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Experimental study of energy transport between two granular gas thermostats.

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Abstract. We report on the energy transport between two coupled probes in contact with granular thermostats at different temperatures. In our experiment, two identical blades, which are electromechanically coupled, are immersed in two granular gases maintained in different non-equilibrium stationary states, characterized by different temperatures. First, we show that the energy flux from one probe to another is, in temporal average, proportional to the temperature difference, as in the case of equilibrium thermostats. Second, we observe that the instantaneous flux is highly intermittent and that fluctuations exhibit an asymmetry which increases with the temperature difference. Interestingly, this asymmetry, related to irreversibility, is correctly accounted for by the Fluctuation Theorem. Thus, as is, our experiment is a simple macroscopic realisation, suitable for the study of energy exchanges between systems in non-equilibrium steady states.

1. Introduction

The comparison between dissipative steady states and equilibrium states is a meaningful question in building the statistical physics of systems slowly relaxing toward equilibrium, or maintained in Non-Equilibrium Steady State (NESS) by external forces [1, 2]. Fluxes are essential in this situation, where steady states result often from the balance between boundary excitation and bulk dissipation [3]. However, fluxes in this context often involve complex inhomogeneous and non-stationary transport processes [4]. The theoretical treatment of irreversibility in transport is possible only in simplified situations [5]. Elements in such toy models are very simple if compared to real phenomena. Besides, fluxes are very difficult to assess experimentally because conditions are often not properly constrained. In order to clarify the situation, we address experimentally the question of the flux between NESS in a very simplified configuration. Two independent thermostats are produced by vibrating at different amplitudes vessels filled with beads. They are coupled thanks to a simple electromechanical system. The resulting granular gases can be seen as two heat reservoirs at different temperatures, provided that the notion of temperature is properly extended to NESS. The temperature and flux are here accounted for by immersing in each vessel, a blade free to rotate around

its vertical axis (the probe) and by measuring the fluctuations of its angular velocity induced by the collisions with the grains. The working definition adopted is thus an effective temperature, resulting from the application of the Fluctuation Theorem (FT) to the blade [6], linked to the usual granular temperature. If the systems, maintained at different temperatures T_1 and T_2 , are coupled, energy is expected to flow from one to the other. The situation is experimentally achieved by an electromechanical coupling between the blades. Thus, the probes which are used to account for the temperature are also used to connect the heat reservoirs. Doing so, we produce an analog of heat conduction between equilibrium heat reservoirs at fixed temperatures. However, crucial differences are that the thermostats are dissipative and far from the thermodynamic limit, *i.e.* the number of particles and volume are not *very* large. Fluctuations are essential, like in micro or nanoscale systems.

The principle of the measurements and the experimental device are described in Sec. 2. After a short discussion on the notion of temperature in Sec. 3, we report in Sec. 4 that the temporal average of the energy flux $\bar{\phi}$ is proportional to the temperature difference ΔT . The proportionality coefficient is derived explicitly from the system parameters. Sec. 4.2 is devoted to the statistics of the temporal fluctuations of the instantaneous flux, $\phi(t)$. Distributions of ϕ are highly non-Gaussian: their kurtosis is large, and their skewness increases with ΔT , which accounts for the increase of the mean. A striking observation is that the most probable value of ϕ is always zero, whatever the average $\bar{\phi}$. We then show in Sec. 5 that the stationary energy flux $\phi(t)$ obeys the Fluctuation Theorem. These results are discussed in Sec. 6.

2. Principle of the experiment and experimental setup

The core principle of the experiment consists in coupling two identical granular gases maintained at different granular temperatures and in measuring the resulting energy flux from one to the other. Each of the granular gases consists of $N = 300$ stainless steel beads (diameter 3 mm and mass 0.1 g) (Fig. 1, left). The beads are placed in a cylindrical vessel (diameter 5 cm and height 6 cm). The gaseous phase is obtained by vibrating the vessel vertically with a sinusoidal acceleration ranging from 2 to 16 g at 40 Hz. (Shaker: Bruel & Kjaer 4809.) In order to improve the mixing of the beads in the gas, we chose a vessel with a conical bottom. Each of the granular gas systems is a duplication of that used in an earlier work [7]. They are identical and only differ in the amplitude of the vibration, thus in their temperature.

In order to assess its temperature, a probe consisting of a thin blade is immersed in each granular gas (Fig. 1, left). The blade is a square, 2 cm \times 2 cm, made out of a stainless steel plate (thickness 0.25 mm). It is attached to the vertical axis of a DC-motor, its lower edge positioned a few mm above the bottom of the vessel. Small brushed DC-motors of nominal power 0.75 W are used (Maxon RE 10 118386). The rotors are ironless to minimise inertia, and precious metal brushes improve the electrical contact with the commutator and reduce solid friction. From all points of

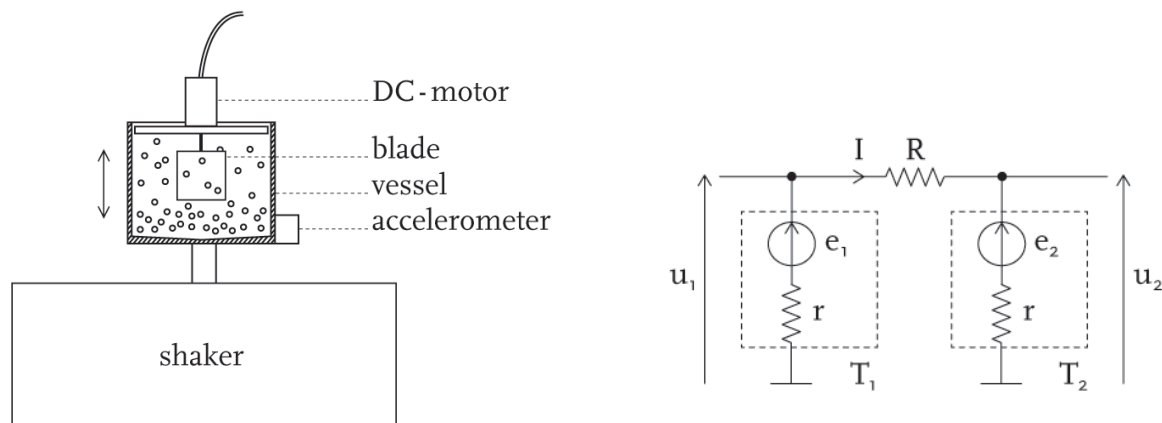


Figure 1. On the left, one of the two ensembles shaker + vessel + motor + blade. The motor + blade ensemble is fixed on a cover. On the right is the electrical circuit of the compound system. The motors are schematized by voltage sources ($e_i \propto \dot{\theta}_i$) and the internal resistances r .

view, the two subsystems are identical. Moreover, we checked that the rotors do not turn spontaneously like ratchets. The temperature can be assessed from the fluctuations of the angular velocity $\dot{\theta}(t)$ of the blade. Indeed, the *granular temperature* is commonly defined, by simple analogy with the kinetic theory, as the variance of the particles velocity fluctuations [8]. Using the same analogy, we write that the temperature of the blade, *thermalized* with the granular heat bath, is proportional to its average kinetic energy. Thus, for a system having a single degree of freedom, we write:

$$\frac{1}{2}kT \equiv \frac{1}{2}M\overline{\dot{\theta}^2}. \quad (1)$$

where M stands for the total moment of inertia of the parts in solid rotation (blade + rotor of the motor, $M = 3.33 \cdot 10^{-8} \text{ kg m}^2$). Eq. (1) defines the energy scale kT which accounts for the *temperature* of the system, here the blade in contact with a stationary granular heat bath. 'Temperature' or 'thermal energy' will be used indeterminately for kT in the following. We choose not to express temperature in Kelvins, because the Boltzmann constant k_B has no meaning in such macroscopic system.

The granular gas is very dilute, in the sense that the mean free path is not much less than the size of the vessel of radius ρ , or even the blade's dimension. It is difficult to investigate experimentally the bulk of granular matter. But, assuming in first approximation and for small acceleration of the vessel that most of the beads are in the lower $h \simeq 1 \text{ cm}$ because of stratification, an average distance between particles is calculated from the average density $n = N/(\pi R^2 h)$, of the order of 4 mm. Thanks to kinetic theory, the mean free path is: $\lambda = 1/(n\pi r^2)$, where $r = 1.5 \text{ mm}$ is the radius of a bead. This gives a rough lower estimate $\lambda \sim 9 \text{ mm}$, as discussed already in [7]. This value depends very much on the acceleration given to the vessel, *i.e.* temperature, but confirms the assertion that the granular is dilute in any case of interest here.

The coupling relies on an electromechanical symmetry. In practice, the two

subsystems are coupled electrically by connecting the motors with one another by a resistor $R = 23.3 \Omega$, approximately equal to the internal resistance r of the motors (Fig. 1, right). The internal resistance of the motors, $r \simeq 22 \Omega$, is simply measured with an ohmmeter, at room temperature in the absence of granular gas (We checked that, in what follows, the inductance of the motors can be neglected.) The principle of the coupling is the following. A DC-motor can be used reversibly as motor or generator. As a motor, the torque, $\Gamma = \alpha I$, is proportional to the current, I (The factor α represents all the physical characteristics of the motor, like number of poles, coils, and turns, magnetic field of the permanent magnets, and geometric factors). As a generator, a voltage, $e = \alpha \dot{\theta}$, proportional to the angular velocity, $\dot{\theta}$, is induced (Notice here that the coefficient $\alpha = 4.27 \cdot 10^{-3}$ Vs/rad is unique). Thus, when momentum is transferred to one blade by the surrounding beads in one reservoir, a voltage is induced in the circuit. This voltage causes a current to circulate in the other motor. A torque is then produced on the second blade, that transfers momentum to the other reservoir. As the electromechanical devices are reversible, this process occurs randomly in both directions.

Assessment of the temperatures and of the energy flux rely on the measurement of electrical quantities, only. The voltages u_1 and u_2 are monitored (NI PXI-4462) at 1 kHz for 1 hour samples. Considering the electrical circuit in Fig. 1, we write:

$$\begin{aligned} e_1 &= u_1 + rI, \\ e_2 &= u_2 - rI, \end{aligned} \tag{2}$$

and:

$$I = \frac{1}{R_{\text{tot}}} (e_1 - e_2) = \frac{1}{R} (u_1 - u_2) \tag{3}$$

where R_{tot} is the sum of all resistances in the circuit: $R_{\text{tot}} = 2r + R$. Notice that the knowledge of u_1 and u_2 is enough to assess, for instance, the temperature kT_1 . Indeed, from Eq. (1), $kT_1 = M\dot{\theta}_1^2 = \frac{M}{\alpha^2} e_1^2$ with, from Eqs. (2) and (3), $e_1 = u_1 + \frac{r}{R} (u_1 - u_2)$. A similar relation exists for kT_2 . Thus, the temperature difference can be obtained from u_1 and u_2 by using:

$$kT_1 - kT_2 = \frac{M}{\alpha^2} \frac{R_{\text{tot}}}{R} \left(\overline{u_1^2} - \overline{u_2^2} \right) \tag{4}$$

In Sec. 4, we report on the dependence of the energy flux in the electrical circuit as consequence of such temperature difference.

3. Energy flux

Let us first discuss thoroughly the quantities we shall measure. Notice that, although energy is dissipated in the resistors, the current is conserved in the circuit that insures the coupling. This is a decisive difference with classical heat conduction between thermostats in presence of losses. As a consequence of this conservation, the magnetic torques Γ_1 and Γ_2 resulting from the current I applied respectively to the blade 1 and 2 are equal at all times ($\Gamma_1 = \Gamma_2 = \Gamma$). Thus, the energy exchange between the blades is related to the difference in their instantaneous velocity of rotation.

Indeed the instantaneous electromagnetic power on blade 1 can be written $\dot{w}_1 = \Gamma\dot{\theta}_1 = e_1 I$, whereas, for blade 2, $\dot{w}_2 = \Gamma\dot{\theta}_2 = -e_2 I$ (The change in sign is only due to the orientation of the current I). Therefore, the current I , even if it imposes the same torque Γ to both blades, is associated to a difference in the electromagnetic power transferred to the blades, $\Delta\dot{w} \equiv \dot{w}_1 - \dot{w}_2$, which is only due to the difference ($\dot{\theta}_1 - \dot{\theta}_2$) in their instantaneous velocity of rotation.

Considering now that the current I originates itself from the fluctuations of the angular positions of the blades, one can regard the difference $-\Delta\dot{w}$ as the difference between the power transmitted by the blade 1 to the blade 2 and the power transmitted by the blade 2 to the blade 1, thus as the energy flux, ϕ , between them. In electrical variables, we have:

$$\phi = -\Delta\dot{w} = I(e_1 + e_2) = \frac{1}{R_{\text{tot}}} (e_1^2 - e_2^2). \quad (5)$$

4. Temperature and mean energy flux

4.1. Temporal average

Recalling the definition of the temperature Eq. (4), we write the temporal average of Eq. (5) in the form:

$$\bar{\phi} = \frac{\alpha^2}{MR_{\text{tot}}} (kT_1 - kT_2). \quad (6)$$

The energy flux is proportional to the temperature difference, like in the Fourier law of heat conduction [9]. The transport coefficient only depends on the characteristics, α , M and R_{tot} of the coupling.

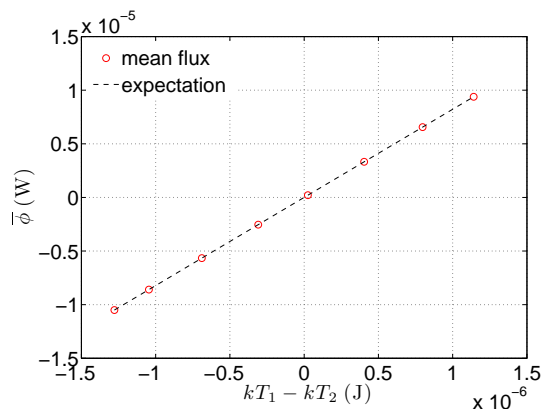


Figure 2. Average energy flux, $\bar{\phi}$, vs. thermal energy difference $kT_1 - kT_2$ (\circ). The associated transport coefficient, *i.e.* the slope, is $\alpha^2/(MR_{\text{tot}})$.

The result is exact, provided that kT_1 and kT_2 are measured in place on the coupled systems, or that the systems are not altered by the coupling, *i.e.* that the blades remain in equilibrium with unaltered heat baths. This assumption is reasonable as $\bar{\phi}$ is

extremely small (of the order of 10^{-5} W) if compared to the injected power (of order 10 W) needed to keep the granular gases in motion and compensate its dissipation. We checked experimentally that the difference with free fluctuations, assessed independently by measuring the voltages e_1 and e_2 in open circuit, or in the system of coupled motors is indiscernible. In order to illustrate the Eq. (6), we plot in Fig. 2 the average flux $\bar{\phi}$ against the temperature difference $kT_1 - kT_2$, calculated from the same data set, taking into account the value $\alpha^2/(MR_{\text{tot}}) = 8.13$ Hz. This value is known precisely. The parameters α and the moment of inertia of the motor are read in Maxon's data sheet, and the moment of inertia of the blade is calculated on the basis of precise dimension and weight measurements. The measurement of the set $\{u_1(t), u_2(t)\}$ give on one hand $I(t)$ provided R , and in the other hand $\{e_1(t), e_2(t)\}$ provided internal resistance of the motors r . R is easily and precisely obtained, so the quality of the measurement lean on the reliability of the measurement of r . Being the main source of errors, it concentrates our attention and the difficulty of this measurement, otherwise rather easy, as long as large but homogeneous time series can be acquired. The values r chosen take into account the temperature elevation after a few hours of operation, and the quality of the resulting measurements are attested by the plot of Fig. 2. Another source of error inherent to DC-motors is friction losses. They have been evaluated to be about an order of magnitude less than a typical power $\bar{e}_i I$.

It is remarkable that a non-zero average energy flux circulates from one bath to the other, although torques, Γ_i , as well as angular velocities, $\dot{\theta}_i$, are always 0 in average. This non-intuitive feature results from the fact that correlations exist between the voltages e_i and the current I (and thus between Γ and $\dot{\theta}_i$), caused by the conservation of the current I over the whole circuit. The resulting average energy flux between the blades in contact with granular thermostats is proportional to the temperature difference. This is similar to what would be observed for the transport between equilibrium thermostats. However, as shown in the following, the fluctuations of the instantaneous energy flux, $\phi(t)$, exhibits significant differences with equilibrium systems.

4.2. Temporal fluctuations

We report in Fig. 3 histograms of the instantaneous energy flux, ϕ . We observe that the histograms are highly non-Gaussian and, in general, asymmetric.

For vanishing small temperature difference, $kT_1 - kT_2 \simeq 0$ (Fig. 3, left), $\bar{\phi} \simeq 0$ (Eq. 5). We observe, in this case, that the histogram is symmetric and, accordingly, that the most probable value of ϕ is zero. In contrast, when the temperature difference, $kT_1 - kT_2$ departs significantly from 0 (Fig. 3, right), $\bar{\phi} \neq 0$ (Eq. 5) and the histogram is significantly asymmetric: for $kT_1 > kT_2$, the probability of a given (positive) flux ϕ from the *hot* source to the *cold* one is larger than the probability of the same (negative) flux in the opposite direction. In average, the *heat* flows from the *hot* to the *cold* blade. However, strikingly, in spite of the asymmetry of this flow, the most probable value of the flux ϕ remains zero.

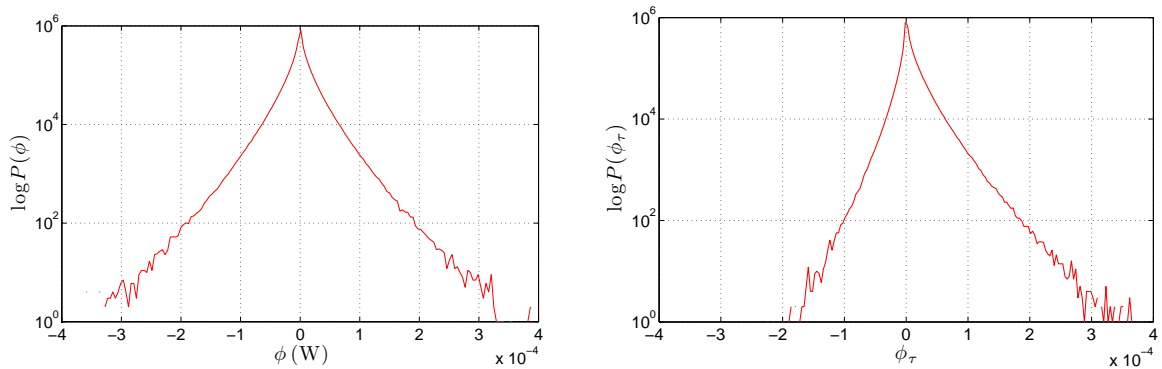


Figure 3. Histograms of the instantaneous energy flux, ϕ , for two different values of the temperature difference, $kT_1 - kT_2$ (Left: $kT_1 - kT_2 \simeq 1.7 \cdot 10^{-8}$ J, Right: $kT_1 - kT_2 \simeq 7.9 \cdot 10^{-7}$ J).

To characterize these statistics at the simplest level, we compute the skewness, S , and kurtosis, K , coefficients of the distributions of ϕ for different values of temperature difference $kT_1 - kT_2$ (Fig. 4). The coefficients, S and K , are defined to be the adimensional third and fourth moments of the distribution: $S = \overline{\Delta\phi^3} / \overline{\Delta\phi^2}^{3/2}$ and $K = \overline{\Delta\phi^4} / \overline{\Delta\phi^2}^2$, with $\Delta\phi = \phi - \bar{\phi}$. Note that S and K are respectively odd and even functions of $kT_1 - kT_2$, as required by the symmetry of the experimental configuration.

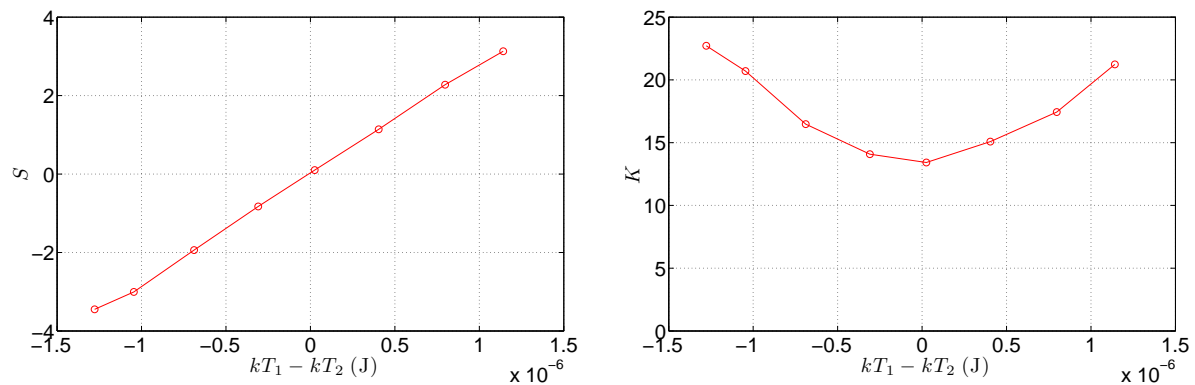


Figure 4. Skewness S (left) and kurtosis K (right) vs. temperature difference $kT_1 - kT_2$.

For $kT_1 \simeq kT_2$, we have $S = 0$ and K reaches a minimum of about 14. Such value of K is unusually large if compared to the values 3 and 6 associated with Gaussian and exponential distributions.

5. Fluctuation Theorem

As shown in Sec. 4.2, the distribution of ϕ , characterized by its moments, S and K , happens to be, in general, asymmetric and, always, very broad. We can go a significant

step further by verifying that the flux ϕ follows the Gallavotti-Cohen Fluctuation Theorem [10]. The Fluctuation Theorem (FT), which generalizes the 2nd principle, compares the probability that the entropy increase or decrease by the same amount during a certain (long) duration. For the present purpose, it relates the asymmetry of the distribution to the irreversibility of the dynamics and, thus, to dissipation. In our experimental configuration, dissipation mainly refers to energy losses by Joule effect in the resistors. If expressed in terms of the energy flux ϕ_τ exchanged during a time-lag τ , the FT reads in the limit of large τ :

$$\log \frac{P(\phi_\tau)}{P(-\phi_\tau)} = \mu\tau\phi_\tau, \quad (7)$$

with $\phi_\tau = \frac{1}{\tau} \int_\tau \phi(t)dt = \frac{1}{\tau} \int_\tau (e_1 + e_2)Idt$. In this detailed form, the FT states that the relative statistical weight of positive over negative fluxes increases exponentially with ϕ . In Fig. 5 (left), the so-called asymmetry function, $\frac{1}{\tau} \log \frac{P(\phi_\tau)}{P(-\phi_\tau)}$, is plotted against ϕ_τ for increasing values of τ , with the best linear fit at the origin.

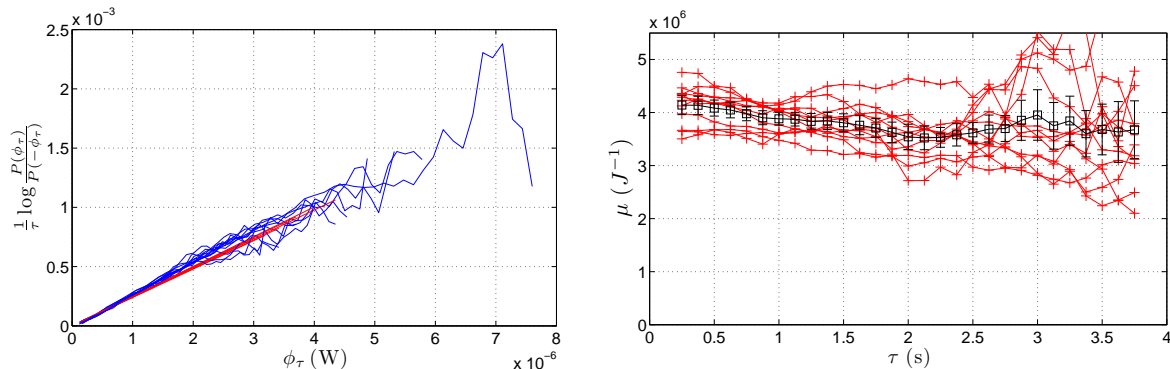


Figure 5. Left: Asymmetry function vs. coarse-grained energy flux ϕ_τ . Right: coefficient μ vs. τ (crosses $+$, 1-hour time series; squares \square , average)

The experimental data reveal a good agreement with the linear dependence expected from Eq. (7) when τ is large enough. For each 1-hour time series, we determine the slope and report the associated values of μ as function of τ (crosses $+$, Fig. 5, right). Note that, in order to limit the influence of statistical noise, we consider the slope at the origin). Finally, we average 12 curves of $\mu(\tau)$, calculated over 12 consecutive 1-hour time series and estimate the error bars from the standard deviation (squares \square , Fig. 5, right). We observe that, the coefficient μ , determined as explained above, indeed reaches an asymptotic value plateau for large τ , before the increase of the statistical noise. It can be seen in figure 5 (right) that the typical time of convergence is several seconds, whereas the correlation time of the exchange with the gas is a few dozen of ms. This is a consequence of the intermittency of ϕ (revealed by the large value of the kurtosis). The slow convergence of any statistical quantities makes the measurement of μ especially difficult. We point out that the existence an asymptotic value of μ implies the validity of the FT for the energy flux.

It is now interesting to assess the dependence of the asymptotic value of μ on the temperature difference $kT_1 - kT_2$. Jarzynski *et. al.* have derived a version of the FT for the heat flux between two (equilibrium) thermostats, named *exchange fluctuation theorem* (XFT) [11]. They obtained a relation, similar to Eq. (7), in which $\mu = \Delta\beta$, with $\Delta\beta = \frac{1}{kT_1} - \frac{1}{kT_2}$. We observe in Fig. 6 that our experimental value of μ is indeed proportional to $\Delta\beta$ defined, we remind, from the fluctuations of the angular velocity of the blades. We mention, however, that slope of the best linear fit is experimentally 5.69 and not 1, as derived from the XFT.

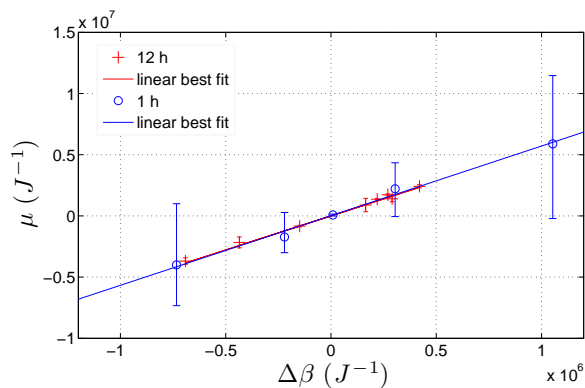


Figure 6. Coefficient μ vs. $\Delta\beta$. Crosses and circles correspond to two different measurement campaigns, with 1 or 12 hour-long time series.

6. Discussion and conclusion

We achieved an experimental situation which makes possible to study the stationary transport of energy between two granular heat reservoirs. The experimental set-up presented here is technically very simple, thanks to small electromechanical devices. The crafty use of DC-motors simultaneously as actuator and probe avoids calibration difficulties.

Let us first comment that we do not assess the properties of the gases directly, but rather the dynamical properties of the probes in contact with the granular system. Thus, as is, ϕ is the instantaneous energy flux between the two probes and not between the gases. These two instantaneous energy fluxes differ from one another by the changes in the kinetic energy of the blades. However, in temporal average, in the stationary regime, the kinetic energy of the blades remains constant and $\bar{\phi}$ is indeed the energy flux between the two reservoirs.

We reported that $\bar{\phi}$ is proportional to the temperature difference between the two probes. This means, as the kT measure is proportional to the temperature of the granular gas, that the mean energy flux between the two granular gas baths is linear. The result is similar to the Fourier law for heat conduction and the *transport coefficient*

is related to the characteristics of the coupling. This feature is similar to what would be observed for equilibrium systems.

Interestingly, in our dissipative system, the fluctuations exhibit surprising features: they are very intermittent, in the sense that intense events are highly represented in the statistics (this is shown by the large value of the kurtosis, $K \geq 14$); the most probable value of ϕ remains 0 even when, in average, $\bar{\phi} \neq 0$ (this is shown by the fact that the skewness departs from 0 when the temperature difference is increased). This feature differs from the recent observation of the transport between Nyquist resistors at equilibrium at various temperatures coupled by a capacitor [12]. When comparing these two studies, one must keep in mind that the conditions are distinct in several ways. In particular, our thermostats are dissipative and far from the thermodynamic limit (small number of grains, relatively large mean free path). It is difficult at this point to give the origin of the peculiarities observed here. It would certainly be interesting to build an experiment that would allow to vary either the number of degrees of freedom alone, or the dissipation alone.

Focusing further on the fluctuations of the flux, we observed that our experimental systems follows reasonably the FT in spite of several discrepancies between its underlying assumptions and our experimental conditions: the thermostats are out-of-equilibrium; they are far from the thermodynamic limit.

For comparison, our system exhibits a qualitative agreement, but quantitative discrepancies with the *exchange fluctuation theorem*. But one remember that the hypothesis used to derive the XFT are not fulfilled.

It remains that our system is possibly the first experimental evidence that the FT holds for a transport process in dissipative systems.

Acknowledgments

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