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# About the relative entropy method for hyperbolic systems of conservation laws

Denis Serre*, Alexis F. Vasseur†

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**Abstract**

We review the relative entropy method in the context of first-order hyperbolic systems of conservation laws, in one space-dimension. We prove that contact discontinuities in full gas dynamics are uniformly stable. Generalizing this calculus, we derive an infinite-dimensional family of Lyapunov functions for the system of full gas dynamics.

## 1 Systems of conservation laws and entropies

We are interested in vector fields $u(x,t)$ obeying first-order PDEs. The space variable $x$ and time $t$ run over the physical domain $\mathbb{R}^d \times (0,T)$. The field takes values in a convex open subset $\mathcal{U}$ of $\mathbb{R}^n$.

A *conservation law* is a first-order PDE of the form

$$\partial_t a + \mathrm{div}_x \vec{b} = 0.$$

The terminology refers to the fact that weak solutions obey the identity

$$\frac{d}{dt}\int_\Omega a(x,t)\,dx + \int_{\partial\Omega} \vec{b}\cdot\nu\,ds(x) = 0,$$

for every regular open subdomain $\Omega \subset \mathbb{R}^d$. Hereabove $\nu$ is the outer normal and $ds$ is the area element over the boundary. Actually, the PDE is often derived from the latter identity, which expresses a physical principle such as conservation of mass, momentum, species, charge, energy, ... See C. Dafermos's book [9] for a thorough description of this correspondance.

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A given physical process is modelled by one or several conservation laws, which are written in a compact form as

$$\partial_t u + \mathrm{Div}_x F = 0,$$

where $u$ is the field of conserved quantities, taking values in some open subset $\mathcal{U}$ of $\mathbb{R}^n$, and $F$, the flux, takes values in $(n \times d)$-matrices. Mind that the operator $\mathrm{Div}_x$ with a capital letter represents the row-wise divergence of a matrix-valued field. The system is closed when $F$ is given as a function of $u$, in the form $F = f(u)$ where $f : \mathcal{U} \to \mathbb{R}^{n\times d}$ is a given smooth field. We then have

$$\partial_t u + \mathrm{Div}_x f(u) = 0, \tag{1}$$

which must be understood in the sense of distributions. One speaks of a *scalar* equation if $n = 1$, and of a *system* if $n \geq 2$. Mind that such a closure excludes dissipation processes such as viscous effects or thermal diffusion. Instead, it assumes a thermal equilibrium everywhere at any time.

The processes that we have in mind obey a kind of second principle of thermodynamics. Mathematically, this can be expressed in terms of one additional conservation law

$$\partial_t \eta(u) + \mathrm{div}\vec{q}(u) = 0. \tag{2}$$

We say that a function $\eta$ satisfying (2) for every $C^1$-solution of (1) is a *mathematical entropy* of system (1), and $\vec{q}$ is its *entropy flux*. In order that (2) be compatible with (1), we have the linear differential relations

$$\frac{\partial q_\alpha}{\partial u_j} = \sum_k \frac{\partial \eta}{\partial u_k} \frac{\partial f_\alpha^k}{\partial u_j}, \qquad \alpha = 1, \ldots, d,\ j = 1, \ldots, n. \tag{3}$$

In order that (2) brings some new information, it has to follow from (1) in a non-trivial way, by multiplying the system by the differential form $\mathrm{d}\eta(u)$. In particular, $\eta$ should not be affine ; affine functions are called *trivial* entropies. When $n \geq 3$, or when $n, d \geq 2$, the number of constraints in (3) exceeds that of the unknown functions $\eta, q_\alpha$, and therefore non-trivial entropies are exceptional objects : if a system (1) is chosen at random, it should not admit any non-trivial entropy. However, it is a well documented fact that systems modelling physics do admit at least one non-trivial entropy $\eta$ ; somehow, they are exceptional. It turns out that most of them admit only this entropy, in the sense that the solution set of (3) is spanned by $\eta$ and the trivial entropies. Mind that this is not true if either $n = 1$, or when $n = 2$ and $d = 1$, because then the number of constraints does not exceed the number of unknowns.

The additional entropy $\eta$ is often a convex smooth function (and then the phase space $\mathcal{U}$ is convex), strongly convex in the sense that the Hessian matrix $\mathrm{D}^2\eta_a$ is positive definite at every $a \in \mathcal{U}$. It may happen that $\eta$ be non convex, when (1) is an artificial first-order form of a

system of second-order PDEs in several space dimensions, but then $\eta$ has a related *quasi-convex* property. For the sake of simplicity, we shall assume throughout this paper that $\eta$ is strongly convex. For a more general situation, one refers to the works [6, 17].

As mentionned above, if $m$ is affine and $\lambda$ is a positive constant, then $u \mapsto \lambda\eta(u) + m(u)$ is another strongly convex entropy, associated with the flux $\lambda\vec{q} + m \circ f$. This observation is at the basis of the notion of relative entropy. Given $a \in \mathcal{U}$, we define another entropy, still strongly convex, by

$$u \mapsto \eta(u|a) := \eta(u) - \eta(a) - \mathrm{d}\eta_a \cdot (u - a).$$

This is an affine correction of $\eta$. The corresponding entropy flux is

$$u \mapsto \vec{q}(u; a) := \vec{q}(u) - \vec{q}(a) - \mathrm{d}\eta(a) \cdot (f(u) - f(a)).$$

We warn the reader that the latter field is not an affine correction of $u \mapsto \vec{q}(u)$. This is why we employ a semi-colon, where we put a bar instead in the definition of $\eta(u|a)$. The notation $\vec{q}(u; a)$ is actually a bit misleading, suggesting that it depends linearly on $\vec{q}$. As a matter of fact, it may happen that $\vec{q}(u) \equiv 0$, while $\vec{q}(u; a)$ is non-trivial. We shall give later on an example of such a paradox.

The function $(u, a) \mapsto \eta(u|a)$ is the *relative entropy*. Because $\eta$ is convex, $\eta(u|a)$ is positive away from the diagonal $u = a$. We point out that the partial function $a \mapsto \eta(u|a)$ is not an entropy, unless $\eta$ was quadratic. The strong convexity tells us that

$$\eta(u|a) \geq \omega_a |u - a|^2 \tag{4}$$

for some $\omega_a > 0$ when $u$ belongs to a compact neighbourhood of $a$. Therefore, estimating $\int \eta(u|a)\, dx$ is a way to estimate $u(\cdot, t) - a$ in $L^2$. Suppose for instance that $u(\cdot, t) - a$ has compact support, then the conservation law (2) tells us

$$\int_{\mathbb{R}^d} \eta(u(x,t)|a)\, dx = \int_{\mathbb{R}^d} \eta(u(x,0)|a)\, dx,$$

whence

$$\omega_a \|u(\cdot, t) - a\|^2_{L^2(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} \eta(u(x,0)|a)\, dx.$$

This expresses the $L^2$-stability of constant states.

The primary role of the entropy identity is therefore to provide an *a priori* estimate in $L^2$, or at least in some appropriate Orlicz space. It is at work in the existence theory for the Cauchy problem too, although it is far from being sufficient. Because the flux is nonlinear, the existence theory requires estimates of some derivatives. These cannot be achieved directly, but it was found by Kato, Gårding and others that $H^s$-estimates are valid, at least locally in time, whenever $s > 1 + \frac{d}{2}$, for systems admitting a *symmetrization* in Friedrichs' sense. A symmetrization is a change of unknowns $u \mapsto z$ by which (1) is transformed into

$$A^0(z)\partial_t z + \sum_{\alpha=1}^{d} A^\alpha(z)\partial_\alpha z = 0,$$

where the matrices $A^0(z), \dots, A^d(z)$ are symmetric, and $A^0(z)$ is positive definite. It was observed by Godunov [13] and independently by Friedrichs & Lax [12] that systems of conservation laws admitting a strongly convex entropy can be symmetrized. One can either choose the conjugate unknowns $z = \nabla_u \eta$, or take $z = u$ and multiply the system by $A^0 = \mathrm{D}^2\eta(u)$. We therefore derive the fundamental local existence result, see for instance Dafermos' book [9],

**Theorem 1.1** *If (1) is endowed with a strongly convex entropy, then the Cauchy problem is locally (in time) well-posed in the space $H^s_{uloc}(\mathbb{R}^d)$ whenever $s > 1 + \frac{d}{2}$.*

The subscript "uloc" means "locally uniformly" : the norm is the supremum over all $h \in \mathbb{R}^d$ of the $H^s$-norm of the restriction to the unit ball centered at $h$. We point out that the threshold $s > d + \frac{1}{2}$ means that $H^s_{uloc}(\mathbb{R}^d) \subset \mathcal{C}^1(\mathbb{R}^d)$, by Sobolev embedding. Therefore the theorem speaks about classical solutions of (1).

# 2 The question of global solutions

Let $a \in \mathcal{U}$ be given. The field $\bar{a}(x,t) :\equiv a$ is a particular solution of (1). Its stability can be analyzed, in a first instance, by linearizing the system about $a$. One finds the constant coefficient system

$$\partial_t U + \sum_\alpha \mathrm{d}f_\alpha(a)\partial_\alpha U = 0,$$

or equivalently

$$A^0(a)\partial_t Z + \sum_{\alpha=1}^d A^\alpha(a)\partial_\alpha Z = 0.$$

This linear system propagates planar waves in direction $\xi \in S^{d-1}$ at finite velocities $\lambda_1(a;\xi) \leq \cdots \leq \lambda_n(a;\xi)$ that are the eigenvalues of the matrix

$$M(a;\xi) := A^0(a)^{-1} \sum_\alpha \xi_\alpha A^\alpha(a).$$

The fact that this matrix-valued symbol is diagonalisable with real eigenvalues is classical. The diagonalisation can be performed uniformly with respect to $\xi$. This is referred to as the *hyperbolicity* of the linearized system. By extension, we say also that (1) is *hyperbolic*. Therefore every closed system of first-order conservation laws endowed with a strongly convex entropy is hyperbolic.

By linear superposition, one may design linear waves that are non-planar. They still propagate at finite velocity in the direction normal to the front.

Because of the non-linearity, the velocities depend on the state $a \in \mathcal{U}$. This usually causes a loss of regularity in finite time for the solutions of (1). This can be seen even in the simplest

nonlinear equation, named after Burgers,

$$\partial_t u + \partial_x \frac{u^2}{2} = 0, \qquad (n = d = 1).$$

Then the space derivative $y := \partial_x u$ satisfies

$$(\partial_t + u\partial_x)y + y^2 = 0.$$

The former equation, written as $(\partial_t + u\partial_x)u = 0$, is a transport equation, telling us that $u(X(t), t) \equiv u(X(0), 0)$ along the integral curves of $\frac{dX}{dt} = u(X, t)$ (the wave velocity is therefore $u$ itself). The latter is a Riccati's equation along the same curves, whose solution blows up in finite time whenever $y(X(0), 0)$ is negative. Therefore the solution of the forward Cauchy problem with initial data $u_0$ blows up in finite time in the $C^1$-norm, unless $u_0$ is monotone non-decreasing.

We infer that for most systems of the form (1), and for most initial data, the classical solution exists only on some finite time interval. For practical applications it is however necessary to consider large-time, say global, solutions. This requires considering non-differentiable solution, for which (1) is interpreted in the distributional sense. To this end, it sufficies that $u$ be a locally measurable bounded field. Within this framework, the blow-up described above is usually resolved by the development of discontinuities, which propagate along hypersurfaces. Let $\Sigma$ be such a discontinuity front in $\mathbb{R}^d \times (0, T)$, with unit normal $\mathbf{n} = (n_1, \ldots, n_d, n_0)$, and let $u_\pm$ be the limits of $u(x, t)$ at some point $(\bar{x}, \bar{t}) \in \Sigma$ from either side. Then it is well-known that $(u_-, u_+; \mathbf{n})$ satisfies the *Rankine–Hugoniot condition*

$$n_0(u_+ - u_-) + \sum_\alpha n_\alpha (f(u_+) - f(u_-)) = 0. \tag{5}$$

Notice that for a genuine discontinuity, one has $(n_1, \ldots, n_d) \neq 0$. Denoting $[h] := h(u_+) - h(u_-)$, Rankine–Hugoniot can therefore be written in the compact form

$$[f(u)] \cdot \nu = \sigma[u],$$

where $\nu \in S^{d-1}$ is the normal of the spatial trace of the front and $\sigma$ is the normal velocity.

The possibility for solutions to (1) to admit discontinuities is not the end of the story. Although this extension makes up for the lack of classical solutions, it has the flaw that rough solutions are way too many. This is where the second principle comes into play, with the role of selecting the physically admissible solutions. Mathematically speaking, one first observes that the additional conservation law (2) is not any more a consequence of (1), because multiplying (1) by $\mathrm{d}\eta(u)$ does not make sense, and the chain rule does not apply for non-Lipschitz fields. Instead, one argues that first-order conservation laws are actually the limit (as some parameter $\epsilon > 0$ tends to zero) of some second-order system that includes a natural diffusion process. For instance, the Euler equation of a perfect fluid is the limit of the Navier-Stokes-Fourier

system. The latter system has a parabolic type which makes the solution smooth for $t > 0$ ; it is therefore liable to multiply it by $\mathrm{d}\eta(u)$, and apply the chain rule. One finds then that its solutions satisfy some differential inequality

$$\partial_t \eta(u^\epsilon) + \mathrm{div}_x \vec{q}(u^\epsilon, \epsilon \nabla_x u^\epsilon) \leq 0.$$

Passing to the limit, we obtain, at least formally, the distributional inequality

$$\partial_t \eta(u) + \mathrm{div}_x \vec{q}(u) \leq 0, \tag{6}$$

called the *entropy inequality*. We therefore declare that an admissible solution (1) must satisfy (6). We speak of an *entropy solution*. Remark that (6) does not yield any additional information in zones where $u$ is a Lipschitz function. However, it tells us that across a discontinuity $\Sigma$, $(u_-, u_+; \mathbf{n})$ satisfies

$$[\vec{q}(u)] \cdot \nu \leq \sigma[\eta(u)], \tag{7}$$

where

$$u_\pm = \lim_{h \to 0^+} u(x \pm h\nu, t).$$

The role of the entropy condition is therefore to restore the uniqueness in the Cauchy problem, while leaving the door open to the existence. We don't know so far if this role is successful, except in a few instances. We have a complete and satisfactory theory for scalar equations in any dimension, due to Kruzkhov. A. Bressan and collaborators showed that the Cauchy problem for one-dimensional systems is well-posed as long as the initial data has small total variation, see [1]. When $d = 1$ and $n = 2$, R. DiPerna succeeded to prove the existence of an entropy solution for large initial data, if the system is sufficiently non-linear, by using the compensated compactness method initiated by L. Tartar. When $n \geq 3$, the mere existence of an entropy solution for large data is still unknown. The uniqueness of entropy solution is doubtful in absence of BV regularity ; C. de Lellis and L. Székelyhidi provide actually counter-examples [10].

# 3 The method of relative entropy

The method of relative entropy is a technique devised by R. DiPerna [11] and C. Dafermos [7, 8] in order to estimate the $L^2$-distance between two solutions, one of both being smooth or at least not too weak. It adapts to the context of first-order conservation laws the *weak-strong* estimates that is well-known for 3-D Navier-Stokes equation, for instance.

If $u$ and $v$ are smooth solutions of (1), say Lipschitz continuous, then the relative entropy, evaluated at $(u, v)$, satisfies the identity

$$\partial_t \eta(u|v) + \mathrm{div}_x \vec{q}(u; v) = -\mathcal{R}$$

where

$$\mathcal{R} := \sum_\alpha \partial_\alpha(\mathrm{d}\eta(v)) \cdot f_\alpha(u|v),$$

with the obvious definition for $f(u|v)$ (affine correction of $f(u)$). If $u$ and $v$ agree outside of some relatively compact domain $\Omega \subset \mathbb{R}^d$, we deduce

$$\frac{d}{dt}\int_\Omega \eta(u|v)\,dx = -\int_\Omega \mathcal{R}\,dx.$$

The source term $\mathcal{R}$ can be bounded by

$$C\|\nabla_x v\|_\infty |u-v|^2.$$

Using (4), we infer that

$$\frac{d}{dt}\int_\Omega \eta(u|v)\,dx \le C'\int_\Omega \eta(u|v)\,dx, \qquad C' := \frac{C\|\nabla_x v\|_\infty}{\omega}.$$

The Gronwall inequality thus gives an estimate

$$\int_\Omega \eta(u|v)\,dx \le e^{C't}\int_\Omega \eta(u_0|v_0)\,dx,$$

where $u_0, v_0$ are the initial data. This is the way uniqueness is proved for Lipschitz solutions, but also stability : if $u_0$ and $v_0$ are close to each other, then so are $u(t)$ and $v(t)$. Mind however that the distance between $u(t)$ and $v(t)$ may increase unboundedly as $t \to +\infty$. We speak of *finite-time stability*. In addition, because of the propagation of the information at finite velocity, the correct estimate is instead (again, see Dafermos' book)

$$\int_\Omega \eta(u|v)\,dx \le e^{C't}\int_{\Omega+B(0;V)} \eta(u_0|v_0)\,dx,$$

where $V$ is a suitable constant, larger than the wave velocities.

When $v$ is Lipschitz but $u$ is only a bounded entropy solution, one still has

$$\partial_t\eta(u|v) = \partial_t\eta(u) - \partial_t\eta(v) - \mathrm{d}\eta(v)\cdot(\partial_t u - \partial_t v) - \mathrm{D}^2\eta_v(\partial_t v, u-v),$$

with a analogous formula for $\mathrm{div}_x\vec{q}(u;v)$. Using then the system for $u$ and $v$, the entropy inequality for $u$ and the entropy *equality* for $v$, we obtain the distributional inequality

$$\partial_t\eta(u|v) + \mathrm{div}_x\vec{q}(u;v) \le -\mathcal{R}. \tag{8}$$

From (8), we can reach the same conclusion as when $u$ and $v$ were both Lipschitz. We therefore have uniqueness and finite-time stability as long as the Cauchy problem admits a Lipschitz continuous solution $v$. This leaves open the uniqueness question beyond the blow-up time

(except in the cases covered by Bressan's work). Such a result is very similar to the weak-strong stability/uniqueness statement for the incompressible Navier-Stokes in dimension three.

When $v$ is globally defined, that is for all $t > 0$, and is Lipschitz, one may wonder whether the source term $\mathcal{R}$ remains non-negative, in which case the total relative entropy

$$\int_{\mathbb{R}^d} \eta(u|v)\, dx$$

is a non-increasing function of time. We then say that $v$ is *uniformly stable*. This happens in gas dynamics for solutions of the 1-d Riemann problem when only rarefaction waves occur (G.-Q. Chen & coll. [2, 3, 4, 5]), and also for multi-dimensional expanding flows (D. Serre [19]).

## 3.1 Stability of discontinuities

From now on, we restrict to the one-dimensional situation, but we consider the stability of a one-dimensional simple discontinuity

$$v(x,t) = \begin{cases} u_\ell, & \text{if} \quad x < \sigma t, \\ u_r, & \text{if} \quad x > \sigma t. \end{cases}$$

Without loss of generality, we may assume $\sigma = 0$, up to replacing $x$ by $x - \sigma t$ and $f(u)$ by $f(u) - \sigma u$. We thus restrict to a steady discontinuity, meaning that $f(u_\ell) = f(u_r)$ and $q(u_r) \leq q(u_\ell)$.

Away from $x = 0$, the inequality (8) is still valid, while $\mathcal{R} \equiv 0$, because $v$ is constant on either sides of $x = 0$. We therefore obtain

$$\frac{d}{dt} \int_{\mathbb{R}} \eta(u|v)\, dx \leq q(u_+; u_r) - q(u_-; u_\ell),$$

where $u_\pm$ stand for the right and left limits of $u(x,t)$ as $x \to 0$. The discontinuity will therefore be stable if the right-hand side, a numerical function of $(u_-, u_+)$, is non-positive. Unfortunately, this is usually not the case; the interested reader might compute a Taylor expansion and find that its sign is non constant, but let us present a more illuminating argument. Suppose (1) is a scalar equation with $f'' > 0$. Take $u^0 = v^0 + \phi$, where $\phi$ is bounded, compactly supported. It is classical that the initial disturbance is absorbed by the shock: after some finite time $T > 0$, $u$ coincides with the shifted shock $v(\cdot - h)$. The conservation of mass determines $h = \frac{1}{[u]} \int_{\mathbb{R}} \phi\, dx$. With $\eta(s) = \frac{1}{2}s^2$, that is $\eta(a|b) = \frac{1}{2}(a-b)^2$, we deduce that

$$\int_{\mathbb{R}} \eta(u|v)\, dx = \frac{1}{2} \left| [u] \int_{\mathbb{R}} \phi\, dx \right|.$$

On the other hand, we have $\eta(u^0|v^0) = \frac{1}{2}\phi^2$. Because the inequality

$$\left| [u] \int_{\mathbb{R}} \phi\, dx \right| \leq \int_{\mathbb{R}} \phi^2 dx$$

is violated for some (many) disturbances $\phi$, we see that $t \mapsto \int_{\mathbb{R}} \eta(u|v)\,dx$ is not a decreasing function in general.

N. Leger & A. Vasseur [14, 15] introduced the following refinement of the method. The example above suggests to estimate the distance between $u(t)$ and $v$ by the relative entropy, *up to a shift*. We thus define a functional

$$E(t) := \inf_{h\in\mathbb{R}} \int_{\mathbb{R}} \eta(u(x,t)|v(x+h))\,dx = \inf_{h\in\mathbb{R}} \left( \int_{-\infty}^{h} \eta(u|u_\ell)\,dx + \int_{h}^{+\infty} \eta(u|u_r)\,dx \right).$$

The quantity in parenthesis is a continuous function of $h$, which tends to $+\infty$ as $h \to \pm\infty$. Therefore the infimum is attained at some $\bar{h}$. If $u$ has left and right limits at $\bar{h}$, the left and right $h$-derivatives at $\bar{h}$ must be negative and positive respectively:

$$\eta(u_-|u_\ell) \le \eta(u_-|u_r), \qquad \eta(u_+|u_\ell) \ge \eta(u_+|u_r).$$

If $u$ is continuous at $\bar{h}$, this tells us $\eta(u|u_\ell) = \eta(u|u_r)$ ; in other words, $\bar{u} := u(\bar{h})$ belongs to the hyperplane defined by $[\mathrm{d}\eta] \cdot \bar{u} = [\mathrm{d}\eta(u) \cdot u - \eta(u)]$, which separates $u_\ell$ from $u_r$. If $u_+ \neq u_-$, Rankine–Hugoniot gives instead $f(u_+) - f(u_-) = X'(u_+ - u_-)$, where we may suppose that $X \equiv h$ locally (isolated times are not meaningful for the derivative), and therefore $X' = h'$. In conclusion, $(u_-, u_+)$ obey to severe constraints in both situations. Because

$$E'(t) \le q(u_+; u_r) - q(u_-; u_\ell) - h'(\eta(u_+|u_r) - \eta(u_-|u_\ell)) =: D(u_\pm; u_{\ell,r}),$$

we see that the discontinuity $v$ is stable up to a shift if $D \le 0$ for every pair $(u_-, u_+, h')$ satisfying the constraints above. We point out that the definition of $D$ is not ambiguous: $h'$ is well-defined (by R.-H.) in the discontinuous case, while $D = q(u; u_r) - q(u; u_\ell)$ in the continuous case. With this approach, Leger and Vasseur proved

**Theorem 3.1** *For a scalar equation with a convex flux, shocks are uniformly stable up to a shift.*

In terms of the entropy $\frac{1}{2}u^2$, this can be viewed as an $L^2$-contraction property: $t \mapsto \inf_h \|u - v(\cdot - h)\|_{L^2}$ is non-increasing. We warn the reader that this is true only if $v$ is a pure shock. When $u, v$ are arbitrary entropy solutions, we only know the contraction in the $L^1$-norm, which is a part of Kruzkhov's theory.

# 4 Using two entropies

In [20], the authors applied the idea described above to systems ($n \ge 2$). Unfortunately, most systems resist to the method. A new tool was therefore needed, which has been elaborated

recently by Vasseur. Instead of one entropy $\eta$, one uses two entropies $\eta_\pm$ to measure the distance from $u$ to $v$. Let us consider the quantity

$$E(t) := \int_{-\infty}^{0} \eta_-(u|u_\ell)\,dx + \int_{0}^{+\infty} \eta_+(u|u_r)\,dx.$$

Of course, we may incorporate a shift:

$$E_{\min}(t) := \inf_h \left( \int_{-\infty}^{h} \eta_-(u|u_\ell)\,dx + \int_{h}^{+\infty} \eta_+(u|u_r)\,dx \right).$$

In practice, $\eta_+$ and $\eta_-$ are proportional, because most systems admit only one independent non-trivial entropy. Say $\eta_\pm = a_\pm \eta$ with $a_\pm$ positive constants. Remark that the uniform stability (in terms of the decay of $E$) implies the stability up to a shift, due to the space-shift invariance of the Cauchy problem.

A calculation analogous to that in the scalar case can be made. If $v^0$ is a steady Lax shock associated with a genuinely nonlinear field, and if $u^0 = v^0 + \phi$ is a compactly supported perturbation where $\phi$ is small, then the asymptotic behaviour has been described by T.-P. Liu [16]. It consists of a superposition of so-called $N$-waves (which decay in $L^2$-norm like $t^{-1/4}$), of linear waves that just propagate at constant velocity, and of a shift of the shock from $x = 0$ to $x = h$. To determine $h$, we split the mass $m = \int_{\mathbb{R}} \phi\,dx$ into three parts $X_- + h[u] + X_+$, where $X_-$ (resp. $X_+$) belongs to the stable (resp. unstable) subspace of $\mathrm{d}f(u_\ell)$ (resp. $\mathrm{d}f(u_r)$). All these waves asymptotically separate from each other and therefore the limit of $E(t)$ consists of the sum of contributions of the shift and of the linear waves. In particular, a decay of $E(t)$ for $t > 0$ would imply that $|h|\eta(u_r|u_\ell)$, or $|h|\eta(u_\ell|u_r)$ is not greater than

$$\int_{-\infty}^{0} \eta_-(u_\ell + \phi|u_\ell)\,dx + \int_{0}^{+\infty} \eta_+(u_r + \phi|u_r)\,dx \sim \int_{\mathbb{R}} \phi^2\,dx.$$

This is obviously false, and therefore the uniform stability of shock waves cannot be obtained without involving a shift. As a matter of fact, Vasseur showed recently that the stability up to a shift may hold true for "extreme shocks", that is the slowest and the fastest shocks.

The argument presented above does not settle the case of contact discontinuities, because then the asymptotics does not involve a shift. We have at least a positive result in the case of full gas dynamics. The system consists of conservation of mass, momentum and energy. In Lagrangian variables (where $x$ is mass), it writes

$$\partial_t \tau = \partial_x w, \qquad \partial_t w + \partial_x p(\tau, e) = 0, \qquad \partial_t (e + \frac{1}{2} w^2) + \partial_x (wp) = 0, \tag{9}$$

where $\tau$ is the specific volume, $w$ the flow velocity and $e$ the internal energy. The pressure $p$ is given by an equation of state $p(e, \tau)$. Thermodynamics tells us that there exists a positive

function $\theta(e,\tau)$ (the temperature) and a concave function $s(e,\tau)$ (the entropy) satisfying the Gibbs relation $\theta \mathrm{d}s = \mathrm{d}e + p\mathrm{d}\tau$. The second principle is then that $\partial_t s \geq 0$ for observable flows. In other words, $\eta = -s$ is an entropy in the mathematical sense, and admissible flows should satisfy the entropy inequality. Let us point out that $q \equiv 0$ in this example.

**Theorem 4.1** *Consider the 1-D system of full gas dynamics in Lagrangian variables. Then the contact discontinuity $(u_\ell, u_r)$ is uniformly stable in the sense that*

$$t \mapsto E(t) := -\int_{-\infty}^{0} \theta_\ell s(u|u_\ell)\, dx - \int_{0}^{+\infty} \theta_r s(u|u_r)\, dx$$

*is non-increasing, for every solution $u$ equal to $u_\ell/u_r$ as $x$ is sufficiently large and negative/positive.*

Of course, we may just ask that $u(\cdot,t) - u_\ell \in L^2(\mathbb{R}^-)$ and $u(\cdot,t) - u_r \in L^2(\mathbb{R}^+)$. Our statement has a counterpart when one uses the Eulerian variables, where $x$ is then a space coordinate.

*Proof*

For $x > 0$, we have $\partial_t s(u|u_r) + \partial_x q(u;u_r) \leq 0$, where $q(a;b) = \mathrm{d}s_b \cdot (f(a) - f(b))$. We deduce

$$E'(t) \leq \theta_\ell \mathrm{d}s_\ell \cdot (f_- - f(u_\ell)) - \theta_r \mathrm{d}s_r \cdot (f_+ - f(u_r))$$

where $f_\pm$ denote the right/left traces of $f(u)$ along $x = 0$. The conservation laws (1) tells us, not only that these traces are well defined as bounded measurable functions, but also that they coincide. Because $f(u_r) = f(u_\ell)$, we have $w_r = w_\ell$ and $p_r = p_\ell$, and therefore $\theta_\ell \mathrm{d}s_\ell = \theta_r \mathrm{d}s_r$. We thus have

$$E'(t) \leq (\theta \mathrm{d}s)_{\ell,r} \cdot (f_- - f_\ell - f_+ + f_r) = 0.$$

■

**Remarks.** – Full gas dynamics admits infinitely many independent entropies $g \circ s$, where $g$ is any numerical function. Such entropies are strongly convex when $g' < 0$ and $g'' \geq 0$. We could therefore have chosen $\eta_\pm = g_\pm \circ s$ in the estimate above. However the derivation of the entropy inequality from the Navier-Stokes-Fourier system works only for one entropy, namely $-s$ (see [18]). Therefore our estimates with $\eta_\pm = -\theta_{\ell,r} s$ are the only one with physically relevance. – The calculation above can be adapted to more general systems for which $\lambda \equiv 0$ is a simple eigenvalue of $\mathrm{d}f$. The image of $\mathcal{U}$ under $u \mapsto (f(u), q(u))$ is a manifold of codimension 2. We ask that it be contained in a hyperplane, that is $\alpha q + L \circ f \equiv \mathrm{cst}$, with $\alpha \neq 0$, say $\alpha > 0$. Up to replacing $\eta$ by $\alpha\eta + L(u)$, we may assume that $q \equiv 0$. One shows that there exists an analogue of the temperature $\theta(u)$ such that if $(u_\ell, u_r)$ is a steady contact discontinuity, then $[\theta \mathrm{d}\eta] = 0$. Therefore the analogue of Theorem 4.1 holds true:

$$t \mapsto E(t) := \int_{-\infty}^{0} \theta_\ell \eta(u|u_\ell)\, dx + \int_{0}^{+\infty} \theta_r \eta(u|u_r)\, dx$$

is non-increasing, for every solution $u$ equal to $u_\ell/u_r$ as $x$ is sufficiently large and negative/positive.

# 5 Lyapunov function for full gas dynamics

The calculus above suggests a fruitful generalization. On the one hand, we consider special solutions of (9) of the form $v \equiv v(x)$ where the velocity $\bar{w}$ and the pressure $\bar{p}$ are constant. For instance, $v$ could be a pure contact discontinuity as above. Such fields are parametrized by the choice of a positive measurable $\tau(x)$, bounded by below and above, and by the constants $(\bar{w}, \bar{p})$. The internal energy is obtained by solving the equation $p(\tau(x), e(x)) = \bar{p}$. These solutions are precisely what we called *linear waves* in Section 4.

Given a linear wave $v$, we may define a functional

$$L_v[u] := -\int_{\mathbb{R}} \theta(v) s(u|v)\, dx,$$

provided that $u - v \in L^2(\mathbb{R})^3$. In the periodic case, where the domain is $\mathbb{R}/M\mathbb{Z}$, we define instead

$$L_v[u] := -\int_0^M \theta(v) s(u|v)\, dx.$$

We have a seemingly new result:

**Theorem 5.1** *The functionals $L_v$ are Lyapunov functions: for every entropy solution of (9) such that $u(t) - v \in L^2$, the function $t \mapsto L_v[u(t)]$ is non-increasing.*

*Proof*

We first prove the result when $\tau$ is a smooth function. We have

$$\begin{aligned} \partial_t(\theta(v)s(u|v)) &= \theta(v)(\partial_t s(u) - \mathrm{d}s_v \partial_t u) \\ &\geq \theta(v)\mathrm{d}s_v \partial_x f(u) = (\mathrm{d}(e + \frac{1}{2}w^2) - \bar{w}\mathrm{d}w + \bar{p}\mathrm{d}\tau)\partial_x f(u) =: \omega(\partial_x f(u)). \end{aligned}$$

The differential form has constant coefficients and therefore we have $\partial_t(\theta(v)s(u|v)) \geq \partial_x(\omega \circ f(u))$. Integrating in space, it comes $\frac{d}{dt}L_v[u(t)] \leq 0$.

If $\tau$ is not smooth, we may approach every function $\tau$ by a sequence $\tau_\epsilon = \tau * \rho(\cdot/\epsilon)$, which converges boundedly almost everywhere. Then $L_{v_\epsilon}[u(t)] \to L_v[u(t)]$. The monotony passes to the limit.

■

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