv} in the (t,u,v)-coordinates.

Fact 7.26. *For all* $(t,u,v) \in I_t \times [u^-,u^+] \times I_v$, *we have*

$$\mathrm{d}\psi(t,u,v) = \begin{pmatrix} R_x(\psi(t,u,v)) & 0 & \frac{\partial\psi_x}{\partial v}(t,u,v) \\ R_y(\psi(t,u,v)) & 1 & \frac{\partial\psi_y}{\partial v}(t,u,v) \\ R_z(\psi(t,u,v)) & 0 & \frac{\partial\psi_z}{\partial v}(t,u,v) \end{pmatrix},$$

$$\mathrm{d}\psi(0,u,v) = \begin{pmatrix} R_x(\frac{\pi}{2},u,v) & 0 & 0 \\ R_y(\frac{\pi}{2},u,v) & 1 & 0 \\ R_z(\frac{\pi}{2},u,v) & 0 & 1 \end{pmatrix}.$$

Fact 7.27. *For all* $(t,u,v) \in I_t \times [u^-,u^+] \times I_v$, *we have*

$$\psi^*\alpha_{prec}(t,u,v) = \mathrm{d}t + k_0\mathrm{d}u \text{ and } \begin{cases} k_{inf}(\psi_z)\sin(\psi_x) = k_0 \\ k_{inf}(\psi_z)\sin(\psi_x)\frac{\partial\psi_y}{\partial v} + \cos(\psi_x)\frac{\partial\psi_z}{\partial v} = 0 \end{cases}$$

where $k_0(v) = k_{inf}(v + z_{prod})$.

Fact 7.28. *For all* $(t,u,v) \in I_t \times [u^-,u^+] \times I_v$, *we have*

$$\psi^*\alpha_{conv} = \left(\frac{k_{conv}(\psi_z)}{k_{inf}(\psi_z)}\sin^2(\psi_x) + \cos^2(\psi_x)\right)\mathrm{d}t + k_{conv}(\psi_z)\sin(\psi_x)\mathrm{d}u + \left(k_{conv}(\psi_z)\sin(\psi_x)\frac{\partial\psi_y}{\partial v} + \cos(\psi_x)\frac{\partial\psi_z}{\partial v}\right)\mathrm{d}\tau,$$

$$\psi^*\alpha_{conv} = \left(\left(\frac{k_{conv}(\psi_z)}{k_{inf}(\psi_z)} - 1\right)\sin^2(\psi_x) + 1\right)\mathrm{d}t + \frac{k_{conv}(\psi_z)}{k_{inf}(\psi_z)}k_0\mathrm{d}u + \left(\left(k_{conv}(\psi_z) - k_{inf}(\psi_z)\right)\sin(\psi_x)\frac{\partial\psi_y}{\partial v}\right)\mathrm{d}\tau.$$

In coordinates (t,u,v), $\alpha_{\text{conv}} = f_1(t,v)\mathrm{d}t + g_1(t,v)\mathrm{d}u + h_1(t,v)\mathrm{d}v$ where f_1 g_1 and h_1 do not depend on u. Note that $f_1(-t,v) = f_1(t,v)$, $g_1(-t,v) = g_1(t,v)$ and $h_1(-t,v) = -h_1(t,v)$ (Fact 7.16). Fix $u^- < u'_0 < u'_1 < u^+$ and $p : [u'_0, u'_1] \to \mathbb{R}$ such that $p = 0$ near u'_0 and $p = 1$ near u'_1. Let

$$\alpha = f_1(t,v)\mathrm{d}t + g_1(t,v)\mathrm{d}u + p(u)h_1(t,v)\mathrm{d}v.$$

Lemma 7.29. *For* ρ *and* C_ψ *small and for* z_0 *close to* z_h, α *is a contact form,* $\frac{\partial f_1}{\partial v} \geq 0$ *and* $\frac{\partial f_1}{\partial v} > 0$ *near* $(0, v_h)$*;*

Proof. The differential of α is

$$\mathrm{d}\alpha = \frac{\partial f_1}{\partial v}\mathrm{d}v\wedge\mathrm{d}t + \frac{\partial g_1}{\partial v}\mathrm{d}v\wedge\mathrm{d}u + \frac{\partial g_1}{\partial t}\mathrm{d}t\wedge\mathrm{d}u + p'h_1\mathrm{d}u\wedge\mathrm{d}v + p\frac{\partial h_1}{\partial t}\mathrm{d}t\wedge\mathrm{d}v.$$

The contact condition is

$$\frac{\partial f_1}{\partial v}g_1 - \frac{\partial g_1}{\partial v}f_1 + ph_1\frac{\partial g_1}{\partial t}\mathrm{d}t + p'h_1f_1 - p\frac{\partial h_1}{\partial t}g_1 > 0.$$

Without loss of generality we may assume that the ranges of f_1, g_1 and h_1 are in $\left[\frac{1}{2}, 2\right]$. In what follows, the bounds associated to the notation O are uniform for all

convexification models. By definition, it holds that

$$\begin{aligned} k_{\mathrm{conv}}(z) - k_{\mathrm{inf}}(z) &= A\Phi_{z_0}(z) = (z-z_0)^2 A\Phi'_{z_0}(z), \\ \frac{k_{\mathrm{conv}}}{k_{\mathrm{inf}}}(z) - 1 &= (z-z_0)^2 A\Phi'_{z_0}(z) O(1), \\ \left(\frac{k_{\mathrm{conv}}}{k_{\mathrm{inf}}}\right)'(z) &= A\Phi'_{z_0}(z)\left(\frac{1}{k_{\mathrm{inf}}(z)} + \frac{\rho(z-z_0)^2}{\left(k_{\mathrm{inf}}(z)\right)^2}\right). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} f_1(t,v) &= 1 + A\Phi'_{z_0}(\psi_z)(\psi_z - z_0)^2 O(1), \\ \frac{\partial f_1}{\partial v} &= A\Phi'_{z_0}(\psi_z)\left(1 + O(\rho) + O(C_\psi) + (\psi_z - z_0)^2 O(1)\right). \end{aligned}$$

Therefore for ρ, C_ψ, $z_h - z_0$ small enough, we have $\frac{\partial f_1}{\partial v} \geq 0$ and $\frac{\partial f_1}{\partial v}(0, v_h) > 0$. Similarly, we obtain

$$\begin{aligned} g_1(t,v) &= 1 + O(\rho) + (z-z_0)^2 O(1) + O(C_\psi), \\ \frac{\partial g_1}{\partial v} &= -\rho + A\Phi'_{z_0}(\psi_z)\left(1 + (\psi_z - z_0)^2 O(1) + O(\rho) + O(C_\psi)\right), \\ \frac{\partial g_1}{\partial t}(t,v) &= A\Phi'_{z_0}(\psi_z) O(C_\psi), \\ h_1(t,v) &= A\Phi'_{z_0}(\psi_z)(\psi_z - z_0)^2 O(1), \\ \frac{\partial h_1}{\partial t}(t,v) &= A\Phi'_{z_0}(\psi_z)\left((\psi_z - z_0)^2 O(1) + O(C_\psi)\right), \end{aligned}$$

and the contact condition is

$$\rho + A\Phi'_{z_0}(\psi_z)\left(O(\rho) + (\psi_z - z_0)^2 O(1) + O(C_\psi)\right) > 0. \tag{14}$$

Yet $A\Phi'_{z_0}(z) < 2\rho$ as $k'_{\mathrm{conv}}(z_h) = \rho_0 < \rho$. Thus the contact condition is satisfied for ρ, C_ψ and $z_h - z_0$ small enough. □

Lemma 7.30. *For ρ, ρ_0, C_ψ, $z_h - z_0$ small, $R_u \geq 0$ and R is positively transverse to S_h for $t < 0$, negatively for $t > 0$ and points toward the half-plane $t < 0$ for $t = 0$.*

Proof. The component R_u is positively collinear to

$$\frac{\partial f_1}{\partial v} - p\frac{\partial h_1}{\partial t} = A\Phi'_{z_0}(\psi_z)\left(1 + O(\rho) + O(C_\psi) + (\psi_z - z_0)^2 O(1)\right).$$

Thus $R_u \geq 0$. By u-invariance, we study the transversality properties in the planes $u = \mathrm{cst}$. The Reeb vector field is positively collinear to

$$Y = \begin{pmatrix} -\frac{\partial g_1}{\partial v} + p'(u)h_1 \\ \frac{\partial g_1}{\partial t} \end{pmatrix}$$

in the coordinates (t, v). The tangency condition for $t = 0$ is automatically satisfied as $h_1(0, v) = 0$. On the non flat part of h, the transversality conditions are satisfied for $\rho_0 < \varepsilon_h^{\mathrm{pert}}$ as $\|k_{\mathrm{conv}} - k_{\mathrm{inf}}\|_{\mathcal{C}^1} \leq \rho_0$ (Fact 7.23). We now prove the result in the flat part of h. The transversality condition is

$$-\frac{\partial g_1}{\partial v} + p'(u)h_1 - l'_\pm \frac{\partial g_1}{\partial t} > 0 \tag{15}$$

for $t \neq 0$ where l_+ and l_- parametrise S_h. Let $\rho_0 = s_0\rho$. Then $A\Phi'_{z_0}(\psi_z) = (1+s_0)\rho$. For $p = 1$, Y satisfies the desired transversality conditions (Fact 7.24). For $p' = 0$ the transversality condition is

$$-s_0\rho + \rho(1+s_0)a(t,u,v) > 0 \tag{16}$$

where a does not depend on s_0. There exists $s_{\max}$ such that (16) is satisfied for $s_0 \in (0, s_{\max}]$. Thus $a > \frac{s_{\max}}{1+s_{\max}}$. For z_0 close to z_h,

$$\left| p'(u)(\psi_z - z_0)^2 \sin(\psi_x) \frac{\partial \psi_y}{\partial v} \right| < \frac{s_{\max}}{2(1+s_{\max})}.$$

The general transversality condition is

$$a(t,u,v) + p'(u)(\psi_z - z_0)^2 \sin(\psi_x) \frac{\partial \psi_y}{\partial v} + s(a(t,u,v) - 1) > 0. \tag{17}$$

For $s \leq \frac{s_{\max}}{2}$, we obtain $a(t,u,v) + s(a(t,u,v)-1) > \frac{s_{\max}}{2(1+s_{\max})}$ and (17) is satisfied. □

Proof of Proposition 7.25. We choose ρ, ρ_0, C_ψ, $z_h - z_0$ small enough to apply Lemma 7.29 and Lemma 7.30. We extend α to $\mathcal{U}$ by $\alpha = f_1 \mathrm{d}t + g_1 \mathrm{d}u$ for $u \in J_u$ and $-\sigma^* \alpha$ for $u \in [u_{\min}, u_{\min} + u_{\max} - u^+]$. It remains to prove that $\alpha = \alpha_{\mathrm{prec}}$ for $v \geq v'_{\max}$ and $v \leq v_\Delta$. The set where

$$(f_1(t,v), g_1(t,v)) \neq (1, 1 - \rho z_{\mathrm{prod}} - \rho v)$$

is contained between the surface S_{z_0} associated to the equation $z = z_0$ and S_h. The surface S_{z_0} has properties similar to S_h. In particular, its v-coordinates are greater than $z_0 - z_{\mathrm{prod}}$. As $z_h - z_{\mathrm{prod}} > v_\Delta$ (Construction 7.22) $z_0 - z_{\mathrm{prod}} > v_\Delta$ for $z_h - z_0$ small enough. Additionally, S_{z_0} intersects S_h in its non-flat part. Yet for $z_h - z_0$ small enough the intersection points are arbitrarily close to the endpoints of the flat part and the v-coordinates of the intersection points are smaller than $v'_{\max}$ (Construction 7.22). □

7.5.3. *Perturbed convexification.* The contact form α described in Proposition 7.25 is adapted to the boundary but does not give us the desired control on the Reeb flow. Let $\Sigma_\pm = \{(\pm t_{\max}, u, v), u \in I_u, v \in I_v\}$ and Φ be the map induced by the Reeb flow of α_{conv} between Σ_- and Σ_+.

Proposition 7.31. *Let (f,g) be a convexification pair given in Proposition 7.25. Upon perturbing α near $I_t \times [u_1, u_5] \times I_v$ and we may also assume that*

1. $\Phi_* \left(\mathcal{C}_p(V,B)\right) \subset \mathcal{C}_{\Phi(p)}(H, \varepsilon_B)$ *and* $\Phi_*^{-1}\left(\mathcal{C}_{\Phi(p)}(V,B)\right) \subset \mathcal{C}_p(H, \varepsilon_B)$ *for all* $p \in [u_2, u_3] \times I_v \cap \Phi^{-1}([u_4, u_5] \times I_v)$*;*
2. $\|\mathrm{d}\Phi(p,v)\| > \frac{L^2}{\sqrt{\eta}} \|v\|$ *and* $\|\mathrm{d}\Phi^{-1}(\Phi(p), w)\| > \frac{L^2}{\sqrt{\eta}} \|w\|$ *for all* $p \in [u_2, u_3] \times I_v \cap \Phi^{-1}([u_4, u_5] \times I_v)$ *and* $v \in \mathcal{C}_p(V,B)$*;*
3. *the return time in* $[u_2, u_3] \times I_v$ *is bounded by* τ*.*

Construction 7.32. Apply Proposition 7.25 with $\rho_0 \leq \varepsilon^h_{\mathrm{pert}}$ and Proposition 7.31.

The end of Section 7.5.3 is devoted to the proof of Proposition 7.31. The contact form from Proposition 7.25 is $\alpha = f(t,u,v)\mathrm{d}t + g(t,u,v)\mathrm{d}u$. Thus

$$R_\alpha = \frac{1}{\frac{\partial f}{\partial v} g - \frac{\partial g}{\partial v} f} \begin{pmatrix} -\frac{\partial g}{\partial v} \\ \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial t} \end{pmatrix}. \tag{18}$$

We progressively modify f and g so that the difference between the u-coordinates of two Reeb orbits which contribute to Φ widens when the Reeb orbits cross the convexification area.

Remark 7.33. If (f_1, g_1) and (f_1, g_2) are two convexification pairs satisfying conditions (1), (3) and (4) of Proposition 7.25 on $I_t \times J_u \times I_v$ then there exists a convexification pair (f_1, g) satisfying the same conditions such that $g = g_1$ outside a neighbourhood of $I_t \times [u_1, u_5] \times I_v$ and $g = g_2$ in a neighbourhood of $I_t \times [u_1, u_5] \times I_v$. There exists a analogous statement to interpolate between f-coordinates in a convexification pair if $f_1 = f_2$ near S_h.

If (f, g) is a convexification pair given in Proposition 7.25, without loss of generality, we have $1 \leq f \leq \frac{3}{2}$. Let Γ_v denote the g-level intersecting $(0, v)$ in a plane $u = \text{cst}$ for $u \in [u_1, u_5]$ and $\Gamma_{[v,v']}$ be the set between Γ_v and $\Gamma_{v'}$ if $v < v'$. Let $m_f > 1024\frac{u_5-u_1}{u_4-u_3}$ and $M_g = \frac{m_f t_1}{8(u_5-u_1)}$ for $t_1 > 0$. Without loss of generality $\rho < \frac{1}{4}$.

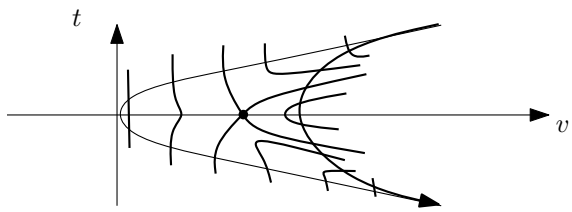

FIGURE 31. The g-levels

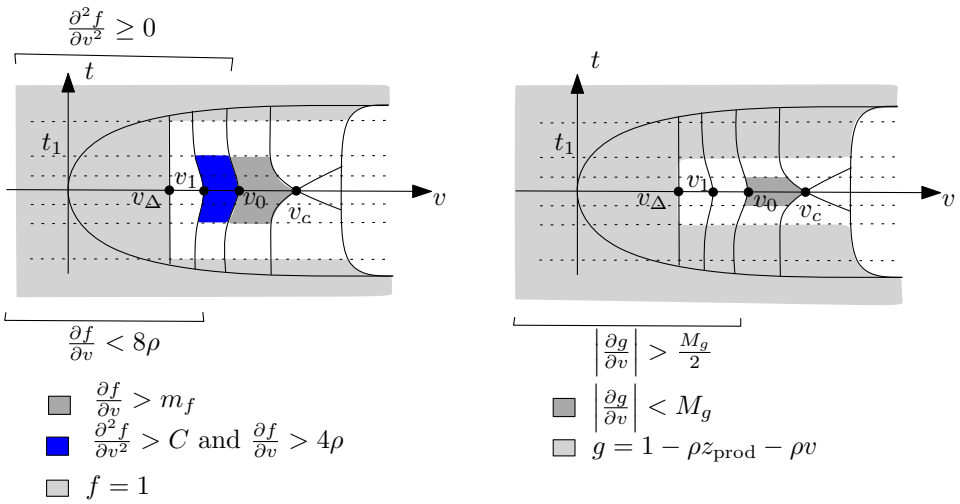

FIGURE 32. Conditions on f and g

Lemma 7.34. *Let $C > 0$. We may assume that there exist $t_1 > 0$, v_c, v_0 and v_1 such that $v_\Delta < v_1 < v_0 < v_c < v_h$ and for all $u \in [u_1, u_5]$ (see Figure 32)*

- $\frac{1}{2} \leq g \leq 1$ *and* $\left|\frac{\partial g}{\partial v}\right| \leq \rho$;
- $g(-t, v) = g(t, v)$ *and* $f(-t, v) = f(t, v)$;
- $(0, v_c)$ *is a saddle for g and the g-level intersecting $\{0\} \times [v_c, v_h]$ do not intersect Σ_+ or Σ_- (see Figure 31);*
- $g(t, u, v) = 1 - \rho z_{prod} - \rho v$ *for all (t, u, v) such that $|t| \geq t_1$ or (t, v) on $\Gamma_{[0,v_\Delta]}$;*
- $f(t, u, v) = 1$ *for all (t, u, v) such that $|t| \geq 2t_1$ or (t, v) on $\Gamma_{[0,v_\Delta]}$;*
- $\frac{\partial^2 g}{\partial v^2} \geq 0$ *and* $\frac{\partial g}{\partial v} < 0$ *on $\Gamma_{[0,v_c]}$;*
- $\left|\frac{\partial g}{\partial v}(t, v)\right| > \frac{M_g}{2}$ *and* $\frac{\partial^2 f}{\partial v^2}(t, u, v) \geq 0$ *on $\Gamma_{[0,v_0]}$;*
- $\frac{\partial f}{\partial v}(t, u, v) < 8\rho$ *on $\Gamma_{[0,v_1]}$;*
- $\frac{\partial f}{\partial v}(t, u, v) > m_f$ *for all (t, u, v) such that $|t| \leq t_1$ and $(t, v) \in \Gamma_{[v_0,v_c]}$;*
- $\left|\frac{\partial g}{\partial v}(t, v)\right| < M_g$ *for all (t, u, v) such that $|t| \leq \frac{t_1}{2}$ and $(t, v) \in \Gamma_{[v_0,v_c]}$;*

- $\frac{\partial^2 f}{\partial v^2}(t,u,v) > C$ *and* $\frac{\partial f}{\partial v}(t,u,v) > 4\rho$ *for all* (t,u,v) *such that* $|t| \leq t_1$ *and* $(t,v) \in \Gamma_{[v_1,v_0]}$*;*

Proof. We extensively use Remark 7.33 to modify f and g. We first modify f and choose v_c so that $\frac{\partial f}{\partial v} > 2m_f$ in a neighbourhood of $\{0\} \times [u_1, u_5] \times \{v_c\}$. To achieve this condition, we modify f near $(0, v_h)$ in a neighbourhood that does not intersect S_h and such that $\frac{\partial f}{\partial v} > 0$. The only non-trivial condition on the perturbed f is the contact condition $\frac{\partial f}{\partial v} g - \frac{\partial g}{\partial v} f > 0$. There exists $\varepsilon > 0$ such that for any f_1 with $|f - f_1| < \varepsilon$ and $\frac{\partial f_1}{\partial v} > \frac{\partial f}{\partial v} - \varepsilon$, then $\frac{\partial f_1}{\partial v} g - \frac{\partial g}{\partial v} f_1 > 0$. We choose f_1 such that $\frac{\partial f_1}{\partial v} > 2m_f$ near $(0, v_c)$, $|f - f_1| < \varepsilon$ and $\frac{\partial f_1}{\partial v} > \frac{\partial f}{\partial v} - \varepsilon$.

We now choose t_1 and v_0 such that $\rho > \frac{M_g}{2}$ and $\left|\frac{\partial f}{\partial v}(t,v)\right| > 2m_f$ for all (t,u,v) such that $|t| \leq t_1$ and $(t,v) \in \Gamma_{[v_0,v_c]}$. We modify g (and change v_0 if necessary) so that g satisfies the desired conditions. To obtain the contact condition we choose g so that $\frac{\partial g}{\partial v} > 0$ implies $\frac{\partial f}{\partial v} > 0$ and $\max\left(\frac{\partial g}{\partial v}\right) \ll 1$. Finally we modify f. □

Lemma 7.35. *The projection in the* (t,v)*-plane of any Reeb orbit which contributes to* Φ *is contained in* $\Gamma_{[v_1,v_0]}$*.*

Proof. The projection in the (t,v)-plane of a Reeb orbit is contained in a g-level. Thus no Reeb orbit contributes to Φ and intersects $\{0\} \times [u_1, u_5] \times [v_c, v_h]$.

If a Reeb orbit intersects $\{0\} \times [u_1, u_5] \times [v_0, v_c]$ then this orbit crosses the strip $|t| \leq t_1$ and is contained in $\Gamma_{[v_0,v_c]}$. In this strip

$$\frac{m_f}{2} \leq \frac{\partial f}{\partial v} g \leq \frac{\partial f}{\partial v} g - \frac{\partial g}{\partial v} f \leq \frac{5}{2}\left|\frac{\partial f}{\partial v}\right|,$$

$|R_t| \leq \frac{2M_g}{m_f}$ and $|R_u| \geq \frac{2}{5}$. Therefore, the time spent in the strip is bounded below by

$$\frac{2t_1}{\max(R_t)} \geq \frac{t_1 m_f}{M_g} = 8(u_5 - u_1)$$

and the u-interval swept out by the orbit is bounded below by $\min|R_u| \times 8(u_5 - u_1) > u_5 - u_1$. The orbit does not contribute to Φ.

We now consider a Reeb orbit which intersects $\{0\} \times [u_1, u_5] \times [-v_{\max}, v_1]$ and crosses the strip $|t| \leq 2t_1$. In this strip $\frac{1}{2}\left|\frac{\partial f}{\partial v}\right| \leq \frac{\partial f}{\partial v} g - \frac{\partial g}{\partial v} f \leq 10\rho \leq 4$, $|R_t| \geq \frac{M_g}{8}$ and $|R_u| \leq 2$. The return time between $-2t_1$ and $2t_1$ is bounded by $\frac{4t_1}{\min(R_t)} \leq \frac{32t_1}{M_g}$ and the u-interval swept out by the orbit is bounded by $\frac{64t_1}{M_g} < u_4 - u_3$. The orbit does not contribute to Φ as $R_u = 0$ for $|t| \geq 2t_1$. □

Proof of Proposition 7.31. We prove that Proposition 7.31 is satisfied for C big enough. We first study the difference between the u-coordinates of two Reeb orbits which contribute to Φ. Let $(-t_{\max}, u, v)$ and $(-t_{\max}, \tilde{u}, \tilde{v})$ be the endpoints of two Reeb chord which contribute to Φ. Without loss of generality $\tilde{v} > v$. Their projections on the (t,v)-plane are contained in $\Gamma_{[v_1,v_0]}$ (Lemma 7.35). Let

$$Y = \left(1, -\frac{\frac{\partial f}{\partial v}}{\frac{\partial g}{\partial v}}, -\frac{\frac{\partial g}{\partial t}}{\frac{\partial g}{\partial v}}\right)$$

be a renormalisation of the Reeb vector field and $t \mapsto \big(-t_{\max} + t, u(t), v(t)\big)$ and $t \mapsto \big(-t_{\max} + t, \tilde{u}(t), \tilde{v}(t)\big)$ be the Y-orbits with endpoints $(-t_{\max}, u, v)$ and $(-t_{\max}, \tilde{u}, \tilde{v})$. Then

$$\frac{\partial Y_u}{\partial v} = -\frac{\frac{\partial^2 f}{\partial v^2}\frac{\partial g}{\partial v} - \frac{\partial^2 g}{\partial v^2}\frac{\partial f}{\partial v}}{\left(\frac{\partial g}{\partial v}\right)^2} \geq -\frac{\frac{\partial^2 f}{\partial v^2}}{\frac{\partial g}{\partial v}} \geq 0$$

and $\tilde{u}-u$ is non-decreasing. In addition, $\frac{\partial Y_u}{\partial v} \geq \frac{C}{\rho}$ for $|t| \leq \frac{t_1}{2}$. Thus, we have

$$\tilde{u}(t_{\max}) - u(t_{\max}) \geq \tilde{u} - u + \frac{C}{\rho} \min_{|t| \leq \frac{t_1}{2}} (\tilde{v}(t) - v(t)).$$

Our orbits are contained in g-levels, therefore it holds that

$$(\tilde{v}(t) - v(t)) \min \left(\left| \frac{\partial g}{\partial v} \right| \right) \leq g(-t_{\max}, v) - g(-t_{\max}, \tilde{v}) \leq (\tilde{v}(t) - v(t)) \max \left(\left| \frac{\partial g}{\partial v} \right| \right)$$

and $\tilde{v}(t) - v(t) \geq \frac{M_g(\tilde{v}-v)}{2\rho}$ as $\frac{M_g}{2} \leq \left| \frac{\partial g}{\partial v} \right| \leq \rho$. Thus we obtain

$$\tilde{u}(t_{\max}) - u(t_{\max}) \geq \tilde{u} - u + \frac{C M_g t_1 (\tilde{v} - v)}{2\rho^2}. \tag{19}$$

Let γ be a curve in $\{-t_{\max}\} \times [u_2, u_3] \times I_v$ such that $|\gamma'(v)| \leq B$ (the v-coordinate is the vertical coordinate). Let δ be its image on $\{t_{\max}\} \times [u_4, u_5] \times I_v$. By symmetry, if it is well-defined, the image of $(-t_{\max}, \gamma(v), v)$ is $(t_{\max}, \delta(v), v)$. Using equation (19), we obtain

$$\delta'(v) \geq \gamma'(v) + \frac{C M_g}{2\rho^2} \geq -B + \frac{C M_g}{2\rho^2} = D. \tag{20}$$

Therefore, we have $D \geq \frac{1}{\varepsilon_B}$ for C big enough. A similar proof shows the symmetric result. Additionally, if $w = (1, \gamma'(v))$ then $\|w\| \leq \sqrt{1+B^2}$ and $\|\mathrm{d}\Phi(p,w)\| \geq \sqrt{1+D^2}$. Thus the dilatation condition is satisfied for C big .

Finally the return time between $\{-t_{\max}\} \times [u_2, u_3] \times I_v$ and $\{t_{\max}\} \times [u_4, u_5] \times I_v$ is bounded by $2(u_5 - u_1) + 2t_{\max}$. Indeed, for $|t| \geq t_1$, we have $|R_t| \leq 1$ and the return time is bounded by $2t_{\max}$. Additionally, for $|t| \leq t_1$, we have $|R_u| =\geq \frac{8}{11}$ and the return time is bounded by $\frac{11}{8}(u_5 - u_1)$ as the u-interval is bounded by $u_5 - u_1$. As $u_5 - u_1 \leq \left| \int \Gamma_{\lambda_D} \alpha \right| < \tau$ by definition of pre-convex bypass, we obtain the desired condition on the return time. □

7.5.4. *Convexification smoothing.* In this section we interpolate between α_{conv} and α_{prec} for $y \geq \frac{2}{5} y_{\mathrm{std}}$.

Construction 7.36. For $y \in \left[\frac{2}{5} y_{\mathrm{std}}, y_{\mathrm{std}}\right]$, let $\alpha = \alpha_{\mathrm{prec}} + a l(y) \Phi_{z_0}(z) \sin(x) \mathrm{d}y$ where l is non increasing, $l = 1$ in $\left[\frac{2}{5} y_{\mathrm{std}}, \frac{1}{2} y_{\mathrm{std}}\right]$ and $l = 0$ for $y \geq \frac{2}{3} y_{\mathrm{std}}$. We extend this construction to the other non-convex areas by symmetry.

The 1-form α is a contact form as $a l(y) \Phi_{z_0}(z) \sin(x)$ is $\mathcal{C}^1$-close to 0. Let $\mathcal{B}_{\mathrm{conv}}$ be the convexified bypass and α_{conv} the associated contact structure. We call $\mathcal{C} = \mathcal{B}_{\mathrm{conv}} \backslash \mathring{\mathcal{B}}$ the *convexification area* and we denote by $\mathcal{P}$ the set where $\alpha_{\mathrm{conv}} \neq \alpha_{\mathrm{prec}}$. In coordinates (x, y, z), the connected component of $\mathcal{P}$ containing Γ_0 is the set $z \geq z_0$. In coordinates (t, u, v), it is contained between $S_{\mathcal{B}}$ and S_h and its v-coordinates are in $(v_\Delta, v'_{\max})$ (Proposition 7.25).

Corollary 7.37 (Corollary of Lemma 7.21)**.** *Let γ be a Reeb orbit intersecting $\mathcal{P}$. If γ enters $\mathcal{C}$ in $p_{in} = (x_{in}, y_{in}, z_{in})$ such that $x_{in} \in \left[\frac{\pi}{2} - \frac{\lambda}{4}, \frac{\pi}{2} + \frac{\lambda}{4}\right]$ and $z_{in} > 0$ then the exiting point $p_{out} = (x_{out}, y_{out}, z_{out})$ satisfies $x_{out} \in \left[\frac{\pi}{2} - \frac{\lambda}{3}, \frac{\pi}{2} + \frac{\lambda}{3}\right]$ and $y_{out} > y_{in}$.*

Proof. If γ intersects the set $y \geq \frac{2}{5} y_{\mathrm{std}}$, we obtain the desired result as $R_y > 0$ and h is defined for $x \in \left[\frac{\pi}{2} - \frac{\lambda}{3}, \frac{\pi}{2} + \frac{\lambda}{3}\right]$. We now assume that γ is contained in the set $y \leq \frac{2}{5} y_{\mathrm{std}}$. As γ intersects $\mathcal{P}$, we have $t_{\mathrm{out}} - t_{\mathrm{in}} \geq \Delta$, and $p_{\mathrm{in}}, p_{\mathrm{out}} \in I'_t \times I_u \times I'_v$ (Lemma 7.21 and Construction 7.22). In addition $u_{\mathrm{out}} - u_{\mathrm{in}} \geq 0$ as $R_u \geq 0$. Lemma 7.21 gives the desired result. □

By symmetry, there exist analogous statements near any endpoint of $\Gamma_{\mathrm{prec}}^{\leq y_{\mathrm{std}}}$.

7.6. **Conditions (B1) to (B8).** We now prove that our construction satisfies conditions (B1) to (B8) and is adapted to the boundary.

The contact form is adapted to the boundary. By the definition of pre-convex bypass, the contact form is adapted to the boundary outside $\mathcal{C}$. The contact form is adapted for $y \geq \frac{2}{3}y_{\text{std}}$ as $\alpha_{\text{conv}} = \alpha_{\text{prec}}$ and $\|h\|_{\mathcal{C}^\infty} < \varepsilon'_{\text{stab}}$ (Lemma 7.14). For $y \leq \frac{2}{5}y_{\text{std}}$ and $z \leq \frac{2}{5}y_{\text{std}} + z_{\text{prod}}$, Lemma 7.25 gives the desired result. For $y \in \left[\frac{2}{5}y_{\text{std}}, \frac{2}{3}y_{\text{std}}\right]$ and $x^h_{\text{flat}} \leq |x - \frac{\pi}{2}|$, we apply Lemma 7.23. Finally, for $y \in \left[\frac{2}{5}y_{\text{std}}, \frac{2}{3}y_{\text{std}}\right]$ and $|x - \frac{\pi}{2}| < x^h_{\text{flat}}$, we have $h = h(0)$ and R_z is positively collinear to

$$\left(k_{\text{inf}}(z)k_{\text{sup}}(z) + al(y)\Phi_{z_0}(z)\right)\cos(x).$$

Thus

- $R_z = 0$ for $x = \frac{\pi}{2}$;
- $R_z > 0$ for $x < \frac{\pi}{2}$;
- $R_z < 0$ for $x > \frac{\pi}{2}$.

The tangency points between $\tilde{S}_h$ and $R_{\alpha_{\text{conv}}}$ are the segment $x = \frac{\pi}{2}$. Along this segment and for $y \leq \frac{1}{2}y_{\text{std}}$, R_x is positively collinear to $f_{\text{inf}}(y)\rho - a\Phi'_{z_0}(z_h)$ and

$$f_{\text{inf}}(y)\rho - a\Phi'_{z_0}(z_h) \leq \rho - a\Phi'_{z_0}(z_h) < 0.$$

For $y > \frac{1}{2}y_{\text{std}}$, R_x is positively collinear to

$$-k'_{\text{sup}}(z_h) - cg(y)\Phi'_{z_0}(z_h) < 0.$$

By symmetry we obtain the desired result in the other convexified areas.

Condition (B6). Let γ be a Reeb chord of S_Z in $\mathcal{B}^{\leq y_{\text{std}}}_{\text{conv}}$. If γ does not meet $\mathcal{P}$, condition (B6) is given by Lemma 7.6. We now assume that γ intersects $\mathcal{P}$. Let p^S_{in} and p^S_{out} denote the endpoints of γ and p_{in} and p_{out} the first entering and exiting point of $\mathcal{C}$. We assume that p_{in} is in the connected component of $\mathcal{C}$ containing Γ_0. The proof of the other cases is similar.

If γ does not intersect $\mathcal{P}$ after p_{out}, then γ is contained in $\left[\pi - \frac{\lambda}{4}, \pi + \frac{\lambda}{4}\right] \times [0, y_{\text{std}}] \times I_{\max}$ after p_{out} (Lemma 7.8 and Equation (7)). Thus we have $x^S_{\text{out}} \in \left[\pi - \frac{\lambda}{2}, \pi + \frac{\lambda}{2}\right]$ (Lemma 7.5). Then $x_{\text{in}} \in \left[\pi - \frac{\lambda}{3}, \pi + \frac{\lambda}{3}\right]$ and $z_{\text{in}} > z_{\text{out}}$ (Corollary 7.37). Therefore $z^S_{\text{in}} > z_{\text{in}} > z_{\text{out}} > z^S_{\text{out}}$ as $R_z < 0$. We obtain $x^S_{\text{in}} \in \left[\pi - \frac{\lambda}{2}, \pi + \frac{\lambda}{2}\right]$ (Lemma 7.5).

If γ intersects $\mathcal{P}$ after p_{out}, then γ meets the connected component associated to $\left[\pi, \frac{3\pi}{2}\right] \times [0, y_{\text{std}}] \times [-z_{\text{std}}, 0]$ (Lemma 7.8). Let p'_{in} and p'_{out} denote the second entering and exiting point. Between p_{out} and p'_{in}, γ is contained $\left[\pi - \frac{\lambda}{4}, \pi + \frac{\lambda}{4}\right] \times [0, y_{\text{std}}] \times I_{\max}$ (Lemma 7.8). Thus we have $x_{\text{in}} \in \left[\pi - \frac{\lambda}{3}, \pi + \frac{\lambda}{3}\right]$ and $z_{\text{in}} > z_{\text{out}}$ (Corollary 7.37). Therefore $x^S_{\text{in}} \in \left[\pi - \frac{\lambda}{2}, \pi + \frac{\lambda}{2}\right]$ (Lemma 7.5). As γ does not intersect $\mathcal{P}$ before p_{in} and $R_z < 0$, we obtain $z^S_{\text{in}} > z_{\text{in}} > z_{\text{out}} > z'_{\text{out}}$. In addition $x'_{\text{out}} \in \left[\pi - \frac{\lambda}{3}, \pi + \frac{\lambda}{3}\right]$ and $z'_{\text{in}} > z'_{\text{out}}$ (Corollary 7.37). As γ does not meet $\mathcal{P}$ after p'_{out} (Lemma 7.8) and $R_z < 0$, we obtain $z^S_{\text{in}} > z_{\text{in}} > z_{\text{out}} > z'_{\text{in}} > z'_{\text{out}} > z^S_{\text{out}}$ and $x^S_{\text{out}} \in \left[\pi - \frac{\lambda}{2}, \pi + \frac{\lambda}{2}\right]$ (Lemma 7.5).

Condition (B7). Let γ be a Reeb orbit in $\mathcal{B}^{\leq y_{\text{std}}}_{\text{conv}}$ with endpoints p^S_{in} and p^S_{out} in S_Z and $S_{y_{\text{std}}}$. If γ does not meet $\mathcal{P}$ we obtain the desired result by Lemma 7.6. We now assume that γ meets $\mathcal{P}$. The image of $\mathcal{P}$ on S_Z is contained in $X \cup X + 2\pi$. Thus $p^S_{\text{in}} \in X + 2k\pi$ for $k \in \{0, 1\}$. In addition, there exists $k' \in \{0, 1\}$ such that $p^S_{\text{out}} \in \left[\frac{\pi}{2} - \frac{\lambda}{2} + 2k'\pi, \frac{\pi}{2} + \frac{\lambda}{2} + 2k'\pi\right] \times I_{\max}$ (Lemma 7.6). It remains to prove that $k = k'$. If γ meets $\mathcal{P}$ once, then the x-coordinate of the exiting point is in $\left[2k\pi - \frac{\lambda}{8}, (2k+1)\pi + \frac{\lambda}{8}\right]$ as $\mathcal{P} \subset \mathcal{V}$ and thus $k = k'$ (Lemma 7.5). If γ meets $\mathcal{P}$ twice, then the first exiting point has a x-coordinate in $\left[2k\pi - \frac{\lambda}{8}, (2k+1)\pi + \frac{\lambda}{8}\right]$. Thus the x-coordinate of the second entering point is in $\left[2k\pi - \frac{\lambda}{4}, (2k+1)\pi + \frac{\lambda}{4}\right]$

(Lemma 7.5) and the x-coordinate of the second exiting point is contained in $\left[2k\pi - \frac{\lambda}{3}, (2k+1)\pi + \frac{\lambda}{3}\right]$ (Corollary 7.37). Thus $k = k'$. The proof of *condition (B8)* is similar.

Condition (B5). By Remark 7.7, if γ is a Reeb chord of $S_{y_{\text{std}}}$ in $\mathcal{B}^{\leq y_{\text{std}}}_{\text{conv}}$ then γ intersects $S_{\frac{2}{3}y_{\text{std}}}$. Let p^S_{in} and p^S_{out} denote the endpoints of γ. By Lemma 7.6, $x^S_{\text{in}} \in \left[\frac{3\pi}{2} - \frac{\lambda}{2} + 2k\pi, \frac{3\pi}{2} + \frac{\lambda}{2} + 2k\pi\right]$ and $x^S_{\text{out}} \in \left[\frac{\pi}{2} - \frac{\lambda}{2} + 2k'\pi, \frac{\pi}{2} + \frac{\lambda}{2} + 2k'\pi\right]$. In addition γ intersects $\mathcal{P}$ (Lemma 7.5). Yet γ intersects only one connected component of $\mathcal{P}$ (Lemma 7.8). This leads to a contradiction.

Condition (B4). This condition is a consequence from Lemma 7.9.

Conditions (B1), (B2) and (B3). These conditions derive from Lemma 7.17 and Lemma 7.31. Indeed, by Lemma 7.6, all the Reeb chords which contribute to the map between R_λ and S_R intersect $\mathcal{P}$ and thus Σ_+ and Σ_-. In addition, the intersection points with Σ_- are in $[u_2, u_3] \times I_v$ (Equation (11)) and the intersection points with Σ_+ in $[u_4, u_5] \times I_v$ (Equations (9) and (10) and Lemma 7.6). Let δ be a curve in $\left[\frac{\pi}{2} + \lambda, \pi - \lambda\right] \times I_{\max}$ with tangents in $\mathcal{C}(V, A)$. Then the tangents of the image of δ in Σ_- are in $\mathcal{C}(V, B)$ (Lemma 7.17) and the tangents of the image of δ in Σ_+ are in $\mathcal{C}(H, \varepsilon_B)$ (Lemma 7.31). Thus the image of δ on S_R is ε_R close to δ_1. Similarly, the tangent of the image of an horizontal segment in S_R are in $\mathcal{C}(H, \nu)$ (Lemma 7.17). Condition (B3) is a consequence of the definition of pre-convex bypasses. We obtain the rectangle structures on $\text{dom}(\varphi_i)$ and $\text{im}(\varphi_i)$ by considering the images of vertical curves in S_Z and the inverse images of horizontal curves in S_R. These curves are transverse (definition of pre-convex bypasses).

8. Conley-Zehnder index

In this section we prove Theorem 2.6: we compute the Conley-Zehnder index μ of the periodic orbit $\gamma_{\mathbf{a}}$ described in Theorem 2.1.

8.1. **Two technical lemmas.**

Lemma 8.1. *Let $(R_t)_{t\in[0,1]}$ be a path of symplectic matrices in $\mathbb{R}^2$ such that $R_0 = \text{Id}$ and $R_1 \in Sp^*$. Let $R_t e_1 = r(t)e^{i\alpha(t)}$. If $\alpha(1) \in \left[2k\pi + \frac{\pi}{2}, 2k\pi + \frac{3\pi}{2}\right]$ and $\mu(R)$ is odd, then then $\mu(R) = 2k+1$. Similarly if $\mu(R)$ is even and $\alpha(1) \in \left[2k\pi - \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}\right]$, then $\mu(R_t) = 2k$.*

Proof. We extend R_t, α_t and r_t to $t \in [1, 2]$ (see Section 3.3.2). Let θ_t denote the rotation angle associated to the polar decomposition $R_t = S_t O_t$. Without loss of generality $\theta_0 = 0$. As S_t is positive-definite, $\theta_t - \frac{\pi}{2} < \alpha_t < \theta_t + \frac{\pi}{2}$. Additionally, if there exists $t \in [1, 2]$ such that $\theta_t = 0[2\pi]$ then $R_t \in \text{Sp}^-(2)$. Similarly if $\theta_t = \pi[2\pi]$ then we have $R_t \in \text{Sp}^+(2)$. Therefore if $\mu(R)$ is odd, $\theta_t \neq 0[2\pi]$ for all $t \in [1, 2]$. Thus $\theta_1 - \pi < \theta_2 < \theta_1 + \pi$ and $\alpha_1 - \frac{3\pi}{2} < \theta_2 < \alpha_1 + \frac{3\pi}{2}$. Therefore $\theta_2 \in ((2k-1)\pi, (2k+3)\pi)$. The proof of the other case is similar. □

Lemma 8.2. *Let $\theta_0 > 0$. There exists $\nu(\theta_0) > 0$ such that if $R \in \text{Sp}(2)$ and*

- *$Re_1 \in \mathcal{C}(e_1, \tan(\nu(\theta_0)))$;*
- *$\|Re_1\| \geq 3$;*
- *there exists $f \in C(e_2, \tan(\theta_0))$ such that $Rf \in C(e_2, \tan(\theta_0))$;*

then R is $\mathbb{R}$-diagonalizable and its eigenvalues are of the sign of $\langle e_1, Re_1\rangle$.

Proof. We prove that $|\text{tr}(R)| > 2$. Without loss of generality, (e_1, f) is a direct basis. The matrix associated to the change of basis from (e_1, e_2) to (e_1, f) is

$$P = \begin{pmatrix} 1 & \cos(\theta) \\ 0 & \sin(\theta) \end{pmatrix}$$

where $\theta \in \left[\frac{\pi}{2} - \theta_0, \frac{\pi}{2} + \theta_0\right]$. In the basis (e_1, e_2), the matrix of R is

$$P^{-1}RP = \frac{1}{\sin(\theta)} \begin{pmatrix} \mu_1 \sin(\theta - \theta_1) & \mu_2 \sin(\theta - \theta_2) \\ \mu_1 \sin(\theta_1) & \mu_2 \sin(\theta_2) \end{pmatrix}$$

where $\theta_1 \in [-\nu(\theta_0), \nu(\theta_0)]$, $\theta_2 \in \left[\frac{\pi}{2} - \theta_0, \frac{\pi}{2} + \theta_0\right]$ and $|\mu_1| \geq 3$. As $\mu_1\mu_2 > 0$, we obtain $\mathrm{tr}(R) > 2$ for $\nu(\theta_0)$ small enough. □

8.2. **Computation of the Conley-Zehnder index.** Let $\mathbf{a} = a_{i_1} \dots a_{i_k}$ be a word such that $l(\mathbf{a}) < K$. Let $p_{\mathbf{a}}$ be an intersection point between $\gamma_{\mathbf{a}}$ and S_Z^-. We denote by $T(\mathbf{a})$ the period of $\gamma_{\mathbf{a}}$. Let R_t be the path of symplectic matrices along $\gamma_{\mathbf{a}}$ associated to the trivialisation described in Section 2.4. Let $\chi_{\mathbf{a}}$ be the map induced by the Reeb flow between S_Z and $S_{\mathbf{a}}$, a surface tangent to $\xi(p_{\mathbf{a}})$ at $p_{\mathbf{a}}$. Let $G_{\mathbf{a}} = \varphi_B \circ \psi_{i_k} \cdots \circ \varphi_B \circ \psi_{i_1}$. By definition of φ_B and ψ,

- $\mathrm{dom}(G_{\mathbf{a}})$ and $\mathrm{im}(G_{\mathbf{a}})$ are rectangles with respectively horizontal and vertical fibres and $G_{\mathbf{a}}$ preserves the fibres;
- $\mathrm{d}G_{\mathbf{a}}\left(p_{\mathbf{a}}, \frac{\partial}{\partial x}\right) \in \mathcal{C}(H, \nu)$;
- $\left\|\mathrm{d}G_{\mathbf{a}}\left(p_{\mathbf{a}}, \frac{\partial}{\partial x}\right)\right\| \geq \frac{1}{(\eta M)}$.

Lemma 8.3. *There exists θ_0 such that for μ, ν and η small enough, $R_{T(\mathbf{a})}$ satisfies the hypothesis of Lemma 8.2.*

Proof. Note that $\mathrm{d}\chi_{\mathbf{a}}(p_{\mathbf{a}})\frac{\partial}{\partial x} = \pm e_1$. We choose θ_0 such that $\chi_{\mathbf{a}*}(\mathcal{C}(V, A)) \subset \mathcal{C}(e_2, \tan(\theta_0))$. For μ small enough, we have $\chi_{\mathbf{a}*}(\mathcal{C}(H, \mu)) \subset \mathcal{C}(e_1, \tan(\nu(\theta_0))$. Let l be such that $\|\mathrm{d}\chi_{p_{\mathbf{a}}}(p_{\mathbf{a}})\| < l$ and $\|\mathrm{d}\chi_{p_{\mathbf{a}}}(p_{\mathbf{a}})^{-1}\| < l$. Then

$$\|R_{\mathbf{a},T(\mathbf{a})}e_1\| \geq \left\|d\chi_{p_{\mathbf{a}}}(p_{\mathbf{a}})^{-1}\right\|^{-1} \left\|\mathrm{d}G_{\mathbf{a}}\frac{\partial}{\partial x}\right\| \geq \frac{1}{l\eta M}$$

and $\|Re_1\| \geq 3$ for η small enough. □

Lemma 8.4. *For all $p \in \mathrm{dom}(G_{\mathbf{a}})$, $\left\langle \mathrm{d}G_{\mathbf{a}}(p)\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right\rangle$ is of the sign of $\prod_{j=1}^{k}(-1)^{\tilde{\mu}(a_{i_j})}$.*

Proof. Note that $\left\langle \mathrm{d}\varphi_B(p)\frac{\partial}{\partial z}, \frac{\partial}{\partial x}\right\rangle > 0$ for all $p \in \mathrm{dom}(\varphi_B)$. Therefore, we have

$$\left\langle \mathrm{d}\varphi_B(p)v, \frac{\partial}{\partial x}\right\rangle > 0$$

for all $v \in \mathcal{C}_p(V, A)$ such that $\left\langle v, \frac{\partial}{\partial z}\right\rangle > 0$.

We prove the desired result by induction on k. If $k = 1$ then $\mathrm{d}\psi_{a_1}(p)\frac{\partial}{\partial x} \in \mathcal{C}(V, A)$. If $\tilde{\mu}(a_1)$ is even, $\left\langle -\mathrm{d}\psi_{a_1}(p)\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right\rangle < 0$ (see Figure 33) and we obtain

$$\left\langle \mathrm{d}G_{a_1}(p)\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right\rangle > 0.$$

Similarly, if $\mu(a_1)$ is odd, we obtain $\left\langle -\mathrm{d}\psi_{a_1}(p)\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right\rangle > 0$ and

$$\left\langle \mathrm{d}G_{a_1}(p)\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right\rangle < 0.$$

We now prove the result for $\mathbf{a} = a_{i_1} \dots a_{i_{k+1}}$. Let $p \in \mathrm{dom}(G_{\mathbf{a}})$ and $v = \mathrm{d}G_{a_{i_1} \dots a_{i_k}}(p)\frac{\partial}{\partial x}$. By induction, $\left\langle v, \frac{\partial}{\partial x}\right\rangle$ is of the sign of $\prod_{j=1}^{k}(-1)^{\tilde{\mu}(a_{i_j})}$. Then, $\left\langle \mathrm{d}\psi_{a_{i_{k+1}}}v, \frac{\partial}{\partial z}\right\rangle$ is of the sign of $\prod_{j=1}^{k+1}(-1)^{\tilde{\mu}(a_{i_j})}$ and so is $\left\langle \mathrm{d}G_{a_{i_{k+1}}}v, \frac{\partial}{\partial x}\right\rangle$. □

Corollary 8.5. *For μ, ν and η small enough, $\sum_{j=1}^{k}\tilde{\mu}(a_{i_j}) = \mu(\gamma_{\mathbf{a}})$[2].*

Proof. The signs of $\left\langle \mathrm{d}G_{\mathbf{a}}\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right\rangle$ and $\left\langle R_{T(\mathbf{a})}\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right\rangle$ coincide and $R_{T(\mathbf{a})}$ is hyperbolic (Lemmas 8.2 and 8.3). Its eigenvalues are positive if $\sum_{j=1}^{k}\tilde{\mu}(a_{i_j})$ is even and negative if $\sum_{j=1}^{k}\tilde{\mu}(a_{i_j})$ is odd (Lemma 8.4). □

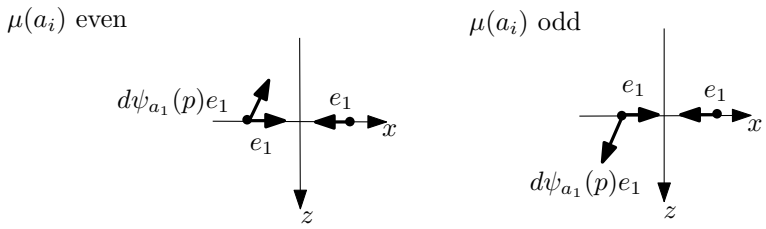

FIGURE 33. The vector $\mathrm{d}\psi_{a_1}(p)e_1$

Lemma 8.6. *There exists a collar neighbourhood $S_{\gamma_{\mathbf{a}}}$ of $\gamma_{\mathbf{a}}$ in the trivialisation class given in Section 2.4 such that if R_t is the associated path of symplectic matrices along $\gamma_{\mathbf{a}}$ and $R_t\frac{\partial}{\partial x} = r(t)e^{i\theta(t)}$ then $\theta(t) \neq 0[\pi]$ when $\gamma_{\mathbf{a}}(t)$ is in the bypass and $t > 0$.*

Proof. Let c be a Reeb chord in the bypass contained in $\gamma_{\mathbf{a}}$ with endpoints c_+ and c_- on S_Z. We construct a strip S_c along c such that no Reeb chord with one endpoint on the vertical segment containing c_+ and close to c_+ intersects S_c. We then glue together the half of S_c and the collar associated to the Reeb chords to obtain $S_{\gamma_{\mathbf{a}}}$. We choose S_c such that

(1) between S_Z and the convexification, S_c is tangent to $\frac{\partial}{\partial x}$;
(2) in the convexification area, S_c is tangent to $\frac{\partial}{\partial u}$;
(3) is the upper part of the bypass, S_c is tangent $\frac{\partial}{\partial r}$.

We smooth the resulting surface. Figure 34 shows c and a Reeb chord with one endpoint on the vertical segment contained c_+ (dotted curve). In this figure S_c is transverse to the projection.

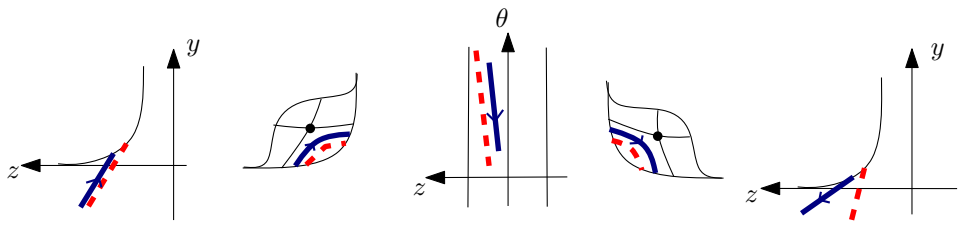

FIGURE 34. The surface S_c and the Reeb chords

□

Proof of Theorem 2.6. We consider the trivialisation from Lemma 8.6. Without loss of generality $\theta(0) = 0$. Let $0 = t_1 < t_1' < \cdots < t_k < t_k' = T(\mathbf{a})$ be the times associated to the intersection points between $\gamma_{\mathbf{a}}$ and S_Z ($\gamma_{\mathbf{a}}(0)$ is the fixed point of $G_{\mathbf{a}}$).

We prove by induction that for all $j = 1, \ldots, k$

$$\theta(t_j) \in \left[\left(\sum_{l=1}^{j-1} \mu(a_{i_l})\pi\right) - \nu(\theta_0), \left(\sum_{l=1}^{j-1} \mu(a_{i_l})\pi\right) + \nu(\theta_0)\right]. \tag{21}$$

As $\theta(0) = 0$, the condition (21) is satisfied for $j = 1$. We now suppose that the equation (21) stands for $i \in 1 \dots j-1$. By definition of $\mu(a_{i_j})$,

$$\theta(t'_j) \in \left[\left(\sum_{l=1}^{j} \mu(a_{i_l})\pi \right), \left(\sum_{l=1}^{j} \mu(a_{i_l})\pi \right) + \pi \right].$$

We obtain (Lemma 8.6)

$$\theta(t_{j+1}) \in \left[\left(\sum_{l=1}^{j} \mu(a_{i_l})\pi \right) - \nu(\theta_0), \left(\sum_{l=1}^{j} \mu(a_{i_l})\pi \right) + \nu(\theta_0) \right]$$

$$\text{or} \qquad \theta(t_{j+1}) \in \left[\left(\sum_{l=1}^{j} \mu(a_{i_l})\pi \right) + \pi - \nu(\theta_0), \left(\sum_{l=1}^{j} \mu(a_{i_l})\pi \right) + \pi + \nu(\theta_0) \right].$$

By Lemma 8.4, we obtain the equation (21) for $i = j$. Lemma 8.1 provides us with the desired Conley-Zehnder index. □

Acknowledgements. I am deeply grateful to my advisor, Vincent Colin, for his guidance and support. I would also like to thank Samuel Tapie, Paolo Ghiggini and François Laudenbach for stimulating discussions. Thanks also go to Chris Wendl for his help with holomorphic curves, Jean-Claude Sikorav for suggesting numerous improvements and corrections and Marc Mezzarobba for proofreading this text.

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