

# Extracting Herbrand trees in classical realizability using forcing

## Derivations of some proof terms

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$$\begin{aligned}
\perp &:= \forall Z^o. Z \\
\neg A &:= A \Rightarrow \perp \\
a =_{\tau} b &:= \forall Z^{\tau \rightarrow o}. Z a \Rightarrow Z b \\
A \wedge \dots \wedge B &:= \forall Z^o. (A \Rightarrow \dots \Rightarrow B \Rightarrow Z) \Rightarrow Z \\
A \vee \dots \vee B &:= \forall Z^o. ((A \Rightarrow Z) \Rightarrow \dots \Rightarrow (B \Rightarrow Z) \Rightarrow Z) \Rightarrow Z \\
\exists x^{\tau}. A \&\dots \& B &:= \neg(\forall x^{\tau}. A \Rightarrow \dots \Rightarrow B \Rightarrow \perp)
\end{aligned}$$

Figure 1: Encoding of usual connectives into minimal logic

$$\begin{array}{c}
\frac{}{\mathcal{E}; \Gamma, x : A \vdash x : A} \qquad \frac{\mathcal{E}; \Gamma \vdash t : A}{\mathcal{E}; \Gamma \vdash t : A'} \text{ } A \approx_{\mathcal{E}} A' \\
\frac{\mathcal{E}; \Gamma, x : A \vdash t : B}{\mathcal{E}; \Gamma \vdash \lambda x. t : A \Rightarrow B} \\
\frac{\mathcal{E}; \Gamma \vdash t : A \Rightarrow B \quad \mathcal{E}; \Gamma \vdash u : A}{\mathcal{E}; \Gamma \vdash t u : B} \\
\frac{\mathcal{E}, M = N; \Gamma \vdash t : A}{\mathcal{E}; \Gamma \vdash t : M \doteq_{\tau} N \mapsto A} \qquad \frac{\mathcal{E}; \Gamma \vdash t : M \doteq_{\tau} M \mapsto A}{\mathcal{E}; \Gamma \vdash t : A} \\
\frac{\mathcal{E}; \Gamma \vdash t : A}{\mathcal{E}; \Gamma \vdash t : \forall x^{\tau}. A} \text{ } x \notin FV(\Gamma) \qquad \frac{\mathcal{E}; \Gamma \vdash t : \forall x^{\tau}. A}{\mathcal{E}; \Gamma \vdash t : A[N^{\tau}/x^{\tau}]} \\
\hline
\mathcal{E}; \Gamma \vdash \text{callcc} : ((A \Rightarrow B) \Rightarrow A) \Rightarrow A
\end{array}$$

Figure 2: Deduction rules for  $\text{PA}\omega^+$

## 1 Foreword

This appendix is not meant to be read but is rather given as a reference for the definitions and derivations of the proof terms whose existence is claimed in the paper.

The presentation of formal derivation we use is inspired from Fitch-style natural deduction, in the sense that we use a linear presentation and not proof trees. (This is necessary when proof-terms become big).

The bracket on the left side of proofs denote the scope of assumptions that can be of three kinds:

- abstractions, denoted by  $[\lambda]$ , that introduce proof variables (that thus modify both the term and its type);
- universal quantifications, denoted by  $[\forall]$ , that do not alter the proof term but only introduce term variables;
- equations, denoted by  $[\text{Eq.}]$ , that do not effect the proof term but introduce equations for the congruence  $\approx_{\mathcal{E}}$ . (The subscript  $\mathcal{E}$  will never be written explicitly).

Whenever we write  $[Y/Z^*r]$ , it stands for the substitution  $[(\lambda \vec{x}r. Y \vec{x})/Z^*]$ . Note that  $Y$  and  $Z$  will often be the same letter. As in the paper,  $\equiv$  stands for syntactic identity,  $\approx$  stands for congruence and  $=$  stand for Leibniz equality.

Let us first remind the reader of the definition of the connectives in second-order encodings, the full set of inference rules of the congruence  $\approx_{\mathcal{E}}$  and the deduction rules of system  $\text{PA}\omega^+$ .

### Extending the proof system with new constructions

#### Notations

In order to ease the presentation of the proof and make it both more understandable and program-like, we will use the following abbreviations:

**Reflexivity, symmetry, transitivity and base case**

$$\frac{}{M \simeq_{\mathcal{E}} M} \quad \frac{}{M \simeq_{\mathcal{E}} N} \quad (M = N) \in \mathcal{E} \quad \frac{M \simeq_{\mathcal{E}} N}{N \simeq_{\mathcal{E}} M} \quad \frac{M \simeq_{\mathcal{E}} N \quad N \simeq_{\mathcal{E}} P}{M \simeq_{\mathcal{E}} P}$$

**Context closure**

$$\frac{M \simeq_{\mathcal{E}} N}{\lambda x. M \simeq_{\mathcal{E}} \lambda x. N} \quad \frac{M \simeq_{\mathcal{E}} N \quad P \simeq_{\mathcal{E}} Q}{MP \simeq_{\mathcal{E}} NQ} \quad \frac{A \simeq_{\mathcal{E}} B \quad C \simeq_{\mathcal{E}} D}{A \Rightarrow C \simeq_{\mathcal{E}} B \Rightarrow D}$$

$$\frac{A \simeq_{\mathcal{E}} B}{\forall x^{\tau}. A \simeq_{\mathcal{E}} \forall x^{\tau}. B} \quad \frac{M \simeq_{\mathcal{E}} N \quad P \simeq_{\mathcal{E}} Q \quad A \simeq_{\mathcal{E}, M=P} B}{M \doteq P \mapsto A \simeq_{\mathcal{E}} N \doteq Q \mapsto B}$$

**$\beta\eta$ -conversion**

$$\frac{}{(\lambda x. M) N \simeq_{\mathcal{E}} M[N^{\tau}/x^{\tau}]} \quad \frac{}{\lambda x. M x \simeq_{\mathcal{E}} M} \quad x \notin FV(M)$$

$$\frac{}{\text{rec}_{\tau} M N 0 \simeq_{\mathcal{E}} M} \quad \frac{}{\text{rec}_{\tau} M N (S P) \simeq_{\mathcal{E}} N P (\text{rec}_{\tau} M N P)}$$

**Semantically equivalent propositions**

$$\frac{}{\forall x^{\tau} \forall y^{\sigma}. A \simeq_{\mathcal{E}} \forall y^{\sigma} \forall x^{\tau}. A} \quad \frac{}{\forall x^{\tau}. A \simeq_{\mathcal{E}} A} \quad x \notin FV(A)$$

$$\frac{}{A \Rightarrow \forall x^{\tau}. B \simeq_{\mathcal{E}} \forall x^{\tau}. A \Rightarrow B} \quad x \notin FV(A)$$

$$\frac{}{M \doteq M \mapsto A \simeq_{\mathcal{E}} A} \quad \frac{}{M \doteq N \mapsto A \simeq_{\mathcal{E}} N \doteq M \mapsto A}$$

$$\frac{}{M \doteq N \mapsto P \doteq Q \mapsto A \simeq_{\mathcal{E}} P \doteq Q \mapsto M \doteq N \mapsto A}$$

$$\frac{}{A \Rightarrow M \doteq N \mapsto B \simeq_{\mathcal{E}} M \doteq N \mapsto A \Rightarrow B}$$

$$\frac{}{\forall x^{\tau}. M \doteq N \mapsto A \simeq_{\mathcal{E}} M \doteq N \mapsto \forall x^{\tau}. A} \quad x \notin FV(M, N)$$

Figure 3: Inference rules for the relation  $\simeq_{\mathcal{E}}$

$\lambda^* x. M$		$\gamma_1 (\lambda x. M)$
$M @ N$		$(\gamma_3 (M) N)$
$\langle a, b \rangle$		$\lambda f. f a b$
let $(x, y) := c$ in $M$		$c (\lambda xy. M)$
true		$\lambda xy. x$
false		$\lambda xy. y$
if $b$ then $f$ else $g$		$b f g$
true*		$\gamma_1 (\lambda u. \gamma_1 (\lambda v. \beta_3 (\beta_4 u)))$
false*	becomes	$\gamma_1 (\lambda u. \gamma_1 (\lambda v. \beta_4 v))$
Leaf $f$		$\lambda x_L x_N. x_L f$
Node $a t_1 t_2$		$\lambda x_L x_N. x_N a (t_1 x_L x_N) (t_2 x_L x_N)$
match <sub>Tree</sub> $t$ with   Leaf $f \Rightarrow M$   Node $a t_1 t_2 \Rightarrow N$		$t (\lambda f. M (\lambda at_1 t_2. N))$
$\emptyset$		$\lambda x_{\emptyset} x_+ x_- . x_{\emptyset}$
consT $a p$		$\lambda x_{\emptyset} x_+ x_- . x_+ a (p x_{\emptyset} x_+ x_-)$
consF $a p$		$\lambda x_{\emptyset} x_+ x_- . x_- a (p x_{\emptyset} x_+ x_-)$
match <sub>Val</sub> $c$ with   $\emptyset \Rightarrow M$   consT $a p \Rightarrow N$   consF $a p \Rightarrow P$		$c M (\lambda ap. N) (\lambda ap. P)$

**Admissible derivation rules**

Using the definitions of the previous constructions, we can prove the admissibility in  $\text{PA}\omega^+$  of the inference rules given figure 4.

$$\begin{array}{c}
\frac{\mathcal{E}; \Gamma, x : q\mathbb{F}A \vdash M : pq\mathbb{F}B}{\mathcal{E}; \Gamma \vdash \lambda^*x. M : p\mathbb{F}A \Rightarrow B} \quad q \notin FV(\Gamma) \qquad \frac{\mathcal{E}; \Gamma \vdash M : p\mathbb{F}A \Rightarrow B \quad \mathcal{E}; \Gamma \vdash N : p\mathbb{F}B}{\mathcal{E}; \Gamma \vdash M@N : p\mathbb{F}B} \\
\frac{\mathcal{E}; \Gamma \vdash M : A \quad \mathcal{E}; \Gamma \vdash N : B}{\mathcal{E}; \Gamma \vdash \langle M, N \rangle : A \wedge B} \qquad \frac{\mathcal{E}; \Gamma \vdash M : A \wedge B \quad \mathcal{E}; \Gamma, x : A, y : B \vdash N : C}{\mathcal{E}; \Gamma \vdash \text{let } (x, y) := M \text{ in } N : C} \quad x, y \notin FV(\Gamma) \cup FV(M) \\
\frac{\mathcal{E}; \Gamma \vdash \text{true} : 1 \in \text{Bool} \qquad \mathcal{E}; \Gamma \vdash \text{false} : 0 \in \text{Bool}}{\mathcal{E}; \Gamma \vdash M : b \in \text{Bool} \quad \mathcal{E}, b = 1; \Gamma \vdash N : A \quad \mathcal{E}, b = 0; \Gamma \vdash P : A} \\
\mathcal{E}; \Gamma \vdash \text{if } M \text{ then } N \text{ else } P : A \\
\frac{\mathcal{E}; \Gamma \vdash \text{true}^* : p\mathbb{F}1 \in \text{Bool} \qquad \mathcal{E}; \Gamma \vdash \text{false}^* : p\mathbb{F}0 \in \text{Bool}}{\mathcal{E}; \Gamma \vdash M : a \in \text{Atom} \quad \mathcal{E}; \Gamma \vdash N : p \in \text{FVal}} \quad \text{mem } ap \approx_{\varepsilon} 0 \\
\frac{\mathcal{E}; \Gamma \vdash M : a \in \text{Atom} \quad \mathcal{E}; \Gamma \vdash N : p \in \text{FVal}}{\mathcal{E}; \Gamma \vdash \text{consF } MN : pa^0 \in \text{FVal}} \quad \text{mem } ap \approx_{\varepsilon} 0 \\
\frac{\mathcal{E}; \Gamma \vdash M : p \in \text{FVal} \qquad \mathcal{E}; \Gamma \vdash p \doteq \emptyset \mapsto N : A}{\mathcal{E}; \Gamma \vdash \text{match}_{\text{FVal}} M \text{ with } | \emptyset \Rightarrow N | \text{consT } xy \Rightarrow P | \text{consF } xy \Rightarrow Q : A} \\
\mathcal{E}, p = qa^1; \Gamma \vdash \lambda xy. P : a \in \text{Atom} \Rightarrow q \in \text{FVal} \Rightarrow A \\
\mathcal{E}, p = qa^0; \Gamma \vdash \lambda xy. Q : a \in \text{Atom} \Rightarrow q \in \text{FVal} \Rightarrow A \\
\mathcal{E}; \Gamma \vdash \text{match}_{\text{FVal}} M \text{ with } | \emptyset \Rightarrow N | \text{consT } xy \Rightarrow P | \text{consF } xy \Rightarrow Q : A
\end{array}$$

Figure 4: Admissible inference rules for macros

## 2 Absoluteness of connectives

The derivations for the primitive connectives (false, implication, universal quantification, Leibniz equality) and for the datatype  $\mathbb{N}$  can be found in the appendix of [13].

### 2.1 Conjunction

We deal only with binary conjunctions but the extension to bigger arities is straightforward. Assuming that both propositions  $A$  and  $B$  are absolute, we want to show that  $A \wedge B$  is absolute.

$$\begin{array}{l}
[\lambda] z : p \text{ IF } A \wedge B \simeq \forall (Z^*)^{k \rightarrow o}. p \text{ IF } (A \Rightarrow B \Rightarrow Z) \Rightarrow Z \\
[\lambda] c : C[p] \\
[\forall] Z^o \\
[\lambda] x : A \Rightarrow B \Rightarrow Z \\
z : (p \text{ IF } A \Rightarrow B \Rightarrow Z) \Rightarrow Z[Z/Z^* r] \\
\gamma_3 z : (p \text{ IF } A \Rightarrow B \Rightarrow Z)[Z/Z^* r] \Rightarrow (p \text{ IF } Z)[Z/Z^* r] \\
[\forall] q_A^k \\
[\lambda] u : q_A \text{ IF } A \\
\xi_A u : C[q_A] \Rightarrow A \\
[\forall] q_B^k \\
[\lambda] v : q_B \text{ IF } B \\
\xi_B v : C[q_B] \Rightarrow B \\
[\forall] r^k \\
[\lambda] c : C[pq_A q_B r] \\
\alpha_2 (\alpha_1^2 c) : C[q_A] \\
\xi_A u (\alpha_2 (\alpha_1^2 c)) : A \\
x (\xi_A u (\alpha_2 (\alpha_1^2 c))) : B \Rightarrow Z \\
\alpha_2 (\alpha_1 c) : C[q_B] \\
\xi_B v (\alpha_2 (\alpha_1 c)) : B \\
x (\xi_A u (\alpha_2 (\alpha_1^2 c))) (\xi_B v (\alpha_2 (\alpha_1 c))) : Z \\
[\lambda c. x (\xi_A u (\alpha_2 (\alpha_1^2 c))) (\xi_B v (\alpha_2 (\alpha_1 c))) : \forall r^k. C[pq_A q_B r] \Rightarrow Z \equiv (pq_A q_B \text{ IF } Z)[Z/Z^* r] \\
\lambda v c. x (\xi_A u (\alpha_2 (\alpha_1^2 c))) (\xi_B v (\alpha_2 (\alpha_1 c))) : \forall q_B^k. (q_B \text{ IF } B) \Rightarrow (pq_A q_B \text{ IF } Z)[Z/Z^* r] \\
\gamma_1 (\lambda v c. x (\xi_A u (\alpha_2 (\alpha_1^2 c))) (\xi_B v (\alpha_2 (\alpha_1 c)))) : (pq_A \text{ IF } B \Rightarrow Z)[Z/Z^* r] \\
\lambda u. \gamma_1 (\lambda v c. x (\xi_A u (\alpha_2 (\alpha_1^2 c))) (\xi_B v (\alpha_2 (\alpha_1 c)))) : \forall q_A^k. (q_A \text{ IF } A) \Rightarrow (pq_A \text{ IF } B \Rightarrow Z)[Z/Z^* r] \\
\gamma_1 (\lambda u. \gamma_1 (\lambda v c. x (\xi_A u (\alpha_2 (\alpha_1^2 c))) (\xi_B v (\alpha_2 (\alpha_1 c)))) : (p \text{ IF } A \Rightarrow B \Rightarrow Z)[Z/Z^* r] \\
\gamma_3 z (\gamma_1 (\lambda u. \gamma_1 (\lambda v c. x (\xi_A u (\alpha_2 (\alpha_1^2 c))) (\xi_B v (\alpha_2 (\alpha_1 c)))))) : (p \text{ IF } Z)[Z/Z^* r] \equiv \forall r^k. C[pr] \Rightarrow Z \\
\alpha_7 c : C[p1] \\
[\gamma_3 z (\gamma_1 (\lambda u. \gamma_1 (\lambda v c. x (\xi_A u (\alpha_2 (\alpha_1^2 c))) (\xi_B v (\alpha_2 (\alpha_1 c)))))) (\alpha_7 c) : Z \\
\lambda x. \gamma_3 z (\gamma_1 (\lambda u. \gamma_1 (\lambda v c. x (\xi_A u (\alpha_2 (\alpha_1^2 c))) (\xi_B v (\alpha_2 (\alpha_1 c)))))) (\alpha_7 c) : \forall Z^o. (A \Rightarrow B \Rightarrow Z) \Rightarrow Z \equiv A \wedge B \\
\lambda z c x. \gamma_3 z (\gamma_1 (\lambda u. \gamma_1 (\lambda v c. x (\xi_A u (\alpha_2 (\alpha_1^2 c))) (\xi_B v (\alpha_2 (\alpha_1 c)))))) (\alpha_7 c) : (p \text{ IF } A \wedge B) \Rightarrow (C[p] \Rightarrow A \wedge B)
\end{array}$$

$$\begin{array}{l}
[\lambda] z : C[p] \Rightarrow A \wedge B \\
[\forall] (Z^*)^{k \rightarrow o} \\
[\forall] q^k \\
[\lambda] x : q \text{ IF } A \Rightarrow B \Rightarrow Z \\
\gamma_3 x : (q \text{ IF } A) \Rightarrow (q \text{ IF } B \Rightarrow Z) \\
[\forall] r^k \\
[\lambda] c : C[pqr] \\
\alpha_1^2 c : C[p] \\
z(\alpha_1^2 c) : A \wedge B \\
z(\alpha_1^2 c) : (A \Rightarrow B \Rightarrow Z^* r) \Rightarrow Z^* r \\
[\lambda] u : A \\
[\lambda] v : B \\
\lambda_{\cdot} u : C[q] \Rightarrow A \\
\xi'_A(\lambda_{\cdot} u) : q \text{ IF } A \\
\gamma_3 x(\xi'_A(\lambda_{\cdot} u)) : q \text{ IF } B \Rightarrow Z \\
\gamma_3(\gamma_3 x(\xi'_A(\lambda_{\cdot} u))) : (q \text{ IF } B) \Rightarrow (q \text{ IF } Z) \\
\lambda_{\cdot} v : C[q] \Rightarrow B \\
\xi'_B(\lambda_{\cdot} v) : q \text{ IF } B \\
\gamma_3(\gamma_3 x(\xi'_A(\lambda_{\cdot} u)))(\xi'_B(\lambda_{\cdot} v)) : q \text{ IF } Z \equiv \forall r^k. C[qr] \Rightarrow Z^* r \\
\alpha_{10} c : C[qr] \\
\gamma_3(\gamma_3 x(\xi'_A(\lambda_{\cdot} u)))(\xi'_B(\lambda_{\cdot} v))(\alpha_{10} c) : Z^* r \\
\lambda uv. \gamma_3(\gamma_3 x(\xi'_A(\lambda_{\cdot} u)))(\xi'_B(\lambda_{\cdot} v))(\alpha_{10} c) : A \Rightarrow B \Rightarrow Z^* r \\
z(\alpha_1^2 c)(\lambda uv. \gamma_3(\gamma_3 x(\xi'_A(\lambda_{\cdot} u)))(\xi'_B(\lambda_{\cdot} v))(\alpha_{10} c)) : Z^* r \\
\lambda c. z(\alpha_1^2 c)(\lambda uv. \gamma_3(\gamma_3 x(\xi'_A(\lambda_{\cdot} u)))(\xi'_B(\lambda_{\cdot} v))(\alpha_{10} c)) : \forall r^k. C[pqr] \Rightarrow Z^* r \equiv pq \text{ IF } Z \\
\lambda xc. z(\alpha_1^2 c)(\lambda uv. \gamma_3(\gamma_3 x(\xi'_A(\lambda_{\cdot} u)))(\xi'_B(\lambda_{\cdot} v))(\alpha_{10} c)) : \forall q^k. (q \text{ IF } A \Rightarrow B \Rightarrow Z) \Rightarrow (pq \text{ IF } Z) \\
\gamma_1(\lambda xc. z(\alpha_1^2 c)(\lambda uv. \gamma_3(\gamma_3 x(\xi'_A(\lambda_{\cdot} u)))(\xi'_B(\lambda_{\cdot} v))(\alpha_{10} c))) : p \text{ IF } (A \Rightarrow B \Rightarrow Z) \Rightarrow Z \\
\gamma_1(\lambda xc. z(\alpha_1^2 c)(\lambda uv. \gamma_3(\gamma_3 x(\xi'_A(\lambda_{\cdot} u)))(\xi'_B(\lambda_{\cdot} v))(\alpha_{10} c))) : \forall Z^*. p \text{ IF } (A \Rightarrow B \Rightarrow Z) \Rightarrow Z \simeq p \text{ IF } A \wedge B \\
\lambda z. \gamma_1(\lambda xc. z(\alpha_1^2 c)(\lambda uv. \gamma_3(\gamma_3 x(\xi'_A(\lambda_{\cdot} u)))(\xi'_B(\lambda_{\cdot} v))(\alpha_{10} c))) : (C[p] \Rightarrow A \wedge B) \Rightarrow (p \text{ IF } A \wedge B)
\end{array}$$

Abstracting over the proof terms  $\xi_A$ ,  $\xi'_A$ ,  $\xi_B$  and  $\xi'_B$ , we get two combinators  $\xi_{\wedge}$  and  $\xi'_{\wedge}$  which can be used to prove absoluteness of more complex propositions.

## 2.2 Disjunction

Again, we deal only with the binary case but the extension to bigger arities is straightforward.

$$\begin{array}{l}
[\lambda] z : p \text{ IF } A \vee B \\
[\lambda] c : C[p] \\
[\forall] Z^o \\
[\lambda] x : A \Rightarrow Z \\
[\lambda] y : B \Rightarrow Z \\
z : (p \text{ IF } (A \Rightarrow Z) \Rightarrow (B \Rightarrow Z) \Rightarrow Z)[Z/Z^* r] \\
\gamma_3 z : (p \text{ IF } A \Rightarrow Z)[Z/Z^* r] \Rightarrow (p \text{ IF } (B \Rightarrow Z) \Rightarrow Z)[Z/Z^* r] \\
[\forall] q^k \\
[\lambda] u : q \text{ IF } A \\
\xi_A u : C[q] \Rightarrow A \\
[\forall] r^k \\
[\lambda] c' : C[pqr] \\
\alpha_1 c' : C[pq] \\
\alpha_2 (\alpha_1 c') : C[q] \\
\xi_A u (\alpha_2 (\alpha_1 c')) : A \\
x (\xi_A u (\alpha_2 (\alpha_1 c'))) : Z \\
\lambda c'. x (\xi_A u (\alpha_2 (\alpha_1 c'))) : \forall r^k. C[pqr] \Rightarrow Z \equiv (pq \text{ IF } Z)[Z/Z^* r] \\
\lambda uc'. x (\xi_A u (\alpha_2 (\alpha_1 c'))) : \forall q^k. (q \text{ IF } A) \Rightarrow (pq \text{ IF } Z)[Z/Z^* r] \\
\gamma_1 (\lambda uc'. x (\xi_A u (\alpha_2 (\alpha_1 c')))) : (p \text{ IF } A \Rightarrow Z)[Z/Z^* r] \\
idem \\
\gamma_1 (\lambda uc'. x (\xi_B u (\alpha_2 (\alpha_1 c')))) : (p \text{ IF } B \Rightarrow Z)[Z/Z^* r] \\
\gamma_3 z (\gamma_1 (\lambda uc'. x (\xi_A u (\alpha_2 (\alpha_1 c'))))) : (p \text{ IF } (B \Rightarrow Z) \Rightarrow Z)[Z/Z^* r] \\
\gamma_3 (\gamma_3 z (\gamma_1 (\lambda uc'. x (\xi_A u (\alpha_2 (\alpha_1 c')))))) : (p \text{ IF } B \Rightarrow Z)[Z/Z^* r] \Rightarrow (p \text{ IF } Z)[Z/Z^* r] \\
\gamma_3 (\gamma_3 z (\gamma_1 (\lambda uc'. x (\xi_A u (\alpha_2 (\alpha_1 c')))))) (\gamma_1 (\lambda uc'. x (\xi_B u (\alpha_2 (\alpha_1 c'))))) : (p \text{ IF } Z)[Z/Z^* r] \equiv \forall r^k. C[pr] \Rightarrow Z \\
\alpha_7 c : C[p1] \\
\gamma_3 (\gamma_3 z (\gamma_1 (\lambda uc'. x (\xi_A u (\alpha_2 (\alpha_1 c')))))) (\gamma_1 (\lambda uc'. x (\xi_B u (\alpha_2 (\alpha_1 c'))))) (\alpha_7 c) : Z \\
\lambda xy. \gamma_3 (\gamma_3 z (\gamma_1 (\lambda uc'. x (\xi_A u (\alpha_2 (\alpha_1 c')))))) (\gamma_1 (\lambda uc'. x (\xi_B u (\alpha_2 (\alpha_1 c'))))) (\alpha_7 c) : \forall Z^o. (A \Rightarrow Z) \Rightarrow (B \Rightarrow Z) \Rightarrow Z \equiv A \vee B \\
\lambda zcxy. \gamma_3 (\gamma_3 z (\gamma_1 (\lambda uc'. x (\xi_A u (\alpha_2 (\alpha_1 c')))))) (\gamma_1 (\lambda uc'. x (\xi_B u (\alpha_2 (\alpha_1 c'))))) (\alpha_7 c) : (p \text{ IF } A \vee B) \Rightarrow (C[p] \Rightarrow A \vee B)
\end{array}$$

$$\begin{array}{l}
[\lambda] z : C[p] \Rightarrow A \vee B \\
[[\forall] (Z^*)^{k \rightarrow o} \\
[[\forall] q_A^k \\
[\lambda] x : q_A \text{ IF } A \Rightarrow Z \\
\beta_3 x : (q_A q_B \text{ IF } A \Rightarrow Z) \\
\gamma_3 (\beta_3 x) : (q_A q_B \text{ IF } A) \Rightarrow (q_A q_B \text{ IF } Z) \\
[[\forall] q_B^k \\
[\lambda] y : q_B \text{ IF } B \Rightarrow Z \\
\beta_4 y : (q_A q_B \text{ IF } B \Rightarrow Z) \\
\gamma_3 (\beta_4 y) : (q_A q_B \text{ IF } B) \Rightarrow (q_A q_B \text{ IF } Z) \\
[[\forall] r^k \\
[\lambda] c : C[pq_A q_B r] \\
\alpha_1^3 c : C[p] \\
z(\alpha_1^3 c) : A \vee B \\
z(\alpha_1^3 c) : (A \Rightarrow (q_A q_B \text{ IF } Z)) \Rightarrow (B \Rightarrow (q_A q_B \text{ IF } Z)) \Rightarrow (q_A q_B \text{ IF } Z) \\
[[\lambda] u : A \\
\lambda_{\cdot} u : C[q_A q_B] \Rightarrow A \\
\xi'_A (\lambda_{\cdot} u) : q_A q_B \text{ IF } A \\
\gamma_3 (\beta_3 x) (\xi'_A (\lambda_{\cdot} u)) : q_A q_B \text{ IF } Z \\
\lambda u. \gamma_3 (\beta_3 x) (\xi'_A (\lambda_{\cdot} u)) : A \Rightarrow (q_A q_B \text{ IF } Z) \\
z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_A (\lambda_{\cdot} u))) : (B \Rightarrow (q_A q_B \text{ IF } Z)) \Rightarrow (q_A q_B \text{ IF } Z) \\
[[\lambda] v : B \\
\lambda_{\cdot} v : C[q_A q_B] \Rightarrow B \\
\xi'_B (\lambda_{\cdot} v) : q_A q_B \text{ IF } B \\
\gamma_3 (\beta_4 y) (\xi'_B (\lambda_{\cdot} v)) : q_A q_B \text{ IF } Z \\
\lambda v. \gamma_3 (\beta_4 y) (\xi'_B (\lambda_{\cdot} v)) : B \Rightarrow (q_A q_B \text{ IF } Z) \\
z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_A (\lambda_{\cdot} u))) (\lambda v. \gamma_3 (\beta_4 y) (\xi'_B (\lambda_{\cdot} v))) : q_A q_B \text{ IF } Z \equiv \forall r^k. C[q_A q_B r] \Rightarrow Z^* r \\
\alpha_{16} c : C[q_A q_B r] \\
z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_A (\lambda_{\cdot} u))) (\lambda v. \gamma_3 (\beta_4 y) (\xi'_B (\lambda_{\cdot} v))) (\alpha_{16} c) : Z^* r \\
[\lambda c. z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_A (\lambda_{\cdot} u))) (\lambda v. \gamma_3 (\beta_4 y) (\xi'_B (\lambda_{\cdot} v))) (\alpha_{16} c) : \forall r^k. C[pq_A q_B r] \Rightarrow Z^* r \equiv pq_A q_B \text{ IF } Z \\
\lambda yc. z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_A (\lambda_{\cdot} u))) (\lambda v. \gamma_3 (\beta_4 y) (\xi'_B (\lambda_{\cdot} v))) (\alpha_{16} c) : \forall q_B^k. (q_B \text{ IF } B \Rightarrow Z) \Rightarrow (pq_A q_B \text{ IF } Z) \\
\gamma_1 (\lambda yc. z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_A (\lambda_{\cdot} u))) (\lambda v. \gamma_3 (\beta_4 y) (\xi'_B (\lambda_{\cdot} v))) (\alpha_{16} c)) : pq_A \text{ IF } (B \Rightarrow Z) \Rightarrow Z \\
\lambda x. \gamma_1 (\lambda yc. z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_A (\lambda_{\cdot} u))) (\lambda v. \gamma_3 (\beta_4 y) (\xi'_B (\lambda_{\cdot} v))) (\alpha_{16} c)) : \forall q_A^k. (q_A \text{ IF } A \Rightarrow Z) \Rightarrow (pq_A \text{ IF } (B \Rightarrow Z) \Rightarrow Z) \\
\gamma_1 (\lambda x. \gamma_1 (\lambda yc. z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_A (\lambda_{\cdot} u))) (\lambda v. \gamma_3 (\beta_4 y) (\xi'_B (\lambda_{\cdot} v))) (\alpha_{16} c))) : p \text{ IF } (A \Rightarrow Z) \Rightarrow (B \Rightarrow Z) \Rightarrow Z \\
\gamma_1 (\lambda x. \gamma_1 (\lambda yc. z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_A (\lambda_{\cdot} u))) (\lambda v. \gamma_3 (\beta_4 y) (\xi'_B (\lambda_{\cdot} v))) (\alpha_{16} c))) \\
: \forall Z^*. p \text{ IF } (A \Rightarrow Z) \Rightarrow (B \Rightarrow Z) \Rightarrow Z \simeq p \text{ IF } A \vee B \\
\lambda z. \gamma_1 (\lambda x. \gamma_1 (\lambda yc. z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_A (\lambda_{\cdot} u))) (\lambda v. \gamma_3 (\beta_4 y) (\xi'_B (\lambda_{\cdot} v))) (\alpha_{16} c))) : (C[p] \Rightarrow A \vee B) \Rightarrow (p \text{ IF } A \vee B)
\end{array}$$

Abstracting over the proof terms  $\xi_A$ ,  $\xi'_A$ ,  $\xi_B$  and  $\xi'_B$ , we get two combinators  $\xi_{\vee}$  and  $\xi'_{\vee}$  which can be used to prove absoluteness of more complex propositions.



### 2.3 Classical existential quantifier (on multiple propositions)

Remember that  $\exists x^\tau. A_1 \& \dots \& A_k$  is defined as  $\neg(\forall x^\tau. A_1 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp)$ .

Here, we assume that  $\tau$  is a  $T$ -sort, that is we have  $\tau^* = \tau$ . Otherwise, the result does not hold.

$$\begin{array}{l}
[\lambda] z : p \text{ IF } \exists x^\tau. A_1 \& \dots \& A_k \equiv p \text{ IF } (\forall x^\tau. A_1 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp) \Rightarrow \perp \\
[\lambda] c : C[p] \\
[\lambda] x : \forall x^\tau. A_1 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp \\
\gamma_3 z : (p \text{ IF } \forall x^\tau. A_1 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp) \Rightarrow (p \text{ IF } \perp) \\
\begin{array}{l}
[\forall] x^\tau \\
[\forall] q_1^k \\
[\lambda] x_1 : q_1 \text{ IF } A_1 \\
\xi_1 x_1 : C[q_1] \Rightarrow A_1 \\
\vdots \\
[\forall] q_k^k \\
[\lambda] x_k : q_k \text{ IF } A_k \\
\xi_k x_k : C[q_k] \Rightarrow A_k \\
[\lambda] c' : C[pq_1 \dots q_k] \\
\alpha_2 (\alpha_1^{k-1} c') : C[q_1] \\
\xi_1 x_1 (\alpha_2 (\alpha_1^{k-1} c')) : A_1 \\
\vdots \\
\alpha_2 c' : C[q_k] \\
\xi_k x_k (\alpha_2 c') : A_k \\
x (\xi_1 x_1 (\alpha_2 (\alpha_1^{k-1} c'))) \dots (\xi_k x_k (\alpha_2 c')) : \perp \\
\lambda c'. x (\xi_1 x_1 (\alpha_2 (\alpha_1^{k-1} c'))) \dots (\xi_k x_k (\alpha_2 (\alpha_1 c'))) : C[pq_1 \dots q_k] \Rightarrow \perp \\
\xi'_1 (\lambda c'. x (\xi_1 x_1 (\alpha_2 (\alpha_1^{k-1} c'))) \dots (\xi_k x_k (\alpha_2 (\alpha_1 c')))) : pq_1 \dots q_k \text{ IF } \perp \\
\lambda x_k. \xi'_1 (\lambda c'. x (\xi_1 x_1 (\alpha_2 (\alpha_1^{k-1} c'))) \dots (\xi_k x_k (\alpha_2 (\alpha_1 c')))) : \forall q_k^k (q_k \text{ IF } A_k) \Rightarrow (pq_1 \dots q_k \text{ IF } \perp) \\
\gamma_1 (\lambda x_k. \xi'_1 (\lambda c'. x (\xi_1 x_1 (\alpha_2 (\alpha_1^{k-1} c'))) \dots (\xi_k x_k (\alpha_2 (\alpha_1 c')))) : pq_1 \dots q_{k-1} \text{ IF } A_k \Rightarrow \perp \\
\vdots \\
\lambda x_1. \gamma_1 \dots (\gamma_1 (\lambda x_k. \xi'_1 (\lambda c'. x (\xi_1 x_1 (\alpha_2 (\alpha_1^{k-1} c'))) \dots (\xi_k x_k (\alpha_2 (\alpha_1 c')))) \dots)) : \forall q_1^k. (q_1 \text{ IF } A_1) \Rightarrow (pq_1 \text{ IF } A_2 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp) \\
\gamma_1 (\lambda x_1. \gamma_1 \dots (\gamma_1 (\lambda x_k. \xi'_1 (\lambda c'. x (\xi_1 x_1 (\alpha_2 (\alpha_1^{k-1} c'))) \dots (\xi_k x_k (\alpha_2 (\alpha_1 c')))) \dots)) : p \text{ IF } A_1 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp \\
\gamma_1 (\lambda x_1. \gamma_1 \dots (\gamma_1 (\lambda x_k. \xi'_1 (\lambda c'. x (\xi_1 x_1 (\alpha_2 (\alpha_1^{k-1} c'))) \dots (\xi_k x_k (\alpha_2 (\alpha_1 c')))) \dots)) \\
: \forall x^\tau. p \text{ IF } A_1 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp \simeq p \text{ IF } \forall x^\tau. A_1 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp \\
\gamma_3 z (\gamma_1 (\lambda x_1. \gamma_1 \dots (\gamma_1 (\lambda x_k. \xi'_1 (\lambda c'. x (\xi_1 x_1 (\alpha_2 (\alpha_1^{k-1} c'))) \dots (\xi_k x_k (\alpha_2 (\alpha_1 c')))) \dots)) : p \text{ IF } \perp \\
\xi'_1 (\gamma_3 z (\gamma_1 (\lambda x_1. \gamma_1 \dots (\gamma_1 (\lambda x_k. \xi'_1 (\lambda c'. x (\xi_1 x_1 (\alpha_2 (\alpha_1^{k-1} c'))) \dots (\xi_k x_k (\alpha_2 (\alpha_1 c')))) \dots)) : C[p] \Rightarrow \perp \\
\xi'_1 (\gamma_3 z (\gamma_1 (\lambda x_1. \gamma_1 \dots (\gamma_1 (\lambda x_k. \xi'_1 (\lambda c'. x (\xi_1 x_1 (\alpha_2 (\alpha_1^{k-1} c'))) \dots (\xi_k x_k (\alpha_2 (\alpha_1 c')))) \dots)) : \perp \\
\lambda z c x. \xi'_1 (\gamma_3 z (\gamma_1 (\lambda x_1. \gamma_1 \dots (\gamma_1 (\lambda x_k. \xi'_1 (\lambda c'. x (\xi_1 x_1 (\alpha_2 (\alpha_1^{k-1} c'))) \dots (\xi_k x_k (\alpha_2 (\alpha_1 c')))) \dots)) : (p \text{ IF } \exists x^\tau. A_1 \& \dots \& A_k) \Rightarrow C[p] \Rightarrow (\forall x^\tau. A_1 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp) \Rightarrow \perp \equiv (p \text{ IF } \exists x^\tau. A_1 \& \dots \& A_k) \Rightarrow (C[p] \Rightarrow \exists x^\tau. A_1 \& \dots \& A_k)
\end{array}
\end{array}$$

Abstracting over the  $k$  proof term  $\xi_1, \dots, \xi_k$ , we get the proof term

$$\xi_{\exists k} := \lambda \xi_1 \dots \xi_k z c x. \xi'_1 (\gamma_3 z (\gamma_1 (\lambda x_1. \gamma_1 \dots (\gamma_1 (\lambda x_k. \xi'_1 (\lambda c'. x (\xi_1 x_1 (\alpha_2 (\alpha_1^{k-1} c'))) \dots (\xi_k x_k (\alpha_2 (\alpha_1 c')))) \dots)) \dots)) c$$

which can be used as a combinator to prove the absoluteness of the proposition  $\exists x^\tau. A_1 \& \dots \& A_k$  from the proof terms  $\xi_1, \dots, \xi_k$  of absoluteness of  $A_1, \dots, A_k$ .

$$\begin{array}{l}
[\lambda] z : C[p] \Rightarrow \exists x^\tau. A_1 \& \dots \& A_k \\
[\forall] q^k \\
[\lambda] x : q \text{ IF } \forall x^\tau. A_1 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp \simeq \forall x^\tau. q \text{ IF } A_1 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp \\
[\lambda] c : C[pq] \\
\alpha_1 c : C[p] \\
z(\alpha_1 c) : \exists x^\tau. A_1 \& \dots \& A_k \equiv (\forall x^\tau. A_1 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp) \Rightarrow \perp \\
[\forall] x^\tau \\
[\lambda] y_1 : A_1 \\
\vdots \\
[\lambda] y_k : A_k \\
\lambda_{\perp}. y_1 : C[pq] \Rightarrow A_1 \\
\xi'_1(\lambda_{\perp}. y_1) : pq \text{ IF } A_1 \\
\beta_4 x : \forall x^\tau. pq \text{ IF } A_1 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp \\
\gamma_3(\beta_4 x) : \forall x^\tau. (pq \text{ IF } A_1) \Rightarrow (pq \text{ IF } A_2 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp) \\
\gamma_3 x(\xi'_1(\lambda_{\perp}. y_1)) : pq \text{ IF } A_2 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp \\
\vdots \\
\gamma_3(\dots(\gamma_3(\beta_4 x)(\xi'_1(\lambda_{\perp}. y_1)))\dots) : pq \text{ IF } A_k \Rightarrow \perp \\
\lambda_{\perp}. y_k : C[pq] \Rightarrow A_k \\
\xi'_k(\lambda_{\perp}. y_k) : pq \text{ IF } A_k \\
\gamma_3(\gamma_3 \dots (\gamma_3(\beta_4 x)(\xi'_1(\lambda_{\perp}. y_1)))\dots) : (pq \text{ IF } A_k) \Rightarrow (pq \text{ IF } \perp) \\
\gamma_3(\gamma_3 \dots (\gamma_3(\beta_4 x)(\xi'_1(\lambda_{\perp}. y_1)))\dots) (\xi'_k(\lambda_{\perp}. y_k)) : pq \text{ IF } \perp \\
\xi_{\perp}(\gamma_3(\gamma_3 \dots (\gamma_3(\beta_4 x)(\xi'_1(\lambda_{\perp}. y_1)))\dots) (\xi'_k(\lambda_{\perp}. y_k))) : C[pq] \Rightarrow \perp \\
\xi_{\perp}(\gamma_3(\gamma_3 \dots (\gamma_3(\beta_4 x)(\xi'_1(\lambda_{\perp}. y_1)))\dots) (\xi'_k(\lambda_{\perp}. y_k))) c : \perp \\
\lambda y_1 \dots y_k. \xi_{\perp}(\gamma_3(\gamma_3 \dots (\gamma_3(\beta_4 x)(\xi'_1(\lambda_{\perp}. y_1)))\dots) (\xi'_k(\lambda_{\perp}. y_k))) c : \forall x^\tau. A_1 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp \\
z(\alpha_1 c)(\lambda y_1 \dots y_k. \xi_{\perp}(\gamma_3(\gamma_3 \dots (\gamma_3(\beta_4 x)(\xi'_1(\lambda_{\perp}. y_1)))\dots) (\xi'_k(\lambda_{\perp}. y_k))) c) : \perp \\
\lambda c. z(\alpha_1 c)(\lambda y_1 \dots y_k. \xi_{\perp}(\gamma_3(\gamma_3 \dots (\gamma_3(\beta_4 x)(\xi'_1(\lambda_{\perp}. y_1)))\dots) (\xi'_k(\lambda_{\perp}. y_k))) c) : C[pq] \Rightarrow \perp \\
\xi'_{\perp}(\lambda c. z(\alpha_1 c)(\lambda y_1 \dots y_k. \xi_{\perp}(\gamma_3(\gamma_3 \dots (\gamma_3(\beta_4 x)(\xi'_1(\lambda_{\perp}. y_1)))\dots) (\xi'_k(\lambda_{\perp}. y_k))) c)) : pq \text{ IF } \perp \\
\lambda x. \xi'_{\perp}(\lambda c. z(\alpha_1 c)(\lambda y_1 \dots y_k. \xi_{\perp}(\gamma_3(\gamma_3 \dots (\gamma_3(\beta_4 x)(\xi'_1(\lambda_{\perp}. y_1)))\dots) (\xi'_k(\lambda_{\perp}. y_k))) c)) \\
\quad : \forall q^k. (q \text{ IF } \forall x^\tau. A_1 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp) \Rightarrow (pq \text{ IF } \perp) \\
\gamma_1(\lambda x. \xi'_{\perp}(\lambda c. z(\alpha_1 c)(\lambda y_1 \dots y_k. \xi_{\perp}(\gamma_3(\gamma_3 \dots (\gamma_3(\beta_4 x)(\xi'_1(\lambda_{\perp}. y_1)))\dots) (\xi'_k(\lambda_{\perp}. y_k))) c))) \\
\quad : p \text{ IF } (\forall x^\tau. A_1 \Rightarrow \dots \Rightarrow A_k \Rightarrow \perp) \Rightarrow \perp \equiv p \text{ IF } \exists x^\tau. A_1 \& \dots \& A_k \\
\lambda z. \gamma_1(\lambda x. \xi'_{\perp}(\lambda c. z(\alpha_1 c)(\lambda y_1 \dots y_k. \xi_{\perp}(\gamma_3(\gamma_3 \dots (\gamma_3(\beta_4 x)(\xi'_1(\lambda_{\perp}. y_1)))\dots) (\xi'_k(\lambda_{\perp}. y_k))) c))) \\
\quad : (C[p] \Rightarrow \exists x^\tau. A_1 \& \dots \& A_k) \Rightarrow (p \text{ IF } \exists x^\tau. A_1 \& \dots \& A_k)
\end{array}$$

Again, we build a combinator  $\xi'_{\exists_k}$  to use in bigger propositions.

### 3 Properties of the generic set

We recall that by definition,  $(p \in G)^* q := C[pq]$  so that  $q \Vdash p \in G \equiv \forall r^k. C[qr] \Rightarrow C[pr] \equiv q \leq p$ .

#### 3.1 $G$ is a filter

Saying that  $G$  is a filter amounts to proving four properties:

- $G$  is non empty:  $1 \in G$
- $G$  is a subset of  $\forall C$ :
- $G$  is upward-closed:
- $G$  is closed under product:

##### 3.1.1 $G$ is non-empty

We want to prove  $1 \Vdash 1 \in G \equiv 1 \leq 1 \equiv \forall r^k. C[1r] \Rightarrow C[1r]$ .

$$\left[ \begin{array}{l} [\forall] r^k \\ [\lambda] x : C[1r] \\ x : C[1r] \\ \lambda x. x : \forall r^k. C[1r] \Rightarrow C[1r] \end{array} \right.$$

##### 3.1.2 $G$ is a subset of $C$

We want to prove that for all  $p^k$ ,  $1 \Vdash p \in G \Rightarrow C[p]$ .

$$\left[ \begin{array}{l} [\forall] q^k \\ [\lambda] x : q \Vdash p \in G \equiv q \leq p \equiv \forall r^k. C[qr] \Rightarrow C[pr] \\ [\forall] r^k \\ [\lambda] c : C[1q] \\ \alpha_3 c : C[q1] \\ x(\alpha_3 c) : C[p1] \\ \alpha_1(x(\alpha_3 c)) : C[p] \\ \alpha_1 \circ x \circ \alpha_3 \equiv \lambda c. \alpha_1(x(\alpha_3 c)) : C[1q] \Rightarrow C[p] \\ \xi'_C(\alpha_1 \circ x \circ \alpha_3) : 1q \Vdash C[p] \\ \lambda x. \xi'_C(\alpha_1 \circ x \circ \alpha_3) : \forall q^k. (q \Vdash p \in G) \Rightarrow (1q \Vdash C[p]) \\ \gamma_1(\lambda x. \xi'_C(\alpha_1 \circ x \circ \alpha_3)) : 1 \Vdash p \in G \Rightarrow C[p] \end{array} \right.$$

##### 3.1.3 $G$ is upward closed

We want to prove that for all  $p^k$  and  $q^k$ ,  $1 \Vdash pq \in G \Rightarrow p \in G$ .

$$\left[ \begin{array}{l} [\forall] r^k \\ [\lambda] x : r \Vdash pq \in G \equiv r \leq pq \equiv \forall s^k. C[rs] \Rightarrow C[pqs] \\ [\forall] s^k \\ [\lambda] c : C[1rs] \\ \alpha_{10} c : C[rs] \\ x(\alpha_{10} c) : C[pqs] \\ \alpha_9(x(\alpha_{10} c)) : C[ps] \\ \alpha_9 \circ x \circ \alpha_{10} \equiv \lambda c. \alpha_9(x(\alpha_{10} c)) : \forall s^k. C[1rs] \Rightarrow C[ps] \equiv 1r \leq p \equiv 1r \Vdash p \in G \\ \lambda x. \alpha_9 \circ x \circ \alpha_{10} : \forall r^k. r \Vdash pq \in G \Rightarrow 1r \Vdash p \in G \\ \gamma_1(\lambda x. \alpha_9 \circ x \circ \alpha_{10}) : 1 \Vdash pq \in G \Rightarrow p \in G \end{array} \right.$$

### 3.1.4 $G$ is stable by product

We want to prove that for all  $p^k$  and  $q^k$ ,  $1 \Vdash p \in G \Rightarrow q \in G \Rightarrow pq \in G$ .

$$\begin{array}{l}
[\forall] r_p^k \\
[\lambda] x : r_p \Vdash p \in G \equiv \forall s^k. C[r_p s] \Rightarrow C[ps] \\
[\forall] r_q^k \\
[\lambda] y : r_q \Vdash q \in G \equiv \forall s^k. C[r_q s] \Rightarrow C[qs] \\
[\forall] s^k \\
[\lambda] c : C[1r_p r_q s] \\
\alpha_5 c : C[1r_p(r_q s)] \\
\alpha_5^2 c : C[1(r_p(r_q s))] \\
\alpha_2(\alpha_5^2 c) : C[r_p(r_q s)] \\
x(\alpha_2(\alpha_5^2 c)) : C[p(r_q s)] \\
\alpha_{12}(x(\alpha_2(\alpha_5^2 c))) : C[r_q(sp)] \\
y(\alpha_{12}(x(\alpha_2(\alpha_5^2 c)))) : C[q(sp)] \\
\alpha_{13}(y(\alpha_{12}(x(\alpha_2(\alpha_5^2 c)))))) : C[pqs] \\
[\alpha_{13} \circ y \circ \alpha_{12} \circ x \circ \alpha_2 \circ \alpha_5^2 \equiv \lambda c. \alpha_{13}(y(\alpha_{12}(x(\alpha_2(\alpha_5^2 c)))))] : \forall s^k. C[1r_p r_q s] \Rightarrow C[pqs] \equiv 1r_p r_q \Vdash pq \in G \\
\lambda y. \alpha_{13} \circ y \circ \alpha_{12} \circ x \circ \alpha_2 \circ \alpha_5^2 : \forall r_q^k. r_q \Vdash q \in G \Rightarrow 1r_p r_q \Vdash pq \in G \\
\gamma_1(\lambda y. \alpha_{13} \circ y \circ \alpha_{12} \circ x \circ \alpha_2 \circ \alpha_5^2) : 1r_p \Vdash q \in G \Rightarrow pq \in G \\
\lambda x. \gamma_1(\lambda y. \alpha_{13} \circ y \circ \alpha_{12} \circ x \circ \alpha_2 \circ \alpha_5^2) : \forall r_p^k. r_p \Vdash p \in G \Rightarrow 1r_p \Vdash q \in G \Rightarrow pq \in G \\
\gamma_1(\lambda x. \gamma_1(\lambda y. \alpha_{13} \circ y \circ \alpha_{12} \circ x \circ \alpha_2 \circ \alpha_5^2)) : 1 \Vdash p \in G \Rightarrow q \in G \Rightarrow pq \in G
\end{array}$$

### 3.2 $G$ is generic

We want to prove for all absolute set  $D^{\rightarrow o}$ :

$$1 \Vdash (\forall p^k. C[p] \Rightarrow \exists q^k. C[pq] \ \& \ pq \in D) \Rightarrow \exists p^k. p \in G \ \& \ p \in D.$$

Recall that with the proof terms of absoluteness of existential quantification (with two formulas), we can get proof terms for the absoluteness of the formula  $\exists x. A \ \& \ B$ :

- for all  $x$  and  $p$ ,  $\xi_{\exists_2} \xi_A \xi_B : (p \Vdash \exists x. A \ \& \ B) \Rightarrow (C[p] \Rightarrow \exists x. A \ \& \ B)$
- for all  $x$  and  $p$ ,  $\xi'_{\exists_2} \xi'_A \xi'_B : (C[p] \Rightarrow \exists x. A \ \& \ B) \Rightarrow (p \Vdash \exists x. A \ \& \ B)$

Because  $D$  is absolute, we have for all  $p^k$  and  $r^k$ :

$$\begin{array}{l}
\xi_D : (r \Vdash p \in D) \Rightarrow (C[r] \Rightarrow p \in D) \\
\xi'_D : (C[r] \Rightarrow p \in D) \Rightarrow (r \Vdash p \in D)
\end{array}$$

We will use the fact that if  $D^{\rightarrow o}$  is absolute, then for all  $p^k$ ,  $q \mapsto pq \in D$  is also absolute and the proof terms are the same. This comes from the fact that the deduction rules for  $\forall$  are computationally transparent.

$$\begin{array}{l}
[\forall] r_1^k \\
[\lambda] x : r_1 \text{ IF } \forall p^k. C[p] \Rightarrow \exists q^k. C[pq] \& pq \in D \\
[\forall] r_2^k \\
[\lambda] y : r_2 \text{ IF } \forall p^k. p \in G \Rightarrow p \in D \Rightarrow \perp \simeq \forall p^k. r_2 \text{ IF } p \in G \Rightarrow p \in D \Rightarrow \perp \\
[\lambda] c : C[1r_1r_2] \\
x : r_1 \text{ IF } C[1r_1r_2] \Rightarrow \exists q^k. C[1r_1r_2q] \& 1r_1r_2q \in D \\
\gamma_3 x : (r_1 \text{ IF } C[1r_1r_2]) \Rightarrow (r_1 \text{ IF } \exists q^k. C[1r_1r_2q] \& 1r_1r_2q \in D) \\
\lambda_. c : C[r_1] \Rightarrow C[1r_1r_2] \\
\xi'_C(\lambda_. c) : r_1 \text{ IF } C[1r_1r_2] \\
\gamma_3 x(\xi'_C(\lambda_. c)) : r_1 \text{ IF } \exists q^k. C[1r_1r_2q] \& 1r_1r_2q \in D \\
\xi_{\exists_2} \xi_C \xi_D(\gamma_3 x(\xi'_C(\lambda_. c))) : C[r_1] \Rightarrow \exists q^k. C[1r_1r_2q] \& 1r_1r_2q \in D \\
\alpha_1 c : C[1r_1] \\
\alpha_2(\alpha_1 c) : C[r_1] \\
\xi_{\exists_2} \xi_C \xi_D(\gamma_3 x(\xi'_C(\lambda_. c)))(\alpha_2(\alpha_1 c)) : \exists q^k. C[1r_1r_2q] \& 1r_1r_2q \in D \equiv \neg(\forall q^k. C[1r_1r_2q] \Rightarrow 1r_1r_2q \in D \Rightarrow \perp) \\
[\forall] q^k \\
[\lambda] c' : C[1r_1r_2q] \\
[\lambda] d : 1r_1r_2q \in D \\
y : r_2 \text{ IF } 1r_1r_2q \in G \Rightarrow 1r_1r_2q \in D \Rightarrow \perp \\
\beta_4 y : 1r_1r_2 \text{ IF } 1r_1r_2q \in G \Rightarrow 1r_1r_2q \in D \Rightarrow \perp \\
\beta_3(\beta_4 y) : 1r_1r_2q \text{ IF } 1r_1r_2q \in G \Rightarrow 1r_1r_2q \in D \Rightarrow \perp \\
\gamma_3(\beta_3(\beta_4 y)) : (1r_1r_2q \text{ IF } 1r_1r_2q \in G) \Rightarrow (1r_1r_2q \text{ IF } 1r_1r_2q \in D \Rightarrow \perp) \\
\lambda x. x : 1r_1r_2q \leq 1r_1r_2q \equiv 1r_1r_2q \text{ IF } 1r_1r_2q \in G \\
\gamma_3(\beta_3(\beta_4 y))(\lambda x. x) : 1r_1r_2q \text{ IF } 1r_1r_2q \in D \Rightarrow \perp \\
\gamma_3(\gamma_3(\beta_3(\beta_4 y))(\lambda x. x)) : (1r_1r_2q \text{ IF } 1r_1r_2q \in D) \Rightarrow (1r_1r_2q \text{ IF } \perp) \\
\lambda_. d : C[1r_1r_2q] \Rightarrow 1r_1r_2q \in D \\
\xi'_D(\lambda_. d) : 1r_1r_2q \text{ IF } 1r_1r_2q \in D \\
\gamma_3(\gamma_3(\beta_3(\beta_4 y))(\lambda x. x))(\xi'_D(\lambda_. d)) : 1r_1r_2q \text{ IF } \perp \\
\xi_{\perp}(\gamma_3(\gamma_3(\beta_3(\beta_4 y))(\lambda x. x))(\xi'_D(\lambda_. d))) : C[1r_1r_2q] \Rightarrow \perp \\
[\xi_{\perp}(\gamma_3(\gamma_3(\beta_3(\beta_4 y))(\lambda x. x))(\xi'_D(\lambda_. d))) c' : \perp \\
\lambda c'. d. \xi_{\perp}(\gamma_3(\gamma_3(\beta_3(\beta_4 y))(\lambda x. x))(\xi'_D(\lambda_. d))) c' : \forall q^k. C[1r_1r_2q] \Rightarrow 1r_1r_2q \in D \Rightarrow \perp \\
\xi_{\exists_2} \xi_C \xi_D(\gamma_3 x(\xi'_C(\lambda_. c)))(\alpha_2(\alpha_1 c))(\lambda c'. d. \xi_{\perp}(\gamma_3(\gamma_3(\beta_3(\beta_4 y))(\lambda x. x))(\xi'_D(\lambda_. d))) c' : \perp \\
\lambda c. \xi_{\exists_2} \xi_C \xi_D(\gamma_3 x(\xi'_C(\lambda_. c)))(\alpha_2(\alpha_1 c))(\lambda c'. d. \xi_{\perp}(\gamma_3(\gamma_3(\beta_3(\beta_4 y))(\lambda x. x))(\xi'_D(\lambda_. d))) c' : C[1r_1r_2] \Rightarrow \perp \\
[\xi'_{\perp}(\lambda c. \xi_{\exists_2} \xi_C \xi_D(\gamma_3 x(\xi'_C(\lambda_. c)))(\alpha_2(\alpha_1 c))(\lambda c'. d. \xi_{\perp}(\gamma_3(\gamma_3(\beta_3(\beta_4 y))(\lambda x. x))(\xi'_D(\lambda_. d))) c')] : 1r_1r_2 \text{ IF } \perp \\
\lambda y. \xi'_{\perp}(\lambda c. \xi_{\exists_2} \xi_C \xi_D(\gamma_3 x(\xi'_C(\lambda_. c)))(\alpha_2(\alpha_1 c))(\lambda c'. d. \xi_{\perp}(\gamma_3(\gamma_3(\beta_3(\beta_4 y))(\lambda x. x))(\xi'_D(\lambda_. d))) c')) \\
: \forall r_2^k. (r_2 \text{ IF } \forall p^k. p \in G \Rightarrow p \in D \Rightarrow \perp) \Rightarrow (1r_1r_2 \text{ IF } \perp) \\
\gamma_1(\lambda y. \xi'_{\perp}(\lambda c. \xi_{\exists_2} \xi_C \xi_D(\gamma_3 x(\xi'_C(\lambda_. c)))(\alpha_2(\alpha_1 c))(\lambda c'. d. \xi_{\perp}(\gamma_3(\gamma_3(\beta_3(\beta_4 y))(\lambda x. x))(\xi'_D(\lambda_. d))) c')) \\
: 1r_1 \text{ IF } (\forall p^k. p \in G \Rightarrow p \in D \Rightarrow \perp) \Rightarrow \perp \equiv 1r_1 \text{ IF } \exists p^k. p \in G \& p \in G \\
\lambda x. \gamma_1(\lambda y. \xi'_{\perp}(\lambda c. \xi_{\exists_2} \xi_C \xi_D(\gamma_3 x(\xi'_C(\lambda_. c)))(\alpha_2(\alpha_1 c))(\lambda c'. d. \xi_{\perp}(\gamma_3(\gamma_3(\beta_3(\beta_4 y))(\lambda x. x))(\xi'_D(\lambda_. d))) c')) \\
: \forall r_1^k. (r_1 \text{ IF } \forall p^k. C[p] \Rightarrow \exists q^k. C[pq] \& pq \in D) \Rightarrow (1r_1 \text{ IF } \exists p^k. p \in G \& p \in G) \\
\gamma_1(\lambda x. \gamma_1(\lambda y. \xi'_{\perp}(\lambda c. \xi_{\exists_2} \xi_C \xi_D(\gamma_3 x(\xi'_C(\lambda_. c)))(\alpha_2(\alpha_1 c))(\lambda c'. d. \xi_{\perp}(\gamma_3(\gamma_3(\beta_3(\beta_4 y))(\lambda x. x))(\xi'_D(\lambda_. d))) c')) \\
: 1 \text{ IF } (\forall p^k. C[p] \Rightarrow \exists q^k. C[pq] \& pq \in D) \Rightarrow \exists p^k. p \in G \& p \in G
\end{array}$$

## 4 Finite relations and valuations

### 4.1 Bool and Bool'

Let  $b \in \text{Bool}' := \forall Z^o. (b \doteq 1 \mapsto Z) \Rightarrow (b \doteq 0 \mapsto Z) \Rightarrow Z$ .

We show the equivalence with  $b \in \text{Bool} := \forall Z^{l \rightarrow o}. Z 1 \Rightarrow Z 0 \Rightarrow Z b$ .

Since the denotation in classical realizability are the same, realizers are also the same and from a computational point of view, we can consider Bool and Bool' to be equal. Nevertheless, they are not exactly the same from the point of view of proofs.

$$\left[ \begin{array}{l} [\lambda] z : b \in \text{Bool}' \\ [\forall] Z^{l \rightarrow o} \\ [\lambda] x : Z 1 \\ [\lambda] y : Z 0 \\ z : (b \doteq 1 \mapsto Z b) \Rightarrow (b \doteq 0 \mapsto Z b) \Rightarrow Z b \\ x : b \doteq 1 \mapsto Z 1 \simeq b \doteq 1 \mapsto Z b \\ y : b \doteq 0 \mapsto Z 0 \simeq b \doteq 0 \mapsto Z b \\ z x y : Z b \\ [\lambda x y. z x y : \forall Z^{l \rightarrow o}. Z 1 \Rightarrow Z 0 \Rightarrow Z b \equiv b \in \text{Bool} \\ \lambda z x y. z x y : b \in \text{Bool}' \Rightarrow b \in \text{Bool} \end{array} \right.$$

If we had subtyping in  $\text{PA}\omega^+$ , the conversion would be the identity and not an  $\eta$ -expansion thereof.

$$\left[ \begin{array}{l} [\lambda] z : b \in \text{Bool} \\ [\forall] Z^o \\ \text{We set } Z x := x \doteq b \mapsto Z \text{ in } z \\ z : (1 \doteq b \mapsto Z) \Rightarrow (0 \doteq b \mapsto Z) \Rightarrow (b \doteq b \mapsto Z) \simeq (b \doteq 1 \mapsto Z) \Rightarrow (b \doteq 0 \mapsto Z) \Rightarrow Z \\ z : b \in \text{Bool}' \\ \lambda z. z : b \in \text{Bool} \Rightarrow b \in \text{Bool}' \end{array} \right.$$

Yet, it is enough to have the deduction rule we want for the if – then – else construction:

$$\frac{\mathcal{E}; \Gamma \vdash M : b \in \text{Bool}}{\mathcal{E}; \Gamma \vdash M : (b \doteq 1 \mapsto A) \Rightarrow (b \doteq 0 \mapsto A) \Rightarrow (b \doteq b \mapsto A)}}{\frac{\mathcal{E}; \Gamma \vdash M : (b \doteq 1 \mapsto A) \Rightarrow (b \doteq 0 \mapsto A) \Rightarrow A \quad \mathcal{E}, b = 1; \Gamma \vdash N : A}{\mathcal{E}; \Gamma \vdash N : b \doteq 1 \mapsto A} \quad \frac{\mathcal{E}, b = 0; \Gamma \vdash P : A}{\mathcal{E}; \Gamma \vdash P : b \doteq 0 \mapsto A}}{\mathcal{E}; \Gamma \vdash M N : (b \doteq 0 \mapsto A) \Rightarrow A} \quad \frac{\mathcal{E}; \Gamma \vdash M N : (b \doteq 0 \mapsto A) \Rightarrow A \quad \mathcal{E}; \Gamma \vdash P : b \doteq 0 \mapsto A}{\mathcal{E}; \Gamma \vdash M N P : A}}$$

## 4.2 Totality of mem

We want to prove that the function  $\text{mem}$  defined by  $\text{mem } a p := \text{test } p a 1 \parallel \text{test } p a 0$  is total from the proof  $\text{Tot}_{\text{test}}$  of totality of test. That is, we assume

$$\text{Tot}_{\text{test}} : \forall p \in \text{FVal}. \forall a \in \text{Atom}. \forall b \in \text{Bool}. \text{test } p a b \in \text{Bool}$$

and we want to prove

$$\text{Tot}_{\text{mem}} : \forall p \in \text{FVal}. \forall a \in \text{Atom}. \text{mem } a p \in \text{Bool}$$

for a suitable proof term  $\text{Tot}_{\text{mem}}$ .

$$\left[ \begin{array}{l} [\forall] p^k \\ [\lambda] x : p \in \text{FVal} \\ [\forall] a^t \\ [\lambda] y : a \in \text{Atom} \\ [\forall] Z^{t \rightarrow o} \\ [\lambda] u : Z 1 \\ [\lambda] v : Z 0 \\ \text{Tot}_{\text{test}} \ x y : \forall b \in \text{Bool}. \text{test } a b p \in \text{Bool} \\ \text{true} : 1 \in \text{Bool} \\ \text{Tot}_{\text{test}} \ x y \text{ true} : \text{test } a 1 p \in \text{Bool} \equiv \forall Y^o. Y 1 \Rightarrow Y 0 \Rightarrow Y (\text{test } a 1 p) \\ \text{Tot}_{\text{test}} \ x y \text{ true} : Z(1 \parallel \text{test } a 0 p) \Rightarrow Z(0 \parallel \text{test } a 0 p) \Rightarrow Z(\text{test } a 1 p \parallel \text{test } a 0 p) \simeq Z 1 \Rightarrow Z(\text{test } a 0 p) \Rightarrow Z(\text{test } a 1 p \parallel \text{test } a 0 p) \\ \text{Tot}_{\text{test}} \ x y \text{ true } u : Z(\text{test } a 0 p) \Rightarrow Z(\text{test } a 1 p \parallel \text{test } a 0 p) \\ \text{false} : 0 \in \text{Bool} \\ \text{Tot}_{\text{test}} \ x y \text{ false} : \text{test } a 0 p \in \text{Bool} \equiv \forall Y^o. Y 1 \Rightarrow Y 0 \Rightarrow Y (\text{test } a 0 p) \\ \text{Tot}_{\text{test}} \ x y \text{ false} : Z 1 \Rightarrow Z 0 \Rightarrow Z(\text{test } a 0 p) \\ \text{Tot}_{\text{test}} \ x y \text{ false } u v : Z(\text{test } a 0 p) \\ [\text{Tot}_{\text{test}} \ x y \text{ true } u (\text{Tot}_{\text{test}} \ x y \text{ false } u v) : Z(\text{test } a 1 p \parallel \text{test } a 0 p) \simeq Z(\text{mem } a p) \\ \lambda uv. \text{Tot}_{\text{test}} \ x y \text{ true } u (\text{Tot}_{\text{test}} \ x y \text{ false } u v) : \forall Z. Z 1 \Rightarrow Z 0 \Rightarrow Z(\text{mem } a p) \equiv \text{mem } a p \in \text{Bool} \\ \lambda xyuv. \text{Tot}_{\text{test}} \ x y \text{ true } u (\text{Tot}_{\text{test}} \ x y \text{ false } u v) : \forall p \in \text{FVal}. \forall a \in \text{Atom}. \text{mem } a p \in \text{Bool} \end{array} \right.$$

### 4.3 Totality of eval

We want to find a proof term  $\text{Tot}_{\text{eval}}$  such that

$$\text{Tot}_{\text{eval}} : \forall p \in \text{FVal}. \forall c \in \text{Comp}. \forall b \in \text{Bool}. \text{eval } c b p \in \text{Bool}$$

$$\begin{array}{l}
[\forall] p^t \\
[\lambda] x_p : p \in \text{FVal} \\
[\forall] c^t \\
[\lambda] x_c : c \in \text{Comp} \\
[\forall] b^t \\
[\lambda] y : b \in \text{Bool} \\
\text{false} : 0 \in \text{Bool} \simeq \text{eval } \perp 1 p \in \text{Bool} \\
\text{true} : 1 \in \text{Bool} \simeq \text{eval } \perp 0 p \in \text{Bool} \\
[y \text{ false true} : \text{eval } \perp b p \in \text{Bool} \\
\lambda y. y \text{ false true} : \forall b \in \text{Bool}. \text{eval } \perp b p \in \text{Bool} \\
[\forall] a^t \\
[\lambda] z : a \in \text{Atom} \\
[\forall] b^t \\
[\lambda] y : b \in \text{Bool} \\
[\text{Tot}_{\text{test}} x_p z y : \text{test } p a b \in \text{Bool} \simeq \text{eval } a b p \in \text{Bool} \\
\lambda z y. \text{Tot}_{\text{test}} x_p z y : \forall b \in \text{Bool}. \forall a \in \text{Atom}. \text{eval } a b p \in \text{Bool} \\
[\forall] c_1^t \\
[\forall] c_2^t \\
[\lambda] y_1 : \forall b \in \text{Bool}. \text{eval } c_1 b p \in \text{Bool} \\
[\lambda] y_2 : \forall b \in \text{Bool}. \text{eval } c_2 b p \in \text{Bool} \\
[\forall] b^t \\
[\lambda] x : b \in \text{Bool} \\
y_2 x : \text{eval } c_2 b p \in \text{Bool} \\
\lambda uv. x v u : \text{not } b \in \text{Bool} \\
y_1 (\lambda uv. x v u) : \text{eval } c_1 (\text{not } b) p \in \text{Bool} \\
[[\forall] Z^{t \rightarrow o} \\
[\lambda] z_1 : Z 1 \\
[\lambda] z_2 : Z 0 \\
y_1 (\lambda uv. x v u) z_1 z_2 : Z (\text{eval } c_1 (\text{not } b) p) \simeq Z (\text{eval } c_1 (\text{not } b) p \parallel 0) \\
z_1 : Z 1 \simeq Z (\text{eval } c_1 (\text{not } b) p \parallel 1) \\
[y_2 x z_1 (y_1 (\lambda uv. x v u) z_1 z_2) : Z (\text{eval } c_1 (\text{not } b) p \parallel \text{eval } c_2 b p) \simeq Z (\text{eval } (c_1 \Rightarrow c_2) b p) \\
\lambda z_1 z_2. y_2 x z_1 (y_1 (\lambda uv. x v u) z_1 z_2) : \text{eval } (c_1 \Rightarrow c_2) b p \in \text{Bool} \\
\lambda x z_1 z_2. y_2 x z_1 (y_1 (\lambda uv. x v u) z_1 z_2) : \forall b \in \text{Bool}. \text{eval } (c_1 \Rightarrow c_2) b p \in \text{Bool} \\
\lambda y_1 y_2 x z_1 z_2. y_2 x z_1 (y_1 (\lambda uv. x v u) z_1 z_2) \\
: \forall c_1^t c_2^t. (\forall b \in \text{Bool}. \text{eval } c_1 b p \in \text{Bool}) \Rightarrow (\forall b \in \text{Bool}. \text{eval } c_2 b p \in \text{Bool}) \Rightarrow (\forall b \in \text{Bool}. \text{eval } (c_1 \Rightarrow c_2) b p \in \text{Bool}) \\
x_c (\lambda y. y \text{ false true}) (\lambda z y. \text{Tot}_{\text{test}} x_p z y) (\lambda y_1 y_2 x z_1 z_2. y_2 x z_1 (y_1 (\lambda uv. x v u) z_1 z_2)) : \forall b \in \text{Bool}. \text{eval } c b p \in \text{Bool} \\
\lambda x_p x_c. x_c (\lambda y. y \text{ false true}) (\lambda z y. \text{Tot}_{\text{test}} x_p z y) (\lambda y_1 y_2 x z_1 z_2. y_2 x z_1 (y_1 (\lambda uv. x v u) z_1 z_2)) \\
: \forall p \in \text{FVal}. \forall c \in \text{Comp}. \forall b \in \text{Bool}. \text{eval } c b p \in \text{Bool}
\end{array}$$

The proof for  $\text{Tot}_{\text{subHtree}}$  is similar to this one and is omitted here.



## 4.4 Merging of subH

We want to prove the following result

$$\forall p. \forall a \in \text{Atom}. \text{mem } a \ p \doteq 0 \mapsto \text{subH}(pa^1) \Rightarrow \text{subH}(pa^0) \Rightarrow \text{subH } p$$

```

[[[V] p'
  [V] a'
  [λ] xa : a ∈ Atom
  [Eq.] mem a p = 0
  [λ] x : subH(pa1)
  [λ] y : subH(pa0)
  [[Let] x1 : t1 ∈ Tree
   [Let] x2 : subHtree(pa1) t1 = 1
   [[Let] y1 : t2 ∈ Tree
    [Let] y2 : subHtree(pa0) t2 = 1
    Node xa x1 y1 : Node a t1 t2 ∈ Tree
    [[V] Zo
     [λ] u : Z(subHtree p(Node a t1 t2)) ≈ Z(subHtree(pa1) t1 && subHtree(pa0) t2)
     x2 : Z(subHtree(pa1) t1 && subHtree(pa0) t2) ⇒ Z(1 && subHtree(pa0) t2)
     x2 u : Z(1 && subHtree(pa0) t2) ≈ Z(subHtree(pa0) t2)
     y2 : Z(subHtree(pa0) t2) ⇒ Z 1
     y2 (x2 u) : Z 1
     y2 ∘ x2 ≡ λu. y2 (x2 u) : ∀Zo. Z(subHtree p(Node a t1 t2)) ⇒ Z 1 ≡ subHtree p(Node a t1 t2) = 1
     ⟨Node xa x1 y1, y2 ∘ x2⟩ : subH p
     let (y1, y2) := y in ⟨Node xa x1 y1, y2 ∘ x2⟩ : subH p
     let (x1, x2) := x in let (y1, y2) := y in ⟨Node xa x1 y1, y2 ∘ x2⟩ : subH p
     λxa.xy. let (x1, x2) := x in let (y1, y2) := y in ⟨Node xa x1 y1, y2 ∘ x2⟩
       : ∀p'. ∀a ∈ Atom. mem a p ≐ 0 ↦ subH(pa1) ⇒ subH(pa0) ⇒ subH p

```

We let

$$\text{merge} := \lambda x_a xy. \text{let } (x_1, x_2) := x \text{ in let } (y_1, y_2) := y \text{ in } \langle \text{Node } x_a x_1 y_1, y_2 \circ x_2 \rangle .$$

## 4.5 Monotonicity of test

We want to prove that

$$\forall p^t \forall q^t \forall a^t \forall b^t. \text{test } p a b = 1 \Rightarrow \text{test}(p \cup q) a b = 1$$

$$\left[ \begin{array}{l} [\forall] p^t \\ [\forall] q^t \\ [\forall] a^t \\ [\forall] b^t \\ [\lambda] x : \text{test } p a b = 1 \\ [[\forall] Z^{t \rightarrow o} \\ [\lambda] y : Z(\text{test}(p \cup q) a b) \simeq \text{test } a b p \parallel \text{test } a b q \\ [x y : Z(1 \parallel \text{test } a b q) \simeq Z 1 \\ \lambda y. x y : \forall Z^{t \rightarrow o}. Z(\text{test}(p \cup q) a b) \Rightarrow Z 1 \equiv \text{test}(p \cup q) a b = 1 \\ \lambda x y. x y : \forall p^t \forall q^t \forall a^t \forall b^t. \text{test } a b p = 1 \Rightarrow \text{test}(p \cup q) a b = 1 \end{array} \right.$$

and we write  $\text{Mon}_{\text{test}} := \lambda x y. x y$ .

## 4.6 Monotonicity of eval and subHtree

We want to prove that

$$\text{Mon}_{\text{eval}} : \forall p^t. \forall c \in \text{Comp}. \forall b \in \text{Bool}. \text{eval } c b p = 1 \Rightarrow \text{eval } c b (p \cup q) = 1$$

and

$$\text{Mon}_{\text{subHtree}} : \forall p^t. \forall t \in \text{Tree}. \forall b \in \text{Bool}. \text{subHtree } p t = 1 \Rightarrow \text{subHtree}(p \cup q) t = 1$$

for suitable proof terms  $\text{Mon}_{\text{eval}}$  and  $\text{Mon}_{\text{subHtree}}$ . Since both proofs follow the same pattern, we will focus only on the first one.

**Induction principle on quantifier-free formulas** To prove that eval is monotonic, we will need to use the *induction principle* of quantifier-free proposition. It is very similar to the relativization predicate  $c \in \text{Comp}$  except that recursive arguments are guarded by relativization. More precisely, it is a proof term  $\text{ind}_{\text{Comp}}$  of the formula

$$\begin{aligned} c \in \text{Comp} \Rightarrow \forall Z^{t \rightarrow o}. Z \perp \Rightarrow \\ (\forall a \in \text{Atom}. Z a) \Rightarrow \\ (\forall c_1 \in \underline{\text{Comp}}. \forall c_2 \in \underline{\text{Comp}}. Z c_1 \Rightarrow Z c_2 \Rightarrow Z(c_1 \Rightarrow c_2)) \Rightarrow \\ Z c \end{aligned}$$

(The differences with Comp are underlined.)

It can be easily built using the proof term of  $c \in \text{Comp}$  on the proposition  $P c := c \in \text{Comp} \Rightarrow Z c$ .

$[\forall] p^t$   
 $[\forall] c^t$   
 $[\lambda] x_c : c \in \text{Comp}$   
 $[\forall] b^t$   
 $[\lambda] x_b : b \in \text{Bool}$   
 $[\lambda] y : \text{eval } \perp b p = 1$   
 $y : \text{eval } \perp b p = 1 \simeq \text{not } b = 1 \simeq \text{eval } \perp b (p \cup q) = 1$   
 $\lambda x_b y. y : \forall b \in \text{Bool}. \text{eval } \perp b p = 1 \Rightarrow \text{eval } \perp b (p \cup q) = 1$   
 $[\forall] b^t$   
 $[\lambda] x_b : b \in \text{Bool}$   
 $[\forall] p^t$   
 $[\forall] a^t$   
 $[\lambda] u : a \in \text{Atom}$   
 $[\lambda] v : \text{eval } a b p = 1 \simeq \text{test } a b p = 1$   
 $[\text{Mon}_{\text{test}}] v : \text{test}(p \cup q) a b = 1 \simeq \text{eval } a b (p \cup q) = 1$   
 $\lambda x_b u v. \text{Mon}_{\text{test}} v : \forall b \in \text{Bool}. \forall a \in \text{Atom}. \text{eval } a b p = 1 \Rightarrow \text{eval } a b (p \cup q) = 1$   
 $[\forall] c_1^t$   
 $[\lambda] x_1 : c_1 \in \text{Comp}$   
 $[\forall] c_2^t$   
 $[\lambda] x_2 : c_2 \in \text{Comp}$   
 $[\lambda] u : \forall b \in \text{Bool}. \forall p^t. \text{eval } c_1 b p = 1 \Rightarrow \text{eval } c_1 b (p \cup q) = 1$   
 $[\lambda] v : \forall b \in \text{Bool}. \forall p^t. \text{eval } c_2 b p = 1 \Rightarrow \text{eval } c_2 b (p \cup q) = 1$   
 $[\forall] b^t$   
 $[\lambda] x_b : b \in \text{Bool}$   
 $[\lambda] w : \text{eval}(c_1 \Rightarrow c_2) b p = 1 \simeq \text{eval } c_1 (\text{not } b) p \parallel \text{eval } c_2 b p = 1$   
 $\lambda u v. x_n v u : \text{not } b \in \text{Bool}$   
 $\text{Tot}_{\text{eval}} x_1 (\lambda u v. x_n v u) : \text{eval } c_1 (\text{not } b) p \in \text{Bool}$   
 $\text{I Tot}_{\text{eval}} x_1 (\lambda u v. x_n v u) : \text{eval } c_1 (\text{not } b) p \in \text{Bool}'$   
 $[[\text{Eq.}] \text{eval } c_1 (\text{not } b) p = 1$   
 $u x_b : \text{eval } c_1 (\text{not } b) p = 1 \Rightarrow \text{eval } c_1 (\text{not } b) (p \cup q) = 1 \simeq 1 = 1 \Rightarrow \text{eval } c_1 (\text{not } b) (p \cup q) = 1$   
 $u x_b \text{I} : \text{eval } c_1 (\text{not } b) (p \cup q) = 1$   
 $[[\forall] Z^{\rightarrow o}$   
 $[\lambda] z : Z(\text{eval}(c_1 \Rightarrow c_2) b (p \cup q)) \simeq Z(\text{eval } c_1 (\text{not } b) (p \cup q) \parallel \text{eval } c_1 b (p \cup q))$   
 $[u x_b \text{I} z : Z(1 \parallel \text{eval } c_1 b (p \cup q)) \simeq Z 1$   
 $\lambda z. u x_b \text{I} z : \text{eval}(c_1 \Rightarrow c_2) b (p \cup q) = 1$   
 $\lambda z. u x_b \text{I} z : \text{eval } c_1 (\text{not } b) p \doteq 1 \mapsto \text{eval}(c_1 \Rightarrow c_2) b (p \cup q) = 1$   
 $[[\text{Eq.}] \text{eval } c_1 (\text{not } b) p = 0$   
 $w : \text{eval } c_1 (\text{not } b) p \parallel \text{eval } c_2 b p = 1 \simeq 0 \parallel \text{eval } c_2 b p = 1 \simeq \text{eval } c_2 b p = 1$   
 $v x_b w : \text{eval } c_2 b (p \cup q) = 1$   
 $[[\forall] Z^{\rightarrow o}$   
 $[\lambda] z : Z(\text{eval}(c_1 \Rightarrow c_2) b (p \cup q)) \simeq Z(\text{eval } c_1 (\text{not } b) (p \cup q) \parallel \text{eval } c_2 b (p \cup q))$   
 $[v x_b w z : Z(\text{eval } c_1 (\text{not } b) (p \cup q) \parallel 1) \simeq Z 1$   
 $\lambda z. v x_b w z : \text{eval}(c_1 \Rightarrow c_2) b (p \cup q) = 1$   
 $\lambda z. v x_b w z : \text{eval } c_1 (\text{not } b) p \doteq 0 \mapsto \text{eval}(c_1 \Rightarrow c_2) b (p \cup q) = 1$   
 $\text{I Tot}_{\text{eval}} x_1 (\lambda u v. x_n v u) (\lambda z. u x_b \text{I} z) (\lambda z. v x_b w z) : \text{eval}(c_1 \Rightarrow c_2) b (p \cup q) = 1$   
 $\lambda x_1 x_2 u v x_b w. \text{I Tot}_{\text{eval}} x_1 (\lambda u v. x_n v u) (\lambda z. u x_b \text{I} z) (\lambda z. v x_b w z) : \dots$   
 $\text{ind}_{\text{Comp}} x_c (\lambda x_b y. y) (\lambda x_b u v. \text{Mon}_{\text{test}} v) (\lambda x_1 x_2 u v x_b w. \text{I Tot}_{\text{eval}} x_1 (\lambda u v. x_n v u) (\lambda z. u x_b \text{I} z) (\lambda z. v x_b w z))$   
 $: \forall b \in \text{Bool}. \text{eval } c b p = 1 \Rightarrow \text{eval } c b (p \cup q) = 1$   
 $\lambda x_c. \text{ind}_{\text{Comp}} x_c (\lambda x_b y. y) (\lambda x_b u v. \text{Mon}_{\text{test}} v) (\lambda x_1 x_2 u v x_b w. \text{I Tot}_{\text{eval}} x_1 (\lambda u v. x_n v u) (\lambda z. u x_b \text{I} z) (\lambda z. v x_b w z))$   
 $: \forall c \in \text{Comp}. \forall b \in \text{Bool}. \text{eval } c b p = 1 \Rightarrow \text{eval } c b (p \cup q) = 1$

## 4.7 A proof of upward closure of FVal

We need to enrich the interface for finite relations over  $\text{Atom} \times \text{Bool}$  with an intersection operation  $\_ \cap \_$  with the following properties: for all  $a, b, p$  and  $q$ ,

$$p \cap \emptyset \simeq \emptyset \quad p \cap (p \cup q) \simeq p \quad p \cap (qa^b) \simeq_{\text{mem } a \ p=1} (p \cap q)a^b \quad p \cap (qa^b) \simeq_{\text{mem } a \ p=0} p \cap q$$

These congruences could be derived from more primitive ones like the one the have for  $\cup$  but these are exactly the ones we need to prove the upward closure of FVal.

We let

$$T^b := \forall r^k. \forall a \in \text{Atom}. \text{mem } a \ r = 0 \mapsto Z(p \cap r) \Rightarrow Z(p \cap (ra^b))$$

$$\left[ \begin{array}{l} [\forall] p^k \\ [\forall] q^k \\ [\lambda] z : pq \in \text{FVal} \\ [\forall] Z^o \\ [\lambda] e_0 : Z \emptyset \\ [\lambda] e_+ : \forall r^k. \forall a \in \text{Atom}. \text{mem } a \ r = 0 \mapsto Zr \Rightarrow Z(ra^1) \\ [\lambda] e_- : \forall r^k. \forall a \in \text{Atom}. \text{mem } a \ r = 0 \mapsto Zr \Rightarrow Z(ra^0) \\ \delta_{20} e_0 : Z(p \cap \emptyset) \\ z(\delta_{20} e_0) : T^1 \Rightarrow T^0 \Rightarrow Z(p \cap (p \cup q)) \\ [\forall] r^k \\ [\forall] a^t \\ [\lambda] x_a : a \in \text{Atom} \\ [\text{Eq.}] \text{mem } a \ r = 0 \\ [\lambda] y : Z(p \cap r) \\ e_+ : \text{mem } a \ p = 0 \mapsto Z(p \cap r) \Rightarrow Z((p \cap r)a^1) \simeq Z(p \cap r) \Rightarrow Z((p \cap r)a^1) \\ e_+ y : Z((p \cap r)a^1) \\ [\text{Eq.}] \text{mem } a \ r = 1 \\ \delta_{22} : \text{mem } a \ r \doteq 1 \mapsto (p \cap r)a^1 = p \cap (ra^1) \simeq (p \cap r)a^1 = p \cap (ra^1) \equiv \forall Z. Z((p \cap r)a^1) \Rightarrow Z(p \cap (ra^1)) \\ \delta_{22}(e_+ y) : Z(p \cap (ra^1)) \\ \delta_{22}(e_+ y) : \text{mem } a \ r \doteq 1 \mapsto Z(p \cap (ra^1)) \\ [\text{Eq.}] \text{mem } a \ r = 0 \\ \delta_{23} : \text{mem } a \ r \doteq 0 \mapsto p \cap r = p \cap (ra^1) \simeq p \cap r = p \cap (ra^1) \equiv \forall Z. Z(p \cap r) \Rightarrow Z(p \cap (ra^1)) \\ \delta_{23} y : Z(p \cap (ra^1)) \\ \delta_{23} y : \text{mem } a \ r \doteq 0 \mapsto Z(p \cap (ra^1)) \\ \delta_{10} x_a : \text{mem } a \ p \in \text{Bool} \\ \text{I}(\delta_{10} x_a) : \text{mem } a \ p \in \text{Bool}' \equiv \forall Z^o. (\text{mem } a \ p \doteq 1 \mapsto Z) \Rightarrow (\text{mem } a \ p \doteq 0 \mapsto Z) \Rightarrow Z \\ \text{I}(\delta_{10} x_a)(\delta_{22}(e_+ y))(\delta_{23} y) : Z(p \cap (ra^1)) \\ \lambda x_a y. \text{I}(\delta_{10} x_a)(\delta_{22}(e_+ y))(\delta_{23} y) : \forall r. \forall a \in \text{Atom}. \text{mem } a \ r = 0 \mapsto Z(p \cap r) \Rightarrow Z(p \cap (ra^1)) \equiv T^1 \\ \text{idem} \\ \lambda x_a y. \text{I}(\delta_{10} x_a)(\delta_{22}(e_- y))(\delta_{23} y) : \forall r. \forall a \in \text{Atom}. \text{mem } a \ r = 0 \mapsto Z(p \cap r) \Rightarrow Z(p \cap (ra^0)) \equiv T^0 \\ z(\delta_{20} e_0)(\lambda x_a y. \text{I}(\delta_{10} x_a)(\delta_{22}(e_+ y))(\delta_{23} y))(\lambda x_a y. \text{I}(\delta_{10} x_a)(\delta_{22}(e_- y))(\delta_{23} y)) : Z(p \cap (p \cup q)) \\ \delta_{21}(z(\delta_{20} e_0)(\lambda x_a y. \text{I}(\delta_{10} x_a)(\delta_{22}(e_+ y))(\delta_{23} y))(\lambda x_a y. \text{I}(\delta_{10} x_a)(\delta_{22}(e_- y))(\delta_{23} y))) : Zp \\ \lambda e_0 e_+ e_- . \delta_{21}(z(\delta_{20} e_0)(\lambda x_a y. \text{I}(\delta_{10} x_a)(\delta_{22}(e_+ y))(\delta_{23} y))(\lambda x_a y. \text{I}(\delta_{10} x_a)(\delta_{22}(e_- y))(\delta_{23} y))) : p \in \text{FVal} \\ \lambda z e_0 e_+ e_- . \delta_{21}(z(\delta_{20} e_0)(\lambda x_a y. \text{I}(\delta_{10} x_a)(\delta_{22}(e_+ y))(\delta_{23} y))(\lambda x_a y. \text{I}(\delta_{10} x_a)(\delta_{22}(e_- y))(\delta_{23} y))) : \forall p \forall q. pq \in \text{FVal} \Rightarrow p \in \text{FVal} \end{array} \right.$$

## 4.8 A proof of C[1]

$$\left[ \begin{array}{l} [\forall] Z^o \\ [\lambda] x : Z1 \\ [\lambda] y : \forall p^k. \forall a \in \text{Atom}. \text{mem } a \ p = 0 \mapsto Zp \Rightarrow Z(pa^1) \\ [\lambda] z : \forall p^k. \forall a \in \text{Atom}. \text{mem } a \ p = 0 \mapsto Zp \Rightarrow Z(pa^0) \\ x : Z1 \\ \lambda xyz. z : \forall Z^o. Z1 \Rightarrow (\dots) \Rightarrow (\dots) \Rightarrow Z1 \equiv 1 \in \text{FVal} \\ \langle \lambda xyz. z, \text{I} \rangle : 1 \in \text{FVal} \wedge (\text{subH } 1 \Rightarrow \text{subH } 1) \equiv \text{C}[1] \end{array} \right.$$

## 4.9 A proof of $\forall p^k q^k. C[pq] \Rightarrow C[p]$

Recall that we have the following proof terms:

$$\begin{aligned} \text{Mon}_{\text{subHtree}} &: \forall p^t q^t t'. \text{subHtree } p t = 1 \Rightarrow \text{subHtree}(p \cup q) t = 1 \\ \text{Up}_{\text{FVal}} &: \forall p^t q^t. (p \cup q) \in \text{FVal} \Rightarrow p \in \text{FVal} \end{aligned}$$

$$\begin{array}{l} \left[ \begin{array}{l} [\forall] p^k \\ [\forall] p^k \\ [\lambda] c : C[pq] := pq \in \text{FVal} \wedge \text{subH}(pq) \Rightarrow \text{subH } \emptyset \\ \left[ \begin{array}{l} [\text{Let}] c_1 : pq \in \text{FVal} \\ [\text{Let}] c_2 : \text{subH}(pq) \Rightarrow \text{subH } \emptyset \\ \text{Up}_{\text{FVal}} c_1 : p \in \text{FVal} \\ \left[ \begin{array}{l} [\lambda] x : \text{subH } p \\ \left[ \begin{array}{l} [\text{Let}] x_1 : t \in \text{Tree} \\ [\text{Let}] x_2 : \text{subHtree } p t = 1 \\ \text{Mon}_{\text{subHtree}} x_2 : \text{subHtree}(p \cup q) t = 1 \\ \langle x_1, \text{Mon}_{\text{subHtree}} x_2 \rangle : \text{subH}(pq) \\ c_2 \langle x_1, \text{Mon}_{\text{subHtree}} x_2 \rangle : \text{subH } \emptyset \\ \text{let } (x_1, x_2) := x \text{ in } c_2 \langle x_1, \text{Mon}_{\text{subHtree}} x_2 \rangle : \text{subH } \emptyset \\ \lambda x. \text{let } (x_1, x_2) := x \text{ in } c_2 \langle x_1, \text{Mon}_{\text{subHtree}} x_2 \rangle : \text{subH } p \Rightarrow \text{subH } \emptyset \\ \langle \text{Up}_{\text{FVal}} c_1, \lambda x. \text{let } (x_1, x_2) := x \text{ in } c_2 \langle x_1, \text{Mon}_{\text{subHtree}} x_2 \rangle \rangle : C[p] \end{array} \right] \\ \text{let } (c_1, c_2) := c \text{ in } \langle \text{Up}_{\text{FVal}} c_1, \lambda x. \text{let } (x_1, x_2) := x \text{ in } c_2 \langle x_1, \text{Mon}_{\text{subHtree}} x_2 \rangle \rangle : C[p] \end{array} \right] \\ \lambda c. \text{let } (c_1, c_2) := c \text{ in } \langle \text{Up}_{\text{FVal}} c_1, \lambda x. \text{let } (x_1, x_2) := x \text{ in } c_2 \langle x_1, \text{Mon}_{\text{subHtree}} x_2 \rangle \rangle : C[pq] \Rightarrow C[p] \end{array} \right] \end{array} \right. \end{array}$$

## 5 Absoluteness of datatypes

### 5.1 Tree is absolute

The ideas are the same as for  $\mathbb{N}$ . We recall the definition of  $t \in \text{Tree}$ :

$$\begin{aligned} t \in \text{Tree} &:= \forall Z^{t \rightarrow o}. (\forall \vec{v} \in \text{Term}. Z(\text{Leaf } \vec{v})) \Rightarrow \\ &\quad (\forall t_1^t \forall t_2^t. \forall a \in \text{Atom}. Z t_1 \Rightarrow Z t_2 \Rightarrow Z(\text{Node } a t_1 t_2)) \Rightarrow \\ &\quad Z t \end{aligned}$$

To ease writing, we set

$$\text{clos}Z := \forall t_1^t \forall t_2^t. \forall a \in \text{Atom}. Z t_1 \Rightarrow Z t_2 \Rightarrow Z(\text{Node } a t_1 t_2)$$

Because of its length, the proof is split in two parts. The first one is the main proof and the second one is meant to be plugged into it.

$[\lambda] z : p \Vdash t \in \text{Tree}$   
 $[\lambda] c : C[p]$   
 $[\forall] Z^{t \rightarrow o}$   
 $[\lambda] x : \forall \vec{v} \in \text{Term}. Z(\text{Leaf } \vec{v})$   
 $[\lambda] y : \text{closZ}$   
 $z : (p \Vdash (\forall \vec{v} \in \text{Term}. Z(\text{Leaf } \vec{v})) \Rightarrow \text{closZ} \Rightarrow Z t)[Z/Z^* r]$   
 $\gamma_3 z : (p \Vdash \forall \vec{v} \in \text{Term}. Z(\text{Leaf } \vec{v}))[Z/Z^* r] \Rightarrow (p \Vdash \text{closZ} \Rightarrow Z t)[Z/Z^* r]$   
 $[\forall] (\vec{v})^t$   
 $[\forall] q_{\vec{v}}^k$   
 $[\lambda] w : q_{\vec{v}} \Vdash \vec{v} \in \text{Term}$   
 $\xi_{\text{Term}} w : C[q_{\vec{v}}] \Rightarrow \vec{v} \in \text{Term}$   
 $[\forall] r^k$   
 $[\lambda] c' : C[pq_{\vec{v}}r]$   
 $\alpha_2(\alpha_1 c') : C[q_{\vec{v}}]$   
 $\xi_{\text{Term}} w(\alpha_2(\alpha_1 c')) : \vec{v} \in \text{Term}$   
 $x(\xi_{\text{Term}} w(\alpha_2(\alpha_1 c'))) : Z(\text{Leaf } \vec{v})$   
 $\lambda c'. x(\xi_{\text{Term}} w(\alpha_2(\alpha_1 c'))) : \forall r. C[pq_{\vec{v}}r] \Rightarrow Z(\text{Leaf } \vec{v}) \equiv (pq_{\vec{v}} \Vdash Z(\text{Leaf } \vec{v}))[Z/Z^* r]$   
 $\lambda wc'. x(\xi_{\text{Term}} w(\alpha_2(\alpha_1 c'))) : \forall \vec{v} q_{\vec{v}}. (q_{\vec{v}} \Vdash \vec{v} \in \text{Term}) \Rightarrow (pq_{\vec{v}} \Vdash Z(\text{Leaf } \vec{v}))[Z/Z^* r]$   
 $\gamma_1(\lambda wc'. x(\xi_{\text{Term}} w(\alpha_2(\alpha_1 c')))) : \forall \vec{v}. (p \Vdash \vec{v} \in \text{Term} \Rightarrow Z(\text{Leaf } \vec{v}))[Z/Z^* r] \simeq (p \Vdash \forall \vec{v} \in \text{Term}. Z(\text{Leaf } \vec{v}))[Z/Z^* r]$   
 $\gamma_3 z(\gamma_1(\lambda wc'. x(\xi_{\text{Term}} w(\alpha_2(\alpha_1 c'))))) : (p \Vdash \text{closZ} \Rightarrow Z t)[Z/Z^* r]$   
 $\gamma_3(\gamma_3 z(\gamma_1(\lambda wc'. x(\xi_{\text{Term}} w(\alpha_2(\alpha_1 c')))))) : (p \Vdash \text{closZ})[Z/Z^* r] \Rightarrow (p \Vdash Z t)[Z/Z^* r]$   
 $[\forall] t_1^t$   
 $[\forall] t_2^t$   
 $[\forall] a^t$   
 $[\text{See below}]$   
 $\gamma_1(\lambda u. \gamma_1(\lambda v_1. \gamma_1(\lambda v_2 c'. y(\xi_{\text{Atom}} u(\alpha_2(\alpha_1^3 c')))(v_1(\alpha_9(\alpha_{16} c')))(v_2(\alpha_{10}(\alpha_{16} c')))))) : p \Vdash \text{closZ}$   
 $\text{Write } T \text{ the term just above so that } T : p \Vdash \text{closZ}$   
 $\gamma_3(\gamma_3 z(\gamma_1(\lambda wc'. x(\xi_{\text{Term}} w(\alpha_2(\alpha_1 c')))))) T : (p \Vdash Z t)[Z/Z^* r] \equiv \forall r^k. C[pr] \Rightarrow Z t$   
 $\alpha_7 c : C[p1]$   
 $\gamma_3(\gamma_3 z(\gamma_1(\lambda wc'. x(\xi_{\text{Term}} w(\alpha_2(\alpha_1 c')))))) T(\alpha_7 c) : Z t$   
 $\lambda xy. \gamma_3(\gamma_3 z(\gamma_1(\lambda wc'. x(\xi_{\text{Term}} w(\alpha_2(\alpha_1 c')))))) T(\alpha_7 c) : \forall Z. (\forall \vec{v} \in \text{Term}. Z(\text{Leaf } \vec{v})) \Rightarrow \text{closZ} \Rightarrow Z t \equiv t \in \text{Tree}$   
 $\lambda z c x y. \gamma_3(\gamma_3 z(\gamma_1(\lambda wc'. x(\xi_{\text{Term}} w(\alpha_2(\alpha_1 c')))))) T(\alpha_7 c) : (p \Vdash t \in \text{Tree}) \Rightarrow (C[p] \Rightarrow t \in \text{Tree})$

For the second part of the proof, recall that we have the following context:

$z : p \Vdash t \in \text{Tree}$   
 $c : C[p]$   
 $x : \forall \vec{v} \in \text{Term}. Z(\text{Leaf } \vec{v})$   
 $y : \text{closZ}$

$$\begin{array}{l}
[\forall] t_1^t \\
[\forall] t_2^t \\
[\forall] a^t \\
[\forall] q_0^k \\
[\lambda] u : q_0 \text{ IF } a \in \text{Atom} \\
\xi_{\text{Atom}} u : C[q_0] \Rightarrow a \in \text{Atom} \\
[\forall] q_1^k \\
[\lambda] v_1 : (q_1 \text{ IF } Z t_1)[Z/Z^*r] \simeq \forall r^k. C[q_1 r] \Rightarrow Z t_1 \\
[\forall] q_2^k \\
[\lambda] v_2 : (q_2 \text{ IF } Z t_2)[Z/Z^*r] \simeq \forall r^k. C[q_2 r] \Rightarrow Z t_2 \\
[\forall] r^k \\
[\lambda] c' : C[pq_0 q_1 q_2 r] \\
\alpha_2 (\alpha_1^3 c') : C[q_0] \\
\xi_{\text{Atom}} u (\alpha_2 (\alpha_1^3 c')) : a \in \text{Atom} \\
y (\xi_{\text{Atom}} u (\alpha_2 (\alpha_1^3 c'))) : Z t_1 \Rightarrow Z t_2 \Rightarrow Z (\text{Node } a t_1 t_2) \\
\alpha_9 (\alpha_{16} c') : C[q_1 r] \\
v_1 (\alpha_9 (\alpha_{16} c')) : Z t_1 \\
y (\xi_{\text{Atom}} u (\alpha_2 (\alpha_1^3 c'))) (v_1 (\alpha_9 (\alpha_{16} c'))) : Z t_2 \Rightarrow Z (\text{Node } a t_1 t_2) \\
\alpha_{10} c' : C[q_2 r] \\
v_2 (\alpha_{10} c') : Z t_2 \\
y (\xi_{\text{Atom}} u (\alpha_2 (\alpha_1^3 c'))) (v_1 (\alpha_9 (\alpha_{16} c'))) (v_2 (\alpha_{10} c')) : Z (\text{Node } a t_1 t_2) \\
[\lambda c'. y (\xi_{\text{Atom}} u (\alpha_2 (\alpha_1^3 c'))) (v_1 (\alpha_9 (\alpha_{16} c'))) (v_2 (\alpha_{10} (\alpha_{16} c')))] : (pq_0 q_1 q_2 \text{ IF } Z (\text{Node } a t_1 t_2))[Z/Z^*r] \\
\lambda v_2 c'. y (\xi_{\text{Atom}} u (\alpha_2 (\alpha_1^3 c'))) (v_1 (\alpha_9 (\alpha_{16} c'))) (v_2 (\alpha_{10} (\alpha_{16} c'))) : (\forall q_2^k. (q_2 \text{ IF } Z t_2) \Rightarrow (pq_0 q_1 q_2 \text{ IF } Z (\text{Node } a t_1 t_2)))[Z/Z^*r] \\
\gamma_1 (\lambda v_2 c'. y (\xi_{\text{Atom}} u (\alpha_2 (\alpha_1^3 c'))) (v_1 (\alpha_9 (\alpha_{16} c'))) (v_2 (\alpha_{10} (\alpha_{16} c')))) : (pq_0 q_1 \text{ IF } Z t_2 \Rightarrow Z (\text{Node } a t_1 t_2))[Z/Z^*r] \\
\lambda v_1. \gamma_1 (\lambda v_2 c'. y (\xi_{\text{Atom}} u (\alpha_2 (\alpha_1^3 c'))) (v_1 (\alpha_9 (\alpha_{16} c'))) (v_2 (\alpha_{10} (\alpha_{16} c')))) \\
: (\forall q_1^k. (q_1 \text{ IF } Z t_1) \Rightarrow (pq_0 q_1 \text{ IF } Z t_2 \Rightarrow Z (\text{Node } a t_1 t_2)))[Z/Z^*r] \\
\gamma_1 (\lambda v_1. \gamma_1 (\lambda v_2 c'. y (\xi_{\text{Atom}} u (\alpha_2 (\alpha_1^3 c'))) (v_1 (\alpha_9 (\alpha_{16} c'))) (v_2 (\alpha_{10} (\alpha_{16} c'))))) : (pq_0 \text{ IF } Z t_1 \Rightarrow Z t_2 \Rightarrow Z (\text{Node } a t_1 t_2))[Z/Z^*r] \\
\lambda u. \gamma_1 (\lambda v_1. \gamma_1 (\lambda v_2 c'. y (\xi_{\text{Atom}} u (\alpha_2 (\alpha_1^3 c'))) (v_1 (\alpha_9 (\alpha_{16} c'))) (v_2 (\alpha_{10} (\alpha_{16} c'))))) \\
: \forall q_0^k. (q_0 \text{ IF } a \in \text{Atom}) \Rightarrow (pq_0 \text{ IF } Z t_1 \Rightarrow Z t_2 \Rightarrow Z (\text{Node } a t_1 t_2))[Z/Z^*r] \\
\gamma_1 (\lambda u. \gamma_1 (\lambda v_1. \gamma_1 (\lambda v_2 c'. y (\xi_{\text{Atom}} u (\alpha_2 (\alpha_1^3 c'))) (v_1 (\alpha_9 (\alpha_{16} c'))) (v_2 (\alpha_{10} (\alpha_{16} c')))))) \\
: (p \text{ IF } a \in \text{Atom} \Rightarrow Z t_1 \Rightarrow Z t_2 \Rightarrow Z (\text{Node } a t_1 t_2))[Z/Z^*r] \\
\gamma_1 (\lambda u. \gamma_1 (\lambda v_1. \gamma_1 (\lambda v_2 c'. y (\xi_{\text{Atom}} u (\alpha_2 (\alpha_1^3 c'))) (v_1 (\alpha_9 (\alpha_{16} c'))) (v_2 (\alpha_{10} (\alpha_{16} c')))))) : p \text{ IF } \text{clos}Z
\end{array}$$

$$\begin{array}{l}
[\lambda] z : C[p] \Rightarrow t \in \text{Tree} \\
[\forall] (Z^*)^{t \rightarrow \kappa \rightarrow o} \\
[\forall] q_L^{\kappa} \\
[\lambda] x : q_L \text{ IF } \forall \vec{v} \in \text{Term}. Z(\text{Leaf } \vec{v}) \\
[\forall] q_N^{\kappa} \\
[\lambda] y : q_N \text{ IF } \text{closZ} \\
\beta_3 x : \forall (\vec{v})^t. q_L q_N \text{ IF } \vec{v} \in \text{Term} \Rightarrow Z(\text{Leaf } \vec{v}) \\
\gamma_3 (\beta_3 x) : \forall (\vec{v})^t. (q_L q_N \text{ IF } \vec{v} \in \text{Term}) \Rightarrow (q_L q_N \text{ IF } Z(\text{Leaf } \vec{v})) \\
[\forall] (\vec{v})^t \\
[\lambda] u : \vec{v} \in \text{Term} \\
\lambda_. u : C[q_L q_N] \Rightarrow \vec{v} \in \text{Term} \\
\xi'_{\text{Term}}(\lambda_. u) : q_L q_N \text{ IF } \vec{v} \in \text{Term} \\
\gamma_3 (\beta_3 x) (\xi'_{\text{Term}}(\lambda_. u)) : q_L q_N \text{ IF } Z(\text{Leaf } \vec{v}) \\
\lambda u. \gamma_3 (\beta_3 x) (\xi'_{\text{Term}}(\lambda_. u)) : \forall \vec{v} \in \text{Term}. q_L q_N \text{ IF } Z(\text{Leaf } \vec{v}) \\
\beta_4 y : \forall t_1^t \forall t_2^t \forall a^t. q_L q_N \text{ IF } a \in \text{Atom} \Rightarrow Z t_1 \Rightarrow Z t_2 \Rightarrow Z(\text{Node } a t_1 t_2) \\
\gamma_3 (\beta_4 y) : \forall t_1^t \forall t_2^t \forall a^t. (q_L q_N \text{ IF } a \in \text{Atom}) \Rightarrow (q_L q_N \text{ IF } Z t_1 \Rightarrow Z t_2 \Rightarrow Z(\text{Node } a t_1 t_2)) \\
[\forall] t_1^t \\
[\forall] t_2^t \\
[\forall] a^t \\
[\lambda] u : a \in \text{Atom} \\
[\lambda] v : q_L q_N \text{ IF } Z t_1 \\
[\lambda] w : q_L q_N \text{ IF } Z t_2 \\
\lambda_. u : C[q_L q_N] \Rightarrow a \in \text{Atom} \\
\xi'_{\text{Atom}}(\lambda_. u) : q_L q_N \text{ IF } a \in \text{Atom} \\
\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u)) : q_L q_N \text{ IF } Z t_1 \Rightarrow Z t_2 \Rightarrow Z(\text{Node } a t_1 t_2) \\
\gamma_3 (\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u))) : (q_L q_N \text{ IF } Z t_1) \Rightarrow (q_L q_N \text{ IF } Z t_2 \Rightarrow Z(\text{Node } a t_1 t_2)) \\
\gamma_3 (\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u))) v : q_L q_N \text{ IF } Z t_2 \Rightarrow Z(\text{Node } a t_1 t_2) \\
\gamma_3 (\gamma_3 (\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u)))) v : (q_L q_N \text{ IF } Z t_2) \Rightarrow (q_L q_N \text{ IF } Z(\text{Node } a t_1 t_2)) \\
\gamma_3 (\gamma_3 (\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u)))) v w : q_L q_N \text{ IF } Z(\text{Node } a t_1 t_2) \\
\lambda uvw. \gamma_3 (\gamma_3 (\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u)))) v w : \forall t_1^t \forall t_2^t \forall a \in \text{Atom}. (q_L q_N \text{ IF } Z t_1) \Rightarrow (q_L q_N \text{ IF } Z t_2) \Rightarrow q_L q_N \text{ IF } Z(\text{Node } a t_1 t_2) \\
[\forall] r^{\kappa} \\
[\lambda] c : C[p q_L q_N r] \\
\alpha_1^3 c : C[p] \\
z(\alpha_1^3 c) : t \in \text{Tree} \\
z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_{\text{Term}}(\lambda_. u))) (\lambda uvw. \gamma_3 (\gamma_3 (\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u)))) v w) : q_L q_N \text{ IF } Z t \\
z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_{\text{Term}}(\lambda_. u))) (\lambda uvw. \gamma_3 (\gamma_3 (\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u)))) v w) : C[q_L q_N r] \Rightarrow Z^* t r \\
\alpha_{16} c : C[q_L q_N r] \\
z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_{\text{Term}}(\lambda_. u))) (\lambda uvw. \gamma_3 (\gamma_3 (\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u)))) v w) (\alpha_{16} c) : Z^* t r \\
\lambda c. z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_{\text{Term}}(\lambda_. u))) (\lambda uvw. \gamma_3 (\gamma_3 (\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u)))) v w) (\alpha_{16} c) \\
: \forall r^{\kappa}. C[p q_L q_N r] \Rightarrow Z^* t r \equiv p q_L q_N \text{ IF } Z t \\
\lambda yc. z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_{\text{Term}}(\lambda_. u))) (\lambda uvw. \gamma_3 (\gamma_3 (\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u)))) v w) (\alpha_{16} c) \\
: \forall q_N^{\kappa}. (q_N \text{ IF } \text{closZ}) \Rightarrow (p q_L q_N \text{ IF } Z t) \\
\gamma_1 (\lambda yc. z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_{\text{Term}}(\lambda_. u))) (\lambda uvw. \gamma_3 (\gamma_3 (\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u)))) v w) (\alpha_{16} c)) : p q_L \text{ IF } \text{closZ} \Rightarrow Z t \\
\lambda x. \gamma_1 (\lambda yc. z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_{\text{Term}}(\lambda_. u))) (\lambda uvw. \gamma_3 (\gamma_3 (\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u)))) v w) (\alpha_{16} c)) \\
: \forall q_L^{\kappa}. (q_L \text{ IF } \forall \vec{v} \in \text{Term}. Z(\text{Leaf } \vec{v})) \Rightarrow (p q_L \text{ IF } \text{closZ} \Rightarrow Z t) \\
\gamma_1 (\lambda x. \gamma_1 (\lambda yc. z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_{\text{Term}}(\lambda_. u))) (\lambda uvw. \gamma_3 (\gamma_3 (\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u)))) v w) (\alpha_{16} c))) \\
: p \text{ IF } (\forall \vec{v} \in \text{Term}. Z(\text{Leaf } \vec{v})) \Rightarrow \text{closZ} \Rightarrow Z t \\
\gamma_1 (\lambda x. \gamma_1 (\lambda yc. z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_{\text{Term}}(\lambda_. u))) (\lambda uvw. \gamma_3 (\gamma_3 (\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u)))) v w) (\alpha_{16} c))) \\
: \forall Z^*. p \text{ IF } (\forall \vec{v} \in \text{Term}. Z(\text{Leaf } \vec{v})) \Rightarrow \text{closZ} \Rightarrow Z t \simeq p \text{ IF } t \in \text{Tree} \\
\lambda z. \gamma_1 (\lambda x. \gamma_1 (\lambda yc. z(\alpha_1^3 c) (\lambda u. \gamma_3 (\beta_3 x) (\xi'_{\text{Term}}(\lambda_. u))) (\lambda uvw. \gamma_3 (\gamma_3 (\gamma_3 (\beta_4 y) (\xi'_{\text{Atom}}(\lambda_. u)))) v w) (\alpha_{16} c))) \\
: (C[p] \Rightarrow t \in \text{Tree}) \Rightarrow (p \text{ IF } t \in \text{Tree})
\end{array}$$



## 5.2 subH is absolute

Remember that

$$\text{subH } p := \exists t \in \text{Tree}. \text{subHtree } p t = 1 \equiv (\forall t. t \in \text{Tree} \Rightarrow \text{subHtree } p t = 1 \Rightarrow \perp) \Rightarrow \perp$$

$$\begin{array}{l}
[\lambda] z : q \text{ IF subH } p \\
[\lambda] c : C[q] \\
[\lambda] x : \forall t. t \in \text{Tree} \Rightarrow \text{subHtree } p t = 1 \Rightarrow \perp \\
\gamma_3 z : (q \text{ IF } \forall t. t \in \text{Tree} \Rightarrow \text{subHtree } p t = 1 \Rightarrow \perp) \Rightarrow (q \text{ IF } \perp) \\
[\forall] t' \\
[\forall] q_T^k \\
[\lambda] u : q_T \text{ IF } t \in \text{Tree} \\
[\forall] q_p^k \\
[\lambda] v : q_p \text{ IF subHtree } p t = 1 \\
[\lambda] c' : C[q_T q_p] \\
\xi_{\text{Tree}} u : C[q_T] \Rightarrow t \in \text{Tree} \\
\alpha_2 (\alpha_1 c') : C[q_T] \\
\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c')) : t \in \text{Tree} \\
x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) : \text{subHtree } p t = 1 \Rightarrow \perp \\
\alpha_2 c' : C[q_p] \\
\xi_{\text{eq}} v : C[q_p] \Rightarrow \text{subHtree } p t = 1 \\
\xi_{\text{eq}} v (\alpha_2 c') : \text{subHtree } p t = 1 \\
[x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 (\alpha_1 c')))] : \perp \\
\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')) : C[q_T q_p] \Rightarrow \perp \\
\xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c'))) : q_T q_p \text{ IF } \perp \\
\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c'))) : \forall q_p^k. (q_p \text{ IF subHtree } p t = 1) \Rightarrow (q_T q_p \text{ IF } \perp) \\
\gamma_1 (\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')))) : q_T \text{ IF subHtree } p t = 1 \Rightarrow \perp \\
\lambda u. \gamma_1 (\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')))) : \forall q_T^k. (q_T \text{ IF } t \in \text{Tree}) \Rightarrow (q_T \text{ IF subHtree } p t = 1 \Rightarrow \perp) \\
\gamma_1 (\lambda u. \gamma_1 (\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')))) : q \text{ IF } t \in \text{Tree} \Rightarrow \text{subHtree } p t = 1 \Rightarrow \perp \\
\gamma_3 z (\gamma_1 (\lambda u. \gamma_1 (\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')))))) : q \text{ IF } \perp \\
\xi_{\perp} (\gamma_3 z (\gamma_1 (\lambda u. \gamma_1 (\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')))))) : C[q] \Rightarrow \perp \\
[\xi_{\perp} (\gamma_3 z (\gamma_1 (\lambda u. \gamma_1 (\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c'))))))] c : \perp \\
[\lambda x. \xi_{\perp} (\gamma_3 z (\gamma_1 (\lambda u. \gamma_1 (\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c'))))))] c : \text{subH } p \\
\lambda z c. z @ (\lambda^* u. (\lambda^* v. (\lambda c'. \langle \xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c')), \xi_{\text{eq}} v (\alpha_2 (\alpha_1 c')) \rangle)) (\alpha_7 c)) : (q \text{ IF subH } p) \Rightarrow (C[q] \Rightarrow \text{subH } p)
\end{array}$$

$$\begin{array}{l}
[\lambda] z : q \text{ IF subH } p \\
[\lambda] c : C[q] \\
[\lambda] x : \forall t. t \in \text{Tree} \Rightarrow \text{subHtree } p t = 1 \Rightarrow \perp \\
\gamma_3 z : (q \text{ IF } \forall t. t \in \text{Tree} \Rightarrow \text{subHtree } p t = 1 \Rightarrow \perp) \Rightarrow (q \text{ IF } \perp) \\
[\forall] t' \\
[\forall] q_T^k \\
[\lambda] u : q_T \text{ IF } t \in \text{Tree} \\
[\forall] q_p^k \\
[\lambda] v : q_p \text{ IF subHtree } p t = 1 \\
[\lambda] c' : C[qq_T q_p] \\
\xi_{\text{Tree}} u : C[q_T] \Rightarrow t \in \text{Tree} \\
\alpha_2 (\alpha_1 c') : C[q_T] \\
\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c')) : t \in \text{Tree} \\
x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) : \text{subHtree } p t = 1 \Rightarrow \perp \\
\alpha_2 c' : C[q_p] \\
\xi_{\text{eq}} v : C[q_p] \Rightarrow \text{subHtree } p t = 1 \\
\xi_{\text{eq}} v (\alpha_2 c') : \text{subHtree } p t = 1 \\
x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')) : \perp \\
\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')) : C[qq_T q_p] \Rightarrow \perp \\
\xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c'))) : qq_T q_p \text{ IF } \perp \\
\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c'))) : \forall q_p^k. (q_p \text{ IF subHtree } p t = 1) \Rightarrow (qq_T q_p \text{ IF } \perp) \\
\gamma_1 (\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')))) : qq_T \text{ IF subHtree } p t = 1 \Rightarrow \perp \\
\lambda u. \gamma_1 (\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')))) : \forall q_T^k. (q_T \text{ IF } t \in \text{Tree}) \Rightarrow (qq_T \text{ IF subHtree } p t = 1 \Rightarrow \perp) \\
\gamma_1 (\lambda u. \gamma_1 (\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')))) : q \text{ IF } t \in \text{Tree} \Rightarrow \text{subHtree } p t = 1 \Rightarrow \perp \\
\gamma_3 z (\gamma_1 (\lambda u. \gamma_1 (\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')))) : q \text{ IF } \perp \\
\xi_{\perp} (\gamma_3 z (\gamma_1 (\lambda u. \gamma_1 (\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')))) : C[q] \Rightarrow \perp \\
\xi_{\perp} (\gamma_3 z (\gamma_1 (\lambda u. \gamma_1 (\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')))) : \perp \\
\lambda x. \xi_{\perp} (\gamma_3 z (\gamma_1 (\lambda u. \gamma_1 (\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')))) : \text{subH } p \\
\lambda z c x. \xi_{\perp} (\gamma_3 z (\gamma_1 (\lambda u. \gamma_1 (\lambda v. \xi'_{\perp} (\lambda c'. x (\xi_{\text{Tree}} u (\alpha_2 (\alpha_1 c'))) (\xi_{\text{eq}} v (\alpha_2 c')))) : (q \text{ IF subH } p) \Rightarrow (C[q] \Rightarrow \text{subH } p)
\end{array}$$

$$\begin{array}{l}
[\lambda] z : C[q] \Rightarrow \text{subH } p \\
[\forall] r^k \\
[\lambda] x : r \text{ IF } \forall t. t \in \text{Tree} \Rightarrow \text{subHtree } p t = 1 \Rightarrow \perp \simeq \forall t. r \text{ IF } t \in \text{Tree} \Rightarrow \text{subHtree } p t = 1 \Rightarrow \perp \\
\beta_4 x : \forall t. qr \text{ IF } t \in \text{Tree} \Rightarrow \text{subHtree } p t = 1 \Rightarrow \perp \\
\gamma_3 (\beta_4 x) : \forall t. (qr \text{ IF } t \in \text{Tree}) \Rightarrow (qr \text{ IF } \text{subHtree } p t = 1 \Rightarrow \perp) \\
[\lambda] c : C[qr] \\
\alpha_1 c : C[q] \\
z (\alpha_1 c) : \text{subH } p \equiv (\forall t. t \in \text{Tree} \Rightarrow \text{subHtree } p t = 1 \Rightarrow \perp) \Rightarrow \perp \\
[\forall] t' \\
[\lambda] u : t \in \text{Tree} \\
[\lambda] v : \text{subHtree } p t = 1 \\
\lambda_{\cdot} u : C[qr] \Rightarrow t \in \text{Tree} \\
\xi'_{\text{Tree}} (\lambda_{\cdot} u) : qr \text{ IF } t \in \text{Tree} \\
\gamma_3 (\beta_4 x) (\xi'_{\text{Tree}} (\lambda_{\cdot} u)) : qr \text{ IF } \text{subHtree } p t = 1 \Rightarrow \perp \\
\gamma_3 (\gamma_3 (\beta_4 x) (\xi'_{\text{Tree}} (\lambda_{\cdot} u))) : (qr \text{ IF } \text{subHtree } p t = 1) \Rightarrow (qr \text{ IF } \perp) \\
\lambda_{\cdot} v : C[qr] \Rightarrow \text{subHtree } p t = 1 \\
\xi_{\text{eq}} (\lambda_{\cdot} v) : qr \text{ IF } \text{subHtree } p t = 1 \\
\gamma_3 (\gamma_3 (\beta_4 x) (\xi'_{\text{Tree}} (\lambda_{\cdot} u))) (\xi_{\text{eq}} (\lambda_{\cdot} v)) : qr \text{ IF } \perp \\
\xi_{\perp} (\gamma_3 (\gamma_3 (\beta_4 x) (\xi'_{\text{Tree}} (\lambda_{\cdot} u))) (\xi_{\text{eq}} (\lambda_{\cdot} v))) : C[qr] \Rightarrow \perp \\
\xi_{\perp} (\gamma_3 (\gamma_3 (\beta_4 x) (\xi'_{\text{Tree}} (\lambda_{\cdot} u))) (\xi_{\text{eq}} (\lambda_{\cdot} v))) c : \perp \\
\lambda uv. \xi_{\perp} (\gamma_3 (\gamma_3 (\beta_4 x) (\xi'_{\text{Tree}} (\lambda_{\cdot} u))) (\xi_{\text{eq}} (\lambda_{\cdot} v))) c : \forall t. (t \in \text{Tree}) \Rightarrow (\text{subHtree } p t = 1) \Rightarrow \perp \\
z (\alpha_1 c) (\lambda uv. \xi_{\perp} (\gamma_3 (\gamma_3 (\beta_4 x) (\xi'_{\text{Tree}} (\lambda_{\cdot} u))) (\xi_{\text{eq}} (\lambda_{\cdot} v))) c) : \perp \\
\lambda c. z (\alpha_1 c) (\lambda uv. \xi_{\perp} (\gamma_3 (\gamma_3 (\beta_4 x) (\xi'_{\text{Tree}} (\lambda_{\cdot} u))) (\xi_{\text{eq}} (\lambda_{\cdot} v))) c) : C[qr] \Rightarrow \perp \\
\xi_{\perp} (\lambda c. z (\alpha_1 c) (\lambda uv. \xi_{\perp} (\gamma_3 (\gamma_3 (\beta_4 x) (\xi'_{\text{Tree}} (\lambda_{\cdot} u))) (\xi_{\text{eq}} (\lambda_{\cdot} v))) c)) : qr \text{ IF } \perp \\
\lambda x. \xi_{\perp} (\lambda c. z (\alpha_1 c) (\lambda uv. \xi_{\perp} (\gamma_3 (\gamma_3 (\beta_4 x) (\xi'_{\text{Tree}} (\lambda_{\cdot} u))) (\xi_{\text{eq}} (\lambda_{\cdot} v))) c)) \\
: \forall r. (r \text{ IF } \forall t. t \in \text{Tree} \Rightarrow \text{subHtree } p t = 1 \Rightarrow \perp) \Rightarrow (qr \text{ IF } \perp) \\
\gamma_1 (\lambda x. \xi_{\perp} (\lambda c. z (\alpha_1 c) (\lambda uv. \xi_{\perp} (\gamma_3 (\gamma_3 (\beta_4 x) (\xi'_{\text{Tree}} (\lambda_{\cdot} u))) (\xi_{\text{eq}} (\lambda_{\cdot} v))) c))) \\
: q \text{ IF } (\forall t. t \in \text{Tree} \Rightarrow \text{subHtree } p t = 1 \Rightarrow \perp) \Rightarrow \perp \simeq q \text{ IF } \text{subH } p \\
\lambda z. \gamma_1 (\lambda x. \xi_{\perp} (\lambda c. z (\alpha_1 c) (\lambda uv. \xi_{\perp} (\gamma_3 (\gamma_3 (\beta_4 x) (\xi'_{\text{Tree}} (\lambda_{\cdot} u))) (\xi_{\text{eq}} (\lambda_{\cdot} v))) c))) : (C[q] \Rightarrow \text{subH } p) \Rightarrow (q \text{ IF } \text{subH } p)
\end{array}$$

### 5.3 FVal is absolute

The definition of FVal is

$$\begin{aligned}
p \in \text{FVal} &:= \forall Z^{l \rightarrow o}. Z \emptyset \Rightarrow \\
&(\forall p^u. \forall a \in \text{Atom}. \text{mem } a p^u =_l 0 \mapsto Z p^u \Rightarrow Z (p^u a^1)) \Rightarrow \\
&(\forall p^u. \forall a \in \text{Atom}. \text{mem } a p^u =_l 1 \mapsto Z p^u \Rightarrow Z (p^u a^0)) \Rightarrow \\
&Z p
\end{aligned}$$

To ease writing, we set

$$\begin{aligned}
\text{closZ}^+ &:= \forall r^u \forall a \in \text{Atom}. \text{mem } a r =_l 0 \mapsto Z r \Rightarrow Z (r a^1) \\
\text{closZ}^- &:= \forall r^u \forall a \in \text{Atom}. \text{mem } a r =_l 0 \mapsto Z r \Rightarrow Z (r a^0).
\end{aligned}$$

$[\lambda] z : q \text{ IF } p \in \text{FVal}$   
 $[\lambda] c : C[q]$   
 $[\forall] Z^{\rightarrow o}$   
 $[\lambda] x : Z \emptyset$   
 $[\lambda] y^+ : \text{clos}Z^+$   
 $[\lambda] y^- : \text{clos}Z^-$   
 $z : (q \text{ IF } Z \emptyset \Rightarrow \text{clos}Z^+ \Rightarrow \text{clos}Z^- \Rightarrow Z p)[Z/Z^* r]$   
 $\gamma_3 z : (q \text{ IF } Z \emptyset)[Z/Z^* r] \Rightarrow (q \text{ IF } \text{clos}Z^+ \Rightarrow \text{clos}Z^- \Rightarrow Z p)[Z/Z^* r]$   
 $\lambda_{\cdot} x : \forall r^k. C[qr] \Rightarrow Z \emptyset \simeq (q \text{ IF } Z \emptyset)[Z/Z^* r]$   
 $\gamma_3 z(\lambda_{\cdot} x) : (q \text{ IF } \text{clos}Z^+ \Rightarrow \text{clos}Z^- \Rightarrow Z p)[Z/Z^* r]$   
 $\gamma_3(\gamma_3 z(\lambda_{\cdot} x)) : (q \text{ IF } \text{clos}Z^+)[Z/Z^* r] \Rightarrow (q \text{ IF } \text{clos}Z^- \Rightarrow Z p)[Z/Z^* r]$   
 $[\forall] (p')^t$   
 $[\forall] a^t$   
 $[\forall] q_0^k$   
 $[\lambda] u : q_0 \text{ IF } a \in \text{Atom}$   
 $\xi_{\text{Atom}} u : C[q_0] \Rightarrow a \in \text{Atom}$   
 $[\forall] q_1^k$   
 $[\lambda] v : (q_1 \text{ IF } Z r)[Z/Z^* r] \simeq \forall s^k. C[q_1 s] \Rightarrow Z r$   
 $[\forall] s^k$   
 $[\lambda] c' : C[q q_0 q_1 s]$   
 $\alpha_2(\alpha_1^2 c') : C[q_0]$   
 $\xi_{\text{Atom}} u(\alpha_2(\alpha_1^2 c')) : a \in \text{Atom}$   
 $y^+(\xi_{\text{Atom}} u(\alpha_2(\alpha_1^2 c')))) : \text{mem } a r =_i 0 \mapsto Z r \Rightarrow Z(r a^1)$   
 $y^+(\xi_{\text{Atom}} u(\alpha_2(\alpha_1^2 c')))) : Z r \Rightarrow Z(r a^1)$   
 $\alpha_{10} c' : C[q_1 s]$   
 $v(\alpha_{10} c') : Z r$   
 $y^+(\xi_{\text{Atom}} u(\alpha_2(\alpha_1^2 c')))(v(\alpha_{10} c')) : Z(r a^1)$   
 $[\lambda c'. y^+(\xi_{\text{Atom}} u(\alpha_2(\alpha_1^2 c')))(v(\alpha_{10} c')) : \forall s^k. C[q q_0 q_1 s] \Rightarrow Z(r a^1) \equiv (q q_0 q_1 \text{ IF } Z(r a^1))[Z/Z^* r]$   
 $\lambda v c'. y^+(\xi_{\text{Atom}} u(\alpha_2(\alpha_1^2 c')))(v(\alpha_{10} c')) : \forall q_1^k. (q_1 \text{ IF } Z r)[Z/Z^* r] \equiv (q q_0 q_1 \text{ IF } Z(r a^1))[Z/Z^* r]$   
 $\gamma_1(\lambda v c'. y^+(\xi_{\text{Atom}} u(\alpha_2(\alpha_1^2 c')))(v(\alpha_{10} c')) : (q q_0 \text{ IF } Z r \Rightarrow Z(r a^1))[Z/Z^* r]$   
 $\gamma_1(\lambda v c'. y^+(\xi_{\text{Atom}} u(\alpha_2(\alpha_1^2 c')))(v(\alpha_{10} c'))$   
 $: \text{mem } a r =_i 0 \mapsto (q q_0 \text{ IF } Z r \Rightarrow Z(r a^1))[Z/Z^* r] \simeq (q q_0 \text{ IF } \text{mem } a r =_i 0 \mapsto Z r \Rightarrow Z(r a^1))[Z/Z^* r]$   
 $\lambda u. \gamma_1(\lambda v c'. y^+(\xi_{\text{Atom}} u(\alpha_2(\alpha_1^2 c')))(v(\alpha_{10} c')) : \forall q_0^k. q_0 \text{ IF } a \in \text{Atom} \Rightarrow (q q_0 \text{ IF } \text{mem } a r =_i 0 \mapsto Z r \Rightarrow Z(r a^1))[Z/Z^* r]$   
 $\gamma_1(\lambda u. \gamma_1(\lambda v c'. y^+(\xi_{\text{Atom}} u(\alpha_2(\alpha_1^2 c')))(v(\alpha_{10} c')))) : (q \text{ IF } a \in \text{Atom} \Rightarrow \text{mem } a r =_i 0 \mapsto Z r \Rightarrow Z(r a^1))[Z/Z^* r]$   
 $\gamma_1(\lambda u. \gamma_1(\lambda v c'. y^+(\xi_{\text{Atom}} u(\alpha_2(\alpha_1^2 c')))(v(\alpha_{10} c')))) : (q \text{ IF } \text{clos}Z^+)[Z/Z^* r]$   
 $\text{Write } T^{+/-} := \gamma_1(\lambda u. \gamma_1(\lambda v c'. y^{+/-}(\xi_{\text{Atom}} u(\alpha_2(\alpha_1^2 c')))(v(\alpha_{10} c'))))$   
 $T^+ : (q \text{ IF } \text{clos}Z^+)[Z/Z^* r]$   
 $\text{idem}$   
 $T^- : (q \text{ IF } \text{clos}Z^-)[Z/Z^* r]$   
 $\gamma_3(\gamma_3 z(\lambda_{\cdot} x)) T^+ : (q \text{ IF } \text{clos}Z^- \Rightarrow Z t)[Z/Z^* r]$   
 $\gamma_3(\gamma_3(\gamma_3 z(\lambda_{\cdot} x)) T^+) : (q \text{ IF } \text{clos}Z^-)[Z/Z^* r] \Rightarrow (q \text{ IF } Z p)[Z/Z^* r]$   
 $\gamma_3(\gamma_3(\gamma_3 z(\lambda_{\cdot} x)) T^+) T^- : (q \text{ IF } Z p)[Z/Z^* r] \equiv \forall s^k. C[q s] \Rightarrow Z p$   
 $\alpha_7 c : C[q_1]$   
 $\gamma_3(\gamma_3(\gamma_3 z(\lambda_{\cdot} x)) T^+) T^-(\alpha_7 c) : Z p$   
 $\lambda x y^+ y^-. \gamma_3(\gamma_3(\gamma_3 z(\lambda_{\cdot} x)) T^+) T^-(\alpha_7 c) : \forall Z^{\rightarrow o}. Z \emptyset \Rightarrow \text{clos}Z^+ \Rightarrow \text{clos}Z^- \Rightarrow Z p \equiv p \in \text{FVal}$   
 $\lambda z c x y^+ y^-. \gamma_3(\gamma_3(\gamma_3 z(\lambda_{\cdot} x)) T^+) T^-(\alpha_7 c) : (q \text{ IF } p \in \text{FVal}) \Rightarrow (C[q] \Rightarrow p \in \text{FVal})$

remembering that  $T^{+/-} := \gamma_1(\lambda u. \gamma_1(\lambda v c'. y^{+/-}(\xi_{\text{Atom}} u(\alpha_2(\alpha_1^2 c')))(v(\alpha_{10} c'))))$

$$\begin{array}{l}
[\lambda] z : C[q] \Rightarrow p \in \text{FVal} \\
[[\forall] (Z^*)^{\iota \rightarrow \kappa \rightarrow o} \\
[[\forall] q_0^\kappa \\
[\lambda] x : q_0 \text{ IF } Z \emptyset \\
[[\forall] (q^+)^\kappa \\
[\lambda] y^+ : q^+ \text{ IF } \text{clos}Z^+ \\
[[\forall] (q^-)^\kappa \\
[\lambda] y^- : q^- \text{ IF } \text{clos}Z^- \\
\beta_3 x : q_0 q^+ q^- \text{ IF } Z \emptyset \\
\beta_4 y^+ : \forall r^t \forall a'. q_0 q^+ q^- \text{ IF } a \in \text{Atom} \Rightarrow \text{mem } ar =_i 0 \mapsto Zr \Rightarrow Z(ra^1) \\
\gamma_3 (\beta_4 y^+) : \forall r^t \forall a'. (q_0 q^+ q^- \text{ IF } a \in \text{Atom}) \Rightarrow (q_0 q^+ q^- \text{ IF } \text{mem } ar =_i 0 \mapsto Zr \Rightarrow Z(ra^1)) \\
[[\forall] r^t \\
[[\forall] a' \\
[\lambda] u : a \in \text{Atom} \\
[\text{Eq.}] \text{mem } ar =_i 0 \\
[\lambda] v : q_0 q^+ q^- \text{ IF } Zr \\
\lambda_. u : C[q_0 q^+ q^-] \Rightarrow a \in \text{Atom} \\
\xi'_{\text{Atom}}(\lambda_. u) : q_0 q^+ q^- \text{ IF } a \in \text{Atom} \\
\gamma_3 (\beta_4 y^+) (\xi'_{\text{Atom}}(\lambda_. u)) : q_0 q^+ q^- \text{ IF } \text{mem } ar =_i 0 \mapsto Zr \Rightarrow Z(ra^1) \simeq \text{mem } ar =_i 0 \mapsto q_0 q^+ q^- \text{ IF } Zr \Rightarrow Z(ra^1) \\
\gamma_3 (\beta_4 y^+) (\xi'_{\text{Atom}}(\lambda_. u)) : q_0 q^+ q^- \text{ IF } Zr \Rightarrow Z(ra^1) \\
\gamma_3 (\gamma_3 (\beta_4 y^+) (\xi'_{\text{Atom}}(\lambda_. u))) : (q_0 q^+ q^- \text{ IF } Zr) \Rightarrow (q_0 q^+ q^- \text{ IF } Z(ra^1)) \\
\gamma_3 (\gamma_3 (\beta_4 y^+) (\xi'_{\text{Atom}}(\lambda_. u))) v : q_0 q^+ q^- \text{ IF } Z(ra^1) \\
\lambda uv. \gamma_3 (\gamma_3 (\beta_4 y^+) (\xi'_{\text{Atom}}(\lambda_. u))) v : \forall r^t \forall a \in \text{Atom}. \text{mem } ar =_i 0 \mapsto (q_0 q^+ q^- \text{ IF } Zr) \Rightarrow (q_0 q^+ q^- \text{ IF } Z(ra^1)) \\
\text{Write } U^{+/-} := \lambda uv. \gamma_3 (\gamma_3 (\beta_4 y^{+/-}) (\xi'_{\text{Atom}}(\lambda_. u))) v \\
U^+ : \forall r^t \forall a \in \text{Atom}. \text{mem } ar =_i 0 \mapsto (q_0 q^+ q^- \text{ IF } Zr) \Rightarrow (q_0 q^+ q^- \text{ IF } Z(ra^1)) \\
\text{idem} \\
U^- : \forall r^t \forall a \in \text{Atom}. \text{mem } ar =_i 0 \mapsto (q_0 q^+ q^- \text{ IF } Zr) \Rightarrow (q_0 q^+ q^- \text{ IF } Z(ra^0)) \\
[[\forall] r^\kappa \\
[\lambda] c : C[q_0 q^+ q^- r] \\
\alpha_1^4 c : C[q] \\
z(\alpha_1^4 c) : p \in \text{FVal} \\
z(\alpha_1^4 c) (\beta_3 x) U^+ U^- : q_0 q^+ q^- \text{ IF } Zp \equiv \forall r^\kappa. C[q_0 q^+ q^- r] \Rightarrow Z^* p r \\
\alpha_6 (\alpha_{16} (\alpha_5 c)) : C[q_0 q^+ q^- r] \\
z(\alpha_1^4 c) (\beta_3 x) U^+ U^- (\alpha_6 (\alpha_{16} (\alpha_5 c))) : Z^* p r \\
[\lambda c. z(\alpha_1^4 c) (\beta_3 x) U^+ U^- (\alpha_6 (\alpha_{16} (\alpha_5 c)))] : \forall r^\kappa. C[q_0 q^+ q^- r] \Rightarrow Z^* p r \equiv p q_0 q^+ q^- \text{ IF } Zp \\
\lambda y^- c. z(\alpha_1^4 c) (\beta_3 x) U^+ U^- (\alpha_6 (\alpha_{16} (\alpha_5 c))) : \forall q^-. q^- \text{ IF } \text{clos}Z^- \Rightarrow q_0 q^+ q^- \text{ IF } Zp \\
[\gamma_1 (\lambda y^- c. z(\alpha_1^4 c) (\beta_3 x) U^+ U^- (\alpha_6 (\alpha_{16} (\alpha_5 c))))] : q_0 q^+ q^- \text{ IF } \text{clos}Z^- \Rightarrow Zp \\
\lambda y^+. (\gamma_1 (\lambda y^- c. z(\alpha_1^4 c) (\beta_3 x) U^+ U^- (\alpha_6 (\alpha_{16} (\alpha_5 c)))) : \forall q^+. (q^+ \text{ IF } \text{clos}Z^+) \Rightarrow (q_0 q^+ q^- \text{ IF } \text{clos}Z^- \Rightarrow Zp) \\
[\gamma_1 (\lambda y^+. (\gamma_1 (\lambda y^- c. z(\alpha_1^4 c) (\beta_3 x) U^+ U^- (\alpha_6 (\alpha_{16} (\alpha_5 c)))))] : q_0 \text{ IF } \text{clos}Z^+ \Rightarrow \text{clos}Z^- \Rightarrow Zp \\
\lambda x. \gamma_1 (\lambda y^+. (\gamma_1 (\lambda y^- c. z(\alpha_1^4 c) (\beta_3 x) U^+ U^- (\alpha_6 (\alpha_{16} (\alpha_5 c)))))] : \forall q_0. (q_0 \text{ IF } Z \emptyset) \Rightarrow (q_0 \text{ IF } \text{clos}Z^+ \Rightarrow \text{clos}Z^- \Rightarrow Zp) \\
[\gamma_1 (\lambda x. \gamma_1 (\lambda y^+. (\gamma_1 (\lambda y^- c. z(\alpha_1^4 c) (\beta_3 x) U^+ U^- (\alpha_6 (\alpha_{16} (\alpha_5 c)))))] : q \text{ IF } Z \emptyset \Rightarrow \text{clos}Z^+ \Rightarrow \text{clos}Z^- \Rightarrow Zp \\
\lambda z. \gamma_1 (\lambda x. \gamma_1 (\lambda y^+. (\gamma_1 (\lambda y^- c. z(\alpha_1^4 c) (\beta_3 x) U^+ U^- (\alpha_6 (\alpha_{16} (\alpha_5 c)))))] : \forall Z^*. q \text{ IF } Z \emptyset \Rightarrow \text{clos}Z^+ \Rightarrow \text{clos}Z^- \Rightarrow Zp \simeq q \text{ IF } p \in \text{FVal} \\
\lambda z. \gamma_1 (\lambda x. \gamma_1 (\lambda y^+. (\gamma_1 (\lambda y^- c. z(\alpha_1^4 c) (\beta_3 x) U^+ U^- (\alpha_6 (\alpha_{16} (\alpha_5 c)))))] : (C[q] \Rightarrow p \in \text{FVal}) \Rightarrow (q \text{ IF } p \in \text{FVal})
\end{array}$$

remembering that  $U^{+/-} := \lambda uv. \gamma_3 (\gamma_3 (\beta_4 y^{+/-}) (\xi'_{\text{Atom}}(\lambda_. u))) v$ .

## 5.4 C is absolute

The set  $C$  is defined by  $C[p] := p \in \text{FVal} \wedge (\text{subH } p \Rightarrow \text{subH } \emptyset)$ .

We have

$$\xi_{\text{FVal}} : (q \text{ IF } p \in \text{FVal}) \Rightarrow (C[q] \Rightarrow p \in \text{FVal})$$

$$\xi'_{\text{FVal}} : (C[q] \Rightarrow p \in \text{FVal}) \Rightarrow (q \text{ IF } p \in \text{FVal})$$

$$\xi_{\text{subH}} : (q \text{ IF } \text{subH } p) \Rightarrow (C[q] \Rightarrow \text{subH } p)$$

$$\xi'_{\text{subH}} : (C[q] \Rightarrow \text{subH } p) \Rightarrow (q \text{ IF } \text{subH } p)$$

but also

$$\begin{aligned}\xi_{\text{subH}} &: (q \Vdash \text{subH } \emptyset) \Rightarrow (C[q] \Rightarrow \text{subH } \emptyset) \\ \xi'_{\text{subH}} &: (C[q] \Rightarrow \text{subH } \emptyset) \Rightarrow (q \Vdash \text{subH } \emptyset)\end{aligned}$$

We then get

$$\begin{aligned}\xi \wedge \xi_{\text{FVal}} (\xi \Rightarrow \xi'_{\text{subH}} \xi_{\text{subH}}) &: (q \Vdash C[p]) \Rightarrow (C[q] \Rightarrow C[p]) \\ \xi' \wedge \xi'_{\text{FVal}} (\xi' \Rightarrow \xi_{\text{subH}} \xi'_{\text{subH}}) &: (C[q] \Rightarrow C[p]) \Rightarrow (q \Vdash C[p])\end{aligned}$$

## 6 Proof of forcing the extra axiom (A)

We want to prove  $1 \Vdash \forall a \in \text{Atom}. (\forall pb. p \in G \Rightarrow b \in \text{Bool} \Rightarrow \text{test } p a b = 1 \Rightarrow \perp) \Rightarrow \perp$ .

### 6.1 A textual proof

We first give a textual proof before the formal one in order to convey the essential ideas.

*Proof.* Given  $a, q_a, q$  and proof terms

$$\begin{aligned}x &: q_a \Vdash a \in \text{Atom} \\ y &: q \Vdash \forall p \forall b. p \in G \Rightarrow b \in \text{Bool} \Rightarrow \text{test } p a b = 1 \Rightarrow \perp,\end{aligned}$$

we want to prove  $1 q_a q \Vdash \perp$ . Since 1 is neutral for the product, this is the same as proving that  $q_a q \Vdash \perp$ . With repeated use of  $\gamma_3$ , we can turn  $y$  into  $y' := \lambda uv. \gamma_3 (\gamma_3 (\gamma_3 (\beta_4 y) u) v)$  which is a proof term for

$$\forall q' pb. (q' q \Vdash p \in G) \Rightarrow (q' q \Vdash b \in \text{Bool}) \Rightarrow (q' q \Vdash \text{test } p a b = 1) \Rightarrow (q' q \Vdash \perp).$$

Since  $\text{test}$  is total (assumed in the interface),  $\text{mem}$  is total too and we have  $\text{mem } a (q_a q) \in \text{Bool}$ . We then proceed by case analysis, noting that  $\text{mem } a (q_a q) = 1$  can itself be decomposed into  $\text{test}(q_a q) a 1 = 1$  or  $\text{test}(q_a q) a 0 = 1$ .

- $\text{test}(q_a q) a 1 = 1$ : We take  $p := q_a q$  and  $b := 1$ . We want to use  $y'$  to prove  $q_a q \Vdash \perp$  so that we have to prove its premises:
  - $\lambda x. x : q_a q \Vdash q_a q \in G \equiv q_a q \leq q_a q$ ,
  - $\gamma_1 (\lambda u. \gamma_1 (\lambda v. \beta_4 (\beta_3 u))) : q_a q \Vdash 1 \in \text{Bool}$ ,
  - $\xi'_{\text{eq}} (\lambda _x. x) : q_a q \Vdash \text{test}(q_a q) a 1 = 1$  because the equality holds in the equational theory.
- $\text{test}(q_a q) a 0 = 1$ : It is similar to the previous case.
- $\text{mem } a (q_a q) = 0$ : We use  $\xi'_\perp$  and we are left with  $C[q_a q] \Rightarrow \perp$  to prove. We first prove that we have  $C[(q_a a^1) q] \Rightarrow \perp$  by using first  $\xi'_\perp$  then  $y'$  with  $q := q_a a^1$ ,  $p := a^1$  and  $b := 1$ .
  - $\alpha_{10} \circ \alpha_9 : (q_a a^1) q \Vdash a^1 \in G \equiv (q_a a^1) q \leq a^1$  because product is the glb for  $\leq$ ,
  - $(q_a a^1) q \Vdash 1 \in \text{Bool}$  proved as before,
  - $(q_a a^1) q \Vdash \text{test } a^1 a 1 = 1$  proved as before.

In a similar fashion, we prove that  $C[(q_a a^0) q] \Rightarrow \perp$ .

Back to the proof of  $C[q_a q] \Rightarrow \perp$ . Assume  $C[q_a q]$ . Applying the proof term of  $C[q_a a^1 q] \Rightarrow \perp$ , we have to prove  $C[q_a a^1 q] \equiv q_a a^1 q \in \text{FVal} \wedge (\text{subH } q_a a^1 q \Rightarrow \text{subH } \emptyset)$ . The first part present no difficulty because we have  $C[q_a q]$  and  $\text{mem } a (q_a q) = 0$ : we jsut need to apply the second constructor of  $\text{FVal}$ . For the second one, we assume  $\text{subH } q_a a^1 q$  and we want to prove  $\text{subH } \emptyset$ . Instead we choose to prove  $\perp$ . Applying again the same method with  $C[q_a a^0 q] \Rightarrow \perp$ , we end up proving  $\text{subH } \emptyset$  with  $\text{subH } q_a a^1 q$  and  $\text{subH } q_a a^0 q$  as extra hypotheses.  $\text{subH } q_a a^1 q$  and  $\text{subH } q_a a^0 q$  allow us to prove  $\text{subH } q_a q$ . Finally, from  $C[q_a q]$ , we deduce  $\text{subH } q_a q \Rightarrow \text{subH } \emptyset$ , which allows us to conclude. ■

## 6.2 The formal proof

Given  $q^k$  and

$$f : \forall pb. q \Vdash p \in G \Rightarrow b \in \text{Bool} \Rightarrow M = N \Rightarrow \perp$$

$$x_b : q \Vdash b \in \text{Bool}$$

we have

$$\left[ \begin{array}{l} [\text{Eq.}] M = N \\ I : q \Vdash q \in G \equiv q \leq q \\ f @ I @ x_b : q \Vdash M = N \Rightarrow \perp \\ I : M = M \simeq M = N \\ I^* : q \Vdash M = N \\ f @ I @ x_b @ I^* : \perp \\ f @ I @ x_b @ I^* : M \doteq N \mapsto q \Vdash \perp \end{array} \right.$$

We will use the lemma four times in the coming proof, always with  $x_b = \text{true}^*$  or  $x_b = \text{false}^*$  and  $M = \text{test } q a b$ .

$[\forall] r^k$   
 $[\forall] q_a^k$   
 $[\forall] q_f^k$   
 $[\lambda] c : C[q(q_a(q_f r))]$   
 $[\forall] a^t$   
 $[\lambda] x_a : q_a \text{ IF } a \in \text{Atom}$   
 $[\lambda] f : q_f \text{ IF } \forall pb. p \in G \Rightarrow b \in \text{Bool} \Rightarrow \text{test } p a b = 1 \Rightarrow \perp \simeq \forall pb. q \text{ IF } p \in G \Rightarrow b \in \text{Bool} \Rightarrow \text{test } p a b = 1 \Rightarrow \perp$   
 $\alpha_1(\alpha_2 c) : C[q_a]$   
 $\alpha_6(\alpha_6 c) : C[(qq_a)q_f r]$   
 $\hat{c} := \alpha_1(\alpha_6(\alpha_6 c)) : C[(qq_a)q_f]$   
 $[[\text{Let}] c_1 : (qq_a)q_f \in \text{FVal}$   
 $[\text{Let}] c_2 : \text{subH}(qq_a)q_f \Rightarrow \text{subH } \emptyset$   
 $\hat{x}_a := \xi_{\text{Atom}} x_a(\alpha_1(\alpha_2 c)) : a \in \text{Atom}$   
 $\text{Tot}_{\text{mem}} c_1 \hat{x}_a : \text{mem } a (qq_a)q_f \in \text{Bool}$   
**First case: mem a (qq\_a)q\_f = 1**  
 $\text{Tot}_{\text{test}} c_1 \hat{x}_a \text{ true} : \text{test } (qq_a)q_f a 1 \in \text{Bool}$   
 $[[\text{Eq.}] \text{mem } a (qq_a)q_f = 1$   
**First sub-case: test (qq\_a)q\_f a 1 = 1**  
 $[[\text{Eq.}] \text{test } (qq_a)q_f a 1 = 1$   
 $\beta_4 f : (qq_a)q_f \text{ IF } (qq_a)q_f \in G \Rightarrow 1 \in \text{Bool} \Rightarrow \text{test } (qq_a)q_f a 1 = 1 \Rightarrow \perp$   
 $\text{true}^* : (qq_a)q_f \text{ IF } 1 \in \text{Bool}$   
 $(\beta_4 f) @ I @ \text{true}^* @ I^* : \text{test } (qq_a)q_f a 1 \doteq 1 \mapsto (qq_a)q_f \text{ IF } \perp \simeq (qq_a)q_f \text{ IF } \perp$   
 $[(\beta_4 f) @ I @ \text{true}^* @ I^*] c : \perp$   
**Second sub-case: test (qq\_a)q\_f a 1 = 0**  
 $[[\text{Eq.}] \text{test } (qq_a)q_f a 1 = 0$   
From  $\text{mem } a (qq_a)q_f =_{\varepsilon} 1$  and  $\text{test } (qq_a)q_f a 1 =_{\varepsilon} 0$ , we deduce  $\text{test } (qq_a)q_f a 0 =_{\varepsilon} 1$   
 $\beta_4 f : (qq_a)q_f \text{ IF } (qq_a)q_f \in G \Rightarrow 0 \in \text{Bool} \Rightarrow \text{test } (qq_a)q_f a 0 = 1 \Rightarrow \perp$   
 $\text{false}^* : (qq_a)q_f \text{ IF } 0 \in \text{Bool}$   
 $(\beta_4 f) @ I @ \text{false}^* @ I^* : \text{test } (qq_a)q_f a 0 \doteq 1 \mapsto (qq_a)q_f \text{ IF } \perp \simeq (qq_a)q_f \text{ IF } \perp$   
 $[(\beta_4 f) @ I @ \text{false}^* @ I^*] : \perp$   
 $\text{if } \text{Tot}_{\text{test}} c_1 \hat{x}_a \text{ true then } ((\beta_4 f) @ I @ \text{true}^* @ I^*) c \text{ else } ((\beta_4 f) @ I @ \text{false}^* @ I^*) c : \perp$   
**Second case: mem a (qq\_a)q\_f = 0**  
 $[\text{See below}$   
 $(\beta_3 (\beta_4 f) @ I @ \text{true}^* @ I^*) (\alpha_7 \langle \text{const } \hat{x}_a c_1, \lambda u. B \rangle) : C[(qq_a)q_f a^1 1] : \perp$   
 $\text{if } \text{Tot}_{\text{mem}} c_1 \hat{x}_a$   
 $\text{then if } \text{Tot}_{\text{test}} c_1 \hat{x}_a \text{ true then } ((\beta_4 f) @ I @ \text{true}^* @ I^*) c \text{ else } ((\beta_4 f) @ I @ \text{false}^* @ I^*) c$   
 $\text{else } (\beta_3 (\beta_4 f) @ I @ \text{true}^* @ I^*) (\alpha_7 \langle \text{const } \hat{x}_a c_1, \lambda u. B \rangle) : \perp$   
 $\text{let } (c_1, c_2) := \hat{c} \text{ in if } \text{Tot}_{\text{mem}} c_1 \hat{x}_a$   
 $\text{then if } \text{Tot}_{\text{test}} c_1 \hat{x}_a \text{ true then } ((\beta_4 f) @ I @ \text{true}^* @ I^*) c \text{ else } ((\beta_4 f) @ I @ \text{false}^* @ I^*) c$   
 $\text{else } (\beta_3 (\beta_4 f) @ I @ \text{true}^* @ I^*) (\alpha_7 \langle \text{const } \hat{x}_a c_1, \lambda u. B \rangle) : \perp$   
 $\lambda c x_a f.$   
 $\text{let } (c_1, c_2) := \hat{c} \text{ in if } \text{Tot}_{\text{mem}} c_1 \hat{x}_a$   
 $\text{then if } \text{Tot}_{\text{test}} c_1 \hat{x}_a \text{ true then } ((\beta_4 f) @ I @ \text{true}^* @ I^*) c \text{ else } ((\beta_4 f) @ I @ \text{false}^* @ I^*) c$   
 $\text{else } (\beta_3 (\beta_4 f) @ I @ \text{true}^* @ I^*) (\alpha_7 \langle \text{const } \hat{x}_a c_1, \lambda u. B \rangle)$   
 $: q \text{ IF } \forall a \in \text{Atom}. (\forall pb. p \in G \Rightarrow b \in \text{Bool} \Rightarrow \text{test } p a b = 1 \Rightarrow \perp) \Rightarrow \perp$



For the second case, recall that we have

$$\begin{aligned}
& \text{mem } a (qq_a)q_f =_{\varepsilon} 0 \\
& \quad c : C[q(q_a(q_f r))] \\
& \quad c_1 : (qq_a)q_f \in \text{FVal} \\
& \quad c_2 : \text{subH}(qq_a)q_f \Rightarrow \text{subH } \emptyset \\
& \quad x_a : q_a \Vdash a \in \text{Atom} \\
& \hat{x}_a := \xi_{\text{Atom}} x_a (\alpha_1 (\alpha_2 c)) : a \in \text{Atom} \\
& \quad f : \forall pb. q_f \Vdash p \in G \Rightarrow b \in \text{Bool} \Rightarrow \text{test } p a b = 1 \Rightarrow \perp
\end{aligned}$$

$$\begin{aligned}
& \hat{x}_a : a \in \text{Atom} \\
& \beta_3 (\beta_4 f) : ((qq_a)q_f)a^1 \Vdash ((qq_a)q_f)a^1 \in G \Rightarrow 1 \in \text{Bool} \Rightarrow \text{test}((qq_a)q_f)a^1 a 1 = 1 \Rightarrow \perp \\
& \beta_3 (\beta_4 f) @ I @ \text{true}^* @ \Gamma^* : ((qq_a)q_f)a^1 \Vdash \perp \\
& \beta_3 (\beta_4 f) : ((qq_a)q_f)a^0 \Vdash ((qq_a)q_f)a^0 \in G \Rightarrow 0 \in \text{Bool} \Rightarrow \text{test}((qq_a)q_f)a^0 a 1 = 1 \Rightarrow \perp \\
& \beta_3 (\beta_4 f) @ I @ \text{false}^* @ \Gamma^* : ((qq_a)q_f)a^0 \Vdash \perp \\
& \text{consT } \hat{x}_a c_1 : ((qq_a)q_f)a^1 \in \text{FVal} \\
& \left[ \begin{array}{l}
[\lambda] u : \text{subH}((qq_a)q_f)a^1 \\
\text{consF } \hat{x}_a c_1 : ((qq_a)q_f)a^0 \in \text{FVal} \\
[\lambda] v : \text{subH}((qq_a)q_f)a^0 \\
\text{merge } \hat{x}_a u v : \text{subH}(qq_a)q_f \\
c_2 (\text{merge } \hat{x}_a u v) : \text{subH } \emptyset \\
\lambda v. c_2 (\text{merge } \hat{x}_a u v) : \text{subH}((qq_a)q_f)a^0 \Rightarrow \text{subH } \emptyset \\
\langle \text{consF } \hat{x}_a c_1, \lambda v. c_2 (\text{merge } \hat{x}_a u v) \rangle : C[((qq_a)q_f)a^0] \\
\alpha_7 \langle \text{consF } \hat{x}_a c_1, \lambda v. c_2 (\text{merge } \hat{x}_a u v) \rangle : C[((qq_a)q_f)a^0 1] \\
B := (\beta_3 (\beta_4 f) @ I @ \text{false}^* @ \Gamma^*) (\alpha_7 \langle \text{consF } \hat{x}_a c_1, \lambda v. c_2 (\text{merge } \hat{x}_a u v) \rangle) : \perp \\
B : \text{subH } \emptyset \\
\lambda u. B : \text{subH}((qq_a)q_f)a^1 \Rightarrow \text{subH } \emptyset \\
\langle \text{consT } \hat{x}_a c_1, \lambda u. B \rangle : C[((qq_a)q_f)a^1] \\
\alpha_7 \langle \text{consT } \hat{x}_a c_1, \lambda u. B \rangle : C[((qq_a)q_f)a^1 1] \\
(\beta_3 (\beta_4 f) @ I @ \text{true}^* @ \Gamma^*) (\alpha_7 \langle \text{consT } \hat{x}_a c_1, \lambda u. B \rangle) : C[((qq_a)q_f)a^1 1] : \perp
\end{array} \right.
\end{aligned}$$

### 6.3 Simplifying the combinators

We now use a canonical implementation for finite valuations featuring associativity, commutativity and idempotency of union whose neutral element is  $\emptyset$ . For instance, one can take ordered list without repetition. This implies that most primitive combinators of the forcing structure are the identity. More precisely, only  $\alpha_1$  and  $\alpha_2$  are not the identity among the first 8 combinators which triggers a similar result for derived ones. All in all, there is only one real combinator, which we call  $\alpha$ , that remove a part of the finite list (and thus is idempotent).

$$\begin{aligned}
& \alpha_1 = \alpha_2 = \alpha_9 = \alpha_{10} = \alpha_{14} = \alpha_{15} = \alpha_{16} = \alpha \\
& \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_{11} = \alpha_{12} = \alpha_{13} = \text{I} \\
& \beta_2 = \beta_3 = \beta_4 = \lambda x c. x (\alpha c) \\
& \gamma_1 = \lambda x c y. x y c \quad \gamma_2 = \gamma_3 = \lambda x y c. x c y \quad \gamma_4 = \lambda x c y. x (y (\alpha c))
\end{aligned}$$

In this context, the previous program becomes

$$\begin{aligned}
& \lambda c a f. \text{let } (c_1, c_2) := \alpha c \text{ in} & \hat{x}_a := \xi_{\text{Atom}} a (\alpha c) : a \in \text{Atom} \\
& \text{if } \text{Tot}_{\text{mem}} c_1 \hat{x}_a \text{ then} \\
& \quad \text{if } \text{Tot}_{\text{test}} c_1 \hat{x}_a \text{ true then } f (\alpha c) \text{I true}^* \Gamma^* \text{ else } f (\alpha c) \text{I false}^* \Gamma^* \\
& \quad \text{else } f (\alpha \langle \text{consT } \hat{x}_a c_1, \lambda u. f (\alpha \langle \text{consF } \hat{x}_a c_1, \lambda v. c_2 (\text{merge } \hat{x}_a u v) \rangle) \text{I false}^* \Gamma^* \rangle) \text{I true}^* \Gamma^*
\end{aligned}$$

Let us now look at the implementation of  $\alpha$ :

$$\lambda c. \text{let } (c_1, c_2) := c \text{ in } \langle \text{Up}_{\text{FVal}} c_1, \lambda x. \text{let } (x_1, x_2) := x \text{ in } c_2 \langle x_1, \text{Mon}_{\text{SubHtree}} x_2 \rangle \rangle$$

The term  $\text{Mon}_{\text{subHtree}} : \forall pqt. \text{subHtree } pt = 1 \Rightarrow \text{subHtree } pqt = 1$  proves a true Horn formula that can be realized by the identity. Indeed, any Herbrand tree below  $p$  is also a Herbrand tree below  $pq$ . Thus, from a realizer point of view, we have

$$\begin{aligned}
\alpha &:= \lambda c. \text{let } (c_1, c_2) := c \text{ in } \langle \text{Up}_{\text{FVal}} c_1, \lambda x. \text{let } (x_1, x_2) := x \text{ in } c_2 \langle x_1, \text{Mon}_{\text{subHtree}} x_2 \rangle \rangle \\
&\equiv \lambda c. \text{let } (c_1, c_2) := c \text{ in } \langle \text{Up}_{\text{FVal}} c_1, \lambda x. \text{let } (x_1, x_2) := x \text{ in } c_2 \langle x_1, x_2 \rangle \rangle \\
&\equiv \lambda c. \text{let } (c_1, c_2) := c \text{ in } \langle \text{Up}_{\text{FVal}} c_1, \lambda x. c_2 x \rangle \\
&\equiv \lambda c. \text{let } (c_1, c_2) := c \text{ in } \langle \text{Up}_{\text{FVal}} c_1, c_2 \rangle
\end{aligned}$$

This shows that the second component of  $C[p]$  is never touched by  $\alpha$  and the last line of the program becomes

$$f \langle \text{Up}_{\text{FVal}} \text{const } \hat{x}_a c_1, \lambda u. f \langle \text{Up}_{\text{FVal}} \text{consF } \hat{x}_a c_1, \lambda v. c_2 (\text{merge } \hat{x}_a u v) \rangle \text{I false}^* \text{I}^* \rangle \text{I true}^* \text{I}^*$$