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# On monodromy eigenfunctions of Heun equations and boundaries of phase-lock areas in a model of overdamped Josephson effect

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## Abstract

We study a family of double confluent Heun equations that are linearizations of nonlinear equations on two-torus modeling the Josephson effect in superconductivity. They have the form  $\mathcal{L}E = 0$ , where  $\mathcal{L} = \mathcal{L}_{\lambda, \mu, n}$  is a family of differential operators of order two acting on germs of holomorphic functions in one complex variable. They depend on parameters  $\lambda, n \in \mathbb{R}$ ,  $\mu > 0$ ,  $\lambda + \mu^2 \equiv \text{const} > 0$ . We show that for every  $b \in \mathbb{C}$  and  $n \in \mathbb{R}$  satisfying a certain “non-resonance condition” and every parameter values  $\lambda, \mu$  there exists a unique entire function  $f_{\pm} : \mathbb{C} \rightarrow \mathbb{C}$  (up to multiplicative constant) such that  $z^{-b}\mathcal{L}(z^b f_{\pm}(z^{\pm 1})) = d_{0\pm} + d_{1\pm}z$  for some  $d_{0\pm}, d_{1\pm} \in \mathbb{C}$ . The latter  $d_{j,\pm}$  are expressed as functions of the parameters. This result has several applications. First of all, it gives the description of those parameter values for which the monodromy operator of the corresponding Heun equation has given eigenvalues. This yields the description of the non-integer level curves of the rotation number of the family of equations on two-torus as a function of parameters. In the particular case, when the monodromy is parabolic (has multiple eigenvalue), we get the complete

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description of those parameter values that correspond to the boundaries of the phase-lock areas: integer level sets of the rotation number, which have non-empty interiors.

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## 1 Introduction: statement of results, sketch of proof and plan of the paper

We study the problem to find solutions of *non-homogeneous* double-confluent Heun equations that are monodromy eigenfunctions. Our study is motivated by applications to nonlinear equations modeling the Josephson effect in superconductivity. The main results, their existence and uniqueness (Theorems 1.1 and 1.3) are stated in Subsection 1.1. Applications to monodromy eigenfunctions and eigenvalues of homogeneous double confluent Heun equations and to nonlinear equations modeling Josephson effect are presented in Subsection 1.1, Sections 4 and 5.

Each eigenfunction is the product of a monomial  $z^b$  and a function  $f(z)$  holomorphic on  $\mathbb{C}^*$ . The Heun equation is equivalent to recurrence relations on the Laurent coefficients of the function  $f$ . The proofs of the above-mentioned results are based on studying the latter recurrence relations. We prove existence and uniqueness Theorem 1.7 for converging solutions of a more general class of recurrence relations (stated in Subsection 1.2 and proved in Section 2). Its proof is based on ideas from hyperbolic dynamics and a fixed point argument for appropriate contracting mapping.

### 1.1 Main result: existence and uniqueness of monodromy eigenfunctions of non-homogeneous double confluent Heun equations

We consider the family of double confluent Heun equations

$$\mathcal{L}E = z^2 E'' + (nz + \mu(1 - z^2))E' + (\lambda - \mu n z)E = 0; \quad (1.1)$$

$$\lambda = \frac{1}{4\omega^2} - \mu^2, \quad n = l + 1, \quad \lambda, n \in \mathbb{R}, \quad \omega, \mu > 0.$$

Our goal is to study existence of its eigenfunctions with a given eigenvalue  $e^{2\pi i b}$ ,  $b \in \mathbb{C}$ : solutions of the form

$$E(z) = z^b f(z), \quad f(z) \text{ is holomorphic on } \mathbb{C}^*. \quad (1.2)$$

To do this, we study the non-homogeneous equations of type

$$z^{-b} \mathcal{L}(z^b f_{\pm}(z^{\pm 1})) = d_{0\pm} + d_{1\pm} z \quad (1.3)$$

One of our main results is the next theorem

**Theorem 1.1** For every  $(n, \lambda, \mu, b) \in U$ ,

$$U = \{(n, \lambda, \mu, b) \in \mathbb{C}^4 \mid \mu \neq 0, b, b+n \notin \mathbb{Z}\}, \quad (1.4)$$

there exist holomorphic functions  $f_{\pm}(z)$  on a neighborhood of zero,  $f_{+}(0) = 0$  such that the functions  $f_{\pm}(z^{\pm 1})$  satisfy equations (1.3) for appropriate  $d_{0\pm}(n, \lambda, \mu, b)$ ,  $d_{1\pm}(n, \lambda, \mu, b)$ . The functions  $f_{\pm}$  are unique up to constant factor (depending on the parameters), and they are entire functions: holomorphic on  $\mathbb{C}$ . For every sign index  $\pm$  the corresponding vector  $(d_{0\pm}, d_{1\pm})$  is uniquely defined up to scalar factor depending on parameters. The above constant factors can be chosen so that both  $f_{\pm}$  and  $d_{j\pm}$  depend holomorphically on  $(n, \lambda, \mu, b) \in U$  and  $f_{\pm}(z)$  are real-valued in  $z \in \mathbb{R}$  for real parameter values.

**Corollary 1.2** Let  $(n, \lambda, \mu, b) \in U$ . The corresponding equation (1.1) has a monodromy eigenfunction with eigenvalue  $e^{2\pi i b}$ ,  $b \in \mathbb{C}$ , if and only if the corresponding vectors  $d_{\pm} = (d_{0\pm}, d_{1\pm})$  are proportional:

$$d_{0+}d_{1-} - d_{0-}d_{1+} = 0. \quad (1.5)$$

Theorem 1.1 will be proved in the next subsection and Section 2. The corollary will be proved in the next subsection. The explicit formulas for the functions  $f_{\pm}$  and  $d_{j\pm}$ , together with an explicit form for equation (1.5) will be given in Section 3.

**Theorem 1.3** For every  $n \in \mathbb{N}$  and  $(\lambda, \mu) \in \mathbb{C}^2$  there exists a unique function  $E(z) \not\equiv 0$  (up to constant factor) holomorphic on a neighborhood of zero such that  $\mathcal{L}E = \text{const}(z)$ . The latter constant  $\text{const}(z) = \xi_{n-1}(\lambda, \mu)$  depends only on the parameters  $(n, \lambda, \mu)$ .

Theorem 1.3 will be proved in the next subsection.

**Remark 1.4** Theorem 1.3 is closely related to the question of the existence of a solution holomorphic at 0 of equation (1.1) (such a solution is automatically entire, i.e., holomorphic on  $\mathbb{C}$ ). This question was studied by V.M.Buchstaber and S.I.Tertychnyi in [8]. The existence of a solution  $E$  from Theorem 1.3 and explicit expressions for  $E$  and the corresponding function  $\xi_{n-1}(\lambda, \mu)$  (analytic in  $(\lambda, \mu) \in \mathbb{C}^2$ ) were given in [8, pp. 337–338]. The existence result implies that if  $\xi_{n-1}(\lambda, \mu) = 0$ , then the homogeneous equation (1.1), i.e.,  $\mathcal{L}E = 0$  has a solution holomorphic on  $\mathbb{C}$ . A conjecture stated by V.M.Buchstaber and S.I.Tertychnyi in loc. cit. said that the converse is true: if equation (1.1) has a holomorphic solution at 0, then

$\xi_{n-1}(\lambda, \mu) = 0$ . This conjecture was studied in loc. cit and [9], where it was reduced to a series of conjectures on polynomial solutions of auxiliary Heun equations and modified Bessel functions of the first kind. All these conjectures were solved in [11]. As the next corollary shows, Theorem 1.3 implies the conjecture of Buchstaber and Tertychnyi immediately, without using neither polynomial solutions, nor modified Bessel functions.

**Corollary 1.5** [11, theorem 3.5] *Equation (1.1) has an entire solution, if and only if  $\xi_{n-1}(\lambda, \mu) = 0$ , where  $\xi_{n-1}(\lambda, \mu)$  is the function from loc. cit., introduced in [8, formula (31), p. 337]; see also formula (4.11) in Subsection 4.3 below.*

**Proof** Let  $\xi_{n-1}(\lambda, \mu) = 0$ . Then the function  $E$  from Theorem 1.3 is an entire solution of equation (1.1):  $\mathcal{L}E = 0$ . Conversely, let equation (1.1) have a solution  $E$  holomorphic at 0. If  $\xi_{n-1}(\lambda, \mu) \neq 0$ , then there exists a holomorphic function  $\hat{E}$  on a neighborhood of zero satisfying the equation  $\mathcal{L}\hat{E} = \xi_{n-1}(\lambda, \mu) \neq 0$ , by Theorem 1.3. This together with the uniqueness statement of Theorem 1.3 implies that  $\hat{E} = E$  up to constant factor, hence  $\mathcal{L}\hat{E} = 0$ . The contradiction thus obtained proves the corollary.  $\square$

## 1.2 Solutions of three-term recurrence relations

Equation (1.1) is equivalent to the recurrence relations

$$((k+b)(k+b+n-1)+\lambda)a_k - \mu(k+b+n-1)a_{k-1} + \mu(k+b+1)a_{k+1} = 0, \quad (1.6)$$

which can be written in the matrix form

$$\begin{pmatrix} a_k \\ a_{k+1} \end{pmatrix} = A_k \begin{pmatrix} a_{k-1} \\ a_k \end{pmatrix},$$

$$A_k = \frac{k+b+n-1}{k+b+1} \begin{pmatrix} 0 & \frac{k+b+1}{k+b+n-1} \\ 1 & -\frac{\lambda+(k+b)(k+b+n-1)}{\mu(k+b+n-1)} \end{pmatrix}. \quad (1.7)$$

**Remark 1.6** A function  $f_+(z) = \sum_{k \geq 1} a_k z^k$  satisfies equation (1.3) for some  $d_{j+}$ , if and only if its Taylor coefficients  $a_k$  satisfy (1.6), or equivalently, (1.7) for  $k \geq 2$ . Similarly, a function  $f_-(z^{-1})$ ,  $f_-(w) = \sum_{k \leq 0} a_k w^{-k}$  satisfies (1.3), if and only if its coefficients satisfy (1.6) for  $k \leq -1$ .

**Proof of Corollary 1.2.** Let  $E(z) = z^b \sum_{k \in \mathbb{Z}} a_k z^k$  be a solution of equation (1.1) having type (1.2). Then

$$E(z) = z^b (f_+(z) + f_-(z^{-1})), \quad f_+(z) = \sum_{k \geq 1} a_k z^k, \quad f_-(z) = \sum_{k \geq 0} a_{-k} z^k. \quad (1.8)$$

The coefficients  $a_k$  satisfy (1.6) for all  $k$ . This together with the above remark implies that the functions  $f_{\pm}(z^{\pm 1})$  satisfy (1.3). The corresponding expressions  $d_{\pm} = d_{0\pm} + d_{1\pm}z$  should cancel out, since  $E$  is a solution of the homogeneous equation. This implies (1.5). Conversely, let  $f_{\pm}(z^{\pm 1})$  be solutions of (1.3), and let (1.5) hold: that is, the vectors  $d_+$  and  $d_-$  are proportional. Then we can normalize the latter vectors, and hence, the corresponding solutions by constant factors (not both vanishing simultaneously) so that  $d_+ + d_- = 0$ . Then the function  $E$  given by (1.8) is a solution of equation (1.1). This proves the corollary.  $\square$

As it is shown below, Theorem 1.1 is implied by the following general theorem

**Theorem 1.7** *Consider recurrence relations*

$$g_k a_k + f_k a_{k-1} + h_k a_{k+1} = 0 \quad (1.9)$$

in unknown sequence  $a_k$ , where sequences  $g_k, f_k, h_k \in \mathbb{C}$  numerated by  $k \in \mathbb{N}$  satisfy the following conditions:

$$f_k, h_k \neq 0, \text{ for every } k, \quad (1.10)$$

$$f_k, h_k = o(g_k), \text{ as } k \rightarrow \infty. \quad (1.11)$$

Then there exists a unique series  $\sum_{k=0}^{+\infty} a_k z^k \neq 0$  with  $a_k$  satisfying (1.9) for  $k \geq 1$  and having positive converging radius. It converges on all of  $\mathbb{C}$ .

Theorem 1.7 will be proved in the next section.

**Remark 1.8** In the series  $\sum_k a_k z^k$  from Theorem 1.7 for every  $k \geq 0$  the two neighbor coefficients  $a_k, a_{k+1}$  do not vanish simultaneously: hence, they present a point  $(a_k : a_{k+1}) \in \mathbb{C}\mathbb{P}^1$ . Each pair of neighbor coefficients  $(a_k, a_{k+1})$  determines a unique sequence satisfying (1.9). Both statements follow from the fact that for every  $k \geq 1$  the coefficient  $a_{k\pm 1}$  is expressed as a linear combination of  $a_{k\mp 1}$  and  $a_k$  by (1.9), since  $f_k, h_k \neq 0$ . Hence, if some two neighbor coefficients  $a_{k-1}, a_k$  vanish, then all the coefficients vanish, and the series is zero, – a contradiction.

**Theorem 1.9** *Let  $b, n \in \mathbb{C}$ , Then for every  $k_0 \in \mathbb{Z}$  such that*

$$k + b + n - 1, k + b + 1 \neq 0 \text{ for every } k > k_0, \quad (1.12)$$

*for every  $\lambda, \mu \in \mathbb{C}$ ,  $\mu \neq 0$  there exists and unique nonzero one-sided series  $\sum_{k \geq k_0} a_k z^k$  (up to multiplicative constant) converging on some punctured disk centered at 0 that satisfies recurrence relations (1.6) (or equivalently, (1.7)) for  $k > k_0$ . Similarly, for every  $k_0 \in \mathbb{Z}$  such that*

$$k + b + n - 1, k + b + 1 \neq 0 \text{ for every } k < k_0 \quad (1.13)$$

*there exists and unique one-sided series  $\sum_{k \leq k_0} a_k z^k$  (up to multiplicative constant) that satisfies recurrence relations (1.6) for  $k < k_0$  and converges outside some disk centered at 0. Both series converge on the whole punctured complex line  $\mathbb{C}^*$ .*

**Example 1.10** Let in the conditions of Theorem 1.9 one have  $b, n + b \notin \mathbb{Z}$  (cf. (1.4)). Then its statements hold for all  $k_0 \in \mathbb{Z}$ , since inequalities (1.12) hold for all  $k \in \mathbb{Z}$ . Otherwise, if either  $b \in \mathbb{Z}$ , or  $b + n \in \mathbb{Z}$ , then the statements of Theorem 1.9

$$\text{hold for } k > k_0 \text{ whenever } k_0 \geq \max\{m \in \{-1 - b, 1 - b - n\} \mid m \in \mathbb{Z}\} \quad (1.14)$$

$$\text{hold for } k < k_0 \text{ whenever } k_0 \leq \min\{m \in \{-1 - b, 1 - b - n\} \mid m \in \mathbb{Z}\} \quad (1.15)$$

Theorem 1.9 together with Remark 1.6 and the first statement of Example 1.10 imply Theorem 1.1.

**Proof of Theorems 1.9 and 1.1.** The coefficients

$$g_k = (k + b)(k + b + n - 1) + \lambda, \quad f_k = -\mu(k + b + n - 1), \quad h_k = \mu(k + b + 1)$$

of recurrence relations (1.6) satisfy the conditions of Theorem 1.7 for  $k > k_0$  ( $k < k_0$ ). Indeed, the asymptotics (1.11) is obvious. Inequalities  $f_k, h_k \neq 0$  follow from (1.12) (respectively, (1.13)). This together with Theorem 1.7 proves Theorem 1.9, and hence, Theorem 1.1.  $\square$

**Proof of Theorem 1.3.** Let  $b = 0$ ,  $n \in \mathbb{N}$ . Inequalities (1.12) hold for  $k > k_0 = 0$ . Therefore, there exists a unique series  $E(z) = \sum_{k \geq 0} a_k z^k$  converging on a neighborhood of the origin, whose coefficients satisfy (1.6) for  $k \geq 1$ , and it converges on all of  $\mathbb{C}$  (Theorem 1.9). The system of relations (1.6) for  $k \geq 1$  is equivalent to the statement that  $\mathcal{L}E = \text{const}$ . This proves Theorem 1.3.  $\square$



### 1.3 Historical remarks, applications and plan of the paper

Our results are motivated by applications to the family

$$\frac{d\phi}{dt} = -\sin \phi + B + A \cos \omega t, \quad A, \omega > 0, \quad B \geq 0. \quad (1.16)$$

of nonlinear equations, which arises in several models in physics, mechanics and geometry: in a model of the Josephson junction in superconductivity (our main motivation), see [18, 23, 12, 2, 21]; in planimeters, see [13, 14]. Here  $\omega$  is a fixed constant, and  $(B, A)$  are the parameters. Set

$$\tau = \omega t, \quad l = \frac{B}{\omega}, \quad \mu = \frac{A}{2\omega}.$$

The variable change  $t \mapsto \tau$  transforms (1.16) to a non-autonomous ordinary differential equation on the two-torus  $\mathbb{T}^2 = S^1 \times S^1$  with coordinates  $(\phi, \tau) \in \mathbb{R}^2/2\pi\mathbb{Z}^2$ :

$$\dot{\phi} = \frac{d\phi}{d\tau} = -\frac{\sin \phi}{\omega} + l + 2\mu \cos \tau. \quad (1.17)$$

The graphs of its solutions are the orbits of the vector field

$$\begin{cases} \dot{\phi} = -\frac{\sin \phi}{\omega} + l + 2\mu \cos \tau \\ \dot{\tau} = 1 \end{cases} \quad (1.18)$$

on  $\mathbb{T}^2$ . The *rotation number* of its flow, see [1, p. 104], is a function  $\rho(B, A) = \rho(B, A; \omega)$  of parameters. It is given by the formula

$$\rho(B, A; \omega) = \lim_{n \rightarrow +\infty} \frac{\phi(2\pi n)}{n},$$

where  $\phi(\tau)$  is an arbitrary solution of equation (1.17).

The *phase-lock areas* are the level subsets of the rotation number in the  $(B, A)$ -plane with non-empty interior. They have been studied by V.M.Buchstaber, O.V.Karpov, S.I.Tertychnyi et al, see [8, 11, 15] and references therein. Each phase-lock area is an infinite chain of adjacent domains separated by *adjacency points*. The description of their coordinates as solutions of analytic functional equations was conjecturally stated by V.M.Bushstaber and S.I.Tertychnyi in [8] and proved by the authors of the present paper in [11]. Namely, the family of non-linear equations was reduced in [7, 24] to two families of second order linear differential equations of double confluent Heun type: equation (1.1) and the equation

$$\mathcal{L}E = z^2 E'' + ((-l + 1)z + \mu(1 - z^2))E' + (\lambda + \mu(l - 1)z)E = 0. \quad (1.19)$$

The latter equation is obtained by writing equation (1.1) in terms of the parameter  $l = n - 1$  and changing sign at  $l$ . In the case, when  $l \geq 0$ , Buchstaber and Tertychnyi have shown that the adjacencies correspond exactly to those parameter values, for which equation (1.1) has a non-trivial holomorphic solution at 0 (which is automatically an entire solution: holomorphic on  $\mathbb{C}$ ); see the statement in [8, p.332, paragraph 2] and the proof in [11, theorem 3.3 and subsection 3.2]. They have explicitly constructed a family of holomorphic solutions for parameters satisfying an explicit functional equation  $\xi_l(\lambda, \mu) = 0$ , see Corollary 1.5. They have conjectured that the latter functional equation describes the adjacencies completely. They have reduced this conjecture to another one saying that if equation (1.19) has a polynomial solution, then equation (1.1) does not have an entire solution. Later they have reduced their second conjecture on polynomial solutions to the third one saying that appropriate determinants formed by modified Bessel functions of the first type do not vanish on the positive semiaxis. The latter conjecture together with the other ones were proved in [11]. The above-mentioned conjecture of Buchstaber and Tertychnyi on functional equation describing the adjacencies follows from Corollary 1.5 and their correspondence to entire solutions of Heun equations.

V.M.Buchstaber and S.I.Tertychnyi have constructed symmetries of double confluent Heun equation (1.1) [8, 10]. The symmetry  $\# : E(z) \mapsto 2\omega z^{-l-1}(E'(z^{-1}) - \mu E(z^{-1}))$ , which is an involution of its solution space, was constructed in [25, equations (32), (34)]. It corresponds to the symmetry  $(\phi, t) \mapsto (\pi - \phi, -t)$  of the nonlinear equation (1.16); the latter symmetry was found in [19]. In [10] they have found new nontrivial symmetries in the case, when  $l \in \mathbb{Z}$  and equation (1.19) does not have polynomial solutions.

**Convention 1.11** Everywhere in the paper by *formal* solution  $(a_k)_{k \geq k_0}$  (or  $(a_k)_{k \leq k_0}$ ) of linear recurrence relation  $f_k a_k + g_k a_{k-1} + h_k a_{k+1} = 0$  we mean a (one- or two-sided) sequence of complex numbers  $a_k$  satisfying the relation for all  $k > k_0$  (respectively,  $k < k_0$ ). (Here one may have two-sided infinite sequences.) If in addition, the power series  $\sum_k a_k z^k$  converges on some annulus centered at 0 (for all the relations under consideration, this would automatically imply convergence on all of  $\mathbb{C}^*$ ) then the formal solution under question is called simply *a solution*: the adjective “converging” is omitted for simplicity.

In Section 3 we write down explicit formulas for solutions of recurrence relations (1.6) using the proof of Theorem 1.7. Then in Section 4 we deduce explicit functional equations satisfied by monodromy eigenvalues of double

confluent Heun equations (explicit versions of Corollary 1.2).

In Section 5 we apply results of Sections 3 and 4 to phase-lock areas in the model of Josephson effect. It is known that the ratio of the monodromy eigenvalues of equation (1.1) equals  $e^{\pm 2\pi i \rho(A,B)}$  and their product equals  $e^{-2\pi i l}$ . This together with results of Section 4 implies an explicit functional equation satisfied by non-integer level curves of the rotation number as a function of parameters (Theorem 5.7 in Subsection 5.1). The union of boundaries of the phase-lock areas coincides with the set where the monodromy has multiple eigenvalue and is described by the condition that the monodromy of equation (1.1) has eigenvalue  $\pm e^{-\pi i l}$ . This together with results of Sections 3, 4 implies an explicit description of the boundaries (Corollary 5.13 in Subsection 5.2). Open problems on phase-lock areas and possible approaches to some of them using the above description of boundaries are discussed in Subsections 5.5 and 5.6.

The following new result will be also proved in Section 5 using results of Section 4.

**Theorem 1.12** *Let  $\omega > 0$ ,  $(B, A) \in \mathbb{R}^2$ ,  $B, A > 0$ ,  $l = \frac{B}{\omega}$ ,  $\mu = \frac{A}{2\omega}$ ,  $\lambda = \frac{1}{4\omega^2} - \mu^2$ ,  $\rho = \rho(B, A)$ . The double confluent Heun equation (1.19) corresponding to the latter  $\lambda$ ,  $\mu$  and  $l$  has a polynomial solution, if and only if  $l, \rho \in \mathbb{Z}$ ,  $\rho \equiv l \pmod{2}$  and  $0 \leq \rho \leq l$ , and in addition, the point  $(B, A)$  lies in the boundary of the phase-lock area number  $\rho$  and is not an adjacency. In other terms, the points  $(B, A) \in \mathbb{R}_+^2$  corresponding to equations (1.19) with polynomial solutions lie in boundaries of phase-lock areas and are exactly their intersection points with the lines  $l = \frac{B}{\omega} \equiv \rho \pmod{2}$ ,  $0 \leq \rho \leq l$ , except for the adjacencies.*

**Remark 1.13** V.M.Buchstaber and S.I.Tertychnyi have shown in [7] that if a point  $(B, A) \in \mathbb{R}_+^2$  corresponds to equation (1.19) with a polynomial solution, then  $l, \rho$  are integers,  $0 \leq \rho \leq l$  and  $\rho \equiv l \pmod{2}$ .

#### 1.4 A sketch of proof of Theorem 1.7.

For every initial condition  $(a_0, a_1)$  there exists a unique sequence  $(a_k)_{k \geq 0}$  satisfying recurrence relations (1.9), by Remark 1.8. But in general, the series  $\sum_k a_k z^k$  may diverge. We have to prove that it converges for appropriately chosen unknown initial condition. To do this, we use the following trick: we run the recursion in the opposite direction, “from infinity to zero”. That is, take a big  $k$  and a given “final condition”  $q_k = (a_k, a_{k+1})$ . Then the inverse recursion gives all  $a_j = a_j(q_k)$ ,  $0 \leq j \leq k$ . It appears that the initial

condition  $(a_0, a_1)$  we are looking for can be obtained as a limit of the initial conditions  $(a_0(q_k), a_1(q_k))$  obtained by the above inverse recursion (after rescaling), as  $k \rightarrow \infty$ ; the only condition on  $q_k$  is that its projectivization  $[q_k] = (a_k : a_{k+1}) \in \mathbb{CP}^1$  should avoid some small explicitly specified “bad region”, which contracts to the point  $(0 : 1)$ , as  $k \rightarrow \infty$ .

The projectivized inverse recursion

$$P_k : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 : [q_k] = (a_k : a_{k+1}) \mapsto [q_{k-1}] = (a_{k-1} : a_k)$$

defined by (1.9) can be considered as the dynamical system

$$T : (\mathbb{N}_{\geq 2} \cup \{\infty\}) \times \mathbb{CP}^1 \rightarrow (\mathbb{N} \cup \{\infty\}) \times \mathbb{CP}^1, \quad \mathbb{N}_{\geq 2} = \mathbb{N} \cap [2, +\infty),$$

where for every  $x \in \mathbb{CP}^1$  and  $k \in \mathbb{N}_{\geq 2}$  one has

$$T : (k, x) \mapsto (k-1, P_k(x)); \quad T : \infty \times \mathbb{CP}^1 \mapsto \infty \times (1 : 0).$$

It appears that for every  $k$  large enough  $P_k$  has a strongly attracting fixed point tending to  $(1 : 0)$  and a strongly repelling fixed point tending to  $(0 : 1)$ , as  $k \rightarrow \infty$ . This together with the ideas from basic theory of hyperbolic dynamics implies that the fixed point  $p_\infty = \infty \times (1 : 0)$  of the transformation  $T$  should have a unique unstable manifold: an invariant sequence  $(k, [q_k])$  converging to  $p_\infty$ . We show that a solution  $(a_k)$  of recurrence relations (1.9) gives a converging Taylor series  $\sum_k a_k z^k$  on some neighborhood of zero, if and only if  $(a_k : a_{k+1}) = [q_k]$  for all  $k$ , and then the series converge everywhere. This will prove Theorem 1.7.

The existence and uniqueness of the above-mentioned unstable manifold is implied by the following discrete analogue of the classical Hadamard–Perron Theorem on the unstable manifold of a dynamical system at a hyperbolic fixed point.

**Theorem 1.14** *Let  $E_1, E_2, \dots$  be a sequence of complete metric spaces with uniformly bounded diameters. For brevity, the distance on each of them will be denoted  $d$ . Let  $P_k : E_k \rightarrow E_{k-1}$  be a sequence of uniformly contracting mappings: there exists a  $\lambda$ ,  $0 < \lambda < 1$  such that  $d(P_k(x), P_k(y)) < \lambda d(x, y)$  for every  $x, y \in E_k$  and  $k \geq 2$ . Then there exists a unique sequence of points  $x_k \in E$  such that  $x_{k-1} = P_k(x_k)$  for all  $k \geq 2$ . One has*

$$x_{k-1} = \lim_{m \rightarrow \infty} P_k \circ \dots \circ P_m(x), \quad (1.20)$$

*and the convergence is uniform in  $x$ : for every  $\varepsilon > 0$  there exists some  $l \in \mathbb{N}$  such that for every  $m \geq l$  and every  $x \in E_m$  one has  $d(P_k \circ \dots \circ$*

$P_m(x), x_{k-1}) < \varepsilon$ . If in addition the spaces  $E_k$  coincide with one and the same space  $E$  and the fixed points of the mappings  $P_k$  tend to some  $x_\infty \in E$ , as  $k \rightarrow \infty$ , then

$$\lim_{k \rightarrow \infty} x_k = x_\infty. \quad (1.21)$$

**Proof** The proof repeats the argument of the classical proof of Hadamard–Perron Theorem. Consider the space  $S$  of all sequences  $X = (x_k)_{k \in \mathbb{N}}$ ,  $x_k \in E_k$ , equipped with the distance

$$D(X, Y) = \sup_k d(x_k, y_k).$$

The transformation

$$T : S \rightarrow S, (x_1, x_2, \dots) \mapsto (P_2(x_2), P_3(x_3), \dots)$$

is a contraction. Therefore, it has a unique fixed point, which is exactly the sequence we are looking for. The second statement of the theorem on the uniform convergence of compositions to  $x_{k-1}$  follows from the uniform convergence of iterations of the contracting map  $T$  to its fixed point. In the last condition of Theorem 1.14 statement (1.21) follows by the above fixed point argument in the subspace in  $S$  of the sequences  $(x_k)$  tending to  $x_\infty$ , as  $k \rightarrow \infty$ : this is a complete  $T$ -invariant metric subspace in  $S$ , and hence,  $T$  has a fixed point there, which coincides with the previous sequence  $(x_k)$  by uniqueness. Theorem 1.14 is proved.  $\square$

## 2 Proof of Theorem 1.7

Recurrence relations (1.9) can be written in the matrix form

$$\begin{pmatrix} a_k \\ a_{k+1} \end{pmatrix} = \Lambda_k \begin{pmatrix} a_{k-1} \\ a_k \end{pmatrix}, \quad \Lambda_k = h_k^{-1} \begin{pmatrix} 0 & h_k \\ -f_k & -g_k \end{pmatrix}. \quad (2.1)$$

Consider the inverse matrices

$$\Lambda_k^{-1} = \begin{pmatrix} -\frac{g_k}{f_k} & -\frac{h_k}{f_k} \\ 1 & 0 \end{pmatrix} \quad (2.2)$$

and their projectivizations  $P_k : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$  acting on the projective line  $\mathbb{C}\mathbb{P}^1 = \overline{\mathbb{C}}$  with homogeneous coordinates  $(z_1 : z_2)$ . Let us introduce the affine coordinate  $w = \frac{z_2}{z_1}$  on it. For every  $C > 0$  we denote

$$D_C = \{|w| < C\} \subset \mathbb{C} \subset \overline{\mathbb{C}}.$$

**Proposition 2.1** *The transformations  $P_k$  converge to the constant mapping  $\mathbb{CP}^1 \mapsto \{w = 0\}$  uniformly on every closed disk  $\overline{D}_C$ ,  $C > 0$ , as  $k \rightarrow \infty$ . Their inverses converge to the constant mapping  $\mathbb{CP}^1 \mapsto \{w = \infty\}$  uniformly on the complement of every disk  $D_C$ .*

**Proof** The image of a vector  $(1, w)$  with  $|w| \leq C$  under the matrix  $\Lambda_k^{-1}$  is the vector

$$(u_k(w), v_k(w)) = -\frac{g_k}{f_k} \left(1 + \frac{h_k}{g_k} w, -\frac{f_k}{g_k}\right).$$

Recall that  $\frac{h_k}{g_k}, \frac{f_k}{g_k} \rightarrow 0$ , see (1.11), hence,  $g_k \neq 0$  for all  $k$  large enough. The latter asymptotics and formula together imply that  $\frac{v_k(w)}{u_k(w)} \rightarrow 0$  uniformly on  $\overline{D}_C$  and prove the first statement of the proposition. Let us prove its second statement. For every fixed  $C > 0$  and every  $k$  large enough (dependently on  $C$ ) one has  $P_k^{-1}(\mathbb{CP}^1 \setminus D_C) \subset \mathbb{CP}^1 \setminus D_C$ , by the first statement of the proposition. The image of a vector  $(1, w)$  under the matrix  $\Lambda_k$  is  $(s_k(w), t_k(w)) = (w, -\frac{g_k w + f_k}{h_k})$ . This together with (1.11) implies that  $\frac{s_k(w)}{t_k(w)} \rightarrow 0$  uniformly on  $|w| \geq C$ , as  $k \rightarrow \infty$ ; or equivalently,  $P_k \rightarrow \infty$  uniformly on  $\mathbb{CP}^1 \setminus D_C$ . The proposition is proved.  $\square$

**Proof of Theorem 1.7.** Let  $C > 1$ ,  $E_C$  denote the closed disk  $\overline{D}_C \subset \mathbb{CP}^1$  equipped with the Euclidean distance. There exist a  $0 < \lambda < 1$  and a  $N = N(\lambda, C) \in \mathbb{N}$  such that for every  $k > N$  one has  $P_k(E_C) \subset E_C$  and the mapping  $P_k : E_C \rightarrow E_C$  is a  $\lambda$ -contraction:  $|P_k(x) - P_k(y)| < \lambda|x - y|$ . This follows from the first statement of the proposition and Schwarz Lemma. The fixed point of the mapping  $P_k$  tends to 0, as  $k \rightarrow \infty$ , by uniform convergence (Proposition 2.1). This together with Theorem 1.14 implies that there exists a unique sequence  $(x_k)_{k \geq N}$  such that  $P_k(x_k) = x_{k-1}$  for all  $k > N$  and  $|x_k| \leq C$ . The latter sequence corresponds to a unique sequence  $(a_k)_{k \geq N}$  (up to multiplicative constant) such that  $x_k = (a_k : a_{k+1})$ ; one has  $|w(x_k)| = |\frac{a_{k+1}}{a_k}| \leq C$  for every  $k \geq N$ . The sequence  $(a_k)$  satisfies relations (1.9) for  $k > N$ , which are equivalent to the equalities  $P_k(x_k) = x_{k-1}$ . It extends to a unique sequence  $(a_k)_{k \geq 0}$  satisfying (1.9) for  $k \geq 1$ , as in Remark 1.8. In addition,  $x_k \rightarrow 0$ , i.e.,  $\frac{a_{k+1}}{a_k} \rightarrow 0$ , as  $k \rightarrow \infty$ , by (1.21) and since the attracting fixed points of the mappings  $P_k$  converge to 0, by Proposition 2.1. Therefore, the series  $\sum_{k \geq 0} a_k z^k$  converges on the whole complex line  $\mathbb{C}$ . The existence is proved. Now let us prove the uniqueness. Let, by contradiction, there exist a series  $\sum a_k z^k$  satisfying relations (1.9), having a positive convergence radius and not coinciding with the one constructed above. Then there exists a  $k > N$  such that  $|\frac{a_{k+1}}{a_k}| > C$ , i.e.,  $x_k \notin \overline{D}_C$ . For

every  $l > k$  one has  $x_l = P_l^{-1} \circ \dots \circ P_{k+1}^{-1}(x_k) \rightarrow \infty$ , that is,  $\frac{a_{l+1}}{a_l} \rightarrow \infty$ , as  $l \rightarrow \infty$ , by the second statement of Proposition 2.1. Hence the series diverges everywhere: has zero convergence radius. The contradiction thus obtained proves Theorem 1.7.  $\square$

### 3 Explicit formulas for solutions and the coefficients $d_{j\pm}$

Here we present explicit formulas for the unique converging series from Theorem 1.9 solving recurrence relations (1.6). First in Subsection 3.1 we provide a general method for writing them, which essentially repeats and slightly generalizes the method from [8, section 3, pp. 337–338]. Then we write them for  $k \rightarrow +\infty$ , and afterwards for  $k \rightarrow -\infty$ .

#### 3.1 Solution of recurrence relation via infinite matrix product: a general method

Here we consider a solution of general recurrence relations (1.9) from Theorem 1.7. Let  $g_k, f_k, h_k$  be the coefficients in (1.9). Let  $P_k : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$  be the projectivizations of the transformations  $\Lambda_k^{-1}$ , see (2.2). Let  $\sum_k a_k z^k$  be a solution to (1.9). Recall that we denote  $x_k = (a_k : a_{k+1}) \in \mathbb{CP}^1 \simeq \overline{\mathbb{C}}$ , in the standard coordinate  $w$  on  $\overline{\mathbb{C}}$  one has  $x_k = \frac{a_{k+1}}{a_k}$ . We have  $x_{k-1} = P_k(x_k)$ , and for every  $k$  the infinite product  $P_k P_{k+1} \dots$  converges to  $x_{k-1}$ . More precisely,  $P_k \circ \dots \circ P_m(z) \rightarrow x_{k-1}$ , as  $m \rightarrow \infty$  uniformly on compact subsets in  $\mathbb{C} = \overline{\mathbb{C}} \setminus \{\infty\}$ , as in the proof of Theorem 1.7.

One can then deduce that there exists a number sequence  $r_k$  such that for every  $k$  the infinite matrix product  $(r_k \Lambda_k^{-1})(r_{k+1} \Lambda_{k+1}^{-1}) \dots$  converges to a rank 1 matrix  $\mathcal{R}_k$  such that

$$\begin{pmatrix} a_{k-1} \\ a_k \end{pmatrix} = \mathcal{R}_k \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (3.1)$$

This would be an explicit formula for the solution  $\sum_k a_k z^k$ .

However the infinite product of the matrices  $\Lambda_k^{-1}$  themselves diverges, since their terms  $-\frac{g_k}{f_k}$  tend to infinity: one has to find a priori unknown normalizing constants  $r_k$ . To construct a converging matrix product explicitly, we will consider a rescaled sequence  $a_k$ , that is

$$c_k = q_k a_k, \quad q_k \in \mathbb{C}, \quad \frac{q_{k-1}}{q_k} \simeq -\frac{f_k}{g_k} \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

Rewriting relations (1.9) in terms of the new sequence  $c_k$  yields

$$\begin{pmatrix} c_{k-1} \\ c_k \end{pmatrix} = M_k \begin{pmatrix} c_k \\ c_{k+1} \end{pmatrix}, \quad (3.2)$$

$$M_k = \begin{pmatrix} -\frac{q_{k-1}}{q_k} \frac{g_k}{f_k} & -\frac{q_{k-1}}{q_{k+1}} \frac{h_k}{f_k} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 + o(1) & o(1) \\ 1 & 0 \end{pmatrix}.$$

The matrices  $M_k$  converge to the projector

$$P : \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad P = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Our goal is to choose the above rescaling so that the infinite products

$$R_k = M_k M_{k+1} \dots$$

converge: then the limit is a one-dimensional operator  $R_k$  with  $\ker R_k$  being generated by the vector  $(0, 1)$ . It appears that one can achieve the latter convergence by appropriate choice of normalizing constants  $q_k$ .

We use the following sufficient conditions of convergence of products of almost projectors  $M_k$ .

**Lemma 3.1** *Let  $H$  be either a finite dimensional, or a Hilbert space. Let  $M_k : H \rightarrow H$  be a sequence of bounded operators that tend (in the norm) to an orthogonal projector  $P : H \rightarrow H$ . Let*

$$M_k = P + S_k, \quad \sum_k \|S_k\| < \infty. \quad (3.3)$$

*Then the infinite product  $R_k = M_k M_{k+1} \dots$  converges in the norm, and  $\ker P \subset \ker R_k$ . One has  $R_k \rightarrow P$ , as  $k \rightarrow \infty$ , in the operator norm, and  $\ker R_k = \ker P$  for every  $k$  large enough.*

**Proof** Fix a  $k$  and set  $T_n = T_{k,n} = M_k \dots M_n$  for  $n \geq k$ ;  $T_k = M_k$ . One has

$$T_{n+1} = T_n M_{n+1} = T_n (P + S_{n+1}), \quad T_{n+1} - T_n P = T_n S_{n+1}.$$

The latter equality implies that

$$\|T_{n+1}\| \leq \|T_n\| (1 + \|S_{n+1}\|) \leq e^{\|S_{n+1}\|} \|T_n\|.$$

This implies that

$$\|T_{k,n}\| \leq C_k, \quad C_k = e^{\sum_{j \geq k} \|S_j\|} \|M_k\|. \quad (3.4)$$



Now one has

$$\begin{aligned} T_{k,n} &= T_{k,n-1}P + T_{k,n-1}S_n, \\ T_{k,n+1} &= T_{k,n}P + T_{k,n}S_{n+1} = T_{k,n-1}P^2 + T_{k,n-1}S_nP + T_{k,n}S_{n+1} \\ &= T_{k,n-1}P + T_{k,n-1}S_nP + T_{k,n}S_{n+1} : \end{aligned}$$

here we have used the equality  $P^2 = P$ , which holds for all the projectors. The two latter formulas together with (3.4) imply that

$$\|T_{k,n+1} - T_{k,n}\| \leq C_k(2\|S_n\| + \|S_{n+1}\|).$$

The latter right-hand side being a converging series in  $n$ , the sum of the left-hand sides in  $n$  converges and so does  $T_{k,n}$ , as  $n \rightarrow \infty$ , in the operator norm. This also implies that the norm distance of each  $T_{k,n}$  to the limit  $R_k = \lim_{n \rightarrow \infty} T_{k,n}$ , and in particular,  $\text{dist}(M_k, R_k)$  is bounded from above by  $\Delta_{k,n} = 3C_k \sum_{j \geq n-1} \|S_j\|$ . One has  $\Delta_{k,n} \rightarrow 0$ , as  $n \rightarrow \infty$  uniformly in  $k$ , and also  $\Delta_{k,n} \rightarrow 0$ , as  $k, n \rightarrow \infty$  so that  $k \leq n$ . This implies that  $R_k$  and  $M_k$  converge to the same limit  $P$  in the operator norm, as  $k \rightarrow \infty$ . For every  $v \in \ker P$  one has  $M_nv = S_nv \rightarrow 0$ . Hence,  $T_{k,n}v = T_{k,n-1}(S_nv) \rightarrow 0$ , as  $n \rightarrow \infty$ . Therefore,  $R_kv = 0$  and  $\ker P \subset \ker R_k$ . Let  $N > 0$  be such that for every  $k > N$  one has  $\|R_k - P\| < 1$ . Let us show that  $\ker R_k = \ker P$  for these  $k$ . Indeed, suppose the contrary:  $\ker R_k$  is strictly bigger than  $\ker P$ . Note that  $H = \ker P \oplus P(H)$  (orthogonal decomposition), since  $P$  is an orthogonal projector. Therefore, there exists a vector  $u_k \in P(H)$  such that  $R_k(u_k) = 0$ . Hence,

$$\|P(u_k)\| = \|(P - R_k)(u_k)\| < \|u_k\|,$$

while  $P(u_k) = u_k$ , since  $P$  is a projector. The contradiction thus obtained proves the lemma.  $\square$

**Addendum to Lemma 3.1.** *Let in Lemma 3.1 the operators  $S_n$  depend holomorphically on some parameters so that the series  $\sum_n \|S_n\|$  converges uniformly on compact subsets in the parameter space. Then the infinite products  $R_k$  are also holomorphic in the parameters.*

**Proof** The above proof implies that the sequence  $T_{k,n}$  converges uniformly on compact subsets in the parameter space. This together with the Weierstrass Theorem implies the holomorphicity of the limit.  $\square$

**Corollary 3.2** *Let*

$$M_k = \begin{pmatrix} 1 + \delta_{11,k} & \delta_{12,k} \\ 1 & 0 \end{pmatrix}, \quad \sum_k |\delta_{ij,k}| < \infty \text{ for } (ij) = (11), (12). \quad (3.5)$$

Then the infinite product  $R_k = M_k M_{k+1} \dots$  converges, and the right column of the limit product matrix  $R_k$  vanishes. In the case, when  $\delta_{12,k} \neq 0$  for all  $k$ , the limit matrix  $R_k$  has rank 1 for all  $k$ : its kernel is generated by the vector  $(0, 1)$ . In the case, when  $\delta_{ij,k}$  depend holomorphically on some parameters and the convergence of the corresponding series is uniform on compact sets, the limit  $R_k$  is also holomorphic.

**Addendum 1 to Corollary 3.2.** In the conditions of the corollary set

$$c_k = (0, 1) R_k \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (3.6)$$

Then the sequence  $c_k$  is a solution of recurrence relations (3.2) such that  $\frac{c_k}{c_{k-1}} \rightarrow 1$ , and one has

$$\begin{pmatrix} c_{k-1} \\ c_k \end{pmatrix} = R_k \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (3.7)$$

**Proof of Corollary 3.2.** This is the direct application of the lemma and its addendum for the norm induced by appropriate scalar product: the latter product should make the matrix

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

an orthogonal projector. The kernel  $\ker R_k$  contains the kernel  $\ker P$ , which is generated by the vector  $(0, 1)$ ;  $\ker R_k = \ker P$ , i.e.,  $rk(R_k) = 1$  for all  $k$  large enough, by the lemma. In particular, the right column in each  $R_k$  vanishes. Now it remains to note that  $rk(R_k) = 1$  for all  $k$ , since the matrices  $M_k$  are all non-degenerate:  $\delta_{12,k} \neq 0$ . The corollary is proved.  $\square$

**Proof of Addendum 1.** Consider the affine chart  $\mathbb{C} = \mathbb{CP}^1 \setminus \{(1 : -1)\}$  with the coordinate  $w = \frac{z_1 - z_2}{z_1 + z_2}$  centered at  $(1 : 1)$ . The projectivizations  $P_k$  of the linear operators  $M_k : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  converge to  $(1 : 1)$  uniformly on compact subsets in  $\mathbb{C}$ . Hence, for every  $C > 0$  there exist a  $N = N(C) > 0$  and an  $0 < \lambda < 1$  such that for every  $k \geq N$  one has  $P_k(\overline{D_C}) \Subset D_C$ , and  $P_k$  is a  $\lambda$ -contraction of the disk  $\overline{D_C}$ , as in the proof of Theorem 1.7 in the previous section. This together with Theorem 1.14 implies that there exists a sequence  $(x_k)_{k \geq N(C)}$ ,  $x_k \in \overline{\mathbb{C}} = \mathbb{CP}^1$ ,  $w(x_k) \rightarrow 0$ , as  $k \rightarrow \infty$ , such that  $P_k(x_k) = x_{k-1}$  and  $P_k \circ \dots \circ P_m$  converges to  $x_{k-1}$  uniformly on compact subsets in  $\mathbb{C}$ , as  $m \rightarrow \infty$ . Convergence at  $(1 : 0)$  implies that  $x_{k-1} = (R_{k,11} : R_{k,21}) = (c_{k-1} : c_k)$ . Moreover,  $R_{k,21} = R_{k+1,11}$ , since  $R_k = M_k R_{k+1}$  and the matrix  $M_k$  has lower row  $(1, 0)$ . The two last

statements together imply that the sequence  $c_k = R_{k,21}$  satisfies recurrence relations (3.2). One has  $\frac{c_{k+1}}{c_k} \rightarrow 1$ , since  $x_k = (c_k : c_{k+1}) \rightarrow (1 : 1)$ . This proves the addendum.  $\square$

**Corollary 3.3** *Consider recurrence relations (1.9). Let  $q_k \in \mathbb{C}$  be a sequence such that the rescaling  $c_k = q_k a_k$  transforms (1.9) to (3.2). Let the corresponding matrices  $M_k$  from (3.2) be the same, as in (3.5). Let  $c_k$  be the same, as in (3.6). Then the sequence*

$$a_k = q_k^{-1} c_k$$

*is a solution of relations (1.9) such that the series  $\sum_{k \geq 0} a_k z^k$  converges on all of  $\mathbb{C}$ .*

**Proof** The sequence  $(a_k)$  is a solution of (1.9), by construction and Addendum 1. One has  $\frac{q_{k-1}}{q_k} \simeq -\frac{f_k}{g_k} \rightarrow 0$ , as  $k \rightarrow \infty$ , since the above sequence rescaling transforms (1.9) to (3.2). Therefore,  $\frac{a_k}{a_{k-1}} \rightarrow 0$ , by the latter statement and since  $\frac{c_k}{c_{k-1}} \rightarrow 1$ , as was shown above. This implies the convergence of the series  $\sum_{k \geq 0} a_k z^k$  on  $\mathbb{C}$  and proves the corollary.  $\square$

### 3.2 Forward solutions from Theorems 1.1 and 1.9

Here we give explicit formulas for the solution  $\sum_k a_k z^k$  of recurrence relations (1.6) converging, as  $k \rightarrow +\infty$ .

**Case 1):  $b, b+n \notin \mathbb{Z}$  (i.e., the conditions of Theorem 1.1 hold).** Let us invert matrix relation (1.7). We get

$$\begin{pmatrix} a_{k-1} \\ a_k \end{pmatrix} = W_k \begin{pmatrix} a_k \\ a_{k+1} \end{pmatrix}, \quad (3.8)$$

$$W_k = \begin{pmatrix} \frac{k+b}{\mu} \left(1 + \frac{\lambda}{(k+b)(k+b+n-1)}\right) & \frac{k+b+1}{k+b+n-1} \\ 1 & 0 \end{pmatrix}.$$

To obtain an explicit formula for solution of relation (1.6), we will use results of Subsection 3.1. To do this, we reduce equation (3.8) to a similar equation with the matrix in the right-hand side converging to a projector. This is done by renormalizing the sequence  $a_k$  by multiplication by appropriate constants depending on  $k \geq 0$ . Namely, set

$$c_k = \frac{a_k(b)_{k+1}}{\mu^k}, \quad (b)_l := b \dots (b+l-1) = \frac{\Gamma(b+l)}{\Gamma(b)}.$$

Recall that the symbol  $(b)_l$  is called the *Pochhammer symbol*. Translating relations (3.8) in terms of the sequence  $c_k$  yields

$$\begin{aligned} \begin{pmatrix} c_{k-1} \\ c_k \end{pmatrix} &= M_k \begin{pmatrix} c_k \\ c_{k+1} \end{pmatrix}, \\ M_k &= \begin{pmatrix} 1 + \frac{\lambda}{(k+b)(k+b+n-1)} & \frac{\mu^2}{(k+b)(k+b+n-1)} \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{(b)_k}{\mu^{k-1}} & 0 \\ 0 & \frac{(b)_{k+1}}{\mu^k} \end{pmatrix} W_k \begin{pmatrix} \frac{\mu^k}{(b)_{k+1}} & 0 \\ 0 & \frac{\mu^{k+1}}{(b)_{k+2}} \end{pmatrix}. \end{aligned} \quad (3.9)$$

The infinite matrix product

$$R_k = M_k M_{k+1} \dots \quad (3.10)$$

converges and depends analytically on  $(\lambda, \mu, n, b)$  whenever the denominators in its definition do not vanish, by Corollary 3.2.

**Theorem 3.4** *Let  $b, b+n \notin \mathbb{Z}$ . Set*

$$c_k = (0 \ 1) R_k \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (3.11)$$

$$a_k = \frac{\mu^k}{(b)_{k+1}} c_k \quad (3.12)$$

*The coefficients  $a_k$  satisfy recurrence relations (1.6), for all  $k$ , and the series*

$$f_+(z) = \sum_{k \geq 1} a_k z^k \quad (3.13)$$

*converges on all of  $\mathbb{C}$ .*

**Proof** The sequence  $c_k$  satisfies relations (3.9), and  $\frac{c_k}{c_{k-1}} \rightarrow 1$ , as  $k \rightarrow \infty$ , by Addendum 1 to Corollary 3.2. This implies that  $a_k$  satisfy (1.6). The series (3.13) converges on all of  $\mathbb{C}$ , by Corollary 3.3. This proves the theorem.  $\square$

**Case 2): some of the numbers  $b$  or  $b+n$  is an integer.** Set

$$k_{0+} = \max\{m \in \{-1-b, 1-b-n\} \mid m \in \mathbb{Z}\}. \quad (3.14)$$

Note that now the product  $(b)_l = b(b+1)\dots(b+l-1)$  can be equal to zero, and thus, the sequence  $a_k$  defined by (3.12) is not necessarily well-defined.

Let us modify the above rescaling coefficients relating  $a_k$  and  $c_k$  as follows. For every  $s \leq l + 1$  set

$$(b)_{s,l+1} = (b+s) \dots (b+l) = \frac{(b)_{l+1}}{(b)_s}; \quad (b)_{0,l+1} = (b)_{l+1}; \quad (b)_{s,s} = 1. \quad (3.15)$$

Set

$$c_k = a_k \mu^{k_{0+}+1-k} (b)_{k_{0+}+2,k+1} \text{ for every } k > k_{0+}; \quad c_{k_{0+}+1} = a_{k_{0+}+1}. \quad (3.16)$$

The sequence  $(a_k)$  satisfies (1.7), if and only if the sequence  $(c_k)$  satisfies (3.9). The above formulas remain valid with the same matrices  $M_k$ , which are well-defined for  $k \geq k_{0+} + 2$ : the denominators in its fractions do not vanish. Therefore, the infinite product  $R_k = M_k M_{k+1} \dots$  is well-defined for the same  $k$  in the case under consideration.

**Theorem 3.5** *Let  $b, n, k_{0+}$  be as above,  $M_k$  be as in (3.9),  $R_k = M_k M_{k+1} \dots$ ,*

$$c_k = R_{k,21} \text{ for } k \geq k_{0+} + 2, \quad c_{k_{0+}+1} = R_{k_{0+}+2,11}, \quad (3.17)$$

$$a_k = \mu^{k-k_{0+}-1} \frac{c_k}{(b)_{k_{0+}+2,k+1}} \text{ for } k \geq k_{0+} + 1, \quad (3.18)$$

$$a_{k_{0+}} = \frac{((k_{0+} + b + 1)(k_{0+} + b + n) + \lambda)a_{k_{0+}+1} + \mu(k_{0+} + b + 2)a_{k_{0+}+2}}{\mu(k_{0+} + b + n)}. \quad (3.19)$$

The sequence  $(a_k)_{k \geq k_{0+}}$  satisfies recurrence relations (1.6) for  $k > k_{0+}$ . The series  $\sum_{k=k_{0+}}^{+\infty} a_k z^k$  converges on all of  $\mathbb{C}$ .

**Proof** The sequence  $c_k$  satisfies relations (3.9) for  $k \geq k_{0+} + 2$ , by Corollary 3.2. Therefore,  $a_k$  satisfy relations (1.7), which are equivalent to (1.6), see the previous discussion. Formula (3.19) is equivalent to relation (1.6) for  $k = k_{0+} + 1$ . The denominator  $\mu(k_{0+} + b + n)$  in (3.19) does not vanish. In the case, when  $b + n \notin \mathbb{Z}$ , this is obvious. In the case, when  $b + n \in \mathbb{Z}$ , one has  $k_{0+} + b + n \geq 1$ , by (3.14). The series  $\sum_{k \geq k_{0+}} a_k z^k$  converges on  $\mathbb{C}$ , as in the previous subsection. The theorem is proved.  $\square$

### 3.3 Backward solutions

Here we give explicit formulas for the solution  $\sum_k a_k z^{-k}$  of recurrence relations (1.6) with  $k \rightarrow -\infty$ . Set

$$m = -k, \quad \hat{a}_m = a_{-m}.$$

Relation (1.7) in new variables  $m$  and  $\hat{a}_m$  takes the matrix form

$$\begin{pmatrix} \hat{a}_m \\ \hat{a}_{m-1} \end{pmatrix} = A_{-m} \begin{pmatrix} \hat{a}_{m+1} \\ \hat{a}_m \end{pmatrix}.$$

Writing the latter equation with permuted order of vector components (we place  $\hat{a}_s$  having smaller indices above) yields the same equation with the new matrix obtained from  $A_{-m}$  by permutation of lines and columns:

$$\begin{pmatrix} \hat{a}_{m-1} \\ \hat{a}_m \end{pmatrix} = \hat{A}_m \begin{pmatrix} \hat{a}_m \\ \hat{a}_{m+1} \end{pmatrix}, \quad (3.20)$$

$$\hat{A}_m = \frac{b+n-m-1}{b-m+1} \begin{pmatrix} -\frac{\lambda+(b-m)(b-m+n-1)}{\mu(b-m+n-1)} & 1 \\ \frac{b-m+1}{b-m+n-1} & 0 \end{pmatrix}. \quad (3.21)$$

**Case 1):**  $b, b+n \notin \mathbb{Z}$ , as in the conditions of Theorem 1.1. Let us renormalize the sequence  $\hat{a}_m$ : set

$$\hat{c}_m = \frac{\hat{a}_m(2-n-b)_{m+1}}{\mu^m}.$$

Translating equation (3.21) in terms of the sequence  $\hat{c}_m$  yields

$$\begin{pmatrix} \hat{c}_{m-1} \\ \hat{c}_m \end{pmatrix} = S_m \begin{pmatrix} \hat{c}_m \\ \hat{c}_{m+1} \end{pmatrix}, \quad (3.22)$$

$$\begin{aligned} S_m = S_m(b, n) &= \begin{pmatrix} 1 + \frac{\lambda-n+2}{(b-m+1)(b-m+n-2)} & \frac{\mu^2(b-m+n-1)}{(b-m+1)(b-m+n-2)(b-m+n-3)} \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{(2-n-b)_m}{\mu^{m-1}} & 0 \\ 0 & \frac{(2-n-b)_{m+1}}{\mu^m} \end{pmatrix} \hat{A}_m \begin{pmatrix} \frac{\mu^m}{(2-n-b)_{m+1}} & 0 \\ 0 & \frac{\mu^{m+1}}{(2-n-b)_{m+2}} \end{pmatrix}. \end{aligned} \quad (3.23)$$

**Theorem 3.6** *Let  $b, b+n \notin \mathbb{Z}$ . Let the matrices  $S_m$  be as above,*

$$T_m = S_m S_{m+1} \dots, \quad (3.24)$$

$$\hat{c}_m = \begin{pmatrix} 0 & 1 \end{pmatrix} T_m \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{a}_m = \frac{\hat{c}_m \mu^m}{(2-n-b)_{m+1}}, \quad a_k = \hat{a}_{-k}. \quad (3.25)$$

*The sequence  $(a_k)$  satisfies recurrence relations (1.6), and the series*

$$f_-(z) = \sum_{k \leq 0} a_k z^{-k} \quad (3.26)$$

*converges on  $\mathbb{C}$ .*

**Proof** The above matrix product converges, and the sequence  $\hat{c}_m$  satisfies equation (3.22), by Corollary 3.2. This implies that the corresponding sequence  $\hat{a}_m$  satisfies (3.20), the sequence  $a_k$  satisfies (1.6) and the series  $f_-(z)$  converges, as in the previous subsection. This proves the theorem.  $\square$

**Case 2): some of the numbers  $b$  or  $b + n$  is an integer.** Let

$$k_{0-} = \min\{r \in \{-1 - b, 1 - b - n\} \mid r \in \mathbb{Z}\}, \quad m_0 = -k_{0-}. \quad (3.27)$$

The above Pochhammer symbol may be not well-defined in the case, when  $n + b \in \mathbb{Z}$ ,  $2 - n - b < 0$ . We use the inequalities

$$b - m + 1 \neq 0 \text{ for every } m > m_0;$$

$$2 - n - b + m, 3 - n - b + m \neq 0 \text{ for every } m \geq m_0, \quad (3.28)$$

which follow immediately from (3.27). The sequence rescaling

$$\hat{c}_m = \frac{\hat{a}_m(2 - n - b)_{m_0, m+1}}{\mu^m}$$

is well-defined and invertible for all  $m \geq m_0$ , by (3.28). It differs from the previous one by multiplication by constant independent on  $m$ , and hence, transforms (3.20) to (3.22), as above. The matrices  $S_m$  are well-defined for  $m > m_0$ : the denominators in their fractions do not vanish, by (3.28). Let  $T_m$  be their products (3.24) defined for  $m > m_0$ .

**Theorem 3.7** *Let  $S_m$  and  $T_m$  be the same, as in (3.24),*

$$\hat{c}_m = (0 \quad 1) T_m \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for } m > m_0, \quad c_{m_0} = T_{m_0+1}, \quad 11,$$

$$\hat{a}_m = \frac{\hat{c}_m \mu^m}{(2 - n - b)_{m_0, m+1}} \text{ for } m \geq m_0, \quad a_k = \hat{a}_{-k}. \quad (3.29)$$

*The sequence  $(a_k)_{k \leq k_{0-}}$  satisfies recurrence relations (1.6), and the series  $\sum_{k \leq k_{0-}} a_k z^{-k}$  converges on  $\mathbb{C}$ .*

The proof of Theorem 3.7 repeats the proof of Theorem 3.6 with obvious changes.

### 3.4 Theorem 1.1: formulas for $d_{0\pm}$ and $d_{1\pm}$

**Lemma 3.8** *Let  $f_+(z) = \sum_{k \geq 1} a_k z^k$  and  $f_-(z^{-1}) = \sum_{k \leq 0} a_k z^{-k}$  be the functions from (3.13) and (3.26) constructed in the two previous subsections, case 1). Then*

$$z^{-b} \mathcal{L}(z^b f_{\pm}(z^{\pm 1})) = d_{0\pm} + d_{1\pm} z, \quad (3.30)$$

$$d_{0+} = \mu(b+1)a_1; \quad d_{1+} = ((b+1)(b+n) + \lambda)a_1 + \mu(b+2)a_2, \quad (3.31)$$

where  $a_1$  and  $a_2$  are the same, as in (3.12);

$$d_{0-} = (b(b+n-1) + \lambda)a_0 - \mu(b+n-1)a_{-1}; \quad d_{1-} = -\mu(b+n)a_0, \quad (3.32)$$

where  $a_{-1}$  and  $a_0$  are the same, as in (3.25).

**Proof** The left-hand side in (3.30) with index “+” is a Taylor series with coefficients at  $z^k$  being equal to the left-hand side of the corresponding recurrence relation (1.6). The latter relation holds for all  $k \geq 2$ , by construction. This implies (3.30) with  $d_{0+}$ ,  $d_{1+}$  being equal to the left-hand sides of relations (1.6) for  $k = 0$  and  $k = 1$  respectively. This implies (3.31). The proof for the index “-” is analogous.  $\square$

## 4 Application: monodromy eigenvalues

Here we study the eigenfunctions of the monodromy operator of Heun equation (1.1). This is the operator acting on the space of germs of solutions at a point  $z_0 \neq 0$  by analytic extension along a positive circuit around zero. Each monodromy eigenfunction with eigenvalue  $e^{2\pi i b}$  has the form of a series

$$E(z) = \sum_{k \in \mathbb{Z}} a_k z^{k+b}, \quad b \in \mathbb{C}, \quad (4.1)$$

converging on  $\mathbb{C}^*$ . Here we write down an explicit analytic equation on those  $b$ , for which the latter solution  $E(z)$  of equation (1.1) exists, i.e., there exists a bi-infinite sequence  $(a_k)_{k \in \mathbb{Z}}$  satisfying recurrence relations (1.6) such that the bi-infinite series  $\sum_{k \in \mathbb{Z}} a_k z^k$  converges on  $\mathbb{C}^*$ .

We consider different cases, but the method of finding the above  $b$  is general for all of them. The coefficients  $a_k$  with  $k \rightarrow +\infty$  should form a unique converging series (up to multiplicative constant) that satisfies recurrence relations (1.6). Similarly, its coefficients with  $k \rightarrow -\infty$  should form a unique converging series satisfying (1.6). Finally, the above positive and negative parts of the series should paste together and form a solution of



Heun equation. In the simplest, non-resonant case, when  $b, b + n \notin \mathbb{Z}$ , the pasting equation is given by (1.5). The coefficients  $a_k$ ,  $k \geq 1$  satisfying (1.6) for  $k \geq 2$  and forming a converging series are given by formulas (3.12); the sequence  $(a_k)_{k \leq 0}$ ,  $a_k = \hat{a}_{-k}$ , satisfying (1.6) for  $k < 0$  and forming a converging series is given by formula (3.25).

It appears that substituting the above-mentioned formulas for  $a_k$  to formulas (3.31) and (3.32) for  $d_{j\pm}$  and then substituting the latter formulas to (1.5) yields a rather complicated pasting equation. To obtain a simpler formula, we proceed as follows. In the non-resonant case we extend the sequence  $(a_k)_{k \geq 1}$  to  $k = 0$  by putting appropriate  $\alpha \in \mathbb{C}$  instead  $a_0$  (we get  $\alpha, a_1, a_2, \dots$ ) so that the longer sequence thus obtained satisfies (1.6) also for  $k = 0$ . Similarly, we extend the sequence  $(a_k)_{k \leq 0}$  to  $k = 1$  by putting appropriate  $\beta \in \mathbb{C}$  instead of  $a_1$  (we get  $\dots a_{-1}, a_0, \beta$ ) in order to satisfy equation (1.6) for  $k = 0$ . The positive and negative series thus constructed paste together to a converging bi-infinite series  $\sum_{k \in \mathbb{Z}} a_k z^k$  satisfying (1.6) (after their rescaling by multiplicative constants), if and only if

$$\alpha\beta = a_0 a_1. \quad (4.2)$$

We obtain an explicit expression for equation (4.2).

In what follows, we use the two next propositions.

**Proposition 4.1** *The determinant of the monodromy operator of Heun equation (1.1) equals  $e^{-2\pi in}$ .*

**Proof** The monodromy matrix is the product of the formal monodromy matrix  $\text{diag}(e^{-2\pi in}, 1)$  and a pair of unipotent matrices: the inverse to the Stokes matrices, cf. [15, formulas (2.15) and (3.2)]. Therefore, its determinant equals  $e^{-2\pi in}$ . Another possible proof would be to use the formula for Wronskian of two linearly independent solutions of equation (1.1) from [8, p. 339, proof of theorem 4]. It shows that the Wronskian equals  $z^{-n}$  times a function holomorphic on  $\mathbb{C}^*$ , and hence, it gets multiplied by  $e^{-2\pi in}$  after analytic continuation along a positive circuit around zero.  $\square$

Recall [8, p. 336, lemma 1] that the transformation  $\# : E \mapsto \#E$ :

$$(\#E)(z) := 2\omega z^{-n}(E'(z^{-1}) - \mu E(z^{-1})), \quad \lambda + \mu^2 = \frac{1}{4\omega^2}, \quad \omega > 0 \quad (4.3)$$

is an involution acting on the space of solutions of equation (1.1).

**Proposition 4.2** *Let the monodromy operator of Heun equation have distinct eigenvalues. Then the involution  $\#$  permutes the corresponding eigenfunctions.*

**Proof** The involution under question is a composition of transformation of a function to its linear combination with its derivative, the variable change  $z \mapsto z^{-1}$  and multiplication by  $z^{-n}$ . Let now  $E$  be a monodromy eigenfunction with eigenvalue  $\lambda_1$ . The composition of the first and second operations transforms  $E$  to a function whose monodromy extension along positive circuit around the origin multiplies it by  $\lambda_1^{-1}$ : the second operation inverses the direction of the circuit. The multiplication by  $z^{-n}$  multiplies the above result of analytic extension by  $e^{-2\pi in}$ . Therefore,  $\#E$  is a monodromy eigenfunction with the eigenvalue  $\lambda_2 = \lambda_1^{-1}e^{-2\pi in}$ . It coincides with the second monodromy eigenvalue, since it is found by the condition that  $\lambda_1\lambda_2 = e^{-2\pi in}$ , see Proposition 4.1. This proves the proposition.  $\square$

#### 4.1 Nonresonant case: $b, b+n \notin \mathbb{Z}$

In this case the denominators in formulas (3.9) and (3.23) for the matrices  $M_k$  and  $S_m$  respectively are nonzero for all integer  $k$  and  $m$ , and hence, the matrices are well-defined together with the infinite products  $R_k = M_k M_{k+1} \dots$ ,  $T_m = S_m S_{m+1} \dots$ .

**Theorem 4.3** *Let  $b, b+n \notin \mathbb{Z}$ . Equation (1.1) has a monodromy eigenfunction with eigenvalue  $e^{2\pi ib}$ ,  $b \in \mathbb{C}$ , if and only if*

$$(b+1)(b+n-2)R_{1,11}T_{0,11} + \mu^2 R_{1,21}T_{0,21} = 0. \quad (4.4)$$

**Proof** Let  $f_+(z) = \sum_{k \geq 1} a_k z^k$  be a converging series satisfying (1.6) for  $k \geq 2$ . Recall that

$$a_1 = \frac{\mu}{b(b+1)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Set

$$\alpha = \frac{1}{b} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{b} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} R_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The sequence  $\alpha, a_1, a_2, \dots$  satisfies (1.6) for  $k \geq 1$ , by Theorem 3.4. Recall that

$$\hat{c}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{a}_0 = a_0 = \frac{\hat{c}_0}{2-n-b}$$

Set

$$\beta = \mu^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} T_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The sequence  $\dots, a_{-2}, a_{-1}, a_0, \beta$  satisfies (1.6) for all  $k \leq 0$ , by Theorem 3.6. Substituting the above formulas for  $a_1, \alpha, a_0, \beta$  to pasting equation (4.2) yields (4.4). The theorem is proved.  $\square$

## 4.2 One-resonant case: only one of the numbers $b$ , $b + n$ is integer

Note that in the case under consideration one has  $n \notin \mathbb{Z}$ .

**Proposition 4.4** *Let  $n \notin \mathbb{Z}$ . Then Heun equation (1.1) has a solution of type  $E(z) = z^b f(z)$  with  $b$  satisfying the assumption of the subsection and the function  $f(z)$  being holomorphic on  $\mathbb{C}^*$ , if and only if it has a solution holomorphic on  $\mathbb{C}^*$ , i.e., corresponding to  $b = 0$ . In this case the monodromy eigenvalues are 1 and  $e^{-2\pi i n} \neq 1$ .*

**Proof** Let the above solution  $E$  exist. Then it is a monodromy eigenfunction with the eigenvalue  $e^{2\pi i b}$ . The other eigenvalue equals  $e^{-2\pi i(b+n)}$ , by Proposition 4.1. Exactly one eigenvalue equals one, by assumption. The monodromy eigenfunction corresponding to unit eigenvalue is holomorphic on  $\mathbb{C}^*$ . Conversely, a solution holomorphic on  $\mathbb{C}^*$  is a solution  $E$  as above with  $b = 0$ . The last statement of the proposition follows from the existence of unit eigenvalue and Proposition 4.1. This proves proposition 4.4.  $\square$

**Corollary 4.5** *Let  $n \notin \mathbb{Z}$ . A solution  $E$  as in Proposition 4.4 exists, if and only if the recurrence relations (1.6) with  $b = 0$ :*

$$(k(k+l) + \lambda)a_k - \mu(k+l)a_{k-1} + \mu(k+1)a_{k+1} = 0, \quad l = n - 1 \quad (4.5)$$

have a solution  $(a_k)_{k \in \mathbb{Z}}$  such that the series  $\sum_{k \in \mathbb{Z}} a_k z^k$  converges on  $\mathbb{C}^*$ .

**Proposition 4.6** *Every semiinfinite sequence  $(a_k)_{k \geq -2}$  satisfying equations (4.5) for  $k \geq -1$  (without convergence condition) satisfies the relation*

$$(1 - l + \lambda)a_{-1} - \mu(l - 1)a_{-2} = 0. \quad (4.6)$$

**Proof** Equation (4.6) coincides with (4.5) for  $k = -1$ .  $\square$

**Corollary 4.7** *Let  $l \notin \mathbb{Z}$ . A solution  $(a_k)_{k \in \mathbb{Z}}$  to (4.5) with the series  $\sum_k a_k z^k$  converging on  $\mathbb{C}^*$  exists, if and only if the unique semiinfinite sequence  $(a_k)_{k \leq -1}$  solving (4.5) for  $k \leq -2$  with series  $\sum_{k=-\infty}^{-1} a_k z^k$  converging on  $\mathbb{C}^*$  satisfies relation (4.6).*

**Proof** Let

$$f(z) = \sum_{k=-\infty}^{-1} a_k z^k \quad (4.7)$$

be a semiinfinite solution of recurrence relations (4.5) for  $k \leq -2$ . Note that for every  $k \leq -1$  its coefficient  $a_k$  is uniquely determined as a linear combination of the two previous ones  $a_{k-2}$  and  $a_{k-1}$ , see (4.5) written for  $k \leq -2$ . The same holds in the opposite sense: for every  $k \leq -3$  the coefficient  $a_k$  is expressed as a linear combination of the coefficients  $a_{k+1}$  and  $a_{k+2}$  by (4.5), since  $l \notin \mathbb{Z}$ . The two latter statements together imply that  $a_{-2}$  and  $a_{-1}$  do not both vanish. Therefore, the above negative semiinfinite series can be extended to positive  $k$  as a (may be just formal) two-sided solution of (4.5) only in the case, when relation (4.6) holds. Let us show that in this case it does extend to a true (not just formal) two-sided solution.

Note that  $a_{-1} \neq 0$ , by relation (4.6) and since  $a_{-2}$ ,  $a_{-1}$  do not vanish both and  $l \neq 1$ . Equation (4.5) with  $k = -1$  has zero multiplier at  $a_0$  and hence, holds for arbitrary  $a_0$ . The same equation with  $k = 0$  yields

$$\lambda a_0 - l\mu a_{-1} + \mu a_1 = 0. \quad (4.8)$$

This is a linear non-homogeneous equation on the pair  $(a_0, a_1)$ . Hence, its solutions form a line  $L_1 \subset \mathbb{C}^2$  that does not pass through the origin:  $a_{-1} \neq 0$ . The pairs  $(a_0, a_1)$  extendable to semiinfinite solutions in positive  $k$  are all proportional (uniqueness of solution up to constant factor and since for every  $k \geq 0$  the coefficient  $a_{k+1}$  is uniquely determined by  $a_k$  and  $a_{k-1}$  via relations (4.5)). Hence, they form a line  $L_0$  through the origin. Let us choose  $(a_0, a_1)$  to be the intersection of the above lines  $L_0$  and  $L_1$ , provided they are not parallel (the case of parallel lines is discussed below). Then the pair  $(a_0, a_1)$  extends to a semiinfinite solution of relations (4.5) in positive  $k$ , by construction. The complete Laurent series  $\sum_{k=-\infty}^{+\infty} a_k z^k$  thus constructed is a solution to equations (4.5) and hence, to Heun equation (1.1).

**Case, when  $L_0$  and  $L_1$  are parallel.** In this case  $(a_0, a_1)$  defines a solution to (4.5) with positive  $k$ , if and only if  $\lambda a_0 + \mu a_1 = 0$ . This solution extends to negative  $k$  by putting  $a_k = 0$  for  $k < 0$ , since relation (5.11) for  $k = -1$  is equivalent to (4.6). Finally we obtain a *converging Taylor series* satisfying (4.5) and hence, presenting a solution of Heun equation (1.1) holomorphic on  $\mathbb{C}$ . This implies that  $n \in \mathbb{Z}$ . Indeed, if  $n \notin \mathbb{Z}$ , then the corresponding Heun equation has no entire solution, or equivalently, the corresponding recurrence relations (1.6) have no solution  $(a_k)$  with  $a_k = 0$  for all  $k < 0$ . This follows from the fact that if  $n \notin \mathbb{Z}$ , then for every  $k$  the coefficients at  $a_{k\pm 1}$  in the corresponding relation (1.6) do not vanish: hence, no two neighbor coefficients  $a_k$  and  $a_{k+1}$  of a solution vanish simultaneously. The contradiction thus obtained shows that the case of parallel lines is impossible, if  $n \notin \mathbb{Z}$ . Corollary 4.7 is proved.  $\square$

**Theorem 4.8** *Let  $n \notin \mathbb{Z}$ ,*

$$S_m = \begin{pmatrix} 1 + \frac{\lambda - n + 2}{(1-m)(n-m-2)} & \frac{\mu^2(n-m-1)}{(1-m)(n-m-2)(n-m-3)} \\ 1 & 0 \end{pmatrix} \text{ for } m \geq 2,$$

$$T_m = S_m S_{m+1} \dots$$

*Heun equation (1.1) has a solution holomorphic on  $\mathbb{C}^*$ , if and only if*

$$(2 - n + \lambda)(4 - n)T_{2,11} - \mu^2(n - 2)T_{2,21} = 0 \quad (4.9)$$

**Proof** Set

$$\begin{aligned} \hat{c}_m &= T_{m,21} = T_{m+1,11}, \quad \hat{c}_1 = T_{2,11}, \\ \hat{a}_m &= \frac{\hat{c}_m \mu^m}{(2-n)_{m+1}} = a_{-m}, \quad m \geq 1. \end{aligned}$$

The sequence  $(a_k)_{k \leq -1}$  satisfies recurrence relations (4.5) for  $k \leq -2$ , as in Subsection 3.3, and the series  $\sum_{k \leq -1} a_k z^{-k}$  converges on  $\mathbb{C}$ : here we have rewritten the formulas from Subsection 3.3 for  $b = 0$ . One has

$$a_{-2} = \frac{\mu^2 T_{2,21}}{(2-n)(3-n)(4-n)}, \quad a_{-1} = \frac{\mu T_{2,11}}{(2-n)(3-n)},$$

by definition. Substituting the latter formulas and  $l = n - 1$  to (4.6) yields (4.9). This together with Corollary 4.7 proves the theorem.  $\square$

### 4.3 Double resonant case: $n, b \in \mathbb{Z}$

Recall that we study the existence of solution (4.1) of Heun equation (1.1). In the case under consideration  $b \in \mathbb{Z}$ , and without loss of generality we can and will consider that  $b = 0$ . In this case a solution we are looking for is holomorphic on  $\mathbb{C}^*$  and presented by a Laurent series  $E(z) = \sum_{k \in \mathbb{Z}} a_k z^k$  converging on  $\mathbb{C}^*$ . Without loss of generality we will also consider that  $l = n - 1 \geq 0$ . One can achieve this by applying the transformation

$$\diamond : E(z) \mapsto e^{\mu(z+z^{-1})} E(-z^{-1}),$$

which is an isomorphism of the solution space of equation (1.1) (written in terms of the parameter  $l = n - 1$ ) onto the solution space of the same equation with  $l = n - 1$  replaced by  $-l$ , see [8, section 4, formula (39)]:

$$\mathcal{L}E = z^2 E'' + ((-l + 1)z + \mu(1 - z^2))E' + (\lambda + \mu(l - 1)z)E = 0 \quad (4.10)$$

It sends solutions of equation (1.1) holomorphic on  $\mathbb{C}^*$  onto solutions of equation (4.10) holomorphic on  $\mathbb{C}^*$ .

**Theorem 4.9** *Let  $l \in \mathbb{Z}$ ,  $l \geq 0$ . Equation (1.1) with  $n = l+1$  has a solution holomorphic on  $\mathbb{C}^*$ , if and only if its monodromy is unipotent. This happens, if and only if equation (1.1) satisfies one of the two following incompatible statements:*

- 1) *either it has an entire solution, i.e., holomorphic on  $\mathbb{C}$ ;*
- 2) *or the corresponding equation (4.10) has a nontrivial polynomial solution.*

Theorem 4.9 will be proved below. The sets of parameter values for which statements 1) or 2) hold were already described in [7, 8, 11]. Let us recall this description. To do this, consider the following matrices  $M_k$ ,  $R_k$  and numbers  $a_k$ ,  $\xi_l$ :

$$M_k = \begin{pmatrix} 1 + \frac{\lambda}{k(k+l)} & \frac{\mu^2}{k(k+l)} \\ 1 & 0 \end{pmatrix}, \quad R_k = M_k M_{k+1} \dots \quad \text{for } k \geq 1,$$

$$a_k = \frac{\mu^k}{k!} R_{k,21} \quad \text{for } k \geq 1, \quad a_0 = R_{1,11},$$

$$\xi_l = \xi_l(\lambda, \mu) = \lambda a_0 + \mu a_1 = \lambda R_{1,11} + \mu^2 R_{1,21}. \quad (4.11)$$

**Theorem 4.10** *A Heun equation (1.1) with  $l = n - 1 \in \mathbb{Z}$ ,  $l \geq 0$  has an entire solution, if and only if  $\xi_l(\lambda, \mu) = 0$ .*

Theorem 4.10 is equivalent to Corollary 1.5. It was partly proved and conjectured in [8, section 3, theorem 2] and proved completely in [11, subsection 3.1, theorem 3.5]. For completeness of presentation let us give its direct proof without using results of loc. cit.

**Proof of Theorem 4.10.** The above matrices  $M_k$  and numbers  $a_k$  coincide with those from (3.9) and (3.18) respectively constructed for recurrence relations (1.6) with  $b = 0$ ,  $n = l + 1$ ,

$$(k(k+l) + \lambda)a_k - \mu(k+l)a_{k-1} + \mu(k+1)a_{k+1} = 0; \quad (4.12)$$

here  $k_{0+} = -1$ . (If  $l = 0$ , then  $k_{0+} = 0$ , but the corresponding sequence  $a_k$  from (3.18) remains the same, as in (4.11), up to constant factor.) This together with Theorem 3.5 implies that the sequence  $(a_k)$  satisfies (4.12) for  $k \geq 1$  and the series  $E(z) = \sum_{k=0}^{+\infty} a_k z^k$  converges on  $\mathbb{C}$ . Therefore,  $\mathcal{L}E = \text{const}$ , and the latter constant is the left-hand side of the relation (4.12) corresponding to  $k = 0$ : that is,  $\lambda a_0 + \mu a_1 = \xi_l(\lambda, \mu)$ . This together with the uniqueness of an entire function  $E$  for which  $\mathcal{L}E = \text{const}$  (Theorem 1.3) implies the statement of Theorem 4.10.  $\square$

To describe equations (4.10) with polynomial solutions, consider the three-diagonal matrix

$$H = \begin{pmatrix} 0 & \mu & 0 & 0 & 0 & 0 \dots & 0 \\ \mu(l-1) & 1-l & 2\mu & 0 & 0 & \dots & 0 \\ 0 & \mu(l-2) & -2(l-2) & 3\mu & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 2\mu & -2(l-2) & (l-1)\mu \\ 0 & \dots & 0 & 0 & 0 & \mu & 1-l \end{pmatrix} : \quad (4.13)$$

$$H_{ij} = 0 \text{ if } |i-j| \geq 2; \quad H_{jj} = -j(l-j); \quad H_{j,j+1} = \mu j; \quad H_{j,j-1} = \mu(l-j+1).$$

**Theorem 4.11** [7, section 3] *A Heun equation (4.10) with  $l \in \mathbb{Z}$ ,  $l \geq 0$  has a polynomial solution, if and only if the three-diagonal matrix  $H + \lambda Id$  has zero determinant.*

**Proof of Theorem 4.9.** Let equation (1.1) have a solution  $E(z) = \sum_{k \in \mathbb{Z}} a_k z^k$  holomorphic on  $\mathbb{C}^*$ . Equivalently, its series converges on  $\mathbb{C}^*$  and the coefficients  $a_k$  satisfy recurrence relations (4.12). For  $k = -l$  and  $k = -1$  respectively these relations take the form

$$\lambda a_{-l} + \mu(1-l)a_{1-l} = 0, \quad (4.14)$$

$$(1-l+\lambda)a_{-1} - \mu(l-1)a_{-2} = 0. \quad (4.15)$$

In particular, they do not contain  $a_j$ ,  $j \notin \{-l, \dots, -1\}$ . Therefore, given a solution holomorphic on  $\mathbb{C}^*$  of Heun equation (1.1), its Laurent coefficients  $a_k$  with  $-l \leq k \leq -1$  should form a vector  $(a_{-l}, \dots, a_{-1})$  satisfying equations (4.14), (4.15) and the  $l-2$  recurrence equations (4.12) for intermediate  $k = -l+1, \dots, -2$ . In other terms, the latter vector should be in the kernel of the three-diagonal  $l \times l$ -matrix  $\tilde{H}$  of equations (4.12) with  $k = -1, \dots, -l$ : its line number  $-k$  consists of the coefficients of the  $k$ -th relation; the coefficient at  $a_{-j}$  stands at the column number  $j$ .

**Proposition 4.12** *Let  $\tilde{H}$  be the latter matrix, and let  $H^t$  be the transposed matrix (4.13). One has*

$$\tilde{H} = Q(H^t + \lambda Id)Q^{-1}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & \dots & 1 \\ 0 & \dots & 0 & -1 & 0 \\ \dots & 0 & 1 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ (-1)^{l-1} & 0 & 0 & \dots & 0 \end{pmatrix}.$$

The proposition follows from definition.

**Case 1).** **There exists a solution of Heun equation (1.1) holomorphic on  $\mathbb{C}^*$  with  $a_j = 0$  for all  $j \in \{-l, \dots, -1\}$ , and  $l \geq 1$ .** Then the series  $\sum_{k \geq 0} a_k z^k$  is an entire solution, i.e., holomorphic on  $\mathbb{C}$ : it satisfies relations (4.12) for all  $k$  by assumption and (4.15). It is known that in this case each solution of equation (1.1) is holomorphic on  $\mathbb{C}^*$  and its Laurent series does not contain monomials  $z^j$ ,  $j \in \{-l, \dots, -1\}$  [9, lemma 3, statement 6].

**Case 2).** **There exists a solution of Heun equation holomorphic on  $\mathbb{C}^*$  with  $a_k \neq 0$  for some  $k \in \{-l, \dots, -1\}$ .** In this case the three-diagonal matrix  $H$  of relations (4.12) with  $k = -l, \dots, -1$  has zero determinant, by the above arguments. Hence, the matrix  $H + \lambda Id$ , whose transposed is conjugated to  $\tilde{H}$  (Proposition 4.12), also has zero determinant. Therefore, equation (4.10) has a polynomial solution (Theorem 4.11). It is known that if (4.10) has a polynomial solution, then the corresponding equation (1.1) does not have entire solution [11, theorem 3.10]. Therefore, cases 1) and 2) are incompatible. Theorem 4.9 is proved.  $\square$

## 5 Applications to phase-lock areas in the model of Josephson effect

### 5.1 Phase-lock areas in Josephson effect: brief description of results and plan of the section

Here we apply the above results to the family of nonlinear equations (1.16):

$$\dot{\phi} = \frac{d\phi}{dt} = -\sin \phi + B + A \cos \omega t, \quad A, \omega > 0, \quad B \geq 0. \quad (5.1)$$

We fix an arbitrary  $\omega > 0$  and consider family (5.1) depending on two variable parameters  $(B, A)$ . The variable change  $\tau = \omega t$  transforms (5.1) to differential equation (1.17) on the two-torus  $\mathbb{T}^2 = S^1 \times S^1$  with coordinates  $(\phi, \tau) \in \mathbb{R}^2 / 2\pi\mathbb{Z}^2$ . Its solutions are tangent to the vector field

$$\begin{cases} \dot{\phi} = -\frac{\sin \phi}{\omega} + l + 2\mu \cos \tau, & l = \frac{B}{\omega}, \quad \mu = \frac{A}{2\omega} \\ \dot{\tau} = 1 \end{cases} \quad (5.2)$$

on the torus. The *rotation number* of the equation (5.1) is, by definition, the rotation number of the flow of the field (5.2), see [1, p. 104]. It is a function  $\rho(B, A)$  of parameters. (Normalization convention: the rotation number of



a usual circle rotation equals the rotation angle divided by  $2\pi$ .) The  $B$ -axis will be called the *abscissa*, and the  $A$ -axis will be called the *ordinate*.

**Definition 5.1** (cf. [15, definition 1.1]) The  $r$ -th *phase-lock area* is the level set  $\{(B, A) \mid \rho(B, A) = r\} \subset \mathbb{R}^2$ , provided it has a non-empty interior.

**Remark 5.2 : phase-lock areas and Arnold tongues.** The behavior of phase-lock areas for small  $A$  demonstrates the Arnold tongues effect [1, p. 110]. The phase-lock areas are called “Arnold tongues” in [15, definition 1.1].

Recall that the rotation number of system (5.1) has the physical meaning of the mean voltage over a long time interval. The segments in which the phase-lock areas intersect horizontal lines correspond to the Shapiro steps on the voltage-current characteristic.

It has been shown earlier that

- the phase-lock areas exist only for integer values of the rotation number (a “quantization effect” observed in [5] and later also proved in [17, 16]);
- the boundary of each phase-lock area  $\{\rho = r\}$  consists of two analytic curves, which are the graphs of two functions  $B = g_{r,\pm}(A)$  (see [6]; this fact was later explained by A.V.Klimenko via symmetry, see [19]);
- the latter functions have Bessel asymptotics (observed and proved on physical level in [22], see also [20, chapter 5], [2, section 11.1], [4]; proved mathematically in [19]).
- each phase-lock area is an infinite chain of bounded domains going to infinity in the vertical direction, each two subsequent domains are separated by one point, the separation points lying outside the horizontal  $B$ -axis are called the *adjacency points* (or briefly *adjacencies*), see Fig.1;
- for every  $r \in \mathbb{Z}$  the  $r$ -th phase-lock area is symmetric to the  $-r$ -th one with respect to the  $A$ -axis (symmetry of equation (5.1)).

In the present section we obtain functional equations satisfied by non-integer level curves  $\{\rho(B, A) = r\}$  of the rotation number (Subsection 5.1) and the boundaries of the phase-lock areas (Subsection 5.2) using relation of equation (5.1) to Heun equation (1.1) (recalled below) and the results on monodromy eigenvalues of Heun equations from the previous section. The above-mentioned functional equations will be written in the complement to the adjacencies and the algebraic set of the parameters corresponding to the existence of a polynomial solution of equation (4.10). Afterwards we discuss open problems and possible approaches to them using the same results on Heun equations.

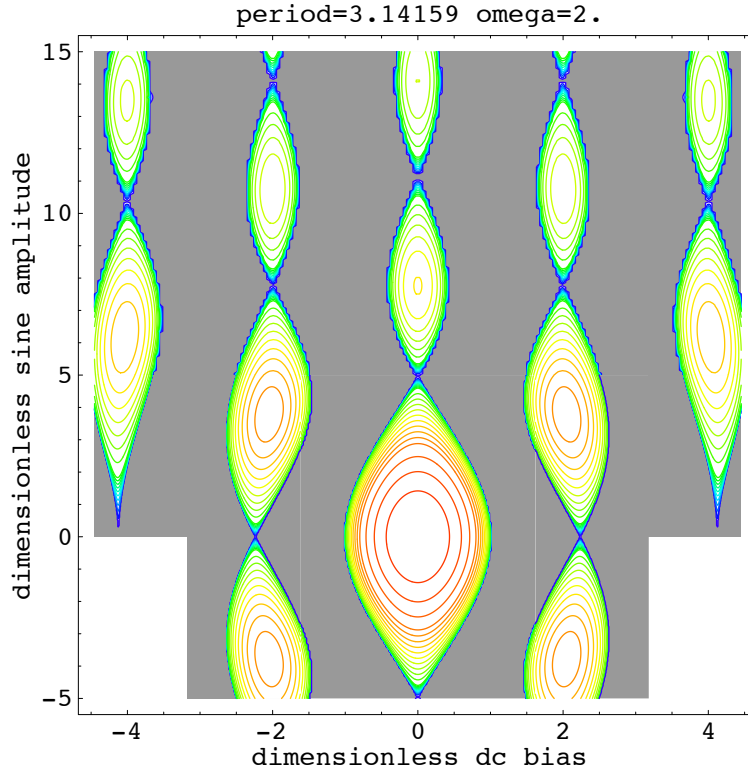


Figure 1: Phase-lock areas and their adjacencies for  $\omega = 2$ . The abscissa is  $B$ , the ordinate is  $A$ . Figure taken from [8, p. 331].

## 5.2 Reduction to double confluent Heun equations. Entire and polynomial solutions: special boundary points of phase-lock areas

Set

$$\Phi = e^{i\phi}, \quad z = e^{i\tau} = e^{i\omega t}.$$

The complexified equation (5.1) is equivalent to the Riccati equation

$$\frac{d\Phi}{dz} = z^{-2}((lz + \mu(z^2 + 1))\Phi - \frac{z}{2i\omega}(\Phi^2 - 1)).$$

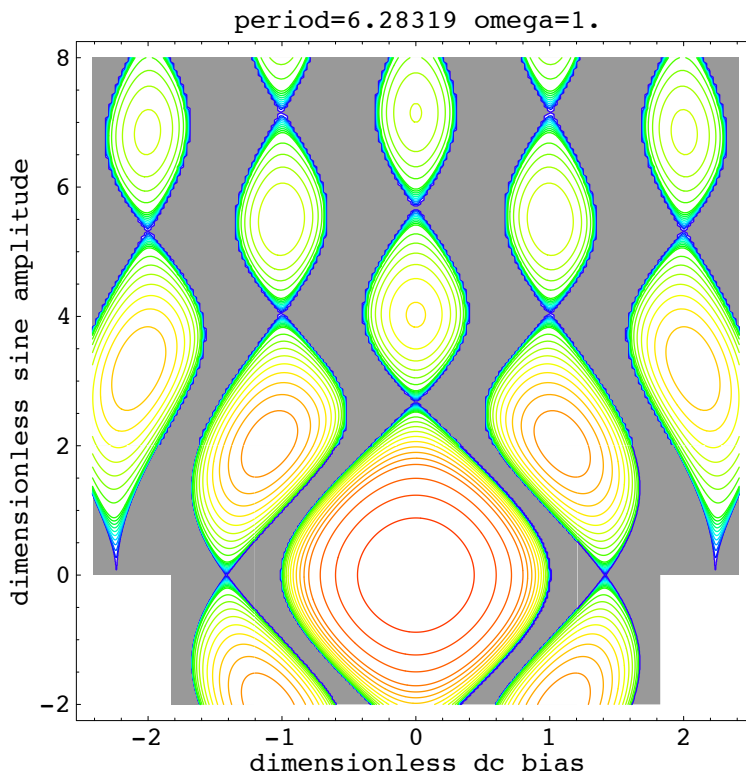


Figure 2: Phase-lock areas and their adjacencies for  $\omega = 1$ . The abscissa is  $B$ , the ordinate is  $A$ .

The latter is the projectivization of the following linear equation in vector function  $(u, v)$  with  $\Phi = \frac{v}{u}$ , see [11, subsection 3.2]:

$$\begin{cases} v' = \frac{1}{2i\omega z} u \\ u' = z^{-2}(-lz + \mu(1 + z^2))u + \frac{z}{2i\omega} v \end{cases} \quad (5.3)$$

This reduction to a system of linear equations was earlier obtained in slightly different terms in [3, 13, 8, 17]. It is easy to check that a function  $v(z)$  is the component of a solution of system (5.3), if and only if the function  $E(z) = e^{\mu z} v(z)$  satisfies equation (1.1) with  $n = l + 1$  and

$$\lambda = \left(\frac{1}{2\omega}\right)^2 - \mu^2. \quad (5.4)$$

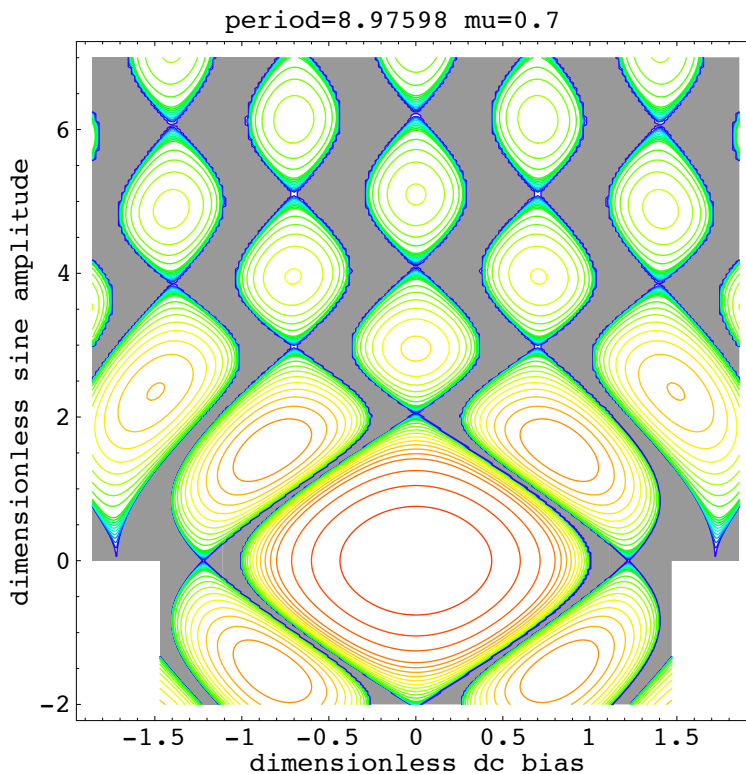


Figure 3: Phase-lock areas and their adjacencies for  $\omega = 0.7$ .

**Theorem 5.3** (see [11, theorems 3.3, 3.5]). For every  $\omega > 0$ ,  $l \geq 0$  a point  $(B, A) \in \mathbb{R}^2$  with  $A \geq 0$ ,  $B = l\omega$  is an adjacency for family of equations (5.1), if and only if  $l \in \mathbb{Z}$  and the corresponding equation (1.1) with  $n = l+1$  and  $\mu, \lambda$  as in (5.2) and (5.4) has a nontrivial entire solution, i.e., if and only if equation  $\xi_l(\lambda, \mu) = 0$  holds;  $\xi_l$  is the same, as in (4.11).

**Remark 5.4** The statement of Theorem 5.3 holds, if and only if the monodromy operator of Heun equation (1.1) with  $n = l + 1$  is trivial. This follows from [8, theorem 3] and [15, proposition 3.2].

In what follows we will use the next proposition.

**Proposition 5.5** Let  $\omega > 0$ ,  $(B, A) \in \mathbb{R}^2$ , and let  $\rho = \rho(B, A)$  denote the corresponding rotation number. If  $(B, A)$  does not lie in the interior of a phase-lock area, then the monodromy operator of the corresponding Heun

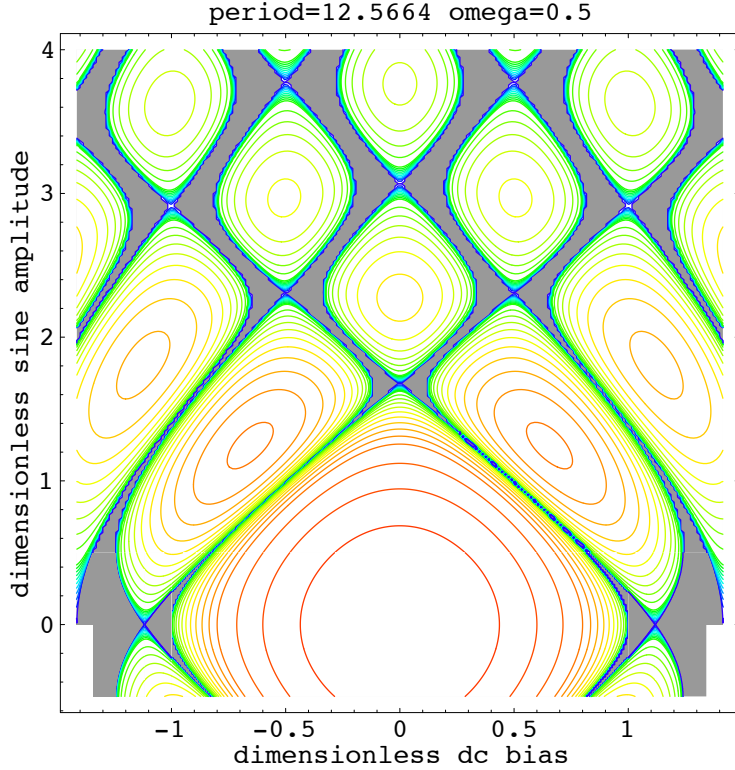


Figure 4: Phase-lock areas and their adjacencies for  $\omega = 0.5$ .

equation (1.1) with  $n = l + 1$ ,  $l = \frac{B}{\omega}$  has eigenvalues  $e^{\pi i(\rho-l)}$  and  $e^{-\pi i(\rho+l)}$ . In particular, the latter pair of eigenvalues is the same for all other points  $(B, A')$  with the same  $B$  and  $\rho(B, A') = \pm \rho(B, A) \pmod{2}$ .

**Proof** Let  $\lambda_1, \lambda_2$  denote the eigenvalues of the above monodromy operator of Heun equation. The point  $(B, A)$  does not lie in the interior of a phase-lock area. If  $r = \rho(B, A) \notin \mathbb{Z}$ , then the monodromy of the corresponding Riccati equation is an elliptic Möbius transformation conformally conjugated to the rotation by angle  $2\pi r$ . Therefore, it has two fixed points with multipliers  $e^{\pm 2\pi i r}$ . The latter multipliers are ratios of the eigenvalues  $\lambda_j$ , and without loss of generality we consider that  $\frac{\lambda_1}{\lambda_2} = e^{2\pi i r}$ . On the other hand,  $\lambda_1 \lambda_2 = e^{-2\pi i l}$ , by Proposition 4.1. This implies that the eigenvalues under question are equal to  $\pm e^{\pi i(r-l)}$ ,  $\pm e^{-\pi i(r+l)}$ . In the case, when  $r \in \mathbb{Z}$ , the point  $(B, A)$  lies in the boundary of a phase-lock area and the monodromy of the Riccati

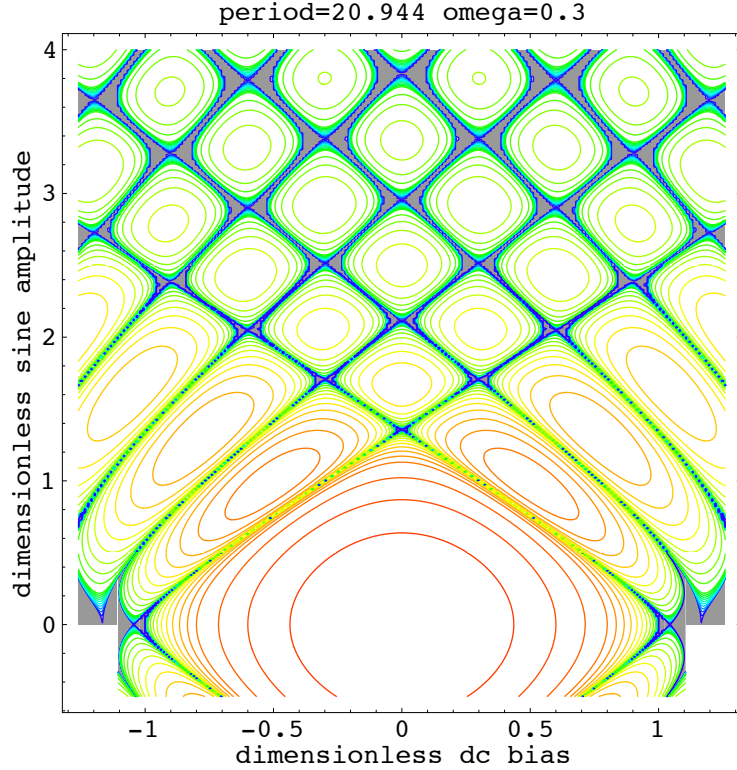


Figure 5: Phase-lock areas and their adjacencies for  $\omega = 0.3$ .

equation is parabolic, The monodromy of the Heun equation has multiple eigenvalue given by same (now coinciding) formulas. The correct sign should be the same for all the points  $(B, A)$  in the complement of the parameter plane to the union of the interiors of phase-lock areas, by path connectivity of the latter complement and continuity. The sign is “+” at each adjacency, since the corresponding monodromy is trivial (Remark 5.4). Hence, it is “+” everywhere. This proves the proposition.  $\square$

**Theorem 5.6** *Let  $\omega > 0$ ,  $(B, A) \in \mathbb{R}^2$ ,  $B, A > 0$ ,  $l = \frac{B}{\omega}$ ,  $\mu = \frac{A}{2\omega}$ ,  $\lambda = \frac{1}{4\omega^2} - \mu^2$ ,  $\rho = \rho(B, A)$ . The double confluent Heun equation (4.10) corresponding to the latter  $\lambda$ ,  $\mu$  and  $l$  has a polynomial solution, if and only if  $l, \rho \in \mathbb{Z}$ ,  $0 \leq \rho \equiv l \pmod{2}$  and  $\rho \leq l$ , and in addition, the point  $(B, A)$  lies in the boundary of the phase-lock area number  $\rho$  and is not an adjacency. In other terms, the points  $(B, A) \in \mathbb{R}_+^2$  corresponding to equations (4.10) with*

polynomial solutions lie in boundaries of phase-lock areas and are exactly their intersection points with the lines  $l = \frac{B}{\omega} \equiv \rho(\text{mod}2)$ ,  $0 \leq \rho \leq l$ , except for the adjacencies, see Fig. 6.

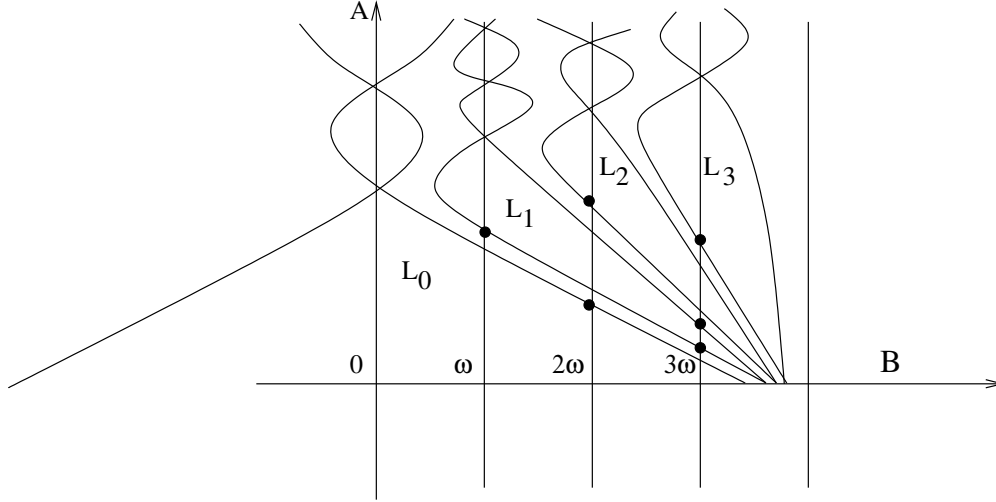


Figure 6: Approximate phase-lock areas for  $\omega \simeq 0.27$ ; the marked points correspond to equations (4.10) with polynomial solutions.

**Proof** It is known that every point  $(B, A) \in \mathbb{R}_+^2$  corresponding to equation (4.10) with a polynomial solution lies in the boundary of the phase-lock area number  $\rho$ , and one has  $l, \rho \in \mathbb{Z}$ ,  $l = \frac{B}{\omega} \equiv \rho(\text{mod}2)$ ,  $0 \leq \rho \leq l$  [7, corollary 6 and theorem 5]. In addition,  $(B, A)$  is not an adjacency [11, theorem 3.10]. Let us prove the converse: if  $(B, A)$  satisfy all the latter statements, then the corresponding equation (4.10) has a polynomial solution. Indeed, if  $(B, A) \in \mathbb{R}_+^2$  lies in the boundary of the phase-lock area number  $\rho$ ,  $l \in \mathbb{Z}$  and  $\rho \equiv l(\text{mod}2)$ , then the monodromy of Heun equation (1.1) is unipotent, by Proposition 5.5: the corresponding eigenvalues are equal to  $e^{-\pi i(\rho+l)} = 1$ . Let us now suppose that  $(B, A)$  is not an adjacency, or equivalently, equation (1.1) does not have an entire solution. Then equation (4.10) has a polynomial solution, by Theorem 4.9. Theorem 5.6 is proved.  $\square$

### 5.3 Equation on non-integer level sets of rotation number

For given  $\omega > 0$  and  $r > 0$  set

$$L_r = \{(B, A) \in \mathbb{R}^2 \mid \rho(B, A) = r\}, \quad L_{[r]} = \sqcup_{v \equiv \pm r \pmod{2}} L_v.$$

We consider that  $r \notin \mathbb{Z}$ : then each  $L_v$ ,  $v \equiv \pm r \pmod{2}$  is an analytic curve, the graph of an analytic function  $B = g_v(A)$ . Here we write down analytic equations defining the set  $L_{[r]}$  in the complement  $\mathbb{R}^2 \setminus \Sigma_{[r]}$ , where

$$\Sigma_{[r]} = \cup_{\pm} \{(B, A) \in \mathbb{R}^2 \mid l = \frac{B}{\omega} \equiv \pm r \pmod{2}\}. \quad (5.5)$$

For every  $(B, A) \in L_{[r]}$  the corresponding Heun equation (1.1) has a monodromy eigenfunction of the type

$$E(z) = z^b \sum_{k \in \mathbb{Z}} a_k z^k, \quad b = \frac{r-l}{2}, \quad l = \frac{B}{\omega},$$

by Proposition 5.5. One has  $b, b+l \notin \mathbb{Z}$ , if  $(B, A) \notin \Sigma_{[r]}$ . Therefore, the analytic subset  $L_{[r]} \setminus \Sigma_{[r]} \subset (\mathbb{R}^2 \setminus \Sigma_{[r]})$  is described by equation (4.4). Let us write it down explicitly. The corresponding matrices  $M_k, R_k, S_m, T_m$ , see (3.9) and (3.23) are

$$M_k = \begin{pmatrix} 1 + \frac{\lambda}{(k+\frac{r}{2})^2 - \frac{l^2}{4}} & \frac{\mu^2}{(k+\frac{r}{2})^2 - \frac{l^2}{4}} \\ 1 & 0 \end{pmatrix}, \quad R_k = M_k M_{k+1} \dots,$$

$$S_m = \begin{pmatrix} 1 + \frac{\lambda-l+1}{(\frac{r-l}{2}-m+1)(\frac{r+l}{2}-m-1)} & \frac{\mu^2(\frac{r+l}{2}-m)}{(\frac{r-l}{2}-m+1)(\frac{r+l}{2}-m-1)(\frac{r+l}{2}-m-2)} \\ 1 & 0 \end{pmatrix},$$

$$T_m = S_m S_{m+1} \dots$$

**Theorem 5.7** *Let  $\omega > 0$ ,  $r \in \mathbb{R}$ . The set  $L_{[r]} \cap (\mathbb{R}^2 \setminus \Sigma_{[r]})$  is defined by the following equation in  $(B, A) \in \mathbb{R}^2 \setminus \Sigma_{[r]}$ , where  $B = l\omega$ ,  $A = 2\mu$ ,  $\lambda + \mu^2 = \frac{1}{4\omega^2}$ :*

$$\left(\frac{r-l}{2} + 1\right)\left(\frac{r+l}{2} - 1\right)R_{1,11}T_{0,11} + \mu^2 R_{1,21}T_{0,21} = 0. \quad (5.6)$$

**Proof** A point  $(B, A) \in \mathbb{R}^2 \setminus \Sigma_{[r]}$  is contained in  $L_{[r]}$ , if and only if  $\rho = \rho(A, B) \equiv \pm r \pmod{2}$ . Or equivalently, some of the corresponding monodromy eigenvalues  $e^{\pi i(\pm \rho - l)}$  equals  $e^{2\pi i b} = e^{\pi i(r-l)}$ . The latter statement is equivalent to (5.6), by Theorem 4.3. This proves Theorem 5.7.  $\square$



## 5.4 Description of boundaries of phase-lock areas

**Proposition 5.8** *A point in the parameter space of equation (5.1) lies in the boundary of a phase-lock area, if and only if the monodromy of the corresponding Heun equation (1.1) is parabolic: has multiple eigenvalue.*

**Proof** The point under question lies in the boundary of a phase-lock area, if and only if the flow mapping of the vector field (5.2) for the period  $2\pi$  (restricted to the coordinate  $\phi$ -circle) is parabolic: has a fixed point with unit derivative. The period mapping is the restriction to the unit circle of the monodromy of the corresponding Riccati equation: the projectivized monodromy. Parabolicity of the projectivization of a two-dimensional linear operator is equivalent to its own parabolicity. The proposition is proved.  $\square$

**Proposition 5.9** *Let a Heun equation (1.1) have a parabolic monodromy. Then the monodromy either has Jordan cell type, or is the identity.*

**Proof** The monodromy matrix is the product of the formal monodromy matrix  $\text{diag}(e^{-2\pi in}, 1)$  and a pair of unipotent matrices: the inverse to the Stokes matrices at 0, cf. [15, formulas (2.15) and (3.2)]. Therefore, if the monodromy of a Heun equation (1.1) is a multiplication by scalar number, then the Stokes matrices are trivial, and the monodromy coincides with the formal one. Hence, both monodromies are scalar and given by the above diagonal matrix with unit eigenvalue. Thus, they are trivial. The proposition is proved.  $\square$

The condition saying that the monodromy has multiple eigenvalue is equivalent to the statement that it has eigenvalue  $\pm e^{-\pi il}$ , by Proposition 4.1. This is equivalent to the statement that there exists a multivalued solution  $z^b \sum_{k \in \mathbb{Z}} a_k z^k$  of Heun equation with  $b \in \{-\frac{l}{2}, -\frac{l+1}{2}\}$ : a monodromy eigenfunction with the above eigenvalue. Heun equations (1.1) satisfying the latter statement will be described below by using the following proposition. Afterwards we deduce immediately the description of boundaries of phase-lock areas.

**Proposition 5.10** *Let a Heun equation (1.1) have a Jordan cell monodromy. Then its eigenfunction is either invariant, or anti-invariant under the involution  $\#$ :*

$$(\#E)(z) = 2\omega z^{-l-1}(E'(z^{-1}) - \mu E(z^{-1})).$$

The proposition follows from the fact that the involution  $\#$  sends monodromy eigenfunctions to eigenfunctions (Proposition 4.2).

**Case 1:**  $l \notin 2\mathbb{Z}$ ,  $b = -\frac{l}{2}$ : **the monodromy eigenvalue equals**  $e^{-\pi il} \neq 1$ . Then the monodromy operator of Heun equation (1.1) is a Jordan cell, by Proposition 5.9. Consider the matrices

$$M_k = \begin{pmatrix} 1 + \frac{\lambda}{k^2 - \frac{l^2}{4}} & \frac{\mu^2}{k^2 - \frac{l^2}{4}} \\ 1 & 0 \end{pmatrix}, \quad R_k = M_k M_{k+1} \dots \quad (5.7)$$

**Theorem 5.11** *Let  $\lambda + \mu^2 = \frac{1}{4\omega^2}$ ,  $\omega, \mu \neq 0$ ,  $n = l + 1$ ,  $l \notin 2\mathbb{Z}$ . The monodromy operator of the corresponding Heun equation (1.1) has eigenvalue  $e^{-\pi il}$ , if and only if*

$$R_{0,21} \pm \omega l (R_{0,21} - R_{0,11}) = 0. \quad (5.8)$$

with some choice of sign.

**Proof** The corresponding eigenfunction  $E$  has the form

$$E(z) = z^{-\frac{l}{2}} f(z), \quad f(z) = \sum_{k \in \mathbb{Z}} a_k z^k \text{ is holomorphic on } \mathbb{C}^*. \quad (5.9)$$

Equation (1.1) is equivalent to recurrence equations (1.6) with  $b = -\frac{l}{2}$ :

$$(k^2 - \frac{l^2}{4} + \lambda)a_k - \mu(k + \frac{l}{2})a_{k-1} + \mu(k - \frac{l}{2} + 1)a_{k+1} = 0. \quad (5.10)$$

The series  $f(z)$  should converge on  $\mathbb{C}^*$ . The above matrices  $M_k$  and  $R_k$  coincide with those constructed in (3.9), and they are well-defined for all  $k \in \mathbb{Z}$ . Therefore, the coefficients  $a_k$ ,  $k \geq 0$  are given by formulas (3.12) up to common constant factor, by Theorem 1.9:

$$a_0 = -\frac{2}{l} R_{0,21}, \quad a_{-1} = \mu^{-1} R_{0,11}. \quad (5.11)$$

Now we will use the condition of (anti-) invariance  $\#E = \pm E$  (Proposition 5.10), which takes the form

$$\sum_{k \in \mathbb{Z}} a_k z^{k - \frac{l}{2}} = \pm 2\omega \left( \sum_{k \in \mathbb{Z}} (k - \frac{l}{2}) a_k z^{-k - \frac{l}{2}} - \mu \sum_{k \in \mathbb{Z}} a_k z^{-\frac{l}{2} - k - 1} \right),$$

or equivalently,

$$\sum a_k z^k = \pm 2\omega \left( \sum (k - \frac{l}{2}) a_k z^{-k} - \mu \sum a_k z^{-k-1} \right). \quad (5.12)$$

The free (zero power) term of the latter equation is equivalent to the relation

$$(1 \pm l\omega)a_0 \pm 2\omega\mu a_{-1} = 0, \quad (5.13)$$

which is in its turn equivalent to (5.8), by (5.11). Therefore, existence of the above solution  $E$  implies (5.8).

Let us prove the converse: each equation (5.8) implies the existence of a solution (5.9) of Heun equation. To do this, consider the action of the transformation  $\#$  on the *formal* series (5.9) (with  $f$  not necessarily converging). It sends formal solutions of Heun equation (equivalently, formal solutions of (5.10)) to formal solutions. (The proof of symmetry of Heun equation under the transformation  $\#$  uses only Leibniz differentiation rule and remains valid for formal series.) The space of formal solutions is two-dimensional, and it is identified with the space of its initial conditions  $(a_{-1}, a_0)$ . The transformation  $\#$  is its involution. Its eigenvalues are equal to  $\pm 1$ , and the corresponding eigenspaces are defined by initial conditions that satisfy (5.13). Therefore, both eigenspaces are one-dimensional and are exactly characterized by equations (5.13), since both equations (5.13) are nontrivial. Thus, a *formal solution*  $(a_k)_{k \in \mathbb{Z}}$  of recurrence relations (5.10) is  $\#$ -(anti)-invariant, if and only if its coefficients  $a_{-1}, a_0$  satisfy (5.13) with the corresponding sign.

Fix the one-sided solution  $\sum_{k \geq -1} a_k z^k$  of recurrence relations (5.10) for  $k \geq 0$ . It satisfies (5.13), by (5.8). The sequence  $(a_k)_{k \geq -1}$  extends uniquely to a two-sided formal solution  $(a_k)_{k \in \mathbb{Z}}$  of (5.10) (a priori, not necessarily presenting a converging series for  $k \rightarrow -\infty$ ), since the coefficients at  $a_{k \pm 1}$  in (5.10) do not vanish. The latter formal solution should be  $\#$ -(anti)-invariant, by (5.13) and the previous statement. Hence,

$$a_k = \pm 2\omega \left( -k - \frac{l}{2} \right) a_{-k} - \mu a_{-(k+1)}$$

by (5.12). The series  $\sum_{k < 0} a_k z^k$  converges on  $\mathbb{C}^*$ : it is bounded from above by converging series  $2\omega \sum_{k \geq 0} (k + |l| + \mu + 1) |a_k z^k|$ , by the latter formula. This together with the above argument proves the theorem.  $\square$

**Case 2:**  $l \notin 2\mathbb{Z} + 1$  and  $b = -\frac{l+1}{2}$ : **the monodromy eigenvalue equals**  $-e^{-\pi i l} \neq 1$ . Then the monodromy of equation (1.1) is a Jordan cell, as above. Consider the matrices

$$M_k = \begin{pmatrix} 1 + \frac{\lambda}{(k-\frac{1}{2})^2 - \frac{l^2}{4}} & \frac{\mu^2}{(k-\frac{1}{2})^2 - \frac{l^2}{4}} \\ 1 & 0 \end{pmatrix}, \quad R_k = M_k M_{k+1} \dots \quad (5.14)$$

**Theorem 5.12** *Let  $\lambda + \mu^2 = \frac{1}{4\omega^2}$ ,  $\omega, \mu \neq 0$ ,  $n = l + 1$ ,  $l \notin 2\mathbb{Z} + 1$ . The monodromy operator of the corresponding Heun equation (1.1) has eigenvalue  $-e^{-\pi il}$ , if and only if*

$$R_{1,11} \pm 2\omega\mu(R_{1,11} - R_{1,21}) = 0 \quad (5.15)$$

with some choice of sign.

**Proof** We are looking for a double-infinite solution

$$E(z) = z^{-\frac{l+1}{2}} f(z), \quad f(z) = \sum_{k \in \mathbb{Z}} a_k z^k \quad (5.16)$$

of Heun equation (1.1) with  $f$  holomorphic on  $\mathbb{C}^*$ . That is, with  $a_k$  satisfying recurrence relations (1.6) for  $b = -\frac{l+1}{2}$ , which take the form

$$\left((k - \frac{1}{2})^2 - \frac{l^2}{4} + \lambda\right)a_k - \mu\left(k + \frac{l-1}{2}\right)a_{k-1} + \mu\left(k - \frac{l-1}{2}\right)a_{k+1} = 0. \quad (5.17)$$

The above matrices  $M_k$  and  $R_k$  coincide with those constructed in (3.9), and they are well-defined for all  $k \in \mathbb{Z}$ . Therefore, the coefficients  $a_k$ ,  $k \geq 0$  are given by formulas (3.12) up to common constant factor, by Theorem 1.9. In particular,

$$a_0 = \frac{1}{b}R_{0,21} = \frac{1}{b}R_{1,11} = -\frac{2}{l+1}R_{1,11}, \quad a_1 = \frac{\mu}{b(b+1)}R_{1,21} = \frac{4\mu}{l^2-1}R_{1,21}.$$

The condition of (anti-) invariance under the involution  $\#$  of the solution takes the form

$$\sum a_k z^{k - \frac{l+1}{2}} = \pm 2\omega \left( \sum \left(k - \frac{l+1}{2}\right) a_k z^{-k - \frac{l-1}{2}} - \mu \sum a_k z^{-k - \frac{l+1}{2}} \right),$$

or equivalently,

$$\sum a_k z^k = \pm 2\omega \left( \sum \left(k - \frac{l+1}{2}\right) a_k z^{-k+1} - \mu \sum a_k z^{-k} \right).$$

The free term (zero power term) of the latter equation is

$$(1 \pm 2\omega\mu)a_0 \pm \omega(l-1)a_1 = 0, \quad (5.18)$$

which is equivalent to (5.15). The rest of proof of Theorem 5.12 is analogous to the proof of Theorem 5.11.  $\square$

**Corollary 5.13** *Let  $\omega, \mu > 0$ ,  $\lambda + \mu^2 = \frac{1}{4\omega^2}$ ,  $l \geq 0$ ,  $n = l + 1$ ,  $B = l\omega$ ,  $A = 2\mu\omega$ . The point  $(B, A)$  lies in the boundary of a phase-lock area, if and only if one of the following four incompatible statements holds:*

- 1)  $(B, A)$  is an adjacency:  $\xi_l(\lambda, \mu) = 0$ ;
- 2) Heun equation (4.10) has a polynomial solution:  $\det(H + \lambda Id) = 0$ , where  $H$  is the  $l \times l$ -matrix from (4.13);
- 3)  $l \notin 2\mathbb{Z}$  and equation (5.8) holds;
- 4)  $l \notin 2\mathbb{Z} + 1$  and equation (5.15) holds.

**Proof** If one of the above statements holds, then  $(B, A)$  lies in the boundary of a phase-lock area, by Proposition 5.8 and Theorems 4.9, 5.11, 5.12. Conversely, let  $(B, A)$  lie in the boundary of a phase-lock area. The monodromy of the corresponding Heun equation (1.1) is parabolic, by Proposition 5.8. It is unipotent, if and only if some of the two incompatible statements 1) or 2) holds, by Theorems 4.9 and 5.3. Otherwise, the monodromy has Jordan cell type with eigenvalue  $\pm e^{-\pi il} \neq 1$ . Therefore, one of the statements 3) or 4) holds, by Theorems 5.11 and 5.12. Statements 3) and 4) are incompatible: they correspond to Heun equation (1.1) with monodromy having multiple eigenvalue  $e^{-\pi il}$  or  $-e^{-\pi il}$  respectively. This proves the corollary.  $\square$

**Proposition 5.14** *Let  $l \in \mathbb{Z}$ . For given  $\omega, \mu > 0$  and  $n = l + 1$  the corresponding Heun equation (1.1) has a monodromy eigenfunction with eigenvalue  $-1$ , if and only if the corresponding point  $(B, A) \in \mathbb{R}^2$  lies in the boundary of a phase-lock area with a rotation number  $\rho \equiv l + 1 \pmod{2}$ .*

**Proof** For  $l \in \mathbb{Z}$  the monodromy has unit determinant (Proposition 4.1). Therefore, if it has eigenvalue  $-1$ , then its other eigenvalue is also  $-1$ . Hence, the point  $(B, A)$  lies in the boundary of the phase-lock area number  $\rho = \rho(B, A)$  (Proposition 5.8). Thus,  $e^{\pi i(\pm\rho - l)} = -1$  for both signs, by Proposition 5.5. The latter equality holds if and only if  $\rho \equiv l + 1 \pmod{2}$ . Conversely, if a point  $(B, A)$  with  $l = \frac{B}{\omega} \in \mathbb{Z}$  lies in the boundary of a phase-lock area, and  $\rho(B, A)$  satisfied the above equality, then the monodromy eigenvalues are equal to  $-1$ , by Proposition 4.1. The proposition is proved.  $\square$

## 5.5 Conjectures on geometry of phase-lock areas

Here we state conjectures that are motivated by numerical simulations and theoretical results of [7, 8, 11, 15]. In what follows for every  $r \in \mathbb{Z}$  we denote

$$L_r = \text{the phase-lock area number } r.$$

The next five conjectures are due to the first author (V.M.Buchstaber) and S.I.Tertychnyi.

**Conjecture 5.15** Each phase-lock area  $L_r$  is a garland of infinitely many connected components separated by adjacencies  $\mathcal{A}_{r,1}, \mathcal{A}_{r,2} \dots$  lying in the line  $\{B = r\omega\}$  and ordered by their  $A$ -coordinates.

**Remark 5.16** It was shown in [15, theorems 1.2, 3.17] that for every  $r \in \mathbb{Z}$  the abscissa of each adjacency in  $L_r$  equals  $l\omega$ ,  $l \in \mathbb{Z}$ ,  $l \equiv r \pmod{2}$ ;  $0 \leq l \leq r$  if  $r \geq 0$ ;  $r \leq l \leq 0$  if  $r \leq 0$ .

**Conjecture 5.17** For every  $k \geq 2$  the  $k$ -th component in  $L_r$  contains the interval  $(A_{r,k-1}, A_{r,k})$ .

**Conjecture 5.18** For every given  $k \in \mathbb{N}$  all the adjacencies  $\mathcal{A}_{r,k}$ ,  $r = 1, 2, \dots$ , lie on the same line, see Fig. 1–5; its azimuth depends on  $k$ .

**Conjecture 5.19** The first component of the zero phase-lock area lies in the square with vertices  $(0, \pm 1)$ ,  $(\pm 1, 0)$ .

For  $l \in \mathbb{N}$  let  $\mathcal{P}_l \in \{B = l\omega\}$  be the point with maximal  $A$ -coordinate that corresponds to Heun equation (1.19) with polynomial solution.

**Conjecture 5.20** All  $\mathcal{P}_l$  lie on the same line, see Fig. 7.

**Conjecture 5.21** As  $\omega \rightarrow 0$ , for every  $r$  the set  $L_{r+} := L_r \cap \{A \geq \mathcal{A}_{r,1}\}$  accumulates to the  $A$ -axis.

In what follows we will discuss in detail the next two conjectures that are closely related to Conjectures 5.15 and 5.17.

**Conjecture 5.22** For  $r \in \mathbb{N}$  the phase-lock area with rotation number  $r+1$  does not intersect the line

$$\Lambda_r = \{B = \omega r\} \subset \mathbb{R}^2.$$

**Conjecture 5.23** For  $r \in \mathbb{N}$  the phase-lock area with rotation number  $r$  does not intersect the line  $\Lambda_{r-2}$ .

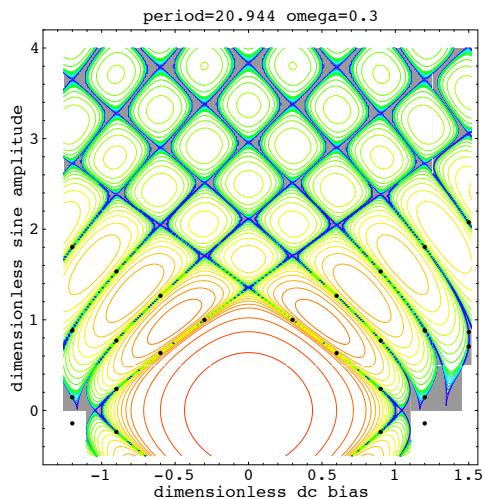


Figure 7: Phase-lock areas for  $\omega = 0.3$ ; the marked points correspond to Heun equations (1.19) with polynomial solutions. They are described by Theorem 5.6.

**Remark 5.24** Conjecture 5.22 implies Conjecture 5.23. Indeed, the points  $(B, A)$  with  $A > 0$  large enough of the phase-lock area  $L_r$ ,  $r \in \mathbb{Z}$  lie close to  $\Lambda_r$ , i.e., they are separated from the line  $\Lambda_{r-2}$  by  $\Lambda_{r-1}$ . This follows from the fact that its boundary consists of graphs of two functions  $B = g_{\pm r}(A)$  and  $g_{\pm r}(A) \rightarrow r\omega$ , as  $A \rightarrow +\infty$  (follows from results of [19]). Each one of Conjectures 5.22, 5.23 together with [15, theorems 1.2, 3.17] (see Remark 5.16) imply Conjecture 5.15.

**A possible strategy for Conjecture 5.22.** If the boundary of the phase-lock area with rotation number  $r + 1$  intersects the line  $\Lambda_r$ , then the intersection points correspond to parabolic monodromy operator of Jordan cell type with both eigenvalues equal to -1 (Proposition 5.14). That is, some of equations (5.8) or (5.15) should hold at each intersection point.

**Conjecture 5.25** Let  $l \in \mathbb{N}$ , and let the parameter  $\mu$  satisfy some of equations (5.8) if  $l \notin 2\mathbb{Z}$ , or (5.15) if  $l \in 2\mathbb{Z}$ . Then the double-valued eigenfunction  $E$  from Proposition 5.14 gives a periodic solution of the corresponding Riccati equation having rotation number between 0 and  $l$ .

Conjecture 5.25 would imply Conjecture 5.22.

**A possible strategy for Conjecture 5.23.** We know that for  $\omega \geq 1$  the statements of Conjecture 5.22 and hence Conjecture 5.23 hold (Chaplygin Theorem argument, see [6, lemma 4] and [15, proposition 3.4]). The adjacencies of a phase-lock area with rotation number  $\rho$  cannot lie on lines  $\Lambda_l$  with  $l \not\equiv \rho \pmod{2}$ , see [15, theorem 3.17]; this also follows from Proposition 5.14. Suppose that for a certain “critical” value  $\omega = \omega_0 < 1$  the boundary of the phase-lock area number  $l + 2 > 0$  moves from the right to the left, as  $\omega$  decreases to  $\omega_0$ , and touches the line  $\Lambda_l$  at some point  $(B, A)$ , as  $\omega = \omega_0$ . Then there are two possibilities for the corresponding Heun equation:

- the associated Heun equation (4.10) (equation (1.1) with  $l$  replaced by  $-l$ ) has a polynomial solution. But this case is forbidden by Buchstaber–Tertychnyi result [7, theorem 4], which states that then the corresponding rotation number cannot be greater than  $l$ .

- the point  $(B, A)$  an adjacency: Heun equation (1.1) has a solution holomorphic on  $\mathbb{C}$ . This together with the above-mentioned known fact that the boundaries of phase-lock areas are graphs of functions (Remark 5.24) implies that both boundary components of the phase-lock area with rotation number  $l + 2$  are tangent to the line  $\Lambda_l$  at the point  $(B, A)$ .

**Conjecture 5.26** For every  $\omega > 0$  for every adjacency  $(B_0, A_0) \in \mathbb{R}_+^2$  of any phase-lock area the branches of its boundary at  $(A, B)$  cannot be both tangent to the vertical line  $B = B_0$ .

**Proposition 5.27** *Conjecture 5.26 implies Conjecture 5.23, and hence, 5.15.*

The proposition follows from the above argument and Remark 5.24.

A possible approach to Conjecture 5.26 could be studying equations (5.8) and (5.15) defining the boundaries and to see what happens with them when the “non-resonant” parameters approach the resonant ones. A first step is done below.

## 5.6 Description of boundaries of phase-lock areas near adjacencies. Relation to Conjecture 5.23

Let us write down equation (5.8) on the boundaries in a neighborhood of a line  $\Lambda_{l_0}$ ,  $l_0 \in 2\mathbb{Z}$ . Let us recall the formulas for the corresponding matrices:

$$M_k = M_k(\lambda, \mu, l) = \begin{pmatrix} 1 + \frac{\lambda}{k^2 - \frac{l^2}{4}} & \frac{\mu^2}{k^2 - \frac{l^2}{4}} \\ 1 & 0 \end{pmatrix}, \quad R_k = M_k M_{k+1} \dots$$



Equation (5.8) for the boundaries is

$$R_{0,21} \pm \omega l (R_{0,21} - R_{0,11}) = 0.$$

Note that the matrices  $M_k$  are analytic in a neighborhood of the line  $l = l_0$  except for the matrix  $M_{\frac{l_0}{2}}$ , which has pole of order one along the latter line.

One has

$$\frac{l_0^2 - l^2}{4} M_{\frac{l_0}{2}} = \left( \begin{pmatrix} \lambda & \mu^2 \\ 0 & 0 \end{pmatrix} + \frac{l_0^2 - l^2}{4} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right).$$

Set

$$\mathcal{R} = \frac{l_0^2 - l^2}{4} R_0, \quad X = M_0 \dots M_{\frac{l_0}{2}-1}.$$

One has

$$\begin{aligned} \mathcal{R} &= X \left( \begin{pmatrix} \lambda & \mu^2 \\ 0 & 0 \end{pmatrix} + \frac{l_0^2 - l^2}{4} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right) R_{\frac{l_0}{2}+1}, \\ \begin{pmatrix} \lambda & \mu^2 \\ 0 & 0 \end{pmatrix} R_{\frac{l_0}{2}+1} \Big|_{l=l_0} &= \begin{pmatrix} \xi_{l_0}(\lambda, \mu) & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

by (4.11) and since the matrices  $M_{\frac{l_0}{2}+k}(\lambda, \mu, l_0)$ ,  $R_{\frac{l_0}{2}+k}(\lambda, \mu, l_0)$  coincide with the matrices  $M_k$ ,  $R_k$  preceding (4.11) with  $l = l_0$ . Therefore,

$$\mathcal{R} = \xi_{l_0}(\lambda, \mu) \begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix} + O(l - l_0). \quad (5.19)$$

Now equation (5.8) can be rewritten as

$$\mathcal{R}_{21} \pm \omega l (\mathcal{R}_{21} - \mathcal{R}_{11}) = 0. \quad (5.20)$$

Taking into account asymptotics (5.19) one gets asymptotic form of equation (5.20):

$$\xi_{l_0}(\lambda, \mu) (X_{21} \pm \omega l (X_{21} - X_{11})) + O(l - l_0) = 0. \quad (5.21)$$

Now let us consider the case, when  $l_0 \in 2\mathbb{Z} + 1$ , and let us write down equation (5.15) in a neighborhood of the line  $\Lambda_{l_0}$ . The corresponding matrices from (5.14) are

$$M_k = \begin{pmatrix} 1 + \frac{\lambda}{(k-\frac{1}{2})^2 - \frac{l^2}{4}} & \frac{\mu^2}{(k-\frac{1}{2})^2 - \frac{l^2}{4}} \\ 1 & 0 \end{pmatrix}, \quad R_k = M_k M_{k+1} \dots$$

Set

$$\mathcal{R} = \frac{l_0^2 - l^2}{4} R_1, \quad X = M_1 \dots M_{\frac{l_0+1}{2}-1} : X = Id, \text{ if } l_0 = 1.$$

Analogously to the above calculations, we get asymptotic relation (5.19). Together with (5.15), it implies

$$\xi_{l_0}(\lambda, \mu)(X_{11} \pm 2\omega l(X_{11} - X_{21})) + O(l - l_0) = 0 \quad (5.22)$$

Set

$$\zeta_l(\omega, \mu) = \xi_l(\lambda, \mu) = \xi_l\left(\frac{1}{4\omega^2} - \mu^2, \mu\right).$$

**Conjecture 5.28**  $\frac{\partial \zeta_l}{\partial \mu} \neq 0$  at zeros of the function  $\xi_l$  for every  $l \in \mathbb{Z}$ ,  $l \geq 0$ .

**Remark 5.29** The above matrices  $X$  are both non-degenerate for  $l = l_0$ . This implies that in formulas (5.21) and (5.22) the multiplier at  $\xi_{l_0}$  is non-zero for at least one choice of sign.

Conjecture 5.28 together with the above remark would imply that for every  $l \in \mathbb{Z}$  at every adjacency in the line  $\Lambda_l$  at least one boundary component of the corresponding phase-lock area (depending on the above-chosen sign) is transversal to the line  $\Lambda_l$ .

**Proposition 5.30** *Conjecture 5.28 implies Conjectures 5.26 and 5.23.*

**Proof** Conjecture 5.28 implies that no adjacency can be born from a tangency of both boundary components with a line  $l = l_0$ ,  $l_0 \in \mathbb{Z}$ , by transversality (the above statement). In other words, it implies Conjecture 5.26, and hence, Conjecture 5.23, by Proposition 5.27.  $\square$

**Open Question 6.** Study the degeneracy of equation (5.6) on non-integer level curves of rotation number, as the latter number tends to an integer value. The level curves should degenerate to boundaries of phase-lock areas. How to retrieve equations (5.21) and (5.22) on the boundaries and equation  $\xi_l(\lambda, \mu) = 0$  on the adjacencies from asymptotics of degenerating equation (5.6)?

## 5.7 Asymptotic behavior of phase-lock areas for small $\omega$

D.A.Filimonov, V.A.Kleptsyn, I.V.Schurov, V.M.Buchstaber and S.I.Tertychnyi have done numerical simulations studying what happens to the phase-lock areas, as  $\omega \rightarrow 0$ . They have observed that after appropriate rescaling of the variables  $(B, A)$ , the phase-lock areas tend to open sets (which we will call the *limit rescaled phase-lock areas*) whose components partition the whole plane somewhat similarly to a chess table turned by  $\frac{\pi}{4}$ , see Fig. 5 for  $\omega = 0.3$ .

**Open Question 7.** Study what happens with the phase-lock areas, as  $\omega \rightarrow 0$ . Study the separation curves and adjacencies of the limit rescaled phase-lock areas. To do this, it would be helpful to study the asymptotics of the functions  $\zeta_l(\omega, \mu)$  in variable  $\mu$  with fixed  $\omega$  and their zeros, as  $\omega \rightarrow 0$ .

## References

- [1] Arnold, V. I. *Geometrical Methods in the Theory of Ordinary Differential Equations*, Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 250. Springer-Verlag, New York, 1988.
- [2] Barone, A.; Paterno, G. *Physics and Applications of the Josephson Effect*, John Wiley and Sons, New York–Chichester–Brisbane–Toronto–Singapore, 1982.
- [3] Buchstaber, V.M.; Karpov, O.V.; Tertychnyi, S.I. *On properties of the differential equation describing the dynamics of an overdamped Josephson junction*, Russian Math. Surveys, **59:2** (2004), 377–378.
- [4] Buchstaber, V.M.; Karpov, O.V.; and Tertychnyi, S.I. *Peculiarities of dynamics of a Josephson junction shifted by a sinusoidal SHF current* (in Russian). Radiotekhnika i Elektronika, **51:6** (2006), 757–762.
- [5] Buchstaber, V.M.; Karpov, O.V.; Tertychnyi, S.I. *The rotation number quantization effect*, Theoret and Math. Phys., **162** (2010), No. 2, 211–221.
- [6] Buchstaber, V.M.; Karpov, O.V.; Tertychnyi, S.I. *The system on torus modeling the dynamics of Josephson junction*, Uspekhi Mat. Nauk, **67:1(403)** (2012), 181–182; English transl.: Russ. Math. Surveys, **67** (2012), 178–180.
- [7] Buchstaber, V.M.; Tertychnyi, S.I. *Explicit solution family for the equation of the resistively shunted Josephson junction model.*, Theoret. and Math. Phys., **176** (2013), No.2, 965–986.
- [8] Buchstaber, V.M.; Tertychnyi, S.I. *Holomorphic solutions of the double confluent Heun equation associated with the RSJ model of the Josephson junction*, Theoret. and Math. Phys., **182:3** (2015), 329–355.
- [9] Buchstaber, V.M.; Tertychnyi, S.I. *A remarkable sequence of Bessel matrices*, Mathematical Notes, **98** (2015), No. 5, 714–724.

- [10] Buchstaber, V.M.; Tertychnyi, S.I. *Automorphisms of solution space of special double confluent Heun equations*, To appear in *Funct. Analysis and Applications*, No 3 (2016).
- [11] Buchstaber, V.M.; Glutsyuk, A.A. *On determinants of modified Bessel functions and entire solutions of double confluent Heun equations*. Preprint <https://arxiv.org/abs/1509.01725>.
- [12] McCumber, D.E. *Effect of ac impedance on dc voltage-current characteristics of superconductor weak-link junctions*, *J. Appl. Phys.*, **39** (1968), No.7, 3113–3118.
- [13] Foote, R.L., *Geometry of the Prytz Planimeter*, *Reports on Math. Phys.* **42:1/2** (1998), 249–271.
- [14] Foote, R.L.; Levi, M.; Tabachnikov, S. *Tractrices, bicycle tire tracks, hatchet planimeters, and a 100-year-old conjecture*, *Amer. Math. Monthly*, **103** (2013), 199–216.
- [15] Glutsyuk, A.A.; Kleptsyn, V.A.; Filimonov, D.A.; Schurov, I.V. *On the adjacency quantization in an equation modeling the Josephson effect*, *Funct. Analysis and Appl.*, **48** (2014), No.4, 272–285.
- [16] Ilyashenko, Yu.S. *Lectures of the summer school “Dynamical systems”*, Poprad, Slovak Republic, 2009.
- [17] Ilyashenko, Yu.S.; Filimonov, D.A.; Ryzhov, D.A. *Phase-lock effect for equations modeling resistively shunted Josephson junctions and for their perturbations*, *Funct. Analysis and its Appl.* **45** (2011), No. 3, 192–203.
- [18] Josephson, B.D., *Possible new effects in superconductive tunnelling*, *Phys. Lett.*, **1** (1962), No. 7, 251–253.
- [19] Klimenko, A.V.; Romaskevich, O.L. *Asymptotic properties of Arnold tongues and Josephson effect*, *Mosc. Math. J.*, **14:2** (2014), 367–384.
- [20] Likharev, K.K.; Ulrikh, B.T. *Systems with Josephson junctions: Basic Theory*, Izdat. MGU, Moscow, 1978.
- [21] Schmidt, V.V., *Introduction to physics of superconductors* (in Russian), MCCME, Moscow, 2000.
- [22] Shapiro, S.; Janus, A.; Holly, S. *Effect of microwaves on Josephson currents in superconducting tunneling*, *Rev. Mod. Phys.*, **36** (1964), 223–225.

- [23] Stewart, W.C., *Current-voltage characteristics of Josephson junctions*. Appl. Phys. Lett., **12** (1968), No. 8, 277–280.
- [24] Tertychnyi, S.I. *Long-term behavior of solutions of the equation  $\dot{\phi} + \sin \phi = f$  with periodic  $f$  and the modeling of dynamics of overdamped Josephson junctions*, Preprint <https://arxiv.org/abs/math-ph/0512058>.
- [25] Tertychnyi, S.I. *The modelling of a Josephson junction and Heun polynomials*, Preprint <https://arxiv.org/abs/math-ph/0601064>