



The risk of divergence

Pierre Lescanne

► To cite this version:

Pierre Lescanne. The risk of divergence. 3rd International Workshop on Strategic Reasoning, Dec 2015, Oxford, United Kingdom. ensl-01246889

HAL Id: ensl-01246889

<https://ens-lyon.hal.science/ensl-01246889>

Submitted on 20 Dec 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The risk of divergence

Pierre Lescanne

University of Lyon, École normale supérieure de Lyon, CNRS (LIP),
46 allée d'Italie, 69364 Lyon, France

We present infinite extensive strategy profiles with perfect information and we show that replacing finite by infinite changes the notions and the reasoning tools. The presentation uses a formalism recently developed by logicians and computer science theoreticians, called coinduction. This builds a bridge between economic game theory and the most recent advance in theoretical computer science and logic. The key result is that rational agents may have strategy leading to *divergence*.

Keywords: divergence, decision, infinite game, sequential game, coinduction.

1 Introduction

Strategies are well described in the framework of sequential games, aka. games in extensive forms with perfect information. In this paper, we describe rational strategies leading to divergence.¹ Indeed divergence understands that the games, the strategies and the strategy profiles are infinite. We present the notion of infinite strategy profiles together with the logical framework to reason on those objects, namely coinduction.

2 Decisions in Finite Strategy Profiles

To present strategy reasoning, we use one of the most popular framework, namely *extensive games with perfect information* ([12] Chapter 5 or [3]) and we adopt its terminology. In particular we call *strategy profile* an *organized* set of strategies, merging the decisions of the agents. This organization mimics this of the game and has the same structure as the game itself. They form the set² StratProf. By “*organized*”, we mean that the strategic decisions are associated with the nodes of a tree which correspond to positions where agents have to take decisions. In our approach strategy profiles are first class citizens and games are byproduct. In other words, strategy profiles are defined first and extensive games are no more than strategy profiles where all the decisions have been erased. Therefore we will only speak about strategy profiles, keeping in mind the underlying extensive game, but without giving them a formal definition³. For simplicity and without loss of generality, we consider only dyadic strategy profiles (i.e.; double choice strategy profiles), that are strategy profiles with only two choices at each position. Indeed it is easy to figure out how multiple choice extensive

¹In this paper we use “divergence” instead of “escalation” since it is somewhat dual convergence a concept which plays a key role in what follows.

²To be correct, we should say the “*they form the coalgebra*”.

³A direct definition of games is possible, but is not necessary in this paper.

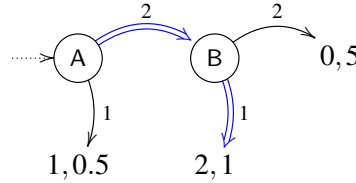
strategy profiles can be represented by double choice extensive strategy profiles. We let the reader imagine such an embedding. Therefore, we consider a set of choices: $\text{Choice} = \{1, 2\}$.

Along the paper, our examples need only a set of two agents: $\text{Agent} = \{A, B\}$. In this paper we use coinduction and corecursion as basic tools for reasoning correctly about and defining properly infinite objects. Readers who want to know more about those concepts are advised to read introductory papers [5, 14], while specific applications to infinite strategy profiles and games are introduced in [9].

Definition 1 A finite strategy profile is defined by induction as follows:

- either given a utility assignment u (i.e., a function $u : \text{Agent} \rightarrow \mathbb{R}$) $\langle\langle u \rangle\rangle$ is a finite strategy profile, which corresponds to an ending position.
- or given an agent a , a choice c and two finite strategy profiles s_1 and s_2 , $\langle\langle a, c, s_1, s_2 \rangle\rangle$ is a finite strategy profile.

For instance, a strategy profile can be drawn easily with the convention that 1 is represented by going down and 2 is represented by going right. The chosen transition is represented by a double arrow \Rightarrow . The other transition is represented by a simple arrow \rightarrow . For instance



is a graphic representation of the strategy profile

$$s_\alpha = \langle\langle A, 2, \langle\langle A \mapsto 1, B \mapsto 0.5 \rangle\rangle, \langle\langle B, 1, \langle\langle A \mapsto 2, B \mapsto 1 \rangle\rangle, \langle\langle A \mapsto 0, B \mapsto 5 \rangle\rangle \rangle\rangle\rangle.$$

From a finite strategy profile, say s , we can define a *utility assignment*, which we write \widehat{s} and which we define as follows:

- $\widehat{\langle\langle u \rangle\rangle} = u$
- $\widehat{\langle\langle a, c, s_1, s_2 \rangle\rangle} = \text{case } c \text{ of } 1 \rightarrow \widehat{s}_1 \mid 2 \rightarrow \widehat{s}_2$

For instance $\widehat{s}_\alpha(A) = 2$ and $\widehat{s}_\alpha(B) = 1$.

We define an equivalence $s =_g s'$ among finite strategy profiles, which we read as “ s and s' have the same (underlying) game”.

Definition 2 We say that two strategy profiles s and s' have the same game and we write $s =_g s'$ iff by induction

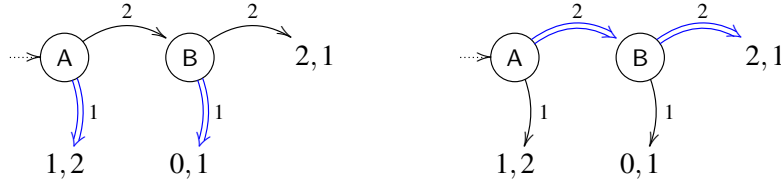
- either $s = \langle\langle u \rangle\rangle$ and $s' = \langle\langle u \rangle\rangle$
- or $s = \langle\langle a, c, s_1, s_2 \rangle\rangle$ and $s' = \langle\langle a', c', s'_1, s'_2 \rangle\rangle$ and $a = a'$, $s_1 =_g s'_1$ and $s_2 =_g s'_2$.

We can define a family of finite strategy profiles that are of interest for decisions. First we start with *backward induction*. Following [20], we consider ‘backward induction’, not as a reasoning method, but as a predicate that specifies some strategy profiles.

Definition 3 (Backward induction) A finite strategy profile s is backward induction if it satisfies the predicate BI , where BI is defined recursively as follows:

- $\text{BI}(\langle\langle u \rangle\rangle)$, i.e., by definition an ending position is ‘backward induction’.
- $\text{BI}(\langle\langle a, 1, s_1, s_2 \rangle\rangle) \Leftrightarrow \text{BI}(s_1) \wedge \text{BI}(s_2) \wedge \hat{s}_1 \geq \hat{s}_2$.
- $\text{BI}(\langle\langle a, 2, s_1, s_2 \rangle\rangle) \Leftrightarrow \text{BI}(s_1) \wedge \text{BI}(s_2) \wedge \hat{s}_2 \geq \hat{s}_1$.

In other words, a strategy profile which is not an ending position is ‘backward induction’ if both its direct strategy subprofiles are and if the choice leads to a better utility, as shown by the comparison of the utility assignments to the direct strategy subprofiles. The two following strategy profiles are ‘backward induction’ [12](Example 158.1)



An agent is rational if she makes a choice dictated by backward induction and if she keeps being rational in the future. We write this predicate Rat_f where the index f insists on finiteness making it distinct from the predicate Rat_∞ on infinite strategy profiles.

Definition 4 (Rationality for finite strategy profiles) The predicate Rat_f is defined recursively as follows:

- $\text{Rat}_f(\langle\langle u \rangle\rangle)$,
- $\text{Rat}_f(\langle\langle a, c, s_1, s_2 \rangle\rangle) \Leftrightarrow \exists \langle\langle a, c, s'_1, s'_2 \rangle\rangle \in \text{StratProf},$
 - $\langle\langle a, c, s'_1, s'_2 \rangle\rangle =_g \langle\langle a, c, s_1, s_2 \rangle\rangle$
 - $\text{BI}(\langle\langle a, c, s'_1, s'_2 \rangle\rangle)$
 - $\text{Rat}_f(s_c)$

Then we can state a variant of Aumann theorem [1] saying that backward induction coincides with rationality.

Theorem 5 $\forall s \in \text{StratProf}, \text{Rat}_f(s) \Leftrightarrow \text{BI}(s)$.

3 Decisions in Infinite Strategy Profiles

We extend the concept of backward induction and the concept of rationality to infinite strategy profiles. For that, we replace induction by coinduction.⁴ Notice that we mix up recursive and corecursive definitions, and that we reason sometime by induction and sometime by coinduction. Therefore we advise the reader to be cautious and to

⁴For readers not familiar with coinduction and not willing to read [5] or [14], we advise her to pretend just that corecursive definitions define infinite objects and coinduction allows reasoning specifically on their infinite aspects, whereas recursive definition define finite objects and induction allows reasoning on their finite aspects.

pay attention to when we use one or the other.⁵ We write InfStratProf the set of finite or infinite strategy profiles.

Definition 6 *The set finite or infinite strategy profiles InfStratProf is defined corecursively as follows:*

- *either given a utility assignment u , then $\langle\langle u \rangle\rangle \in \text{InfStratProf}$, which corresponds to an ending position.*
- *or given an agent a , a choice c and two strategy profiles $s_1 \in \text{InfStratProf}$ and $s_2 \in \text{InfStratProf}$, then $\langle\langle a, c, s_1, s_2 \rangle\rangle \in \text{InfStratProf}$.*

We cannot define the utility assignments on all infinite strategy profiles, only on those on which the utility can be “computed”. The strategy profiles on which utility assignments are defined are called *convergent*, since when one follows the path indicated by the choices one “converges”, that is that one gets to an ending position, i.e., a position where utilities are actually attributed. The predicate *convergent* is defined by induction, meaning that, on s , after finitely many steps following the choices of s an ending position is reached. “Finitely many steps” is a finite aspect and this is why we use an inductive definition.

Definition 7 (Convergent) *Saying that s is convergent is written $\downarrow s$. $\downarrow s$ is defined by induction as follows:*

- $\downarrow \langle\langle u \rangle\rangle$ or
- if $\downarrow s_1$ then $\downarrow \langle\langle a, 1, s_1, s_2 \rangle\rangle$ or
- if $\downarrow s_2$ then $\downarrow \langle\langle a, 2, s_1, s_2 \rangle\rangle$ or

On convergent strategy profiles we can assign utilities. The resulting function is written \hat{s} when applied to a strategy profile s .

Definition 8 (Utility assignment) \hat{s} is defined corecursively on every strategy profile.

$$\begin{array}{lll} \text{when } s = \langle\langle u \rangle\rangle & \hat{s} & = f \\ \text{when } s = \langle\langle a, 1, s_1, s_2 \rangle\rangle & \hat{s} & = \hat{s}_1 \\ \text{when } s = \langle\langle a, 2, s_1, s_2 \rangle\rangle & \hat{s} & = \hat{s}_2 \end{array}$$

The function $\hat{\cdot}$ has to be specified on an infinite object and this is why we use a corecursive definition.

Proposition 9 *If $\downarrow s$, then \hat{s} returns a value.*

Actually convergent strategy profiles are not enough as we need to know the utility assignment not only on the whole strategy profile but also on strategy subprofiles. For that, we need to insure that from any internal position we can reach an ending position, which yields that on any position we can assign a utility. We call *always-convergent* such a predicate⁶ and we write it $\Box\downarrow$.

Definition 10 (Always-convergent)

⁵Notice that not all the authors are as cautious. For instance, Hargreaves-Heap and Varoufakis write ([3] p.27) “The idea [of common knowledge] reminds one what happens when a camera is pointing to a television screen that conveys the same image recorded by the same camera: an infinite self-reflection”. Indeed *common knowledge* is typically inductive whereas *infinite self-reflection* is typically coinductive.

⁶Traditionally \Box is the notation for the modality (i.e., the predicate transformer) *always*.

- $\Box \downarrow \langle\langle u \rangle\rangle$ that is that for whatever u , $\langle\langle u \rangle\rangle$ is always-convergent
- $\Box \downarrow \langle\langle a, c, s_1, s_2 \rangle\rangle$ if
 - $\langle\langle a, c, s_1, s_2 \rangle\rangle$ is convergent (i.e., $\downarrow \langle\langle a, c, s_1, s_2 \rangle\rangle$), and
 - s_1 is always-convergent (i.e., $\Box \downarrow s_1$), and
 - s_2 is always-convergent (i.e., $\Box \downarrow s_2$).

Proposition 11 $\Box \downarrow s \Rightarrow \downarrow s$.

$s_{\Box 2}$ in Figure 1 is a typically non convergent strategy profile, wherever $s_{1\Box 2}$ in the same figure is a typically convergent and not always-convergent strategy profile.

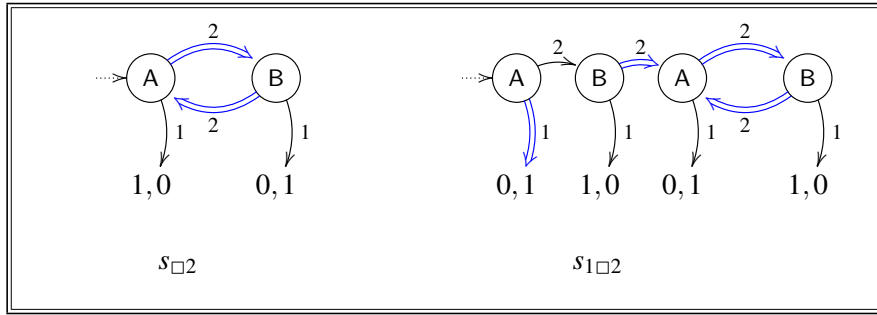


Figure 1: Two examples of strategy profiles

Using the concept of always-convergence we can generalize the notion of backward induction to this that the tradition calls *subgame perfect equilibrium* [15] and which we write here SPE. In short SPE is a corecursive generalization of BI. First we define an auxiliary predicate.

Definition 12 (PE)

$$\text{PE}(s) \Leftrightarrow \Box \downarrow s \wedge s = \langle\langle a, 1, s_1, s_2 \rangle\rangle \Rightarrow \widehat{s}_1(a) \geq \widehat{s}_2(a) \\ \wedge s = \langle\langle a, 2, s_1, s_2 \rangle\rangle \Rightarrow \widehat{s}_2(a) \geq \widehat{s}_1(a)$$

We define *SPE* as *always-PE*. In other words, a strategy profile s is a subgame perfect equilibrium if $\Box \text{PE}(s)$. \Box applies to a predicate.

Definition 13 (Always) Given a predicate P , the predicate $\Box P$ is defined corecursively as follows.

- if $P(\langle\langle u \rangle\rangle)$ then $\Box P(\langle\langle u \rangle\rangle)$ and
- if $\Box P(s_1)$, $\Box P(s_2)$ and $P(\langle\langle a, c, s_1, s_2 \rangle\rangle)$ then $\Box P(\langle\langle a, c, s_1, s_2 \rangle\rangle)$

Formally *SPE* is $\Box \text{PE}$. Besides we may notice that the notation used for always-convergence (Definition 10) is consistent with Definition 13. Now thanks to SPE we can give a notion of rationality for infinite strategy profiles. Like for finite strategy profiles we define corecursively, this time, an equivalence $s =_g s'$ on infinite strategy profiles (read s and s' have the same game). Two strategy profiles have the same game if at each step, they have the same agent and their respective direct strategy subprofiles have the same game and only the choices differ.

Definition 14 We say that two strategy profiles s and s' have the same game and we write $s =_g s'$ iff corecursively

- either $s = \langle\langle u \rangle\rangle$ and $s' = \langle\langle u \rangle\rangle$
- or $s = \langle\langle a, c, s_1, s_2 \rangle\rangle$ and $s' = \langle\langle a', c', s'_1, s'_2 \rangle\rangle$ and $a = a'$, $s_1 =_g s'_1$ and $s_2 =_g s'_2$.

Definition 15 (Rationality for finite or infinite strategy profiles) Rat_∞ is defined corecursively as follows.

- $\text{Rat}_\infty(\langle\langle u \rangle\rangle)$,
- $\text{Rat}_\infty(\langle\langle a, c, s_1, s_2 \rangle\rangle) \Leftrightarrow \exists \langle\langle a, c, s'_1, s'_2 \rangle\rangle \in \text{InfStratProf},$
 $\langle\langle a, c, s'_1, s'_2 \rangle\rangle =_g \langle\langle a, c, s_1, s_2 \rangle\rangle \wedge \text{SPE}(\langle\langle a, c, s'_1, s'_2 \rangle\rangle) \wedge \text{Rat}_\infty(s_c)$

The reader may notice the similarity with Definition 4 of rationality for finite games. The difference is twofold: the definition is corecursive instead of recursive and BI has been replaced by SPE. Let us now define a predicate that states the opposite of convergence ⁷

Definition 16 (Divergence) $\uparrow s$ is defined corecursively as follows:

- if $\uparrow s_1$ then $\uparrow \langle\langle a, 1, s_1, s_2 \rangle\rangle$,
- if $\uparrow s_2$ then $\uparrow \langle\langle a, 2, s_1, s_2 \rangle\rangle$.

$s_{\square 2}$ in Figure 1 is a typical divergent strategy profile. The main theorem of this paper can then be stated, saying that there exists a strategy profile that is both divergent and rational.

Theorem 17 (Risk of divergence) $\exists s \in \text{InfStratProf}, \text{Rat}_\infty(s) \wedge \uparrow s$.

Witnesses of divergent and rational strategy profiles will be given in Section 4.2 and Section 5.

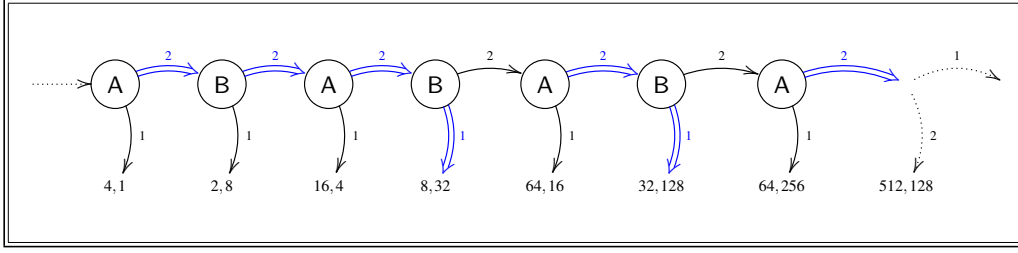
4 Extrapolating the centipede

As an illustration of the above concepts, we show, in this section, two simple extensions to infinity of a folklore example. The centipede has been proposed by Rosenthal [13]. Starting from a wording suggested by Aumann [1] we study two infinite generalization⁸. Wikipedia [24] says:

Consider two players: Alice and Bob. Alice moves first. At the start of the game, Alice has two piles of coins in front of her: one pile contains 4 coins and the other pile contains 1 coin. Each player has two moves available: either **”take”** the larger pile of coins and give the smaller pile to the other player or **”push”** both piles across the table to the other player. Each time the piles of coins pass across the table, the quantity of coins in each pile doubles.

⁷People used to coinduction know why it is better to define *divergence* directly instead of defining it as the negation of convergence.

⁸The reason why we call them *∞pede* and *ωpede*.

Figure 2: A sketch of a strategy profile of the ∞ pede.

4.1 The ∞ pede

$P_\infty(s)$ is a set of strategy profiles extending the strategy profiles of the centipede. Such an infinite strategy profile can only be sketched on Figure 2. Actually proposing an infinite extension of the centipede is quite natural for two reasons. First there is no natural way to make the game finite. Indeed in the definition of the game, nothing precise is said about its end, when no player decides to take a pile. For instance, Wikipedia [24] says:

The game continues for a fixed number of rounds or until a player decides to end the game by pocketing a pile of coins.

We do not know what the utilities are in the end position described as “a fixed number of rounds”. Since A started, we can assume that the end after a fixed number of rounds is B’s turn and that there are outcomes like:

1. B receives 2^{n+1} coins and A receives 2^{n+3} coins like for the previous B rounds and that is all.
2. B chooses between
 - (a) receiving 2^{n+1} coins whereas A receives 2^{n+3} or
 - (b) sharing with A, each one receiving 2^{n+2} .
3. Both A and B receive nothing.

Moreover the statement “Each player has two moves available: either “take” ... or push...” is not true, in the ending position. We are not hair-splitting since the end positions are the initializations of the (backward) induction and must be defined as precisely as the induction step. Ending with 2.(b) does not produce the same backward induction as the others. Let us consider the strategy profiles

$$\begin{aligned} p_n &= \langle \langle A, 1, \langle \langle A \mapsto 2^{2n+2}, B \mapsto 2^{2n} \rangle \rangle, \pi_n \rangle \rangle \\ \pi_n &= \langle \langle B, 1, \langle \langle A \mapsto 2^{2n+1}, B \mapsto 2^{2n+3} \rangle \rangle, p_{n+1} \rangle \rangle \end{aligned}$$

In words, the p_n ’s and the π_n ’s are the strategy subprofiles of the ∞ pede in which Alice and Bob stop always. Notice that

$$\begin{aligned} \hat{p}_n(A) &= 2^{2n+2} & \hat{p}_n(B) &= 2^{2n} \\ \hat{\pi}_n(A) &= 2^{2n+1} & \hat{\pi}_n(B) &= 2^{2n+3} \end{aligned}$$

Theorem 18

1. $\forall n \in \mathbb{N}, \text{SPE}(p_n) \wedge \text{SPE}(\pi_n),$

payroll. In other words, after ω , the piles keep the same size 2^ω . An example of strategy profile is sketched on Figure 4. In this family of strategy profiles, which we write P_ω , the utilities stay stable after the ω^{th} positions. Every *always-convergent* strategy profile of P_ω , such that agents **push** until ω is in SPE. We conclude the existence of rational divergent strategy profiles in P_ω . In other words in the ω pede there is a risk of divergence.

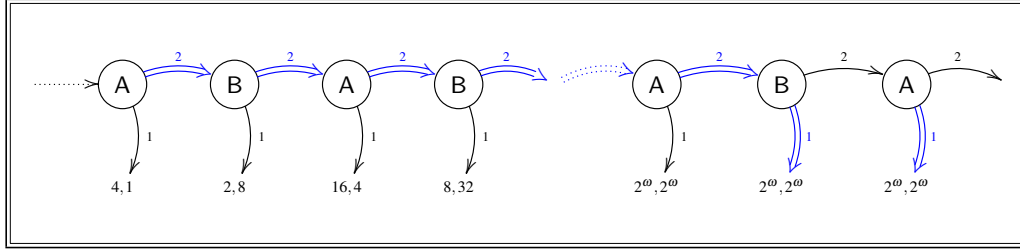


Figure 4: A ‘backward induction’ strategy profile for the ω pede.

Theorem 19 $\exists s \in P_\omega, \text{Rat}_\infty(s) \wedge \uparrow s$.

One may imagine that divergence is when optimistic agents hope a reverse of the tendency.

Comments: The ω pede example is degenerated, but it is interesting in two respects. First, it shows a very simple and naive case of rational divergence. Second it shows that cutting the infinite game, case 2. (b) is the most natural way, with an equilibrium in which agents **take** until the end.

5 Two examples

0, 1 strategy profiles 0, 1 strategy profiles are strategy profiles with the shape of an infinite “comb” in which the utilities are 0 for the agent who quits and 1 for the other agent. It can be shown [8] that strategy profiles where one agent continues always and the other quits infinitely often (in other words the other agent never continues always) are in SPE. For this reason, the strategy profile where both agents continue always is in Rat_∞ , which shows that divergence is rational.

The dollar auction The *dollar auction* is a well known game [16, 6, 11]. Its strategy profiles have the same infinite comb shape as the 0, 1 strategy profiles, the ∞ pede and the ω pede with the sequence of pairs of utilities:

$$(0, 100) (95, 0) (-5, 95) (90, -5) (-10, 90) (85, -10) \dots (-5n, 100 - 5n) (100 - 5(n+1), -5n) \dots$$

and corresponds to an auction in which the bet of the looser is not returned to her. We have shown [10] that the dollar auction may diverge with rational agents. People speak of *escalation* in this case. The divergent strategy profile of the dollar auction is in Rat_∞ .

6 Reflection

Je est un autre (I is another)
Arthur Rimbaud (1854-1891)

Examples like the dollar auction or the 0,1 raise the following question: “How is it possible in an escalation that the agents do not see that they are entering a hopeless process?”. The answer is “reflection”. Indeed, when reasoning, betting and choosing, the agents should leave the world where they live and act in order to observe the divergence. If they are wise, they change their beliefs in an infinite world as soon as they realize that they go nowhere [17]. This ability is called reflection and is connected to observability, from the theoretical computer science point of view, which is itself connected to coalgebras and to coinduction [5]. In other words, agents should leave the environment in which they are enclosed and observe themselves. Like the poet, they should be able to claim “*I is another*” whom I consider as an object.

7 Singularities and divergence

Divergence is called singularity, bubble, crash, escalation, or turbulence according to the context or the scientific field. In mechanics this is considered as a topic by itself. Leonardo da Vinci’s drawings (Fig. 5 left) show that he considered early turbulence and vortices and only Reynolds during the XIXth century studied it from a scientific point of view. In many other domains, phenomena of this family are rejected from the core of the field, despite they have been observed experimentally. Scientists, among them mainstream economists [2], prefer smoothness, continuity and equilibria [23] and they often claim that departing from this leads to “paradoxes” [16]. In [7], we surveyed Zeno of Elea’s paradox from the point of view of coinduction, as well as Weierstrass function [22], the first mathematical example showing discontinuity at the infinite. Here we would like to address two other cases. In 1935, that is one year before his



Figure 5: Da Vinci’s drawings (left) and a artist view of a blackhole (right) [Wikimedia commons](#)

famous article in the *Proceedings of the London Mathematical Society* [19], Alan Turing wrote a paper [18] presenting his result for a publication in the *Proceedings of the French Academy of Science*. In this paper he calls “nasty” a machine that terminates and “nice” a machine that does not terminate, showing his positive view of non ter-

minating computations.¹⁰ In 1795, Laplace published his book *Exposition du Système du Monde* and proposed the first clear vision of the notion of blackhole (Fig. 5 right), but probably in order not to hurt his contemporaries, he found wiser to remove this presentation from the third edition of his book. Then we had to wait Schwarzschild in 1915, few months after the publication by Einstein of the general theory of relativity, for a second proposal of the concept of blackhole. But at that time the general relativity was not yet fully accepted as were not blackholes. Only recently, at the end of the last century, the general relativity has been considered as “the” theory of gravitation and there is no more doubt on the existence of blackholes. Since blackholes are singularities in gravitation, they are for the general theory of relativity the equivalent of divergent strategy profiles for game theory.

Contribution of this paper

Unlike previous presentations of similar results [10, 8, 21] here we focus on the concept of strategy profile which is central for the those of convergence, of divergence and of equilibrium and is more targeted for a workshop on *strategy reasoning*. Moreover we introduce the *ωpede* (a new infinite version of the centipede) and “*divergent*” *strategy profiles* are those that were called “escalation” in previous literature. This terminology seems better fitted for its duality with convergence.

8 Conclusion

We have shown that strategy profiles in which no fixed limit is set must be studied as infinite objects using coinduction and corecursion. In these infinite objects, the risk of divergence is real and should be considered seriously.

References

- [1] Robert J. Aumann (1995): *Backward Induction and Common Knowledge of Rationality*. *Games and Economic Behavior* 8, pp. 6–19.
- [2] David Colander, Richard P. F. Holt & Jr Rosser, Barkley J. (2004): *The Changing Face of Mainstream Economics*. *Review of Political Economy* 16(4), p. 485–499, doi:10.1080/0953825042000256702.
- [3] Shaun Hargreaves-Heap & Yanis Varoufakis (2004): *Game Theory: A Critical Introduction*. Routledge.
- [4] Furio Honsell, Marina Lenisa & Rekha Redamalla (2012): *Categories of Coalgebraic Games*. In Branislav Rován, Vladimiro Sassone & Peter Widmayer, editors: *Mathematical Foundations of Computer Science 2012 - 37th International Symposium, MFCS 2012, Bratislava, Slovakia, August 27–31, 2012. Proceedings, Lecture Notes in Computer Science* 7464, Springer, pp. 503–515, doi:10.1007/978-3-642-32589-2_45. Available at http://dx.doi.org/10.1007/978-3-642-32589-2_45.
- [5] Bart Jacobs (2012): *Introduction to Coalgebra. Towards Mathematics of States and Observations*. Online book. Version 2.0.

¹⁰Notice that he changed his terminology in [18] and calls “circular” the terminating machine and “circular-free” the non terminating machine.

- [6] Wolfgang Leininger (1989): *Escalation and cooperation in conflict situations*. *J. of Conflict Resolution* 33, pp. 231–254.
- [7] Pierre Lescanne (2011): *Rationality and Escalation in Infinite Extensive Games*. CoRR abs/1112.1185. Available at <http://arxiv.org/abs/1112.1185>.
- [8] Pierre Lescanne (2013): *A Simple Case of Rationality of Escalation*. In Reiko Heckel & Stefan Milius, editors: *CALCO, Lecture Notes in Computer Science* 8089, Springer, pp. 191–204. Available at http://dx.doi.org/10.1007/978-3-642-40206-7_15.
- [9] Pierre Lescanne (2014): *Intelligent escalation and the principle of relativity*. Technical Report ensl-01096264v4, LIP - ENS Lyon.
- [10] Pierre Lescanne & Matthieu Perrinel (2012): *"Backward" coinduction, Nash equilibrium and the rationality of escalation*. *Acta Inf.* 49(3), pp. 117–137. Available at <http://dx.doi.org/10.1007/s00236-012-0153-3>.
- [11] Barry O'Neill (1986): *International escalation and the dollar auction*. *J. of Conflict Resolution* 30(33-50).
- [12] Martin J. Osborne (2004): *An Introduction to Game Theory*. Oxford.
- [13] R. W. Rosenthal (1981): *Games of perfect information, predatory pricing and the chain-store paradox*. *Journal of Economy Theory* 25(1), pp. 92–100.
- [14] Jan J. M. M. Rutten (2000): *Universal coalgebra: a theory of systems*. *Theor. Comput. Sci.* 249(1), pp. 3–80. Available at [http://dx.doi.org/10.1016/S0304-3975\(00\)00056-6](http://dx.doi.org/10.1016/S0304-3975(00)00056-6).
- [15] Reinhard Selten (1965): *Spieltheoretische Behandlung eines Oligopolmodells mit Nachfragerträglichkeit*. *Zeitschrift für gesamte Staatswissenschaft* 121.
- [16] Martin Shubik (1971): *The Dollar auction game: A paradox in noncooperative behavior and escalation*. *Journal of Conflict Resolution* 15(1), pp. 109–111.
- [17] Keith E. Stanovich (2009): *Two minds: Dual processis and beyond*, J. Evans and K. Frankish eds., chapter Distinguishing the reflective, algorithmic, and autonomous minds: Is it time for a tri-process theory? Oxford University Press.
- [18] A. M. Turing (1935): *First draft of précis of 'Computable Numbers', made for Comptes Rendues*. <http://www.turingarchive.org/browse.php/K/4>.
- [19] A. M. Turing (1936): *On Computable Numbers, with an Application to the Entscheidungsproblem*. *Proceedings of the London Mathematical Society. Second Series* 42, pp. 230–265.
- [20] René Vestergaard (2006): *A Constructive Approach to Sequential Nash Equilibria*. *Inf. Process. Lett.* 97, pp. 46–51.
- [21] René Vestergaard (2007): *Cascaded Games* 4545, pp. 185–201. Available at http://dx.doi.org/10.1007/978-3-540-73433-8_14.
- [22] Karl Weierstrass (1872): *Über continuirliche Funktionen eines reellen Arguments, die für keinen Werth des letzteren einen bestimmten Differentialquotienten besitzen*. in *Karl Weierstrass Mathematische Werke, Abhandlungen II Gelesen in der Königl. Akademie der Wissenschaften, am 18 Juli 1872*, pp. 71–74.
- [23] E. Roy Weintraub (2002): *Neoclassical Economics*. Library of Economics and Liberty. In David R. Henderson (ed.). *Concise Encyclopedia of Economics* (1st ed.).
- [24] Wikipedia (2015): *Centipede game* — *Wikipedia, The Free Encyclopedia*. Available at http://en.wikipedia.org/w/index.php?title=Centipede_game&oldid=663077006. [Online; accessed 25-May-2015].