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# PARAMETRIZING ELLIPTIC CURVES BY MODULAR UNITS

by

François Brunault

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**Abstract.** — It is well-known that every elliptic curve over the rationals admits a parametrization by means of modular functions. In this short note, we show that only finitely many elliptic curves over  $\mathbf{Q}$  can be parametrized by modular units. This answers a question raised by Zudilin in a recent work on Mahler measures. Further, we give the list of all elliptic curves  $E$  of conductor up to 1000 parametrized by modular units supported in the rational torsion subgroup of  $E$ . Finally, we raise several open questions.

Since the work of Boyd [3], Deninger [6] and others, it is known that there is a close relationship between Mahler measures of polynomials and special values of  $L$ -functions. Although this relationship is still largely open, some strategies have been identified in several instances. Specifically, let  $P \in \mathbf{Q}[x, y]$  be a polynomial whose zero locus defines an elliptic curve  $E$ . If the polynomial  $P$  is tempered, then the Mahler measure of  $P$  can be expressed in terms of a regulator integral

$$(1) \quad \int_{\gamma} \log |x| d \arg(y) - \log |y| d \arg(x)$$

where  $\gamma$  is a (non necessarily closed) path on  $E$  (see [6, 12]). If the curve  $E$  happens to have a parametrization by *modular units*  $x(\tau)$ ,  $y(\tau)$ , then we may change to the variable  $\tau$  in (1) and try to compute the regulator integral using [12, Thm 1]. In favourable cases, this leads to an identity between the Mahler measure of  $P$  and  $L(E, 2)$ : see for example [12, §3] and the references therein. The following natural question, raised by Zudilin, thus arises:

*Which elliptic curves can be parametrized by modular units?*

We show in Section 1 that only finitely many elliptic curves over  $\mathbf{Q}$  can be parametrized by modular units. The proof uses Watkins' lower bound on the modular degree of elliptic curves. Further, we give in Section 2 the list of all elliptic curves  $E$  of conductor up to 1000 parametrized by modular units supported in the rational torsion subgroup of  $E$ . It turns out that there are 30 such elliptic curves. Finally, we raise in Section 3 several open questions.

## 1. A finiteness result

**Definition 1.** — Let  $E/\mathbf{Q}$  be an elliptic curve of conductor  $N$ . We say that  $E$  can be *parametrized by modular units* if there exist two modular units  $u, v \in \mathcal{O}(Y_1(N))^\times$  such that the function field  $\mathbf{Q}(E)$  is isomorphic to  $\mathbf{Q}(u, v)$ .

**Theorem 2.** — *There are only finitely many elliptic curves over  $\mathbf{Q}$  which can be parametrized by modular units.*

Let  $E/\mathbf{Q}$  be an elliptic curve of conductor  $N$ . Assume that  $E$  can be parametrized by two modular units  $u, v$  on  $Y_1(N)$ . Then there is a finite morphism  $\varphi : X_1(N) \rightarrow E$  and two rational functions  $f, g \in \mathbf{Q}(E)^\times$  such that  $\varphi^*(f) = u$  and  $\varphi^*(g) = v$ .

Let  $E_1$  be the  $X_1(N)$ -optimal elliptic curve in the isogeny class of  $E$ , and let  $\varphi_1 : X_1(N) \rightarrow E_1$  be an optimal parametrization. By [9, Prop 1.4], there exists an isogeny  $\lambda : E_1 \rightarrow E$  such that  $\varphi = \lambda \circ \varphi_1$ . Consider the functions  $f_1 = \lambda^*(f)$  and  $g_1 = \lambda^*(g)$ . Note that  $u = \varphi_1^*(f_1)$  and  $v = \varphi_1^*(g_1)$ . Theorem 2 is now a consequence of the following result.

**Theorem 3.** — *If  $N$  is sufficiently large, then  $\varphi_1^*(\mathbf{Q}(E_1)) \cap \mathcal{O}(Y_1(N)) = \mathbf{Q}$ .*

*Proof.* — Let  $C_1(N)$  be the set of cusps of  $X_1(N)$ . Let  $f \in \mathbf{Q}(E_1) \setminus \mathbf{Q}$  be such that  $\varphi_1^*(f) \in \mathcal{O}(Y_1(N))$ . Let  $P$  be a pole of  $f$ . Then  $\varphi_1^{-1}(P)$  must be contained in  $C_1(N)$ , and we have

$$\deg \varphi_1 = \sum_{Q \in \varphi_1^{-1}(P)} e_{\varphi_1}(Q) \leq \sum_{Q \in C_1(N)} e_{\varphi_1}(Q).$$

Let  $g_N$  be the genus of  $X_1(N)$ . By the Riemann-Hurwitz formula for  $\varphi_1$ , we have

$$2g_N - 2 = \sum_{Q \in X_1(N)} (e_{\varphi_1}(Q) - 1).$$

It follows that

$$\begin{aligned} \deg \varphi_1 &\leq \#C_1(N) + \sum_{Q \in C_1(N)} (e_{\varphi_1}(Q) - 1) \\ &\leq \#C_1(N) + 2g_N - 2. \end{aligned}$$

By the classical genus formula [8, Prop 1.40], and since  $X_1(N)$  has no elliptic points for  $N \geq 4$ , we have

$$\#C_1(N) + 2g_N - 2 = \frac{1}{12} [\mathrm{SL}_2(\mathbf{Z}) : \Gamma_1(N)] = \frac{\phi(N)\nu(N)}{12} \quad (N \geq 4)$$

where  $\phi(N)$  denotes Euler's function, and  $\nu(N)$  is defined by

$$\nu(N) = N \prod_{i=1}^k \left(1 + \frac{1}{p_i}\right) \quad \text{if } N = \prod_{i=1}^k p_i^{\alpha_i}.$$

We thus get

$$(2) \quad \deg \varphi_1 \leq \frac{\phi(N)\nu(N)}{12}.$$

We are now going to show that (2) contradicts lower bounds of Watkins [11] on the modular degree if  $N$  is sufficiently large. Let  $E_0$  be the strong Weil curve in the isogeny class of  $E$ . We have a commutative diagram

$$(3) \quad \begin{array}{ccc} X_1(N) & \xrightarrow{\pi} & X_0(N) \\ \downarrow \varphi_1 & & \downarrow \varphi_0 \\ E_1 & \xrightarrow{\lambda_0} & E_0. \end{array}$$

We deduce that

$$\deg \varphi_1 = \frac{\deg \pi \cdot \deg \varphi_0}{\deg \lambda_0}.$$

We have  $\deg \pi = \frac{\phi(N)}{2}$ . For every  $\alpha \in (\mathbf{Z}/N\mathbf{Z})^\times / \pm 1$ , there exists a unique point  $A(\alpha) \in E_1(\mathbf{Q})_{\mathrm{tors}}$  such that  $\varphi_1 \circ \langle \alpha \rangle = t_{A(\alpha)} \circ \varphi_1$ , where  $t_{A(\alpha)}$  denotes translation by  $A(\alpha)$ . The map  $\alpha \mapsto A(\alpha)$  is a morphism of groups and its image is  $\ker(\lambda_0)$ . It follows that  $\deg(\lambda_0) \leq \#E_1(\mathbf{Q})_{\mathrm{tors}} \leq 16$ . By [11], we have  $\deg \varphi_0 \gg N^{7/6-\varepsilon}$  for any  $\varepsilon > 0$ . It follows that  $\deg \varphi_1 \gg \phi(N)N^{7/6-\varepsilon}$ . Since  $\nu(N) \ll N^{1+\varepsilon}$  for any  $\varepsilon > 0$ , this contradicts (2) for  $N$  sufficiently large.  $\square$

**Remark 4.** — It would be interesting to determine the complete list of elliptic curves over  $\mathbf{Q}$  parametrized by modular units. Unfortunately, the bound on the conductor  $N$  provided by Watkins' result, though effective, is too large to permit an exhaustive search. However, we observed numerically in [4] that the ramification index of  $\varphi_0$  at a cusp of  $X_0(N)$  seems to be always a divisor of 24. If this observation is true, then we can replace (2) by the better bound  $\deg \varphi_1 \leq 12\phi(N) \sum_{d|N} \phi((d, N/d))$ . Combining this with known linear lower bounds on  $\deg \varphi_0$  (see [11]), this yields a better (but still large) bound on  $N$ . Furthermore, if we restrict to semistable elliptic curves, then  $\varphi_0$ ,  $\pi$  and  $\varphi_1$  are unramified at the cusps; in this case we get that  $N$  has at most 6 prime factors and  $N \leq 233310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 101$ .

## 2. Preimages of torsion points under modular parametrizations

In order to find elliptic curves parametrized by modular units, we consider the following related problem. Let  $E$  be an elliptic curve over  $\mathbf{Q}$  of conductor  $N$ , and let  $\varphi : X_1(N) \rightarrow E$  be a modular parametrization sending the 0-cusp to 0. By the Manin-Drinfeld theorem, the image by  $\varphi$  of a cusp of  $X_1(N)$  is a torsion point of  $E$ . Conversely, given a point  $P \in E_{\text{tors}}$ , when does the preimage of  $P$  under  $\varphi$  consist only of cusps? The link between this question and parametrizations by modular units is given by the following easy lemma.

**Lemma 5.** — *Suppose that there exists a subset  $S$  of  $E(\mathbf{Q})_{\text{tors}}$  satisfying the following two conditions:*

- (1) *We have  $\varphi^{-1}(S) \subset C_1(N)$ .*
- (2) *There exist two functions  $f, g$  on  $E$  supported in  $S$  such that  $\mathbf{Q}(E) = \mathbf{Q}(f, g)$ .*

*Then  $E$  can be parametrized by modular units.*

*Proof.* — By condition (1), the functions  $u = \varphi^*(f)$  and  $v = \varphi^*(g)$  are modular units of level  $N$ , and by condition (2), we have  $\mathbf{Q}(E) \cong \mathbf{Q}(u, v)$ .  $\square$

We are therefore led to search for elliptic curves  $E/\mathbf{Q}$  admitting sufficiently many torsion points  $P$  such that  $\varphi^{-1}(P) \subset C_1(N)$ .

We first give an equivalent form of condition (2) in Lemma 5.

**Proposition 6.** — *Let  $S$  be a subset of  $E(\mathbf{Q})_{\text{tors}}$ . Let  $\mathcal{F}_S$  be the set of nonzero functions  $f$  on  $E$  which are supported in  $S$ . The following conditions are equivalent:*

- (a) *There exist two functions  $f, g \in \mathcal{F}_S$  such that  $\mathbf{Q}(E) = \mathbf{Q}(f, g)$ .*
- (b) *The field  $\mathbf{Q}(E)$  is generated by  $\mathcal{F}_S$ .*
- (c) *We have  $\#S \geq 3$ , and there exist two points  $P, Q \in S$  such that  $P - Q$  has order  $\geq 3$ .*

In order to prove Proposition 6, we show the following lemma.

**Lemma 7.** — *Let  $P \in E(\mathbf{Q})_{\text{tors}}$  be a point of order  $n \geq 2$ . Let  $f_P$  be a function on  $E$  such that  $\text{div}(f_P) = n(P) - n(0)$ . Then the extension  $\mathbf{Q}(E)/\mathbf{Q}(f_P)$  has no intermediate subfields. Moreover, if  $P, P' \in E(\mathbf{Q})_{\text{tors}}$  are points of order  $n \geq 4$  such that  $\mathbf{Q}(f_P) = \mathbf{Q}(f_{P'})$ , then  $P = P'$ .*

*Proof.* — Let  $K$  be a field such that  $\mathbf{Q}(f_P) \subset K \subset \mathbf{Q}(E)$ . If  $K$  has genus 1, then  $K$  is the function field of an elliptic curve  $E'/\mathbf{Q}$  and  $f_P$  factors through an isogeny  $\lambda : E \rightarrow E'$ . Then  $\text{div}(f_P)$  must be invariant under translation by  $\ker(\lambda)$ . This obviously implies  $\ker(\lambda) = 0$ , hence  $K = \mathbf{Q}(E)$ . If  $K$  has genus 0, then we have  $K = \mathbf{Q}(h)$  for some function  $h$  on  $E$ , and we may factor  $f_P$  as  $g \circ h$  with  $g : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ . We may assume  $h(P) = 0$  and  $h(0) = \infty$ . Then  $g^{-1}(0) = \{0\}$  and  $g^{-1}(\infty) = \{\infty\}$ , which implies  $g(t) = at^m$  for some  $a \in \mathbf{Q}^\times$  and  $m \geq 1$ .

Thus  $\operatorname{div}(f) = m \operatorname{div}(h)$ . Since  $\operatorname{div}(h)$  must be a principal divisor, it follows that  $m = 1$  and  $K = \mathbf{Q}(f_P)$ .

Let  $P, P' \in E(\mathbf{Q})$  be points of order  $n \geq 4$  such that  $\mathbf{Q}(f_P) = \mathbf{Q}(f_{P'})$  and  $P \neq P'$ . Then  $f_{P'} = (af_P + b)/(cf_P + d)$  for some  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{GL}_2(\mathbf{Q})$ . Considering the divisors of  $f_P$  and  $f_{P'}$ , we must have  $f_{P'} = af_P + b$  for some  $a, b \in \mathbf{Q}^\times$ . Then the ramification indices of  $f_P : E \rightarrow \mathbf{P}^1$  at  $P, P', 0$  are equal to  $n$ , which contradicts the Riemann-Hurwitz formula for  $f_P$ .  $\square$

*Proof of Proposition 6.* — It is clear that (a) implies (b). Let us show that (b) implies (c). If  $\#S \leq 2$ , then  $\mathcal{F}_S/\mathbf{Q}^\times$  has rank at most 1 and cannot generate  $\mathbf{Q}(E)$ . Assume that for every points  $P, Q \in S$ , we have  $P - Q \in E[2]$ . Translating  $S$  if necessary, we may assume  $0 \in S$ . It follows that  $S \subset E[2]$  and  $\mathcal{F}_S \subset \mathbf{Q}(x) \not\subset \mathbf{Q}(E)$ .

Finally, let us assume (c). Translating  $S$  if necessary, we may assume  $0 \in S$ . Let us first assume that  $S$  contains a point  $P$  of order 2. Then  $\mathbf{Q}(f_P) = \mathbf{Q}(x)$  has index 2 in  $\mathbf{Q}(E)$  and is the fixed field with respect to the involution  $\sigma : p \mapsto -p$  on  $E$ . By assumption, there exist two points  $Q, R \in S$  such that  $Q - R$  has order  $n \geq 3$ . Let  $g$  be a function on  $E$  such that  $\operatorname{div}(g) = n(Q) - n(R)$ . Then it is easy to see that  $\operatorname{div}(g)$  is not invariant under  $\sigma$ . It follows that  $g \notin \mathbf{Q}(f_P)$  and  $\mathbf{Q}(f_P, g) = \mathbf{Q}(E)$ . Let us now assume that  $S \cap E[2] = \{0\}$ . By assumption,  $S$  contains two distinct points  $P, Q$  having order  $\geq 3$ . If  $P$  or  $Q$  has order  $\geq 4$ , then Lemma 7 implies that  $\mathbf{Q}(f_P, f_Q) = \mathbf{Q}(E)$ . If  $P$  and  $Q$  have order 3, then we must have  $Q = -P$  because  $\mathbf{Q}(E[3])$  contains  $\mathbf{Q}(\zeta_3)$ . It follows that the function  $g$  on  $E$  defined by  $\operatorname{div}(g) = (P) + (-P) - 2(0)$  has degree 2, so we have  $g \notin \mathbf{Q}(f_P)$  and  $\mathbf{Q}(f_P, g) = \mathbf{Q}(E)$ .  $\square$

Let  $E/\mathbf{Q}$  be an elliptic curve of conductor  $N$ . Fix a Néron differential  $\omega_E$  on  $E$ , and let  $f_E$  be the newform of weight 2 and level  $N$  associated to  $E$ . We define  $\omega_{f_E} = 2\pi i f_E(z) dz$ . Let  $\varphi_E : X_1(N) \rightarrow E$  be a modular parametrization of minimal degree. We have  $\varphi_E^* \omega_E = c_E \omega_{f_E}$  for some integer  $c_E \in \mathbf{Z} - \{0\}$  [9, Thm 1.6], and we normalize  $\varphi_E$  so that  $c_E > 0$ . Conjecturally, we have  $c_E = 1$  [9, Conj. I].

We now describe an algorithm to compute the set  $S_E$  of points  $P \in E(\mathbf{Q})_{\text{tors}}$  such that  $\varphi_E^{-1}(P) \subset C_1(N)$ . Let  $P \in E(\mathbf{Q})_{\text{tors}}$ . We define an integer  $e_P$  by

$$e_P = \sum_{\substack{x \in C_1(N) \\ \varphi_E(x) = P}} e_{\varphi_E}(x).$$

It is clear that  $\varphi_E^{-1}(P) \subset C_1(N)$  if and only if  $e_P = \deg \varphi_E$ . Let  $d$  be a divisor of  $N$ , and let  $C_d$  be the set of cusps of  $X_1(N)$  of denominator  $d$  (that is, the set of cusps  $\frac{a}{b}$  satisfying  $(b, N) = d$ ). Every cusp  $x \in C_d$  can be written (non uniquely) as  $x = \langle \alpha \rangle \sigma(\frac{1}{d})$  with  $\alpha \in (\mathbf{Z}/N\mathbf{Z})^\times / \pm 1$  and  $\sigma \in \operatorname{Gal}(\mathbf{Q}(\zeta_d)/\mathbf{Q})$ . Since  $e_{\varphi_E}(x) = e_{\varphi_1}(x) = e_{\varphi_1}(1/d)$ , we get

$$e_P = \sum_{d|N} e_{\varphi_1}(1/d) \cdot \#\{x \in C_d : \varphi_E(x) = P\}.$$

Recall that for each  $\alpha \in (\mathbf{Z}/N\mathbf{Z})^\times$ , there exists a unique point  $A(\alpha) \in E(\mathbf{Q})_{\text{tors}}$  such that  $\varphi_E \circ \langle \alpha \rangle = t_{A(\alpha)} \circ \varphi_E$ , where  $t_{A(\alpha)}$  denotes translation by  $A(\alpha)$ . We let  $A_E \subset E(\mathbf{Q})_{\text{tors}}$  be the image of the map  $\alpha \mapsto A(\alpha)$ . Note that the set  $\{x \in C_d : \varphi_E(x) = P\}$  is empty unless  $\varphi_E(1/d) \in P + A_E$ , in which case we have  $\varphi_E(C_d) = P + A_E$  and the number of cusps  $x \in C_d$  such that  $\varphi_E(x) = P$  is given by  $\#C_d/\#A_E$ . Thus we get

$$e_P = \frac{1}{\#A_E} \sum_{\substack{d|N \\ \varphi_E(1/d) \in P + A_E}} e_{\varphi_1}(1/d) \cdot \#C_d.$$

Furthermore, let  $\pi : X_1(N) \rightarrow X_0(N)$  and  $\varphi_0 : X_0(N) \rightarrow E_0$  be the maps as in (3). The ramification index of  $\pi$  at  $\frac{1}{d}$  is equal to  $(d, N/d)$ . Thus  $e_{\varphi_1}(1/d) = (d, N/d) \cdot e_{\varphi_0}(1/d)$ . The quantity  $e_{\varphi_0}(1/d)$  is equal to the order of vanishing of  $\omega_{f_E}$  at the cusp  $1/d$ , and may be computed numerically (see [4, §7]). Moreover, the number of cusps of  $X_0(N)$  of denominator  $d$  is given by  $\phi((d, N/d))$ . It follows that  $\#C_d = \phi((d, N/d)) \cdot \phi(N)/(2(d, N/d))$  and we get

$$(4) \quad e_P = \frac{\phi(N)}{2\#A_E} \sum_{\substack{d|N \\ \varphi_E(1/d) \in P+A_E}} e_{\varphi_0}(1/d) \cdot \phi((d, N/d)).$$

Finally, using notations from Section 1, the modular degree of  $E$  may be computed as

$$(5) \quad \deg \varphi_E = \frac{\phi(N)}{2} \cdot \frac{\text{covol}(\Lambda_{E_0})}{\text{covol}(\Lambda_E)} \cdot \deg \varphi_0$$

where  $\Lambda_{E_0}$  and  $\Lambda_E$  denote the Néron lattices of  $E_0$  and  $E$ . We read off the modular degree  $\deg \varphi_0$  from Cremona's tables [5, Table 5]. Formulas (4) and (5) lead to the following algorithm.

- (1) Compute generators  $\alpha_1, \dots, \alpha_r$  of  $(\mathbf{Z}/N\mathbf{Z})^\times$ .
- (2) For each  $j$ , compute numerically  $\int_{z_0}^{(\alpha_j)z_0} \omega_{f_E}$  for  $z_0 = (-\alpha_j + i)/N$ .
- (3) Deduce  $A_j = A(\alpha_j) \in E(\mathbf{Q})_{\text{tors}}$ .
- (4) Compute the subgroup  $A_E$  generated by  $A_1, \dots, A_r$ .
- (5) Compute the list  $(P_1, \dots, P_n)$  of all rational torsion points on  $E$ .
- (6) Initialize a list  $(e_{P_1}, \dots, e_{P_n}) = (0, \dots, 0)$ .
- (7) For each  $d$  dividing  $N$ , do the following:
  - (a) Compute numerically  $z_d = \int_0^{1/d} \omega_{f_E}$ .
  - (b) Check whether the point  $Q_d = \varphi_E(1/d)$  is rational or not.
  - (c) If  $Q_d$  is rational, then do the following:
    - (i) Compute numerically  $e_{\varphi_0}(1/d)$ .
    - (ii) For each  $B \in A_E$ , do  $e_{Q_d+B} \leftarrow e_{Q_d+B} + e_{\varphi_0}(1/d)\phi((d, N/d))$ .
- (8) Output  $S_E = \{P \in E(\mathbf{Q})_{\text{tors}} : e_P = \#A_E \cdot \frac{\text{covol}(\Lambda_{E_0})}{\text{covol}(\Lambda_E)} \cdot \deg \varphi_0\}$ .

The following table gives all elliptic curves  $E$  of conductor  $\leq 1000$  such that  $S_E$  satisfies condition (c) of Proposition 6. Computations were done using Pari/GP [10] and the Modular Symbols package of Magma [2].

**Remarks 8.** — (1) In order to compute the points  $A_j$  in step (3) and  $Q_d$  in step (7b), we implicitly make use of Stevens' conjecture that  $c_E = 1$ . This conjecture is known for all elliptic curves of conductor  $\leq 200$  [9].

(2) Of course, steps (2), (7a) and (7ci) are done only once for each isogeny class.

(3) If  $x$  is a cusp of  $X_1(N)$ , then the order of  $\varphi_E(x)$  is bounded by the exponent of the cuspidal subgroup of  $J_1(N)$ . Hence we may ascertain that  $\varphi_E(x)$  is rational or not by a finite computation.

(4) We compute  $e_{\varphi_0}(\frac{1}{d})$  by a numerical method. It would be better to use an exact method.

### 3. Further questions

Note that in Lemma 5, we considered functions on  $E$  which are supported in  $E(\mathbf{Q})_{\text{tors}}$ . In general, the image by  $\varphi_E$  of a cusp of  $X_1(N)$  is only rational over  $\mathbf{Q}(\zeta_N)$ , and we may use functions on  $E$  supported in these non-rational points. In fact, let  $S'_E$  denote the set of points  $P \in E(\mathbf{Q}(\zeta_N))_{\text{tors}}$  such that  $\varphi_E^{-1}(P) \subset C_1(N)$ . The set  $S'_E$  is stable under the action of  $\text{Gal}(\mathbf{Q}(\zeta_N)/\mathbf{Q})$ . Then  $E$  can be parametrized by modular units if *and only if* there exist two

$E$	$E(\mathbf{Q})_{\text{tors}}$	$S_E$	$E$	$E(\mathbf{Q})_{\text{tors}}$	$S_E$
11a3	$\mathbf{Z}/5\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$	26a3	$\mathbf{Z}/3\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$
14a1	$\mathbf{Z}/6\mathbf{Z}$	$\{0, (9, 23), (1, -1), (2, -5)\}$	27a3	$\mathbf{Z}/3\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$
14a4	$\mathbf{Z}/6\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$	27a4	$\mathbf{Z}/3\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$
14a6	$\mathbf{Z}/6\mathbf{Z}$	$\{0, (2, -2), (2, -1)\}$	30a1	$\mathbf{Z}/6\mathbf{Z}$	$\{0, (3, 4), (-1, 0), (0, -2)\}$
15a1	$\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$	$\{0, (-2, 3), (-1, 0), (8, 18)\}$	32a1	$\mathbf{Z}/4\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$
15a3	$\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$	$\{0, (0, 1), (1, -1), (0, -2)\}$	32a4	$\mathbf{Z}/4\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$
15a8	$\mathbf{Z}/4\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$	35a3	$\mathbf{Z}/3\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$
17a4	$\mathbf{Z}/4\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$	36a1	$\mathbf{Z}/6\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$
19a3	$\mathbf{Z}/3\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$	36a2	$\mathbf{Z}/6\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$
20a1	$\mathbf{Z}/6\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$	40a3	$\mathbf{Z}/4\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$
20a2	$\mathbf{Z}/6\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$	44a1	$\mathbf{Z}/3\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$
21a1	$\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$	$\{0, (-1, -1), (-2, 1), (5, 8)\}$	54a3	$\mathbf{Z}/3\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$
24a1	$\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$	56a1	$\mathbf{Z}/4\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$
24a3	$\mathbf{Z}/4\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$	92a1	$\mathbf{Z}/3\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$
24a4	$\mathbf{Z}/4\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$	108a1	$\mathbf{Z}/3\mathbf{Z}$	$E(\mathbf{Q})_{\text{tors}}$

TABLE 1. Some elliptic curves parametrized by modular units

functions  $f, g \in \mathbf{Q}(E)^\times$  supported in  $S'_E$  such that  $\mathbf{Q}(E) = \mathbf{Q}(f, g)$ . As the next example shows, this yields new elliptic curves parametrized by modular units.

**Example 9.** — Consider the elliptic curve  $E = X_0(49) = 49a1 : y^2 + xy = x^3 - x^2 - 2x - 1$ . The group  $E(\mathbf{Q})_{\text{tors}}$  has order 2 and is generated by the point  $Q = (2, -1)$ , which is none other than the cusp  $\infty$  (recall that the cusp 0 is the origin of  $E$ ). The set  $S'_E$  consists of all cusps of  $X_0(49)$ . Let  $P$  be the cusp  $\frac{1}{7}$ . It is defined over  $\mathbf{Q}(\zeta_7)$  and its Galois conjugates are given by  $\{P^\sigma\}_\sigma = \{P, 3P + Q, -5P, -P + Q, -3P, 5P + Q\}$ . There exists a function  $v \in \mathbf{Q}(E)$  of degree 7 such that  $\text{div}(v) = \sum(P^\sigma) + (Q) - 7(0)$ . Since  $x - 2$  and  $v$  have coprime degrees, the curve  $E$  can be parametrized by the modular units  $u = x - 2$  and  $v$ .

**Example 10.** — Consider the elliptic curve  $E = 64a1 : y^2 = x^3 - 4x$ . Its rational torsion subgroup is given by  $E(\mathbf{Q})_{\text{tors}} \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ . There is a degree 2 morphism  $\varphi_0 : X_0(64) \rightarrow E$ , and we have  $S_E = E(\mathbf{Q})_{\text{tors}}$ . However, the image of the cusp  $\frac{1}{8}$  is given by  $P = \varphi_0(\frac{1}{8}) = (2i, -2\sqrt{2} + 2i\sqrt{2})$ . This point is defined over  $\mathbf{Q}(\zeta_8)$  and we have  $S'_E = S_E \cup \{P^\sigma\}_\sigma$ . We can check that  $\mathcal{F}_{S'_E}/\mathbf{Q}^\times$  is generated by  $x, x \pm 2$  and  $x^2 + 4$ , hence it cannot generate  $\mathbf{Q}(E)$ . However, if we base change to the field  $\mathbf{Q}(\sqrt{2})$ , then we find that the function  $v = y - \sqrt{2}x + 2\sqrt{2}$  is supported in  $S'_E$  and has degree 3. Hence  $E/\mathbf{Q}(\sqrt{2})$  can be parametrized by the modular units  $u = x$  and  $v$ .

Example 10 suggests the following question : which elliptic curves  $E/\mathbf{Q}$  of conductor  $N$  can be parametrized by modular units *defined over*  $\mathbf{Q}(\zeta_N)$ ? The argument in Section 1, which is of geometrical nature, shows that  $S'_E$  is empty if  $N$  is sufficiently large; however, it crucially uses the fact that the modular parametrization  $X_1(N) \rightarrow E$  is defined over  $\mathbf{Q}$ .

Finally, here are several questions to which I don't know the answer.

**Question 11.** — Let  $E/\mathbf{Q}$  be an elliptic curve of conductor  $N$ . Assume  $E$  can be parametrized by modular units of some level  $N'$  (not necessarily equal to  $N$ ). Then we have a non-constant morphism  $X_1(N') \rightarrow E$  and  $N$  must divide  $N'$ . Does it necessarily follow that  $E$  admits a parametrization by modular units of level  $N$ ? In other words, does it make a difference if we

allow modular units of arbitrary level in Definition 1? Similarly, does it make a difference if we replace  $Y_1(N)$  by  $Y(N)$  or  $Y(N')$  in Definition 1?

**Question 12.** — Does it make a difference if we allow the function field of  $E$  to be generated by more than two modular units in Definition 1?

**Question 13.** — What about elliptic curves over  $\mathbf{C}$ ? It is not hard to show that if  $E/\mathbf{C}$  can be parametrized by modular functions, then  $E$  must be defined over  $\overline{\mathbf{Q}}$ . In fact, by the proof of Serre’s conjecture due to Khare and Wintenberger, it is known that the elliptic curves over  $\overline{\mathbf{Q}}$  which can be parametrized by modular functions are precisely the  $\mathbf{Q}$ -curves [7]. Which  $\mathbf{Q}$ -curves can be parametrized by modular units?

**Question 14.** — It is conjectured in [1] that only finitely many smooth projective curves over  $\mathbf{Q}$  of given genus  $g \geq 2$  can be parametrized by modular functions. Is it possible to prove, at least, that only finitely many smooth projective curves over  $\mathbf{Q}$  of given genus  $g \geq 2$  can be parametrized by modular units?

**Question 15.** — According to [1], there are exactly 213 curves of genus 2 over  $\mathbf{Q}$  which are new and modular, and they can be explicitly listed. Which of them can be parametrized by modular units?

**Question 16.** — Let  $u$  and  $v$  be two multiplicatively independent modular units on  $Y_1(N)$ . Assume that  $u$  and  $v$  do not come from modular units of lower level. Can we find a lower bound for the genus of the function field generated by  $u$  and  $v$ ?

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