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Veronika Ertl, Wieslawa Niziol

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Abstract. We prove a mixed characteristic analog of the Beilinson-Lichtenbaum Conjecture for p-adic motivic cohomology. It gives a description, in the stable range, of p-adic motivic cohomology (defined using algebraic cycles) in terms of differential forms. This generalizes a result of Geisser [10] from small Tate twists to all twists and uses as a critical new ingredient the comparison theorem between syntomic complexes and p-adic nearby cycles proved recently in [8].

Contents

1. Introduction

For a smooth variety over a field of characteristic zero, the Beilinson-Lichtenbaum Conjecture states that, in a certain stable range, the p-adic motivic cohomology is equal to the étale cohomology:

$$H^i_{\mathrm{M}}(X, \mathbf{Z}/p^n(r)) \xrightarrow{\sim} H^i_{\text{ét}}(X, \mathbf{Z}/p^n(r)), \quad i \leq r.$$

Here motivic cohomology is defined as the hypercohomology of the Bloch's cycle complex $\mathbf{Z}/p^n(r)_{\mathrm{M}}$. This conjecture follows [31] from the Bloch-Kato Conjecture that was proved by Voevodsky and Rost [35].

For a smooth variety over a field of positive characteristic p, the analog of the Beilinson-Lichtenbaum Conjecture states that, in the same stable range, the p-adic motivic cohomology is equal to the logarithmic de Rham-Witt cohomology:

$$H^i_{\mathrm{M}}(X, \mathbf{Z}/p^n(r)) \xrightarrow{\sim} H^{i-r}_{\text{ét}}(X, W_n\Omega^r_{X,\log}).$$

It was proved by Geisser-Levine [11].

The purpose of this note is to prove a mixed characteristic analog of the Beilinson-Lichtenbaum Conjecture for p-adic motivic cohomology. Let X be a semistable scheme over $\mathcal{O}_K$ – a complete discrete valuation ring with fraction field K of characteristic 0 and with perfect residue field k of characteristic p. We fix a uniformizer ϖ of K. Let F be the fraction field of the ring of Witt vectors $W(k)$. We assume that the special fiber X_0 of X is smooth and treat X as a log-scheme. We show that, in the same stable range as above, the p-adic motivic cohomology of X_{tr} – the open set where the log-structure is trivial – is equal to the (logarithmic) syntomic-étale cohomology of X. This relates algebraic cycles to differential forms.

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Corollary 1.1. *We have the following natural isomorphism*[1]

$$H^i_{\mathrm{M}}(X_{\mathrm{tr}}, \mathbf{Q}_p(r)) \xrightarrow{\sim} H^i_{\text{ét}}(X, \mathcal{E}(r))_{\mathbf{Q}}, \quad i \leq r,$$

where $\mathcal{E}(\cdot)$ denotes the syntomic-étale cohomology complex. If X is proper, this yields the following natural isomorphism

$$H^i_{\mathrm{M}}(X_{\mathrm{tr}}, \mathbf{Q}_p(r)) \xrightarrow{\sim} H^i_{\text{ét}}(X, \mathcal{S}(r))_{\mathbf{Q}}, \quad i \leq r,$$

where $\mathcal{S}(\cdot)$ denotes the syntomic cohomology complex.

The rational syntomic cohomology $H^*_{\text{ét}}(X, \mathcal{S}(r))_{\mathbf{Q}}$ above is that defined in [9] as filtered Frobenius eigenspace of crystalline cohomology[2]. We show in the appendix that it is isomorphic to the logarithmic version of the convergent syntomic cohomology defined in [25] as well as to the rigid syntomic cohomology defined in [2, 13].

The above corollary is a simple consequence of the following theorem which is the main result of this paper.

Theorem 1.2. *Let $r \geq 0$. Let $j'_* : X_{\mathrm{tr}} \to X$ be the natural open immersion. Then there are natural cycle class maps between complexes of sheaves on the Nisnevich site of X and X_0, respectively,*

$$\mathrm{cl}^{\mathrm{syn}}_r : \mathrm{R}j'_*\mathbf{Z}/p^n(r)_{\mathrm{M}} \to \mathcal{E}'_n(r)_{\mathrm{Nis}}, \quad \mathrm{cl}^{\mathrm{syn}}_r : i^*\mathrm{R}j'_*\mathbf{Z}/p^n(r)_{\mathrm{M}} \to \mathcal{S}'_n(r)_{\mathrm{Nis}},$$

where $i : X_0 \hookrightarrow X$ is the special fiber of X. They are compatible with the étale cycle class maps and are p^N-quasi-isomorphisms, i.e., the kernels and cokernels of the maps induced on the cohomology sheaves are annihilated by p^N for a constant $N = N(e, p, r)$, which depends on the absolute ramification index e of K, r, but not on X or n[3].

The syntomic-étale cohomology $\mathcal{E}'_n(r)$ was defined by Fontaine-Messing [9] by gluing syntomic cohomology $\mathcal{S}'_n(r)$ on X_0 with étale cohomology on the generic fiber via the relative fundamental exact sequence of p-adic Hodge Theory. It is a complex of sheaves on the étale site of X. We extend this definition to logarithmic schemes (where one replaces syntomic cohomology by logarithmic syntomic cohomology). The Nisnevich version that appears in the above theorem is defined by projecting to the Nisnevich site and truncating at r:

$$\mathcal{E}'_n(r)_{\mathrm{Nis}} := \tau_{\leq r}\mathrm{R}\varepsilon_*\mathcal{E}'_n(r), \quad \mathcal{S}'_n(r)_{\mathrm{Nis}} := \tau_{\leq r}\mathrm{R}\varepsilon_*\mathcal{S}'_n(r),$$

where $\varepsilon : X_{\text{ét}} \to X_{\mathrm{Nis}}$ is the natural projection.

The syntomic part of the above theorem (hence of the above corollary as well), for twists $r \leq p-2$ (where no constants are needed) was proved by Geisser[4] [10, Theorem 1.3]. The key ingredient in his proof is the exact sequence of Kurihara [20] that links syntomic cohomology with p-adic nearby cycles coupled with the Beilinson-Lichtenbaum Conjecture over fields of characteristic zero and p. Our proof of Theorem 1.2 proceeds in a similar manner using as the main new ingredient the relation between syntomic complexes and p-adic nearby cycles proved recently in [8].

We will now describe it briefly in the case when there is no horizontal log-structure. First, we show that we have the p^{Nr}-distinguished triangle (on the étale site of X_0), for a universal constant N,

$$\mathcal{E}'_n(r)_X \to \mathcal{E}'_n(r)_{X^\times} \to W_n\Omega^{r-1}_{X_0,\log}[-r], \tag{1.1}$$

where $W_n\Omega^{r-1}_{X_0,\log}[-r]$ denotes the logarithmic de Rham-Witt sheaf and $X^\times$ denotes the scheme X with added log-structure coming from the special fiber. The syntomic-étale cohomology $\mathcal{E}'_n(r)_{X^\times}$ comes equipped with a period map

$$\alpha_r : \mathcal{E}'_n(r)_{X^\times} \to \mathrm{R}j_*\mathbf{Z}/p^n(r)'_{X_K},$$

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[1] For a smooth scheme Y, we set $H^*_{\mathrm{M}}(Y, \mathbf{Q}_p(r)) := H^* \operatorname{holim}_n \mathrm{R}\Gamma(Y_{\mathrm{Zar}}, \mathbf{Z}/p^n(r)_M) \otimes \mathbf{Q}$.

[2] It differs from the one defined in [22] by the absence of log-structure associated to the special fiber.

[3] If K has enough roots of unity then $N = N'r$ for a universal constant N' (not depending on p, X, K, n or r). See Section (2.1.1) of [8] for what it means for a field to contain enough roots of unity. The field F contains enough roots of unity and for any K, the field $K(\zeta_{p^n})$, for $n \geq c(K)+3$, where $c(K)$ is the conductor of K, contains enough roots of unity.

[4] Geisser's result was conditional on the Bloch-Kato Conjecture which at the time of the publication of his paper was not a theorem yet.

where $j_* : X_K \hookrightarrow X$ and $\mathbf{Z}/p^n(r)' = (p^a a!)^{-1}\mathbf{Z}/p^n(r)$ for $r = (p-1)a + b, a, b \in \mathbf{Z}, 0 \leq b < p-1$. Projecting it to the Nisnevich site and truncating at r we obtain the Nisnevich syntomic-étale period map

$$\alpha_r : \mathcal{E}'_n(r)_{X^\times,\mathrm{Nis}} \to \tau_{\leq r}\mathrm{R}\varepsilon_*\mathrm{R}j_*\mathbf{Z}/p^n(r)'_{X_K}.$$

The computations of p-adic nearby cycles via syntomic cohomology from [8] imply that this is a p^N-quasi-isomorphism, for a constant N as in the theorem. Hence, from (1.1), we obtain the p^N-distinguished triangle

$$\mathcal{E}'_n(r)_{X,\mathrm{Nis}} \xrightarrow{\alpha_r} \tau_{\leq r}\mathrm{R}j_*\tau_{\leq r}\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)'_{X_K} \to i_*W_n\Omega^{r-1}_{X_0,\log}[-r]. \tag{1.2}$$

Next, we note that the localization sequence in motivic cohomology yields the following distinguished triangle (on the Nisnevich site of X)

$$\mathbf{Z}/p^n(r)_{\mathrm{M}} \to j_*\mathbf{Z}/p^n(r)_{\mathrm{M}} \to i_*\mathbf{Z}/p^n(r-1)_{\mathrm{M}}[-1].$$

By the Beilinson-Lichtenbaum Conjecture and the computations of Geisser-Levine [11] of motivic cohomology in characteristic p, we have the cycle class map quasi-isomorphisms

$$\mathbf{Z}/p^n(r)_{\mathrm{M}} \xrightarrow{\sim} \tau_{\leq r}\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)_{X_K}, \quad \mathbf{Z}/p^n(r)_{\mathrm{M}} \xrightarrow{\sim} W_n\Omega^r_{X_0,\log}[-r].$$

The above triangle becomes

$$\mathbf{Z}/p^n(r)_{\mathrm{M}} \to j_*\tau_{\leq r}\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)_{X_K} \to i_*W_n\Omega^{r-1}_{X_0,\log}[-r] \tag{1.3}$$

Since $j_*\mathbf{Z}/p^n(r)_{\mathrm{M}} \xrightarrow{\sim} \mathrm{R}j_*\mathbf{Z}/p^n(r)_{\mathrm{M}}$, $\tau_{\leq r}\mathbf{Z}/p^n(r)_{\mathrm{M}} \xrightarrow{\sim} \mathbf{Z}/p^n(r)_{\mathrm{M}}$, the cycle class map of Theorem 1.2 can now be obtained by comparing sequences (1.2) and (1.3).

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1.0.1. *Notation and Conventions.* We assume all the schemes to be locally noetherian. We work in the category of fine log-schemes.

Definition 1.3. Let $N \in \mathbf{N}$. For a morphism $f : M \to M'$ of $\mathbf{Z}_p$-modules, we say that f is p^N-*injective* (resp. p^N-*surjective*) if its kernel (resp. its cockernel) is annihilated by p^N and we say that f is p^N-*isomorphism* if it is p^N-injective and p^N-surjective. We define in the same way the notion of p^N-*distinguished triangle* or p^N-*acyclic complex* (a complex whose cohomology groups are annihilated by p^N) as well as the notion of p^N-*quasi-isomorphism* (map in the derived category that induces a p^N-isomorphism on cohomology).

We will use a shorthand for certain homotopy limits. Namely, if $f : C \to C'$ is a map in the dg derived category of abelian groups, we set

$$[\ C \xrightarrow{f} C'\ ] := \mathrm{holim}(C \to C' \leftarrow 0).$$

And we set

$$\begin{bmatrix} C_1 & \xrightarrow{f} & C_2 \\ \downarrow & & \downarrow \\ C_3 & \xrightarrow{g} & C_4 \end{bmatrix} := [[C_1 \xrightarrow{f} C_2] \to [C_3 \xrightarrow{g} C_4]],$$

for a commutative diagram (the one inside the large bracket) in the dg derived category of abelian groups.

2. Syntomic cohomology

Let $\mathcal{O}_K$ be a complete discrete valuation ring with fraction field K of characteristic 0 and with perfect residue field k of characteristic p. Let ϖ be a uniformizer of $\mathcal{O}_K$; we will keep it fixed throughout the paper[5]. Let $W(k)$ be the ring of Witt vectors of k with fraction field F (i.e, $W(k) = \mathcal{O}_F$); let e be the ramification index of K over F. Let $\sigma = \varphi$ be the absolute Frobenius on $W(\overline{k})$. For a $\mathcal{O}_K$-scheme X, let X_0 denote the special fiber of X and let X_n denote the reduction modulo p^n of X. We will denote by $\mathcal{O}_K$, ${\mathcal{O}_K}^{\times}$, and ${\mathcal{O}_K}^{0}$ the scheme $\mathrm{Spec}(\mathcal{O}_K)$ with the trivial, canonical (i.e., associated to the closed point), and $(\mathbf{N} \to \mathcal{O}_K, 1 \mapsto 0)$ log-structure respectively.

In this section we will briefly review the definitions of syntomic and syntomic-étale cohomologies and their basic properties. We refer the reader for details to [33, 2].

2.1. **Syntomic cohomology.** For a log-scheme X we denote by X_{syn} the small syntomic site of X. It is built from log-syntomic morphisms $f : Y \to Z$ in the sense of Kato [18, 2.5] (see also [7, 6.1]), i.e., the morphism f is integral, the underlying morphism of schemes is flat and locally of finite presentation, and, étale locally on Y, there is a factorization $Y \overset{i}{\hookrightarrow} W \overset{h}{\to} Z$ where h is log-smooth and i is an exact closed immersion that is transversally regular over Z.

For a log-scheme X log-syntomic over $\mathrm{Spec}(W(k))$, define

$$\mathcal{O}_n^{\mathrm{cr}}(X) = H^0_{\mathrm{cr}}(X_n, \mathcal{O}_{X_n}), \qquad \mathcal{J}_n^{[r]}(X) = H^0_{\mathrm{cr}}(X_n, \mathcal{J}^{[r]}_{X_n}),$$

where $\mathcal{O}_{X_n}$ is the structure sheaf of the absolute crystalline site (i.e., over $W_n(k)$), $\mathcal{J}_{X_n} = \mathrm{Ker}(\mathcal{O}_{X_n/W_n(k)} \to \mathcal{O}_{X_n})$, and $\mathcal{J}^{[r]}_{X_n}$ is its r'th divided power of $\mathcal{J}_{X_n}$. Set $\mathcal{J}^{[r]}_{X_n} = \mathcal{O}_{X_n}$ if $r \leq 0$. We know [9, II.1.3] that the presheaves $\mathcal{J}_n^{[r]}$ are sheaves on $X_{n,\mathrm{syn}}$, flat over $\mathbf{Z}/p^n$, and that $\mathcal{J}_{n+1}^{[r]} \otimes \mathbf{Z}/p^n \simeq \mathcal{J}_n^{[r]}$. There is a natural, compatible with Frobenius, and functorial isomorphism

$$H^*(X_{\mathrm{syn}}, \mathcal{J}_n^{[r]}) \simeq H^*_{\mathrm{cr}}(X_n, \mathcal{J}^{[r]}_{X_n}).$$

It is easy to see that $\varphi(\mathcal{J}_n^{[r]}) \subset p^r \mathcal{O}_n^{\mathrm{cr}}$ for $0 \leq r \leq p-1$. This fails in general and we modify $\mathcal{J}_n^{[r]}$:

$$\mathcal{J}_n^{<r>} := \{x \in \mathcal{J}_{n+s}^{[r]} \mid \varphi(x) \in p^r \mathcal{O}_{n+s}^{\mathrm{cr}}\}/p^n,$$

for some $s \geq r$. This definition is independent of s. We check that $\mathcal{J}_n^{<r>}$ is flat over $\mathbf{Z}/p^n$ and $\mathcal{J}_{n+1}^{<r>} \otimes \mathbf{Z}/p^n \simeq \mathcal{J}_n^{<r>}$. This allows us to define the divided Frobenius $\varphi_r = "\varphi/p^r" : \mathcal{J}_n^{<r>} \to \mathcal{O}_n^{\mathrm{cr}}$.

Set

$$\mathcal{S}_n(r) := \mathrm{Cone}(\mathcal{J}_n^{<r>} \overset{1-\varphi_r}{\longrightarrow} \mathcal{O}_n^{\mathrm{cr}})[-1].$$

Since the following sequence is exact

$$0 \longrightarrow \mathcal{S}_n(r) \longrightarrow \mathcal{J}_n^{<r>} \overset{1-\varphi_r}{\longrightarrow} \mathcal{O}_n^{\mathrm{cr}} \longrightarrow 0,$$

we actually have

$$\mathcal{S}_n(r) := \mathrm{Ker}(\mathcal{J}_n^{<r>} \overset{1-\varphi_r}{\longrightarrow} \mathcal{O}_n^{\mathrm{cr}}).$$

In the same way we can define syntomic sheaves $\mathcal{S}_n(r)$ on $X_{m,\mathrm{syn}}$ for $m \geq n$. Abusing notation, we set $\mathcal{S}_n(r) = i_* \mathcal{S}_n(r)$ for the natural map $i : X_{m,\mathrm{syn}} \to X_{\mathrm{syn}}$. Since i_* is exact, $H^*(X_{m,\mathrm{syn}}, \mathcal{S}_n(r)) = H^*(X_{\mathrm{syn}}, \mathcal{S}_n(r))$. Because of that we will write $\mathcal{S}_n(r)$ for the syntomic sheaves on $X_{m,\mathrm{syn}}$ as well as on X_{syn}. We will also need the "undivided" version of syntomic complexes of sheaves:

$$\mathcal{S}'_n(r) := \mathrm{Cone}(\mathcal{J}_n^{[r]} \overset{p^r-\varphi}{\longrightarrow} \mathcal{O}_n^{\mathrm{cr}})[-1].$$

For $r, i \geq 0$, we have the long exact sequences

$$\begin{aligned} &\to H^i(X_{\text{ét}}, \mathcal{S}_n(r)) \to H^i_{\mathrm{cr}}(X_n, J_{X_n}^{<r>}) \overset{1-\varphi_r}{\longrightarrow} H^i_{\mathrm{cr}}(X_n, \mathcal{O}_{X_n}) \to \\ &\to H^i(X_{\text{ét}}, \mathcal{S}'_n(r)) \to H^i_{\mathrm{cr}}(X_n, J_{X_n}^{[r]}) \overset{p^r-\varphi}{\longrightarrow} H^i_{\mathrm{cr}}(X_n, \mathcal{O}_{X_n}) \to \end{aligned} \tag{2.1}$$

[5]This is necessary to fix an embedding of $\mathrm{Spec}(\mathcal{O}_K)$ into a smooth scheme over $\mathbf{Z}_p$.

Proposition 2.1. *([8, Prop. 3.12]) For X a fine and saturated log-smooth log-scheme over $\mathcal{O}_K$ and $0 \leq r \leq p-2$, the natural map of complexes of sheaves on the étale site of X_0*

$$\tau_{\leq r}\mathcal{S}_n(r) \to \mathcal{S}_n(r)$$

is a quasi-isomorphism. For X semistable over $\mathcal{O}_K$[6] and $r \geq 0$, the natural map of complexes of sheaves on the étale site of X_0

$$\tau_{\leq r}\mathcal{S}'_n(r) \to \mathcal{S}'_n(r)$$

is a p^{Nr}-quasi-isomorphism for a universal constant N.

The natural map $\omega : \mathcal{S}'_n(r) \to \mathcal{S}_n(r)$ induced by the maps $p^r : \mathcal{J}_n^{[r]} \to \mathcal{J}_n^{<r>}$ and $\mathrm{Id} : \mathcal{O}_n^{\mathrm{cr}} \to \mathcal{O}_n^{\mathrm{cr}}$ has kernel and cokernel killed by p^r. So does the map $\tau : \mathcal{S}_n(r) \to \mathcal{S}'_n(r)$ induced by the maps $\mathrm{Id} : \mathcal{J}_n^{<r>} \to \mathcal{J}_n^{[r]}$ and $p^r : \mathcal{O}_n^{\mathrm{cr}} \to \mathcal{O}_n^{\mathrm{cr}}$. We have $\tau\omega = \omega\tau = p^r$.

If it does not cause confusion, we will write $\mathcal{S}_n(r)$, $\mathcal{S}'_n(r)$ also for $\mathrm{R}\varepsilon_*\mathcal{S}_n(r)$, $\mathrm{R}\varepsilon_*\mathcal{S}'_n(r)$, respectively, where $\varepsilon : X_{n,\mathrm{syn}} \to X_{n,\text{ét}}$ is the natural projection to the étale site (or sometimes to the Nisnevich site)

2.1.1. *Syntomic cohomology and differential forms.* Let X be a syntomic scheme over $W(k)$. Recall the differential definition [16] of syntomic cohomology. Assume first that we have an immersion $\iota : X \hookrightarrow Z$ over $W(k)$ such that Z is a smooth $W(k)$-scheme endowed with a compatible system of liftings of the Frobenius $\{F_n : Z_n \to Z_n\}$. Let $D_n = D_{X_n}(Z_n)$ be the PD-envelope of X_n in Z_n (compatible with the canonical PD-structure on $pW_n(k)$) and J_{D_n} the ideal of X_n in D_n. Set $J_{D_n}^{<r>} := \{a \in J_{D_{n+s}}^{[r]} | \varphi(a) \in p^r\mathcal{O}_{D_{n+s}}\}/p^n$ for some $s \geq r$. For $0 \leq r \leq p-1$, $J_{D_n}^{<r>} = J_{D_n}^{[r]}$. This definition is independent of s. Consider the following complexes of sheaves on $X_{\text{ét}}$.

$$\begin{aligned} S_n(r)_{X,Z} &:= \mathrm{Cone}(J_{D_n}^{<r-\bullet>} \otimes \Omega^\bullet_{Z_n} \xrightarrow{1-\varphi_r} \mathcal{O}_{D_n} \otimes \Omega^\bullet_{Z_n})[-1], \\ S'_n(r)_{X,Z} &:= \mathrm{Cone}(J_{D_n}^{[r-\bullet]} \otimes \Omega^\bullet_{Z_n} \xrightarrow{p^r-\varphi} \mathcal{O}_{D_n} \otimes \Omega^\bullet_{Z_n})[-1], \end{aligned} \tag{2.2}$$

where $\Omega^\bullet_{Z_n} := \Omega^\bullet_{Z_n/W_n(k)}$ and φ_r is "φ/p^r" (see [33, 2.1] for details). The complexes $S_n(r)_{X,Z}$, $S'_n(r)_{X,Z}$ are, up to canonical quasi-isomorphisms, independent of the choice of ι and $\{F_n\}$ (and we will omit the subscript Z from the notation). Again, the natural maps $\omega : S'_n(r)_X \to S_n(r)_X$ and $\tau : S_n(r)_X \to S'_n(r)_X$ have kernels and cokernels annihilated by p^r.

In general, immersions as above exist étale locally, and we define $S_n(r)_X \in \mathbf{D}^+(X_{\text{ét}}, \mathbf{Z}/p^n)$ by gluing the local complexes. We define $S'_n(r)_X$ in a similar way.

Let now X be a log-syntomic scheme over $W(k)$. Using log-crystalline cohomology, the above construction of syntomic complexes goes through almost verbatim (see [33, 2.1] for details) to yield the logarithmic analogs $S_n(r)$ and $S'_n(r)$ on $X_{\text{ét}}$. In this paper we are often interested in log-schemes coming from a regular syntomic scheme X over $W(k)$ and a relative simple (i.e., with no self-intersections) normal crossing divisor D on X. In such cases we will write $S_n(r)_X(D)$ and $S'_n(r)_X(D)$ for the syntomic complexes and use the Zariski topology instead of the étale one.

2.1.2. *Products.* We need to discuss products. Assume that we are in the lifted situation (2.2). Then we have a product structure

$$\cup : S'_n(r)_{X,Z} \otimes S'_n(r')_{X,Z} \to S'_n(r+r')_{X,Z}, \quad r, r' \geq 0,$$

defined by the following formulas

$$\begin{aligned} (x,y) \otimes (x',y') &\mapsto (xx', (-1)^a p^r xy' + y\varphi(x')) \\ (x,y) \in S'_n(r)^a_{X,Z} &= (J_{D_n}^{[r-a]} \otimes \Omega^a_{Z_n}) \oplus (\mathcal{O}_{D_n} \otimes \Omega^{a-1}_{Z_n}), \\ (x',y') \in S'_n(r')^b_{X,Z} &= (J_{D_n}^{[r'-b]} \otimes \Omega^b_{Z_n}) \oplus (\mathcal{O}_{D_n} \otimes \Omega^{b-1}_{Z_n}). \end{aligned}$$

[6] A scheme X over $\mathcal{O}_K$ is called semistable if it is surjective on $\mathrm{Spec}\,\mathcal{O}_K$, regular, and there is a distinguished divisor "at infinity" D_∞ which is a strict relative normal crossing divisor and which together with the special fiber forms a strict normal crossing divisor.

Globalizing, we obtain the product structure

$$\cup : S'_n(r)_X \otimes^{\mathbb{L}} S'_n(r')_X \to S'_n(r+r')_X, \quad r, r' \geq 0.$$

This product is clearly compatible with the crystalline product.

Similarly, we have the product structures

$$\cup : S_n(r)_{X,Z} \otimes S_n(r')_{X,Z} \to S_n(r+r')_{X,Z}, \quad r, r' \geq 0,$$

defined by the formulas

$$(x,y) \otimes (x',y') \mapsto (xx', (-1)^a xy' + y\varphi_{r'}(x'))$$
$$(x,y) \in S_n(r)^a_{X,Z} = (J_{D_n}^{<r-a>} \otimes \Omega^a_{Z_n}) \oplus (\mathcal{O}_{D_n} \otimes \Omega^{a-1}_{Z_n}),$$
$$(x',y') \in S_n(r')^b_{X,Z} = (J_{D_n}^{<r'-b>} \otimes \Omega^b_{Z_n}) \oplus (\mathcal{O}_{D_n} \otimes \Omega^{b-1}_{Z_n}).$$

Globalizing, we obtain the product structure

$$\cup : S_n(r)_X \otimes^{\mathbb{L}} S_n(r')_X \to S_n(r+r')_X, \quad r, r' \geq 0.$$

This product is also clearly compatible with the crystalline product.

The above product structures are compatible with the maps ω. On the other hand the maps τ are, in general, not compatible with products.

2.1.3. *Symbol maps.* Let X be a regular syntomic scheme over $W(k)$ with a divisor D with relative simple normal crossings. Recall that there are symbol maps defined by Kato and Tsuji [33, 2.2]

$$(M^{\mathrm{gp}}_{X,n})^{\otimes r} \to H^r(S'_n(r)_X(D)), \quad (M^{\mathrm{gp}}_{X,n+1})^{\otimes r} \to H^r(S_n(r)_X(D)), \quad r \geq 0, \tag{2.3}$$

where, for a log-scheme X, M_X denotes its log-structure. For $r = 1$, we get the first Chern class maps (recall that $M^{\mathrm{gp}}_X = j_*\mathcal{O}^*_{X\setminus D}$, where $j : X \setminus D \hookrightarrow X$ is the natural immersion)

$$c_1^{\mathrm{syn}} : \quad j_*\mathcal{O}^*_{X\setminus D}[-1] \to i_*j_*\mathcal{O}^*_{(X\setminus D)_{n+1}}[-1] \to S_n(1)_X(D),$$
$$c_1^{\mathrm{syn}} : \quad j_*\mathcal{O}^*_{X\setminus D}[-1] \to i_*j_*\mathcal{O}^*_{(X\setminus D)_n}[-1] \to S'_n(1)_X(D),$$

that are compatible, i.e., the following diagram commutes

$$\begin{array}{ccc} j_*\mathcal{O}^*_{X\setminus D}[-1] & \xrightarrow{c_1^{\mathrm{syn}}} & S'_n(1)_X(D) \\ \Big\downarrow{\scriptstyle pc_1^{\mathrm{syn}}} & \swarrow{\scriptstyle \omega} & \\ S_n(1)_X(D) & & \end{array}$$

In the lifted situation these classes are defined in the following way. Let C_n be the complex

$$(1 + J_{D_n} \to M^{\mathrm{gp}}_{D_n}) \simeq j_*\mathcal{O}^*_{(X\setminus D)_n}[-1].$$

The Chern class maps

$$c_1^{\mathrm{syn}} : j_*\mathcal{O}^*_{(X\setminus D)_n}[-1] \to S'_n(1)_X(D), \quad c_1^{\mathrm{syn}} : j_*\mathcal{O}^*_{(X\setminus D)_{n+1}}[-1] \to S_n(1)_X(D),$$

are defined by the morphisms of complexes

$$C_n \to S'_n(1)_{X,Z}, \quad C_{n+1} \to S_n(1)_{X,Z}$$

given by the formulas

$$1 + J_{D_n} \to (S'_n(1)_{X,Z})^0 = J_{D_n}; \quad a \mapsto \log a;$$
$$1 + J_{D_{n+1}} \to (S_n(1)_{X,Z})^0 = J_{D_n}; \quad a \mapsto \log a \mod p^n;$$

and

$$M^{\mathrm{gp}}_{D_n} \to (S'_n(1)_{X,Z})^1 = (\mathcal{O}_{D_n} \otimes \Omega^1_{Z_n}) \oplus \mathcal{O}_{D_n}; \quad b \mapsto (d\log b, \log(b^p\varphi_{D_n}(b)^{-1}));$$
$$M^{\mathrm{gp}}_{D_{n+1}} \to (S_n(1)_{X,Z})^1 = (\mathcal{O}_{D_n} \otimes \Omega^1_{Z_n}) \oplus \mathcal{O}_{D_n}; \quad b \mapsto (d\log b \mod p^n, p^{-1}\log(b^p\varphi_{D_{n+1}}(b)^{-1})).$$

The symbol maps (2.3) for general r are obtained from $r = 1$ using the product structure on syntomic cohomology.

2.2. **Syntomic-étale cohomology.** We will now recall the definition and basic properties of syntomic-étale cohomology. The relationship between syntomic cohomology and syntomic-étale cohomology mirrors the one between étale nearby cycles and étale cohomology. Let X be a log-scheme, log-syntomic over $\mathrm{Spec}(W(k))$. We will need the logarithmic version of the syntomic-étale site of Fontaine-Messing [9]. We say that a morphism $\mathcal{Z} \to \mathcal{Y}$ of p-adic formal log-schemes over $\mathrm{Spf}(W(k))$ is (small) log-syntomic if every $Z_n \to Y_n$ is (small) log-syntomic. For a formal log-scheme $\mathcal{Z}$ the syntomic-étale site $\mathcal{Z}_{\mathrm{sé}}$ is defined by taking as objects morphisms $f : \mathcal{Y} \to \mathcal{Z}$ that are small log-syntomic and have log-étale generic fiber. This last condition means that, étale locally on $\mathcal{Y}$, f has a factorization $\mathcal{Y} \xrightarrow{i} \mathcal{X} \xrightarrow{g} \mathcal{Z}$ with $\mathcal{X}$ affine, i an exact closed immersion, and g log-smooth such that the map $F \otimes_{W(k)} \Gamma(\mathcal{Y}, I/I^2) \to F \otimes_{W(k)} \Gamma(\mathcal{Y}, i^*\Omega^1_{\mathcal{X}/\mathcal{Z}})$ is an isomorphism, where I is the ideal of $\mathcal{O}_{\mathcal{X}}$ defining $\mathcal{Y}$. For a log-scheme Z, we also have the syntomic-étale site $Z_{\mathrm{sé}}$. Here the objects are morphisms $U \to Z$ that are small log-syntomic with the generic fiber U_K log-étale over Z_K.

Let $\widehat{X}$ be the p-adic completion of X. Let $i : X_{n,\text{ét}} \to X_{\text{ét}}$ and $j : X_{\mathrm{tr},K,\text{ét}} \to X_{\text{ét}}$ be the natural maps. Here X_{tr} is the open set of X where the log-structure is trivial. We have the following commutative diagram of maps of topoi

$$\begin{array}{ccccc} \widehat{X}_{\mathrm{sé}} & \xrightarrow{i_{\mathrm{sé}}} & X_{\mathrm{sé}} & \xleftarrow{j_{\mathrm{sé}}} & X_{K,\mathrm{sé}} \\ \downarrow{\scriptstyle \widehat{\varepsilon}} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon_K} \\ \widehat{X}_{\text{ét}} & \xrightarrow{i_{\text{ét}}} & X_{\text{ét}} & \xleftarrow{j_{\text{ét}}} & X_{K,\text{ét}} \end{array}$$

Assume first that $0 \leq r \leq p-2$. Abusively, let $\mathcal{S}_n(r)$ denote also the direct image of $\mathcal{S}_n(r)$ under the canonical morphism $X_{n,\mathrm{syn}} \to \widehat{X}_{\mathrm{sé}}$. By [9, III.5], for $j' : X_{\mathrm{tr},K,\text{ét}} \to X_{K,\mathrm{sé}}$, there is a canonical homomorphism

$$\alpha_r : \mathcal{S}_n(r) \to i^*_{\mathrm{sé}} j_{\mathrm{sé}*} j'_* G\mathbf{Z}/p^n(r),$$

where G denotes the Godement resolution of a sheaf (or a complex of sheaves). Similarly, for any $r \geq 0$, we get a natural map

$$\tilde{\alpha}_r : \mathcal{S}_n(r) \to i^*_{\mathrm{sé}} j_{\mathrm{sé}*} j'_* G\mathbf{Z}/p^n(r)',$$

where $\mathbf{Z}/p^n(r)' = (p^a a!)^{-1}\mathbf{Z}/p^n(r)$ for $r = (p-1)a + b, a, b \in \mathbf{Z}, 0 \leq b < p-1$ [9, III.5]. Composing with the map $\mathcal{S}'_n(r) \to \mathcal{S}_n(r)$ we get a natural morphism

$$\alpha_r : \mathcal{S}'_n(r) \to i^*_{\mathrm{sé}} j_{\mathrm{sé}*} j'_* G\mathbf{Z}/p^n(r)'.$$

2.2.1. *Syntomic complexes and p-adic nearby cycles.* For log-schemes over $\mathcal{O}_K^{\times}$, in a stable range, syntomic cohomology tends to compute (via the period morphism) p-adic nearby cycles . We will briefly recall the relevant theorems. For $0 \leq r \leq p-2$, there is a natural homomorphism on the étale site of X_n

$$\alpha_r : \mathcal{S}_n(r) \to i^* \mathrm{R}j_* \mathbf{Z}/p^n(r).$$

To define it, we apply $\mathrm{R}\widehat{\varepsilon}_*$ to the map $\mathcal{S}_n(r) \to i^*_{\mathrm{sé}} \mathrm{R}j_{\mathrm{sé}*} \mathrm{R}j'_* \mathbf{Z}/p^n(r)$ induced from the map α_r described above and get

$$\mathrm{R}\varepsilon_* \mathcal{S}_n(r) = \mathrm{R}\widehat{\varepsilon}_* \mathcal{S}_n(r) \to \mathrm{R}\widehat{\varepsilon}_* i^*_{\mathrm{sé}} \mathrm{R}j_{\mathrm{sé}*} \mathrm{R}j'_* \mathbf{Z}/p^n(r) = i^*_{\text{ét}} \mathrm{R}\varepsilon_* \mathrm{R}j_{\mathrm{sé}*} \mathrm{R}j'_* \mathbf{Z}/p^n(r) = i^* \mathrm{R}j_* \mathbf{Z}/p^n(r).$$

The first equality follows from the fact that the morphism $X_{n,\mathrm{syn}} \to \widehat{X}_{\mathrm{sé}}$ is exact [9, III.4.1]. The second equality was proved in [19, 2.5], [32, 5.2.3]. One checks that α_r is compatible with products.

Theorem 2.2. *(*[34, Theorem 5.1]*) For $i \leq r \leq p-2$ and for a fine and saturated log-scheme X log-smooth over $\mathcal{O}_K^{\times}$ the period map*

$$\alpha_r : \quad \mathcal{S}_n(r)_X \xrightarrow{\sim} \tau_{\leq r} i^* \mathrm{R}j_* \mathbf{Z}/p^n(r)_{X_{\mathrm{tr}}}. \tag{2.4}$$

is an isomorphism.

Similarly, for any $r \geq 0$, we get a natural map

$$\tilde{\alpha}_r : \mathcal{S}_n(r) \to i^* \mathrm{R}j_* \mathbf{Z}/p^n(r)'.$$

Composing with the map $\omega : \mathcal{S}'_n(r) \to \mathcal{S}_n(r)$ we get a natural, compatible with products, morphism

$$\alpha_r : \mathcal{S}'_n(r) \to i^* \mathrm{R}j_* \mathbf{Z}/p^n(r)'.$$

Theorem 2.3. *(*[8, Theorem 1.1]*) For $0 \leq i \leq r$ and for a semistable scheme X over $\mathcal{O}_K$, consider the period map*

$$\alpha_r : \quad \mathcal{H}^i(\mathcal{S}'_n(r)_X) \to i^* \mathrm{R}^i j_* \mathbf{Z}/p^n(r)'_{X_{\mathrm{tr}}}. \tag{2.5}$$

If K has enough roots of unity then the kernel and cokernel of this map are annihilated by p^{Nr} for a universal constant N (not depending on p, X, K, n or r)*. In general, the kernel and cokernel of this map are annihilated by p^N for an integer $N = N(e, p, r)$, which depends on e, r, but not on X or n.*

2.2.2. *Syntomic-étale cohomology.* Recall [9, III.4.4], [32, 5.2.2] that the functor $\mathcal{F} \mapsto (i^*_{\mathrm{sé}}\mathcal{F}, j^*_{\mathrm{sé}}\mathcal{F}, i^*_{\mathrm{sé}}\mathcal{F} \to i^*_{\mathrm{sé}} j_{\mathrm{sé}\,*} j^*_{\mathrm{sé}}\mathcal{F})$ from the category of sheaves on $X_{\mathrm{sé}}$ to the category of triples $(\mathcal{G}, \mathcal{H}, \mathcal{G} \to i^*_{\mathrm{sé}} j_{\mathrm{sé}\,*}\mathcal{H})$, where $\mathcal{G}$ (resp. $\mathcal{H}$) are sheaves on $\widehat{X}_{\mathrm{sé}}$ (resp. $X_{K,\mathrm{sé}}$) is an equivalence of categories. It follows that we can glue the complexes of sheaves $\mathcal{S}_n(r)$ and $\mathcal{S}'_n(r)$ and the complexes of sheaves $j'_* G\mathbf{Z}/p^n(r)$ and $j'_* G\mathbf{Z}/p^n(r)'$ by the maps α_r and obtain complexes of sheaves $\mathcal{E}_n(r)$ and $\mathcal{E}'_n(r)$ on $X_{\mathrm{sé}}$. We have the exact sequences

$$\begin{aligned} 0 \to & j_{\mathrm{sét}!} j'_* G\mathbf{Z}/p^n(r) \to \mathcal{E}_n(r) \to i_* \mathcal{S}_n(r) \to 0, && 0 \leq r \leq p-2; \\ 0 \to & j_{\mathrm{sét}!} j'_* G\mathbf{Z}/p^n(r)' \to \mathcal{E}'_n(r) \to i_* \mathcal{S}'_n(r) \to 0, && r \geq 0. \end{aligned}$$

Remark 2.4. The syntomic-étale complexes $\mathcal{E}_n(r)$ that we described here are the same (in the derived category) as those defined by Fontaine-Messing in [9, 5] in the case when $X_{\mathrm{tr}} = X$ but differ from those defined by Tsuji in [32, 5.2] in the general situation. More specifically, we have

$$\mathcal{E}^T_n(r) = \mathcal{H}^0(\mathcal{E}_n(r)),$$

where we wrote $\mathcal{E}^T_n(r)$ for the syntomic-étale sheaves of Tsuji.

If it does not cause confusion, we will denote by $\mathcal{E}_n(r)$ and $\mathcal{E}'_n(r)$ also the derived pushforwards of $\mathcal{E}_n(r)$ and $\mathcal{E}'_n(r)$ to $X_{\mathrm{ét}}$. Notice that they are quasi-isomorphic to the complexes obtained by gluing the complexes of sheaves $\mathcal{S}_n(r)$ and $\mathcal{S}'_n(r)$ and the complexes of sheaves $j'_* G\mathbf{Z}/p^n(r)'$ by the maps $\tilde{\alpha}_r$ and α_r. Hence we have the distinguished triangles

$$j_{\mathrm{ét}!}\mathrm{R}j'_*\mathbf{Z}/p^n(r)' \to \mathcal{E}_n(r) \to i_*\mathcal{S}_n(r), \quad j_{\mathrm{ét}!}\mathrm{R}j'_*\mathbf{Z}/p^n(r)' \to \mathcal{E}'_n(r) \to i_*\mathcal{S}'_n(r), \tag{2.6}$$

where $j' : X_{\mathrm{tr},K} \to X_K$, as well as the natural maps

$$\tilde{\alpha}_r : \mathcal{E}_n(r) \to \mathrm{R}j_*\mathbf{Z}/p^n(r)', \quad \alpha_r : \mathcal{E}_n(r)' \to \mathrm{R}j_*\mathbf{Z}/p^n(r)'$$

compatible with the maps $\tilde{\alpha}_r$ and α_r. For $a \geq 0$, we have the truncated version of the above - the distinguished triangles

$$j_{\mathrm{ét}!}\tau_{\leq a}\mathrm{R}j'_*\mathbf{Z}/p^n(r)' \to \tau_{\leq a}\mathcal{E}_n(r) \to i_*\tau_{\leq a}\mathcal{S}_n(r), \quad j_{\mathrm{ét}!}\tau_{\leq a}\mathrm{R}j'_*\mathbf{Z}/p^n(r)' \to \tau_{\leq a}\mathcal{E}'_n(r) \to i_*\tau_{\leq a}\mathcal{S}'_n(r). \tag{2.7}$$

2.2.3. *Syntomic-étale cohomology and étale cohomology of the generic fiber.* For a log-scheme over $\mathcal{O}_K^\times$, in a stable range, syntomic-étale cohomology tends to compute étale cohomology of the generic fiber.

Theorem 2.5. *Let X be a log-scheme log-smooth over ${\mathcal{O}_K}^\times$. Let $j : X_{\mathrm{tr}} \hookrightarrow X$ be the natural open immersion. Then*

(1) *we have a natural quasi-isomorphism*

$$\tilde{\alpha}_r : \tau_{\leq r}\mathcal{E}_n(r) \simeq \tau_{\leq r}\mathrm{R}j_*\mathbf{Z}/p^n(r), \quad 0 \leq r \leq p-2.$$

(2) *if X is semistable, there is a constant N as in Theorem 2.3 and a natural morphism*

$$\alpha_r : \mathcal{E}'_n(r) \to \mathrm{R}j_*\mathbf{Z}/p^n(r)', \quad r \geq 0,$$

such that the induced map on cohomology sheaves in degree $q \leq r$ has kernel and cokernel annihilated by p^N.

Proof. Assume that $0 \leq r \leq p-2$. Consider the following commutative diagram of distinguished triangles

$$\begin{array}{ccccc} j_{\text{ét}!}\tau_{\leq r}\mathrm{R}j'_*\mathbf{Z}/p^n(r) & \longrightarrow & \tau_{\leq r}\mathcal{E}_n(r) & \longrightarrow & i_*\mathcal{S}_n(r) \\ \wr\downarrow \mathrm{Id} & & \downarrow \tilde{\alpha}_r & & \wr\downarrow \alpha_r \\ j_{\text{ét}!}\tau_{\leq r}\mathrm{R}j'_*\mathbf{Z}/p^n(r) & \longrightarrow & \tau_{\leq r}\mathrm{R}j_*\mathbf{Z}/p^n(r) & \longrightarrow & i_*i^*\tau_{\leq r}\mathrm{R}j_*\mathbf{Z}/p^n(r) \end{array}$$

The top triangle is distinguished because we have the distinguished triangle from (2.7) and the natural map $\tau_{\leq r}\mathcal{S}_n(r) \xrightarrow{\sim} \mathcal{S}_n(r)$ is a quasi-isomorphism. The map α_r is a quasi-isomorphism by the main theorem of [34]. The first part of the theorem follows.

For the second part consider the following commutative diagram of distinguished triangles

$$\begin{array}{ccccc} j_{\text{ét}!}\tau_{\leq r}\mathrm{R}j'_*\mathbf{Z}/p^n(r)' & \longrightarrow & \tau_{\leq r}\mathcal{E}'_n(r) & \longrightarrow & i_*\tau_{\leq r}\mathcal{S}'_n(r) \\ \wr\downarrow \mathrm{Id} & & \downarrow \alpha_r & & \downarrow \alpha_r \\ j_{\text{ét}!}\tau_{\leq r}\mathrm{R}j'_*\mathbf{Z}/p^n(r)' & \longrightarrow & \tau_{\leq r}\mathrm{R}j_*\mathbf{Z}/p^n(r)' & \longrightarrow & i_*i^*\tau_{\leq r}\mathrm{R}j_*\mathbf{Z}/p^n(r)' \end{array}$$

By [8, Theorem 1.1], the right period map α_r on the level of cohomology has kernels and cokernels killed by N for a constant N as in the theorem. Hence the same is true of the left map α_r, as wanted. □

The above theorem implies that the logarithmic syntomic-étale cohomology is close to the logarithmic syntomic-étale cohomology of the complement of the divisor at infinity.

Corollary 2.6. *Let X be a semistable scheme over $\mathcal{O}_K$ with a divisor at infinity D_∞. We treat it is as a log-scheme over $\mathcal{O}_K^\times$. Let $Y := X \setminus D_\infty$ and let $j_1 : Y \hookrightarrow X$.*

(1) *we have a natural quasi-isomorphism*

$$\tilde{\alpha}_r : \tau_{\leq r}\mathcal{E}_n(r)_X \xrightarrow{\sim} \tau_{\leq r}\mathrm{R}j_{1*}\mathcal{E}_n(r)_Y, \quad 0 \leq r \leq p-2.$$

(2) *there is a constant N as in Theorem 2.3 and a natural morphism*

$$\alpha_r : \mathcal{E}'_n(r)_X \to \mathrm{R}j_{1*}\mathcal{E}'_n(r)_Y, \quad r \geq 0,$$

such that the induced map on cohomology sheaves in degree $q \leq r$ has kernel and cokernel annihilated by p^N.

Proof. Note that $X_{\text{tr}} = Y_K$ and set $j_2 : Y_K \hookrightarrow Y$. We have $j = j_1 j_2$. By Theorem 2.5, both terms in the first claim are quasi-isomorphic to

$$\tau_{\leq r}\mathrm{R}j_*\mathbf{Z}/p^n(r)_{X_{\text{tr}}} = \tau_{\leq r}\mathrm{R}j_{1*}\tau_{\leq r}\mathrm{R}j_{2*}\mathbf{Z}/p^n(r)_{Y_K}.$$

Hence they are quasi-isomorphic. The second claim of the corollary is proved in the same way. □

2.2.4. *Nisnevich syntomic-étale cohomology.* We will pass now to the Nisnevich topos of X. Denote by $\varepsilon : X_{\text{ét}} \to X_{\text{Nis}}$ the natural projection. For $r \geq 0$, by applying $\mathrm{R}\varepsilon_*$ to the étale period map above and using that $\mathrm{R}\varepsilon_* i^* = i^*\mathrm{R}\varepsilon_*$[7] (c.f. [10, 2.2.b]), we obtain a natural map

$$\tilde{\alpha}_r : \mathrm{R}\varepsilon_*\mathcal{S}_n(r) \to i^*\mathrm{R}j_*\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)'.$$

Composing with the map $\omega : \mathrm{R}\varepsilon_*\mathcal{S}'_n(r) \to \mathrm{R}\varepsilon_*\mathcal{S}_n(r)$ we get a natural, compatible with products, morphism

$$\alpha_r : \mathrm{R}\varepsilon_*\mathcal{S}'_n(r) \to i^*\mathrm{R}j_*\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)'.$$

Write, for simplicity, $\mathcal{S}_n(r)$ and $\mathcal{S}'_n(r)$ for the derived pushforwards of $\mathcal{S}_n(r)$ and $\mathcal{S}'_n(r)$ from $X_{\text{ét}}$ to X_{Nis}. Same for $\mathcal{E}_n(r)$ and $\mathcal{E}'_n(r)$. Notice that they are quasi-isomorphic to the complexes obtained by gluing the

[7] This equality fails for the projection to Zariski topology and is the reason we use Nisnevich topology instead of Zariski.

complexes of sheaves $\mathcal{S}_n(r)$ and $\mathcal{S}'_n(r)$ on $X_{0,\mathrm{Nis}}$ and the complexes of sheaves $\varepsilon_* j'_* G\mathbf{Z}/p^n(r)'$ on $X_{K,\mathrm{Nis}}$ by the maps $\tilde{\alpha}_r$ and α_r. Hence we have the distinguished triangles

$$j_{\mathrm{Nis}!}\mathrm{R}j'_*\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)' \to \mathcal{E}_n(r) \to i_*\mathcal{S}_n(r), \quad j_{\mathrm{Nis}!}\mathrm{R}j'_*\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)' \to \mathcal{E}'_n(r) \to i_*\mathcal{S}'_n(r), \tag{2.8}$$

as well as the natural maps

$$\tilde{\alpha}_r : \mathcal{E}_n(r) \to \mathrm{R}j_*\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)', \quad \alpha_r : \mathcal{E}_n(r)' \to \mathrm{R}j_*\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)'$$

compatible with the maps $\tilde{\alpha}_r$ and α_r. For $a \geq 0$, we have the truncated version of the above - the distinguished triangles

$$j_{\mathrm{Nis}!}\tau_{\leq a}\mathrm{R}j'_*\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)' \to \tau_{\leq a}\mathcal{E}_n(r) \to i_*\tau_{\leq a}\mathcal{S}_n(r), \quad j_{\mathrm{Nis}!}\tau_{\leq a}\mathrm{R}j'_*\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)' \to \tau_{\leq a}\mathcal{E}'_n(r) \to i_*\tau_{\leq a}\mathcal{S}'_n(r). \tag{2.9}$$

Define the following complexes of sheaves on X_{Nis}

$$\mathcal{S}_n(r)_{\mathrm{Nis}} := \tau_{\leq r}\mathcal{S}_n(r), \quad \mathcal{S}'_n(r)_{\mathrm{Nis}} := \tau_{\leq r}\mathcal{S}'_n(r);$$
$$\mathcal{E}_n(r)_{\mathrm{Nis}} := \tau_{\leq r}\mathcal{E}_n(r), \quad \mathcal{E}'_n(r)_{\mathrm{Nis}} := \tau_{\leq r}\mathcal{E}'_n(r).$$

Example 2.7. For $X = \mathrm{Spec}(W(k))$ we have

$$H^i(W(k), \mathcal{S}_n(r)_{\mathrm{Nis}}) = \begin{cases} \mathbf{Z}/p^n & i = r = 0, \\ W_n(k) & i = 1, r \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the morphism $H^i(W(k), \mathcal{E}_n(r)_{\mathrm{Nis}}) \to H^i(W(k), \mathcal{S}_n(r)_{\mathrm{Nis}})$ is an isomorphism.

To see the first claim, note that we have

$$\mathcal{S}_n(0)_{\text{ét}} : W_n(k) \xrightarrow{1-\varphi} W_n(k), \quad \mathcal{S}_n(r)_{\text{ét}} : 0 \to W_n(k), \quad r \geq 1.$$

It follows that

$$\mathcal{S}_n(0)_{\mathrm{Nis}} = \mathbf{Z}/p^n, \quad \mathcal{S}_n(r)_{\text{ét}} := W_n(k)[-1], \quad r \geq 1.$$

For the second claim use the distinguished triangle (2.9) and the fact that $H^i(W(k), j_{\mathrm{Nis}!}\tau_{\leq a}\mathrm{R}j'_*\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)) = 0$, $i \geq 0$, because $W(k)$ is henselian.

3. Syntomic cohomology and motivic cohomology

Let X be a smooth scheme over $\mathcal{O}_K$. Let $\mathbf{Z}(r)_{\mathrm{M}}$ denote the complex of motivic sheaves $\mathbf{Z}(r)_{\mathrm{M}} := X \mapsto z^r(X, 2r - *)$ in the étale topology of X. Let $\mathbf{Z}/p^n(r)_{\mathrm{M}} := \mathbf{Z}(r)_{\mathrm{M}} \otimes \mathbf{Z}/p^n$. Recall how the complex $z^r(X, *)$ is defined [3]. Denote by $\triangle^n$ the algebraic n-simplex $\mathrm{Spec}\,\mathbf{Z}[t_0, \ldots, t_n]/(\sum t_i - 1)$. Let $z^r(X, i)$ be the free abelian group generated by closed integral subschemas of codimension r of $X \times \triangle^i$ meeting all faces properly. Then $z^r(X, *)$ is the chain complex thus defined with boundaries given by pullbacks of cycles along face maps. This complex is covariant for proper morphisms (with a shift in weight and degree) and contravariant for flat morphisms.

We know that in the Zariski topology $H^j(X_{\mathrm{Zar}}, \mathbf{Z}/p^n(i)_{\mathrm{M}}) = H^j\Gamma(X_{\mathrm{Zar}}, \mathbf{Z}/p^n(r)_{\mathrm{M}})$ is the Bloch higher Chow group [10, Theorem 3.2] and that this is also the case for the Nisnevich topology [10, Prop. 3.6]. Locally, in the étale topology, when p is invertible, the étale cycle class map defines a quasi-isomorphism $\mathbf{Z}/p^n(r)_{\mathrm{M}} \simeq \mathbf{Z}/p^n(r)$; when X is of characteristic p, then the logarithmic de Rham-Witt cycle class map defines a quasi-isomorphism $\mathbf{Z}/p^n(r)_{\mathrm{M}} \simeq W_n\Omega^r_{X,\log}[-r]$ [11], where, for a log-scheme Y, $W_n\Omega^*_{Y,\log}$ denotes the sheaf of logarithmic de Rham-Witt differential forms [21]. Moreover, if $i : Z \hookrightarrow X$ is a closed subscheme of codimension c with open complement $j : U \hookrightarrow X$ then the exact sequence

$$0 \to i_*\mathbf{Z}(r - c)_{M,Z}[-2c] \to \mathbf{Z}(r)_{M,X} \to j_*\mathbf{Z}(r)_{M,U}$$

forms a distinguished triangle in the derived category of sheaves on X_*, $*$ denoting the Zariski or Nisnevich topology. We define motivic cohomology as

$$H^*_{\mathrm{M}}(X, \mathbf{Z}/p^n(r)) := H^*(X_{\mathrm{Zar}}, \mathbf{Z}/p^n(r)_M) = H^*(X_{\mathrm{Nis}}, \mathbf{Z}/p^n(r)_M);$$
$$H^*_{\mathrm{M},\text{ét}}(X, \mathbf{Z}/p^n(r)) := H^*(X_{\text{ét}}, \mathbf{Z}/p^n(r)_M).$$

For a smooth scheme Y over $\mathcal{O}_K$, we define its p-adic motivic cohomology as

$$H^*_{\mathrm{M}}(Y, \mathbf{Q}_p(r)) := H^*(\mathrm{holim}_n \mathrm{R}\Gamma(Y_{\mathrm{Zar}}, \mathbf{Z}/p^n(r)_{\mathrm{M}}) \otimes \mathbf{Q}) = H^*(\mathrm{holim}_n \Gamma(Y_{\mathrm{Zar}}, \mathbf{Z}/p^n(r)_{\mathrm{M}}) \otimes \mathbf{Q}).$$

We define its étale version $H^*_{\mathrm{M},\text{ét}}(Y, \mathbf{Q}_p(r))$ in an analogous way.

We list the following corollary of Theorem 2.5.

Corollary 3.1. *Let X be a smooth variety over K. Then there exists a natural syntomic cycle class map*

$$\mathrm{cl}^{\mathrm{syn}}_{i,r} : H^i_{\mathrm{M}}(X, \mathbf{Q}_p(r)) \to H^i_{\mathrm{syn}}(X, \mathbf{Q}_p(r)),$$

where the target group is the syntomic cohomology defined in [22]*. This map is compatible with the étale cycle class map, i.e., the following diagram commutes*

$$\begin{array}{ccc} H^i_{\mathrm{M}}(X, \mathbf{Q}_p(r)) & & \\ \downarrow{\scriptstyle \mathrm{cl}^{\mathrm{syn}}_{i,r}} & \searrow^{\mathrm{cl}^{\text{ét}}_{i,r}} & \\ H^i_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) & \xrightarrow{\alpha^{NN}_{i,r}} & H^i_{\text{ét}}(X, \mathbf{Q}_p(r)), \end{array}$$

where $\alpha^{NN}_{i,r}$ is the period map defined in [22]*. Moreover, the cycle class map $\mathrm{cl}^{\mathrm{syn}}_{i,r}$ is an isomorphism for $i \leq r$.*

Proof. Consider the following diagram

$$\begin{array}{ccc} \mathcal{E}'_n(r)_{\mathrm{Nis}} & \xrightarrow{\alpha_r} & \tau_{\leq r}\mathrm{R}j_*\tau_{\leq r}\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)' \\ & \searrow^{\mathrm{cl}^{\mathrm{syn}}_r} & \uparrow{\scriptstyle \mathrm{cl}^{\text{ét}}_r}\wr \\ & & \mathrm{R}j_*\mathbf{Z}/p^n(r)'_M \end{array}$$

The étale cycle class map $\mathrm{cl}^{\text{ét}}_r$ is a quasi-isomorphism by the Beilinson -Lichtenbaum Conjecture (a corollary [31], [12] of the Bloch-Kato Conjecture proved by Voevodsky and Rost [35]) and by [11] that give the quasi-isomorphism

$$\mathbf{Z}/p^n(r)_M \xrightarrow{\sim} \tau_{\leq r}\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)$$

and by the quasi-isomorphisms $j_*\mathbf{Z}/p^n(r)_M \xrightarrow{\sim} \mathrm{R}j_*\mathbf{Z}/p^n(r)_M$ and $\tau_{\leq r}\mathrm{R}j_*\mathbf{Z}/p^n(r)_M \xrightarrow{\sim} \mathrm{R}j_*\mathbf{Z}/p^n(r)_M$. Since, by Theorem 2.5, the period map α_r is a p^N-quasi-isomorphism, we can define the syntomic cycle class map $\mathrm{cl}^{\mathrm{syn}}_r$ to make the above diagram commute. It induces the syntomic class map into syntomic cohomology

$$\mathrm{cl}^{\mathrm{syn}}_r : \mathrm{R}j_*\mathbf{Z}/p^n(r)_M \xrightarrow{\mathrm{cl}^{\mathrm{syn}}_r} \mathcal{E}'_n(r)_{\mathrm{Nis}} \to \mathcal{S}'_n(r)_{\mathrm{Nis}} \to \mathrm{R}\varepsilon_*\mathcal{S}'_n(r)_{\text{ét}}$$

By construction it is compatible with the étale cycle class map (via the map α_r).

Recall that the syntomic cohomology $H^i_{\mathrm{syn}}(X, \mathbf{Q}_p(r))$ is defined by h-sheafifying the (rational) Fontaine-Messing syntomic cohomology. Everything being natural, the above construction of cycle classes h-sheafifies and gives the syntomic cycle class map

$$\mathrm{cl}^{\mathrm{syn}}_{i,r} : H^i_{\mathrm{M}}(X, \mathbf{Q}_p(r)) \to H^i_{\mathrm{syn}}(X, \mathbf{Q}_p(r)).$$

For compatibility with the étale cycle class, it suffices to check that $\alpha^{NN}_{i,r} = \alpha_{i,r}$ but this was done in [29].

The last claim of the corollary follows from the fact that both $\alpha^{NN}_{i,r}$ and $\mathrm{cl}^{\text{ét}}_{i,r}$ are isomorphisms for $i \leq r$ by [22, Theorem A] and the Beilinson-Lichtenbaum Conjecture, respectively. □

3.1. **Syntomic cohomology and logarithmic de Rham-Witt cohomology.** We will show in this section that adding logarithmic structure at the special fiber changes syntomic cohomology by logarithmic de Rham-Witt cohomology.

Theorem 3.2. *Let X be a semistable scheme over $\mathcal{O}_K$ with a smooth special fiber. For a universal constant N, we have the following p^{Nr}-distinguished triangles of sheaves in the étale and Nisnevich topology of X, respectively.*

$$\mathcal{S}'_n(r)_X \to \mathcal{S}'_n(r)_{X^\times} \to W_n\Omega^{r-1}_{X_0,\log}[-r],$$
$$\mathcal{S}'_n(r)_{X,\mathrm{Nis}} \to \mathcal{S}'_n(r)_{X^\times,\mathrm{Nis}} \to W_n\Omega^{r-1}_{X_0,\log}[-r].$$

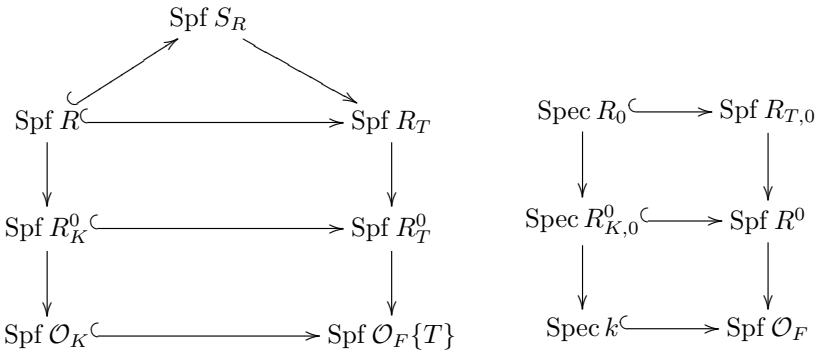
Here we wrote $X^\times$ for the scheme X with added log-structure coming from the special fiber.

Proof. After setting up the local coordinates, we do, as an example, computations in dimension zero, where it becomes clear how to define the map to logarithmic de Rham-Witt differentials. Then we lift this computations to higher dimensions and globalize.

(1) Choice of local coordinates. To construct the first distinguished triangle, we start with local computations. Let d be a positive integer. Let $R^0_K := \mathcal{O}_K\{X_1^{\pm 1}, \dots, X_d^{\pm 1}\}$ be the p-adic completion of $\mathcal{O}_K[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$. Let R be the p-adic completion of an étale algebra over R^0_K. Let R^0_T be the (p,T)-adic completion of $W(k)[T, X_1^{\pm 1}, \dots, X_d^{\pm 1}]$; take the map $R^0_T \mapsto R^0_K$, $T \mapsto \varpi$, and take the (formally) étale lifting R_T of R to R^0_T. Let S_R be the p-adically complete PD-envelope of R in R_T equipped with the PD-filtration F^rS_R. We will write $S_K := S_{\mathcal{O}_K}$. We have $S_R = R_T\widehat{\otimes}_{W(k)\{T\}}S_K$ with filtration $F^rS_R := R_T\widehat{\otimes}_{W(k)\{T\}}F^rS_K$. Let $R^0 := W(k)\{X_1^{\pm 1}, \dots, X_d^{\pm 1}\}$ and let $R_{T,0} := R_T/T$.

We have the following diagram of maps (the right diagram is obtained by reducing the rings modulo T)

(3.1)



Equip R^0 with Frobenius $\varphi_{R^0} : X_i^{\pm 1} \mapsto X_i^{\pm p}$. Equip R^0_T with Frobenius $\varphi_{R^0_T}$ compatible with φ_{S_K} $(T \mapsto T^p)$ and with φ_{R^0}, and equip R_T with a Frobenius φ_{R_T} compatible with $\varphi_{R^0_T}$. We will simply write φ for Frobenius if the domain of action is understood.

Set $\Omega_{S_R} := S_R \otimes_{R_T} \Omega_{R_T}$. For $r \in \mathbf{N}$, we filter the de Rham complex $\Omega^\bullet_{S_R}$ by subcomplexes

$$F^r\Omega^\bullet_{S_R} := F^rS_R \to F^{r-1}S_R \otimes_{R_T} \Omega_{R_T} \to F^{r-2}S_R \otimes_{R_T} \Omega^2_{R_T} \to \dots$$

We define *the syntomic complex* of R as

$$S(R,r) := \mathrm{Cone}(F^r\Omega^\bullet_{S_R} \xrightarrow{p^r-\varphi} \Omega^\bullet_{S_R})[-1] \tag{3.2}$$

Set $\Omega_{S^\times_R} := S_R \otimes_{R_T} \Omega_{R^\times_T}$, where $R^\times_T$ is the ring R_T with log-structure induced by T. We define the *log-syntomic complex* of R as

$$S(R^\times,r) := \mathrm{Cone}(F^r\Omega^\bullet_{S^\times_R} \xrightarrow{p^r-\varphi} \Omega^\bullet_{S^\times_R})[-1]. \tag{3.3}$$

For $n \in \mathbf{N}$, we define the syntomic and log-syntomic complexes modulo p^n as $S(R,r)_n := S(R,r) \otimes_{\mathbf{Z}} \mathbf{Z}/p^n$, $S(R^\times,r) := S(R^\times,r) \otimes_{\mathbf{Z}} \mathbf{Z}/p^n$, respectively. In the case when R is the p-adic completion of an étale algebra over $\mathcal{O}_K[X_1^{\pm 1}, \cdots, X_d^{\pm 1}]$, we have

$$S'_n(r)_R = S(R,r)_n, \quad S'_n(r)_{R^\times} = S(R^\times,r)_n;$$
$$\mathrm{holim}_n\, S'_n(r)_R = S(R,r), \quad \mathrm{holim}_n\, S'_n(r)_{R^\times} = S(R^\times,r).$$

We would like to separate the arithmetic and the geometric variables. Specifically, we remove the differentials connected with the variable T by setting $\Omega_{S'_R} := S_R \otimes_{R^0} \Omega_{R^0}$. Since $\Omega_{W(k)[T]} = W(k)[T]dT$ we can dispose of this module of differentials by writing df as $\partial f dT$ and we can rewrite the above syntomic complex as the following homotopy limit

$$S(R,r) = \left[\begin{array}{ccc} F^r\Omega^\bullet_{S'_R} & \xrightarrow{p^r - p^\bullet \varphi_\bullet} & \Omega^\bullet_{S'_R} \\ \downarrow \partial & & \downarrow \partial \\ F^{r-1}\Omega^\bullet_{S'_R} & \xrightarrow{p^r - p^{\bullet+1}T^{p-1}\varphi_\bullet} & \Omega^\bullet_{S'_R} \end{array} \right]$$

Here the map $\varphi_\bullet : \Omega^\bullet_{S'_R} \to \Omega^\bullet_{S'_R}$ sends $\omega \in \Omega^k_{S'_R}$ to $(\varphi/p^k)(\omega)$. By adding logarithmic differentials dT/T along the special fiber, we get the following *log-syntomic complex*

$$S(R^\times,r) = \left[\begin{array}{ccc} F^r\Omega^\bullet_{S'_R} & \xrightarrow{p^r - p^\bullet \varphi_\bullet} & \Omega^\bullet_{S'_R} \\ \downarrow T\partial & & \downarrow T\partial \\ F^{r-1}\Omega^\bullet_{S'_R} & \xrightarrow{p^r - p^{\bullet+1}\varphi_\bullet} & \Omega^\bullet_{S'_R} \end{array} \right]$$

(2) Dimension 0. For $R = \mathcal{O}_K$, we obtain the following proposition.

Proposition 3.3. *Let $n \geq 1$. We have the following p^{15}-distinguished triangle of sheaves in the étale topology of* Spec k

$$S(\mathcal{O}_K,1)_n \to S(\mathcal{O}_K^\times,1)_n \to \mathbf{Z}/p^n[-1].$$

For $r \neq 1$, the natural map $S(\mathcal{O}_K,r)_n \to S(\mathcal{O}_K^\times,r)_n$ is a p^{15r}-quasi-isomorphism.

Proof. We have the following two syntomic complexes.

$$\begin{aligned} S(\mathcal{O}_K,r): &\quad F^rS_K \xrightarrow{(\partial, p^r-\varphi)} F^{r-1}S_K \oplus S_K \xrightarrow{-(p^r - pT^{p-1}\varphi)+\partial} S_K \\ S(\mathcal{O}_K^\times,r): &\quad F^rS_K \xrightarrow{(T\partial, p^r-\varphi)} F^{r-1}S_K \oplus S_K \xrightarrow{-(p^r-p\varphi)+T\partial} S_K \end{aligned}$$

The residue map: $\Omega^1_{\log, S_K^{[1]}} \to \mathcal{O}_F$ induces the following sequence of complexes:

$$0 \to S(\mathcal{O}_K^{[1]},r) \to S_{\log}(\mathcal{O}_K^{[1]},r) \to \left[0 \to \mathcal{O}_F \xrightarrow{-(p^r-p\varphi)} \mathcal{O}_F\right] \to 0,$$

where $R^{[1]}$ is "the ring of analytic functions over F with integral values on the disk $v_p(T) \geq 1/e$" defined in [8, Remark 2.2] and we define its syntomic complexes by analogous formulas to (3.2) and (3.3) replacing S_R by $R^{[1]}$. We have the natural maps $S(\mathcal{O}_K,r) \to S(\mathcal{O}_K^{[1]},r)$ and $S(\mathcal{O}_K^\times,r) \to S_{\log}(\mathcal{O}_K^{[1]},r)$ that are p^{6r}-quasi-isomorphisms [8, Prop. 3.3]. The above sequence is p-exact because $F^sS_K^{[1]} = \frac{E^s}{p^s}S_K^{[1]}$, for E the minimal polynomial of ϖ over F, which implies that $F^s\Omega^1_{\log, S_K^{[1]}}/F^s\Omega^1_{S_K^{[1]}} \cong S_K^{[1]}/TS_K^{[1]}$ and $S_K^{[1]}/TS_K^{[1]} = \mathcal{O}_F \oplus M$, where M is p-torsion.

Modulo p^n, we have $\mathcal{O}_{F,n} = W_n(k)$ and the exact sequence in the étale topology of Spec k

$$0 \to \mathbf{Z}/p^n \to \mathcal{O}_{F,n} \xrightarrow{1-\varphi} \mathcal{O}_{F,n} \to 0.$$

For $r = 0$, the map $\mathcal{O}_{F,n} \xrightarrow{p^r - p\varphi} \mathcal{O}_{F,n}$ is an isomorphism since $1 - p\varphi$ is invertible. For $r > 1$, the map $\mathcal{O}_{F,n} \xrightarrow{p^{r-1} - \varphi} \mathcal{O}_{F,n}$ is an isomorphism as well since both φ and $p^{r-1}\varphi^{-1} - 1$ are invertible. This proves our proposition. □

(3) Local computations in higher dimensions. The computations in the above example generalize to any ring R.

Lemma 3.4. *There is a p^{12r}-distinguished triangle in the étale topology of* Spec R_0

$$S(R,r)_n \to S(R^\times,r)_n \to W_n\Omega^{r-1}_{R_0,\log}[-r]$$

Proof. First we pass from S_R to $R^{[1]}$ (via a p^{6r}-quasi-isomorphism). Then we compute as in the proof of Proposition 3.3 and obtain the following p-distinguished triangle

$$S(R^{[1]},r)\to S_{\log}(R^{[1]},r)\to \big[\Omega^\bullet_{R_{T,0}}\xrightarrow{p^r-p^{\bullet+1}\varphi_\bullet}\Omega^\bullet_{R_{T,0}}\big][-1].$$

We note that the complex $\Omega^\bullet_{R_{T,0}}$ computes the crystalline cohomology of R_0 over $W(k)$.

Set $S:=R_{T,0}$. We claim that there exists a p^{4r}-quasi-isomorphism on the étale site of Spec R_0

$$\big[\Omega^\bullet_{S,n}\xrightarrow{p^r-p^{\bullet+1}\varphi_\bullet}\Omega^\bullet_{S,n}\big]\simeq W_n\Omega^{r-1}_{R_0,\log}[-r+1].$$

Indeed, for $r=0$, the complex $\big[\Omega^\bullet_{S,n}\xrightarrow{1-p^{\bullet+1}\varphi_\bullet}\Omega^\bullet_{S,n}\big]$ is acyclic because the map $1-p^{\bullet+1}$ is invertible. Assume thus that $r\geq 1$ and take $s=r-1$. Set

$$\mathrm{HK}(S,s)_n:=\big[\Omega^\bullet_{S,n}\xrightarrow{p^s-p^\bullet\varphi_\bullet}\Omega^\bullet_{S,n}\big].$$

This complex is p^2-quasi-isomorphic to the complex $\big[\Omega^\bullet_{S,n}\xrightarrow{p^r-p^{\bullet+1}\varphi_\bullet}\Omega^\bullet_{S,n}\big]$. Using the global Frobenius lift on S we get the following commutative diagram

$$\begin{array}{ccc}\Omega^\bullet_{S,n} & \xrightarrow{p^s-p^\bullet\varphi_\bullet} & \Omega^\bullet_{S,n}\\ \downarrow{\scriptstyle\Phi(\varphi)} & & \downarrow{\scriptstyle\Phi(\varphi)}\\ W_n\Omega^\bullet_{R_0} & \xrightarrow{p^s-p^\bullet\mathrm{F}} & W_n\Omega^\bullet_{R_0}/dV^{n-1}\Omega^{\bullet-1}_{R_0}\end{array}$$

We note here that the de Rham-Witt Frobenius $\mathrm{F}: W_{n+1}\Omega^\bullet_{R_0}\to W_n\Omega^\bullet_{R_0}$ and that $\mathrm{F}:\mathrm{Fil}^n W_{n+1}\Omega^\bullet_{R_0}=V^n\Omega^\bullet_{R_0}+dV^n\Omega^{\bullet-1}_{R_0}\to dV^{n-1}\Omega^{\bullet-1}_{R_0}$. Hence F factorizes as in the above diagram. Moreover, since $pdV^{n-1}\Omega^*_{R_0}=0$, we get the induced map $p\mathrm{F}: W_n\Omega^\bullet_{R_0}\to W_n\Omega^\bullet_{R_0}$.

The first vertical arrow in the above diagram is a quasi-isomorphism. The second one is a p-quasi-isomorphism since $pdV^{n-1}\Omega^*_{R_0}=0$. Hence the complex $\mathrm{HK}(S,s)_n$ is p^2-quasi-isomorphic to the complex $[W_n\Omega^\bullet_{R_0}\xrightarrow{p^s-p^\bullet\mathrm{F}}W_n\Omega^\bullet_{R_0}/dV^{n-1}\Omega^{\bullet-1}_{R_0}]$. We list the following properties of the latter complex.

(1) For $t>s$, the map $W_n\Omega^t_{R_0}\xrightarrow{1-p^{t-s}\mathrm{F}}W_n\Omega^t_{R_0}$ is an isomorphism (since $1-p^{t-s}\mathrm{F}$ is invertible).

(2) For $t<s$, the map

$$W_n\Omega^t_{R_0}\xrightarrow{p^{s-t}-\mathrm{F}}W_n\Omega^t_{R_0}/dV^{n-1}\Omega^{t-1}_{R_0}$$

is a p-isomorphism. Indeed, for p-surjectivity, it suffices to note that $(p^{s-t}-\mathrm{F})(V\alpha)=p^{s-t}V\alpha-p\alpha$, for $\alpha\in W_n\Omega^t_{R_0}/dV^{n-1}\Omega^{t-1}_{R_0}$, $t\leq s-1$. For p-injectivity, we note that if $(p^{s-t}-\mathrm{F})(\alpha)=0$ for $\alpha\in W_n\Omega^t_{R_0}$ then $V(p^{s-t}-\mathrm{F})(\alpha)=p^{s-t}V\alpha-p\alpha=0$. Hence $p^{s-t-1}V\alpha=\alpha$ which implies that $p^{n(s-t-1)}V^n\alpha=\alpha$. Hence $\alpha=0$.

(3) The map

$$ZW_n\Omega^{s-1}_{R_0}\xrightarrow{p-\mathrm{F}}ZW_n\Omega^{s-1}_{R_0}/dV^{n-1}\Omega^{s-2}_{R_0}$$

is an p^3-isomorphism. For p-injectivity we use the point above. For p^3-surjectivity, we note that, for $\alpha\in ZW_n\Omega^{s-1}_{R_0}$ such that $d\alpha=0$ we have

$$p\alpha=-(p-\mathrm{F})\beta,\quad \beta=V\alpha+V^2\alpha+V^3\alpha\cdots\in VW_{n-1}\Omega^{s-1}_{R_0},$$

and $p(1-\mathrm{F})d\beta=0$. By [4, Lemma 4.3], this implies that $pd\beta=0$.

(4) There is an exact sequence

$$0\to W_n\Omega^s_{R_0,\log}\to W_n\Omega^s_{R_0}\xrightarrow{1-\mathrm{F}}W_n\Omega^s_{R_0}/dV^{n-1}\Omega^{s-1}_{R_0}\to 0$$

in the étale topology of Spec R_0 [6, Lemma 1.2], [21, Prop. 2.13]. In the Nisnevich topology it is still exact on the left and in the middle.

The above implies that there is a natural map

$$W_n\Omega^s_{R_0,\log}[-s]\to[W_n\Omega^\bullet_{R_0}\xrightarrow{p^s-p^\bullet\mathrm{F}}W_n\Omega^\bullet_{R_0}/dV^{n-1}\Omega^{\bullet-1}_{R_0}]$$

and that it is a p^{4s}-quasi-isomorphism in the étale topology of Spec R_0, as wanted. □

(4) Globalization. The above local computations can be globalized in the following way. We note that we have actually proved above that we have the following p^{4s}-quasi-isomorphisms of sheaves on the étale site of X_0[8]

$$W_n\Omega^s_{X_0,\log}[-s] \to [W_n\Omega^\bullet_{X_0} \xrightarrow{p^s-p^\bullet \mathrm{F}} W_n\Omega^\bullet_{X_0}/dV^n\Omega^{s-1}_{X_0}] \leftarrow [\mathcal{A}_{\mathrm{cr},n} \xrightarrow{p^s-\varphi} \mathcal{A}_{\mathrm{cr},n}],$$

where $\mathcal{A}_{\mathrm{cr},n}$ is the sheaf $(U \to X_0) \mapsto \mathrm{R}\Gamma_{\mathrm{cr}}(U/W_n(k))$. The second quasi-isomorphism is [14, Sec. II.1]. It suffices thus to construct a map

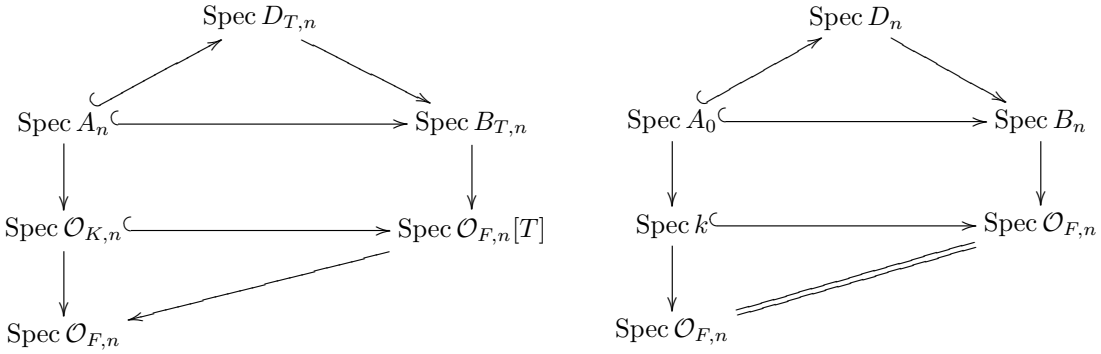
$$\mathcal{S}'_n(r)_{X^\times} \to [\mathcal{A}_{\mathrm{cr},n} \xrightarrow{p^r-p\varphi} \mathcal{A}_{\mathrm{cr},n}][-1]$$

and to show that the triangle

$$\mathcal{S}'_n(r)_X \to \mathcal{S}'_n(r)_{X^\times} \to [\mathcal{A}_{\mathrm{cr},n} \xrightarrow{p^r-p\varphi} \mathcal{A}_{\mathrm{cr},n}][-1]$$

is p^{6r}-distinguished.

For that, consider the following two diagrams of compatible coordinate systems (localize on X if necessary to get $X = \operatorname{Spec} A$).



Here $B_{T,n}$ is smooth over $\mathcal{O}_{F,n}[T]$ and the hooked arrows are closed embeddings. We equip both rings with the log-structure associated to T. The right diagram is obtained by "reducing modulo T" the left diagram. It follows that the residue map $\Omega_{B^\times_{T,n}} \to \mathcal{O}_{B_n}$ induces a map $\Omega^\bullet_{D^\times_{T,n}} \to \Omega^{\bullet-1}_{D_n}$ (we note that the Frobenius φ on the domain is compatible with $p\varphi$ on the target) and the sequence

$$\Omega^\bullet_{D_{T,n}} \to \Omega^\bullet_{D^\times_{T,n}} \to \Omega^{\bullet-1}_{D_n} \tag{3.4}$$

is exact. These constructions glue in the usual way and we obtain a map $\mathcal{J}^{[r]}_{X_n^\times} \to \mathcal{A}_{\mathrm{cr},n}[-1]$ and a sequence of complexes of sheaves on the étale site of X_0

$$\mathcal{J}^{[r]}_{X_n} \to \mathcal{J}^{[r]}_{X_n^\times} \to \mathcal{A}_{\mathrm{cr},n}[-1], \tag{3.5}$$

where we wrote $\mathcal{J}^{[r]}_{X_n^*}$ for the sheaf $(U \to X_n) \mapsto \mathrm{R}\Gamma_{\mathrm{cr}}(U, \mathcal{J}^{[r]}_{X_n^*})$. Hence a sequence

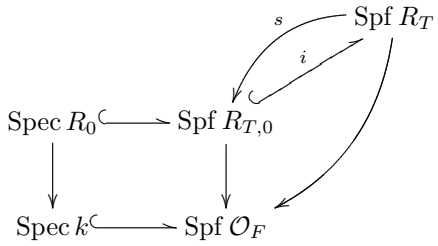
$$\mathcal{S}'_n(r)_X \to \mathcal{S}'_n(r)_{X^\times} \to [\mathcal{A}_{\mathrm{cr},n} \xrightarrow{p^r-p\varphi} \mathcal{A}_{\mathrm{cr},n}][-1] \tag{3.6}$$

It is a p^{6r}-distinguished triangle: this can be checked locally where we can pass to the more convenient coordinate system from (3.1) and use the computations we have done in the proof of Proposition 3.3. This concludes the construction of the first distinguished triangle of our theorem.

For the second triangle in the theorem, take the first triangle and push it down to the Nisnevich site. Since $\tau_{\leq 0}\mathrm{R}\varepsilon_* W_n\Omega^{r-1}_{X_0,\log} \simeq W_n\Omega^{r-1}_{X_0,\log}$ [15], it suffices to check that the map $\mathcal{H}^r(\mathcal{S}'_n(r)_{X^\times,\mathrm{Nis}}) \to$

[8]The notation is slightly abusive here but we hope that this will not lead to confusion.

$W_n\Omega^{r-1}_{X_0,\log}$ is p^{Nr}-surjective, for a universal constant N. For that, Zariski localize and consider the following commutative diagram



Here the map s was chosen to commute with Frobenius and so that $si = \mathrm{Id}$. We can use it to construct a section of the residue map

$$S_{\log}(R^{[1]}, r) \to \big[\Omega^\bullet_{R_{T,0}} \xrightarrow{p^r - p^{\bullet+1}\varphi_\bullet} \Omega^\bullet_{R_{T,0}}\big][-1].$$

It follows that the map $H^i(S_{\log}(R^{[1]}, r)_n) \to H^{i-1}(\big[\Omega^\bullet_{R_{T,0}} \xrightarrow{p^r - p^{\bullet+1}\varphi_\bullet} \Omega^\bullet_{R_{T,0}}\big]_n)$ is surjective. Since, by the above computations, the Nisnevich sheaves $\mathcal{H}^{r-1}(\mathcal{A}_{\mathrm{cr},n} \xrightarrow{p^{r-1}-\varphi} \mathcal{A}_{\mathrm{cr},n})$ and $W_n\Omega^{r-1}_{X_0,\log}$ are p^{Nr}-quasi-isomorphic, for a universal constant N, we are done. □

Corollary 3.5. *Let X be a semistable scheme over $\mathcal{O}_K$ with a smooth special fiber. For a constant $N = N(p, e, r)$, we have the following natural p^N-quasi-isomorphism ($*$ denotes the étale or the Nisnevich topology of X)*

$$\mathcal{S}'_n(r)_{X,*} \oplus W_n\Omega^{r-1}_{X_0,\log}[-r] \to \mathcal{S}'_n(r)_{X^\times,*}$$

Proof. It suffices to argue in the étale topology. Recall that in the proof of Theorem 3.2 we have obtained a p^{6r}-distinguished triangle

$$\mathcal{S}'_n(r)_X \to \mathcal{S}'_n(r)_{X^\times} \xrightarrow{\mathrm{res}_T} [\mathcal{A}_{\mathrm{cr},n} \xrightarrow{p^r - p\varphi} \mathcal{A}_{\mathrm{cr},n}][-1]$$

and a p^{4r} quasi-isomorphism

$$W_n\Omega^{r-1}_{X_0,\log}[-r+1] \simeq [\mathcal{A}_{\mathrm{cr},n} \xrightarrow{p^r - p\varphi} \mathcal{A}_{\mathrm{cr},n}]. \tag{3.7}$$

It suffices thus to construct a section of the residue map. For $r = 0$ there is nothing to prove so we will assume that $r > 0$.

For that, consider the following commutative diagram

$$\begin{array}{ccc} & & \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/W_n(k)) \\ & \overset{\wedge \mathrm{dlog}\, T}{\swarrow} & \downarrow{\scriptstyle i^*} \\ \mathrm{R}\Gamma_{\mathrm{cr}}(X_1^\times/W_n(k))[1] & \xrightarrow{\mathrm{res}_T} & \mathrm{R}\Gamma_{\mathrm{cr}}(X_0/W_n(k)), \end{array}$$

where $i : X_0 \hookrightarrow X_1$ is the natural closed immersion. Since the map $i^* : \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/W_n(k))^{\varphi=p^{r-1}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(X_0/W_n(k))^{\varphi=p^{r-1}}$ is a $p^{N(p,e)}$ quasi-isomorphism [8, Remark 5.9], the maps in the above diagram induce a $p^{N(p,e)}$-map $\mathrm{R}\Gamma_{\mathrm{cr}}(X_0/W_n(k))^{\varphi=p^{r-1}} \to \mathcal{S}'_n(r)_{X^\times}[1]$ that is a $p^{N(p,e,r)}$-section of the residue map on the scheme X_0. It is natural on the étale site of X_0 (it depends only on the uniformizer ϖ) hence gives a section on the level of sheaves that we wanted. □

Let, for $*$ denoting the étale or the Nisnevich topology,

$$\begin{aligned} \mathrm{R}\Gamma(X_*, \mathcal{S}(r))_{\mathbf{Q}} &:= \mathrm{holim}_n\, \mathrm{R}\Gamma(X_*, \mathcal{S}_n(r)) \otimes \mathbf{Q} \xrightarrow{\sim} \mathrm{holim}_n\, \mathrm{R}\Gamma(X_*, \mathcal{S}'_n(r)) \otimes \mathbf{Q}, \\ \mathrm{R}\Gamma(X_*, \mathcal{E}(r))_{\mathbf{Q}} &:= \mathrm{holim}_n\, \mathrm{R}\Gamma(X_*, \mathcal{E}_n(r)) \otimes \mathbf{Q} \xrightarrow{\sim} \mathrm{holim}_n\, \mathrm{R}\Gamma(X_*, \mathcal{E}'_n(r)) \otimes \mathbf{Q}, \\ \mathrm{R}\Gamma(X_*, W\Omega^{r-1}_{X_0,\log})_{\mathbf{Q}} &:= \mathrm{holim}_n\, \mathrm{R}\Gamma(X_*, i_*W_n\Omega^{r-1}_{X_0,\log}). \end{aligned}$$

By (3.7), we have

$$\mathrm{R}\Gamma(X_*, W\Omega^{r-1}_{X_0,\log})_{\mathbf{Q}} \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(X_0/F)^{\varphi=p^{r-1}}_{\mathbf{Q}}[-r+1],$$

where, for a scheme Y over $W(k)$, we set $\mathrm{R}\Gamma_{\mathrm{cr}}(Y/F) := \mathrm{R}\Gamma_{\mathrm{cr}}(Y/W(k))_{\mathbf{Q}} := \mathrm{holim}_n \mathrm{R}\Gamma_{\mathrm{cr}}(Y_1/W_n(k)) \otimes \mathbf{Q}$. The following corollary follows immediately from Corollary 3.5.

Corollary 3.6. *Let X be a semistable scheme over $\mathcal{O}_K$ with a smooth special fiber. We have the following natural quasi-isomorphisms*

$$\mathrm{R}\Gamma(X_*, \mathcal{S}(r))_{\mathbf{Q}} \oplus \mathrm{R}\Gamma(X_*, W\Omega^{r-1}_{X_0,\log})_{\mathbf{Q}}[-r] \xrightarrow{\sim} \mathrm{R}\Gamma(X_*^{\times}, \mathcal{S}(r))_{\mathbf{Q}}.$$

Corollary 3.7. *Let X be a semistable scheme over $\mathcal{O}_K$ with a smooth special fiber. For a constant N as in Theorem 2.3, we have the following p^N-distinguished triangle of sheaves in the étale topology of X_0*

$$\mathcal{S}'_n(r)_X \to \tau_{\leq r} i^* \mathrm{R}j_* \mathbf{Z}/p^n(r)' \to W_n\Omega^{r-1}_{X_0,\log}[-r]. \tag{3.8}$$

Moreover, for a constant $N = N(p, e, r)$ we have the following p^N-quasi-isomorphism

$$\mathcal{S}'_n(r)_X \oplus W_n\Omega^{r-1}_{X_0,\log}[-r] \to \tau_{\leq r} i^* \mathrm{R}j_* \mathbf{Z}/p^n(r)'.$$

Proof. This immediately follows from Theorem 3.2, Theorem 2.3, and Corollary 3.5. □

Remark 3.8. For $r \leq p-2$, the distinguished triangle (3.8) was constructed before by Kurihara. No additional constants are needed in this case.

Theorem 3.9. *([20, 1]) Let X be a smooth scheme over $\mathcal{O}_K$. For $r \leq p-2$, we have the following distinguished triangle of sheaves in the étale topology of X_0*

$$\mathcal{S}_n(r)_X \to \tau_{\leq r} i^* \mathrm{R}j_* \mathbf{Z}/p^n(r) \to W_n\Omega^{r-1}_{X_0,\log}[-r].$$

It is easy to see that the above theorem holds also for schemes X that are semistable over $\mathcal{O}_K$ with a smooth special fiber, i.e., that we have the following distinguished triangle

$$\mathcal{S}_n(r)_X \to \tau_{\leq r} i^* \mathrm{R}j_* \mathbf{Z}/p^n(r) \to W_n\Omega^{r-1}_{X_0,\log}[-r], \quad r \leq p-2.$$

Indeed, it suffices to note that all the terms involved have Gysin sequences [34] and to use the above theorem. In particular, in view of Theorem 2.2, we have the following distinguished triangle

$$\mathcal{S}_n(r)_X \to \mathcal{S}_n(r)_{X^{\times}} \to W_n\Omega^{r-1}_{X_0,\log}[-r], \quad r \leq p-2,$$

a "small twists" analog of the distinguished triangles from Theorem 3.2.

3.2. **Syntomic cohomology and motivic cohomology.** The main theorem of this section shows that, in étale topology, syntomic-étale complexes on smooth schemes over $\mathcal{O}_K$ approximate motivic complexes.

Theorem 3.10. *Let X be a semistable scheme over $\mathcal{O}_K$ with a smooth special fiber. Let $j' : X_{\mathrm{tr}} \hookrightarrow X$ be the natural open immersion. Then*

(1) *there is a natural cycle class map*

$$\mathrm{cl}^{\mathrm{syn}}_r : \mathrm{R}j'_* \mathbf{Z}/p^n(r)_{\mathrm{M}} \to \mathcal{E}_n(r)_{\mathrm{Nis}}, \quad 0 \leq r \leq p-2.$$

It is a quasi-isomorphism.

(2) *there is a natural cycle class map*

$$\mathrm{cl}^{\mathrm{syn}}_r : \mathrm{R}j'_* \mathbf{Z}/p^n(r)_{\mathrm{M}} \to \mathcal{E}'_n(r)_{\mathrm{Nis}}, \quad r \geq 0.$$

It is a p^N-quasi-isomorphism for a constant N as in Theorem 2.3.

We have analogous statements in the étale topology. These cycle class maps are compatible (via the localization map and the period map) with the étale cycle class maps.

Proof. We start with the Nisnevich topology. We will prove the second claim, the proof of the first one being analogous. Consider the following commutative diagram

$$\begin{array}{ccccc} j_{\mathrm{Nis}!}\tau_{\leq r}\mathrm{R}j'_{K,*}\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)' & \longrightarrow & \mathcal{E}'_n(r)_{X,\mathrm{Nis}} & \longrightarrow & i_*\mathcal{S}'_n(r)_{X,\mathrm{Nis}} \\ \downarrow\wr & & \downarrow & & \downarrow \\ j_{\mathrm{Nis}!}\tau_{\leq r}\mathrm{R}j'_{K,*}\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)' & \longrightarrow & \mathcal{E}'_n(r)_{X^\times,\mathrm{Nis}} & \longrightarrow & i_*\mathcal{S}'_n(r)_{X^\times,\mathrm{Nis}} \\ & & & & \downarrow \\ & & & & i_*W_n\Omega^{r-1}_{X_0,\log}[-r] \end{array}$$

The two rows are distinguished triangles; the right column is a p^{Nr}-distinguished triangle, for a universal constant N, by Theorem 3.2. It follows that we have the p^{Nr}-distinguished triangle

$$\mathcal{E}'_n(r)_{X,\mathrm{Nis}} \to \mathcal{E}'_n(r)_{X^\times,\mathrm{Nis}} \to i_*W_n\Omega^{r-1}_{X_0,\log}[-r]. \tag{3.9}$$

Let $Y = X_{\mathrm{tr}}$. By functoriality we get the following map of p^{Nr}-distinguished triangles

$$\begin{array}{ccccc} \mathcal{E}'_n(r)_{X,\mathrm{Nis}} & \longrightarrow & \mathcal{E}'_n(r)_{X^\times,\mathrm{Nis}} & \longrightarrow & i_*W_n\Omega^{r-1}_{X_0,\log}[-r] \\ \downarrow & & \downarrow\wr & & \downarrow\wr \\ \mathrm{R}j'_*\mathcal{E}'_n(r)_{Y,\mathrm{Nis}} & \longrightarrow & \mathrm{R}j'_*\mathcal{E}'_n(r)_{Y^\times,\mathrm{Nis}} & \longrightarrow & \mathrm{R}j'_*i_*W_n\Omega^{r-1}_{Y_0,\log}[-r] \end{array}$$

The right vertical arrow is a quasi-isomorphism since $M_{X_0} = j'_*\mathcal{O}^*_{X_0,\mathrm{tr}}$. The middle vertical arrow is a p^{Nr}-quasi-isomorphism by Corollary 2.6. Hence the left vertical arrow is a p^{Nr}-quasi-isomorphism and we may assume that the horizontal divisor of X is trivial.

Consider the following diagram

$$\begin{array}{ccccc} \mathcal{E}'_n(r)_{X,\mathrm{Nis}} & \longrightarrow & \mathcal{E}'_n(r)_{X^\times,\mathrm{Nis}} & \longrightarrow & i_*W_n\Omega^{r-1}_{X_0,\log}[-r] \\ \vdots & & \wr\downarrow\alpha_r & & \| \\ \mathcal{C}_n(r) & \longrightarrow & \tau_{\leq r}\mathrm{R}j_*\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)'_{X_K} & \xrightarrow{\kappa} & i_*W_n\Omega^{r-1}_{X_0,\log}[-r] \end{array} \tag{3.10}$$

Here the map κ is induced from a map $\tau_{\leq r}i^*\mathrm{R}j_*\mathbf{Z}/p^n(r) \to W_n\Omega^{r-1}_{X_0,\log}[-r]$ of sheaves on the étale site of X_0 defined as the composition of the canonical map $\tau_{\leq r}i^*\mathrm{R}j_*\mathbf{Z}/p^n(r) \to i^*\mathrm{R}^r j_*\mathbf{Z}/p^n(r)[-r]$ and the symbol map $i^*\mathrm{R}^r j_*\mathbf{Z}/p^n(r) \to W_n\Omega^{r-1}_{X_0,\log}$. The latter is defined by observing that $i^*\mathrm{R}^r j_*\mathbf{Z}/p^n(r)$ is locally generated by symbols $\{f_1,\dots,f_r\}$ for $f_i \in i^*j_*\mathcal{O}^*_{X_K}$ [5, Cor. 6.1.1]. By multilinearity, each symbol can be written as a sum of symbols of the form $\{f_1,\dots,f_r\}$ and $\{f_1,\dots,f_{r-1},\varpi\}$ for $f_i \in i^*\mathcal{O}^*_X$. Then κ sends the former to zero and the latter to $\mathrm{dlog}[\overline{f}_1]\wedge\cdots\wedge\mathrm{dlog}[\overline{f}_{r-1}]$ where $\overline{f}_i$ is the reduction of f_i to $\mathcal{O}^*_{X_0}$. We defined $\mathcal{C}_n(r)$ as the mapping fiber of the map κ.

We claim that the right square of the diagram p^N-commutes for a constant as in the statement of the theorem. Indeed, we note that we can pass to the étale site and there it suffices to show that the following diagram of maps of sheaves p^N-commutes

$$\begin{array}{ccc} \mathcal{H}^r(\mathcal{S}'_n(r)_{X^\times}) & \xrightarrow{\beta} & W_n\Omega^{r-1}_{X_0,\log} \\ \downarrow\alpha_r & \nearrow_{\kappa} & \\ i^*\mathrm{R}^r j_*\mathbf{Z}/p^n(r)'_{X_K} & & \end{array}$$

Since the map α_r is a p^N-isomorphism and the sheaf $i^*\mathrm{R}^r j_*\mathbf{Z}/p^n(r)_{X_K}$ is generated locally by symbols it suffices to check that the map β sends the symbol $\{f_1,\dots,f_r\}$, $f_i \in i^*\mathcal{O}^*_X$, to zero and the symbol

$\{f_1, \dots, f_{r-1}, \varpi\}$, $f_i \in i^*\mathcal{O}_X^*$, to $\mathrm{dlog}[\overline{f}_1] \wedge \cdots \mathrm{dlog}[\overline{f}_{r-1}]$. But this follows easily from the definition of the symbol maps (2.3).

It follows that the left vertical map in the diagram 3.10 exists. It is unique because

$$\mathrm{Hom}(\mathcal{E}'_n(r)_{X,\mathrm{Nis}}, W_n\Omega^{r-1}_{X_0,\log}[-r-1]) = 0$$

for degree reasons. It is clearly a quasi-isomorphism. All of the above has to be taken in the p^N-sense.

It remains now to show that there exists a p^{Nr}-quasi-isomorphism $\mathbf{Z}/p^n(r)_M \to \mathcal{C}_n(r)$, for a universal constant N. We proceed as in [10, p. 14]. Consider the following diagram of distinguished triangles (the complex $\mathcal{C}'_n(r)$ is defined by the bottom triangle and p^{Nr}-quasi-isomorphic to the complex $\mathcal{C}_n(r)$)

$$\begin{array}{ccccc} \mathbf{Z}/p^n(r)_{M,X} & \longrightarrow & j_*\mathbf{Z}/p^n(r)_{M,X_K} & \longrightarrow & i_*\mathbf{Z}/p^n(r-1)_{M,X_0}[-1] \\ \downarrow & & \downarrow\wr & & \downarrow\wr \\ \mathcal{C}'_n(r) & \longrightarrow & \tau_{\le r}\mathrm{R}j_*\mathrm{R}\varepsilon_*\mathbf{Z}/p^n(r)_{X_K} & \xrightarrow{\kappa} & i_*W_n\Omega^{r-1}_{X_0,\log}[-r] \end{array} \tag{3.11}$$

The middle and the right vertical maps are induced by the étale and the logarithmic de Rham-Witt cycle class map, respectively. They are quasi-isomorphisms by the Beilinson -Lichtenbaum Conjecture. The right square commutes: pass to the étale site and there this fact was shown in [10, p. 14]. Hence the left vertical map exists, is unique, and a quasi-isomorphism as well. This concludes the proof of our theorem.

For the étale topology, the computations are analogous but the diagram (3.11) has to be replaced with the following one

$$\begin{array}{ccccc} \mathbf{Z}/p^n(r)_{M,X} & \longrightarrow & \tau_{\le r}\mathrm{R}j_*\mathbf{Z}/p^n(r)_{M,X_K} & \longrightarrow & \tau_{\le r}(i_*\mathrm{R}i^!\mathbf{Z}/p^n(r)_{M,X}[1]) \\ \downarrow & & \downarrow\wr & & \downarrow\wr \\ \mathcal{C}'_n(r) & \longrightarrow & \tau_{\le r}\mathrm{R}j_*\mathbf{Z}/p^n(r)_{X_K} & \xrightarrow{\kappa} & i_*W_n\Omega^{r-1}_{X_0,\log}[-r] \end{array}$$

The right vertical arrow is a quasi-isomorphism by [10, p. 14]. □

We list several, more or less immediate, corollaries of the above theorems (we set $\alpha := \text{ét}, \mathrm{Nis}$).

Corollary 3.11. *Let X be a smooth scheme over $\mathcal{O}_K$. We have*

(1) $H^*_\alpha(X, \mathcal{E}_n(r)) \simeq H^*_{\mathrm{M},\alpha}(X, \mathbf{Z}/p^n(r))$, $\quad r \le p-2$;
(2) *the kernel and the cokernel of the cycle class map*

$$H^*_{\mathrm{M},\alpha}(X, \mathbf{Z}/p^n(r)) \to H^*_\alpha(X, \mathcal{E}'_n(r))$$

are annihilated by p^N, where N denotes the constant from Theorem 2.3. Hence

$$H^*_\alpha(X, \mathcal{E}(r))_{\mathbf{Q}} \simeq H^*_{\mathrm{M},\alpha}(X, \mathbf{Q}_p(r)).$$

In a more familiar language of syntomic cohomology, the above theorem and corollary can be stated in the following way.

Corollary 3.12. *Let X be a semistable scheme over $\mathcal{O}_K$ with a smooth special fiber. Let $j' : X_{\mathrm{tr}} \hookrightarrow X$ be the natural open immersion. Then, on the étale site of X_0,*

(1) *there is a natural quasi-isomorphism* [10]

$$\mathcal{S}_n(r) \simeq i^*\mathrm{R}j'_*\mathbf{Z}/p^n(r)_{\mathrm{M}}, \quad 0 \le r \le p-2.$$

(2) *there is a constant N as in Theorem 2.3 and a natural p^N-quasi-isomorphism*

$$\mathcal{S}'_n(r) \simeq i^*\mathrm{R}j'_*\mathbf{Z}/p^n(r)'_{\mathrm{M}}, \quad r \ge 0,$$

Corollary 3.13. *Let X be a proper semistable scheme over $\mathcal{O}_K$ with a smooth special fiber. We have*

(1) $H^*_\alpha(X, \mathcal{S}_n(r)) \simeq H^*_{\mathrm{M},\alpha}(X_{\mathrm{tr}}, \mathbf{Z}/p^n(r))$, $\quad r \le p-2$;

(2) *the kernel and the cokernel of the cycle class map*

$$H^*_{\mathrm{M},\alpha}(X_{\mathrm{tr}}, \mathbf{Z}/p^n(r)) \to H^*_\alpha(X, \mathcal{S}'_n(r))$$

are annihilated by p^N, where N denotes the constant from Theorem 2.3. Hence

$$H^*_\alpha(X, \mathcal{S}(r))_{\mathbf{Q}} \simeq H^*_{\mathrm{M},\alpha}(X_{\mathrm{tr}}, \mathbf{Q}_p(r)).$$

Corollary 3.14. *Let X be a proper semistable scheme over $\mathcal{O}_K$ with a smooth special fiber. Then the claims of Corollary 3.13 hold for $X_{\mathcal{O}_{\overline{K}}}$ (in place of X)*[9]*. Moreover, for $i \leq r$, we have the following commutative diagram*

$$\begin{array}{ccc} H^i_{\mathrm{M}}(X_{\mathcal{O}_{\overline{K}},\mathrm{tr}}, \mathbf{Q}_p(r)) & \xrightarrow{j^*}_{\sim} & H^i_{\mathrm{M}}(X_{\overline{K},\mathrm{tr}}, \mathbf{Q}_p(r)) \\ \wr\downarrow \mathrm{cl}^{\mathrm{syn}}_{i,r} & & \wr\downarrow \mathrm{cl}^{\text{ét}}_{i,r} \\ H^i(X_{\mathcal{O}_{\overline{K}}}, \mathcal{S}(r))_{\mathbf{Q}} & \xrightarrow{\alpha_{i,r}} & H^i_{\text{ét}}(X_{\overline{K},\mathrm{tr}}, \mathbf{Q}_p(r)) \end{array}.$$

Proof. The first claim follows from Corollary 3.13 by passing to limit over finite extensions of K in $\overline{K}$. The fact that the localization map j^* is an isomorphism was proved in [24, Lemma 3.1]. □

Remark 3.15. For X proper the above diagram was studied in [24] (see [26] for a brief survey): it was constructed first for the Chern classes from p-adic K-theory and then for motivic cohomology by studying compatibility of Chern classes with operations on K-theory. This did not use the Fontaine-Messing period map $\alpha_{i,r}$ but instead a period map $\alpha_{i,r} : H^i_{\text{ét}}(X_{\overline{K},\mathrm{tr}}, \mathbf{Q}_p(r)) \to H^i(X_{\mathcal{O}_{\overline{K}}}, \mathcal{S}(r))_{\mathbf{Q}}$ was defined using the above diagram. The fact that it is an isomorphism followed from the proof of the Crystalline Conjecture and implied that so is the syntomic cycle class map $\mathrm{cl}^{\mathrm{syn}}_{i,r}$.

For an open X as above, the situation is, at the moment, reversed. We defined log-syntomic p-adic Chern classes [28] using the (universal) syntomic cycle class maps constructed in this paper.

Appendix A. Comparison of crystalline, convergent, and rigid syntomic cohomologies

We will compare crystalline, convergent, and rigid syntomic cohomologies for smooth schemes over $\mathcal{O}_K$ with normal crossing compactifications. Let X be a smooth scheme over $\mathcal{O}_K$. Recall Besser's definition of rigid syntomic cohomology [2]

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{syn}}(X, r) := [\mathrm{R}\Gamma_{\mathrm{rig}}(X_0/F) \oplus F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X_K) \xrightarrow{f} \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/F) \oplus \mathrm{R}\Gamma_{\mathrm{rig}}(X_0/K)], \quad r \geq 0.$$

Here $\mathrm{R}\Gamma_{\mathrm{rig}}(\cdot)$ denotes the rigid cohomology and $f : (x, y) \mapsto ((p^r - \varphi)(x), \mathrm{sp}(y) - x)$, where sp is the Berthelot's specialization map.

Proposition A.1. *Let X be a proper semistable scheme over $\mathcal{O}_K$ with a smooth special fiber. There is a natural quasi-isomorphism*

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{syn}}(X_{\mathrm{tr}}, r) \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(X, r), \quad r \geq 0.$$

Proof. As usual we consider X as a log-scheme. We can write

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{syn}}(X_{\mathrm{tr}}, r) \simeq [\mathrm{R}\Gamma_{\mathrm{rig}}(X_{0,\mathrm{tr}}/F)^{\varphi=p^r} \to \mathrm{R}\Gamma_{\mathrm{rig}}(X_{0,\mathrm{tr}})/F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X_{K,\mathrm{tr}})]$$

Since we have

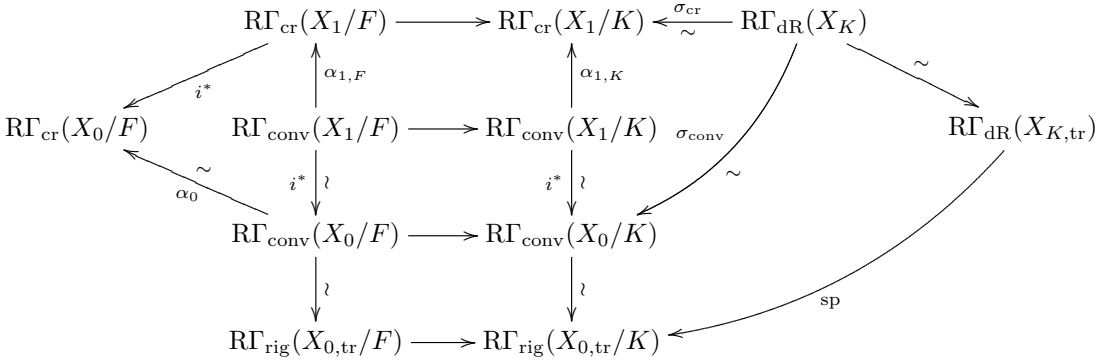
$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, r) \simeq [\mathrm{R}\Gamma_{\mathrm{cr}}(X/F)^{\varphi=p^r} \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)/F^r],$$

it suffices to construct a map

$$\mathrm{R}\Gamma_{\mathrm{cr}}(X/F) \to \mathrm{R}\Gamma_{\mathrm{rig}}(X_{0,\mathrm{tr}}/F)$$

that is compatible (in the dg category sense) with Frobenius and the specialization map from de Rham cohomology. This is accomplished by the following commutative diagram.

[9] Syntomic cohomology of $X_{\mathcal{O}_{\overline{K}}}$ is defined in the same way as the one of X.



Here $\mathrm{R}\Gamma_{\mathrm{conv}}(\cdot)$ denotes the (logarithmic) convergent cohomology [23, 1, 30] that is used classically to connect rigid cohomology with crystalline cohomology. The quasi-isomorphisms between the rigid and the convergent cohomology at the bottom of the diagram are proved in [30, Cor. 2.4.13]. The maps i^* are quasi-isomorphisms by invariance of convergent cohomology under nilpotent thickenings [1, 1.14.3]. The map α_0 is a quasi-isomorphism by [30, Theorem 3.1.1]. The top map i^* is a quasi-isomorphism on φ-eigenspaces [8, Remark 5.9]; hence so is the map $\alpha_{1,F}$. The quasi-isomorphisms $\sigma_{\mathrm{cr}}, \sigma_{\mathrm{conv}}$ are simply the crystalline and the convergent [30, 2.3] Poincaré Lemmas, respectively. It follows that the specialization map sp as well as the map $\alpha_{1,K}$ are quasi-isomorphisms as well. □

Remark A.2. Recall that Besser's definition of rigid syntomic cohomology is modeled on the definition of convergent syntomic cohomology [25]. In its logarithmic form the latter is defined as the following mapping fiber

$$\mathrm{R}\Gamma^{\mathrm{conv}}_{\mathrm{syn}}(X,r) := [\mathrm{R}\Gamma_{\mathrm{conv}}(X_0/F)^{\varphi=p^r} \to \mathrm{R}\Gamma_{\mathrm{conv}}(X_0/K)/F^r\mathrm{R}\Gamma_{\mathrm{conv}}(X_0/K)]$$

The proof of the above proposition shows that, for a proper and semistable scheme over $\mathcal{O}_K$ with a smooth special fiber, we have natural quasi-isomorphisms

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{syn}}(X_{\mathrm{tr}},r) \simeq \mathrm{R}\Gamma^{\mathrm{conv}}_{\mathrm{syn}}(X,r) \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(X,r), \quad r \geq 0. \tag{A.1}$$

In the proper case this was shown in [2, Prop. 9.8].

For a variety Y over K, let $\mathrm{R}\Gamma^{NN}_{\mathrm{syn}}(Y,r)$ denote the syntomic cohomology defined in [22].

Corollary A.3. *Let X be a proper semistable scheme over $\mathcal{O}_K$ with a smooth special fiber. There is a natural distinguished triangle*

$$\mathrm{R}\Gamma^{\mathrm{rig}}_{\mathrm{syn}}(X_{\mathrm{tr}},r) \oplus \mathrm{R}\Gamma(W\Omega^{r-1}_{X_0,\log})_{\mathbf{Q}}[-r] \xrightarrow{\sim} \mathrm{R}\Gamma^{NN}_{\mathrm{syn}}(X_{K,\mathrm{tr}},r).$$

Proof. Since we have a canonical quasi-isomorphism [22, Prop. 3.18]

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X^\times,r)_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma^{NN}_{\mathrm{syn}}(X_{K,\mathrm{tr}},r),$$

this follows immediately from Proposition A.1 and Corollary 3.6. □

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