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SYNTOMIC COMPLEXES AND p-ADIC NEARBY CYCLES.

PIERRE COLMEZ AND WIESŁAWA NIZIOŁ

Abstract. We compute syntomic cohomology of semistable affinoids in terms of cohomology of (φ, Γ)-modules which, thanks to work of Fontaine-Herr, Andreatta-Iovita, and Kedlaya-Liu, is known to compute Galois cohomology of these affinoids. For a semistable scheme over a mixed characteristic local ring this implies a comparison isomorphism, up to some universal constants, between truncated sheaves of p-adic nearby cycles and syntomic cohomology sheaves. This generalizes the comparison results of Kato, Kurihara, and Tsuji for small Tate twists (where no constants are necessary) as well as the comparison result of Tsuji that holds over the algebraic closure of the field. As an application, we combine this local comparison isomorphism with the theory of finite dimensional Banach Spaces and finiteness of étale cohomology of rigid analytic spaces proved by Scholze to prove a Semistable conjecture for formal schemes with semistable reduction.

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1. Introduction

Let $\mathscr{O}_K$ be a complete discrete valuation ring with fraction field K of characteristic 0 and with perfect residue field k of characteristic p. Let $\mathscr{O}_F = W(k)$ and $F = \mathscr{O}_F[\frac{1}{p}]$ so that K is a totally ramified extension of F; let $e = [K : F]$ be the absolute ramification index of K. Let $\mathscr{O}_{\overline{K}}$ denote the integral closure of $\mathscr{O}_K$ in $\overline{K}$. Set $G_K = \mathrm{Gal}(\overline{K}/K)$, and let $\varphi = \varphi_{W(\overline{k})}$ be the absolute Frobenius on $W(\overline{k})$. For a log-scheme X over $\mathscr{O}_K$, X_n will denote its reduction mod p^n, X_0 will denote its special fiber.

1.1. Statement of the main results.

1.1.1. *The Fontaine-Messing map.* Let X be a fine and saturated log-scheme log-smooth over $\mathscr{O}_K$ equipped with the log-structure coming from the closed point. Denote by X_{tr} the locus where the log-structure is trivial. This is an open dense subset of the generic fiber of X. For $r \geq 0$, let $\mathscr{S}_n(r)_X$ be the (log) syntomic sheaf modulo p^n on $X_{0,\text{ét}}$. It can be thought of as a derived Frobenius and filtration eigenspace of crystalline cohomology or as a relative Fontaine functor. Fontaine-Messing [28], Kato [36] have constructed period morphisms $(i : X_0 \hookrightarrow X, j : X_{\mathrm{tr}} \hookrightarrow X)$

$$\alpha^{\mathrm{FM}}_{r,n} : \quad \mathscr{S}_n(r)_X \to i^* R j_* \mathbf{Z}/p^n(r)'_{X_{\mathrm{tr}}}, \quad r \geq 0.$$

from logarithmic syntomic cohomology to logarithmic p-adic nearby cycles. Here we set $\mathbf{Z}_p(r)' := \frac{1}{p^{a(r)}}\mathbf{Z}_p(r)$, for $r = (p-1)a(r) + b(r)$, $0 \leq b(r) \leq p-1$.

Assume now that X has semistable reduction over $\mathscr{O}_K$ or is a base change of a scheme with semistable reduction over the ring of integers of a subfield of K. That is, locally, X can be written as $\mathrm{Spec}(A)$ for a ring A étale over

$$\mathscr{O}_K[X_1^{\pm 1}, \cdots, X_a^{\pm 1}, X_{a+1}, \cdots, X_{a+b}, X_{a+b+1}, \cdots, X_d, X_{d+1}]/(X_{d+1}X_{a+1}\cdots X_{a+b} - \varpi^h), \quad 1 \leq h \leq e.$$

If we put $D := \{X_{a+b+1}\cdots X_d = 0\} \subset \mathrm{Spec}(A)$ then the log-structure on $\mathrm{Spec}\, A$ is associated to the special fiber and to the divisor D. We have $\mathrm{Spec}(A)_{\mathrm{tr}} = \mathrm{Spec}(A_K) \setminus D_K$.

We prove in this paper that the Fontaine-Messing period map $\alpha^{\mathrm{FM}}_{r,n}$, after a suitable truncation, is essentially a quasi-isomorphism. More precisely, we prove the following theorem.

Theorem 1.1. *For $0 \leq i \leq r$, consider the period map*

$$\alpha^{\mathrm{FM}}_{r,n} : \quad \mathscr{H}^i(\mathscr{S}_n(r)_X) \to i^* R^i j_* \mathbf{Z}/p^n(r)'_{X_{\mathrm{tr}}}. \tag{1.2}$$

(i) *If K has enough roots of unity*[1] *then the kernel and cokernel of this map are annihilated by p^{Nr+c_p} for a universal constant N* (not depending on p, X, K, n or r) *and a constant c_p depending only on p (and d if $p = 2$).*

(ii) *In general, the kernel and cokernel of this map are annihilated by p^N for an integer $N = N(e, p, r)$, which depends on e, r, but not on X or n.*

For $i \leq r \leq p-1$, it is known that a stronger statement is true: the period map

$$\alpha^{\mathrm{FM}}_{r,n} : \quad \mathscr{H}^i(\mathscr{S}_n(r)_X) \xrightarrow{\sim} i^* R^i j_* \mathbf{Z}/p^n(r)_{X_{\mathrm{tr}}}. \tag{1.3}$$

is an isomorphism for X a log-scheme log-smooth over a henselian discrete valuation ring $\mathscr{O}_K$ of mixed characteristic. This was proved by Kato [35, 36], Kurihara [40], and Tsuji [63, 64]. In [62] Tsuji generalized

[1] See Section (2.2.1) for what it means for a field to contain enough roots of unity. For any K, the field $K(\zeta_{p^n})$, for $n \geq c(K) + 3$, where $c(K)$ is the conductor of K, contains enough roots of unity.

this result to some étale local systems. As Geisser has shown [30], in the case without log-structure, the isomorphism (1.3) allows to approximate the (continuous) p-adic motivic cohomology (sheaves) of p-adic varieties by their syntomic cohomology; hence to relate p-adic algebraic cycles to differential forms. This was used to study algebraic cycles in mixed characteristic [54], geometric class field theory [41], Beilinson's Tate conjecture [5], variational Hodge conjecture [11], p-adic regulators and special values of p-adic L-functions [58].

We hope that the "isomorphism" (1.2) that generalizes (1.3) will allow to extend the above mentioned applications. Actually, (1.2) was already used to approximate motivic cohomology in mixed characteristic. More precisely, in [24] it is shown that the result of Geisser generalizes to all Tate twists and to the "open" case (i.e., with a possible horizontal divisor at infinity). One gets a higher cycle class map from continuous p-adic motivic cohomology to log-syntomic cohomology that is a quasi-isomorphism (by an application of the p-adic Beilinson-Lichtenbaum conjectures on the special and the generic fibers). This allows to define well-behaved integral universal Chern classes to log-syntomic cohomology [53]. Along similar lines, using (1.2), it is shown in [51] that the p-adic K-theory sheaves localized in the log-syntomic topology coincide with log-syntomic Tate twists (up to a direct factor). This is analogous to what happens ℓ-adically [61, 49].

As an application of Theorem 1.1, one can obtain the following generalization of the Bloch-Kato's exponential map [13]. Let $\mathscr{X}$ be a quasi-compact formal, semistable scheme over $\mathscr{O}_K$ (for example a semi-stable affinoid). For $i \geq 1$, consider the composition

$$\alpha_{r,i}: \quad H^{i-1}_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}) \to H^i(\mathscr{X}, \mathscr{S}(r))_{\mathbf{Q}} \xrightarrow{\alpha_r^{FM}} H^i_{\text{ét}}(\mathscr{X}_{K,\mathrm{tr}}, \mathbf{Q}_p(r)).$$

If X is a proper semistable scheme X over $\mathscr{O}_K$, and $1 \leq i \leq r-1$, then the G_K-representation $V_{i-1} = H^{i-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r))$ is finite dimensional over $\mathbf{Q}_p$, $H^{i-1}_{\mathrm{dR}}(X_K)$ is finite dimensional over K, and $H^{i-1}_{\mathrm{dR}}(X_K) = D_{\mathrm{dR}}(V_{i-1})$. The map $\alpha_{r,i}$ for the formal scheme $\mathscr{X}$ associated to X is then the Bloch-Kato's map [46]

$$D_{\mathrm{dR}}(V_{i-1}) \to H^1(G_K, V_{i-1}) = H^i_{\text{ét}}(X_K, \mathbf{Q}_p(r)).$$

Easy comparison between de Rham and syntomic cohomologies, together with Theorem 1.1, yield the following result.

Corollary 1.4. *For $i \leq r-1$, the map*

$$\alpha_{r,i}: H^{i-1}_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}) \to H^i_{\text{ét}}(\mathscr{X}_{K,\mathrm{tr}}, \mathbf{Q}_p(r))$$

is an isomorphism. The map $\alpha_{r,r}: H^{r-1}_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}) \to H^r_{\text{ét}}(\mathscr{X}_{K,\mathrm{tr}}, \mathbf{Q}_p(r))$ is injective, but its cokernel can be very large if the dimension of $\mathscr{X}_K$ is ≥ 1.

Recall how one shows that the period map $\alpha^{\mathrm{FM}}_{r,n}$ from (1.3) is an isomorphism. Under the stated assumptions one can do dévissage and reduce to $n = 1$. Then one passes to the tamely ramified extension obtained by attaching the p'th root of unity ζ_p. There both sides of the period map (1.3) are invariant under twisting by $t \in \mathbf{A}_{\mathrm{cr}}$ and ζ_p, respectively, so one reduces to the case $r = i$. This is the Milnor case: both sides compute Milnor K-theory modulo p. To see this, one uses symbol maps from Milnor K-theory to the groups on both sides (differential on the syntomic side and Galois on the étale side). Via these maps all groups can be filtered compatibly in a way similar to the filtration of the unit group of a local field. Finally, the quotients can be computed explicitly by symbols [12], [34], [40], [62] and they turn out to be isomorphic. This approach to the computation of p-adic nearby cycles goes back to the work of Bloch-Kato [12] who treated the case of good reduction and whose approach was later generalized to semistable reduction by Hyodo [34].

1.1.2. *Galois cohomology of semistable affinoid algebras.* Our proof is of very different nature: we construct another local (i.e., on affinoids of a special type, see below) period map, that we call $\alpha_r^{\mathscr{L}az}$. Modulo some (φ,Γ)-modules theory reductions, it is a version of an integral Lazard isomorphism between Lie algebra cohomology and continuous group cohomology. We prove directly that it is a quasi-isomorphism and coincides with Fontaine-Messing's map up to constants as in Theorem 1.1. The (hidden) key input is the purity theorem of Faltings [25], Kedlaya-Liu [37], and Scholze [56]: it shows up in the computation of Galois cohomology in terms of (φ,Γ)-modules [4, 37].

More precisely, let R be the p-adic completion of an étale algebra over

$$R_\square := \mathscr{O}_F\{X_1^{\pm 1},\cdots,X_a^{\pm 1},X_{a+1},\cdots,X_{d+1}\}/(X_{d+1}X_{a+1}\cdots X_{a+b}-\varpi),$$

where a,b are integers and $d\geq a+b$. We equip the associated formal schemes with the log-structure induced by the "divisor at infinity": $X_{a+b+1}\cdots X_d=0$ and the special fiber. These are formal log-schemes with semistable reduction over $\mathscr{O}_K$.

To compute the syntomic cohomology of R, we need to choose good crystalline coordinates for R, i.e., we need to write it as a quotient of a log-smooth $\mathscr{O}_F$-algebra R_ϖ^+. The easiest way to do that is to add one variable. We start with $\mathscr{O}_K$. Take the algebra $r_\varpi^+ := \mathscr{O}_F[[X_0]]$ equipped with the log-structure associated to X_0. Sending X_0 to ϖ induces a surjective morphism $r_\varpi^+\to\mathscr{O}_K$.

Let now $R_{\varpi,\square}^+$ be the completion of $\mathscr{O}_F[X_0,X_1^{\pm 1},\cdots,X_a^{\pm 1},X_{a+1},\cdots,X_{d+1},\frac{X_0}{X_{a+1}\cdots X_{a+b}}]$ for the (p,X_0)-adic topology. We add X_0 to the log-structure induced from $R_\square$. Sending X_0 to ϖ induces a surjective morphism $R_{\varpi,\square}^+\to R_\square$ whose kernel is generated by $P=P_\varpi(X_0)$. This provides a closed embedding of $\operatorname{Spf} R_\square$ into a formal log-scheme $\operatorname{Spf} R_{\varpi,\square}^+$ that is log-smooth over $\mathscr{O}_F$. Let R_ϖ^+ be the unique étale lift of R over $R_{\varpi,\square}^+$ complete for the (p,P)-adic topology (which is also the (p,X_0)-adic topology). We equip it with the log-structure induced from $R_{\varpi,\square}^+$. Sending X_0 to ϖ induces a surjective morphism $R_\varpi^+\to R$, whose kernel is generated by $P=P_\varpi(X_0)$. We denote by R_ϖ^{PD} the p-adic PD-envelope of R in R_ϖ^+. We endow it with Frobenius φ_{Kum} induced by $X_i\to X_i^p$, $0\leq i\leq d+1$. This is our PD-coordinate system of R.

The syntomic cohomology of R can then be computed by the complex

$$\operatorname{Syn}(R,r) := \operatorname{Cone}(\ F^r\Omega^\bullet_{R_\varpi^{\mathrm{PD}}}\xrightarrow{p^r-p^\bullet\varphi_{\mathrm{Kum}}}\Omega^\bullet_{R_\varpi^{\mathrm{PD}}}\ )[-1], \tag{1.5}$$

where $\Omega^\bullet_{R_\varpi^{\mathrm{PD}}} := R_\varpi^{\mathrm{PD}}\otimes_{R_\varpi^+}\Omega^\bullet_{R_\varpi^+/\mathscr{O}_F}$: we have $H^i_{\mathrm{syn}}(R,r)=H^i(\operatorname{Syn}(R,r))$.

Now, let $\overline{R}$ be the "maximal extension of R unramified outside the divisor $X_{a+b+1}\cdots X_d=0$ in characteristic 0 (i.e., after inverting p)". Let $G_R=\operatorname{Gal}(\overline{R}/R)$. Modulo the identification of α_r^{FM} and $\alpha_r^{\mathscr{L}az}$, claim (i) of Theorem 1.1 is a consequence of the following more precise statement that relates Galois cohomology of G_R (and étale cohomology of the associated rigid space) with values in $\mathbf{Z}_p(r)$ and syntomic cohomology in degrees $\leq r$. This is the first main result of our paper.

Theorem 1.6. *If K contains enough roots of unity then the maps*

$$\begin{aligned}
\alpha_r^{\mathscr{L}az}:&\quad \tau_{\leq r}\operatorname{Syn}(R,r)\to\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{cont}}(G_R,\mathbf{Z}_p(r)),\\
\alpha_{r,n}^{\mathscr{L}az}:&\quad \tau_{\leq r}\operatorname{Syn}(R,r)_n\to\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{cont}}(G_R,\mathbf{Z}/p^n(r))\to\tau_{\leq r}\mathrm{R}\Gamma((\operatorname{Sp}R[1/p])_{\mathrm{tr},\text{ét}},\mathbf{Z}/p^n(r))
\end{aligned}$$

are quasi-isomorphisms up to p^{Nr}, for a universal constant N.

This statement is more precise than that of Theorem 1.1 because we do not require étale localization. Claim (ii) of Theorem 1.1 follows by a simple descent argument from claim (i). The constants are very crude and no doubt can be improved upon.

Remark 1.7. The same descent argument would allow to remove the condition "K contains enough roots of unity" in Theorem 1.6, at the cost of introducing constants depending on K. It seems likely that one could make the constants depend only on the valuation of the different of K and not on K itself.

In fact, it does not seem unreasonnable to think that a statement analogous to Corollary 1.4, describing étale cohomology in terms of differential forms, should be valid for a general affinoid algebra, smooth in characteristic 0, with constants depending on the "defect of smoothness in mixed characteristic" (i.e., the p-adic valuation of suitable Jacobians). The point is that one can use the purity theorems to pass to a finite cover with very small defect of smoothness where one could try to apply the methods of this paper (suitably modified).

We hope to come back to these questions in a future work.

1.1.3. *Period isomorphisms in the semistable case.* Theorem 1.1 holds also for base changes of semistable schemes and implies that we have a quasi-isomorphism (up to p^{Nr} for a universal constant N)

$$\alpha^{\mathrm{FM}}_{r,n}: \quad \mathscr{H}^i(\mathscr{S}_n(r)_{X_{\mathscr{O}_{\overline{K}}}}) \to \bar{i}^* R^i \bar{j}_* \mathbf{Z}/p^n(r)'_{X_{\overline{K},\mathrm{tr}}}, \quad i \leq r, \tag{1.8}$$

where $\bar{i}: X_{0,\bar{k}} \hookrightarrow X_{\mathscr{O}_{\overline{K}}}$, $\bar{j}: X_{\overline{K},\mathrm{tr}} \hookrightarrow X_{\mathscr{O}_{\overline{K}}}$. Hence an isomorphism

$$\alpha^{\mathrm{FM}}_{r}: \quad H^i_{\mathrm{syn}}(X_{\mathscr{O}_{\overline{K}}}, r)_{\mathbf{Q}} \xrightarrow{\sim} H^i(X_{\overline{K},\mathrm{tr}}, \mathbf{Q}_p(r)), \quad i \leq r. \tag{1.9}$$

This recovers Tsuji's result [63, Theorem 3.3.4] that he proves by similar techniques as his results over K. Namely, in the case when $\zeta_{p^n} \in K$ and $r = i$, Tsuji notices that twisting the nearby cycles by ζ_{p^n} allows to perform dévissage and to reduce to $n = 1$, where one again can use explicit computations by symbols. This allows him to prove Theorem 1.1 in this case [63, Theorem 3.3.2] up to p^N with a constant N that depends only on p and r. To pass to all $i \leq r$ he needs to show that twisting by t does not change the syntomic cohomology sheaves. This he is able to do over $\overline{K}$ [63, Theorem 2.3.2] and up to a constant N that depends only on p, r, and i.

The isomorphism (1.9) is used traditionally to prove p-adic comparison theorems by the syntomic method [28], [36], [63], [65], [68]. Recall how the argument goes in the case of a trivial divisor at infinity, i.e., when $X_{\mathrm{tr}} = X_K$. One composes the map α^{FM}_r with the natural map

$$H^i(X_{\overline{K}}, \mathbf{Q}_p(r)) \to H^i_{\mathrm{HK}}(X) \otimes_F \mathbf{B}_{\mathrm{st}}\{r\},$$

where $H^i_{\mathrm{HK}}(X)$ is the Hyodo-Kato cohomology of X, to obtain the period map

$$\alpha: \quad H^i(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}} \simeq H^i_{\mathrm{HK}}(X) \otimes_F \mathbf{B}_{\mathrm{st}}.$$

This map is shown to be compatible with Poincaré duality. Hence it is an isomorphism.

We have realized that we can prove that the period map α is an isomorphism without evoking Poincaré duality by techniques supplied by the theory of finite dimensional Banach Spaces [20]. This reproves the classical comparison theorem for semistable schemes via a modified syntomic method (including the case of non-trivial divisor at infinity treated in Tsuji [65] and Yamashita-Yasuda [68]). Recall that the semistable comparison theorem for schemes was proved also by different methods in [26], [27], [47], [48], [9], [7] (see [50] for the proof that most of these methods yield the same period map). They all use Poincaré duality. The only proof of a general comparison theorem that does not use Poincaré duality is the proof of the de Rham conjecture for rigid analytic spaces by Scholze [57].

The fact that we do not need Poincaré duality anymore allows us to combine the comparison isomorphism (1.9) with finiteness of étale cohomology of proper rigid analytic spaces proved by Scholze [57] to prove the second main result of this paper – a comparison theorem for semistable formal schemes.

Corollary 1.10. (Semistable conjecture) *Let $\mathscr{X}$ be a proper semistable formal scheme over $\mathscr{O}_K$. There exists a natural $\mathbf{B}_{\mathrm{st}}$-linear Galois equivariant period isomorphism*

$$\alpha: \quad H^i(\mathscr{X}_{\overline{K},\mathrm{tr}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}} \simeq H^i_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}_{\mathrm{st}},$$

that preserves the Frobenius and the monodromy operators, and induces a filtered isomorphism

$$\alpha: \quad H^i(\mathscr{X}_{\overline{K},\mathrm{tr}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}} \simeq H^i_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}) \otimes_K \mathbf{B}_{\mathrm{dR}}.$$

1.2. **Sketch of the proofs of the main results.** We will now sketch the proofs of Theorem 1.6 and Corollary 1.10.

1.2.1. *Local computations.* We will start with Theorem 1.6. If $v > u > 0$, let $r_\varpi^{[u]}$ (resp. $r_\varpi^{[u,v]}$) be the ring of analytic functions over F convergent on the disk $v_p(X_0) \geq u/e$ (resp. the annulus $v/e \geq v_p(X_0) \geq u/e$), where $e = [K : F]$, and let $R_\varpi^{[u]} = r_\varpi^{[u]} \widehat{\otimes}_{r_\varpi^+} R_\varpi^+$ (resp. $R_\varpi^{[u,v]} = r_\varpi^{[u,v]} \widehat{\otimes}_{r_\varpi^+} R_\varpi^+$). Our ring R_ϖ^{PD} is very close to $R_\varpi^{[u]}$, $u = \frac{1}{p-1}$.

Set $u = \frac{p-1}{p}$ and $v = p-1$ in what follows (for $p = 2$, one has to modify slightly the arguments and take $u = \frac{3}{4}$, $v = \frac{3}{2}$, but we will ignore this for the introduction). To define the period map

$$\alpha_r^{\mathscr{L}az} : \quad \tau_{\leq r}\mathrm{Syn}(R, r) \to \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{cont}}(G_R, \mathbf{Z}_p(r)),$$

using (φ, ∂)-modules techniques, we produce a string of "quasi-isomorphisms" (a *"quasi-isomorphism"* is a map of complexes whose associated map on cohomology has kernels and cokernels killed by p^{Cr} for some absolute constant C). We start with quasi-isomorphisms

$$\mathrm{Syn}(R, r) \simeq \mathrm{Kum}(R_\varpi^{[u]}, r) \overset{\tau_{\leq r}}{\simeq} \mathrm{Kum}(R_\varpi^{[u,v]}, r), \tag{1.11}$$

where the Kummer complexes $\mathrm{Kum}(\cdot, r)$ are defined by formulas analogous to (1.5). Here we are forced to truncate the morphism $\mathrm{Kum}(R_\varpi^{[u]}, r) \to \mathrm{Kum}(R_\varpi^{[u,v]}, r)$ because it is a quasi-isomorphism up to too large constants in degrees $> r$. The second quasi-isomorphism in (1.11) is proved using the ψ operator – left inverse to φ – and acyclicity of the $\psi = 0$ eigencomplexes.

We called the complexes in (1.11) Kummer because they are related to the Kummer extension $K(\varpi^{1/p^\infty})$ of K. Let us explain what we mean by that. Let $\mathbf{E}_{\overline{R}}^+$ be the tilt of $\overline{R}$ and set $\mathbf{A}_{\overline{R}}^+ = W(\mathbf{E}_{\overline{R}}^+)$. Choose, inside $\overline{R}$, elements $X_i^{p^{-n}}$, for $i = 1, \dots, d$ and $n \in \mathbf{N}$, satisfying the obvious relations (i.e. $X_i^{p^{-0}} = X_i$ and $(X_i^{p^{-(n+1)}})^p = X_i^{p^{-n}}$ if $n \geq 0$). If $i = 1, \dots, d$, let $x_i = (X_i, X_i^{1/p}, \dots) \in \mathbf{E}_{\overline{R}}^+$.

Sending X_0 to $[\varpi^\flat]$, where $\varpi^\flat$ is a sequence of p'th roots of ϖ, and X_i to $[x_i]$, if $i = 1, \dots, d$, induces an embedding $\iota_{\mathrm{Kum}} : R_\varpi^+ \hookrightarrow \mathbf{A}_{\overline{R}}^+$ which commutes with Frobenius φ and is compatible with filtrations (with filtration on $\mathbf{A}_{\overline{R}}^+$ by powers of the Kernel of the natural map $\theta : \mathbf{A}_{\overline{R}}^+ \to \widehat{\overline{R}}$). By continuity, this extends to embeddings

$$\iota_{\mathrm{Kum}} : R_\varpi^{[u]} \hookrightarrow \mathbf{A}_{\overline{R}}^{[u]}, \quad \iota_{\mathrm{Kum}} : R_\varpi^{[u,v]} \hookrightarrow \mathbf{A}_{\overline{R}}^{[u,v]},$$

which commute with Frobenius and filtration.

But, *if K contains enough roots of unity*, the ring $R_\varpi^{[u,v]}$ can also be embedded into period rings via a cyclotomic embedding (i.e. using the cyclotomic extension of K instead of its Kummer extension). This will however change the Kummer Frobenius into the cyclotomic Frobenius φ_{cycl}. Let us sketch how this is done. Let $i(K)$ be the largest integer i such that K contains ζ_{p^i} and, if $n \in \mathbf{N}$, let $K_n = K(\zeta_{p^{n+i(K)}})$. The assumption that K contains enough roots of unity implies (thanks to the field of norms theory and extra work [20]) the existence of $\pi_K \in \mathbf{A}_{\overline{K}}$, fixed by $\mathrm{Gal}(\overline{K}/K_\infty)$, such that $\varphi(\pi_K) = f(\pi_K)$, with $f(X_0)$ analytic and bounded on the annulus $0 < v_p(X_0) \leq v/e$, and such that modulo $(p, [p^\flat]^{1/p})$ we have $\overline{\pi}_K = (\varpi_{K_n})_{n \in \mathbf{N}} \in \mathbf{E}_{\overline{K}}^+$ – a tower of uniformizers of the fields K_n. We can embed r_ϖ^+ into $\mathbf{A}_{\overline{K}}$, sending X_0 to π_K and this extends to the completion r_ϖ of $r_\varpi^+[X_0^{-1}]$ for the p-adic topology. The image[2] $\mathbf{A}_K$ is stable under the action of φ, hence r_ϖ inherits a Frobenius that we note φ_{cycl}. This Frobenius does not preserve r_ϖ^+, but we have $\varphi_{\mathrm{cycl}}(X_0) \in r_\varpi^{(0,v]+}$ - the ring of analytic functions over F with integral values on the annulus $0 < v_p(X_0) \leq v/e$.

For a general R, first we extend φ_{cycl} to a Frobenius on R_ϖ by setting

$$\varphi_{\mathrm{cycl}}(X_i) = X_i^p, \quad 1 \leq i \leq d; \qquad \varphi_{\mathrm{cycl}}\left(\frac{X_0}{X_{a+1} \cdots X_{a+b}}\right) = \left(\frac{X_0}{X_{a+1} \cdots X_{a+b}}\right)^p \frac{\varphi_{\mathrm{cycl}}(X_0)}{X_0^p}$$

[2]This is the image by $\varphi^{-i(K)}$ of the usual $\mathbf{A}_K$ from the theory of (φ, Γ)-modules.

Then, for $n \in \mathbf{N}$, we consider R_n – the subalgebra of $\overline{R}$ generated by R and $\mathscr{O}_{K_n}$, $X_i^{p^{-n}}$ for $i = 1, \dots, d$, and $\frac{\varpi_{K_n}}{(X_{a+1}\cdots X_{a+b})^{p^{-n}}}$. Set $R_\infty = \cup_{n\in\mathbf{N}} R_n$. The ring $R_\infty[\frac{1}{p}]$ is a Galois extension of $R[\frac{1}{p}]$, with Galois group Γ_R which is the semi-direct product

$$1 \to \Gamma'_R \to \Gamma_R \to \Gamma_K \to 1,$$

where

$$\Gamma'_R = \mathrm{Gal}(R_\infty[\tfrac{1}{p}]/K_\infty \cdot R[\tfrac{1}{p}]) \simeq \mathbf{Z}_p^d, \quad \Gamma_K = \mathrm{Gal}(K_\infty/K) \simeq 1 + p^{i(K)}\mathbf{Z}_p,$$

and $a \in 1 + p^{i(K)}\mathbf{Z}_p$ acts on $\mathbf{Z}_p^d$ by multiplication by a.

We define the cyclotomic embedding[3] $\iota_{\mathrm{cycl}} : R_\varpi^{\mathrm{deco}} \to \mathbf{A}_{\overline{R}}^{\mathrm{deco}}$ using the embedding $\iota_{\mathrm{cycl}} : r_\varpi^{\mathrm{deco}} \to \mathbf{A}_{\overline{K}}^{\mathrm{deco}}$ and sending X_j to $[x_j]$ if $1 \le j \le d$. It is compatible with Frobenius and with filtration. We denote by $\mathbf{A}_R^{\mathrm{deco}}$ the image of R_ϖ^{deco} by ι_{cycl}. Then, the rings $\mathbf{A}_R, \mathbf{A}_R^{[u,v]}, \mathbf{A}_R^{(0,v]+}$ are stable by G_R which acts through Γ_R.

Coming back to cohomology, using standard crystalline techniques, we show that change of Frobenius does not affect syntomic cohomology. That is, we produce quasi-isomorphisms

$$\mathrm{Kum}(R_\varpi^{[u,v]}, r) \simeq \mathrm{Syn}((R_\varpi^{[u,v]} \widehat{\otimes} R_\varpi^{[u,v]})^{\mathrm{PD}}, \varphi_{\mathrm{Kum}} \otimes \varphi_{\mathrm{cycl}}, r) \simeq \mathrm{Cycl}(R_\varpi^{[u,v]}, r),$$

where the last complex is defined as in (1.5) using the ring $R_\varpi^{[u,v]}$ and the cyclotomic Frobenius φ_{cycl}.

Next, choosing a basis of Ω^1 and using the isomorphism $R_\varpi^{[u,v]} \simeq \mathbf{A}_R^{[u,v]}$, we change $\mathrm{Cycl}(R_\varpi^{[u,v]}, r)$ into a Koszul complex:

$$\mathrm{Cycl}(R_\varpi^{[u,v]}, r) \simeq \mathrm{Kos}(\varphi, \partial, F^r\mathbf{A}_R^{[u,v]}).$$

Then, multiplying by suitable powers of t, we can get rid of the filtration (in degrees $\le r$; this is the only place where the truncation is absolutely necessary), which changes the derivatives into the action of the Lie algebra of Γ_R, to obtain:

$$\tau_{\le r}\mathrm{Kos}(\varphi, \partial, F^r\mathbf{A}_R^{[u,v]}) \simeq \tau_{\le r}\mathrm{Kos}(\varphi, \mathrm{Lie}\,\Gamma_R, \mathbf{A}_R^{[u,v]}(r)).$$

Standard analytic arguments à la Lazard change this into a Koszul complex for the group:

$$\mathrm{Kos}(\varphi, \mathrm{Lie}\,\Gamma_R, \mathbf{A}_R^{[u,v]}(r)) \simeq \mathrm{Kos}(\varphi, \Gamma_R, \mathbf{A}_R^{[u,v]}(r)).$$

Then, using (φ, Γ)-module techniques, we get "quasi-isomorphisms"

$$\mathrm{Kos}(\varphi, \Gamma_R, \mathbf{A}_R^{[u,v]}(r)) \simeq \mathrm{Kos}(\varphi, \Gamma_R, \mathbf{A}_R^{(0,v]+}(r)) \simeq \mathrm{Kos}(\varphi, \Gamma_R, \mathbf{A}_R(r)).$$

Here we use the operator ψ_{cycl} – the left inverse to φ_{cycl} and argue by acyclicity of the $\psi = 0$ eigencomplex.

Finally, general nonsense about Koszul complexes gives us a quasi-isomorphism

$$\mathrm{Kos}(\varphi, \Gamma_R, \mathbf{A}_R(r)) \simeq [\ \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{A}_R(r)) \xrightarrow{1-\varphi} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{A}_R(r))\ ];$$

and general relative (φ, Γ)-module theory gives quasi-isomorphisms

$$[\ \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{A}_R(r)) \xrightarrow{1-\varphi} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{A}_R(r))\ ] \simeq$$

$$[\ \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, \mathbf{A}_{\overline{R}}(r)) \xrightarrow{1-\varphi} \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, \mathbf{A}_{\overline{R}}(r))\ ] \simeq \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, \mathbf{Z}_p(r)).$$

The first quasi-isomorphism is proved by the almost étale and decompletion techniques developed in the relative setting by Andreatta-Iovita [4] and by Kedlaya-Liu [37]; the second one follows from the relative Artin-Schreier theory (i.e. the exact sequence $0 \to \mathbf{Z}_p \to \mathbf{A}_{\overline{R}} \xrightarrow{1-\varphi} \mathbf{A}_{\overline{R}} \to 0$). This finishes the definition of the quasi-isomorphism $\alpha_r^{\mathscr{L}az}$ from Theorem 1.6.

[3]Here, and everywhere in the paper, "deco" stands for "decoration", and is one of PD, $[u]$, $[u,v]$, $(0,v]+$, ...

Since, by the p-adic $K(\pi,1)$-Lemma [57] and by Abkhyankar's Lemma, we have

$$\mathrm{R}\Gamma_{\mathrm{cont}}(G_R,\mathbf{Z}/p^n(r))\simeq \mathrm{R}\Gamma((\mathrm{Sp}\,R[1/p])_{\mathrm{tr},\text{ét}},\mathbf{Z}/p^n(r)),$$

the above sequence of quasi-isomorphisms constructs a quasi-isomorphism

$$\alpha^{\mathscr{L}az}_{r,n}:\quad \tau_{\leq r}\mathrm{Syn}(R,r)_n\simeq \tau_{\leq r}\mathrm{R}\Gamma((\mathrm{Sp}\,R[1/p])_{\mathrm{tr},\text{ét}},\mathbf{Z}/p^n(r)).$$

1.2.2. *Finite dimensional Banach Spaces and semi-stable conjecture.* To prove Corollary 1.10, first we show (this is a simplification for the sake of the introduction) that we have the long exact sequence

$$\begin{aligned}\to (H^{i-1}_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}})\otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r\to H^i_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}},r)_{\mathbf{Q}}\to (H^i_{\mathrm{HK}}(\mathscr{X})\otimes_F \mathbf{B}^+_{\mathrm{st}})^{\varphi=p^r,N=0}\\ \to (H^i_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}})\otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r\to\end{aligned}\tag{1.12}$$

For $i<r$, the above long exact sequence yields short exact sequences

$$0\to H^i_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}},r)_{\mathbf{Q}}\to (H^i_{\mathrm{HK}}(\mathscr{X})\otimes_F \mathbf{B}^+_{\mathrm{st}})^{\varphi=p^r,N=0}\to (H^i_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}})\otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r)\to 0$$

To prove this, we observe that $f_i:(H^i_{\mathrm{HK}}(\mathscr{X})\otimes_F \mathbf{B}^+_{\mathrm{st}})^{\varphi=p^r,N=0}\to (H^i_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}})\otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r)$ is the evaluation on $C=\overline{K}^{\wedge}$ of a map of finite dimensional Banach Spaces [19]. Recall that these Spaces are to be thought of as finite dimensional C-vector spaces up to finite dimensional $\mathbf{Q}_p$-vector spaces, and come equipped with a Dimension, which is a pair of numbers (a,b), $a\in\mathbf{N}$, $b\in\mathbf{Z}$, where a is the C-dimension, and b is the $\mathbf{Q}_p$-dimension. Dimension is additive on short exact sequences.

But, $H^i_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}},r)_{\mathbf{Q}}$, $i\leq r$, is a finite dimensional $\mathbf{Q}_p$-vector space: we have the quasi-isomorphism (1.9) with étale cohomology and Scholze proved finite dimensionality of the latter [57]. This implies that the cokernel of f_i, viewed as a map of Banach Spaces, is of Dimension $(0,d_i)$. On the other hand, the Space $(H^i_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}})\otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r$ is a successive extension of C-vector spaces. The theory of Banach Spaces implies that the map $(H^{i-1}_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}})\otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r\to \mathrm{Coker}\,f_i$ is zero, hence $\mathrm{Coker}\,f_i=0$, as wanted.

Now, since we have the Hyodo-Kato isomorphism

$$H^i_{\mathrm{HK}}(\mathscr{X})\otimes_{K_0}K\simeq H^i_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}),$$

the pair $(H^i_{\mathrm{HK}}(\mathscr{X}),H^i_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}))$ is a (φ,N)-filtered module (in the sense of Fontaine). The above short exact sequence and a "weight" argument shows that $\mathbf{V}_{\mathrm{st}}(H^i_{\mathrm{HK}}(\mathscr{X}),H^i_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}))\simeq H^i(X_{\overline{K},\mathrm{tr}},\mathbf{Q}_p)$. Here $\mathbf{V}_{\mathrm{st}}(\cdot)$ is Fontaine's functor from filtered Frobenius modules to Galois representations. The short exact sequence and dimension count give also that $t_N(H^i_{\mathrm{HK}}(\mathscr{X}))=t_H(H^i_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}))$, where $t_N(D)=v_p(\det\varphi)$ and $t_H(D)=\sum_{i\geq 0}i\dim_K(F^iD/F^{i+1}D)$. The theory of Banach Spaces and finite dimensionality of $\mathbf{V}_{\mathrm{st}}(H^i_{\mathrm{HK}}(\mathscr{X}),H^i_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}))$ imply now that the pair $(H^i_{\mathrm{HK}}(\mathscr{X}),H^i_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}))$ is weakly admissible and this proves Corollary 1.10.

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1.2.3. *Notation and Conventions.*

Definition 1.13. Let $N\in\mathbf{N}$. For a morphism $f:M\to M'$ of $\mathbf{Z}_p$-modules, we say that f is *p^N-injective* (resp. *p^N-surjective*) if its kernel (resp. its cockernel) is annihilated by p^N and we say that f is *p^N-isomorphism* if it is p^N-injective and p^N-surjective. We define in the same way the notion of *p^N-exact sequence* or p^N-acyclic complex (complex whose cohomology groups are annihilated by p^N) as

well as the notion of p^N*-quasi-isomorphism* (map in the derived category that induces a p^N-isomorphism on cohomology).

We will use a shorthand for certain homotopy limits. Namely, if $f : C \to C'$ is a map in the dg derived category of an abelian category, we set

$$[\ C \xrightarrow{f} C'\ ] := \mathrm{holim}(C \to C' \leftarrow 0).$$

We also set

$$\left[\begin{array}{ccc} C_1 & \xrightarrow{f} & C_2 \\ \downarrow & & \downarrow \\ C_3 & \xrightarrow{g} & C_4 \end{array}\right] := [[C_1 \xrightarrow{f} C_2] \to [C_3 \xrightarrow{g} C_4]],$$

where the diagram in the brackets is a commutative diagram in the dg derived category. If this diagram is strictly commutative this simply amounts to taking the total complex of the associated double complex. In this paper this is the case everywhere but in Section 5.

2. Formal log-schemes and period rings

This is a preliminary section introducing all the rings that we are going to use in the next three sections. Most of its content consists of variations on standard techniques from the theory of (φ, Γ)-modules [32, 16, 20, 2, 3, 4, 37].

2.1. **The implicit function theorem.** Let $\lambda : \Lambda_1 \to \Lambda_2$ be a continuous morphism of topological rings. Let $\Lambda_1' = \Lambda_1\{Z\}/(Q)$, where[4] $Z = (Z_1, \dots, Z_s)$ and $Q = (Q_1, \dots, Q_s)$. We would like to extend λ to Λ_1'; this amounts to solving the equation $Q^\lambda(Y) = 0$ in Λ_2, where, if $F \in \Lambda_1\{Z\}$, we note $F^\lambda \in \Lambda_2\{Z\}$ the series obtained by applying λ to the coefficients of F.

Let $J = (\frac{\partial Q_j}{\partial Z_i})_{1 \le i,j \le s} \in \mathbf{M}_s(\Lambda_1\{Z_1, \dots, Z_s\})$.

Proposition 2.1. *Assume there exists:*

- $z \in \Lambda_2$,
- *an ideal I of Λ_2 such that $z^{-2}I \subset \Lambda_2$ and Λ_2 is separated and complete with respect to the $z^{-2}I$-adic topology,*
- $Z_\lambda = (Z_{1,\lambda}, \dots, Z_{s,\lambda}) \in \Lambda_2^s$ *and* $H_\lambda \in z^{-1}\mathbf{M}_s(\Lambda_2)$,

such that:

- *the entries of $Q^\lambda(Z_\lambda)$ belong to I,*
- *the entries of $H_\lambda J^\lambda(Z_\lambda) - 1$ belong to $z^{-1}I$.*

Then the equation $Q^\lambda(Y) = 0$ has a unique solution in $Z_\lambda + (z^{-1}I)^s$.

Proof. Consider $f : \Lambda_2^s \to (\Lambda_2[z^{-1}])^s$ defined by

$$f(x) = x - H_\lambda Q^\lambda(x + Z_\lambda).$$

We have $f(0) \in (z^{-1}I)^s$. Now, we can write $f(x) - f(y)$ as:

$$\begin{aligned}&(1 - H_\lambda J^\lambda(Z_\lambda))(x - y)\\ &\quad + H_\lambda\Big(\big(Q^\lambda(y + Z_\lambda) - Q^\lambda(x + Z_\lambda) - J^\lambda(x + Z_\lambda)(y - x)\big) + \big(J^\lambda(x + Z_\lambda) - J^\lambda(Z_\lambda)\big)(y - x)\Big).\end{aligned}$$

The hypothesis implies that, if $x - y \in (z^{-1}I)^s$, then $f(x) - f(y) \in z^{-2}I \cdot (z^{-1}I)^s$. Hence f is contracting on $(z^{-1}I)^s$ and has a unique fixed point in this module. As x is a fixed point if and only if $Q^\lambda(x + Z_\lambda) = 0$, this concludes the proof. □

[4] If Λ is a topological ring, and $Z = (Z_1, \dots, Z_s)$, we let $\Lambda\{Z\}$ design the ring of power series $\sum_{\mathbf{k} \in \mathbf{N}^s} a_{\mathbf{k}} Z^{\mathbf{k}}$, where $a_{\mathbf{k}} \in \Lambda$ goes to 0 when $\mathbf{k} \to \infty$.

Remark 2.2. If Λ_1' is étale over Λ, there exists $H \in \mathbf{M}_s(\Lambda_1\{Z_1,\dots,Z_s\})$ such that $HJ-1$ has its entries in $(Q_1,\dots,Q_s)$, hence if $Q^\lambda(Z_\lambda)$ has coordinates in an ideal I, then $H^\lambda J^\lambda - 1$ has entries in I. If Λ_2 is separated and complete for the I-adic topology, we can take $z=1$ and $H_\lambda = H^\lambda(Z_\lambda)$ in the above proposition, hence the equation $Q^\lambda(Y)$ has a unique solution in $Z_\lambda + I^s$.

2.2. **Kummer theory.**

2.2.1. *Local fields.* Let $F \subset K$ be as in the introduction, so that K is a finite, totally ramified extension of F. Choose a uniformiser ϖ of K, and let P_ϖ be its minimal polynomial over F (hence P_ϖ is an Eisenstein polynomial of degree $e=[K:F]$).

Choose a compatible system $(\zeta_{p^n})_{n\in\mathbf{N}}$ of roots of unity, with $\zeta_{p^0}=1$, $\zeta_p \neq 1$, and $\zeta_{p^n}^p = \zeta_{p^{n-1}}$, and define $F_n = F(\zeta_{p^n})$, and $F_\infty = \cup_n F_n$.

Let $i(K)$ be the largest integer n such that K contains ζ_{p^n}. If $n\in\mathbf{N}$, we set $K_n = K(\zeta_{p^{n+i(K)}})$, so that $K_0=K$, but K is strictly contained in K_n if $n\geq 1$, even if K contains some roots of unity. Set $K_\infty = \cup_{n\in\mathbf{N}} K_n$. Let

$$\delta_K = ev_p(\mathfrak{d}_{K/F_{i(K)}}) \in \mathbf{N}.$$

We say that K *contains enough roots of unity* if

$$v_p(\mathfrak{d}_{K/F_{i(K)}}) < \tfrac{1}{2p} - \tfrac{1}{[F_{i(K)}:F]} \quad \text{or, equivalently,} \quad \delta_K + [K:F_{i(K)}] < \tfrac{e}{2p}.$$

If we fix K, then $K(\zeta_{p^n})$ contains enough roots of unity for all n big enough (this is a restatement of the fact that K_∞/F_∞ is almost étale). More precisely, by [20, Proposition 4.5], it suffices to take $n\geq c(K)+2$ (if $p=2$, one needs $n\geq c(K)+3$), where $c(K)$ is the conductor of K (i.e. $t\geq c(K)$ if and only if K is fixed by the higher ramification subgroup G_F^t of $G_F = \mathrm{Gal}(\overline{F}/F)$).

2.2.2. *Formal log-schemes.* Let $h\in[1,e]$ be an integer (this h is the multiplicity that appears when we base change a semi-stable scheme over $\mathscr{O}_{K'}$, with $K'\subset K$, to $\mathscr{O}_K$; hence it is $\leq e$ in applications, although almost everywhere it could as well be arbitrary).

Let a,b,c be integers and $d=a+b+c$. Set $X=(X_1,\dots,X_d)$ and define

$$\begin{aligned} R_\square &= \mathscr{O}_K\{X, \frac{1}{X_1\cdots X_a}, \frac{\varpi^h}{X_{a+1}\cdots X_{a+b}}\} \\ &= \mathscr{O}_K\{X, X_{d+1}, X_{d+2}\}/(X_{d+2}X_1\cdots X_a - 1, X_{d+1}X_{a+1}\cdots X_{a+b} - \varpi^h). \end{aligned}$$

We endow $R_\square$ with the spectral norm.

Let R be the p-adic completion of an étale algebra over $R_\square$, so that $R_\square$ provides a *system of coordinates* for R (or $\mathrm{Spf}\, R_\square$ a *frame* for $\mathrm{Spf}\, R$). We equip the associated formal schemes with the log-structure induced by the divisor at infinity: $X_{a+b+1}\cdots X_d = 0$ and the special fiber. We obtain that way formal log-schemes that are log-smooth over $\mathscr{O}_K^\times$ (in fact, both schemes have semistable reduction over $\mathscr{O}_K$).

2.2.3. *Adding an arithmetic variable.* To compute the syntomic cohomology of R, we need to write it as a quotient of a log-smooth algebra over $\mathscr{O}_F$ (it is already log-smooth over $\mathscr{O}_K^\times$). The cheapest way to do so is to add an extra variable.

We denote by r_ϖ^+ and r_ϖ the algebras $\mathscr{O}_F[[X_0]]$ and $\mathscr{O}_F[[X_0]]\{X_0^{-1}\}$ with the log-structure associated to X_0. Sending X_0 to ϖ induces a surjective morphism $r_\varpi^+ \to \mathscr{O}_K$.

Let $X'=(X_0,X)$ and let $R_{\varpi,\square}^+$ be the completion of $\mathscr{O}_F[X', \frac{1}{X_1\cdots X_a}, \frac{X_0^h}{X_{a+1}\cdots X_{a+b}}]$ for the (p,X_0)-adic topology. We add X_0 to the log-structure induced from $R_\square$. Sending X_0 to ϖ induces a surjective morphism $R_{\varpi,\square}^+ \to R_\square$ whose kernel is generated by $P=P_\varpi(X_0)$. This provides a closed embedding of $\mathrm{Spf}\, R_\square$ into a formal log-scheme $\mathrm{Spf}\, R_{\varpi,\square}^+$ that is log-smooth over $\mathscr{O}_F$.

We can write R as

$$R = R_\square\{Z_1,\dots,Z_t\}/(Q_1,\dots,Q_t),$$

with $\det(\frac{\partial Q_i}{\partial Z_j})$ invertible in R. Choose liftings $\tilde{Q}_j$ of the Q_j's in $R^+_{\varpi,\square}$, and let R^+_ϖ be the quotient of the completion of $R^+_{\varpi,\square}[Z_1,\dots,Z_t]$ for the (p,P)-adic topology (which is also the (p,X_0)-adic topology) by $(\tilde{Q}_1,\dots,\tilde{Q}_t)$. We equip it with the log-structure induced from $R^+_{\varpi,\square}$. We have that $\det(\frac{\partial \tilde{Q}_i}{\partial Z_j})$ is invertible in R^+_ϖ (since it is modulo P). Hence R^+_ϖ is étale over $R^+_{\varpi,\square}$ and log-smooth over $\mathscr{O}_F$.

Sending X_0 to ϖ induces a surjective morphism $R^+_\varpi \to R$, whose kernel is generated by $P = P_\varpi(X_0)$.

2.2.4. *Divided powers and localizations on smaller balls or annuli.* We let R^{PD}_ϖ be the p-adic completion of $R^+_\varpi[\frac{P^k}{k!},\ k\in\mathbf{N}]$. As $P = X_0^e$ modulo p, we have $R^+_\varpi[\frac{P^k}{k!},\ k\in\mathbf{N}] = R^+_\varpi[\frac{X_0^k}{[\frac{k}{e}]!},\ k\in\mathbf{N}]$. We equip R^{PD}_ϖ with the log-structure induced from R^+_ϖ and X_0. We have defined the following diagram of formal log-schemes.

(2.3)
$$\begin{tikzcd}
 & \operatorname{Spf} R^{\mathrm{PD}}_\varpi \arrow[dr] & \\
\operatorname{Spf} R \arrow[ur, hook] \arrow[rr, hook] \arrow[d] & & \operatorname{Spf} R^+_\varpi \arrow[d] \\
\operatorname{Spf} R_\square \arrow[rr, hook] \arrow[d] & & \operatorname{Spf} R^+_{\varpi,\square} \arrow[d] \\
\operatorname{Spf} \mathscr{O}_K \arrow[rr, hook] \arrow[d] & & \operatorname{Spf} r^+_\varpi \arrow[dl] \\
\operatorname{Spf} \mathscr{O}_F & &
\end{tikzcd}$$

Crystalline cohomology, hence syntomic cohomology, is defined by means of R^{PD}_ϖ, but we will show that, up to absolute constants depending only on u,v,r, we can replace R^{PD}_ϖ by the rings $R^{[u]}_\varpi$ or $R^{[u,v]}_\varpi$ below. So, if $0<u\le v$, let[5]

- $R^{(0,v]+}_\varpi$ be the p-adic completion of $R^+_\varpi[\frac{p^{\lceil vi/e\rceil}}{X_0^i}, i\in\mathbf{N}]$, and $R^{(0,v]}_\varpi = R^{(0,v]+}_\varpi[\frac{1}{X_0}]$,
- $R^{[u]}_\varpi$ be the p-adic completion of $R^+_\varpi[\frac{X_0^i}{p^{[ui/e]}},\ i\in\mathbf{N}]$,
- $R^{[u,v]}_\varpi$ be the p-adic completion of $R^+_\varpi[\frac{X_0^i}{p^{[ui/e]}}, \frac{p^{\lceil vi/e\rceil}}{X_0^i},\ i\in\mathbf{N}]$,
- R_ϖ be the p-adic completion of $R^+_\varpi[\frac{1}{X_0}]$.

We have $R^{[u]}_\varpi \subset R^{\mathrm{PD}}_\varpi$ if $u\le\frac{1}{p}$ and $R^{\mathrm{PD}}_\varpi\subset R^{[u]}_\varpi$ if $u\ge\frac{1}{p-1}$. If $\frac{1}{p}<u<\frac{1}{p-1}$ then there exists a constant $C(u)$ such that $p^{C(u)}R^{[u]}_\varpi\subset R^{\mathrm{PD}}_\varpi$.

We note $v^{(0,v]}$, $v^{[u]}$, $v^{[u,v]}$ the spectral valuations on $R^{(0,v]}_\varpi$, $R^{[u]}_\varpi$ and $R^{[u,v]}_\varpi$ respectively.

Remark 2.4. Denote by r^{deco}_ϖ the ring R^{deco}_ϖ corresponding to $R=\mathscr{O}_K$. One can describe the rings r^{deco}_ϖ as rings of Laurent series with coefficients satisfying growth conditions. Namely:

- r^{PD}_ϖ is the set of $f=\sum_{i\in\mathbf{N}} a_i\frac{X_0^i}{[\frac{i}{e}]!}$, with $a_i\in\mathscr{O}_F$ going to 0 when $i\to\infty$.
- $r^{[u]}_\varpi$ is the set of $f=\sum_{i\in\mathbf{N}} a_i\frac{X_0^i}{p^{[\frac{iu}{e}]}}$, with $a_i\in\mathscr{O}_F$ going to 0 when $i\to\infty$. It is the ring of analytic functions over F with integral values on the disk $v_p(X_0)\ge u/e$.
- $r^{[u,v]}_\varpi$ is the set of $f=\sum_{i\in\mathbf{Z}} a_iX_0^i$, with $a_i\in F$ and $v_p(a_i)\ge(-iv)/e$ if $i\le 0$ and $v_p(a_i)\ge(-iu)/e$ if $i\ge 0$, and with the differences going to $+\infty$ with $i\to\pm\infty$. It is the ring of analytic functions over F with integral values on the annulus $\frac{u}{e}\le v_p(X_0)\le\frac{v}{e}$.

[5]We will need $R^{[u,v]}_\varpi$, with u,v satisfying a number of inequalities: $\frac{p-1}{p}\le u\le\frac{v}{p}<1<v<pv_K$ and $\big(1+\frac{p+1}{2p(p-1)}\big)u>\frac{1}{p-1}$. For $p\ge 3$, we can take $u=\frac{p-1}{p}$ and $v=p-1$. For $p=2$, we can take $u=\frac{3}{4}$ and $v=\frac{3}{2}$.

We have $r_\varpi^{[u]} \subset r_\varpi^{[u,v]}$ and the quotient $r_\varpi^{[u,v]}/r_\varpi^{[u]}$ involves only negative powers of X_0.

• $r_\varpi^{(0,v]+}$ is the set of $f = \sum_{i\in\mathbf{Z}} a_i X_0^i$, with $a_i \in \mathscr{O}_F$ and $v_p(a_i) \geq (-iv)/e$ if $i \leq 0$, the differences going to $+\infty$ with $i \to -\infty$. It is the ring of analytic functions over F with integral values on the annulus $0 < v_p(X_0) \leq \frac{v}{e}$.

• r_ϖ is the set of $f = \sum_{i\in\mathbf{Z}} a_i X_0^i$, with $a_i \in \mathscr{O}_F$ and $a_i \to 0$ when $i \to -\infty$.

Note that going from R_ϖ^+ to R_ϖ^{PD}, $R_\varpi^{[u]}$, $R_\varpi^{[u,v]}$ or R_ϖ involves only the "arithmetic" variable X_0 (going from R_ϖ^+ to $R_\varpi^{[u]}$ (resp. $R_\varpi^{[u,v]}$) amounts to localization of X_0 on the disk $v_p(X_0) \geq \frac{u}{e}$ (resp. on the annulus $\frac{v}{e} \geq v_p(X_0) \geq \frac{u}{e}$). This can be translated into the following isomorphisms:

$$R_\varpi^{\mathrm{PD}} = r_\varpi^{\mathrm{PD}} \widehat{\otimes}_{r_\varpi^+} R_\varpi^+, \quad R_\varpi = r_\varpi \widehat{\otimes}_{r_\varpi^+} R_\varpi^+,$$
$$R_\varpi^{[u]} = r_\varpi^{[u]} \widehat{\otimes}_{r_\varpi^+} R_\varpi^+, \quad R_\varpi^{(0,v]+} = r_\varpi^{(0,v]+} \widehat{\otimes}_{r_\varpi^+} R_\varpi^+, \quad R_\varpi^{[u,v]} = r_\varpi^{[u,v]} \widehat{\otimes}_{r_\varpi^+} R_\varpi^+,$$

where the $\widehat{\otimes}$ is the tensor product completed for the p-adic topology.

Remark 2.5. If Λ is a topological ring, and $X = \mathbf{N}, \mathbf{Z}$, let $\ell_0(X, \Lambda)$ denote the space of sequences $(x_n)_{n\in X}$ of elements of Λ such that $x_n \to 0$ when $n \to \infty$.

It is plain from the definitions that:

- $(x_n)_{n\in\mathbf{N}} \mapsto \sum_{n\in\mathbf{N}} x_n X_0^{-n}$ induces a surjection $\ell_0(\mathbf{N}, R_\varpi^+) \to R_\varpi$,
- $(x_n)_{n\in\mathbf{N}} \mapsto \sum_{n\in\mathbf{N}} x_n p^{\lceil nv/e\rceil} X_0^{-n}$ induces a surjection $\ell_0(\mathbf{N}, R_\varpi^+) \to R_\varpi^{(0,v]+}$,
- $(x_n)_{n\in\mathbf{Z}} \mapsto \sum_{n\geq 0} x_n p^{\lceil nv/e\rceil} X_0^{-n} + \sum_{n<0} x_n p^{\lceil nu/e\rceil} X_0^{-n}$ induces a surjection $\ell_0(\mathbf{Z}, R_\varpi^+) \to R_\varpi^{[u,v]}$.

If $N \geq 1$ and $u \leq v < \frac{e}{N}$, the same is true with r_ϖ^+ replaced by $R_\varpi^+[[\frac{p}{X_0^N}]]$.

Now, let $\lambda : R_\varpi^+ \to R_\varpi^+[[\frac{p}{X_0^N}]]$ be a continuous morphism such that $\frac{\lambda(X_0)}{X_0^A}$ is a unit in $R_\varpi^+[[\frac{p}{X_0^N}]]$, with $A \geq 1$. The above descriptions of R_ϖ^{deco} allow to extend λ by continuity to morphisms $R_\varpi \to R_\varpi$ and $R_\varpi^{(0,v]+} \to R_\varpi^{(0,v/A]+}$, $R_\varpi^{[u,v]} \to R_\varpi^{[u/A,v/A]}$, if $u \leq v < \frac{e}{N}$.

2.2.5. *Filtration.* We filter all the above rings S by order of vanishing at $X_0 = \varpi$.

• If P is invertible in $S[\frac{1}{p}]$ (this is the case if $S = R_\varpi^{(0,v]+}$ or $S = R_\varpi^{(0,v]}$ and $v < 1$, or if $S = R_\varpi^{[u,v]}$ and $1 \notin [u, v]$, or if $S = R_\varpi$), put the trivial filtration on S (i.e. $F^r S = S$ for all r).

• If P is not invertible in $S[\frac{1}{p}]$ (i.e. in all the cases that are not listed above), we have a natural embedding $S \to R[\frac{1}{p}][[P]] = R[\frac{1}{p}][[X_0 - \varpi]]$ by completing $S[\frac{1}{p}]$ for the P-adic topology (the completion of $\mathscr{O}_F[X_0, \frac{1}{p}]$ for the P-adic topology is the complete discrete valuation ring $K[[P]]$, and P or $X_0 - \varpi$ are uniformizers). We use this embedding to endow S with the natural filtration of $R[[P]]$ (by powers of P or, equivalently, $X_0 - \varpi$).

Remark 2.6. An element $f \in r_\varpi^{\mathrm{PD}}$ (resp. $f \in r_\varpi^{[u]}$) can be written (uniquely) in the form $f = f^+ + f^-$ with $f^+ \in F^r r_\varpi^{\mathrm{PD}}$ and f^- of degree $\leq re - 1$ (this implies $f^- \in \frac{1}{(r-1)!}\mathscr{O}_F[X_0]$, resp. $f^- \in \frac{1}{p^{\lceil ru\rceil}}\mathscr{O}_F[X_0]$).

It follows that we can write any $f \in R_\varpi^{\mathrm{PD}}$ as $f_1 + f_2$ with $f_1 \in F^r R_\varpi^{\mathrm{PD}}$ and $f_2 \in \frac{1}{(r-1)!} R_\varpi^+$ and we have the same statement for $f \in R_\varpi^{[u]}$, with $f_1 \in F^r R_\varpi^{[u]}$ and $f_2 \in \frac{1}{p^{\lceil ru\rceil}} \mathrm{R}_\varpi^+$.

2.2.6. *Frobenius.* A Frobenius φ on R_ϖ is a ring homomorphism lifting $x \mapsto x^p$ on R_ϖ/p (in particular, φ restricts to the absolute Frobenius on $\mathscr{O}_F$). We say that φ is *admissible* if:

- $\varphi(r_\varpi) \subset r_\varpi$,
- there exist $v_0 > 1$ such that $\varphi(R_\varpi^{(0,v_0]+}) \subset R_\varpi^{(0,v_0/p]+}$, and $v^{(0,v_0/p]}\big(\frac{\varphi(X_j)}{X_j^p} - 1\big) > \inf(\frac{1}{p-1}, \frac{1}{2})$, if $0 \leq j \leq d$.

(The second condition ensures that $\varphi(x) - x^p$ has divided powers if $p \neq 2$. If $p = 2$, it ensures that it has higher divided powers: $\frac{(\varphi(x)-x^p)^k}{[k/2]!}$ is defined.) If φ is admissible then, for all $v \leq v_0$ (v_0 as above), we have $\varphi(R_\varpi^{(0,v]+}) \subset R_\varpi^{(0,v/p]+}$, and φ extends by continuity to $R_\varpi^{[u,v]}$, for all $0 < u \leq v \leq v_0$. Moreover, the above minoration is valid with $v^{(0,v_0/p]}$ replaced by $v^{(0,v/p]}$ or $v^{[u/p,v/p]}$, accordingly.

We are going to use two admissible Frobenii: φ_{Kum} and φ_{cycl}. The two maps φ_{Kum} and φ_{cycl} are very similar (they differ only on the arithmetic variable). The advantage of φ_{Kum} is that it is already defined on R^+_{ϖ} (which is not the case of φ_{cycl}) and can be used to compute crystalline and syntomic cohomology. The reason for considering φ_{cycl} is that it is the Frobenius coming from the theory of (φ,Γ)-modules and it is closely related to étale cohomology (see Proposition 4.13), through the works of Herr [32] (in dimension 0), Andreatta-Iovita [4] (in the case of good reduction and without divisor at infinity), and Kedlaya-Liu [37] (for the general case). It is not inconceivable that one could use Breuil-Kisin modules [39] or Caruso's (φ,τ)-modules [15] instead of (φ,Γ)-modules (and hence φ_{Kum} instead of φ_{cycl}) to compute étale cohomology.

2.2.7. *The Frobenius φ_{Kum} and its left inverse ψ_{Kum}.* We define φ_{Kum} on $R^+_{\varpi,\square}$ by $\varphi_{\mathrm{Kum}}(X_i)=X_i^p$ if $0\le i\le d$. As R^+_{ϖ} is étale over $R^+_{\varpi,\square}$, φ_{Kum} extends, uniquely, to R^+_{ϖ} (use Remark 2.2 with $\Lambda_1=R^+_{\varpi,\square}$, $\Lambda_1'=\Lambda_2=R^+_{\varpi}$, $\lambda=\varphi_{\mathrm{Kum}}$, $I=(p)$ and $Z_\lambda=Z^p$). Finally, if $S=R_{\varpi,\square},R_{\varpi}$, φ_{Kum} extends, by continuity, to ring endomorphisms of S, S^{PD}, $S^{[u]}$ and to ring morphisms $S^{[u,v]}\to S^{[u,v/p]}$.

If $0\le j\le d$, set

$$\partial_{\mathrm{Kum},j}=X_j\tfrac{\partial}{\partial X_j}.$$

These are well defined on $R^+_{\varpi,\square}$, hence also on R^+_{ϖ} by étaleness. We extend them by the obvious rule on $R^+_{\varpi}[X_0^{-1}]$, and by linearity and continuity to $R^{\mathrm{deco}}_{\varpi}$, for deco $\in\{\ ,\mathrm{PD},[u],(0,v]+,[u,v]\}$. The resulting operators commute pairwise.

If $\alpha=(\alpha_0,\dots,\alpha_d)\in\{0,\dots,p-1\}^{[0,d]}$, set

$$u_{\mathrm{Kum},\alpha}=X_0^{\alpha_0}\cdots X_d^{\alpha_d}.$$

We have

$$\partial_{\mathrm{Kum},j}u_{\mathrm{Kum},\alpha}=\alpha_j\,u_{\mathrm{Kum},\alpha}\quad\text{and}\quad\varphi^{\mathrm{cycl}}(u_{\mathrm{Kum},\alpha})=u^p_{\mathrm{Kum},\alpha}.$$

Lemma 2.7. (i) *Any $x\in R_{\varpi}/p$ can be written uniquely as $x=\sum_\alpha c_{\mathrm{Kum},\alpha}(x)$, with $\partial_{\mathrm{Kum},j}c_{\mathrm{Kum},\alpha}(x)=\alpha_jc_{\mathrm{Kum},\alpha}(x)$, if $0\le j\le d$.*

(ii) *There exists a unique $x_\alpha\in R_{\varpi}/p$ such that $c_{\mathrm{Kum},\alpha}(x)=x_\alpha^pu_{\mathrm{Kum},\alpha}$.*

(iii) *If $x\in R^+_{\varpi}/p$, then $c_{\mathrm{Kum},\alpha}(x)\in R^+_{\varpi}/p$, $x_\alpha\in X_0^{-h-1}R^+_{\varpi}/p$, and $x_\alpha\in R^+_{\varpi}/p$, if $\alpha_i=0$ for $i\ne0$; in particular, $x_0\in R^+_{\varpi}/p$.*

Proof. Let $S=R_{\varpi}/p$, $S^+=R^+_{\varpi}/p$, and $\partial_j=\partial_{\mathrm{Kum},j}$, for $0\le j\le d$.

If $0\le j\le d$, then $\partial_j(\partial_j-1)\cdots(\partial_j-(p-1))$ is identically 0 on $R_{\varpi,\square}/p$, hence also on S by étaleness. It follows that ∂_j is diagonalizable for all j and, since these operators commute pairwise, that we can decompose S and S^+ into the direct sum of common eigenspaces. This proves (i) as well as the the fact that $c_{\mathrm{Kum},\alpha}(x)\in S^+$ if $x\in S^+$.

Now, $X_0,\dots,X_d$ is a basis of the module of differentials of $R_{\varpi,\square}/p$, hence also of S, since S is obtained as the completion of an étale algebra over $R_{\varpi,\square}/p$. It follows from [66] that $X_0,\dots,X_d$ is a p-basis of S which can be rephrased by saying that any element x of S can be written uniquely as $x=\sum_\alpha x_\alpha^pu_{\mathrm{Kum},\alpha}$. Since $\partial_j(x_\alpha^pu_{\mathrm{Kum},\alpha})=\alpha_jx_\alpha^pu_{\mathrm{Kum},\alpha}$, this proves (ii).

Finally, $(X_0\cdots X_dx_\alpha)^p=X_0^{p-\alpha_0}\cdots X_d^{p-\alpha_d}c_{\mathrm{Kum},\alpha}(x)\in S^+$, hence $X_0\cdots X_dx_\alpha\in S^+$ (because S^+ is integrally closed), which implies $X_0^{h+1}X_{a+b+1}\cdots X_dx_\alpha\in S^+$. But $x_\alpha\in S=S^+[X_0^{-1}]$ and X_0 is not a zero divisor in $S^+/(X_{a+b+k})$, if $1\le k\le d-a-b$; hence $x_\alpha\in X_0^{-h-1}S^+$, as wanted.

In the case $\alpha_i=0$ if $i\ne0$, we get $(X_0x_\alpha)^p=X_0^{p-\alpha_0}c_{\mathrm{Kum},\alpha}(x)\in S^+$, hence $X_0x_\alpha\in S^+$. But $(X_0x_\alpha)^p$ is actually divisible par X_0, since $\alpha_0\le p-1$, hence $X_0x_\alpha\in X_0S^+$ and $x_\alpha\in S^+$. □

Corollary 2.8. (i) *Any $x\in R_{\varpi}$ can be written uniquely as $\sum_\alpha\varphi_{\mathrm{Kum}}(x_\alpha)u_{\mathrm{Kum},\alpha}$, with $x_\alpha\in R_{\varpi}$.*

(ii) *If $x \in R_\varpi^+$, then $x_0 \in R_\varpi^+$ and, if $c_{\mathrm{Kum},\alpha}(x) = \varphi_{\mathrm{Kum}}(x_\alpha)u_{\mathrm{Kum},\alpha}$, then $c_{\mathrm{Kum},\alpha}(x) \in R_\varpi^+$ for all α and*

$$\partial_{\mathrm{Kum},j} c_{\mathrm{Kum},\alpha}(x) - \alpha_j c_{\mathrm{Kum},\alpha}(x) \in pR_\varpi^+, \qquad \text{if } 0 \le j \le d.$$

We define ψ_{Kum} on R_ϖ and R_ϖ^+ by the formula:

$$\psi_{\mathrm{Kum}}(x) = \varphi_{\mathrm{Kum}}^{-1}(c_{\mathrm{Kum},0}(x)).$$

A more conceptual definition of ψ_{Kum} is as follows: $R_\varpi^+[X_0^{-1}]$ is an extension of degree p^{d+1} of $\varphi_{\mathrm{Kum}}(R_\varpi^+[X_0^{-1}])$, with basis the $u_{\mathrm{Kum},\alpha}$'s. and since $\varphi_{\mathrm{Kum}}(u_{\mathrm{Kum},\alpha}) = u_{\mathrm{Kum},\alpha}^p$, we have

$$\psi_{\mathrm{Kum}}(x) = p^{-(d+1)}\varphi_{\mathrm{Kum}}^{-1}\big(\mathrm{Tr}_{S/\varphi_{\mathrm{Kum}}(S)}(x)\big).$$

Note that ψ_{Kum} is not a ring morphism; it is a left inverse to φ_{Kum} and, more generally, we have $\psi_{\mathrm{Kum}}(\varphi_{\mathrm{Kum}}(x)y) = x\psi_{\mathrm{Kum}}(y)$. We have

$$\partial_{\mathrm{Kum},i} \circ \varphi_{\mathrm{Kum}} = p\,\varphi_{\mathrm{Kum}} \circ \partial_{\mathrm{Kum},i} \quad \partial_{\mathrm{Kum},i} \circ \psi_{\mathrm{Kum}} = p^{-1}\,\psi_{\mathrm{Kum}} \circ \partial_{\mathrm{Kum},i} \quad \text{if } i = 0, 1, \dots, d$$

The above explicit formula for ψ_{Kum} extends, by continuity, to $R_\varpi^{(0,v]+}$, $R_\varpi^{(0,v]}$ and to maps $R_\varpi^{[u]} \to R_\varpi^{[pu]}$, $R_\varpi^{[u,v]} \to R_\varpi^{[pu,pv]}$ (surjective in all cases since $\psi_{\mathrm{Kum}} \circ \varphi_{\mathrm{Kum}} = \mathrm{id}$). The maps $x \mapsto c_{\mathrm{Kum},\alpha}(x)$ also extend and lead to decompositions $S = \oplus_\alpha S_\alpha$, for $S = R_\varpi^{\mathrm{deco}}$, with deco $\in \{\ ,+,[u],(0,v]+,[u,v]\}$. Since $\psi_{\mathrm{Kum}}(x) = \varphi_{\mathrm{Kum}}^{-1}(c_{\mathrm{Kum},0}(x))$, we have

$$S^{\psi_{\mathrm{Kum}}=0} = \oplus_{\alpha\neq 0} S_\alpha.$$

Lemma 2.9. *If $S = R_\varpi^{\mathrm{deco}}$, and* deco $\in \{\ ,+,[u],(0,v]+,[u,v]\}$, *then $\partial_{\mathrm{Kum},j} = \alpha_j$ on $S_\alpha/pS_\alpha^{\mathrm{deco}}$.*

Proof. If deco $\in \{\ ,+\}$, this is part of Corollary 2.8. If deco $\in \{[u],(0,v]+,[u,v]\}$, elements of S^{deco} are those of the form $\sum_{k\in\mathbf{Z}} p^{r_k}X_0^k a_k$, where $a_k \in S^+$ goes to 0 when $k \to \infty$ and r_k is determined by "deco". Now, if $x = \sum_{k\in\mathbf{Z}} p^{r_k}X_0^k a_k$, then $c_{\mathrm{Kum},\alpha}(x) = \sum_{k\in\mathbf{Z}} p^{r_k}X_0^k c_{\mathrm{Kum},(\alpha_0-k,\alpha_1,\dots,\alpha_d)}(a_k)$, where $\alpha_0 - k$ is to be understood as its representative modulo p between 0 and $p-1$. So, if $x \in S_\alpha$, we can assume that $a_k \in S^+_{(\alpha_0-k,\alpha_1,\dots,\alpha_d)}$, for all k. A direct computation, using the result for S^+, shows then that $\partial_{\mathrm{Kum},j}(X_0^k a_k) - \alpha_j X_0^k a_k \in pS^+$, which allows to conclude. □

2.3. **Cyclotomic theory.** Suppose that K contains enough roots of unity throughout this subsection.

2.3.1. *The cyclotomic variable T.* Let

$$i = i(K), \quad e_0 = [F_i : F] = (p-1)p^{i(K)-1}, \quad f = [K : F_i] = \tfrac{e}{e_0}, \quad \text{and} \quad \zeta = \zeta_{p^i}.$$

As $\zeta - 1$ is a uniformiser of F_i, we can find polynomials $A_0, \dots, A_{f-1} \in \mathscr{O}_F[T]$ with $v_T(A_i) \ge 1$ and $v_T(A_0) = 1$, such that, if

$$Q(X_0, T) = X_0^f + A_{f-1}(T)X_0^{f-1} + \cdots + A_0(T),$$

then $Q(X_0, \zeta - 1)$ is the minimal polynomial of ϖ over F_i (of course the A_i are not uniquely determined by these requirements).

Let $T \in r_\varpi^+$ be the solution of $Q(X_0, T) = 0$. Since $A_0 = \alpha T + \cdots$, with α a unit in $\mathscr{O}_F^*$, and $v_T(A_i) \ge 1$ if $i \ge 1$, one gets (by successive approximations) that $T = -\alpha^{-1}X_0^f + \alpha_{f+1}X_0^{f+1} + \cdots \in r_\varpi^+$. In particular, $X_0^{-f}T$ *is a unit in* r_ϖ^+. We have

$$r_\varpi^+ = \mathscr{O}_F[[T, X_0]]/Q(X_0, T) = \oplus_{i=0}^{f-1}\mathscr{O}_F[[T]]X_0^i,$$

and r_ϖ is an étale extension of $\mathscr{O}_F[[T]]\{T^{-1}\}$. This is a consequence of the following, more precise, lemma.

Lemma 2.10. *Let $Q'(X_0,T)=\frac{\partial Q}{\partial X_0}(X_0,T)$.*

(i) *There exists $U\in\mathscr{O}_F[X_0]$ which is a unit in $\mathscr{O}_F[[X_0]]$, $V,W\in\mathscr{O}_F[X_0,T]$, such that $Q'(X_0,T)=X_0^{\delta_K}U+pV+Q(X_0,T)W$.*

(ii) *The image Δ of $Q'(X_0,T)$ in r_ϖ^+ is invertible in r_ϖ and $X_0^{-\delta_K}\Delta$ is a unit in $\mathscr{O}_F[[X_0,\frac{p}{X_0^{\delta_K}}]]$.*

Proof. We have $v_p(\mathfrak{d}_{K/F_i})=v_p(Q'(\varpi,\zeta-1))$. Now, since $v_p(\mathfrak{d}_{K/F_i})<1$ by assumption, we have

$$\delta_K=e\,v_p(Q'(\varpi,\zeta-1))=\inf_{(k,p)=1}(f v_T(\overline{A}_k)+(k-1))=v_{X_0}\big(\overline{Q}'(X_0,T)\big),$$

where $\overline{}$ is the reduction modulo p. Hence $Q'(X_0,T)$ is an element of r_ϖ^+ congruent to $\alpha X_0^{\delta_K}$ mod $(p,X_0^{\delta_K+1},Q)$, where α is a unit in $\mathscr{O}_F$. The lemma follows. □

2.3.2. *Filtration.* The morphism $r_\varpi^+\to\mathscr{O}_K$ sends T to $\zeta-1$. Hence the subring $\mathscr{O}_F[[T]]$ of r_ϖ^+ can be thought of as $r_{\zeta-1}^+$ and $\mathscr{O}_F[[T]]\{T^{-1}\}$ as $r_{\zeta-1}$. In particular, the filtrations on r_ϖ^+ and its siblings r_ϖ^{PD}, etc. are also given by the order of vanishing at $T=\zeta-1$, which means that *we can use*

$$P_0=\frac{(1+T)^{p^i}-1}{(1+T)^{p^{i-1}}-1}$$

instead of P to define the filtrations.

Set $t=p^i\log(1+T)$.

Lemma 2.11. *If $\frac{p-1}{p}\le u\le\frac{v}{p}<1<v$ and*[6] *$\frac{1}{p}<u$, then:*

(i) *t belongs to $p\,r_\varpi^{[u,v]}$ and to $p\,r_\varpi^{[u,v/p]}$,*

(ii) *$x\mapsto t^rx$ induces a p^r-isomorphism $r_\varpi^{[u,v]}\simeq F^r r_\varpi^{[u,v]}$ and a p^{2r}-isomorphism $r_\varpi^{[u,v/p]}\simeq r_\varpi^{[u,v/p]}$.*

Proof. For (i), we start from $t=\sum_{k\ge1}\frac{(-1)^{k-1}p^i}{k}T^k$, so we have to estimate $\inf_{s\in[u/e,v/e]}\inf_{k\ge1}(i+ks-v_p(k))$. The minimum is reached for $s=u/e$ and $k=p^i$, and is $\frac{pu}{p-1}\ge1$, which means that each of the terms of the series belongs to $p\,r_\varpi^{[u,v]}$ (and, a fortiori, to $p\,r_\varpi^{[u,v/p]}$).

For (ii), we first check that $f=\big(\frac{(1+T)^{p^i}-1}{(1+T)^{p^{i-1}}-1}\big)^{-1}t$ does not vanish on $\frac{u}{e}\le v_p(T)\le\frac{v}{e}$ and t does not vanish on $\frac{u}{e}\le v_p(T)\le\frac{v}{pe}$ (this is ensured by the inequalities put on u and v, as the valuations of zeroes of t are of the form $p^k\frac{1}{e}$ for $k\in\mathbf{Z}$, $k\le i-1$, or $+\infty$). This implies that f is a unit in $r_\varpi^{[u,v]}[\frac{1}{p}]$ and t is a unit in $r_\varpi^{[u,v/p]}[\frac{1}{p}]$. To conclude we need to show that $f^{-1}\in p^{-1}r_\varpi^{[u,v]}$ and $t^{-1}\in p^{-2}r_\varpi^{[u,v/p]}$. Equivalently, we need to check that, if $s\in[u/e,v/e]$ (resp. $s\in[u/e,v/pe]$), $\inf_{v_p(T)=s}v_p(f(T)^{-1})\ge-1$ (resp. $\inf_{v_p(T)=s}v_p(t^{-1})\ge-2$). By additivity of the valuation on a circle, this amounts to checking that $\inf_{v_p(T)=s}v_p(f(T))\le1$ (resp. $\inf_{v_p(T)=s}v_p(t)\ge2$).

• For t on $s\in[u/e,v/pe]$, we have to estimate $\max_{s\in[u/e,v/pe]}\inf_{k\ge1}(i+ks-v_p(k))$. The maximum is reached for $s=\frac{v}{pe}$ and, taking $k=p^i$, we see that it is $\le\frac{p^iv}{pe}<\frac{p}{p-1}\le2$.

• For f on $s\in[u/e,v/e]$, we use the formula $f=((1+T)^{p^{i-1}}-1)\prod_{n\ge i+1}\frac{(1+T)^{p^n}-1}{p((1+T)^{p^{n-1}}-1)}$. Again the maximum of $\inf_{v_p(T)=s}v_p(f(T))$ is obtained for $s=\frac{v}{pe}$ and, since $1>\frac{v}{p}>\frac{1}{p}$, we have

$$\inf_{v_p(T)=s}v_p\big(\frac{(1+T)^{p^n}-1}{p((1+T)^{p^{n-1}}-1)}\big)=0,\quad\text{if } n\ge i+1$$

(the constant term, i.e. 1, has the minimum valuation because $\frac{v}{p}>\frac{1}{p}$), and (because $\frac{v}{p}<1$)

$$\inf_{v_p(T)=s}v_p\big((1+T)^{p^{i-1}}-1\big)\le p^{i-1}s\le\frac{1}{p-1}\le1.$$

This allows to conclude. □

[6] This is an additional condition only for $p=2$.

2.3.3. *Extension of morphisms.* Let $R^+_{\zeta-1,\square} = \mathscr{O}_F[[T]]\{X_1,\dots,X_d\}$. We have

$$R^+_{\varpi,\square} = R^+_{\zeta-1,\square}\{X_0, X_{d+1}, X_{d+2}\}/\big(Q(X_0,T), X_{d+1}X_{a+1}\cdots X_{a+b} - X_0^h, X_{d+2}X_1\cdots X_a - 1\big).$$

Let $\mathscr{M}$ be the monoid generated by $X_0, X_1, \dots, X_{d+2}$, and let[7] $\delta_R = \lceil \frac{\delta_K}{f} \rceil$, so that

$$f\delta_R - \delta_K \geq 0 \quad \text{and} \quad \delta_R + 1 < \tfrac{e_0}{2p} = \tfrac{1}{2}(p-1)p^{i-2}, \text{ since } K \text{ has enough roots of unity.}$$

Proposition 2.12. *Let:*

- $\lambda : R^+_{\zeta-1,\square} \to \Lambda$, *a continuous ring morphism, such that* $\lambda(T)^{2\delta_R}$ *divides* p *and* Λ *is separated and complete for the* $\big(\frac{p}{\lambda(T)^{2\delta_R}}\big)$*-adic topology.*
- $\beta : \mathscr{M} \to \Lambda$, *a morphism of monoids, with* $\beta(X_i) = u_i\lambda(X_i)$, *if* $1 \leq i \leq d$, $\beta(X_0)^f = \lambda(T)u$, *where* $u_i, u \in \Lambda^*$,
- $\mu \in r^+_{\zeta-1}$ *such that* $\frac{\lambda(\mu)}{\lambda(T)^{2\delta_R}} \in \Lambda$ *and is topologically nilpotent in* Λ.

Suppose that:

- $\lambda(\mu)$ *divides* $Q(\beta(X_0), \lambda(T))$,
- *There exists an ideal* J *of* Λ, *containing* $\frac{\lambda(\mu)}{\lambda(T)^{1+\delta_R}}$, *such that* Λ *is separated and complete for the* J*-adic topology, and such that* $\lambda : R^+_{\zeta-1,\square} \to \Lambda/J$ *extends to* $\lambda_\varpi : R^+_\varpi \to \Lambda/J$, *with* $\lambda_\varpi = \beta$ *on* $\mathscr{M}$.

Then λ *admits a unique extension to* R^+_ϖ, *lifting* λ_ϖ, *with* $\lambda(X_0) \in \beta(X_0) + \frac{\lambda(\mu)}{\lambda(T)^{\delta_R}}\Lambda$.

Proof. We will proceed in three steps, extending (uniquely) λ first from $r^+_{\zeta-1}$ to r^+_ϖ (this boils down to defining $\lambda(X_0)$), then from $r^+_\varpi\{X_1,\dots,X_d\}$ to $R^+_{\varpi,\square}$, and finally from $R^+_{\varpi,\square}$ to R^+_ϖ.

For the first step, we use Proposition 2.1, with:

- $\Lambda_1 = r^+_{\zeta-1}$, $\Lambda'_1 = r^+_\varpi$ (and $J = Q'(X_0,T)$), $\Lambda_2 = \Lambda$,
- $z = \lambda(T)^{\delta_R}$, $Z_\lambda = \beta(X_0)$, $H_\lambda = J^\lambda(Z_\lambda)^{-1} = Q'(\beta(X_0), \lambda(T))^{-1}$, and $I = (\lambda(\mu))$.

(We have $Q^\lambda(Z_\lambda) = Q(\beta(X_0), \lambda(T)) \in I$ by assumption, and $H_\lambda J^\lambda(Z_\lambda) - 1 = 0$, so the requirements of that proposition are satisfied provided we show that $H_\lambda \in z^{-1}\Lambda$. But $H_\lambda = Q'(\beta(X_0), \lambda(T))^{-1}$ and there exists a unit U of $\mathscr{O}_F[[X_0]]$ and $V, W \in \mathscr{O}_F[[X_0,T]]$ such that $Q'(X_0,T) = X_0^{\delta_K}U + pV + Q(X_0,T)W$ (Lemma 2.10). Hence, setting $\alpha = \beta(X_0)^{f\delta_R - \delta_K}u^{\delta_R} \in \Lambda$, one can write zH_λ as

$$zH_\lambda = \alpha\Big(U(\beta(X_0)) + \alpha\big(\tfrac{p}{\lambda(T)^{\delta_R}}V(\beta(X_0),\lambda(T)) + \tfrac{Q(\beta(X_0),\lambda(T))}{\lambda(T)^{\delta_R}}W(\beta(X_0),\lambda(T))\big)\Big)^{-1},$$

and the expression in the big parenthesis is indeed a unit in Λ since $U(\beta(X_0))$ is, and $\frac{Q(\beta(X_0),\lambda(T))}{\lambda(T)^{\delta_R}}$ and $\frac{p}{\lambda(T)^{\delta_R}}$ are divisible by $\frac{\lambda(\mu)}{\lambda(T)^{2\delta_R}}$ and $\frac{p}{\lambda(T)^{2\delta_R}}$ which are assumed to be topologically nilpotent in Λ.) In particular, $\lambda(X_0) \in \beta(X_0) + \frac{\lambda(\mu)}{\lambda(T)^{\delta_R}}\Lambda$.

For the second step, we must (and can) extend λ by setting:

$$\lambda(X_{d+1}) = \big(\tfrac{\lambda(X_0)}{\beta(X_0)}\big)^h(u_{a+1}\cdots u_{a+b})^{-1}\beta(X_{d+1}), \quad \lambda(X_{d+2}) = (u_1\cdots u_a)^{-1}\beta(X_{d+2}).$$

(Note that $\beta(X_0)$ divides $\lambda(T)$ by assumption, hence $\frac{\lambda(X_0)}{\beta(X_0)} \in 1 + \frac{\lambda(\mu)}{\lambda(T)^{1+\delta_R}}\Lambda \subset \Lambda$ makes sense.)

For the third step, we use Remark 2.2 with $\Lambda_1 = R^+_{\varpi,\square}$, $\Lambda'_1 = R^+_\varpi$, $\Lambda_2 = \Lambda$, $Z_\lambda = (Z_{\lambda,1},\dots,Z_{\lambda,t})$ where $Z_{\lambda,i}$ is an arbitrary lift of $\lambda_\varpi(Z_i)$, and $I = J$.

□

[7] One could consider setting $\delta_R = \lceil \frac{\delta_K}{f} \rceil$ if $b = 0$ and $\delta_R = \lceil \frac{\delta_K}{f} \rceil + 1$ if $b \neq 0$ in order to differentiate between good and semi-stable reduction. Such a distinction would only be useful when K is unramified, so we will not bother.

2.3.4. *The Frobenius* φ_{cycl}. We endow $R^+_{\zeta-1,\square}$ with the Frobenius φ_{cycl} sending T to $(1+T)^p-1$ and X_i to X_i^p, if $1\leq i\leq d$. Then $\varphi_{\mathrm{cycl}}(x)-x^p\in pR^+_{\zeta-1,\square}$, if $x\in R^+_{\zeta-1,\square}$. Let

$$v_R=\tfrac{e_0}{2p\delta_R}=\tfrac{e}{2pf\delta_R}\quad\text{and}\quad R_\varpi^{(0,v_R)}=R^+_\varpi[[\tfrac{p}{T^{2p\delta_R}}]]=R^+_\varpi[[\tfrac{p}{X_0^{2pf\delta_R}}]],\ R_\varpi^{(0,pv_R)}=R^+_\varpi[[\tfrac{p}{T^{2\delta_R}}]]=R^+_\varpi[[\tfrac{p}{X_0^{2f\delta_R}}]].$$

Note that $v_R>1$, by assumption.

Proposition 2.13. *φ_{cycl} admits a unique extension to R^+_ϖ such that*

$$\varphi_{\mathrm{cycl}}(X_0)-X_0^p\in\tfrac{p}{T^{p\delta_R}}R_\varpi^{(0,v_R)}\quad\text{and}\quad\varphi_{\mathrm{cycl}}(x)-x^p\in\tfrac{p}{T^{p(1+\delta_R)}}R_\varpi^{(0,v_R)},\ \textit{if } x\in R^+_\varpi.$$

Proof. This is a direct consequence of Proposition 2.12, applied to $\Lambda=R_\varpi^{(0,v_R)}$, $\lambda=\varphi_{\mathrm{cycl}}$, $\beta(x)=x^p$, $\mu=p$ and $\lambda_\varpi(x)=x^p$, $J=\big(\frac{p}{T^{p(1+\delta_R)}}\big)$, taking into account that $\frac{\varphi_{\mathrm{cycl}}(T)}{T^p}=\frac{(1+T)^p-1}{T^p}$ is a unit in Λ. (We have $Q(X_0^p,(1+T)^p-1)\equiv Q(X_0,T)^p=0$ mod p, hence μ divides $Q(\beta(X_0),\lambda(T))$, which shows that the assumptions of Proposition 2.12 are indeed fullfilled.) □

Now, $\frac{\varphi_{\mathrm{cycl}}(X_0)}{X_0^p}$ is a unit in $R_\varpi^{(0,v_R)}$. Hence we can use Remark 2.5 to extend φ_{cycl} to morphisms (still noted φ_{cycl}) $R_\varpi\to R_\varpi$, $R_\varpi^{(0,v]+}\to R_\varpi^{(0,v/p]+}$ and $R_\varpi^{[u,v]}\to R_\varpi^{[u/p,v/p]}$, if $u\leq v<v_R$. This Frobenius is admissible.

Lemma 2.14. *If $v<pv_R$, and if $x\in R_\varpi$ is such that $\varphi_{\mathrm{cycl}}(x)\in R_\varpi^{(0,v/p]+}$, then $x\in R_\varpi^{(0,v]+}$.*

Proof. Write φ for φ_{cycl}. We have $R_\varpi^{(0,v]+}=\sum_{n\geq 0}\frac{p^n}{X_0^{[ne/v]}}R^+_\varpi$. Now, if $\varphi(x)\in R_\varpi^{(0,v/p]+}$, this implies, in particular, that the image $\overline{x}$ of x in R_ϖ/p is such that $\overline{x}^p\in R^+_\varpi/p$; hence $\overline{x}\in R^+_\varpi/p$. This means that we can find $a_0\in R^+_\varpi$ such that $x-a_0\in pR_\varpi$. But, $\varphi(a_0)\in R_\varpi^{(0,v/p]+}$ thanks to the assumption $v<pv_R$, which means that $\varphi(x-a_0)\in\sum_{n\geq 1}\frac{p^n}{X_0^{[ne/v]}}R^+_\varpi$. It follows that, if we write $x=a_0+\frac{p}{X_0^{[e/v]+1}}x_1$, the image of $\varphi(x_1)$ in R_ϖ/p belongs to $X_0R^+_\varpi/p$ (since $p\big([e/v]+1\big)-[pe/v]\geq 1$), hence the image of x_1 belongs to $X_0R^+_\varpi/p$, and we can find $a_1\in R^+_\varpi$ such that $x_1-X_0a_1\in pR_\varpi$. But then $\varphi\big(x-a_0-\frac{p}{X_0^{[e/v]}}a_1\big)\in\sum_{n\geq 2}\frac{p^n}{X_0^{[ne/v]}}R^+_\varpi$, and we can iterate the process, writing $x=a_0+\frac{p}{X_0^{[e/v]}}a_1+\frac{p^2}{X_0^{[2e/v]+1}}x_2$, and deducing that $x_2=X_0a_2+py$, with $a_2\in R^+_\varpi$ and $y\in R_\varpi$. Going to the limit, we find $x=\sum_{n\in\mathbf{N}}a_n\frac{p^n}{X_0^{[ne/v]}}$, with $a_n\in R^+_\varpi$, for all n, which concludes the proof. □

2.3.5. *The operator ψ_{cycl}.* Let

$$u_{\mathrm{cycl},\alpha}=(1+T)^{\alpha_0}X_1^{\alpha_1}\cdots X_d^{\alpha_d},\qquad\text{if } \alpha\in\{0,1,\dots,p-1\}^{[0,d]}.$$

Proposition 2.15. (i) *Any $x\in R_\varpi$ can be written uniquely $x=\sum_\alpha c_{\mathrm{cycl},\alpha}(x)$, with $c_{\mathrm{cycl},\alpha}(x)\in\varphi_{\mathrm{cycl}}(R_\varpi)u_{\mathrm{cycl},\alpha}$.*

(ii) *If $v<v_R$, and $x\in R_\varpi^{(0,v]+}$, then $x_{\mathrm{cycl},\alpha}\in X_0^{-p\delta_K}R_\varpi^{(0,v]+}$, for all α.*

Before turning to the proof of this proposition, let us draw some consequences. We define the left inverse ψ_{cycl} of φ_{cycl}, on R_ϖ, by the formula

$$\psi_{\mathrm{cycl}}(x)=\varphi_{\mathrm{cycl}}^{-1}(c_{\mathrm{cycl},0}(x)).$$

Since $\varphi_{\mathrm{cycl}}(u_{\mathrm{cycl},\alpha})=u_{\mathrm{cycl},\alpha}^p$ for all α, we have $\mathrm{Tr}_{R_\varpi/\varphi_{\mathrm{cycl}}(R_\varpi)}(u_{\mathrm{cycl},\alpha})=0$ if $\alpha\neq 0$, and we can define ψ_{cycl}, intrinsically, by the formula

$$\psi_{\mathrm{cycl}}=p^{-(d+1)}\varphi_{\mathrm{cycl}}^{-1}\circ\mathrm{Tr}_{R_\varpi/\varphi_{\mathrm{cycl}}(R_\varpi)}.$$

Proposition 2.16. *If $v<pv_R$, then:*

(i) $\psi_{\mathrm{cycl}}\big(T^{-pN}R_\varpi^{(0,v/p]+}\big)\subset T^{-N-\delta_R}R_\varpi^{(0,v]+}$.

(ii) *If $\ell_R=\lceil\frac{p}{p-1}\delta_R\rceil$, then $T^{-\ell_R}R_\varpi^{(0,v]+}$ is stable by ψ_{cycl}.*

(iii) *If $v\leq p$, the natural map $\oplus_{\alpha\neq 0}\varphi_{\mathrm{cycl}}(R_\varpi^{(0,v]+})u_{\mathrm{cycl},\alpha}\to(R_\varpi^{(0,v/p]+})^{\psi_{\mathrm{cycl}}=0}$ is a p^{p+1}-isomorphism.*

Proof. (i) follows from (ii) of Proposition 2.15, taking into account that $\psi(\varphi(T)^{-N}x) = T^{-N}\psi(x)$, that $\frac{\varphi(T)}{T^p}$ is a unit in $R_\varpi^{(0,v/p]+}$, and that $X_0^{\delta_K}$ divides T^{δ_R}. (ii) is a direct consequence of (i) and the inclusion $R_\varpi^{(0,v]+} \subset R_\varpi^{(0,v/p]+}$.

Finally, if $x \in (R_\varpi^{(0,v/p]+})^{\psi_{\mathrm{cycl}}=0}$, we can write, using Proposition 2.15, $x = \sum_{\alpha\neq 0}\varphi(a_\alpha)u_\alpha$, with $\varphi(a_\alpha)u_\alpha \in X_0^{-p\delta_K}R_\varpi^{(0,v/p]+}$. But u_α divides $X_0^{p(h+1)}\prod_{i=a+b+1}^d X_i^p$, hence $\varphi(a_\alpha) \in X_0^{-p(h+1+\delta_K)}R_\varpi^{(0,v/p]+}$. (See the proof of Lemma 2.7 for a similar argument.) This implies, thanks to Lemma 2.14, that $a_\alpha \in X_0^{-(h+1+\delta_K)}R_\varpi^{(0,v]+}$. This proves (iii) since $X_0^{h+1+\delta_K}$ divides p^{p+1} in $R_\varpi^{(0,v]+}$, as $h \le e$, $\delta_K + 1 \le \frac{e}{p}$, and $v \le p$. □

We let

$$\partial_{\mathrm{cycl},0} = (1+T)\frac{\partial}{\partial T} \quad \text{and} \quad \partial_{\mathrm{cycl},i} = X_i\frac{\partial}{\partial X_i}, \text{ if } 1 \le i \le d.$$

Then we have

$$\partial_{\mathrm{cycl},i}\circ\partial_{\mathrm{cycl},j} = \partial_{\mathrm{cycl},j}\circ\partial_{\mathrm{cycl},i} \quad \text{and} \quad \partial_{\mathrm{cycl},i}\circ\varphi_{\mathrm{cycl}} = p\,\varphi_{\mathrm{cycl}}\circ\partial_{\mathrm{cycl},i},$$
$$\partial_{\mathrm{cycl},i}\circ\psi_{\mathrm{cycl}} = p^{-1}\,\psi_{\mathrm{cycl}}\circ\partial_{\mathrm{cycl},i}, \quad \text{if } i,j = 0,1,\dots,d.$$

Lemma 2.17. *If $v < 2pv_R$, then $\partial_{\mathrm{cycl},0}X_0 \in X_0^{-\delta_K}r_\varpi^{(0,v]+}$ and $v^{(0,v]}(\partial_0 X_0) \ge -1$.*

Proof. Write ∂_0 for $\partial_{\mathrm{cycl},0}$. As $Q(X_0,T) = 0$, we have $\frac{\partial Q}{\partial X_0}(X_0,T)\partial_0 X_0 + \frac{\partial Q}{\partial T}(X_0,T)\partial_0 T = 0$, hence

$$\partial_0 X_0 = -(1+T)\big(\frac{\partial Q}{\partial X_0}(X_0,T)\big)^{-1}\frac{\partial Q}{\partial T}(X_0,T).$$

Since $\frac{\partial Q}{\partial T}(X_0,T) \in r_\varpi^+$, the result follows from Lemma 2.10. □

2.3.6. *Proof of Proposition 2.15.* Write φ for φ_{cycl} in all that follows. We will first prove the existence of a decomposition $x = \sum_\alpha x_\alpha$ as in Proposition 2.15, but with $x_\alpha \in \varphi(R_\varpi)u_{\mathrm{Kum},\alpha}$, and then use the relationship (Lemma 2.22) between the $u_{\mathrm{Kum},\alpha}$'s and the $u_{\mathrm{cycl},\alpha}$'s to conclude.

Let $I = \{a+1,\dots,a+b,d+1\}$, so that $\prod_{i\in I}X_i = X_0^h$. If $i \in I$, let $R_{\varpi,\square,i}^+$, $R_{\varpi,i}^+$, be the rings obtained by localizing on the polycircle where X_j is a unit if $j \in I-\{i\}$ (hence X_i is X_0^h times a unit). Concretely, if $S = R_{\varpi,\square}^+, R_\varpi^+$, then $S_i = S\{Y\}/(1 - Y\prod_{j\in I-\{i\}}X_i)$. Then $R_{\varpi,i}^+$ is étale over $R_{\varpi,\square,i}^+$, and $R_{\varpi,\square,i}^+$ is smooth over $\mathscr{O}_F$ (the equation $\prod_{i\in I}X_i = X_0^h$ becomes $X_i = YX_0^h$, which allows to remove X_i from the variables and the associated space is just a product of closed circles of radius 1 and unit balls).

From $R_{\varpi,i}^+$ we can construct rings $R_{\varpi,i}^{\mathrm{deco}}$ by the same process we used to construct R_ϖ^{deco} from R_ϖ^+. Note that φ extends naturally to $R_{\varpi,i}$, and induces morphisms $R_{\varpi,i}^{(0,pv_R)} \to R_{\varpi,i}^{(0,v_R)}$, and $R_{\varpi,i}^{(0,pv]+} \to R_{\varpi,i}^{(0,v]+}$, if $v < v_R$.

Lemma 2.18. *If $x \in R_\varpi$, the following conditions are equivalent:*
(i) $x \in R_\varpi^{(0,r_R)}$,
(ii) $x \in R_{\varpi,i}^{(0,r_R)}$, *for all $i \in I$.*

Proof. There is only the implication (ii)⇒(i) to prove. Let us first show that the divisibility of x by X_0 in $R_{\varpi,i}^+/p$ for all $i \in I$ implies its divisibility in R_ϖ^+/p. Geometrically, this is equivalent to $\cup_i Z_i$ Zariski dense in Z, where $Z_i = \mathrm{Spec}(R_{\varpi,i}^+/(p,X_0))$ and $Z = \mathrm{Spec}(R_\varpi^+/(p,X_0))$. The same statement for $R_{\varpi,\square}$ instead of R_ϖ is immediate as each irreducible component of $Z_\square$ contains exactly one of the $Z_{\square,i}$'s (with obvious notations), as a Zariski open subset. The statement for R_ϖ follows by étaleness of $Z \to Z_\square$ (Z_i is the inverse image of $Z_{\square,i}$ in Z). Since $R_\varpi/p = (R_\varpi^+/p)[X_0^{-1}]$, one infers from the above statement that if $x \in R_\varpi/p$ is such that $x \in R_{\varpi,i}^+/p$ for all $i \in I$, then $x \in R_\varpi^+/p$.

Choose a section $s : R_\varpi^+/p \to R_\varpi^+$ of the reduction modulo p such that $s(0) = 0$ and $s(X_0x) = X_0s(x)$ so that s is continuous. If $x \in R_\varpi$ belongs to $R_{\varpi,i}^{(0,r_R)}$ for all i, then its image a_0 in R_ϖ/p belongs to

$R^+_{\varpi,i}/p$, for all i, and hence $a_0 \in R^+_\varpi/p$. Now, by definition of $R^{(0,r_R)}_{\varpi,i}$, if $M = 2pf\delta_R$, the image a_1 of $\frac{X_0^M}{p}(x - s(a_0))$ in R_ϖ/p belongs to $R^+_{\varpi,i}/p$ for all i, hence $a_1 \in R^+_\varpi/p$. One can iterate, and get that the image a_2 of $\frac{X_0^{2M}}{p^2}(x - s(a_0) - \frac{p}{X_0^M}s(a_1))$ belongs to $R^+_{\varpi,i}/p$ for all i, hence $a_2 \in R^+_\varpi/p$. This process gives us a sequence $(a_n)_{n\in\mathbf{N}}$ of elements of R^+_ϖ/p, such that $x = \sum s(a_n)\frac{p^n}{X_0^{nM}}$. This concludes the proof. □

Lemma 2.19. *Let $S = r_\varpi, R_{\varpi,i}$, and let $(u_j)_{j\in J}$ be a family of elements of S^+ such that:*

- *any $x \in S/p$ can be written, uniqueley, $x = \sum_j c_j(x)^p u_j$, with $c_j(x) \in S$, for all $j \in J$,*
- *there exists $N \leq 2f\delta_R$ such that, if $x \in S^+/p$, then $c_j(x) \in X_0^{-N}S^+/p$, for all j.*

Then any $x \in S$ can be written, uniquely, $x = \sum_j \varphi(c_j(x))\, u_j$, with $c_j(x) \in S$, and if $x \in S^{(0,v_R)}$, then $c_j(x) \in X_0^{-N}S^{(0,pv_R)}$ for all j.

Proof. The $c_j(x)$'s are obtained by the following algorithm:

• Choose a section $s : S^+/p \to S^+$ of the reduction $y \mapsto \overline{y}$ modulo p, such that $s(X_0y) = X_0y$, and extend s to S/p using this functional equation.

• Define $f : S^{(0,v_R)}[X_0^{-1}] \to S^{(0,v_R)}[X_0^{-1}]$ by $f(y) = \frac{1}{p}\big(y - \sum_j \varphi(s(c_j(\overline{y})))\, u_j\big)$ (that f exists rests upon the fact that $\varphi(S^+) \subset S^{(0,pv_R)}$ and the observation that $f(\varphi(X_0)^{-N}y) = \varphi(X_0)^{-N}f(y)$ thanks to the functional equation $s(X_0y) = X_0s(y)$).

• Set $x_0 = x$ and let $x_{n+1} = f(x_n)$, if $n \geq 0$, so that $x = p^{n+1}x_{n+1} + \sum_j \varphi\big(\sum_{i=0}^n p^i s(c_j(\overline{x_i}))\big)\, u_j$.

An easy induction, using the fact that $N \leq 2f\delta_R$, shows that $x_n \in X_0^{-2pnf\delta_R}S^{(0,v_R)}$ and $s(c_j(\overline{x}_n)) \in X_0^{-N-2nf\delta_R}S^{(0,pv_R)}$. Hence $c_j(x) = \sum_{i\geq 0} p^i s(c_j(\overline{x_i})) \in X_0^{-N}S^{(0,pv_R)}$. The result follows. □

If $i \in I$, let

$$A_i = \{\beta = (\beta_0, \ldots, \beta_{d+1}) \in \{0, 1, \ldots, p-1\}^{[0,d+1]},\ \beta_i = 0\}.$$

If $\beta \in A_i$, let $u_{i,\beta} = X_0^{\beta_0} \cdots X_{d+1}^{\beta_{d+1}}$. Note that $u_{d+1,\beta} = u_{\mathrm{Kum},(\beta_0,\ldots,\beta_d)}$.

Lemma 2.20. *Let $i \in I$.*

(i) *An element x of $R_{\varpi,i}$ can be written uniquely $x = \sum_{\beta\in A_i} x_{i,\beta}$, with $x_{i,\beta} \in \varphi(R_{\varpi,i})u_{i,\beta}$ for all β.*

(ii) *If $x \in R^{(0,v_R)}_{\varpi,i}$, then $x_{i,\beta} \in R^{(0,v_R)}_{\varpi,i}$ for all β.*

Proof. If $S = R^+_{\varpi,\square,i}/p$, the X_j's, for $j \neq i$, form a basis of Ω^1_S. By étaleness, this is also the case if $S = R^+_{\varpi,i}/p$. Hence, according to [66], the $u_{i,\beta}$'s, for $\beta \in A_i$, form a basis of S over $\varphi(S)$, if $S = R^+_{\varpi,i}/p$ and, as a consequence, if $S = R_{\varpi,i}/p$. It follows that we can use Lemma 2.19 (with $N = 0$) to conclude. □

If $\beta \in A_{d+1}$, let $\alpha = (\beta_0, \ldots, \beta_d)$ be the corresponding element of $\{0, \ldots, p-1\}^{[0,d]}$, and set $x_\alpha = x_{d+1,\beta}$, so that the decomposition $x = \sum_{\beta\in A_{d+1}} x_{d+1,\beta}$ becomes simply $x = \sum_\alpha x_\alpha$.

Lemma 2.21. (i) *If $x \in R_\varpi$, then $x_\alpha \in \varphi(R_\varpi)\, u_{\mathrm{Kum},\alpha}$.*

(ii) *If $v < v_R$, and if $x \in R^{(0,v]+}_\varpi$, then $x_\alpha \in R^{(0,v]+}_\varpi$, for all α.*

Proof. (i) is a direct consequence of the analogous statement modulo p (Lemma 2.7, we have $\varphi(x) = x^p$ modulo p). To prove (ii), an adaptation of Remark 2.5 shows that it is enough to prove the same statement for $R^{(0,v_R)}_\varpi$: write an element of $S^{(0,v]+}$ as $\sum_{n\in\mathbf{N}} x_n \frac{p^n}{\varphi(X_0)^{[ne/v]-d_n}}$, with $x_n \in S^{(0,v_R)}$ and $d_n \to +\infty$, and use the result for $R^{(0,v]+}_\varpi$.

According to Lemma 2.18, it is enough to show that $x_\alpha \in R^{(0,v_R)}_{\varpi,i}$, for all $i \in I$. For this purpose it is enough, considering Lemma 2.21, to show that x_α is a linear combination of the $x_{i,\beta}$'s, with coefficients in $R^{(0,v_R)}_{\varpi,i}$, and we will in fact exhibit such a combination with coefficients in $r^{(0,v_R)}_\varpi$.

First, let us decompose elements of $r^{(0,v_R)}_\varpi$. If $N \in \mathbf{Z}$, if $x \in X_0^N r^{(0,v_R)}_\varpi$, and if $N = pq + r$, with $0 \leq r \leq p-1$, one can write x, modulo X_0^{N+1}, as $X_0^r\varphi(X_0)^q \sum_n \varphi(a_n)\frac{p^n}{\varphi(X_0)^{2f\delta_R}}$, with $a_n \in \mathscr{O}_F$. Hence, there exist $x_{N+1} \in X_0^{N+1}r^{(0,v_R)}_\varpi$, and $a_N \in r^{(0,pv_R)}_\varpi$ such that $x = X_0^r\varphi(X_0^q a_N) + x_{N+1}$. Iterating this

process, one can write x as $\sum_{pq+r\geq N, 0\leq r\leq p-1} X_0^r\varphi(X_0^q a_{pq+r})$, with $a_{pq+r}\in r_\varpi^{(0,pv_R)}$. Setting $c_r(x)=\sum_{pq+r\geq N} X_0^r\varphi(X_0^q a_{pq+r})$, we obtain a decomposition

$$x=\sum_{r=0}^{p-1} c_r(x),\quad \text{with } c_r(x)\in X_0^r\varphi(r_\varpi)\cap X_0^N r_\varpi^{(0,v_R)}.$$

In particular, if $N\in\mathbf{N}$, we can write $X_0^N=\sum_{r=0}^{p-1} X_0^N c_{N,r}$, with $c_{N,r}\in r_\varpi^{(0,v_R)}$, in such a way that $X_0^N c_{N,r}\in X_0^r\varphi(r_\varpi)$, for all r.

Next, notice that $u_{i,\beta}=X_0^{\beta_0+\beta_{d+1}h}X_1^{\beta_1}\cdots X_a^{\beta_a}X_{a+1}^{\beta_{a+1}-\beta_{d+1}}\cdots X_{a+b}^{\beta_{a+b}-\beta_{d+1}}X_{a+b+1}^{\beta_{a+b+1}}\cdots X_d^{\beta_d}$. This implies that

$$(x_{i,\beta})_\alpha=\begin{cases} c_{\beta_0+\beta_{d+1}h,\alpha_0}x_{i,\beta}, & \text{if } (\alpha_1,\dots,\alpha_d)=(\beta_1,\dots,\beta_a,\beta_{a+1}-\beta_{d+1},\dots,\beta_{a+b}-\beta_{d+1},\beta_{a+b+1},\dots,\beta_d) \text{ modulo } p,\\ 0, & \text{otherwise.}\end{cases}$$

Summing over $\beta\in A_i$ gives the desired expression of x_α as a combination of the $x_{i,\beta}$'s. □

Lemma 2.22. *If $k\geq 0$, there exists $a_{k,i}\in T^{-\delta_R}r_\varpi^{(0,pv_R)}$ such that $X_0^k=\sum_{i=0}^{p-1}\varphi(a_{k,i})\,(1+T)^i$.*

Proof. Let us denote by Tr the trace from r_ϖ to $r_{\zeta-1}$. If $0\leq j\leq f-1$, let $f_j=X_0^j$. The f_j's are a basis of r_ϖ/p over $r_{\zeta-1}/p$; let $(g_j)_{0\leq j\leq f-1}$ be the dual basis for the bilinear form $\mathrm{Tr}(xy)$. Now, if Δ is as in Lemma 2.10, we have $\mathrm{Tr}(\Delta^{-1}X_0^j)=0$ if $0\leq j\leq f-2$ and $\mathrm{Tr}(\Delta^{-1}X_0^{f-1})=1$. It follows that $g_j=\Delta^{-1}R_j$, where $R_j\in k[X_0]$ is unitary, of degree $f-1-j$; in particular, $g_j\in X_0^{-\delta_K}r_\varpi^+/p$.

Now the f_j^p's are also a basis of r_ϖ/p over $r_{\zeta-1}/p$ and the g_j^p's are the dual basis. So, if $x\in r_\varpi/p$, we have $x=\sum_{j=0}^{f-1}\mathrm{Tr}(g_j^p x)\,f_j^p$. But $g_j^p x\in X_0^{-p\delta_K}r_\varpi^+/p\subset X_0^{-\delta_K}T^{-p\delta_R}r_\varpi^+/p$ (we could improve this to $X_0^{-\delta_K}T^{-\frac{p-1}{p}\delta_R}$). It follows that if $x\in r_\varpi^+/p$, then $\mathrm{Tr}(g_j^p x)\in T^{-p\delta_R}r_{\zeta-1}^+/p$. Hence one can write $\mathrm{Tr}(g_j^p x)$ as $T^{-p\delta_R}\sum_{i=0}^{p-1}(1+T)^i a_{j,i}^p$, with $a_{j,i}\in r_{\zeta-1}^+/p$. It follows that $x=\sum_{i=0}^{p-1}(1+T)^i c_i(x)^p$, with $c_i(x)=T^{-\delta_R}\sum_{j=0}^{f-1}a_{j,i}f_j\in T^{-\delta_R}r_\varpi^+/p$.

The result follows from Lemma 2.19 with $S=r_\varpi$, $N=f\delta_R$, and taking the $(1+T)^i$'s as the u_j's. □

From this lemma, we get the relation:

$$u_{\mathrm{Kum},\alpha}=\sum_{i=0}^{p-1}\varphi(a_{\alpha_0,i})\,u_{\mathrm{cycl},\beta_i},\quad \text{with } \beta_i=(i,\alpha_1,\dots,\alpha_d).$$

This implies that $c_{\mathrm{cycl},\alpha}(x)=\sum_{i=0}^{p-1}\varphi(a_{i,\alpha_0})x_{(i,\alpha_1,\dots,\alpha_d)}$, and makes it possible to use Lemmas 2.21 and 2.22 to finish the proof of Proposition 2.15.

2.4. **Period rings.** let $\overline{R}$ be the "maximal extension of R unramified outside $X_{a+b+1}\cdots X_d=0$ in characteristic 0 (i.e. after inverting p)". Let $G_R=\mathrm{Gal}(\overline{R}/R)$. Define v_p on $\overline{R}[\frac{1}{p}]$ to be the spectral valuation.

2.4.1. *p-adic Hodge theory rings.* If $S=K,R[\frac{1}{p}]$, denote by $\mathbf{C}(\overline{S})$ the completion of $\overline{S}$ for v_p, and let $\mathbf{C}^+(\overline{S})$ be the sub-ring of x's with $v_p(x)\geq 0$. $\mathbf{C}(\overline{S})$ is a perfectoid algebra. Denote by $\mathbf{E}_{\overline{S}}$ its tilt and set $\mathbf{A}_{\overline{S}}=W(\mathbf{E}_{\overline{S}})$. We can describe $\mathbf{E}_{\overline{S}}$ as the set of sequences $(x_n)_{n\in\mathbf{N}}$, with $x_n\in\mathbf{C}(\overline{S})$ and $x_{n+1}^p=x_n$ for all $n\in\mathbf{N}$. If $x\in\mathbf{C}(\overline{S})$, denote by $x^\flat$ any element $(x_n)_{n\in\mathbf{N}}$ of $\mathbf{E}_{\overline{S}}$ with $x_0=x$.

The inclusion $K\subset R[\frac{1}{p}]$ induce inclusions

$$\overline{K}\subset\overline{R}[\tfrac{1}{p}],\quad \mathbf{E}_{\overline{K}}\subset\mathbf{E}_{\overline{R}},\quad \mathbf{A}_{\overline{K}}\subset\mathbf{A}_{\overline{R}}.$$

Define $v_{\mathbf{E}}$ on $\mathbf{E}_{\overline{S}}$ by $v_{\mathbf{E}}(x)=v_p(x^\sharp)$, $x^\sharp:=x_0$. This is a valuation on $\mathbf{E}_{\overline{S}}$ for which it is complete.

Let $\mathbf{E}_{\overline{S}}^+$ be the subring of $\mathbf{E}_{\overline{S}}$ of x's such that $v_{\mathbf{E}}(x)\geq 0$, and let $\mathbf{A}_{\overline{S}}^+=W(\mathbf{E}_{\overline{S}}^+)$. So $\mathbf{E}_{\overline{S}}^+$ is the tilt of $\mathbf{C}^+(\overline{S})$ and, as a ring, it is the projective limit (over $\mathbf{N}$) of $\mathbf{C}^+(\overline{S})/\mathfrak{a}$ for the transition maps $x\mapsto x^p$,

where $\mathfrak{a}$ is the ideal $\{x,\, v_p(x) \geq \frac{1}{p}\}$ (we could have replaced $\frac{1}{p}$ by any number in $]0,1]$). The inclusion $\mathscr{O}_K \subset R$ induce inclusions

$$\mathbf{E}_{\overline{K}}^+ \subset \mathbf{E}_{\overline{R}}^+, \quad \mathbf{A}_{\overline{K}}^+ \subset \mathbf{A}_{\overline{R}}^+.$$

Let

$$\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathbf{E}_{\overline{K}}^+, \quad \pi = [\varepsilon] - 1 \in \mathbf{A}_{\overline{K}}^+ \quad \text{and} \quad \xi = \tfrac{\pi}{\varphi^{-1}(\pi)}.$$

Any element x of $\mathbf{A}_{\overline{S}}$ (resp. $\mathbf{A}_{\overline{S}}^+$) can be written uniquely as

$$x = \sum_{k \in \mathbf{N}} p^k [x_k],$$

with $x_k \in \mathbf{E}_{\overline{S}}$ (resp. $\mathbf{E}_{\overline{S}}^+$). Then $\theta : \mathbf{A}_{\overline{S}}^+ \to \mathbf{C}^+(\overline{S})$ defined by $\theta(\sum_{k\in\mathbf{N}} p^k[x_k]) = \sum_{k\in\mathbf{N}} p^k x_k^\sharp$ is a surjective ring homomorphism whose kernel is principal, generated by $p - [p^\flat]$ (or by $P_\varpi([\varpi^\flat])$ or ξ). We extend θ, by $\mathbf{Q}_p$-linearity, to $\theta : \mathbf{A}_{\overline{S}}^+[\frac{1}{p}] \to \mathbf{C}(\overline{S})$, and we define $\mathbf{B}_{\mathrm{dR}}^+(\overline{S})$ to be the completion of $\mathbf{A}_{\overline{S}}^+[\frac{1}{p}]$ with respect to the ideal $(p - [p^\flat])$. We filter $\mathbf{B}_{\mathrm{dR}}^+(\overline{S})$ by the powers of the ideal $(p - [p^\flat])$, i.e. we set $F^i\mathbf{B}_{\mathrm{dR}}(\overline{S}) = (p - [p^\flat])^i \mathbf{B}_{\mathrm{dR}}^+(\overline{S})$.

We define $\mathbf{A}_{\mathrm{cr}}(\overline{S})$ as the p-adic completion of $\mathbf{A}_{\overline{S}}^+[\frac{(p-[p^\flat])^k}{k!},\ k \in \mathbf{N}] = \mathbf{A}_{\overline{S}}^+[\frac{[p^\flat]^k}{k!},\ k \in \mathbf{N}]$. It is naturally a subring of $\mathbf{B}_{\mathrm{dR}}^+(\overline{S})$.

2.4.2. (φ, Γ)-*modules theory rings.* If $v > 0$ and $S = K, R$, let $\mathbf{A}_{\overline{S}}^{(0,v]}$ be the subring of $\mathbf{A}_{\overline{S}}$ defined by

$$\mathbf{A}_{\overline{S}}^{(0,v]} = \big\{ \sum_{k\in\mathbf{N}} p^k[x_k],\ v v_{\mathbf{E}}(x_k) + k \to +\infty \text{ when } k \to +\infty \big\},$$

and let $\mathbf{A}_{\overline{S}}^{(0,v]+}$ be the subring of $\mathbf{A}_{\overline{S}}^{(0,v]}$ of the $x = \sum_{k\in\mathbf{N}} p^k[x_k]$ such that $v v_{\mathbf{E}}(x_k) + k \geq 0$ for all $k \in \mathbf{N}$. We have

$$\mathbf{A}_{\overline{S}}^{(0,v]} = \mathbf{A}_{\overline{S}}^{(0,v]+}[\tfrac{1}{[p^\flat]}].$$

If $\alpha \in \mathbf{E}_{\overline{K}}^+$ satisfies $v_{\mathbf{E}}(\alpha) = \frac{1}{v}$, then $\mathbf{A}_{\overline{S}}^{(0,v]+}$ is also the completion of $\mathbf{A}_{\overline{S}}^+[\frac{p}{[\alpha]}]$ for the p-adic topology. If $v \geq 1$, the natural map $\mathbf{A}_{\overline{S}}^+[\frac{p}{[\alpha]}] \to \mathbf{B}_{\mathrm{dR}}^+(\overline{S})$ extends, by continuity, to injections

$$\mathbf{A}_{\overline{S}}^{(0,v]+} \to \mathbf{B}_{\mathrm{dR}}^+(\overline{S}), \quad \mathbf{A}_{\overline{S}}^{(0,v]} \to \mathbf{B}_{\mathrm{dR}}^+(\overline{S}).$$

If $0 < u$, and if $\beta \in \mathbf{E}_{\overline{K}}^+$ satisfies $v_{\mathbf{E}}(\beta) = \frac{1}{u}$, we define $\mathbf{A}_{\overline{S}}^{[u]}$ as the completion of $\mathbf{A}_{\overline{S}}^+[\frac{[\beta]}{p}]$ for the p-adic topology. If $u \leq 1$, the natural map $\mathbf{A}_{\overline{S}}^+[\frac{[\beta]}{p}] \to \mathbf{B}_{\mathrm{dR}}^+(\overline{S})$ extends, by continuity, to an injection

$$\mathbf{A}_{\overline{S}}^{[u]} \to \mathbf{B}_{\mathrm{dR}}^+(\overline{S}).$$

If $0 < u \leq v$, and if $\alpha, \beta \in \mathbf{E}_{\overline{K}}^+$ satisfy $v_{\mathbf{E}}(\alpha) = \frac{1}{v}$ and $v_{\mathbf{E}}(\beta) = \frac{1}{u}$, we define $\mathbf{A}_{\overline{S}}^{[u,v]}$ as the completion of $\mathbf{A}_{\overline{S}}^+[\frac{p}{[\alpha]}, \frac{[\beta]}{p}]$ for the p-adic topology. If $u \leq 1 \leq v$, the natural map $\mathbf{A}_{\overline{S}}^+[\frac{p}{[\alpha]}, \frac{[\beta]}{p}] \to \mathbf{B}_{\mathrm{dR}}^+(\overline{S})$ extends, by continuity, to an injection

$$\mathbf{A}_{\overline{S}}^{[u,v]} \to \mathbf{B}_{\mathrm{dR}}^+(\overline{S}).$$

We use these embeddings into $\mathbf{B}_{\mathrm{dR}}^+(\overline{S})$ to induce filtrations on all the rings $\mathbf{A}_{\overline{S}}^{\mathrm{deco}}$. We have

$$\varphi(\mathbf{A}_{\overline{S}}^{(0,v]+}) = \mathbf{A}_{\overline{S}}^{(0,v/p]+}, \quad \varphi(\mathbf{A}_{\overline{S}}^{(0,v]}) = \mathbf{A}_{\overline{S}}^{(0,v/p]}, \quad \varphi(\mathbf{A}_{\overline{S}}^{[u]}) = \mathbf{A}_{\overline{S}}^{[u/p]}, \quad \varphi(\mathbf{A}_{\overline{S}}^{[u,v]}) = \mathbf{A}_{\overline{S}}^{[u/p,v/p]}.$$

The relative crystalline ring of periods $\mathbf{A}_{\mathrm{cr}}(\overline{R}) = \mathbf{A}_{\overline{R}}^+[\frac{[p^\flat]^k}{k!}, k \in \mathbf{N}]^\wedge$ is related to the above rings. We have

$$\mathbf{A}_{\mathrm{cr}}(\overline{R}) \subset \mathbf{A}_{\overline{R}}^+[\tfrac{[\beta]}{p}]^\wedge, \text{ for } \quad 0 < v_{\mathbf{E}}(\beta) \leq p-1;$$
$$\mathbf{A}_{\mathrm{cr}}(\overline{R}) \supset \mathbf{A}_{\overline{R}}^+[\tfrac{[\beta]}{p}]^\wedge, \text{ for } \quad v_{\mathbf{E}}(\beta) \geq p.$$

(If $p - 1 < v_{\mathbf{E}}(\beta) < p$, there exists $C(v)$, $v = v_{\mathbf{E}}(\beta)$, such that $p^{C(v)}\mathbf{A}_{\overline{R}}^+[\frac{[\beta]}{p}]^\wedge \subset \mathbf{A}_{\mathrm{cr}}(\overline{R})$.)

To see this, write $[\beta]=[p^\flat]^v u$, for a unit $u\in \mathbf{A}_{\overline{K}}^+$, $v=v_{\mathbf{E}}(\beta)$. Since $v_p(k!)=(k-s_k)/(p-1)$, where $s_k\geq 1$ is the sum of digits in the p-adic presentation of k, we have $[p^\flat]^{[k]}=[p^\flat]^k p^{-(k-s_k)/(p-1)}u_0$, for a unit $u_0\in\mathscr{O}_F$. For the first inclusion above, it suffices to show that, for all k, $[p^\flat]^k p^{-(k-s_k)/(p-1)}\in \mathbf{A}_{\overline{R}}^+[\frac{[\beta]}{p}]^\wedge$. But $([\beta]/p)^{(k-s_k)/(p-1)}=[p^\flat]^{v(p-1)^{-1}(k-s_k)}p^{-(k-s_k)/(p-1)}u_1$, for a unit $u_1\in\mathbf{A}_{\overline{K}}^+$. It follows that, if $0<v\leq(p-1)$, then $[p^\flat]^{[k]}\in(([\beta]/p)^{(k-s_k)/(p-1)})$, as wanted.

For the second inclusion, if $v\geq p$, then $([\beta]/p)^k=[p^\flat]^{pk}p^{-k}a_2$, $a_2\in\mathbf{A}_{\mathrm{cr}}(\overline{R})$. It suffices to show that $(pk)!p^{-k}\in\mathbf{N}$. But

$$v_p(\tfrac{(pk)!}{p^k})=v_p((pk)!)-v_p(p^k)=\tfrac{pk-s_{pk}}{p-1}-k$$

and this is nonnegative since $k\geq s_{pk}=s_k$.

Hence $\mathbf{A}_{\mathrm{cr}}(\overline{R})\subset\mathbf{A}_{\overline{R}}^{[u]}$ for $u\geq 1/(p-1)$, and $\mathbf{A}_{\mathrm{cr}}(\overline{R})\supset\mathbf{A}_{\overline{R}}^{[u]}$ for $u\leq 1/p$. That means that, in the case of $u_p=(p-1)/p$, for $p>2$, $\mathbf{A}_{\mathrm{cr}}(\overline{R})\subset\mathbf{A}_{\overline{R}}^{[u_p]}$ and, for $p=2$ ($u_p=1/2$), $\mathbf{A}_{\mathrm{cr}}(\overline{R})\supset\mathbf{A}_{\overline{R}}^{[1/2]}$ (we also have $\mathbf{A}_{\mathrm{cr}}(\overline{R})\subset\varphi^{-1}(\mathbf{A}_{\overline{R}}^{[1/2]})$).

2.4.3. *Fundamental exact sequences.* Recall that, if $r\in\mathbf{N}$, we have *the fundamental exact sequence*

$$0\to\mathbf{Z}_p t^{\{r\}}\to\ F^r\mathbf{A}_{\mathrm{cr}}\xrightarrow{p^r-\varphi}\mathbf{A}_{\mathrm{cr}}$$

where $t^{\{r\}}:=t^{b(r)}(t^{p-1}/p)^{a(r)}$, for $r=(p-1)a(r)+b(r)$, $0\leq b(r)\leq p-1$. Moreover the map $p^r-\varphi$ is p^r-surjective. Set $\mathbf{Z}_p(r)':=\frac{1}{p^{a(r)}}\mathbf{Z}_p(r)$. We will need the following generalizations of the above fundamental exact sequence.

Lemma 2.23. (i) *Let $0<v$. We have the following Artin-Schreier exact sequences*

$$0\to\mathbf{Z}_p\to\ \mathbf{A}_{\overline{R}}\xrightarrow{1-\varphi}\mathbf{A}_{\overline{R}}\ \to 0,\quad 0\to\mathbf{Z}_p\to\ \mathbf{A}_{\overline{R}}^{(0,v]+}\xrightarrow{1-\varphi}\mathbf{A}_{\overline{R}}^{(0,v/p]+}\ \to 0 \tag{2.24}$$

(ii) *The following sequence is p^r-exact.*

$$0\to\mathbf{Z}_p(r)'\to\ F^r\mathbf{A}_{\mathrm{cr}}(\overline{R})\xrightarrow{p^r-\varphi}\mathbf{A}_{\mathrm{cr}}(\overline{R})\ \to 0$$

(iii) *Let $0<u\leq 1\leq v$. The following sequence is p^{4r}-exact*

$$0\to\mathbf{Z}_p(r)\to\ F^r\mathbf{A}_{\overline{R}}^{[u,v]}\xrightarrow{p^r-\varphi}\mathbf{A}_{\overline{R}}^{[u,v/p]}\ \to 0$$

Moreover, all the surjections above have continuous sections.

Proof. The exactness of the first two sequences is proved as in [4, 8.1]. The third sequence was treated in [63, A3.26]. The exactness of the last one will follow from the exactness of the following two sequences

$$0\to\mathbf{Z}_p(r)\to(\mathbf{A}_{\overline{R}}^{[u,v]})^{\varphi=p^r}\to\mathbf{A}_{\overline{R}}^{[u,v]}/F^r\to 0 \tag{2.25}$$

$$0\to(\mathbf{A}_{\overline{R}}^{[u,v]})^{\varphi=p^r}\to\ \mathbf{A}_{\overline{R}}^{[u,v]}\xrightarrow{p^r-\varphi}\mathbf{A}_{\overline{R}}^{[u,v/p]}\ \to 0$$

To treat the first sequence, assume that $p>2$ and map the sequence (p^r-exact)

$$0\to\mathbf{Z}_p(r)\to\mathbf{A}_{\mathrm{cr}}(\overline{R})^{\varphi=p^r}\to\mathbf{A}_{\mathrm{cr}}(\overline{R})/F^r\to 0 \tag{2.26}$$

into it. We claim that the cokernel of this map is killed by p^{2r}: Indeed, note that we have the p^r-isomorphisms

$$\mathbf{A}_{\mathrm{cr}}(\overline{R})^{\varphi=p^r}\xrightarrow{\sim}(\mathbf{A}_{\overline{R}}^{[u]})^{\varphi=p^r}\xrightarrow{\sim}(\mathbf{A}_{\overline{R}}^{[u,v]})^{\varphi=p^r}.$$

For the first map the injection is clear and the p^r-surjection follows from the fact that $\varphi(\mathbf{A}_{\overline{R}}^{[u]})\subset\mathbf{A}_{\mathrm{cr}}(\overline{R})$. For the second map, again the injection is clear and to show surjection it suffices to prove that the map

$$(p^r-\varphi):\quad \mathbf{A}_{\overline{R}}^{[u,v]}/\mathbf{A}_{\overline{R}}^{[u]}\xrightarrow{\sim}\mathbf{A}_{\overline{R}}^{[u,v/p]}/\mathbf{A}_{\overline{R}}^{[u]} \tag{2.27}$$

is an isomorphism. Since we have

$$\mathbf{A}_{\overline{R}}^{(0,v]+}/\mathbf{A}_{\overline{R}}^{+} \simeq \mathbf{A}_{\overline{R}}^{[u,v]}/\mathbf{A}_{\overline{R}}^{[u]}, \quad \mathbf{A}_{\overline{R}}^{(0,v/p]+}/\mathbf{A}_{\overline{R}}^{+} \simeq \mathbf{A}_{\overline{R}}^{[u,v/p]}/\mathbf{A}_{\overline{R}}^{[u]},$$

we are reduced to showing that the map

$$(p^r - \varphi): \quad \mathbf{A}_{\overline{R}}^{(0,v]+}/\mathbf{A}_{\overline{R}}^{+} \to \mathbf{A}_{\overline{R}}^{(0,v/p]+}/\mathbf{A}_{\overline{R}}^{+}$$

is an isomorphism. For $r = 0$ this follows from the third sequence in our lemma and the Artin-Schreier theory as in [4, 8.1.1]:

$$0 \to \mathbf{Z}_p \to \mathbf{A}_{\overline{R}}^{+} \xrightarrow{1-\varphi} \mathbf{A}_{\overline{R}}^{+} \to 0 \tag{2.28}$$

For $r > 0$, we can use the fact that the formal inverse of $p^r - \varphi$ is $-(\varphi^{-1} + p^r\varphi^{-2} + p^{2r}\varphi^{-3} + \cdots)$ and it clearly converges.

Note also that the cokernel of the map $\mathbf{A}_{\mathrm{cr}}(\overline{R})/F^r \hookrightarrow \mathbf{A}_{\overline{R}}^{[u,v]}/F^r$ is killed by p^r. This follows from the fact that already the cokernel of the map $\mathbf{A}_{\overline{R}}^{+}/F^r \hookrightarrow \mathbf{A}_{\overline{R}}^{[u,v]}/F^r$ is killed by p^r. To see this write $\mathbf{A}_{\overline{R}}^{[u,v]} = \mathbf{A}_{\overline{R}}^{+}[p/[\alpha], [\beta]/p]^\wedge$, $v_{\mathbf{E}}(\alpha) = 1/v, v_{\mathbf{E}}(\beta) = 1/u$, and note that

$$\tfrac{[\beta]}{p} = \tfrac{[p^\flat]}{p}[\beta'] = \tfrac{([p^\flat]-p)}{p}[\beta'] + [\beta'], \quad v_{\mathbf{E}}(\beta') = 1/u - 1;$$
$$\tfrac{p}{[\alpha]} = (\tfrac{[p^\flat]}{p})^{-1}[\alpha'] = (1 + \tfrac{([p^\flat]-p)}{p})^{-1}[\alpha'], \quad v_{\mathbf{E}}(\alpha') = 1 - 1/v,$$

and use the fact that the kernel of θ is generated by $[p^\flat] - p$. We have shown that the first sequence in (2.25) is p^{3r}-exact for $p > 2$.

For $p = 2$ the argument is similar. We map the sequence

$$0 \to \mathbf{Z}_p(r) \to (\mathbf{A}_{\overline{R}}^{[u]})^{\varphi=p^r} \to \mathbf{A}_{\overline{R}}^{[u]}/F^r \to 0$$

to both the sequence (2.25) and the sequence (2.26). By the above, the first map has the cokernel annihilated by p^r. The cokernel of the second map is annihilated by p^{2r}: use the fact that the cokernel of the map $\mathbf{A}_{\overline{R}}^{[u]}/F^r \hookrightarrow \mathbf{A}_{\mathrm{cr}}(\overline{R})/F^r$ is killed by p^r and that we have p^r-isomorphisms

$$(\mathbf{A}_{\overline{R}}^{[u]})^{\varphi=p^r} \xrightarrow{\sim} \mathbf{A}_{\mathrm{cr}}(\overline{R})^{\varphi=p^r} \xrightarrow{\sim} \varphi^{-1}(\mathbf{A}_{\overline{R}}^{[u]})^{\varphi=p^r}$$

To show that the second sequence in (2.25) is p^{2r}-exact it suffices to show that it is p^{2r}-exact on the right and for that, using the isomorphism (2.27), we can pass to the following sequence

$$0 \to (\mathbf{A}_{\overline{R}}^{[u]})^{\varphi=p^r} \to \mathbf{A}_{\overline{R}}^{[u]} \xrightarrow{p^r-\varphi} \mathbf{A}_{\overline{R}}^{[u]} \to 0 \tag{2.29}$$

That is, it remains to show that the above sequence is p^{2r}-exact on the right. Write $\mathbf{A}_{\overline{R}}^{[u]} = \mathbf{A}_{\overline{R}}^{+}[[\beta]/p]^\wedge$, $v_{\mathbf{E}}(\beta) = 1/u$. Since the map $\mathbf{A}_{\overline{R}}^{+} + [\beta]^r\mathbf{A}_{\overline{R}}^{[u]} \hookrightarrow \mathbf{A}_{\overline{R}}^{[u]}$ is p^r-surjective it suffices to show that $p^r - \varphi$ is p^r-surjective on $\mathbf{A}_{\overline{R}}^{+}$ and on $[\beta]^r\mathbf{A}_{\overline{R}}^{[u]}$ separately. Surjectivity on $\mathbf{A}_{\overline{R}}^{+}$ is clear since for $r = 0$ this is just the Artin-Schreier theory from (2.28) and, for $r > 0$, the series $-(1 + p^r\varphi^{-1} + p^{2r}\varphi^{-2} + \cdots)$ converges on $\mathbf{A}_{\overline{R}}^{+}$ to an inverse of $(p^r\varphi^{-1} - 1)$. To check p^r-surjectivity on $[\beta]^r\mathbf{A}_{\overline{R}}^{[u]}$, for $x = [\beta]^r y$, $y \in \mathbf{A}_{\overline{R}}^{[u]}$, it is enough to check that $(1 + \frac{\varphi}{p^r} + \frac{\varphi^2}{p^{2r}} + \cdots)$ converges (pointwise), as this gives an inverse to $1 - \frac{\varphi}{p^r}$. We have

$$(1 + \tfrac{\varphi}{p^r} + \tfrac{\varphi^2}{p^{2r}} + \cdots)(x) = \sum_{k\geq 0} \tfrac{\varphi^k([\beta]^r y)}{p^{kr}} = \sum_{k\geq 0} \tfrac{[\beta]^{p^k r}}{p^{kr}}\varphi^k(y) = \sum_{k\geq 0} p^{p^k r - kr}(\frac{[\beta]}{p})^{p^k r}\varphi^k(y).$$

We conclude, noticing that $p^k r - kr \geq 0$ and goes to ∞ when $k \to \infty$.

Concerning the last claim of the lemma – can be proved for the first two sequences as in [4, 8.1,8.1.2,8.1.3]. The last two sequences are sequences of spaces with p-adic topology that are complete for that topology – hence the existence of a continuous section is clear. □

Remark 2.30. The same arguments can be used to prove the following result, where $\mathbf{B}_{\mathrm{cr}}^+ = \mathbf{A}_{\mathrm{cr}}[\frac{1}{p}]$.

Let M be a φ-module[8] *over F, $P \in F[X]$ such that $P(0) \neq 0$. Then $P(\varphi) : \mathbf{B}_{\mathrm{cr}}^+ \otimes_F M \to \mathbf{B}_{\mathrm{cr}}^+ \otimes_F M$ is surjective.*

Since $\mathbf{B}_{\mathrm{cr}}^+$ contains $W(\overline{k})$, Dieudonné-Manin's theorem allows us to assume that M is the standard module of slope $\lambda = \frac{a}{h}$, i.e., $M = Fe_1 \oplus \cdots \oplus Fe_h$, and $\varphi(e_i) = e_{i+1}$, if $i \leq h$, $\varphi(e_h) = p^a e_1$. Also, the result is true for P if and only if it is true for all of its irreducible divisors, which allows to assume that $P(0) = 1$, that all roots of P have the same valuation α and that $P(X) = Q(X^h)$ for some $Q \in F[X]$ (by replacing $P = \prod(1-a_iX)$ by its multiple $\prod(1-a_i^hX^h)$). So, we just have to check that $R(\varphi^h) = Q(p^a\varphi^h)$ is surjective on $\mathbf{B}_{\mathrm{cr}}^+$ (note that all roots of R have valuation $\beta = (\alpha - \lambda)h$).

Write $1/R = 1 + b_1X + b_2X + \cdots$. We have $v_p(b_i) \geq -i\beta$, for all i, which implies (by the arguments above: $p^{p^{hk}} - k\beta \to +\infty$ when $k \to +\infty$) that $1 + b_1\varphi^h + b_2\varphi^{2h} + \cdots$ converges (pointwise) on $[p^\flat]\mathbf{B}_{\mathrm{cr}}^+$, and that $P(\varphi)$ has an inverse (and hence is surjective) on $[p^\flat]\mathbf{B}_{\mathrm{cr}}^+$.

So, we are left to check that $R(\varphi^h)$ is surjective on $\mathbf{A}_{\overline{K}}^+[1/p]$. Let us write R as $1 + a_1X + \cdots + a_dX^d$, with $a_d \neq 0$. There are two cases:

• $\beta \leq 0$, which implies that $R \in \mathscr{O}_F[X]$ and $R(\varphi)$ sends $\mathbf{A}_{\overline{K}}^+$ into itself. Now, modulo p, $R(\varphi^h)$ becomes $x \mapsto x + a_1x^{p^h} + \cdots + a_dx^{p^{dh}}$, and is surjective since $\mathbf{E}_{\overline{K}}$ is algebraically closed and $\mathbf{A}_{\overline{K}}^+/p$ is its ring of integers (if $\beta < 0$, things are even simpler: all a_i are 0 modulo p, and $R(\varphi^h)$ is just $x \mapsto x$ modulo p). Since $\mathbf{A}_{\overline{K}}^+$ is p-adically complete, this implies that $R(\varphi^h) : \mathbf{A}_{\overline{K}}^+ \to \mathbf{A}_{\overline{K}}^+$ is, indeed, surjective.

• $\beta > 0$, in which case $a_d^{-1}R \in \mathscr{O}_F[X]$ and is X^d modulo p. It follows that $a_d^{-1}R(\varphi^h)$ becomes $x \mapsto x^{p^{dh}}$ modulo p, which makes it clear that it is surjective.

2.5. **Embeddings into period rings.**

2.5.1. *Kummer embeddings.* Choose, inside $\overline{R}$, elements $X_i^{p^{-n}}$, for $i = 1, \dots, d$ and $n \in \mathbf{N}$, satisfying the obvious relations (i.e. $X_i^{p^{-0}} = X_i$ and $(X_i^{p^{-(n+1)}})^p = X_i^{p^{-n}}$ if $n \geq 0$). If $i = 1, \dots, d$, let $X_i^\flat = (X_i, X_i^{1/p}, \dots) \in \mathbf{E}_{\overline{R}}^+$.

Sending X_0 to $[\varpi^\flat]$ and X_i to $[X_i^\flat]$, if $i = 1, \dots, d$, induces an embedding ι_{Kum} of $R_{\varpi,\square}^+$ into $\mathbf{A}_{\overline{R}}$ which commutes with Frobenius φ and is compatible with filtrations. As R_ϖ^+ is étale over $R_{\varpi,\square}^+$, one can extend ι_{Kum} to an embedding $R_\varpi^+ \to \mathbf{A}_{\overline{R}}$ and, by continuity, to embeddings

$$R_\varpi^{\mathrm{PD}} \to \mathbf{A}_{\mathrm{cr}}(\overline{R}), \quad R_\varpi^{[u]} \to \mathbf{A}_{\overline{R}}^{[u]}, \quad R_\varpi^{[u,v]} \to \mathbf{A}_{\overline{R}}^{[u,v]},$$

which commute with Frobenius (with φ_{Kum} on R_ϖ^{deco}) and filtration (if $1 \notin [u,v]$, there is no filtration on the corresponding rings).

2.5.2. *Cyclotomic embedding of R_ϖ.* Let $R_\square^{\mathrm{cyl}} = \mathscr{O}_{F_i}\{X_1, \dots, X_d\}$. If $n \in \mathbf{N}$, let $R_{\square,n}^{\mathrm{cycl}} = \mathscr{O}_{F_{i+n}}\{X_1^{p^{-n}}, \dots, X_d^{p^{-n}}\}$, and let R_n be the integral closure of R in the subalgebra of $\overline{R}[\frac{1}{p}]$ generated by R and $R_{\square,n}^{\mathrm{cycl}}$. Set $R_{\square,\infty}^{\mathrm{cyl}} = \cup_{n\in\mathbf{N}} R_{\square,n}^{\mathrm{cyl}}$ and $R_\infty = \cup_{n\in\mathbf{N}} R_n$. Then $R_{\square,\infty}^{\mathrm{cyl}}[\frac{1}{p}]$ and $R_\infty[\frac{1}{p}]$ are Galois extensions of $R_\square^{\mathrm{cyl}}[\frac{1}{p}]$ and $R[\frac{1}{p}]$ respectively, with Galois group Γ_R which is the semi-direct product

$$1 \to \Gamma_R' \to \Gamma_R \to \Gamma_K \to 1,$$

where

$$\Gamma_R' = \mathrm{Gal}(R_\infty[\tfrac{1}{p}]/K_\infty R[\tfrac{1}{p}]) \simeq \mathbf{Z}_p^d, \quad \Gamma_K = \mathrm{Gal}(K_\infty/K) \simeq 1 + p^{i(K)}\mathbf{Z}_p,$$

and $a \in 1 + p^{i(K)}\mathbf{Z}_p$ acts on $\mathbf{Z}_p^d$ by multiplication by a.

[8] A finite rank vector space over F with a semilinear Frobenius isomorphism φ.

We define an embedding $\iota_{\mathrm{cycl}} : R^+_{\zeta-1,\square} \to \mathbf{A}^+_{\overline{R}_\square}$, by sending T to $\pi_i = \varphi^{-i}(\pi)$ and X_i to $[X_i^\flat]$, where $X_i^\flat = (X_i, X_i^{1/p}, \dots)$. This embedding commutes with Frobenius (i.e. $\varphi \circ \iota_{\mathrm{cyl}} = \iota_{\mathrm{cycl}} \circ \varphi_{\mathrm{cycl}}$) and filtration (since P_0 is sent to $P_0(\pi_i) = \xi$).

Proposition 2.31. *ι_{cycl} has a unique extension to an embedding $R^+_\varpi \to \mathbf{A}^+_{\overline{R}}[[\frac{p}{\pi_i^{2\delta_R}}]]$ such that*

$$\iota_{\mathrm{cycl}}(X_0) - [\varpi^\flat] \in \tfrac{\xi}{\pi_i^{\delta_R}} \mathbf{A}^+_{\overline{R}}[[\frac{p}{\pi_i^{2\delta_R}}]], \qquad \text{and } \theta \circ \iota_{\mathrm{cycl}} \text{ is the projection } R^+_\varpi \to R.$$

Proof. This is a consequence of Proposition 2.12, with $\Lambda = \mathbf{A}^+_{\overline{R}}[[\frac{p}{\pi_i^{2\delta_R}}]]$, $\lambda = \iota_{\mathrm{cycl}}$, $\beta : \mathscr{M} \to \Lambda$ defined by $\beta(X_0) = [\varpi^\flat]$, $\beta(X_i) = [X_i^\flat]$ if $1 \le i \le d$, $\beta(X_{d+1}) = \big[\frac{(\varpi^\flat)^h}{X^\flat_{a+1} \cdots X^\flat_{a+b}}\big]$, $\mu = P_0$ (hence $\lambda(\mu) = \xi$), $J = \operatorname{Ker}\theta = \big(\frac{xi}{\pi_i^{2\delta_R}}\big)$ and $\lambda_\varpi : R^+_\varpi \to \Lambda/J = \mathbf{C}^+(\overline{R})$ being the natural map $R^+_\varpi \to R$. (We have $\theta(Q([\pi^\flat], \pi_i)) = Q(\varpi, \zeta - 1) = 0$, hence $Q(\beta(X_0), \lambda(T))$ is divisible by $\xi = \lambda(\mu)$ in $\mathbf{A}^+_K \subset \Lambda$, which shows that we can indeed apply Proposition 2.12.) $\square$

Let

$$\pi_K = \iota_{\mathrm{cycl}}(X_0).$$

By construction $\theta(\pi_K) = \varpi$, and one can show that, if $n \in \mathbf{N}$, then $\varpi_n = \theta(\varphi^{-1}(\pi_K))$ is a uniformizer of K_n. Now, $\frac{\pi_K}{[\varpi^\flat]}$ is a unit in $\mathbf{A}^+_{\overline{R}}[[\frac{p}{\pi_i^{2\delta_R}}]]$. This allows, using an obvious variant of Remark 2.5, to extend ι_{cyl} to embeddings $R_\varpi \hookrightarrow \mathbf{A}_{\overline{R}}$, $R_\varpi^{(0,v]+} \hookrightarrow \mathbf{A}^{(0,v]+}_{\overline{R}}$ and $R_\varpi^{[u,v]} \hookrightarrow \mathbf{A}^{[u,v]}_{\overline{R}}$, for $u \le v < pv_R$, which commute with Frobenius (by unicity) and filtration (by construction).

2.5.3. *The action of* Γ_R. We denote by $\mathbf{A}_R^{\mathrm{deco}}$ (resp. $\mathbf{A}_{R,\square}^{\mathrm{deco}}$) the image of R_ϖ^{deco} (resp. $R_{\zeta-1,\square}^{\mathrm{deco}}$) by ι_{cycl}. It is quite clear that $\mathbf{A}^+_{R,\square}$ is stable under the action of G_R. More precisely, G_R acts through Γ_R, and we can choose topological generators γ_j, $0 \le j \le d$, of Γ_R with the following properties. Let

$$c = \exp(p^i); \qquad \text{in particular } a := p^{-i}(c-1) \in \mathbf{Z}_p^*.$$

Then

$$\begin{cases} \gamma_0(\pi_i) = (1+\pi_i)^c - 1 = (1+\pi)^a(1+\pi_i) - 1 \text{ and } \gamma_j(\pi_i) = \pi_i \text{ if } 1 \le j \le d. \\ \gamma_k([X_k^\flat]) = [\varepsilon]\,[X_k^\flat] = (1+\pi)\,[X_k^\flat] \text{ and } \gamma_j([X_k^\flat]) = [X_k^\flat] \text{ if } j \ne k \text{ and } 1 \le k \le d. \end{cases}$$

It follows that $\gamma_1, \dots, \gamma_d$ are topological generators of Γ'_R.

The induced action of Γ_R on $R^+_{\zeta-1,\square}$ is given by the formulas:

$$\gamma_0(T) = (1+T)^{p^i a}(1+T) - 1, \quad \gamma_j(X_j) = (1+T)^{p^i} X_j, \text{ if } 1 \le j \le d,$$
$$\gamma_k(T) = T, \text{ if } k \ne 0, \quad \gamma_k(X_j) = X_j, \text{ if } k \ne j.$$

Proposition 2.32. *Let $\gamma : R^+_{\zeta-1,\square} \to R^+_{\zeta-1,\square}$ be a continuous ring morphism such that there exists $a_j \in \mathbf{Z}_p$, for $0 \le j \le d$, such that:*

$$\gamma(X_j) = (1+T)^{p^i a_j} X_j, \text{ if } 1 \le j \le d, \quad \gamma(T) = (1+T)^{p^i a_0}(1+T) - 1.$$

Then γ extends uniquely to a continuous ring morphism $R^+_\varpi \to R^+_\varpi[[\frac{p}{T^{2\delta_R}}]]$, such that

$$\gamma(x) - x \in \tfrac{(1+T)^{p^i}-1}{T^{1+\delta_R}} R^+_\varpi[[\tfrac{p}{T^{2\delta_R}}]], \quad \text{if } x \in R^+_\varpi \quad \text{and} \quad \gamma(X_0) - X_0 \in \tfrac{(1+T)^{p^i}-1}{T^{\delta_R}} R^+_\varpi[[\tfrac{p}{T^{2\delta_R}}]].$$

Proof. This is a consequence of Proposition 2.12, with $\Lambda = R^+_\varpi[[\frac{p}{T^{2\delta_R}}]]$, $\lambda = \gamma$, $\beta(x) = x$, $\mu = (1+T)^{p^i} - 1$, $\lambda_\varpi(x) = x$, and $J = \big(\frac{(1+T)^{p^i}-1}{T^{1+\delta_R}}\big)$. To check the requirements of that proposition, note that $\frac{\lambda(\mu)}{\mu}$ is a unit and that $\gamma(T) - T$ is divisible by μ in $r^+_{\zeta-1}$, hence $Q(X_0, \gamma(T)) \equiv Q(X_0, T) \bmod \mu r^+_\varpi$, and $Q(\beta(X_0), \lambda(T)) \in \mu r^+_\varpi \subset \mu\Lambda$. The rest of the requirements are obvious. $\square$

Corollary 2.33. (i) *The action of Γ_R extends uniquely to an action on R_ϖ, which stabilizes $R_\varpi^{(0,v]+}$ if $v < pv_R$ and extends to a continuous action on $R_\varpi^{[u,v]}$, if $u \le v < pv_R$. Moreover this action is trivial modulo $T^{-1-\delta_R}((1+T)^{p^i}-1)$ on all these rings.*

(ii) *The rings $\mathbf{A}_R, \mathbf{A}_R^{(0,v]+}, \mathbf{A}_R^{(0,v]}, \mathbf{A}_R^{[u,v]}$ are stable by G_R which acts through Γ_R, and ι_{cycl} commutes with the action of Γ_R.*

Proof. For (i), just use Remark 2.5. For (ii), use the unicity in Proposiiton 2.32 and 2.31: it implies that $\sigma \circ \iota_{\mathrm{cycl}} = \iota_{\mathrm{cycl}} \circ \overline{\sigma}$, if $\sigma \in G_R$ and $\overline{\sigma}$ is its image in Γ_R. □

The action of Γ_R on $\mathbf{A}_R^{[u,v]}[\frac{1}{p}]$ is analytic, and the action of its Lie algebra $\mathrm{Lie}\,\Gamma_R$ is given by:

$$\log \gamma_j = t\partial_j, \quad \text{if } 0 \le j \le d,$$

as one sees easily using the above formulas for the action of Γ_R (here ∂_j is the derivation of $\mathbf{A}_R^{[u,v]}[\frac{1}{p}]$ deduced from $\partial_{\mathrm{cycl},j}$ on $R_\varpi^{[u,v]}[\frac{1}{p}]$ by transport of structure).

Lemma 2.34. *If $u \le v < p$ and $S = R_\varpi^{(0,v]+}$, $R_\varpi^{[u,v]}$, then*

$$(\gamma-1)\cdot T^n(p,T^{e_0})^k S \subset T^{n+p^{i-1}-\delta_R}(p,T^{e_0})^{k+1}S \quad \text{and} \quad (\gamma-1)^k \cdot S \subset T^{k(p^{i-1}-\delta_R-1)}(p,T^{e_0})^k S.$$

Proof. (ii) follows from the triviality of γ modulo $T^{-1-\delta_R}((1+T)^{p^i}-1)$, the fact that $\gamma-1$ acts as a twisted derivation $[\,(\gamma-1)\cdot xy = ((\gamma-1)\cdot x)y + \gamma(x)((\gamma-1)\cdot y)\,]$, which implies that

$$(\gamma-1)\cdot T^n S \subset T^{n-1-\delta_R}((1+T)^{p^i}-1)S,$$

and the fact that $T^{p^{i-1}}$ divides $(1+T)^{p^{i-1}}-1$ (because $v<p$) and $\frac{(1+T)^{p^i}-1}{(1+T)^{p^{i-1}}-1} \in (p,T^{e_0})$ (see Lemma 2.35 below). □

Lemma 2.35. *If $v<p$, then:*
- *$\pi_i^{-p^{i-1}}\pi_1$ is a unit of $\mathbf{A}_R^{(0,v]+}$.*
- *p is divisible by $\pi_i^{[\frac{(p-1)p^{i-1}}{v}]}$, hence also by $\pi_i^{(p-1)p^{i-2}}$.*
- *$\frac{p^2}{\pi_1} \in \mathbf{A}_R^{(0,v]+}$ and is divisible by $\pi_i^{(2(p-1)-v)p^{i-2}}$.*
- *$\frac{\pi}{\pi_1} \in (p,\pi_i^{(p-1)p^{i-1}})\mathbf{A}_R^{(0,v]+}$ and is divisible by $\pi_i^{(p-1)p^{i-2}}$, hence also by $\pi_i^{\delta_R+1}$.*

Proof. We can work in $r_{\zeta-1}^{(0,v]+}$, in which case π_i becomes T and π_1 becomes $(1+T)^{p^{i-1}}-1$, and we are looking at the annulus $0 < v_p(T) \le \frac{v}{(p-1)p^{i-1}}$ on which $(1+T)^{p^{i-1}}-1$ has no zero and $v_p((1+T)^{p^{i-1}}-1) = p^{i-1}v_p(T)$ since $v<p$. This proves the first point.

The second is basically the definition of $\mathbf{A}_R^{(0,v]+}$.

The third point follows from the first two, which give $\frac{p^2}{\pi_1}$ divisible by $\pi_i^{2[\frac{(p-1)p^{i-1}}{v}]-p^{i-1}}$, and the inequality $2[\frac{(p-1)p^{i-1}}{v}]-p^{i-1} \ge (2(p-1)-v)p^{i-2}$.

The fourth follows from the first two and the identity $\pi = \pi_1^{p-1} + p\pi_1^{p-2} + \cdots + p$. □

2.6. **Fat period rings.** We are going to make period rings fatter (Scholze [57] even call them $\mathscr{O}$B's) by adding to them the variables $X_0,\dots,X_d$ (i.e. by tensoring with R_ϖ^+ and its siblings). This kind of rings [63, 14] originate in linearizations of differential operators.

2.6.1. *Structure theorem.* Let:
- $S = R_\varpi^{\mathrm{PD}}$, $R_\varpi^{[u,v]}$ (and $S_\square = R_{\varpi,\square}^{\mathrm{PD}}$, $R_{\varpi,\square}^{[u,v]}$),
- Λ be a p-adically complete filtered $\mathscr{O}_F$-algebra,
- $\iota : S \to \Lambda$ be a continuous injective morphism of filtered $\mathscr{O}_F$-algebras,
- $f : S \otimes \Lambda \to \Lambda$ be the morphism sending $x \otimes y$ to $\iota(x)y$,

• $S\Lambda_s$ be the p-adically completed log-PD-envelope of $S\otimes\Lambda\to\Lambda$ with respect to $\operatorname{Ker} f$.

(We take partial divided powers of level s: i.e. $x^{[k]}=\frac{x^k}{[k/p^s]!}$.) By definition, $S\Lambda_s$ is the p-adic completion of $S\otimes\Lambda$ adjoined $(x\otimes 1-1\otimes\iota(x))^{[k]}$, for $x\in S$ and $k\in\mathbf{N}$, and $(V_j-1)^{[k]}$, for $0\le j\le d$ and $k\in\mathbf{N}$, where $V_j=\frac{X_j\otimes 1}{1\otimes\iota(X_j)}$. (Adding the $(V_j-1)^{[k]}$'s makes all the difference between the log-PD-envelope and the PD-envelope.) The morphism $f:S\otimes\Lambda\to\Lambda$ extends uniquely to a continuous morphism $f:S\Lambda_s\to\Lambda$.

We consider S and Λ as sub-algebras of $S\Lambda_s$ (by $x\mapsto x\otimes 1$ and $y\mapsto 1\otimes y$). Via these identifications, f induces the identity map on Λ, $V_j=\frac{X_j}{\iota(X_j)}$ if $0\le j\le d$, and $(x\otimes 1-1\otimes\iota(x))^{[k]}$ becomes $(x-\iota(x))^{[k]}$ if $x\in S$.

We filter $S\Lambda_s$ by defining $F^rS\Lambda_s$ to be the topological closure of the ideal generated by products of the form $x_1x_2\prod(V_j-1)^{[k_j]}$, with $x_1\in F^{r_1}R_1$, $x_2\in F^{r_2}R_2$, and $r_1+r_2+\sum k_j\ge r$.

Lemma 2.36. (i) *Any element $x\in S\Lambda_s$ can be written, uniquely, as*

$$x=\sum_{\mathbf{k}\in\mathbf{N}^{d+1}}x_{\mathbf{k}}\prod_{j=0}^{d}\big(1-V_j\big)^{[k_j]},$$

with $x_{\mathbf{k}}\in\Lambda$ for all $\mathbf{k}=(k_0,\dots,k_d)\in\mathbf{N}^{d+1}$, and $x_{\mathbf{k}}\to 0$ when $\mathbf{k}\to\infty$.

(ii) *$x\in F^rS\Lambda_s$ if and only if $x_{\mathbf{k}}\in F^{r-|\mathbf{k}|}\Lambda$ for all $\mathbf{k}\in\mathbf{N}^{d+1}$.*

Proof. Let $\Lambda'=\Lambda[\big(1-V_j\big)^{[k]},\ 0\le j\le d,\ k\in\mathbf{N}]^{\wedge}$ and let $\iota':\Lambda'\to S\Lambda_s$ be the natural map. We want to prove it is bijective.

For injectivity, use the morphism $f:S\Lambda_s\to\Lambda$: it kills the $\prod_{j=0}^d\big(1-V_j\big)^{[k_j]}$ with $|\mathbf{k}|>0$. Now $\partial_j=X_j\frac{\partial}{\partial X_j}$ extends by continuity to $S\Lambda_s$ and Λ', and commutes with ι', so we can prove by induction on $|\mathbf{k}|$ that if $\iota'(x)=0$, then $x_{\mathbf{k}}=0$ for all $\mathbf{k}$ (apply f to $\big(\prod\partial_j^{k_j}\big)\circ\iota'$, using the fact that $\partial_jV_\ell=0$ if $\ell\ne j$ and $\partial_jV_j=V_j=(V_j-1)+1$).

For surjectivity, first note that the kernel of $f\circ\iota':\Lambda'\to\Lambda$ is the PD-ideal generated by the (V_j-1)'s since $f\circ\iota'$ is the identity map on Λ. Let $S_\square\subset S$. There exists a unique morphism $\iota_0:S_\square\to\Lambda'$ such that $\iota'\circ\iota_0$ is the natural injection $S_\square\to S\Lambda_s$: unicity because ι' is injective, existence thanks to the formula

$$\iota_0\big(p^a\prod_{j=0}^dX_j^{k_j}\big)=p^a\prod_{j=0}^d\big(\iota(X_j)^{k_j}\big(1-V_j\big)^{k_j}\big)=p^a\prod_{j=0}^d\big(\iota(X_j)^{k_j}\big(\sum_{n=0}^{+\infty}(-1)^n[\tfrac{n}{p^s}]!\tbinom{k_j}{n}\big(V_j-1\big)^{[n]}\big)\big).$$

The above formula shows that $\iota_0(x)-\iota(x)\in\operatorname{Ker}f\circ\iota'$, for all $x\in S_\square$. By étaleness (Remark 2.2), this implies that ι_0 admits a unique extension to a morphism $\iota_0:S\to\Lambda'$ such that $\iota_0(x)-\iota(x)\in\operatorname{Ker}f\circ\iota'$, and by unicity, $\iota'\circ\iota_0$ is the natural injection $S\to S\Lambda_s$. This shows that $\iota'(\Lambda')$ contains S. Finally, $\operatorname{Ker}f$ is generated by the $(V_j-1)^{[k]}$'s and by elements of the form $x-\iota(x)$, for $x\in S$, but these are in the image of the PD-ideal generated by the (V_j-1)'s. Hence the image of the PD-ideal generated by the (V_j-1)'s by ι' contains the log-PD-ideal generated by $\operatorname{Ker}f$ which shows that ι' is indeed surjective.

To prove the statement about filtrations, we argue by induction on r. Applying ∂_j, for $0\le j\le d$, and arguing by induction on $|\mathbf{k}|$, shows that $x_{\mathbf{k}}\in F^{r-|\mathbf{k}|}\Lambda$, if $\mathbf{k}\ne 0$. Now, this implies that $x_0\in\Lambda\cap F^rS\Lambda_s$. But $x_0=f(x_0)$, because $x_0\in\Lambda$, and $f(F^rS\Lambda_s)\subset F^r\Lambda$ because f is the identity on $F^{r_2}\Lambda$, kills $(V_j-1)^{[k_j]}$ if $k_j\ge 1$, and sends $F^{r_1}S$ to $F^{r_1}\Lambda$ since it coincides with ι on S. So $x_0\in F^r\Lambda$, which concludes the proof. □

2.6.2. *The filtered Poincaré Lemma.* Let, as usual, $\Omega^1=\oplus_{j=0}^d\mathbf{Z}\,\frac{dX_j}{X_j}$ and let $\Omega^n=\wedge^n\Omega^1$. So $\Omega^n_{S\Lambda_s/\Lambda}=S\Lambda_s\otimes\Omega^n$. We filter the de Rham complex of $S\Lambda_s$ as usual by subcomplexes

$$F^r\Omega^\bullet_{S\Lambda_s/\Lambda}:=F^rS\Lambda_s\to F^{r-1}S\Lambda_s\otimes\Omega^1\to F^{r-2}S\Lambda_s\otimes\Omega^2\to\cdots$$

Lemma 2.37. (Filtered Poincaré Lemma) *The natural map*

$$F^r\Lambda \to F^r\Omega^\bullet_{S\Lambda_s/\Lambda}$$

is a p^s-quasi-isomorphim.

Proof. Let $\varepsilon : F^r\Lambda \to F^rS\Lambda_s$ be the natural injection. A contracting (Λ-linear) homotopy can be defined as follows. We define the map

$$h^0 : F^rS\Lambda_s \to F^r\Lambda, \qquad \sum_{\mathbf{k}\in\mathbf{N}^{d+1}} a_{\mathbf{k}} \prod_{j=0}^{d} \big(V_j-1\big)^{[k_j]} \mapsto a_0.$$

Clearly $h^0\varepsilon = \mathrm{Id}$. For $q>0$, we define the map

$$h^q : F^{j-q}S\Lambda_s \otimes \Omega^q \to F^{j-q+1}S\Lambda_s \otimes \Omega^{q-1}$$

by the following formula

$$a\prod_{j=0}^{d}\big(V_j-1\big)^{[k_j]} V_{j_1}\tfrac{dX_{j_1}}{X_{j_1}} \wedge \cdots \wedge V_{j_q}\tfrac{dX_{j_q}}{X_{j_q}} \quad (0\le j_1<\cdots<j_q\le d)$$

$$\mapsto \begin{cases} \frac{k_j!}{(k_j+\delta_{jj_1})!}\,\frac{[(k_j+\delta_{jj_1})/p^s]!}{[k_j/p^s]!}\, a\prod_{j=0}^{d}\big(V_j-1\big)^{[k_j+\delta_{jj_1}]} V_{j_2}\frac{dX_{j_2}}{X_{j_2}} \wedge \cdots \wedge V_{j_q}\frac{dX_{j_q}}{X_{j_q}} & \text{if } k_j=0 \text{ for } 0\le j<j_1,\\ 0 & \text{otherwise.}\end{cases}$$

We have $\varepsilon h^0 + h^1 d = \mathrm{Id}$ and $dh^q + h^{q+1}d = \mathrm{Id}$, as wanted. (The correcting factor $\frac{k_j!}{(k_j+\delta_{jj_1})!}\,\frac{[(k_j+\delta_{jj_1})/p^s]!}{[k_j/p^s]!}$ is due to the fact that we have partial divided powers; its valuation is between 0 and $-s$ depending on the valuation of $k_j+\delta_{jj_1}$, which explains why we only get a p^s-quasi-isomorphism.) □

2.6.3. *Extending the actions of φ and G_R.* We take $s=0$ if $p\ge 3$ and $s=1$ if $p=2$, and define:

- $E_R^{[u,v]}$, $E_{R_\infty}^{[u,v]}$ as $S\Lambda_s$ for $S=R_\varpi^{[u,v]}$, $\Lambda=\mathbf{A}_R^{[u,v]}$ or $\mathbf{A}_{R_\infty}^{[u,v]}$, and $\iota=\iota_{\mathrm{cycl}}$.
- $E_{\overline{R}}^{\mathrm{PD}}$ as $S\Lambda_s$ for $S=R_\varpi^{\mathrm{PD}}$, $\Lambda=\mathbf{A}_{\mathrm{cr}}(\overline{R})$, and $\iota=\iota_{\mathrm{Kum}}$,
- $E_{\overline{R}}^{[u,v]}$ as $S\Lambda_s$ for $S=R_\varpi^{[u,v]}$, $\Lambda=\mathbf{A}_{\overline{R}}^{[u,v]}$ and $\iota=\iota_{\mathrm{Kum}}$.

Lemma 2.38. (i) $E_{\overline{R}}^{\mathrm{PD}} \subset E_{\overline{R}}^{[u,v]}$.
(ii) $E_R^{[u,v]} \subset E_{R_\infty}^{[u,v]} \subset E_{\overline{R}}^{[u,v]}$.

Proof. (i) is obvious from the structure result of Lemma 2.36. To prove (ii), granted Lemma 2.36, we just have to check that $(1-\frac{X_0}{\pi_K})^{[k]} \in E_{\overline{R}}^{[u,v]}$. But we have $1-\frac{X_0}{\pi_K} = \big(1-\frac{X_0}{[\varpi^\flat]}\big) + \frac{X_0}{[\varpi^\flat]}\frac{\pi_K-[\varpi^\flat]}{\pi_K}$. Now $1-\frac{X_0}{[\varpi^\flat]}$ has divided powers of level s in $E_{\overline{R}}^{[u,v]}$ by construction, so we just have to show that $\frac{\pi_K-[\varpi^\flat]}{[\varpi^\flat]}$ admits divided powers of level s in $\mathbf{A}_{\overline{R}}^{[u,v]}$. For that, we are going to use the assumptions $2p(\delta_R+1)<e_0$, $u\ge\frac{p-1}{p}$, $v<p$ (and $u=\frac{3}{4}$, $v=\frac{3}{2}$, if $p=2$). We have

$$\tfrac{\pi_K-[\varpi^\flat]}{[\varpi^\flat]} \in \tfrac{\xi}{\pi_i^{\delta_R+1}}\mathbf{A}_R^+[[\tfrac{p}{\pi_i^{2\delta_R}}]] \quad\text{and}\quad \xi=\pi^{e_0}+p\alpha,\ \text{with } \alpha\in\mathbf{A}_R^+.$$

But, $v^{[u,v]}(\frac{p}{\pi_i^{2\delta_R}})\ge 0$, since $v<p$, and

$$v^{[u,v]}\big(\tfrac{\xi}{\pi_i^{\delta_R+1}}\big) \ge \min\big((e_0-\delta_R-1)\tfrac{u}{e_0}, 1-(\delta_R+1)\tfrac{v}{e_0}\big) \ge \min\big((1-\tfrac{1}{2p})u, 1-\tfrac{1}{2}\big).$$

It follows that $v^{[u,v]}(\frac{\pi_K-[\varpi^\flat]}{[\varpi^\flat]})\ge\frac{1}{(p-1)p^s}$, for all p, which allows to conclude. □

Remark 2.39. It follows from the proof that we could have also defined $E_{\overline{R}}^{[u,v]}$ using ι_{cycl} instead of ι_{Kum}.

If (S,Λ) is any of the pairs above, there are natural, commuting, actions of G_R and φ on $S\otimes\Lambda$ (tensor actions, with G_R acting trivially on S, and $\varphi=\varphi_{\mathrm{Kum}}$ or φ_{cycl} on S, if $\iota=\iota_{\mathrm{Kum}}$ or ι_{cycl}).

Lemma 2.40. (i) *φ extends uniquely to continuous morphisms*

$$E_{\overline{R}}^{\mathrm{PD}} \to E_{\overline{R}}^{\mathrm{PD}}, \quad E_R^{[u,v]} \to E_R^{[u,v/p]}, \quad E_{R_\infty}^{[u,v]} \to E_{R_\infty}^{[u,v/p]}, \quad E_{\overline{R}}^{[u,v]} \to E_{\overline{R}}^{[u,v/p]}.$$

(ii) *The action of G_R extends uniquely to continuous actions on $E_{\overline{R}}^{\mathrm{PD}}$, $E_R^{[u,v]}$, $E_{R_\infty}^{[u,v]}$ and $E_{\overline{R}}^{[u,v]}$, commuting with φ. Moreover $E_{R_\infty}^{[u,v]}$ is the fixed point of $E_{\overline{R}}^{[u,v]}$ by $\mathrm{Gal}(\overline{R}[\frac{1}{p}]/R_\infty[\frac{1}{p}]) \subset G_R$.*

Proof. Lemma 2.36 reduces the statement to the verification that $(1-\varphi(V_j))^{[k]}$ and $(1-\sigma(V_j))^{[k]}$ belong to the relevant ring, for $0 \le j \le d$ and $\sigma \in G_R$.

Let us start with Frobenius. If we use ι_{Kum}, the result is obvious as, in that case, $\varphi(V_j) = V_j^p$. If we use ι_{cycl}, the same argument is still valid for $1 \le j \le d$, and we are left to check that $(1-\frac{X_0^p}{\varphi(\pi_K)})^{[k]} \in E_R^{[u,v/p]}$. But

$$1-\tfrac{X_0^p}{\varphi(\pi_K)} = \big(1-\tfrac{X_0^p}{\pi_K^p}\big)\tfrac{\pi_K^p}{\varphi(\pi_K)} + \big(1-\tfrac{\pi_K^p}{\varphi(\pi_K)}\big),$$

and the result follows from the fact that φ_{cycl} is admissible, hence $1-\frac{\pi_K^p}{\varphi(\pi_K)}$ has divided powers of level s in $\mathbf{A}_R^{[u,v/p]}$.

For the action of G_R, we have to check that $\big(1-\frac{X_j}{\sigma(\iota(X_j))}\big)^{[k]}$ makes sense. If $\iota = \iota_{\mathrm{Kum}}$ or if $j \ge 1$, we have $\sigma(\iota(X_j)) = [\varepsilon_j^c(\sigma)]\iota(X_j)$ for some $c(\sigma) \in \mathbf{Z}_p$. So we can write

$$1-\tfrac{X_j}{\sigma(\iota(X_j))} = \big(1-\tfrac{X_j}{\iota(X_j)}\big) + (1-[\varepsilon]^{-c_j(\sigma)})\tfrac{X_j}{\iota(X_j)},$$

and the results follows from the fact that $1-[\varepsilon]^{-c_j(\sigma)}$ is divisible by π which has divided powers of any level. The case $j=0$ and $\iota = \iota_{\mathrm{cycl}}$ is more subtle: we have to consider

$$1-\tfrac{X_0}{\sigma(\pi_K)} = \big(1-\tfrac{X_0}{\pi_K}\big) - \tfrac{X_0}{\pi_K}\tfrac{\pi_K-\sigma(\pi_K)}{\sigma(\pi_K)}.$$

But $\frac{\pi_K-\sigma(\pi_K)}{\sigma(\pi_K)} \in \pi_i^{-\delta_R-1}\pi\mathbf{A}_R^{[u,v]}$, and $\pi_i^{-\delta_R-1}\pi = \pi_i^{-\delta_R-1}\pi_1\xi$ is divisible by ξ in $\mathbf{A}_R^{[u,v]}$ (by Lemma 2.35), and and ξ has divided powers in $\mathbf{A}_R^{[u,v]}$ since it has already divided powers in $\mathbf{A}_F^{\mathrm{PD}}$.

Finally, the "Moreover" is obvious from Lemma 2.36, Remark 2.39, and the fact that V_j is fixed by $\mathrm{Gal}(\overline{R}[\frac{1}{p}]/R_\infty[\frac{1}{p}])$ if we use ι_{cycl}. □

Example 2.41. If $R = \mathscr{O}_K$, we have $E_{\overline{R}}^{\mathrm{PD}} = \widehat{\mathbf{A}}_{\mathrm{st}}$, where $\widehat{\mathbf{A}}_{\mathrm{st}}$ is the p-adic completion of the ring $\mathbf{A}_{\mathrm{cr}}[\frac{Y^k}{k!},\ k \in \mathbf{N}]$, where $Y = \frac{X_0}{[\varpi^\flat]} - 1$. Since $\varphi(X_0) = X_0^p$, we have $\varphi(Y) = (1+Y)^p - 1$, and since $\sigma([\varpi^\flat]) = [\varepsilon]^{c(\sigma)}[\varpi^\flat]$, with $c(\sigma) \in \mathbf{Z}_p$ if $\sigma \in G_K$, we have $\sigma(Y) = [\varepsilon]^{-c(\sigma)}(1+Y) - 1$.

A more conceptual description would be to define $\widehat{\mathbf{A}}_{\mathrm{st}}$ as the H^0 of $\mathscr{O}_{\overline{K}}$ for the log-crystalline cohomology:

$$\widehat{\mathbf{A}}_{\mathrm{st},n} = H^0_{\mathrm{cr}}(\mathscr{O}_{\overline{K},n}^\times/r_{\varpi,n}^{\mathrm{PD}}), \quad \widehat{\mathbf{A}}_{\mathrm{st}} = \operatorname*{proj\,lim}_n H^0_{\mathrm{cr}}(\mathscr{O}_{\overline{K},n}^\times/r_{\varpi,n}^{\mathrm{PD}}).$$

3. Local syntomic computations

The goal of this section is to construct, *if K contains enough roots of unity*, a natural "quasi-isomorphism" between the complex $\mathrm{Syn}(R,r)$ computing syntomic cohomology of R, and a complex $\mathrm{Cycl}(R_\varpi^{[u,v]},r)$ that is closer to Galois cohomology. For example, in dimension 0, choosing a basis of Ω^1, the two complexes become (with $\partial_{\mathrm{Kum}} = X_0\frac{d}{dX_0}$ and $\partial_{\mathrm{cycl}} = (1+X_0)\frac{d}{dX_0}$):

$$\mathrm{Syn}(\mathscr{O}_K,r): \quad F^r r_\varpi^{\mathrm{PD}} \xrightarrow{(\partial_{\mathrm{Kum}},p^r-\varphi_{\mathrm{Kum}})} F^{r-1}r_\varpi^{\mathrm{PD}} \oplus r_\varpi^{\mathrm{PD}} \xrightarrow{-(p^r-p\varphi_{\mathrm{Kum}})+\partial_{\mathrm{Kum}}} r_\varpi^{\mathrm{PD}},$$

$$\mathrm{Cycl}(r_\varpi^{[u,v]},r): \quad F^r r_\varpi^{[u,v]} \xrightarrow{(\partial_{\mathrm{cycl}},p^r-\varphi_{\mathrm{cycl}})} F^{r-1}r_\varpi^{[u,v]} \oplus r_\varpi^{[u,v/p]} \xrightarrow{-(p^r-p\varphi_{\mathrm{cycl}})+\partial_{\mathrm{cycl}}} r_\varpi^{[u,v/p]}.$$

The main step (Proposition 3.10) is the shift from Frobenius φ_{Kum} to φ_{cycl} (this uses standard crystalline techniques). But, since φ_{cycl} is not defined on R_{ϖ}^{PD}, one must first replace R_{ϖ}^{PD} by $R_{\varpi}^{[u,v]}$. This is done through a string of "quasi-isomorphisms"

$$\mathrm{Syn}(R,r) := \mathrm{Kum}(R_{\varpi}^{\mathrm{PD}},r) \cong \mathrm{Kum}(R_{\varpi}^{[u]},r) \cong \mathrm{Kum}^{\psi}(R_{\varpi}^{[u]},r) \cong \mathrm{Kum}^{\psi}(R_{\varpi}^{[u,v]},r) \cong \mathrm{Kum}(R_{\varpi}^{[u,v]},r),$$

using techniques coming from (φ,Γ)-module theory. (Actually, it is better to truncate in degree r as the constants involved in the "quasi-isomorphism" $\mathrm{Kum}^{\psi}(R_{\varpi}^{[u]},r) \cong \mathrm{Kum}^{\psi}(R_{\varpi}^{[u,v]},r)$ are too big in degrees $\geq r$.)

3.1. **Local complexes computing syntomic cohomology.** If

$$S = R_{\varpi}^{\mathrm{PD}}, R_{\varpi}^{[u]}, R_{\varpi}^{[u,v]},$$

set

$$S' = R_{\varpi}^{\mathrm{PD}}, R_{\varpi}^{[u]}, R_{\varpi}^{[u,v/p]}.$$

We endow all these rings with Frobenius φ_{Kum}, and we have $\varphi(S) \subset S'$ in all cases.

If $i \in \mathbf{N}$, let

$$J_i = \{0 \leq j_1 < \cdots < j_i \leq d\} \quad \text{and} \quad \omega_{\mathbf{j}} = \big(\tfrac{dX_{j_1}}{X_{j_1}} \wedge \cdots \wedge \tfrac{dX_{j_i}}{X_{j_i}}\big) \text{ if } \mathbf{j} = (j_1,\ldots,j_i) \in J_i.$$

We define φ_{Kum} and ψ_{Kum} on Ω_S^i by

$$\varphi_{\mathrm{Kum}}\big(\sum_{\mathbf{j}\in J_i} f_{\mathbf{j}}\omega_{\mathbf{j}}\big) = \sum_{\mathbf{j}\in J_i} \varphi_{\mathrm{Kum}}(f_{\mathbf{j}})\omega_{\mathbf{j}} \quad \psi_{\mathrm{Kum}}\big(\sum_{\mathbf{j}\in J_i} f_{\mathbf{j}}\omega_{\mathbf{j}}\big) = \sum_{\mathbf{j}\in J_i} \psi_{\mathrm{Kum}}(f_{\mathbf{j}})\omega_{\mathbf{j}}.$$

(This is not the natural definition, as we have $d(\varphi_{\mathrm{Kum}}(f)) = p\varphi_{\mathrm{Kum}}(df)$ with this definition. The reason for doing this is that we need the left inverse ψ of φ which makes it necessary, for integrality reasons, to divide the natural Frobenius by suitable powers of p. The price to pay is that these powers of p are showing up in the complexes that we are about to define.)

We define $F^r\Omega_S^i$ as the sub-S-module of Ω_S^i generated by $F^rS \cdot \Omega_S^i$; we have thus

$$F^r\Omega_S^i = \oplus_{\mathbf{j}\in J_i} F^rS \cdot \omega_{\mathbf{j}}.$$

We filter the de Rham complex $\Omega_S^\bullet$ by subcomplexes

$$F^r\Omega_S^\bullet := F^rS \to F^{r-1}\Omega_S^1 \to F^{r-2}\Omega_S^2 \to \ldots$$

We define the complex $\mathrm{Kum}(S,r)$ as

$$\mathrm{Kum}(S,r) := [\ F^r\Omega_S^\bullet \xrightarrow{p^r - p^\bullet\varphi_{\mathrm{Kum}}} \Omega_{S'}^\bullet\ ].$$

We define the *syntomic complex* $\mathrm{Syn}(R,r)$ of R and the *syntomic cohomology* $H_{\mathrm{syn}}^*(R,r)$ of R as:

$$\mathrm{Syn}(R,r) := \mathrm{Kum}(R_{\varpi}^{\mathrm{PD}},r) \quad \text{and} \quad H_{\mathrm{syn}}^*(R,r) := H^*(\mathrm{Syn}(R,r)).$$

For $n \in \mathbf{N}$, we define the syntomic complexes modulo p^n and syntomic cohomology modulo p^n of R as

$$\mathrm{Syn}(R,r)_n := \mathrm{Syn}(R,r) \otimes_{\mathbf{Z}} \mathbf{Z}/p^n, \quad \text{and} \quad H_{\mathrm{syn}}^*(R_n,r) := H^*(\mathrm{Syn}(R,r)_n).$$

If $S = R_{\varpi}^{[u]}, R_{\varpi}^{[u,v]}$, set $S'' = R_{\varpi}^{[pu]}, R_{\varpi}^{[pu,v]}$. We define the complex $\mathrm{Kum}^{\psi}(S,r)$, using ψ_{Kum} instead of φ_{Kum}, as:

$$\mathrm{Kum}^{\psi}(S,r) := [\ F^r\Omega_S^\bullet \xrightarrow{p^r\psi_{\mathrm{Kum}} - p^\bullet} \Omega_{S''}^\bullet\ ].$$

Finally, if $S = R_{\varpi}^{[u,v]}$, we define $\mathrm{Cycl}(S,r)$ using φ_{cycl} in place of φ_{Kum} (we cannot define $\mathrm{Cycl}(S,r)$ for $S = R_{\varpi}^{\mathrm{PD}}, R_{\varpi}^{[u]}$ as φ_{cycl} does not send S into S' in these cases (except if $K = F$, i.e if K is absolutely unramified)):

$$\mathrm{Cycl}(S,r) := [\ F^r\Omega_S^\bullet \xrightarrow{p^r - p^\bullet\varphi_{\mathrm{cycl}}} \Omega_{S'}^\bullet\ ].$$

For the remainder of this section, we write simply φ and ψ for φ_{Kum} and ψ_{Kum}.

3.2. **Change of disk of convergence.** We are going to show that $\mathrm{Kum}(R_{\varpi}^{[u]}, r)$ computes the syntomic cohomology up to a constant depending on r. For that we will need Lemma 3.1 below. For $S = r_{\varpi}^{\mathrm{PD}}$ or $r_{\varpi}^{[u]}$, we denote by $v_{X_0} : S \to \mathbf{N} \cup \{+\infty\}$ the valuation relative to X_0: if $f = \sum a_i {X_0}^i$ then $v_{X_0}(f) = \inf\{i \in \mathbf{N},\ a_i \neq 0\}$ and, if $N \in \mathbf{N}$, we define S_N as $\{f \in S,\ v_{X_0}(f) \geq N\}$. We define $R_{\varpi,N}^{\mathrm{PD}}$ and $R_{\varpi,N}^{[u]}$ as the topological closures of $r_{\varpi,N}^{\mathrm{PD}} \otimes_{r_{\varpi}^+} R_{\varpi}^+$ and $R_{\varpi,N}^{[u]} \otimes_{r_{\varpi}^+} R_{\varpi}^+$ in $R_{\varpi,N}^{\mathrm{PD}}$ and $R_{\varpi,N}^{[u]}$.

Lemma 3.1. *Let $S = R_{\varpi}^{\mathrm{PD}}, R_{\varpi}^{[u]}$. If $s \in \mathbf{Z}$ and $N \geq 1$, then $(1 - p^{-s}\varphi) : S_N[\frac{1}{p}] \to S_N[\frac{1}{p}]$ is bijective. Moreover, if $S = R_{\varpi}^{\mathrm{PD}}$ (resp. $S = R_{\varpi}^{[u]}$),*

- *there exists $N(r,e)$ (resp. $N(r,e,u)$) such that $(1-p^{-s}\varphi)$ is $p^{N(r,e)}$-bijective (resp. $p^{N(r,e,u)}$-bijective) on S_1 for all $s \leq r$,*
- *if $N \geq se$, (resp. $N \geq se/u(p-1)$), then $1 - p^{-s}\varphi$ is bijective on S_N.*

Proof. The estimates being easier for $R_{\varpi}^{[u]}$ (since $[\frac{i}{e}]!$ is replaced by $[\frac{iu}{e}]$ which behaves better as a function of i), we will just treat the case $S = R_{\varpi}^{\mathrm{PD}}$. An element f of S_N can be written as $\sum_{i \geq N} f_i \frac{X_0^i}{[\frac{i}{e}]!}$, where $f_i \in R_{\varpi}^+$ goes to 0 when $i \to \infty$. We have then

$$p^{-ks}\varphi^k(f) = \sum_{i \geq N} p^{-ks} \frac{[\frac{p^k i}{e}]!}{[\frac{i}{e}]!} \varphi^k(f_i) \frac{X_0^{p^k i}}{[\frac{p^k i}{e}]!}.$$

Now,

$$\inf_{i \geq 1} \big(v_p([\frac{p^k i}{e}]!) - v_p([\frac{i}{e}]!) - ks\big) = v_p([\frac{p^k}{e}]!) - ks \geq \frac{p^{k-1}}{e} - 1 - ks$$

goes to $+\infty$ when $k \to +\infty$. We deduce that, for $f \in S_N[\frac{1}{p}]$, the series $\sum_{k \in \mathbf{N}} p^{-ks}\varphi^k(f)$ converges in $S_N[\frac{1}{p}]$. Since the sum g satisfies $(1 - p^{-s}\varphi)g = f$, we deduce that $(1 - p^{-s}\varphi)$ is invertible, with inverse $\sum_{k \in \mathbf{N}} p^{-ks}\varphi^k$, and we can take $N(r,e) = \inf_{k \in \mathbf{N}}(-1 - kr + p^{k-1}/e)$.

Now, if $i \geq se$, then

$$v_p([\frac{p^k i}{e}]!) - v_p([\frac{i}{e}]!) = \sum_{j=0}^{k-1} [\frac{p^j i}{e}] \geq \sum_{j=0}^{k-1} sp^j \geq sk.$$

This implies the last statement of the lemma. □

Lemma 3.2. *Let $r, s \in \mathbf{N}$.*

(i) *If $1/(p-1) \leq u \leq 1$, the map $p^s - \varphi$ induces a p^{s+r}-isomorphism $F^r\Omega^i_{R_{\varpi}^{[u]}}/F^r\Omega^i_{R_{\varpi}^{\mathrm{PD}}} \simeq \Omega^i_{R_{\varpi}^{[u]}}/\Omega^i_{R_{\varpi}^{\mathrm{PD}}}$.*

(ii) *If $u' \leq u \leq pu'$, the map $p^s - \varphi$ induces a p^{s+r}-isomorphism $F^r\Omega^i_{R_{\varpi}^{[u]}}/F^r\Omega^i_{R_{\varpi}^{[u']}} \simeq \Omega^i_{R_{\varpi}^{[u]}}/\Omega^i_{R_{\varpi}^{[u']}}$.*

Proof. If $\mathbf{j} \in J_i$, we have $\varphi(\omega_{\mathbf{j}}) = p^j\omega_{\mathbf{j}}$. This makes it possible, by decomposing everything in the basis of the $\omega_{\mathbf{j}}$'s, to only treat the case $i = 0$ (the map becoming $p^s - p^i\varphi$). Let $A = R_{\varpi}^{\mathrm{PD}}$ or $R_{\varpi}^{[u']}$ and $B = R_{\varpi}^{[u]}$; we have $A \subset B$ and $\varphi(B) \leq A$.

To show p^s-injectivity, since $F^rA = A \cap F^rB$, it suffices to show that $(p^s - p^i\varphi)x \subset A$ implies that $x \in A$. But this follows from the fact that $\varphi(B) \subset A$ and the identity $p^sx = (p^s - p^i\varphi)x + p^i\varphi(x)$.

Now let $f \in B$. Write f as $f_1 + f_2$ with $f_2 \in R_{\varpi,se/u(p-1)}^{[u]}$ and $f_1 \in p^{-[u(se/u)/e(p-1)]}R_{\varpi}^+ \subset p^{-s}A$. By Lemma 3.1, we can write f_2 as $(1 - p^{i-s}\varphi)g$, with $g \in B$. By Remark 2.6, we can write $g = g_1 + g_2$ with $g_1 \in F^rB$ and $g_2 \in p^{-[ru]}R_{\varpi}^+$, hence $(1 - p^{i-s}\varphi)p^{-s}g_2 \in p^{-s-r}A$ and $f - (1 - p^{i-s}\varphi)g_1 \in p^{-s-r}A$. Finally, we obtain $f \in p^{-s-r}A + p^{-s}(p^s - p^i\varphi)F^rB$, which allows to conclude. □

Proposition 3.3. (i) *For $\frac{1}{p-1} \leq u \leq 1$, the morphism of complexes $\mathrm{Kum}(R_{\varpi}^{\mathrm{PD}}, r) \to \mathrm{Kum}(R_{\varpi}^{[u]}, r)$ induced by the injection $R_{\varpi}^{\mathrm{PD}} \subset R_{\varpi}^{[u]}$ is a p^{6r}-quasi-isomorphism.*

(ii) *If $u' \leq u \leq pu'$, the morphism of complexes* $\mathrm{Kum}(R_\varpi^{[u']}, r) \to \mathrm{Kum}(R_\varpi^{[u]}, r)$ *induced by the injection* $R_\varpi^{[u']} \subset R_\varpi^{[u]}$ *is a p^{6r}-quasi-isomorphism.*

Proof. The above lemma, applied with $s = r-1$ and $s = r$, allows to show that the cohomology of the quotient complexes is annihilated by p^{6r}. □

3.3. **(φ, ∂)-modules and (ψ, ∂)-modules.** We are going to show that $\mathrm{Kum}^\psi(R_\varpi^{[u]}, r)$ computes syntomic cohomology.

Lemma 3.4. *The following commutative diagram*

$$\begin{array}{ccc} F^r\Omega^\bullet_{R_\varpi^{[u]}} & \xrightarrow{p^r - p^\bullet\varphi} & \Omega^\bullet_{R_\varpi^{[u]}} \\ \downarrow \mathrm{Id} & & \downarrow \psi \\ F^r\Omega^\bullet_{R_\varpi^{[u]}} & \xrightarrow{p^r\psi - p^\bullet} & \Omega^\bullet_{R_\varpi^{[pu]}} \end{array}$$

defines a quasi-isomorphism from $\mathrm{Kum}(R_\varpi^{[u]}, r)$ *to* $\mathrm{Kum}^\psi(R_\varpi^{[u]}, r)$.

Proof. Let $S = R_\varpi^{[u]}$. Since $\psi : \Omega^\bullet_S \to \Omega^\bullet_{R_\varpi^{[pu]}}$ is surjective, the quotient complex is trivial. It suffices thus to show that the kernel complex

$$0 \to S^{\psi=0} \to (\Omega^1_S)^{\psi=0} \to (\Omega^2_S)^{\psi=0} \to \dots$$

is acyclic. We have

$$\Omega^{j,\psi=0}_S = S^{\psi=0} \otimes \Omega^j,$$

where

$$\Omega^1 = \oplus_{1\leq j\leq d+1}\mathbf{Z}\,\frac{dX_j}{X_j} \quad \text{and} \quad \Omega^i = \wedge^i\Omega^1.$$

Now, $S^{\psi=0} = \oplus_{\alpha\in\{0,\cdots,p-1\}^{d+1},\alpha\neq 0} S_\alpha$, and this decomposition induces a direct sum decomposition of the above de Rham complex, so we can argue for every α separately that the following de Rham complex

$$0 \to S_\alpha \to S_\alpha \otimes \Omega^1 \to S_\alpha \otimes \Omega^2 \to \dots$$

is exact, and it is enough to prove this modulo p. But Lemma 2.9 tells us that this complex has a very simple shape modulo p: if $d = 0$, it is just $S_\alpha \xrightarrow{\alpha_0} S_\alpha$, if $d = 1$, it is the total complex attached to the double complex

$$\begin{array}{ccc} S_\alpha & \xrightarrow{\alpha_0} & S_\alpha \\ \downarrow \alpha_1 & & \downarrow \alpha_1 \\ S_\alpha & \xrightarrow{\alpha_0} & S_\alpha \end{array}$$

and, for general d, it is the total complex attached to a $(d+1)$-dimensional cube, with all vertices equal to S_α and arrows in the i-th direction equal to α_i. As one of the α_i is invertible by assumption, this implies that the cohomology of the total complex is 0. □

3.4. **Change of annulus of convergence.**

Lemma 3.5. *If $u \leq 1 \leq v$, the natural morphism* $\mathrm{F}^r R_\varpi^{[u,v]}/\mathrm{F}^r R_\varpi^{[u]} \to R_\varpi^{[u,v]}/R_\varpi^{[u]}$ *is a p^r-isomorphism.*

Proof. It suffices to treat the case when $R = \mathscr{O}_K$. The above map is clearly injective. To prove p^r-surjectivity, we need to verify that $p^{r+[kv]}X_0^{-ke}$ is in the image. For that, note that p being divisible by ϖ^e in $\mathscr{O}_K$, we can write $p = X_0^e A + BP$, with $A, B \in r_\varpi^+$. This implies that we can write $p^{r+[kv]}$ as $B_k P^r + A_k X_0^{e[kv]}$, and $p^{r+[kv]}X_0^{-ke}$ as $A_k X_0^{([kv]-k)e} + X_0^{-ke} B_k P^r$. But $A_k X_0^{([kv]-k)e} \in r_\varpi^+ \subset r_\varpi^{[u]} \subset r_\varpi^{[u,v]}$, hence $X_0^{-ke} B_k P^r \in r_\varpi^{[u,v]}$, and its image in $r_\varpi^{[u,v]}[\frac{1}{p}]$ being divisible by P^r, we have written $p^{r+[kv]}X_0^{-ke}$ as a sum of an element from $F^r r_\varpi^{[u,v]}$ and an element from $r_\varpi^{[u]}$. This allows us to conclude. □

Lemma 3.6. *Let $u \leq 1 \leq v$. The natural morphism from* $\mathrm{Kum}^{\psi}(R_{\varpi}^{[u]}, r)$ *to* $\mathrm{Kum}^{\psi}(R_{\varpi}^{[u,v]}, r)$ *induces a p^{2r}-quasi-isomorphism*

$$\tau_{\leq r}\mathrm{Kum}^{\psi}(R_{\varpi}^{[u]}, r) \xrightarrow{\sim} \tau_{\leq r}\mathrm{Kum}^{\psi}(R_{\varpi}^{[u,v]}, r).$$

The same is true for the complexes modulo p^n.

Proof. Our map is induced by the following diagram

$$\begin{array}{ccc} F^r\Omega^{\bullet}_{R_{\varpi}^{[u]}} & \xrightarrow{p^r\psi - p^{\bullet}} & \Omega^{\bullet}_{R_{\varpi}^{[pu]}} \\ \downarrow & & \downarrow \\ F^r\Omega^{\bullet}_{R_{\varpi}^{[u,v]}} & \xrightarrow{p^r\psi - p^{\bullet}} & \Omega^{\bullet}_{R_{\varpi}^{[pu,v]}} \end{array}$$

It suffices to show that the mapping fiber

$$[\ F^r\Omega^{\bullet}_{R_{\varpi}^{[u,v]}}/F^r\Omega^{\bullet}_{R_{\varpi}^{[u]}} \xrightarrow{p^r\psi - p^{\bullet}} \Omega^{\bullet}_{R_{\varpi}^{[pu,v]}}/\Omega^{\bullet}_{R_{\varpi}^{[pu]}}\ ] \tag{3.7}$$

is p^{2r}-acyclic. By Lemma 3.5, we can ignore the filtration and, working in the basis of the $\omega_{\mathbf{j}}$ of Ω^i, it is enough to show that $p^r\psi - p^i : R_{\varpi}^{[u,v]}/R_{\varpi}^{[u]} \to R_{\varpi}^{[pu,v]}/R_{\varpi}^{[pu]}$ is a p^r-isomorphism if $i \leq r$. But $R_{\varpi}^{[u,v]}/R_{\varpi}^{[u]} \simeq R_{\varpi}^{[pu,v]}/R_{\varpi}^{[pu]}$ and $1 - p^s\psi$, for $s = r - i \geq 0$, is invertible on $R_{\varpi}^{[u,v]}/R_{\varpi}^{[u]}$ with inverse $1 + p^s\psi + p^{2s}\psi^2 + \cdots$ (this converges even if $s = 0$ because ψ is topologically nilpotent). This allows us to conclude.

For complexes modulo p^n, since the quotients $\mathrm{F}^r R_{\varpi}^{[u,v]}/\mathrm{F}^r R_{\varpi}^{[u]} \hookrightarrow R_{\varpi}^{[u,v]}/R_{\varpi}^{[u]}$ are p-torsion free, it again suffices to show that the mod p^n analog of the mapping fiber (3.7) is p^{2r}-acyclic. Note that the mod-p^n version of Lemma 3.5 holds (though we now only have a p^r-injection). Having that the rest of the argument is the same. □

Remark 3.8. Truncating is not absolutely necessary at this point, as ψ is very rapidly nilpotent, but the constants that come out involve $\log e$.

Corollary 3.9. *For $pu \leq v$, the natural map from* $\mathrm{Kum}(R_{\varpi}^{[u]}, r)$ *to*

$$\mathrm{Kum}(R_{\varpi}^{[u,v]}, r) := [\ F^r\Omega^{\bullet}_{R_{\varpi}^{[u,v]}} \xrightarrow{p^r - p^{\bullet}\varphi} \Omega^{\bullet}_{R_{\varpi}^{[u,v/p]}}\ ]$$

induces a p^{2c_e+2r}-quasi-isomorphism

$$\tau_{\leq r}\mathrm{Kum}(R_{\varpi}^{[u]}, r) \xrightarrow{\sim} \tau_{\leq r}\mathrm{Kum}(R_{\varpi}^{[u,v]}, r)$$

Proof. By Lemma 3.4 we can pass from the complex $\mathrm{Kum}(R_{\varpi}^{[u]}, r)$ to $\mathrm{Kum}^{\psi}(R_{\varpi}^{[u]}, r)$ and, by the above lemma, we can pass from $\tau_{\leq r}\mathrm{Kum}^{\psi}(R_{\varpi}^{[u]}, r)$ to $\tau_{\leq r}\mathrm{Kum}^{\psi}(R_{\varpi}^{[u,v]}, r)$. It remains thus to show that we can pass from $\mathrm{Kum}^{\psi}(R_{\varpi}^{[u,v]}, r)$ to $\mathrm{Kum}(R_{\varpi}^{[u,v]}, r)$. Or that the map induced by the following commutative diagram

$$\begin{array}{ccc} F^r\Omega^{\bullet}_{R_{\varpi}^{[u,v]}} & \xrightarrow{p^r - p^{\bullet}\varphi} & \Omega^{\bullet}_{R_{\varpi}^{[u,v/p]}} \\ \downarrow \mathrm{Id} & & \downarrow \psi \\ F^r\Omega^{\bullet}_{R_{\varpi}^{[u,v]}} & \xrightarrow{p^r\psi - p^{\bullet}} & \Omega^{\bullet}_{R_{\varpi}^{[pu,v]}} \end{array}$$

is a quasi-isomorphism. We can now use the same arguments as in the proof of Lemma 3.4 to conclude. □

3.5. **Change of Frobenius.** To pass from syntomic cohomology to étale cohomology, we will need to change the crystalline Frobenius into the (φ,Γ)-module Frobenius.

Proposition 3.10. *The complexes* $\mathrm{Kum}(R_{\varpi}^{[u,v]},r)$ *and* $\mathrm{Cycl}(R_{\varpi}^{[u,v]},r)$ *are* $2^{2(d+1)}$*-quasi-isomorphic.*

Proof. If $A\subset A'$ are filtered rings with a Frobenius $\varphi: A\to A'$, let

$$\mathrm{S}(A,r,\varphi)=[\ F^r\Omega_A^\bullet \xrightarrow{p^r-p^\bullet\varphi} \Omega_{A'}^\bullet\ ],$$

where φ on Ω_A^n is the divided Frobenius (by p^n). So, for example,

$$\mathrm{Kum}(R_{\varpi}^{[u,v]},r)=\mathrm{S}(R_{\varpi}^{[u,v]},r,\varphi_{\mathrm{Kum}})\quad\text{and}\quad \mathrm{Cycl}(R_{\varpi}^{[u,v]},r)=\mathrm{S}(R_{\varpi}^{[u,v]},r,\varphi_{\mathrm{cycl}}).$$

Since $(\mathbf{A}_R^{[u,v]},\varphi)\cong(R_{\varpi}^{[u,v]},\varphi_{\mathrm{cycl}})$, Lemma 2.40 provides us with morphisms of complexes

$$\mathrm{S}(R_{\varpi}^{[u,v]},r,\varphi_{\mathrm{Kum}})\to \mathrm{S}(E_R^{[u,v]},r,\varphi)\leftarrow \mathrm{S}(R_{\varpi}^{[u,v]},r,\varphi_{\mathrm{cycl}}).$$

Lemma 3.11 below shows that these are 2^{d+1}-quasi-isomorphisms, which allows to conclude. □

Let R_1,R_2 be two copies of $R_{\varpi}^{[u,v]}$: we have isomorphisms $\iota_i: R_{\varpi}^{[u,v]}\to R_i$. We set $X_{i,j}=\iota_i(X_j)$, if $i=1,2$, $0\le j\le d$.

Let $R_3=(R_1\otimes R_2)^{\mathrm{PD}}$, with respect to $\iota=\iota_2\circ\iota_1^{-1}$. Hence $R_3\cong E_R^{[u,v]}$ as a ring, without the actions of G_R or φ.

If $i=1,2$, let $\Omega_i^1=\oplus_{j=0}^d\mathbf{Z}\frac{dX_{i,j}}{X_{i,j}}$. Set $\Omega_3^1=\Omega_1^1\oplus\Omega_2^1$ and, if $i=1,2,3$, let $\Omega_i^n=\wedge^n\Omega_i^1$. Then $\Omega_{R_3}^n=R_3\otimes\Omega_3^n$. We filter the de Rham complex of R_3 as usual by subcomplexes

$$F^r\Omega_{R_3}^\bullet:=F^rR_3\to F^{r-1}R_3\otimes\Omega_3^1\to F^{r-2}R_3\otimes\Omega_3^2\to\cdots$$

Lemma 3.11. (Filtered Poincaré Lemma) *The natural maps*

$$F^r\Omega_{R_1}^\bullet\to F^r\Omega_{R_3}^\bullet\leftarrow F^r\Omega_{R_2}^\bullet$$

are 2^{d+1}*-quasi-isomorphims.*

Proof. By symmetry, it is enough to consider the first map. First, we claim that we have the ∂_{R_1} filtered Poincaré Lemma, i.e., that the following sequence

$$0\longrightarrow F^rR_2\longrightarrow F^rR_3\xrightarrow{\partial_{R_1}}F^{r-1}R_3\otimes\Omega_1^1\xrightarrow{\partial_{R_1}}\cdots$$

is 2-exact. Indeed, this is a special case of Lemma 2.37, with $S=R_1$ and $\Lambda=R_2$.

Now, we can extend the above to a sequence of maps of de Rham complexes

$$0\longrightarrow F^r\Omega_{R_2}^\bullet\longrightarrow F^rR_3\otimes\Omega_2^\bullet\xrightarrow{\partial_{R_1}}F^{r-1}R_3\otimes(\Omega_1^1\wedge\Omega_2^\bullet)\longrightarrow\cdots$$

The contracting homotopy used in the proof of Lemma 2.37, being R_2-linear, extends as well and shows that the rows of the above double complex are 2-exact. Since the total complex of the double complex

$$F^rR_3\otimes\Omega_2^\bullet\xrightarrow{\partial_{R_1}}F^{r-1}R_3\otimes(\Omega_1^1\wedge\Omega_2^\bullet)\longrightarrow\cdots$$

is equal to the de Rham complex $F^r\Omega_{R_3}^\bullet$, we are done. □

3.6. **Syntomic cohomology and de Rham cohomology.** We will show that, up to some universal constants, syntomic cohomology has a simple relation to de Rham cohomology. Let $S = R^{\mathrm{PD}}_{\varpi}$, $\varphi = \varphi_{\mathrm{Kum}}$, $r \geq 0$. Set

$$\mathrm{HK}(S,r) := [\Omega^\bullet_S \xrightarrow{p^r - p^\bullet\varphi} \Omega^\bullet_S], \quad \mathrm{DR}(S,r) := \Omega^\bullet_S/F^r.$$

We note that $\tau_{\leq r-1}\mathrm{DR}(S,r) \xrightarrow{\sim} \mathrm{DR}(S,r)$ and that the natural map $\tau_{\leq r+1}\mathrm{HK}(S,r) \to \mathrm{HK}(S,r)$ is a p^{2r}-quasi-isomorphism (since $1 - p^s\varphi$, $s \geq 1$, is invertible on Ω^{r+s}_S).

Proposition 3.12. (i) *The natural map*

$$\tau_{\leq r+1}\mathrm{Syn}(R,r) \to \mathrm{Syn}(R,r)$$

is a p^{2r}-quasi-isomorphism and $H^{r+1}(\mathrm{Syn}(R,r)) \xrightarrow{\sim} H^{r+1}(\mathrm{HK}(S,r))$.

(ii) *The complex $\tau_{\leq r-1}\mathrm{HK}(S,r)$ is p^N-acyclic, for a constant $N = N(e,d,p,r)$. Hence the natural map $\mathrm{DR}(S,r) \to \tau_{\leq r-1}\mathrm{Syn}(R,r)[1]$ is a p^N-quasi-isomorphism.*

(iii) *The above statements are valid also modulo p^n. Moreover, $H^{r+1}(\mathrm{HK}(S,r)_n)$ is, étale locally on R_n, $p^{N(r)}$-trivial.*

Proof. For (i) and (ii), we are going to argue p-adically: the mod-p^n argument for (iii) is analogous. Since we have

$$\mathrm{Syn}(R,r) = \mathrm{Kum}(R^{\mathrm{PD}}_{\varpi}, r) = [F^r\Omega^\bullet_S \xrightarrow{p^r - p^\bullet\varphi} \Omega^\bullet_S] = [\mathrm{HK}(S,r) \to \mathrm{DR}(S,r)],$$

the first claim of our proposition follows from what we noted just before the proposition.

For the second claim, write X_0S for $X_0S[\frac{1}{p}] \cap S$ and define $X_0\Omega^\bullet_S$ accordingly. Writing differentials in the basis of the $\omega_{\mathbf{j}}$'s and using Lemma 3.1, we obtain that the subcomplex $[X_0\Omega^\bullet_S \xrightarrow{p^r - p^\bullet\varphi} X_0\Omega^\bullet_S]$ is $p^{N_1(e,r,d)}$-acyclic. Hence $\mathrm{HK}(S,r)$ is $p^{N_2(e,r,d)}$-quasi-isomorphic to $[\Omega^\bullet_{\overline{S}} \xrightarrow{p^r - p^\bullet\varphi} \Omega^\bullet_{\overline{S}}]$, where $\overline{S} := S/X_0S \cong R^+_{\varpi}/X_0$.

Now, the following lemma shows that $\overline{S}$ is equipped with a left inverse ψ of φ verifying $\psi \circ \partial_i = p\,\partial_i \circ \psi$, if $1 \leq i \leq d$.

Lemma 3.13. *$\psi = \psi_{\mathrm{Kum}}$ stabilizes $X_0R^+_{\varpi}$.*

Proof. We have $\psi = p^{-d-1}\varphi^{-1} \circ \mathrm{Tr}$, with $\mathrm{Tr} = \mathrm{Tr}_{R^+_{\varpi}/\varphi(R^+_{\varpi})}$, and the trace can be computed by taking the sum over the conjugates: if $\eta = (\eta_0, \ldots, \eta_d)$ is a tuple of p-th roots of unity, we define an automorphism σ_η of $R^+_{\varpi}[\zeta_p]$, sending X_i to $\eta_i X_i$ (this gives us σ_η on $R^+_{\varpi,\square}[\zeta_p]$, and we extend it to $R^+_{\varpi}[\zeta_p]$ by étaleness). The $\sigma_\eta(x)$'s are the conjugates of x, hence $\mathrm{Tr}(x) = \sum_\eta \sigma_\eta(x)$, which makes it plain that, if x is divisible by X_0, then so is $\mathrm{Tr}(x)$. Now, if $\varphi(x) \in X_0R^+_{\varpi}$, then $x \in X_0R^+_{\varpi}$: this can be seen by successive approximations, using the last line of the proof of Lemma 2.7 and the fact that $\varphi(X_0R^+_{\varpi}) \subset X_0R^+_{\varpi}$. Hence, if $x \in X_0R^+_{\varpi}$, then $p^{d+1}\psi(x) \in X_0R^+_{\varpi}$ and, since $X_0R^+_{\varpi}[\frac{1}{p}] \cap R^+_{\varpi} = X_0R^+_{\varpi}$, this implies $\psi(x) \in X_0R^+_{\varpi}$. □

We can then use the arguments in the proof of Lemma 3.4 to deduce that $[\Omega^\bullet_{\overline{S}} \xrightarrow{p^r - p^\bullet\varphi} \Omega^\bullet_{\overline{S}}]$ is quasi-isomorphic to $[\Omega^\bullet_{\overline{S}} \xrightarrow{p^r\psi - p^\bullet} \Omega^\bullet_{\overline{S}}]$. But, in degree $i \leq r-1$, $p^r\psi - p^i$ is a p^r-isomorphism (and even a p^i-isomorphism), with inverse $-1 - p^{r-i}\psi - p^{2(r-i)}\psi^2 - \cdots$. Hence $\tau_{\leq r-1}[\Omega^\bullet_{\overline{S}} \xrightarrow{p^r\psi - p^\bullet} \Omega^\bullet_{\overline{S}}]$ is p^r-acyclic, which proves (ii).

For the very last claim of the proposition, since the map $\Omega^{r+1}_S \xrightarrow{1-p\varphi} \Omega^{r+1}_S$ is injective, it suffices to show that the map $\Omega^r_{S_n} \xrightarrow{1-\varphi} \Omega^r_{S_n}$ is surjective, which amounts to showing that $1 - \varphi$ is, étale locally, surjective on S_n. By dévissage we can assume $n = 1$, and then we are reduced to the statement that $x - x^p$ is, étale locally, a surjection, which is clear as Artin-Schreier extensions are étale. □

The following lemma relates the complex $\mathrm{DR}(S,r)$ which appears in (ii) of the above proposition to the usual de Rham complex.

Lemma 3.14. $\mathrm{DR}(S,r)$ *is quasi-isomorphic to $\sigma_{\leq r-1}\Omega^\bullet_R$ after inverting p. Here $\sigma_{\leq r-1}$ denotes the silly truncation.*

Proof. We have $r^+_{\varpi}[\frac{1}{p}]/F^r \cong K[P]/P^r$. Hence $S[1/p]/F^r \cong R \otimes_{\mathscr{O}_K} (K[P]/P^r)$ (we see that $R^{\mathrm{PD}}_{\varpi,\square}/F^r \cong R_\square \otimes (K[P]/P^r)$ by inspection and deduce the result for S by étaleness).

Writing differentials in the basis of the $\omega_{\mathbf{j}}$'s makes it possible to write $\mathrm{DR}(S,r)$ as the total complex of the double complex

$$\begin{tikzcd}
R\otimes(K[P]/P^r) \arrow[r, "(\partial_j)_{1\le j\le d}"] \arrow[d, "\partial_0"] & \big(R\otimes(K[P]/P^{r-1})\big)^d \arrow[r] \arrow[d, "\partial_0"] & \cdots \arrow[r] & R^{\binom{d}{r-1}} \arrow[r] \arrow[d, "\partial_0"] & 0 \arrow[r] \arrow[d] & \cdots \\
R\otimes(K[P]/P^{r-1}) \arrow[r, "(\partial_j)_{1\le j\le d}"] & \big(R\otimes(K[P]/P^{r-2})\big)^d \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Now, ∂_0 induces an isomorphism $PK[P]/P^r \cong K[P]/P^{r-1}$. Hence the subcomplex obtained by replacing $K[P]/P^s$ by $PK[P]/P^s$ in the first row is acyclic, and the quotient complex is clearly isomorphic to $\sigma_{\le r-1}\Omega^\bullet_R$. □

Remark 3.15. It follows from Lemma 3.14 that, if $i \le r-2$, then $H^i(\mathrm{DR}(S,r)) = H^i_{\mathrm{dR}}(\mathrm{Spf}\, R)$ after inverting p. On the other hand, $H^{r-1}(\mathrm{DR}(S,r))$ is much bigger than $H^{r-1}_{\mathrm{dR}}(\mathrm{Spf}\, R)$, since *we take all differential forms* modulo exact ones instead of closed ones. For example, if $r=1$, $r-1=0$, and $H^0(\mathrm{DR}(S,r))$ is actually R, and not just the constants.

Let $\mathscr{X}$ be a semistable formal scheme over $\mathscr{O}_K$ (i.e., locally of the form described in section 2.1.2 with $h=1$). Set $\mathscr{X}_{K,\mathrm{tr}} = \mathscr{X}_K^{\mathrm{an}} \setminus \mathscr{D}$, $\mathscr{D}$ being the divisor at infinity, and $\mathscr{X}_K^{\mathrm{an}}$ – the rigid analytic space associated to $\mathscr{X}$. Define

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K, \log\mathscr{D}) := \mathrm{R}\Gamma(\mathscr{X}_K^{\mathrm{an}}, \Omega^\bullet_{\mathscr{X}_K^{\mathrm{an}}}(\log\mathscr{D})),$$

where $\Omega^\bullet_{\mathscr{X}_K^{\mathrm{an}}}(\log\mathscr{D})$ denotes the (logarithmic) de Rham complex of $\mathscr{X}_K^{\mathrm{an}}$. If $\mathscr{X}$ is quasi-compact, we have

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K, \log\mathscr{D}) \simeq \mathrm{R}\Gamma(\mathscr{X}, \Omega^\bullet_{\mathscr{X}/\mathscr{O}_K^\times})_{\mathbf{Q}} := (\mathrm{holim}_n\, \mathrm{R}\Gamma(\mathscr{X}_n, \Omega^\bullet_{\mathscr{X}_n/\mathscr{O}_{K,n}^\times}))_{\mathbf{Q}}.$$

Corollary 3.16. *Let $\mathscr{X}$ be a quasi-compact formal semistable scheme over $\mathscr{O}_K$. There is a natural map*

$$\alpha_{r,i} : H^{i-1}_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}) \to H^i_{\mathrm{syn}}(\mathscr{X}, r)_{\mathbf{Q}}.$$

It is an isomorphism for $i \le r-1$ and an injection for $i = r$.

Proof. The very definition of syntomic cohomolgy gives us a natural map

$$(\mathrm{holim}_n\, \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_n, \mathscr{O}_{\mathscr{X}_n/W_n(k)}/\mathscr{J}^{[r]}_{\mathscr{X}_n/W_n(k)}))_{\mathbf{Q}} \to H^i_{\mathrm{syn}}(\mathscr{X}, r)_{\mathbf{Q}}$$

from crystalline cohomology to syntomic cohomology. We claim that the domain of this map can be identified with analytic de Rham cohomology. Indeed, we have a natural map from crystalline cohomology to analytic de Rham cohomology

$$\begin{aligned}(\mathrm{holim}_n\, \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_n, \mathscr{O}_{\mathscr{X}_n/W_n(k)}/\mathscr{J}^{[r]}_{\mathscr{X}_n/W_n(k)}))_{\mathbf{Q}} &\to (\mathrm{holim}_n\, \mathrm{R}\Gamma(\mathscr{X}_n, \Omega^\bullet_{\mathscr{X}_n/\mathscr{O}_{K,n}^\times}/F^r))_{\mathbf{Q}} \\ &\simeq \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K, \log\mathscr{D})/F^r \to \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}})/F^r\end{aligned}$$

This is a quasi-isomorphism: the last map is an quasi-isomorphism by Lemma 3.17 below; the first map is a quasi-isomorphism since it suffices to argue locally and there we can use Lemma 3.14.

Hence we have a natural map

$$H^{i-1}_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}) \to H^i_{\mathrm{syn}}(\mathscr{X}, r)_{\mathbf{Q}}, \quad i \le r.$$

By point (ii) of Proposition 3.12 it is an isomorphism for $i < r$ and an injection for $i = r$. □

Lemma 3.17. *For a semistable formal scheme $\mathscr{X}$ over $\mathscr{O}_K$, the natural morphism*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K, \log\mathscr{D}) \to \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}})$$

is a filtered quasi-isomorphism.

Proof. Let $\Omega^\bullet_{\mathscr{X}_K}(*\mathscr{D})$ be the complex of meromorphic (along $\mathscr{D}$) differentials [38, p.17]. There is a natural morphism

$$\Omega^\bullet_{\mathscr{X}_K^{\mathrm{an}}}(*\mathscr{D}) \to j_*\Omega^\bullet_{\mathscr{X}_{K,\mathrm{tr}}}.$$

By [38, Theorem 2.3], this is a quasi-isomorphism: after some preliminary reductions, this amounts to showing that the usual integration works on essentially singular differentials. Since $j : \mathscr{X}_{K,\mathrm{tr}} \hookrightarrow \mathscr{X}_K^{\mathrm{an}}$ is quasi-Stein, we have $\mathrm{R}j_*\Omega^\bullet_{\mathscr{X}_{K,\mathrm{tr}}} = j_*\Omega^\bullet_{\mathscr{X}_{K,\mathrm{tr}}}$.

We also have a natural morphism

$$r : \Omega^\bullet_{\mathscr{X}_K^{\mathrm{an}}}(\log \mathscr{D}) \to \Omega^\bullet_{\mathscr{X}_K^{\mathrm{an}}}(*\mathscr{D}).$$

Integration causes no convergence issues here. In fact, the computations in [38, p. 18] go through in this case: there is a residue map $\mathrm{Res} : \Omega^\bullet_{\mathscr{X}_K^{\mathrm{an}}}(*\mathscr{D}) \to \Omega^\bullet_{\mathscr{X}_K^{\mathrm{an}}}(\log \mathscr{D})$ such that $\mathrm{Res}\, r = \mathrm{Id}$. Integrating gives a homotopy between $r\,\mathrm{Res}$ and Id. Our lemma follows. □

Remark 3.18. One can compute syntomic cohomology of $R = \mathscr{O}_K^\times$ rather explicitly, and the result shows that $\alpha_{r,r}$ is not necessarily surjective: in the case $r = 1$ below, there is an extra $\mathbf{Z}_p$ appearing in $H^1_{\mathrm{syn}}(\mathscr{O}_K^\times, r)$. In higher dimensions the difference is much more serious (see Remark 3.15).

Denote by $\mathscr{O}_K^{(r)}$ the intersection of $\mathscr{O}_K$ with $R_\varpi^{\mathrm{PD}} + P^r F[X_0]_P$. We have $\mathscr{O}_K^{(1)} = \mathscr{O}_K$. We will state the following proposition without a proof.

Proposition 3.19. (i) *For $r \geq 2$, we have*

$$\begin{cases} H^0_{\mathrm{syn}}(\mathscr{O}_K^\times, r) \text{ is } p^{13r}\text{-isomorphic to } 0, \\ H^1_{\mathrm{syn}}(\mathscr{O}_K^\times, r) \text{ is } p^{N(r,e)}\text{-isomorphic to } \mathscr{O}_K^{(r)}, \\ H^2_{\mathrm{syn}}(\mathscr{O}_K^\times, r) \text{ is } p^{N(r,e)}\text{-isomorphic to } 0. \end{cases}$$

(ii) *For $r = 1$, we have $p^{N(r,e)}$-isomorphisms*

$$\begin{cases} H^0_{\mathrm{syn}}(\mathscr{O}_K^\times, r) \cong 0, \\ H^1_{\mathrm{syn}}(\mathscr{O}_K^\times, r) \cong \mathscr{O}_K^{(r)} \oplus \mathbf{Z}_p \\ H^2_{\mathrm{syn}}(\mathscr{O}_K^\times, r) \cong (\mathscr{O}_F/(\varphi - 1)). \end{cases}$$

4. Syntomic cohomology and (φ, Γ)-modules

Assume that K has enough roots of unity. Let $u = (p-1)/p$, $v = p - 1$ if $p \geq 3$, and $u = \frac{3}{4}$, $v = \frac{3}{2}$ if $p = 2$. In particular, $\big(1 + \frac{p^{i-1} - \delta_R - 1}{e_0}\big)u > \big(1 + \frac{1}{p-1} - \frac{1}{2p}\big)u > \frac{1}{p-1}$, for all p.

We use the isomorphism $R_\varpi^{[u,v]} \cong \mathbf{A}_R^{[u,v]}$ of filtered rings with a Frobenius to add an action of Γ_R on the complexes that we consider. This allows us to relate the complex $\mathrm{Cycl}(R_\varpi^{[u,v]}, r)$, that we showed to be p^{Nr}-quasi-isomorphic, for a universal constant N, to our original syntomic complex, to complexes coming from the theory of (φ, Γ)-modules and known to compute Galois cohomology. In dimension 0, this amounts to relating the complex

$$\mathrm{Kos}(\varphi, \partial, F^r\mathbf{A}_K^{[u,v]}) : \quad F^r\mathbf{A}_K^{[u,v]} \xrightarrow{(\partial, p^r - \varphi)} F^{r-1}\mathbf{A}_K^{[u,v]} \oplus \mathbf{A}_K^{[u,v/p]} \xrightarrow{-(p^r - p\varphi) + \partial} \mathbf{A}_K^{[u,v/p]} \ ,$$

that we obtain from $\mathrm{Cycl}(r_\varpi^{[u,v]}, r)$, to the complex of Herr's thesis [32]:

$$\mathrm{Kos}(\varphi, \Gamma_K, \mathbf{A}_K(r)) : \quad \mathbf{A}_K(r) \xrightarrow{(\tau_0, 1-\varphi)} \mathbf{A}_K(r) \oplus \mathbf{A}_K(r) \xrightarrow{-(1-\varphi) + \tau_0} \mathbf{A}_K(r) \ ,$$

where $\tau_0 = \gamma_0 - 1$, and γ_0 is our topological generator of Γ_K.

There are two main steps:

— One gets rid of the filtration by dividing by suitable powers of t; this turns the differential ∂ into the action of the Lie algebra of Γ_R (Lemma 4.4). This looks similar to the construction in [10], where $[\varepsilon]-1$ appears in the place of t (note that $t/([\varepsilon]-1)$ is a unit in $\mathbf{A}_{\mathrm{cr}}$).

— One uses the analyticity of the action to pass from the Lie algebra of Γ_R to Γ_R itself (Lemma 4.5).

The rest is standard (φ,Γ)-modules techniques to change the domains of convergence and move from $\mathbf{A}_R^{[u,v]}$ to $\mathbf{A}_R$.

4.1. **From complexes of differential forms to Koszul complexes.** To start, we are going to transform $\mathrm{Cycl}(R_{\varpi}^{[u,v]},r)$ into a "Koszul complex" by expressing the differentials $d: F^{r-i}\Omega_S^i \to F^{r-i-1}\Omega_S^{i+1}$ in suitable basis. Namely, we define $\omega_0=\frac{dT}{1+T}$ and $\omega_j=\frac{dX_j}{X_j}$ if $1\le j\le d$. If

$$\mathbf{j}=(j_1,\dots,j_i)\in J_i=\{0\le j_1<\dots<j_i\le d\},\qquad \text{we set } \omega_{\mathbf{j}}=\omega_{j_1}\wedge\dots\wedge\omega_{j_i},$$

so that an element of $F^a\Omega_S^i$ can be written uniquely as $\sum_{\mathbf{j}\in J_i}x_{\mathbf{j}}\omega_{\mathbf{j}}$, with $x_{\mathbf{j}}\in F^aS$. In other words the $\omega_{\mathbf{j}}$, for $\mathbf{j}\in J_i$, form a "basis" of $F^a\Omega_S^i$ over F^aS; these are the basis that we use to describe d. In this way, d becomes a map, involving the differential operators ∂_j, for $0\le j\le d$, from $(F^{r-i}S)^{J_i}$ to $(F^{r-i-1}S)^{J_{i+1}}$. We denote by $\mathrm{Kos}(\partial,F^rS)$ the complex $F^r\Omega_S^\bullet$ expressed in these basis and $\mathrm{Kos}(\varphi,\partial,F^rR_{\varpi}^{[u,v]})$ the complex $\mathrm{Cycl}(R_{\varpi}^{[u,v]},r)$ written in the same way. By definition,

$$\mathrm{Kos}(\partial,F^rS)=F^rS\overset{(\partial_j)}{\to}(F^{r-1}S)^{J_1}\to(F^{r-2}S)^{J_2}\to\cdots$$

$$\mathrm{Kos}(\varphi,\partial,F^rR_{\varpi}^{[u,v]})=[\ \mathrm{Kos}(\partial,F^rR_{\varpi}^{[u,v]})\xrightarrow{p^r-p^\bullet\varphi_{\mathrm{cycl}}}\mathrm{Kos}(\partial,R_{\varpi}^{[u,v/p]})\ ].$$

and

$$\mathrm{Kos}(\varphi,\partial,F^rR_{\varpi}^{[u,v]})\simeq\mathrm{Cycl}(R_{\varpi}^{[u,v]},r).$$

The arithmetic and the geometric variables behave quite differently in what follows, so it is convenient to separate them. Let $J_i'\subset J_i$ be the set of $\mathbf{j}$'s with $j_1\neq 0$, hence

$$J_i'=\{(j_1,\dots,j_d),\ 1\le j_1<\dots<j_d\le d\},$$

and let $\partial'=(\partial_1,\dots,\partial_d)$. We denote by $\mathrm{Kos}(\partial',F^rS)$ the subcomplex of $\mathrm{Kos}(\partial,F^rS)$ made of the $(F^{r-i}S)^{J_i'}$. Then $\mathrm{Kos}(\partial,F^rS)$ is the homotopy limit[9]

$$\mathrm{Kos}(\partial,F^rS)=\big[\ \mathrm{Kos}(\partial',F^rS)\xrightarrow{\partial_0}\mathrm{Kos}(\partial',F^rS)\ \big],$$

and $\mathrm{Kos}(\varphi,\partial,F^rR_{\varpi}^{[u,v]})$ is the homotopy limit

$$\mathrm{Kos}(\varphi,\partial,F^rR_{\varpi}^{[u,v]})=\left[\begin{array}{ccc}\mathrm{Kos}(\partial',F^rR_{\varpi}^{[u,v]}) & \xrightarrow{p^r-p^\bullet\varphi} & \mathrm{Kos}(\partial',R_{\varpi}^{[u,v/p]})\\ \downarrow\partial_0 & & \downarrow\partial_0\\ \mathrm{Kos}(\partial',F^{r-1}R_{\varpi}^{[u,v]}) & \xrightarrow{p^r-p^{\bullet+1}\varphi} & \mathrm{Kos}(\partial',R_{\varpi}^{[u,v/p]})\end{array}\right]$$

We can now use the isomorphisms $\iota_{\mathrm{cycl}}: R_{\varpi}^{[u,v]}\simeq\mathbf{A}_R^{[u,v]}$ and $\iota_{\mathrm{cycl}}: R_{\varpi}^{[u,v/p]}\simeq\mathbf{A}_R^{[u,v/p]}$, which commute with Frobenius and filtration, to obtain a (tautological) quasi-isomorphism

$$\mathrm{Kos}(\varphi,\partial,F^rR_{\varpi}^{[u,v]})\simeq\mathrm{Kos}(\varphi,\partial,F^r\mathbf{A}_R^{[u,v]})=\left[\begin{array}{ccc}\mathrm{Kos}(\partial',F^r\mathbf{A}_R^{[u,v]}) & \xrightarrow{p^r-p^\bullet\varphi} & \mathrm{Kos}(\partial',\mathbf{A}_R^{[u,v/p]})\\ \downarrow\partial_0 & & \downarrow\partial_0\\ \mathrm{Kos}(\partial',F^{r-1}\mathbf{A}_R^{[u,v]}) & \xrightarrow{p^r-p^{\bullet+1}\varphi} & \mathrm{Kos}(\partial',\mathbf{A}_R^{[u,v/p]})\end{array}\right]$$

[9]Strictly speaking, we should multiply ∂_0 by p to have it defined, but we will ignore this as it does not change anything in the arguments that follow.

What we have achieved by this procedure is twofold:

• we moved to the world of period rings,

• we gained an action of Γ_R whose infinitesimal action is related to the differentials ∂ by the very useful formula

$$\nabla_j := \log \gamma_j = t\partial_j, \quad \text{for } 0 \leq j \leq d.$$

4.2. **Continuous group cohomology and Koszul complexes.** Before we proceed, we will make a little digression on Koszul complexes (we will need explicit formulas). Consider the Iwasawa algebra $S = \mathbf{Z}_p[[\tau_1, \ldots, \tau_d]]$. The Koszul complex associated to $(\tau_1, \cdots, \tau_d)$ is the following complex

$$K(\tau_1, \cdots, \tau_d) := K(\tau_1)\widehat{\otimes}_{\mathbf{Z}_p} K(\tau_2)\widehat{\otimes}_{\mathbf{Z}_p} \cdots \widehat{\otimes}_{\mathbf{Z}_p} K(\tau_d),$$
$$K(\tau_i) := (0 \to \mathbf{Z}_p[[\tau_i]] \xrightarrow{\tau_i} \mathbf{Z}_p[[\tau_i]] \to 0)$$

Here the right hand term is placed in degree 0. Degree q of this complex equals the exterior power $\bigwedge_S^q S^d$. In the standard basis $\{e_{i_1 \cdots i_q}\}, 1 \leq i_1 < \cdots < i_q \leq d$, of $\bigwedge_S^q S^d$ the differential $d_q^1 : \bigwedge_S^q S^d \to \bigwedge_S^{q-1} S^d$ is given by the formula

$$d_q^1(a_{i_1 \cdots i_q}) = \sum_{k=1}^{q} (-1)^{k+1} a_{i_1 \cdots \widehat{i_k} \cdots i_q} \tau_{i_k} \tag{4.1}$$

The augmentation map $S \to \mathbf{Z}_p$ makes $K(\tau_1, \ldots, \tau_d)$ into a resolution of $\mathbf{Z}_p$ in the category of topological S-modules.

Let $\mathbf{Z}_p[[\Gamma'_R]]$ denote the Iwasawa algebra of Γ'_R, i.e., the completed group ring

$$\mathbf{Z}_p[[\Gamma'_R]] := \varprojlim \mathbf{Z}_p[\Gamma'_R/H],$$

where the limit is taken over all the open normal subgroups H of Γ'_R and every group ring $\mathbf{Z}_p[\Gamma'_R/H]$ is equipped with the p-adic topology. We have $\mathbf{Z}_p[[\Gamma'_R]] \simeq \mathbf{Z}_p[[\tau_1, \cdots, \tau_d]]$, $\tau_j := \gamma_j - 1$, $j \in \{1, \cdots, d\}$. The Koszul complex $K(\tau_1, \ldots, \tau_d)$ is the complex

$$0 \longrightarrow \mathbf{Z}_p[[\Gamma'_R]]^{J'_d} \xrightarrow{d_{d-1}^1} \cdots\cdots \xrightarrow{d_1^1} \mathbf{Z}_p[[\Gamma'_R]]^{J'_1} \xrightarrow{d_0^1} \mathbf{Z}_p[[\Gamma'_R]]^{J'_0} \longrightarrow 0\ ,$$

with differentials given by formula 4.1. It is easy to see (for example, by induction on d) that this is a resolution of $\mathbf{Z}_p$ in the category of topological $\mathbf{Z}_p[[\Gamma'_R]]$-modules. Similarly, we define the Koszul complex $K(\tau_1^c, \ldots, \tau_d^c)$ (with differentials d_q^c), $\tau_j^c := \gamma_j^c - 1$, $c = \exp(p^i)$. Since (γ_j^c), $1 \leq i \leq d$, is a basis of Γ'_R, this is also a resolution of $\mathbf{Z}_p$.

Set $\Lambda := \mathbf{Z}_p[[\Gamma_R]]$ (recall that we have an exact sequence $1 \to \Gamma'_R \to \Gamma_R \to \Gamma_K \to 1$). Consider the complex $K(\Lambda)$:

$$0 \longrightarrow \Lambda^{J'_d} \xrightarrow{d_{d-1}^1} \cdots\cdots \xrightarrow{d_1^1} \Lambda^{J'_1} \xrightarrow{d_0^1} \Lambda^{J'_0} \longrightarrow 0\ ,$$

By [44, Lemma 4.3], we have the isomorphism $\varprojlim_m (\mathbf{Z}_p[\Gamma_K/(\Gamma_K)^{p^m}] \otimes_{\mathbf{Z}_p} K(\tau_1, \cdots, \tau_d)) \simeq K(\Lambda)$ of left Λ- and right $\mathbf{Z}_p[[\tau_1, \cdots, \tau_d]]$-modules. It follows that the complex $K(\Lambda)$ is a resolution of $\mathbf{Z}_p[[\Gamma_K]]$ in the category of topological left Λ-modules. Similarly, we have the complex $K^c(\Lambda)$ (obtained from $K(\tau_1^c, \cdots, \tau_d^c)$) that is also a resolution of $\mathbf{Z}_p[[\Gamma_K]]$.

We define the map

$$\tau_0 : \quad K^c(\Lambda) \to K(\Lambda)$$

by the following commutative diagram of topological left Λ-modules

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^{J'_d} \arrow[r, "d^c_{d-1}"] \arrow[d, "\tau^d_0"] & \cdots\cdots \arrow[r, "d^c_1"] & \Lambda^{J'_1} \arrow[r, "d^c_0"] \arrow[d, "\tau^1_0"] & \Lambda \arrow[r] \arrow[d, "\tau^0_0"] & \mathbf{Z}_p[[\Gamma_K]] \arrow[r] \arrow[d, "\gamma_0-1"] & 0 \\
0 \arrow[r] & \Lambda^{J'_d} \arrow[r, "d^1_{d-1}"] & \cdots\cdots \arrow[r, "d^1_1"] & \Lambda^{J'_1} \arrow[r, "d^1_0"] & \Lambda \arrow[r] & \mathbf{Z}_p[[\Gamma_K]] \arrow[r] & 0
\end{tikzcd}$$

The vertical maps are defined as follows

$$\tau^0_0 = \gamma_0 - 1, \quad \tau^q_0 : (a_{i_1\cdots i_q}) \mapsto (a_{i_1\cdots i_q}(\gamma_0 - \delta_{i_1\cdots i_q})), \quad 1 \le q \le d, \quad 1 \le i_1 < \cdots < i_q \le d$$
$$\delta_{i_1\cdots i_q} = \delta_{i_q}\cdots\delta_{i_1}, \quad \delta_{i_j} = (\gamma^c_{i_j} - 1)(\gamma_{i_j} - 1)^{-1}$$

Note that

$$\frac{\gamma^c_j - 1}{\gamma_j - 1} = \sum_{n\ge 1}\binom{c}{n}(\gamma_j - 1)^{n-1}$$

makes sense since $\gamma_j - 1$ is topologically nilpotent. It is a unit in $\mathbf{Z}_p[[\Gamma'_R]]$. To see that τ_0 is a map of complexes note that

$$(\gamma_0 - \delta_{i_1\cdots i_q}) = \tau^c_{i_q}\tau^c_{i_{q-1}}\cdots\tau^c_{i_1}\tau^0_0\tau^{-1}_{i_1}\cdots\tau^{-1}_{i_{q-1}}\tau^{-1}_{i_q}$$

as the τ_j's commute for $j \in \{1, \cdots, d\}$.

Set $K(\Lambda, \tau) := [\ K^c(\Lambda) \xrightarrow{\tau_0} K(\Lambda)\ ]$. Since we have the exact sequence

$$0 \to \mathbf{Z}_p[[\Gamma_K]] \xrightarrow{\gamma_0 - 1} \mathbf{Z}_p[[\Gamma_K]] \to 0$$

the complex $K(\Lambda, \tau)$ resolves $\mathbf{Z}_p$ in the category of topological left Λ-modules.

For a topological Γ_R-module M, denote by $\mathrm{Kos}(\Gamma'_R, M)$ the complex

$$\mathrm{Kos}(\Gamma'_R, M) := \mathrm{Hom}_{\Lambda,\mathrm{cont}}(K(\Lambda), M) = \mathrm{Hom}_\Lambda(K(\Lambda), M)$$

(which we will also call the Koszul complex). Similarly, we define the Koszul complex $\mathrm{Kos}^c(\Gamma'_R, M)$ using the resolution $K^c(\Lambda)$ of $\mathbf{Z}_p$. In degree q of these two complexes there are $\binom{d}{q}$ copies of M. If this does not cause confusion we will write the Koszul complexes $\mathrm{Kos}(\Gamma'_R, M)$ and $\mathrm{Kos}^c(\Gamma'_R, M)$ as

$$\mathrm{Kos}(\Gamma'_R, M) = M \xrightarrow{(\tau_j)} M^{J'_1} \to \cdots \to M^{J'_d}, \quad \mathrm{Kos}^c(\Gamma'_R, M) = M \xrightarrow{(\tau^c_j)} M^{J'_1} \to \cdots \to M^{J'_d}.$$

The map $\tau_0 : K^c(\Lambda) \to K(\Lambda)$ induces a map of complexes

$$\tau_0 : \quad \mathrm{Kos}(\Gamma'_R, M) \to \mathrm{Kos}^c(\Gamma'_R, M)$$

which we represent by the following diagram:

$$\begin{tikzcd}
M^{J'_0} \arrow[r, "(\tau_j)"] \arrow[d, "\tau^0_0"] & M^{J'_1} \arrow[r] \arrow[d, "\tau^1_0"] & M^{J'_2} \arrow[r] \arrow[d, "\tau^2_0"] & {} \\
M^{J'_0} \arrow[r, "(\tau^c_j)"] & M^{J'_1} \arrow[r] & M^{J'_2} \arrow[r] & {}
\end{tikzcd}$$

Set

$$\mathrm{Kos}(\Gamma_R, M) := \mathrm{Hom}_{\Lambda,\mathrm{cont}}(K(\Lambda, \tau), M) = [\ \mathrm{Kos}(\Gamma'_R, M) \xrightarrow{\tau_0} \mathrm{Kos}^c(\Gamma'_R, M)\ )]$$

Let $X_\bullet$ denote the standard complex [42, V.1.2.1] computing the continuous cohomology $\mathrm{R}\Gamma_{\mathrm{cont}}(\Lambda, M) := \mathrm{Hom}_{\Lambda,\mathrm{cont}}(X_\bullet, M)$. We have $X_n = \Lambda\widehat{\otimes}T^n\Lambda$, where T^n denotes the completed tensor product. Let $Y_\bullet$ denote the standard complex computing the group cohomology of Γ_R: we have $Y_n = T^{n+1}(\mathbf{Z}_p[\Gamma_R])$ and $\mathrm{R}\Gamma(\Gamma_R, M) = \mathrm{Hom}_{\mathbf{Z}_p[\Gamma_R]}(Y_\bullet, M)$. Continuous group cohomology $\mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, M)$ is computed by the complex $\mathrm{Hom}_{\mathbf{Z}_p[\Gamma_R],\mathrm{cont}}(Y_\bullet, M)$ of continuous cochains. The continuous map $Y_\bullet \to X_\bullet$ induces a

morphism $\mathrm{Hom}_{\Lambda,\mathrm{cont}}(X_\bullet, M) \to \mathrm{Hom}_{\mathbf{Z}_p[\Gamma_R]}(Y_\bullet, M)$ and Lazard shows [42, V.1.2.6] that Mahler's theorem implies that this morphism factors through $\mathrm{Hom}_{\mathbf{Z}_p[\Gamma_R],\mathrm{cont}}(Y_\bullet, M)$ and induces an isomorphism $\mathrm{Hom}_{\Lambda,\mathrm{cont}}(X_\bullet, M) \to \mathrm{Hom}_{\mathbf{Z}_p[\Gamma_R],\mathrm{cont}}(Y_\bullet, M)$. Hence $\mathrm{R}\Gamma_{\mathrm{cont}}(\Lambda, M) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, M)$. By choosing a map between the two projective resolutions $K(\Lambda, \tau)$ and $X_\bullet$ of $\mathbf{Z}_p$ (by [42, V.1.1.5.1] such a map exists between any two projective resolutions and is unique up to a homotopy) we obtain a functorial quasi-isomorphism (unique in the derived category)

$$\mathrm{Kos}(\Gamma_R, M) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cont}}(\Lambda, M)$$

Adding up, we have obtained a quasi-isomorphism

$$\lambda : \quad \mathrm{Kos}(\Gamma_R, M) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, M).$$

4.3. **(φ, ∂)-modules and (φ, Γ)-modules.** If

$$S = \mathbf{A}_R^{[u,v]}, \mathbf{A}_R^{(0,v]+}, \mathbf{A}_R,$$

set

$$S' = \mathbf{A}_R^{[u,v/p]}, \mathbf{A}_R^{(0,v/p]+}, \mathbf{A}_R.$$

Write $S(r), S'(r)$ for the Γ_R-module S, S' with the action of Γ_R twisted by χ^r. Define the complex

$$\mathrm{Kos}(\varphi, \Gamma_R, S(r)) := \left[\begin{array}{ccc} \mathrm{Kos}(\Gamma'_R, S(r)) & \xrightarrow{1-\varphi} & \mathrm{Kos}(\Gamma'_R, S'(r)) \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_0} \\ \mathrm{Kos}^c(\Gamma'_R, S(r)) & \xrightarrow{1-\varphi} & \mathrm{Kos}^c(\Gamma'_R, S'(r)) \end{array} \right]$$

Proposition 4.2. *There exists a universal constant N and a natural*[10] *p^{Nr}-quasi-isomorphism*

$$\tau_{\leq r}\mathrm{Kos}(\varphi, \Gamma_R, \mathbf{A}_R^{[u,v]}(r)) \simeq \tau_{\leq r}\mathrm{Kos}(\varphi, \partial, F^r\mathbf{A}_R^{[u,v]}).$$

Proof. Denote by $\mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v]}(r))$ the complex

$$\mathbf{A}_R^{[u,v]}(r) \to \mathbf{A}_R^{[u,v]}(r)^{J'_1} \to \cdots \to \mathbf{A}_R^{[u,v]}(r)^{J'_d}$$

with differentials dual to those in formula 4.1 (with τ_j replaced by ∇_j). Consider the maps

$$\nabla_0 : \quad \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v]}(r)) \to \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v]}(r))$$

defined in the following way:

$$\begin{array}{ccccccc} \mathbf{A}_R^{[u,v]}(r) & \xrightarrow{(\nabla_j)} & \mathbf{A}_R^{[u,v]}(r)^{J'_1} & \longrightarrow \cdots \longrightarrow & \mathbf{A}_R^{[u,v]}(r)^{J'_q} & \longrightarrow \cdots \\ \downarrow{\scriptstyle \nabla_0} & & \downarrow{\scriptstyle \nabla_0 + p^i} & & \downarrow{\scriptstyle \nabla_0 + qp^i} & \\ \mathbf{A}_R^{[u,v]}(r) & \xrightarrow{(\nabla_j)} & \mathbf{A}_R^{[u,v]}(r)^{J'_1} & \longrightarrow \cdots \longrightarrow & \mathbf{A}_R^{[u,v]}(r)^{J'_q} & \longrightarrow \cdots \end{array}$$

Define the complex

$$\mathrm{Kos}(\varphi, \mathrm{Lie}\,\Gamma_R, \mathbf{A}_R^{[u,v]}(r)) := \left[\begin{array}{ccc} \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v]}(r)) & \xrightarrow{1-\varphi} & \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v/p]}(r)) \\ \downarrow{\scriptstyle \nabla_0} & & \downarrow{\scriptstyle \nabla_0} \\ \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v]}(r)) & \xrightarrow{1-\varphi} & \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v/p]}(r)) \end{array} \right]$$

Our proposition follows from Lemma 4.4 and Proposition 4.5 below. □

[10] We use "natural" to mean "defined by universal formulas".

Remark 4.3. The Lie algebra $\mathrm{Lie}\,\Gamma'_R$ of the p-adic Lie group Γ'_R is a free $\mathbf{Z}_p$-module of rank d: $\mathrm{Lie}\,\Gamma'_R = \mathbf{Z}_p[\nabla_j,\ 1 \le j \le d]$. It is commutative. The Lie algebra $\mathrm{Lie}\,\Gamma_R$ of the p-adic Lie group Γ_R is a free $\mathbf{Z}_p$-module of rank $d+1$: $\mathrm{Lie}\,\Gamma_R = \mathbf{Z}_p[\nabla_j,\ 0 \le j \le d]$. We have $[\nabla_j, \nabla_l] = 0$, $1 \le j, l \le d$, and $[\nabla_j, \nabla_0] = p^i \nabla_j$, $1 \le j \le d$. One easily checks that the above Koszul complexes compute Lie algebra cohomology of $\mathrm{Lie}\,\Gamma'_R$ and $\mathrm{Lie}\,\Gamma_R$ with values in $\mathbf{A}_R^{[u,v]}(r)$. That is, we have quasi-isomorphisms

$$\begin{aligned}
&\mathrm{R}\Gamma(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v]}(r)) \simeq \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v]}(r)),\\
&\mathrm{R}\Gamma(\mathrm{Lie}\,\Gamma_R, \mathbf{A}_R^{[u,v]}(r)) \simeq [\mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v]}(r)) \xrightarrow{\nabla_0} \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v]}(r))]\,.
\end{aligned}$$

This implies that

$$\mathrm{Kos}(\varphi, \mathrm{Lie}\,\Gamma_R, \mathbf{A}_R^{[u,v]}(r)) \simeq [\mathrm{R}\Gamma(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v]}(r)) \xrightarrow{1-\varphi} \mathrm{R}\Gamma(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v]}(r))]\,.$$

By Lemma 2.11, multiplication by t^r induces p^{2r}-isomorphisms

$$F^r\mathbf{A}_R^{[u,v]} \simeq t^r\mathbf{A}_R^{[u,v]} \quad \text{and} \quad t^r\mathbf{A}_R^{[u,v/p]} \simeq \mathbf{A}_R^{[u,v/p]}.$$

If $S = \mathbf{A}_R^{[u,v]}, \mathbf{A}_R^{[u,v/p]}$, and if we twist the Galois action by χ^r on the source (i.e. replace S by $S(r)$), then multiplication by t^r becomes Galois-equivariant.

Lemma 4.4. (i) *There is a natural p^{40r}-quasi-isomorphism*

$$\tau_{\le r}\mathrm{Kos}(\varphi, \mathrm{Lie}\,\Gamma_R, \mathbf{A}_R^{[u,v]}(r)) \simeq \tau_{\le r}\mathrm{Kos}(\varphi, \partial, F^r\mathbf{A}_R^{[u,v]}).$$

(ii) *There is a natural p^{80r}-quasi-isomorphism*

$$\tau_{\le r}\mathrm{Kos}(\varphi, \mathrm{Lie}\,\Gamma_R, \mathbf{A}_R^{[u,v]}(r))_n \simeq \tau_{\le r}\mathrm{Kos}(\varphi, \partial, F^r\mathbf{A}_R^{[u,v]})_n.$$

Proof. We will present the proof of the first claim. The proof for the complexes modulo p^n is similar and we will just point out the key point where it differs.

We first construct p^{4r}-quasi-isomorphisms

$$\tau_{\le r}\mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, S(r)) \simeq \tau_{\le r}\mathrm{Kos}(\partial', F^rS),$$

via the following diagram (with the convention $F^jS = S$ for all j, if $S = \mathbf{A}_R^{[u,v/p]}$):

$$\begin{tikzcd}
S(r) \arrow[r, "(\nabla_j)"] \arrow[d, "t^r", "\wr"'] & S(r)^{J'_1} \arrow[r] \arrow[d, "t^r", "\wr"'] & \cdots \arrow[r] & S(r)^{J'_r} \arrow[r] \arrow[d, "t^r", "\wr"'] & S(r)^{J'_{r+1}} \arrow[r] \arrow[d, "t^r", "\wr"'] & \cdots \\
F^rS \arrow[r, "(\nabla_j)"] & F^rS^{J'_1} \arrow[r] & \cdots \arrow[r] & F^rS^{J'_r} \arrow[r] & F^rS^{J'_{r+1}} \arrow[r] & \cdots \\
F^rS \arrow[r, "(\partial_j)"] \arrow[u, "t^0=\mathrm{Id}", "\wr"'] & (F^{r-1}S)^{J'_1} \arrow[r] \arrow[u, "t^1", "\wr"'] & \cdots \arrow[r] & S^{J'_r} \arrow[r] \arrow[u, "t^r", "\wr"'] & S^{J'_{r+1}} \arrow[r] \arrow[u, "t^{r+1}"] & \cdots
\end{tikzcd}$$

The top vertical map, being multiplication by t^r, is a p^{4r}-quasi-isomorphism. For the same reason, the bottom map is a p^{4r}-isomorphism in degree $\le r$ and injective in higher degrees. (Note, however that *it is not a p^N-isomorphism in degree $\ge r+1$ (for any N), which explains why we have to truncate at degree r.*) In the case of complexes modulo p^n, the above argument works (with double exponents in the error terms) since, by Lemma 2.11, t is divisible in S by at most p^2 (hence multiplication by t^{r+1} is

$p^{2(r+1)}$-injective). Compatibility with Frobenius and with ∂_0 gives us a p^{36r}-quasi-isomorphism between

$$\tau_{\leq r}\left[\begin{array}{ccc} \mathrm{Kos}(\partial', F^r\mathbf{A}_R^{[u,v]}) & \xrightarrow{p^r - p^\bullet\varphi} & \mathrm{Kos}(\partial', \mathbf{A}_R^{[u,v/p]}) \\ \downarrow{\scriptstyle \partial_0} & & \downarrow{\scriptstyle \partial_0} \\ \mathrm{Kos}(\partial', F^{r-1}\mathbf{A}_R^{[u,v]}) & \xrightarrow{p^r - p^{\bullet+1}\varphi} & \mathrm{Kos}(\partial', \mathbf{A}_R^{[u,v/p]}) \end{array}\right]$$

and

$$\tau_{\leq r}\left[\begin{array}{ccc} \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v]}(r)) & \xrightarrow{p^r(1-\varphi)} & \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v/p]}(r)) \\ \downarrow{\scriptstyle \nabla_0} & & \downarrow{\scriptstyle \nabla_0} \\ \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v]}(r)) & \xrightarrow{p^r(1-\varphi)} & \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, \mathbf{A}_R^{[u,v/p]}(r)) \end{array}\right]$$

This last complex being p^r-quasi-isomorphic to $\tau_{\leq r}\mathrm{Kos}(\varphi, \mathrm{Lie}\,\Gamma_R, \mathbf{A}_R^{[u,v]}(r))$, our lemma follows. □

Proposition 4.5. *There exists a natural quasi-isomorphism*

$$\mathscr{L}az: \quad \mathrm{Kos}(\varphi, \Gamma_R, \mathbf{A}_R^{[u,v]}(r)) \xrightarrow{\sim} \mathrm{Kos}(\varphi, \mathrm{Lie}\,\Gamma_R, \mathbf{A}_R^{[u,v]}(r))$$

Proof. If $M = \mathbf{A}_R^{[u,v]}(r), \mathbf{A}_R^{[u,v/p]}(r)$, consider the map $\beta : \mathrm{Kos}(\Gamma'_R, M) \to \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, M)$:

$$\begin{array}{ccccccc} M & \xrightarrow{(\gamma_j - 1)} & M^{J'_1} & \longrightarrow & M^{J'_2} & \longrightarrow \\ \wr\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} & \\ M & \xrightarrow{(\nabla_j)} & M^d & \longrightarrow & M^{d_2} & \longrightarrow \end{array}$$

with

$$\beta_q : (a_{i_1\cdots i_q}) \mapsto (\nabla_{i_q}\cdots\nabla_{i_1}\tau_{i_1}^{-1}\cdots\tau_{i_q}^{-1}(a_{i_1\cdots i_q})); \quad 1 \leq q \leq d.$$

Similarly, we define the map $\beta^c : \mathrm{Kos}^c(\Gamma'_R, M) \to \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, M)$ as:

$$\begin{array}{ccccccc} M & \xrightarrow{(\gamma_j^c - 1)} & M^{J'_1} & \longrightarrow & M^{J'_2} & \longrightarrow \\ \downarrow{\scriptstyle \beta_0^c} & & \downarrow{\scriptstyle \beta_1^c} & & \downarrow{\scriptstyle \beta_2^c} & \\ M & \xrightarrow{(\nabla_j)} & M^{J'_1} & \longrightarrow & M^{J'_2} & \longrightarrow \end{array}$$

Here

$$\beta_0^c = \nabla_0\tau_0^{-1}, \quad \beta_q^c : (a_{i_1\cdots i_q}) \mapsto (\nabla_{i_q}\cdots\nabla_{i_1}\nabla_0\tau_0^{-1}\tau_{i_1}^{c,-1}\cdots\tau_{i_q}^{c,-1}(a_{i_1\cdots i_q})); \quad 1 \leq q \leq d.$$

Lemma 4.6. *The maps β and β^c are well-defined isomorphisms.*

Proof. We will treat the map β first. Since, for $j, k \in \{1, \cdots, d\}$, $\nabla_j\nabla_k\tau_k^{-1}\tau_j^{-1} = (\nabla_j/\tau_j)(\nabla_k/\tau_k)$, it suffices to show that for $S = \mathbf{A}_R^{[u,v]}$ or $\mathbf{A}_R^{[u,v/p]}$ and for $j \in \{1, \cdots, d\}$, the map $\nabla_j/\tau_j : \Lambda \to \Lambda$ is a well-defined isomorphism of $S(r)$. Write

$$\frac{\log(1+X)}{X} = 1 + a_1X + a_2X^2 + \cdots \qquad \frac{X}{\log(1+X)} = 1 + b_1X + b_2X^2 + \cdots$$

We have $v_p(a_k) \geq -\frac{k}{p-1}$ for all k (immediate) which implies that $v_p(b_k) \geq -\frac{k}{p-1}$ for all k.

Now, $\tau_j = (\gamma_j - 1)$ if $1 \leq j \leq d$, and $\tau_0 = c^r\gamma_0 - 1 = (c^r - 1)\gamma_0 + (\gamma_0 - 1)$ (the c^r comes from the fact that we are in $S(r)$, not S). Since $c^r - 1$ is divisible by p^2, one infers from Lemma 2.34, that $\tau_j^k(S(r)) \subset \pi_i^{k(p^{i-1}-\delta_R-1)}(p, \pi_i^{e_0})^kS(r)$. It follows that, since $\big(1 + \frac{p^{i-1}-\delta_R-1}{e_0}\big)u > \frac{1}{p-1}$, the series $1 + a_1\tau_j + a_2\tau_j^2 + \cdots$ and $1 + b_1\tau_j + b_2\tau_j^2 + \cdots$ converge as series of operators on $S(r)$, and the limit

operators are inverse of each other. As the first one is ∇_j/τ_j, this allows to conclude that ∇_j/τ_j is an isomorphism of $S(r)$ for $0 \le j \le d$.

That settles the case of β. Let us show that the map β_q^c, for $1 \le q \le d$, is a well-defined isomorphism, i.e., that the maps $\nabla_{i_q} \cdots \nabla_{i_1} \nabla_0 \tau_0^{-1} \tau_{i_1}^{c,-1} \cdots \tau_{i_q}^{c,-1}$, $1 \le i_1 < \cdots < i_q \le d$, are well-defined isomorphisms. We can write the last map as $(\nabla_{i_q}/\tau_{i_q}) \cdots (\nabla_{i_1}/\tau_{i_1}) \tau_{i_q} \cdots \tau_{i_1} \nabla_0 \tau_0^{-1} \tau_{i_1}^{c,-1} \cdots \tau_{i_1}^{c,-1}$. Hence, by the computations done above, we are reduced to showing that the map $\tau_{i_q} \cdots \tau_{i_1} \nabla_0 \tau_0^{-1} \tau_{i_1}^{c,-1} \cdots \tau_{i_1}^{c,-1}$ is well-defined and an isomorphism.

We have

$$\tau_{i_q} \cdots \tau_{i_1} \nabla_0 \tau_0^{-1} \tau_{i_1}^{c,-1} \cdots \tau_{i_q}^{c,-1} = \sum_{k \in \mathbf{N}} a_k \tau_{i_q} \cdots \tau_{i_1} ((c^r - 1)\gamma_0 + (\gamma_0 - 1))^k \tau_{i_1}^{c,-1} \cdots \tau_{i_q}^{c,-1}$$

The formula

$$(\gamma_j^a - 1)(\gamma_0 - x) = (\gamma_0 - x\delta(\gamma_j^a))(\gamma_j^{a/c} - 1), \quad \delta(\gamma_j^a) := \frac{\gamma_j^a - 1}{\gamma_j^{a/c} - 1},$$

yields

$$(\gamma_j^a - 1)(\gamma_0 - 1)^k = (\gamma_0 - \delta(\gamma_j^a))(\gamma_0 - \delta(\gamma_j^a)\delta(\gamma_j^{a/c})) \cdots (\gamma_0 - \delta(\gamma_j^a) \cdots \delta(\gamma_j^{a/c^{k-1}}))(\gamma_j^{a/c^k} - 1)$$

Hence we can write

$$\begin{aligned} \tau_{i_q} \cdots \tau_{i_1} (\gamma_0 - 1)^k \tau_{i_1}^{c,-1} \cdots \tau_{i_1}^{c,-1} &= (\gamma_0 - \delta_k) \cdots (\gamma_0 - \delta_1) \frac{\gamma_{i_q}^{1/c^k} - 1}{\gamma_{i_q}^c - 1} \cdots \frac{\gamma_{i_1}^{1/c^k} - 1}{\gamma_{i_1}^c - 1} \\ &= (\gamma_0 - \delta_k) \cdots (\gamma_0 - \delta_1) \delta_0, \end{aligned}$$

where $\delta_j \in 1 + (p^2, (\gamma_1 - 1), \cdots, (\gamma_d - 1))$ (because $c - 1$ is divisible by p^2).

Writing $(\gamma_0 - \delta_j) = (\gamma_0 - 1) + (1 - \delta_j)$, one concludes that

$$\tau_{i_q} \cdots \tau_{i_1} (\gamma_0 - 1)^k \tau_{i_1}^{c,-1} \cdots \tau_{i_1}^{c,-1} \in (p^2, \gamma_0 - 1, \ldots, \gamma_d - 1)^k.$$

It follows that the series of operators $\sum_{k \in \mathbf{N}} a_k \tau_{i_q} \cdots \tau_{i_1} ((c^r - 1)\gamma_0 + (\gamma_0 - 1))^k \tau_{i_1}^{c,-1} \cdots \tau_{i_q}^{c,-1}$ converges, which shows that $\nabla_{i_q} \cdots \nabla_{i_1} \nabla_0 \tau_0^{-1} \tau_{i_1}^{c,-1} \cdots \tau_{i_q}^{c,-1}$ is well-defined. The same arguments show that the series of operators $\sum_{k \in \mathbf{N}} b_k \tau_{i_q}^c \cdots \tau_{i_1}^c ((c^r - 1)\gamma_0 + (\gamma_0 - 1))^k \tau_{i_1}^{-1} \cdots \tau_{i_q}^{-1}$ converges and the sum is the inverse of the previous operator.

This proves the lemma. □

Remark 4.7. By Remark 4.3, the above isomorphisms can be written as a quasi-isomorphism

$$(\beta, \beta^c): \quad \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, M) \xrightarrow{\sim} \mathrm{R}\Gamma(\mathrm{Lie}\,\Gamma_R, M).$$

Hence this is a (very simple – almost commutative) example of integral Lazard isomorphism between Lie algebra cohomology and continuous group cohomology for p-adic Lie groups (see [33] for a detailed treatment of integral Lazard isomorphism and [59] for an analytic (rational) version of the Lazard isomorphism).

The above maps yield the following isomorphism of complexes (where $M = \mathbf{A}_R^{[u,v]}(r)$ and $M' = \mathbf{A}_R^{[u,v/p]}(r)$):

$$\left[\begin{array}{ccc} \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, M) & \xrightarrow{1-\varphi} & \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, M') \\ \downarrow{\scriptstyle \nabla_0} & & \downarrow{\scriptstyle \nabla_0} \\ \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, M) & \xrightarrow{1-\varphi} & \mathrm{Kos}(\mathrm{Lie}\,\Gamma'_R, M') \end{array}\right] \xleftarrow{(\beta,\beta^c)} \left[\begin{array}{ccc} \mathrm{Kos}(\Gamma'_R, M) & \xrightarrow{1-\varphi} & \mathrm{Kos}(\Gamma'_R, M') \\ \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_0} \\ \mathrm{Kos}^c(\Gamma'_R, M) & \xrightarrow{1-\varphi} & \mathrm{Kos}^c(\Gamma'_R, M') \end{array}\right]$$

Set $\mathscr{L}az := (\beta, \beta^c)$. This fullfills the requirements of Proposition 4.5. □

4.4. **Change of annulus of convergence.**

Lemma 4.8. *The natural map*

$$\mathrm{Kos}(\varphi,\Gamma_R,\mathbf{A}_R^{(0,v]+}(r))\to\mathrm{Kos}(\varphi,\Gamma_R,\mathbf{A}_R^{[u,v]}(r))$$

is a quasi-isomorphism.

Proof. Use the isomorphism $R_\varpi^{\mathrm{deco}}\simeq\mathbf{A}_R^{\mathrm{deco}}$ which defines $\mathbf{A}_R^{\mathrm{deco}}$ to translate everything into the language of anlytic functions. It suffices to prove that the map $1-\varphi: R_\varpi^{[u,v]}/R_\varpi^{(0,v]+}\to R_\varpi^{[u,v/p]}/R_\varpi^{(0,v/p]+}$ is an isomorphism.

First note that the natural map $R_\varpi^{[u,v]}/R_\varpi^{(0,v]+}\to R_\varpi^{[u,v/p]}/R_\varpi^{(0,v/p]+}$ (induced by the inclusion of $R_\varpi^{[u,v]}$ into $R_\varpi^{[u,v/p]}$) is an isomorphism (injectivity is true because elements of the kernel are analytic functions which take integral values on $\frac{u}{e_0}\le v_p(X_0)\le\frac{v}{e_0}$ and which extend to analytic functions taking integral values on $0<v_p(X_0)\le\frac{v}{pe_0}$, hence belong to $R_\varpi^{(0,v]+}$; and surjectivity - because $r_{\zeta-1}^{[u,v/p]}=r_{\zeta-1}^{[u]}+r_{\zeta-1}^{(0,v/p]+}$ as is clear from the definitions). Denote by M the module $R_\varpi^{[u,v]}/R_\varpi^{(0,v]+}$; by the above $1-\varphi$ can be considered as an endomorphism of M.

Now, an element x of $R_\varpi^{[u,v]}$ can be written as $x=\sum_{k\in\mathbf{N}}x_k\frac{X_0^k}{p^{[ku/e_0]}}$, with $x_k\in R_\varpi^{(0,v]+}$ going to 0 p-adically. So $\varphi(x)=\sum_{k\in\mathbf{N}}p^{[pku/e_0]-[ku/e_0]}\varphi(x_k)\big(\frac{\varphi(X_0)}{X_0^p}\big)^k\frac{X_0^{pk}}{p^{[pku/e_0]}}$ and, since $[pku/e_0]-[ku/e_0]\ge 1$ if $[ku/e_0]\ne 0$, we see that $\varphi(x)\in R_\varpi^{(0,v/p]+}+pR_\varpi^{[u,v/p]}$; hence $\varphi(M)\subset pM$.

To conclude, it remains to check that M does not contain p-divisible elements. Let $(e_i)_{i\in I}$ be a collection of elements of R_ϖ^+ whose images form a basis of $R_\varpi^+/(p,X_0)$ over $k=r_{\zeta-1}^+/(p,X_0)$. Then $(e_i)_{i\in I}$ is a topological basis of $R_\varpi^{[u,v]}$ over $r_{\zeta-1}^{[u,v]}$ and of $R_\varpi^{(0,v]+}$ over $r_{\zeta-1}^{(0,v]+}$. Writing everything in the $(e_i)_{i\in I}$ basis, reduces the question to proving that $r_{\zeta-1}^{[u,v]}/r_{\zeta-1}^{(0,v]+}$ has no divisible element. This is rather obvious if you look at Laurent expansions. □

4.5. **Change of disk of convergence.**

Proposition 4.9. *The natural map*

$$\mathrm{Kos}(\varphi,\Gamma_R,\mathbf{A}_R^{(0,v]+}(r))\to\mathrm{Kos}(\varphi,\Gamma_R,\mathbf{A}_R(r))$$

is a p^8-quasi-isomorphism

Proof. We will use the quasi-isomorphisms with ψ-complexes. The proposition is a direct consequence of Lemmas 4.10 and 4.12 below. □

Set $\ell=p^{i-1}$. From Proposition 2.16, we know that $\pi_i^{-\ell}\mathbf{A}_R^{(0,v]+}$ is stable under the action of ψ (we have $\ell\ge\ell_R$). For $S=\mathbf{A}_R,\mathbf{A}_R^{(0,v]+}$, let $\mathrm{Kos}(\psi,\Gamma_R,S(r))$ be the complex:

$$\mathrm{Kos}(\psi,\Gamma_R,S(r)):=\left[\begin{array}{ccc}\mathrm{Kos}(\Gamma'_R,\pi_i^{-\ell}S(r)) & \xrightarrow{\psi-1} & \mathrm{Kos}(\Gamma'_R,\pi_i^{-\ell}S(r))\\ \downarrow\tau_0 & & \downarrow\tau_0\\ \mathrm{Kos}^c(\Gamma'_R,\pi_i^{-\ell}S(r)) & \xrightarrow{\psi-1} & \mathrm{Kos}^c(\Gamma'_R,\pi_i^{-\ell}S(r))\end{array}\right]$$

(For $S=\mathbf{A}_R$, there is no difference between $\pi_i^{-\ell}S$ and S.)

Lemma 4.10. *If $S=\mathbf{A}_R,\mathbf{A}_R^{(0,v]+}$, the natural map*

$$\mathrm{Kos}(\varphi,\Gamma_R,S(r))\to\mathrm{Kos}(\psi,\Gamma_R,S(r)),$$

induced by the identity on the first column and ψ on the second column, is a p^{p+7}-quasi-isomorphism.

Proof. The arguments are the same in both cases, but the case $S = \mathbf{A}_R^{(0,v]+}$ is a little bit subtler (because of the difference between $\pi_i^{-\ell}S$ and S), so we will only treat that case.

The quotient complex is annihilated by p^2, since so is $\pi_i^{-\ell}\mathbf{A}_R^{(0,v]+}/\mathbf{A}_R^{(0,v]+}$ (by Lemma 2.35). It remains to check that the kernel complex

$$\big[\ \mathrm{Kos}\big(\Gamma'_R, \big(\mathbf{A}_R^{(0,v/p]+}(r)\big)^{\psi=0}\big) \xrightarrow{\tau_0} \mathrm{Kos}^c\big(\Gamma'_R, \big(\mathbf{A}_R^{(0,v/p]+}(r)\big)^{\psi=0}\big)\ \big]$$

is p^4-acyclic. Now, according to Proposition 2.16, we have a p^{p+1}-isomorphism

$$\big(\mathbf{A}_R^{(0,v/p]+}(r)\big)^{\psi=0} \simeq \bigoplus_{\alpha\in\{0,\dots,p-1\}^{[0,d]},\,\alpha\neq 0} \varphi(\mathbf{A}_R^{(0,v]+})[x^\alpha], \quad \text{where } [x^\alpha] = (1+\pi_i)^{\alpha_0}[x_1^{\alpha_1}]\cdots[x_d^{\alpha_d}].$$

Hence, up to p^{p+1}, we can replace the kernel complex by

$$\bigoplus_{\alpha\in\{0,\dots,p-1\}^{[0,d]},\,\alpha\neq 0} \big[\ \mathrm{Kos}\big(\Gamma'_R, \varphi(\mathbf{A}_R^{(0,v]+})(r)[x^\alpha]\big) \xrightarrow{\tau_0} \mathrm{Kos}^c\big(\Gamma'_R, \varphi(\mathbf{A}_R^{(0,v]+})(r)[x^\alpha]\big)\ \big],$$

and we can treat the complexes corresponding to each α separately. There are two cases:

• $\alpha_k \neq 0$ for some $k \neq 0$. We claim that then $\mathrm{Kos}\big(\Gamma'_R, \varphi(\mathbf{A}_R^{(0,v]+})[x^\alpha]\big)$ and $\mathrm{Kos}^c\big(\Gamma'_R, \varphi(\mathbf{A}_R^{(0,v]+})[x^\alpha]\big)$ are p^2-acyclic (the twist by (r) has disappeared because the cyclotomic character is trivial on Γ'_R). As the proof is the same in both cases, we will only treat the first complex. Write it as the double complex:

$$\begin{array}{ccccccc}
\varphi(\mathbf{A}_R^{(0,v]+})[x^\alpha] & \xrightarrow{(\gamma_j-1)} & \varphi(\mathbf{A}_R^{(0,v]+})^{J_1''}[x^\alpha] & \longrightarrow & \varphi(\mathbf{A}_R^{(0,v]+})^{J_2''}[x^\alpha] & \longrightarrow & \cdots \\
\big\downarrow{\scriptstyle \gamma_k-1} & & \big\downarrow{\scriptstyle \gamma_k-1} & & \big\downarrow{\scriptstyle \gamma_k-1} & & \\
\varphi(\mathbf{A}_R^{(0,v]+})[x^\alpha] & \xrightarrow{(\gamma_j-1)} & \varphi(\mathbf{A}_R^{(0,v]+})^{J_1''}[x^\alpha] & \longrightarrow & \varphi(\mathbf{A}_R^{(0,v]+})^{J_2''}[x^\alpha] & \longrightarrow & \cdots
\end{array}$$

where the first horizontal maps involve γ_j's with $j \neq k$, $1 \leq j \leq d$. Now, we have:

$$(\gamma_k - 1)\cdot(\varphi(y)[x^\alpha]) = \varphi(\pi_1 G(y))[x^\alpha],$$

where

$$G(y) = (1+\pi_1)^{\alpha_k}\pi_1^{-1}(\gamma_k - 1)y + \pi_1^{-1}((1+\pi_1)^{\alpha_k} - 1)y.$$

(We use the fact that $\gamma_k([x^\alpha]) = [\varepsilon]^{\alpha_k}[x^\alpha] = \varphi((1+\pi_1)^{\alpha_k})[x^\alpha]$.)

Now, G is π_i-linear and, since $(\gamma_k - 1)$ is trivial modulo $\pi_i^{-\delta_R-1}\pi$ (cf. Corollary 2.33), it follows that, modulo π_1, G is just multiplication by α_k, since $\pi_i^{\delta_R+1}\pi_1$ divides π (Lemma 2.35). Hence G is invertible. It follows, that $\gamma_k - 1$ is injective on $\varphi(\mathbf{A}_R^{(0,v]+})[x^\alpha]$ and, since $\frac{p^2}{\pi_1} \in \mathbf{A}_R^{(0,v]+}$, that the cokernel of $(\gamma_k - 1)$ is killed by p^2.

• $\alpha_k = 0$ for all $k \neq 0$, and $\alpha_0 \neq 0$. Then to prove that the kernel complex is p^2-acyclic, we will prove that $\tau_0 : \mathrm{Kos} \to \mathrm{Kos}^c$ is injective and the cokernel complex is killed by p^2. This amounts to showing the same statement for

$$\gamma_0 - \delta_{i_1}\cdots\delta_{i_q} : \varphi(\mathbf{A}_R^{(0,v]+})[x^\alpha](r) \to \varphi(\mathbf{A}_R^{(0,v]+})[x^\alpha](r), \quad \delta_{i_j} = \tfrac{\gamma_{i_j}^c - 1}{\gamma_{i_j} - 1}.$$

We have

$$(\gamma_0 - \delta_{i_1}\cdots\delta_{i_q})\cdot(\varphi(y)[x^\alpha])(r) = \big(c^r\varphi(\gamma_0(y))(1+\pi)^{p^{-i}(c-1)\alpha_0}[x^\alpha]\big)(r) - \big(\varphi(\delta_{i_1}\cdots\delta_{i_q}\cdot y)[x^\alpha]\big)(r).$$

So, we are lead to study the map F defined by

$$F = c^r(1+\pi_1)^a\gamma_0 - \delta_{i_1}\cdots\delta_{i_q}, \quad \text{where } a = p^{-i}(c-1)\alpha_0 \in \mathbf{Z}_p^*.$$

Now, $c^r - 1$ is divisible by p^2, $(1+\pi_1)^a = 1 + a\pi_1 \bmod \pi_1^2$, and $\delta_{i_j} - 1 \in (\gamma_{i_j} - 1)\mathbf{Z}_p[[\gamma_{i_j} - 1]]$. Hence, we can write $\pi_1^{-1}F$ in the form $\pi_1^{-1}F = a + \pi_1^{-1}F'$, with $F' \in (p^2, \pi_1^2, \gamma_0 - 1, \dots, \gamma_d - 1)\mathbf{Z}_p[[\pi_1, \Gamma_R]]$. It follows from Lemmas 2.35 and 2.34 that there exists $N > 0$ such that $\pi_1^{-1}F' = 0$ on $\pi_i^k\mathbf{A}_R^{(0,v]+}/\pi_i^{k+N}\mathbf{A}_R^{(0,v]+}$ for all

$k \in \mathbf{N}$. Hence $\pi_1^{-1}F$ induces multiplication by a on $\pi_i^k \mathbf{A}_R^{(0,v]+}/\pi_i^{k+N}\mathbf{A}_R^{(0,v]+}$ for all $k \in \mathbf{N}$, which implies that it is an isomorphism of $\mathbf{A}_R^{(0,v]+}$, and proves what we want since π_1 divides p^2 by Lemma 2.35. □

Remark 4.11. If $\alpha \in \{0,1,\dots,p-1\}^{[0,d]}$, let $M_\alpha = x^\alpha \varphi(\mathbf{E}_R^+)$. The above proof shows that there exists $N > 0$ such that $\frac{\gamma_j - 1}{\pi}$ is the multiplication by α_j on $\pi_i^{pk} M_\alpha / \pi_i^{pk+pN} M_\alpha$, for all $k \in \mathbf{Z}$. Hence, if $\alpha_j \neq 0$, then $\gamma_j - 1$ is invertible on $x^\alpha \varphi(\mathbf{E}_R)$ and $(\gamma_j - 1)^{-1} \cdot \pi_i^{pk} M_\alpha \subset \pi^{-1}\pi_i^{pk} M_\alpha$, for all $k \in \mathbf{Z}$. In view of the relationship between Koszul complexes and group cohomology, this implies that $H^i(\Gamma_R, \pi_i^{pk} M_\alpha)$ is killed by π, for all $k \in \mathbf{Z}$, and $\alpha \neq 0$.

Now, we defined maps $c_{\mathrm{cycl},\alpha}$ on R_ϖ (cf. Proposition 2.15), and it follows from Proposition 2.15 (using ι_{cycl} to transfer the result to $\mathbf{A}_R$), that $c_{\mathrm{cycl},\alpha}(\pi_i^{pk}\mathbf{E}_R^+) \subset \pi_i^{pk-\delta_R} M_\alpha$. As $x \mapsto (c_{\mathrm{cycl},\alpha}(x))_{\alpha \neq 0}$ gives an isomorphism between $\mathbf{E}_R^{\psi=0}$ and $\oplus_{\alpha\neq 0} x^\alpha \varphi(\mathbf{E}_R)$, commuting with the action of Γ_R, we infer that $H^i(\Gamma_R, (\pi^{pk}\mathbf{E}_R^+)^{\psi=0})$ is killed by $\pi\pi_i^{\delta_R}$, for all $k \in \mathbf{Z}$ and $i \in \mathbf{N}$.

Lemma 4.12. *The natural map*

$$\mathrm{Kos}(\psi, \Gamma_R, \mathbf{A}_R^{(0,v]+}(r)) \to \mathrm{Kos}(\psi, \Gamma_R, \mathbf{A}_R(r))$$

is a quasi-isomorphism.

Proof. The map is injective, so we just have to check that the quotient complex is acyclic. Since $\mathbf{A}_R$ is the completion of $\mathbf{A}_R^{(0,v]+}[\pi_i^{-1}]$ with respect to the p-adic topology, Proposition 2.16 implies that $\psi : \mathbf{A}_R/\pi_i^{-\ell}\mathbf{A}_R^{(0,v]+} \to \mathbf{A}_R/\pi_i^{-\ell}\mathbf{A}_R^{(0,v]+}$ is (pointwise) topologically nilpotent, hence $1-\psi$ is bijective. It follows that $1-\psi$ is bijective on $\mathrm{Kos}(\Gamma'_R, \mathbf{A}_R/\pi_i^{-\ell}\mathbf{A}_R^{(0,v]+})$ and $\mathrm{Kos}^c(\Gamma'_R, \mathbf{A}_R/\pi_i^{-\ell}\mathbf{A}_R^{(0,v]+})$. This allows to conclude. □

4.6. **Passage to Galois cohomology.**

4.6.1. *Galois cohomology of R.* Let $\widetilde{R}_\infty$ be the integral closure of R in the sub-$R[\frac{1}{p}]$-algebra of $\overline{R}[\frac{1}{p}]$ generated by $\zeta_m, X_{a+b+1}^{m^{-1}}, \cdots, X_d^{m^{-1}}$, for all $m \geq 1$. We have $R_\infty \subset \widetilde{R}_\infty$ and, by Abhyankar's Lemma (Lemma 5.8 below), $\overline{R}$ is the maximal extension of $\widetilde{R}_\infty$ which is étale in characteristic zero.

Let $\widetilde{\Gamma}_R = \mathrm{Gal}(\widetilde{R}_\infty[1/p]/R[1/p])$. Then Γ_R is a quotient of $\widetilde{\Gamma}_R$ and the kernel Γ_1 of the projection is isomorphic to $\prod_{\ell \neq p}(\mathbf{Z}_\ell(1))^c$; in particular, it is of "order" prime to p.

The algebra $R_\infty[\frac{1}{p}]$ is perfectoid; let Let $\mathbf{E}_{\widetilde{R}_\infty}$ be its tilt, and let $\mathbf{E}^+_{\widetilde{R}_\infty}$ be its ring of integers (it is the tilt of $\widetilde{R}_\infty$). Finally, let $\mathbf{A}_{\widetilde{R}_\infty} = W(\mathbf{E}_{\widetilde{R}_\infty})$; this is the fixed points of $\mathbf{A}_{\overline{R}}$ by $\mathrm{Gal}(\overline{R}[\frac{1}{p}]/\widetilde{R}_\infty[\frac{1}{p}])$.

Proposition 4.13. (i) *The exact sequence (2.24) induces a quasi-isomorphism*

$$[\ \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, \mathbf{A}_{\overline{R}}(r)) \xrightarrow{1-\varphi} \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, \mathbf{A}_{\overline{R}}(r))\ ] \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, \mathbf{Z}_p(r)).$$

(ii) *The inclusions $\mathbf{A}_{R_\infty} \subset \mathbf{A}_{R'_\infty} \subset \mathbf{A}_{\overline{R}}$ induce quasi-isomorphisms*

$$\mu_H : \quad \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{A}_{R_\infty}(r)) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cont}}(\widetilde{\Gamma}_R, \mathbf{A}_{\widetilde{R}_\infty}(r)) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, \mathbf{A}_{\overline{R}}(r)).$$

(iii) *The inclusion $\mathbf{A}_R \subset \mathbf{A}_{R_\infty}$ induces a quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{A}_R(r)) \xrightarrow{\mu_\infty} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{A}_{R_\infty}(r)).$$

Proof. (i) is immediate. The first quasi-isomorphism in (ii) follows from the fact that Γ_1 is of order prime to p, and the second follows by almost étale descent as in [18, 3.2,3.3], [3, §2], [4, §7]. Finally, we will prove (iii) by an adaptation of the usual decompletion techniques as in [4, Theorem 7.16], [37].

By dévissage, it is enough to prove the statement modulo p, in which case the twist disappears and $\mathbf{A}_R$, $\mathbf{A}_{R_\infty}$ are replaced by $\mathbf{E}_R$, $\mathbf{E}_{R_\infty}$. Now, $\mathbf{E}_{R_\infty}$ is the completion of the perfectisation $\varphi^{-\infty}(\mathbf{E}_R)$ of $\mathbf{E}_R$. But $\mathbf{E}_R = \varphi(\mathbf{E}_R) \oplus \mathbf{E}_R^{\psi=0} = \varphi^2(\mathbf{E}_R) \oplus \varphi(\mathbf{E}_R^{\psi=0}) \oplus \mathbf{E}_R^{\psi=0} = \cdots$, which gives us that

$$\varphi^{-n}(\mathbf{E}_R) = \mathbf{E}_R \oplus \varphi^{-1}(\mathbf{E}_R^{\psi=0}) \oplus \cdots \oplus \varphi^{-n}(\mathbf{E}_R^{\psi=0}).$$

Hence we can write an element x of $\varphi^{-\infty}(\mathbf{E}_R)$, uniquely, as $x = s_0(x) + \sum_{n\geq 1} s_n^*(x)$, with $s_0(x) \in \mathbf{E}_R$ and $s_n^*(x) \in \varphi^{-n}(\mathbf{E}_R^{\psi=0})$ if $n \geq 1$, with $s_n^*(x) = 0$ for all but a finite number of n's.

The map $\mathbf{E}_R$-linear map $s_n = s_0 + s_1^* + \cdots + s_n^*$ is the normalised trace map from $\varphi^{-\infty}(\mathbf{E}_R)$ to $\varphi^{-n}(\mathbf{E}_R)$. On $\varphi^{-n-k}(\mathbf{E}_R)$ it is given by the formula $\varphi^{-n} \circ \psi^n \circ \varphi^{n+k}$, which shows, using (ii) of Proposition 2.16 (or rather its reduction modulo p), that

$$s_n(\varphi^{-n+k}(\mathbf{E}_R^+)) \subset \varphi^{-n}(\pi_i)^{-\ell_R}\varphi^{-n}(\mathbf{E}_R^+) \subset \pi_i^{-\ell_R}\varphi^{-n}(\mathbf{E}_R^+);$$

hence the s_n's are a uniform family of uniformly continuous maps $\varphi^{-\infty}(\mathbf{E}_R)$, and they extend to $\mathbf{E}_{R_\infty}$, and any element $x \in \mathbf{E}_{R_\infty}$ can be written, uniquely, as $x = s_0(x) + \sum_{n\geq 1} s_n^*(x)$, with $s_0(x) \in \mathbf{E}_R$ and $s_n^*(x) \in \varphi^{-n}(\mathbf{E}_R^{\psi=0})$ if $n \geq 1$, with $s_n^*(x) \to 0$ when $n \to \infty$ (this last condition means that $s_n^*(x) \in \pi^{k_n}\varphi^{-n}((\mathbf{E}_R^+)^{\psi=0})$, with $k_n \to +\infty$). In particular, this gives a Γ_R-equivariant decomposition $\mathbf{E}_{R_\infty} = \mathbf{E}_R \oplus Z$ (sending x to $(s_0(x), x - s_0(x))$).

Now, if c is a continuous cocycle on Γ_R, the same is true of the $s_n^*(c)$, and there exists some N such that $s_n^*(c)$ has values in $\pi_i^{-N+k_n}\varphi^{-n}((\mathbf{E}_R^+)^{\psi=0})$, with $k_n \geq 0$ and $k_n \to +\infty$. But Remark 4.11 tells us that $\varphi^n(s_n^*(c))$ is the coboundary of a cochain with values in $\pi_i^{p^n(-N+k_n)}\pi^{-1}\pi_i^{-\delta_R}(\mathbf{E}_R^+)^{\psi=0}$. Hence if we set $M = N + p^i + \delta_R$, it follows that $s_n^*(c)$ is the coboundary of a cochain c_n with values in $\pi_i^{-M+k_n}\varphi^{-n}((\mathbf{E}_R^+)^{\psi=0})$, and $c - s_0(c)$ is the coboundary of $\sum_{n\geq 1} c_n$; hence $H^i(\Gamma_R, Z) = 0$.

This concludes the proof. □

4.6.2. *The map $\alpha_r^{\mathscr{L}az}$.*

Theorem 4.14. *Assume that K contains enough roots of unity. There exist universal constants N and c_p such that there exist p^{Nr+c_p}-quasi-isomorphisms,*

$$\alpha_r^{\mathscr{L}az}: \quad \tau_{\leq r}\mathrm{Syn}(R,r) \simeq \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{cont}}(G_R, \mathbf{Z}_p(r)),$$
$$\alpha_{r,n}^{\mathscr{L}az}: \quad \tau_{\leq r}\mathrm{Syn}(R,r)_n \simeq \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{cont}}(G_R, \mathbf{Z}/p^n(r)).$$

Proof. We will argue integrally – the case modulo p^n being analogous. Section 3 provides us with a "quasi-isomorphism" (in degrees $\leq r$ as, in larger degrees the constants become too large)

$$\mathrm{Syn}(R,r) \simeq \mathrm{Cycl}(R_\varpi^{[u,v]}, r).$$

Choosing a basis of Ω^1 and using the isomorphism $R_\varpi^{[u,v]} \simeq \mathbf{A}_R^{[u,v]}$, we change $\mathrm{Cycl}(R_\varpi^{[u,v]}, r)$ into a Koszul complex and obtain the isomorphism (see § 4.1):

$$\mathrm{Cycl}(R_\varpi^{[u,v]}, r) \simeq \mathrm{Kos}(\varphi, \partial, F^r\mathbf{A}_R^{[u,v]}).$$

Then, multiplying by suitable powers of t, we can get rid of the filtration (Lemma 4.4) (in degrees $\leq r$; this is the only place where the truncation is absolutely necessary), and change the derivatives into the action of the Lie algebra of Γ_R, to obtain:

$$\tau_{\leq r}\mathrm{Kos}(\varphi, \partial, F^r\mathbf{A}_R^{[u,v]}) \simeq \tau_{\leq r}\mathrm{Kos}(\varphi, \mathrm{Lie}\,\Gamma_R, \mathbf{A}_R^{[u,v]}(r)).$$

Standard analytic arguments change this into a Koszul complex for the group (Lemma 4.5):

$$\mathrm{Kos}(\varphi, \mathrm{Lie}\,\Gamma_R, \mathbf{A}_R^{[u,v]}(r)) \simeq \mathrm{Kos}(\varphi, \Gamma_R, \mathbf{A}_R^{[u,v]}(r)).$$

Then, using (φ, Γ)-module techniques, we get a string of "quasi-isomorphisms" (Lemmas 4.8, 4.10, 4.12):

$$\begin{aligned}\mathrm{Kos}(\varphi, \Gamma_R, \mathbf{A}_R^{[u,v]}(r)) &\simeq \mathrm{Kos}(\varphi, \Gamma_R, \mathbf{A}_R^{(0,v]+}(r)) \simeq \mathrm{Kos}(\psi, \Gamma_R, \mathbf{A}_R^{(0,v]+}(r)) \\ &\simeq \mathrm{Kos}(\psi, \Gamma_R, \mathbf{A}_R(r)) \simeq \mathrm{Kos}(\varphi, \Gamma_R, \mathbf{A}_R(r)).\end{aligned}$$

General nonsense about Koszul complexes (see § 4.2) gives us a quasi-isomorphism:

$$\mathrm{Kos}(\varphi, \Gamma_R, \mathbf{A}_R(r)) \simeq [\ \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{A}_R(r)) \xrightarrow{1-\varphi} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{A}_R(r))\ ].$$

Finally, general (φ, Γ)-module theory (Proposition 4.13) gives a quasi-isomorphism:

$$[\ \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{A}_R(r)) \xrightarrow{1-\varphi} \mathrm{R}\Gamma_{\mathrm{cont}}(\Gamma_R, \mathbf{A}_R(r))\ ] \simeq \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, \mathbf{Z}_p(r)).$$

□

Remark 4.15. Quasi-isomorphisms between continuous Galois cohomology and Koszul complexes of the type

$$\mathrm{Kos}(\varphi, \Gamma_R, \mathbf{A}_R(r)) \simeq \mathrm{R}\Gamma_{\mathrm{cont}}(G_R, \mathbf{Z}_p(r)),$$

that we have proved above, were derived before in the case of local fields with imperfect residue fields by Morita [44] (for torsion representations) and Zerbes [67] (for rational representations), and in the case of perfect residue field, but with a complex similar to our complex for $d = 1$, by Tavares Ribeiro [60]. They are all generalizations of the complex from Herr's thesis [32].

4.7. **Comparison with local Fontaine-Messing period map.** The aim of this section is to prove that our period map coincides with that of Fontaine-Messing. We will first recall the definition of the latter.

Let $E^{\mathrm{PD}}_{\overline{R},n}$ denote the log-PD-envelope of $\overline{R}_n$ in $\mathbf{A}^{\times}_{\mathrm{cr}}(\overline{R})_n \otimes R^+_{\varpi}$ compatible with the PD-structures on $\mathbf{A}_{\mathrm{cr},n}$ and $\mathscr{O}_F$. We have $E^{\mathrm{PD}}_{\overline{R},n} = E^{\mathrm{PD}}_{\overline{R}}/p^n$, where $E^{\mathrm{PD}}_{\overline{R}}$ is the ring appearing in Lemma 2.38, hence $E^{\mathrm{PD}}_{\overline{R},n}$ has a natural action of G_R (trivial on R^+_{ϖ}) and a Frobenius φ compatible with the Kummer Frobenius on R^+_{ϖ}.

Diagram (2.3) extends to the following diagram of maps of schemes.

$$\begin{tikzcd}
 & \operatorname{Spec} E^{\mathrm{PD}}_{\overline{R},n} \arrow[dd] \arrow[dr] & \\
\operatorname{Spec} \overline{R}_n \arrow[ur, hook] \arrow[rr, hook] \arrow[dd] & & \operatorname{Spec}(\mathbf{A}_{\mathrm{cr}}(\overline{R})_n \otimes R^+_{\varpi}) \arrow[dd] \\
 & \operatorname{Spec} R^{\mathrm{PD}}_{\varpi,n} \arrow[dd] \arrow[dr] & \\
\operatorname{Spec} R_n \arrow[ur, hook] \arrow[rr, hook] \arrow[dd] & & \operatorname{Spec} R^+_{\varpi,n} \arrow[dd] \\
 & \operatorname{Spec} r^{\mathrm{PD}}_{\varpi,n} \arrow[dr] & \\
\operatorname{Spec} \mathscr{O}_{K,n} \arrow[ur, hook] \arrow[rr, hook] & & \operatorname{Spec} r^+_{\varpi,n}
\end{tikzcd}$$

As always, the bottom map is defined by $X_0 \mapsto \varpi$.

Set $\Omega_{E^{\mathrm{PD}}_{\overline{R},n}} := E^{\mathrm{PD}}_{\overline{R},n} \otimes_{R^+_{\varpi,n}} \Omega_{R^+_{\varpi,n}}$. For $r \in \mathbf{N}$, we filter the de Rham complex $\Omega^{\bullet}_{E^{\mathrm{PD}}_{\overline{R},n}}$ by subcomplexes

$$F^r\Omega^{\bullet}_{E^{\mathrm{PD}}_{\overline{R},n}} := F^r E^{\mathrm{PD}}_{\overline{R},n} \to F^{r-1}E^{\mathrm{PD}}_{\overline{R},n} \otimes_{R^+_{\varpi,n}} \Omega^1_{R^+_{\varpi,n}} \to F^{r-2}E^{\mathrm{PD}}_{\overline{R},n} \otimes_{R^+_{\varpi,n}} \Omega^2_{R^+_{\varpi,n}} \to \cdots$$

Define the syntomic complexes

$$\mathrm{Syn}(\overline{R}, r)_n := [F^r\Omega^{\bullet}_{E^{\mathrm{PD}}_{\overline{R},n}} \xrightarrow{p^r - p^{\bullet}\varphi} \Omega^{\bullet}_{E^{\mathrm{PD}}_{\overline{R},n}}].$$

For a continuous G_R-module M, let $C(G_R, M)$ denote the complex of continuous cochains of G_R with values in M. One defines the Fontaine-Messing period map

$$\tilde{\alpha}^{\mathrm{FM}}_{r,n} : \quad \mathrm{Syn}(R, r)_n \to C(G_R, \mathbf{Z}/p^n(r)')$$

as the composition

$$\begin{aligned}\mathrm{Syn}(R,r)_n &= [F^r\Omega^\bullet_{R^{\mathrm{PD}}_{\varpi,n}} \xrightarrow{p^r-p^\bullet\varphi} \Omega^\bullet_{R^{\mathrm{PD}}_{\varpi,n}}] \to C(G_R, [F^r\Omega^\bullet_{E^{\mathrm{PD}}_{\overline{R},n}} \xrightarrow{p^r-p^\bullet\varphi} \Omega^\bullet_{E^{\mathrm{PD}}_{\overline{R},n}}])\\ &\xleftarrow{\sim} C(G_R, [F^r\mathbf{A}_{\mathrm{cr}}(\overline{R})_n \xrightarrow{p^r-\varphi} \mathbf{A}_{\mathrm{cr}}(\overline{R})_n]) \xleftarrow{\sim} C(G_R, \mathbf{Z}/p^n(r)')\end{aligned}$$

The second quasi-isomorphism above follows from the Poincaré Lemma, i.e., from the quasi-isomorphism

$$F^r\mathbf{A}_{\mathrm{cr}}(\overline{R})_n \xrightarrow{\sim} F^r\Omega^\bullet_{E^{\mathrm{PD}}_{\overline{R},n}}, \quad r\geq 0,$$

proved in [63, Lemma 3.1.7] (cf. also Lemma 2.37).

The following theorem allows us to pass from the local Fontaine-Messing period map to the Lazard type period map we have defined in sections 3 and 4.

Theorem 4.16. *If K has enough roots of unity, the map $\tilde{\alpha}^{\mathrm{FM}}_{r,n}$ is p^{Nr+c_p}-equal to the map $\tilde{\alpha}^{\mathscr{L}az}_{r,n}$ from Theorem 4.14.*

Proof. Choose u,v as usual. The equality of the two maps follows from the commutative diagram below (where we did the p-adic version for simplicity). The objects and maps of this diagram are described after the diagram. It will be clear from this description that $K_{\partial,\varphi}(F^rR^{\mathrm{PD}}_\varpi) = \mathrm{Syn}(R,r)$ and that $\tilde{\alpha}^{\mathrm{FM}}_r$ is exactly the map obtained form the quasi-isomorphisms in the first row (except for the fact we should get $C_G(\mathbf{Z}_p(r)')$ instead of $C_G(\mathbf{Z}_p(r))$, but the two are p^r-isomorphic), whereas $\tilde{\alpha}^{\mathscr{L}az}_r$ is obtained by composing the quasi-isomorphisms forming the lower boundary of the diagram. Tildas in the diagram denote maps that we have proved above to be p^{Nr}-quasi-isomorphisms or that are known to be such. Little diagram chase shows that this implies Theorem 4.16. □

$$\begin{tikzcd}
K_{\partial,\varphi}(F^rR^{\mathrm{PD}}_\varpi) \arrow[r] \arrow[d, "\tau_{\leq r}"', "\wr"] & C_G(K_{\partial,\varphi}(F^rE^{\mathrm{PD}}_{\overline{R}})) & C_G(K_\varphi(F^r\mathbf{A}_{\mathrm{cr}}(\overline{R}))) \arrow[l, "\mathrm{PL}"', "\sim"] & C_G(\mathbf{Z}_p(r)) \arrow[l, "\mathrm{FES}"', "\sim"] \arrow[dl, "\sim"] \arrow[d, "\wr"] \arrow[dr, "\mathrm{AS}", "\sim"'] & \\
K_{\partial,\varphi}(F^rR^{[u,v]}_\varpi) \arrow[r] \arrow[dr] \arrow[ddd, bend right, "\mathrm{PL}"', "\wr"] & C_G(K_{\partial,\varphi}(F^rE^{[u,v]}_{\overline{R}})) \arrow[u] & C_G(K_\varphi(F^r\mathbf{A}^{[u,v]}_{\overline{R}})) \arrow[l, "\mathrm{PL}"', "\sim"] \arrow[u, "\mu_H"] & C_G(K_\varphi(\mathbf{A}^{(0,v]+}_{\overline{R}}(r))) \arrow[l, "t^r"'] \arrow[r] \arrow[u, "\mu_H"] & C_G(K_\varphi(\mathbf{A}_{\overline{R}}(r))) \arrow[u, "\wr"', "\mu_H"] \\
 & C_\Gamma(K_{\partial,\varphi}(F^rE^{[u,v]}_{R_\infty})) \arrow[u, "\mu_\infty"] & C_\Gamma(K_\varphi(F^r\mathbf{A}^{[u,v]}_{R_\infty})) \arrow[l, "\mathrm{PL}"', "\sim"] \arrow[u, "\mu_\infty"] & C_\Gamma(K_\varphi(\mathbf{A}^{(0,v]+}_{R_\infty}(r))) \arrow[l, "t^r"'] \arrow[r] \arrow[u, "\mu_\infty"] & C_\Gamma(K_\varphi(\mathbf{A}_{R_\infty}(r))) \arrow[u, "\wr"', "\mu_\infty"] \\
 & C_\Gamma(K_{\partial,\varphi}(F^rE^{[u,v]}_R)) \arrow[u, "\wr", "\lambda"'] & C_\Gamma(K_\varphi(F^r\mathbf{A}^{[u,v]}_R)) \arrow[l, "\mathrm{PL}"', "\sim"] \arrow[u, "\wr", "\lambda"'] & C_\Gamma(K_\varphi(\mathbf{A}^{(0,v]+}_R(r))) \arrow[l, "t^r"'] \arrow[r] \arrow[u, "\wr", "\lambda"'] & C_\Gamma(K_\varphi(\mathbf{A}_R(r))) \arrow[u, "\wr"', "\lambda"] \\
 & K_{\partial,\varphi,\Gamma}(F^rE^{[u,v]}_R) \arrow[u, "\wr"] \arrow[d, "\wr", "\mathscr{L}az"'] & K_{\varphi,\Gamma}(F^r\mathbf{A}^{[u,v]}_R) \arrow[l, "\mathrm{PL}"', "\sim"] \arrow[u, "\wr"] \arrow[d, "\wr", "\mathscr{L}az"'] & K_{\varphi,\Gamma}(\mathbf{A}^{(0,v]+}_R(r)) \arrow[l, "t^r"'] \arrow[r, "\sim"] \arrow[u, "\wr"] \arrow[d, "\wr", "\mathscr{L}az\,\mathrm{can}"'] & K_{\varphi,\Gamma}(\mathbf{A}_R(r)) \arrow[u, "\wr"] \\
 & K_{\partial,\varphi,\mathrm{Lie}\,\Gamma}(F^rE^{[u,v]}_R) \arrow[u, "t^\bullet"] & K_{\varphi,\mathrm{Lie}\,\Gamma}(F^r\mathbf{A}^{[u,v]}_R) \arrow[l, "\mathrm{PL}"', "\sim"] \arrow[u, "\wr", "t^\bullet,\tau_{\leq r}"'] & K_{\varphi,\mathrm{Lie}\,\Gamma}(\mathbf{A}^{[u,v]}_R(r)) \arrow[l, "t^r"', "\sim"] & \\
 & K_{\partial,\varphi,\partial_{\mathbf{A}}}(F^rE^{[u,v]}_R) \arrow[u] & K_{\varphi,\partial_{\mathbf{A}}}(F^r\mathbf{A}^{[u,v]}_R) \arrow[l, "\mathrm{PL}"', "\sim"] \arrow[u] & &
\end{tikzcd}$$

In the diagram:

- G and Γ are G_R and Γ_R,
- C_G or C_Γ denotes the complex of continuous cochains of G or Γ,

- K denotes a complex of Koszul type:
  - the indices indicate the operators involved in the complex:
    - ∂ is a shorthand for $\big(X_0\frac{\partial}{\partial X_0},\dots,X_d\frac{\partial}{\partial X_d}\big)$,
    - Γ is a shorthand for $(\gamma_0-1,\dots,\gamma_d-1)$, where the γ_i's are our chosen topological generators of Γ,
    - $\operatorname{Lie}\Gamma$ is a shorthand for $\big(\nabla_0,\dots,\nabla_d\big)$, where $\nabla_i=\log\gamma_i$, so that the ∇_i's are a basis of $\operatorname{Lie}\Gamma$ over $\mathbf{Z}_p$,
    - $\partial_{\mathbf{A}}$ is a shorthand for $\big((1+T)\frac{\partial}{\partial T},X_1\frac{\partial}{\partial X_1},\dots,X_d\frac{\partial}{\partial X_d}\big)$, viewed as operators on $\mathbf{A}_R^{[u,v]}$ or $E_R^{[u,v]}$, via the isomorphism $\iota_{\mathrm{cycl}}:R_{\varpi}^{[u,v]}\cong\mathbf{A}_R^{[u,v]}$,
  - only the first term of the complex is indicated: the rest is implicit and obtained from the first term so that the maps involved make sense: φ does not respect filtration or annulus of convergence, and ∂ or $\partial_{\mathbf{A}}$ decrease the degrees of filtration by 1.

For example, choosing a basis of $\Omega_{R_{\varpi}/\mathscr{O}_F}$ transforms complexes involving differentials into complexes of Koszul type: $\mathrm{Kum}(S,r)=K_{\partial,\varphi}(F^rS)$ if $S=R_{\varpi}^{\mathrm{PD}}$ or $R_{\varpi}^{[u,v]}$.

Let us now turn our attention to the maps between rows.

- The maps AS and FES originating for the upper right corner come from the fundamental exact sequences of Lemma 2.23 and Artin-Schreier theory of Proposition 4.13.
- Going from first row to second row just uses the injection $R_{\varpi}^{\mathrm{PD}}\subset R_{\varpi}^{[u,v]}$.
- Going from third row to second row is the inflation map from Γ_R to G_R, using the injection $R_\infty\subset\overline{R}$. Note that we could use almost étale descent (i.e. Faltings' purity theorem or its extension by Scholze or Kedlaya-Liu) to prove that it is a quasi-isomorphism.
- Going form fourth row to third row just uses the injection of R into R_∞. We could prove that this induces quasi-isomorphisms using the usual decompletion techniques, but we do not need it for the theorem.
- The maps λ connecting the fourth row to the fifth are the maps connecting continuous cohomology of Γ_R to Koszul complexes; they are defined in § 4.2.
- All the maps $\mathscr{L}az$ connecting the sixth row to the fifth are defined as in Lemma 4.6 and Remark 4.7. The same lemma shows that they are well-defined isomorphisms. (In the fourth column, $\mathscr{L}az$ is composed with the canonical map $\mathbf{A}_R^{(0,v]+}\to\mathbf{A}_R^{[u,v]}$; this is just to save space and not add an extra column.)
- The maps $t^\bullet$ connecting the last row to the sixth are the maps appearing in the proof of Lemma 4.4 (multiplication by suitable powers of t).

Finally, let us describe the maps between columns.

- The maps from the first column to the second are induced by the natural injections of rings; the PL-map is a quasi-isomorphism by Lemma 3.11.
- The maps from the third column to the second are also induced by the natural injections of rings; the PL-map are quasi-isomorphisms by Lemma 2.37 (for the first 6 rows) and Lemma 3.11 (for the last row).
- From the fourth to the third the map is multiplication by t^r (as explained before Lemma 4.4), composed with inclusion of rings.
- From the fourth to the last the map is just inclusion of rings.

5. Global applications

Unless otherwise stated, we work in the category of integral quasi-coherent log-schemes. We will denote by $\mathscr{O}_K$, $\mathscr{O}_K^\times$, and $\mathscr{O}_K^0$ the scheme $\mathrm{Spec}(\mathscr{O}_K)$ with the trivial, canonical (i.e., associated to the closed point), and $(\mathbf{N} \to \mathscr{O}_K, 1 \mapsto 0)$ log-structure, respectively.

5.1. Syntomic cohomology and p-adic nearby cycles.

5.1.1. *Definition of syntomic cohomology.* Let X be a log-scheme, log-smooth over $\mathscr{O}_K^\times$. For any $r \geq 0$, consider its absolute (meaning over $\mathscr{O}_F$) log-crystalline cohomology complexes

$$\mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{J}^{[r]})_n := \mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{R}u_{X_n/W_n(k)*}\mathscr{J}^{[r]}_{X_n/W_n(k)}), \quad \mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{J}^{[r]}) := \mathrm{holim}_n\, \mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{J}^{[r]})_n,$$

where $u_{X_n/W_n(k)} : (X_n/W_n(k))_{\mathrm{cr}} \to X_{\text{ét}}$ is the projection from the log-crystalline to the étale topos and $\mathscr{J}^{[r]}_{X_n/W_n(k)}$ is the r'th divided power of the canonical PD-ideal $\mathscr{J}_{X_n/W_n(k)}$ (for $r \leq 0$, by convention, $\mathscr{J}^{[r]}_{X_n/W_n(k)} := \mathscr{O}_{X_n/W_n(k)}$ and we will often omit it from the notation).

We have the mod p^n and completed syntomic complexes

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, r)_n := [\mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{J}^{[r]})_n \xrightarrow{p^r - \varphi} \mathrm{R}\Gamma_{\mathrm{cr}}(X_n)],$$
$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, r) := \mathrm{holim}_n\, \mathrm{R}\Gamma_{\mathrm{syn}}(X, r)_n.$$

Here the Frobenius φ is defined by the composition

$$\varphi : \mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{J}^{[r]})_n \to \mathrm{R}\Gamma_{\mathrm{cr}}(X_n) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/\mathscr{O}_F)_n \xrightarrow{\varphi} \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/\mathscr{O}_F)_n \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X_n)$$

The mapping fibers are taken in the ∞-derived category of abelian groups.

We have $\mathrm{R}\Gamma_{\mathrm{syn}}(X, r)_n \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(X, r) \otimes^L \mathbf{Z}/p^n$. There is a canonical quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, r)_n \xrightarrow{\sim} [\mathrm{R}\Gamma_{\mathrm{cr}}(X)_n \xrightarrow{(p^r - \varphi, \mathrm{can})} \mathrm{R}\Gamma_{\mathrm{cr}}(X)_n \oplus \mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{O}/\mathscr{J}^{[r]})_n].$$

Similarly in the completed case.

The above definitions sheafify. Let $\mathscr{J}^{[r]}_{\mathrm{cr},n}$, $\mathscr{A}_{\mathrm{cr},n}$, and $\mathscr{S}_n(r)$ for $r \geq 0$ be the étale sheafifications of the presheaves sending étale map $U \to X$ to $\mathrm{R}\Gamma_{\mathrm{cr}}(U, J^{[r]})_n$, $\mathrm{R}\Gamma_{\mathrm{cr}}(U)_n$, and $\mathrm{R}\Gamma_{\mathrm{syn}}(U, r)_n$, respectively. We have

$$\mathscr{S}_n(r) \simeq [\mathscr{J}^{[r]}_{\mathrm{cr},n} \xrightarrow{p^r - \varphi} \mathscr{A}_{\mathrm{cr},n}], \quad \mathrm{R}\Gamma_{\mathrm{syn}}(X, r)_n = \mathrm{R}\Gamma(X_{\text{ét}}, \mathscr{S}_n(r)).$$

Remark 5.1. Recall that syntomic cohomology can also be defined as cohomology of the syntomic site with values in syntomic Tate twists. Let us explain briefly how. We will be working with fine log-schemes. For a log-scheme X we denote by X_{syn} the small log-syntomic site of X defined as follows. The objects are morphisms $f : Y \to Z$ that are *log-syntomic* in the sense of Kato, i.e., the morphism f is integral, the underlying morphism of schemes is flat and locally of finite presentation, and, étale locally on Y, there is a factorization $Y \overset{i}{\hookrightarrow} W \overset{h}{\to} Z$ where h is log-smooth and i is an exact closed immersion that is transversally regular over Z. We also require f to be locally quasi-finite on the underlying schemes and the cokernel of the map $(f^* M_Z)^{\mathrm{gp}} \to M_Y^{\mathrm{gp}}$ ($M_?$ being the log-structure sheaf) to be torsion.

For a log-scheme X log-syntomic over $\mathrm{Spec}(W(k))$, define

$$\mathscr{O}_n^{\mathrm{cr}}(X) = H^0_{\mathrm{cr}}(X, \mathscr{O})_n, \qquad \mathscr{J}_n^{[r]}(X) = H^0_{\mathrm{cr}}(X, \mathscr{J}^{[r]})_n.$$

We know that the presheaves $\mathscr{J}_n^{[r]}$ are sheaves on $X_{n,\mathrm{syn}}$, flat over $\mathbf{Z}/p^n$, and that $\mathscr{J}_{n+1}^{[r]} \otimes \mathbf{Z}/p^n \simeq \mathscr{J}_n^{[r]}$. There is a canonical isomorphism

$$\mathrm{R}\Gamma(X_{\mathrm{syn}}, \mathscr{J}_n^{[r]}) \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{J}^{[r]})_n$$

that is compatible with Frobenius. Set

$$\mathscr{S}_n(r) := [\mathscr{J}_n^{[r]} \xrightarrow{p^r - \varphi} \mathscr{O}_n^{\mathrm{cr}}].$$

This is the syntomic Tate twist. In the same way we can define log-syntomic sheaves $\mathscr{S}_n(r)$ on $X_{m,\mathrm{syn}}$ for $m \geq n$. Abusing notation, we define $\mathscr{S}_n(r) = i_*\mathscr{S}_n(r)$ for the natural map $i : X_{m,\mathrm{syn}} \to X_{\mathrm{syn}}$. Since i_* is exact, $\mathrm{R}\Gamma(X_{m,\mathrm{syn}}, \mathscr{S}_n(r)) = \mathrm{R}\Gamma(X_{\mathrm{syn}}, \mathscr{S}_n(r))$. We have

$$\mathscr{S}_n(r) = \mathrm{R}\varepsilon_*\mathscr{S}_n(r), \quad \varepsilon : X_{\mathrm{syn}} \to X_{\text{ét}},$$
$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, r)_n = \mathrm{R}\Gamma(X_{\text{ét}}, \mathscr{S}_n(r)) = \mathrm{R}\Gamma(X_{\mathrm{syn}}, \mathscr{S}_n(r)).$$

Proposition 3.12 gives a simple description, up to some universal torsion, of syntomic cohomology sheaves. Namely, assume that X is semistable over $\mathscr{O}_K$ or a base change of such. We write

$$\mathscr{S}_n(r) \simeq [\mathscr{J}^{[r]}_{\mathrm{cr},n} \xrightarrow{p^r-\varphi} \mathscr{A}_{\mathrm{cr},n}] \simeq [\mathscr{A}^{\varphi=p^r}_{\mathrm{cr},n} \to \mathscr{A}_{\mathrm{cr},n}/\mathscr{J}^{[r]}_{\mathrm{cr},n}],$$

where we set $\mathscr{A}^{\varphi=p^r}_{\mathrm{cr},n} := [\mathscr{A}_{\mathrm{cr},n} \xrightarrow{p^r-\varphi} \mathscr{A}_{\mathrm{cr},n}]$.

Corollary 5.2. (i) *For $i \neq r$, the sheaf $\mathscr{H}^i(\mathscr{A}^{\varphi=p^r}_{\mathrm{cr},n})$ is annihilated by p^N, for a constant $N = N(e, d, p, r)$.*

(ii) *For $i \geq r+1$, the sheaf $\mathscr{H}^i(\mathscr{S}_n(r))$ is annihilated by $p^{N(r)}$.*

(iii) *For $i \leq r-1$, the natural map $\mathscr{H}^{i-1}(\mathscr{A}_{\mathrm{cr},n}/\mathscr{J}^{[r]}_{\mathrm{cr},n}) \to \mathscr{H}^i(\mathscr{S}_n(r))$ is a p^N-quasi-isomorphism, for a constant $N = N(e, d, p, r)$. Moreover we have the short exact sequence*

$$0 \to \mathscr{H}^{r-1}(\mathscr{A}_{\mathrm{cr},n}/\mathscr{J}^{[r]}_{\mathrm{cr},n}) \to \mathscr{H}^r(\mathscr{S}_n(r)) \to \mathscr{H}^r(\mathscr{A}^{\varphi=p^r}_{\mathrm{cr},n}) \to 0$$

5.1.2. *Syntomic complexes and p-adic nearby cycles.* Let X be a fine and saturated log-scheme log-smooth over $\mathscr{O}_K^\times$. Denote by X_{tr} the locus where the log-structure is trivial. This is an open dense subset of the generic fiber of X. Fontaine-Messing [28], Kato [36] have constructed period morphisms $(i : X_0 \hookrightarrow X, j : X_{\mathrm{tr}} \hookrightarrow X)$

$$\alpha^{\mathrm{FM}}_{r,n} : \quad \mathscr{S}_n(r)_X \to i^*Rj_*\mathbf{Z}/p^n(r)'_{X_{\mathrm{tr}}}, \quad r \geq 0.$$

Assume now that X has semistable reduction over $\mathscr{O}_K$ or is a base change of a scheme with semistable reduction over the ring of integers of a subfield of K. That is, locally, X can be written as $\mathrm{Spec}(A)$ for a ring A étale over

(5.3)
$$\mathscr{O}_K[X_1^{\pm 1}, \cdots, X_a^{\pm 1}, X_{a+1}, \cdots, X_{a+b}, X_{a+b+1}, \cdots, X_d, X_{d+1}]/(X_{d+1}X_{a+1}\cdots X_{a+b} - \varpi^h), \quad 1 \leq h \leq e.$$

If we put $D := \{X_{a+b+1}\cdots X_d = 0\} \subset \mathrm{Spec}(A)$ then the log-structure on $\mathrm{Spec}\, A$ is associated to the special fiber and to the divisor D. We have $\mathrm{Spec}(A)_{\mathrm{tr}} = \mathrm{Spec}(A_K) \setminus D_K$.

The purpose of this section is to prove the following theorem.

Theorem 5.4. *Let X be a scheme with semistable reduction over $\mathscr{O}_K$ or a base change of a scheme with semistable reduction over the ring of integers of a subfield of K. For $0 \leq i \leq r$, consider the period map*

$$\alpha^{\mathrm{FM}}_{r,n} : \quad \mathscr{H}^i(\mathscr{S}_n(r)_X) \to i^*R^ij_*\mathbf{Z}/p^n(r)'_{X_{\mathrm{tr}}}.$$

(i) *If K has enough roots of unity then the kernel and cokernel of this map is annihilated by p^{Nr+c_p} for universal constants N and c_p.*

(ii) *In general, the kernel and cokernel of this map is annihilated by p^N for an integer $N = N(K, p, r)$, which depends on K, p, r.*

Proof. It suffices to argue locally. Take an $\mathscr{O}_K$-algebra A as in (5.3) and such that $\mathrm{Spec}(A/p)$ is nonempty and connected. We have

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathrm{Spec}(A), r)_n = \mathrm{Syn}(\widehat{A}, r)_n, \quad \mathrm{R}\Gamma_{\mathrm{syn}}(\mathrm{Spec}(A), r) = \mathrm{Syn}(\widehat{A}, r).$$

The Fontaine-Messing period map

$$\alpha^{\mathrm{FM}}_{r,n} = \alpha^{\mathrm{FM}}_{r,n,A} : \quad \mathrm{Syn}(A, r)_n \to \mathrm{R}\Gamma(Y_{\mathrm{tr},\text{ét}}, \mathbf{Z}/p^n(r)'), \quad Y := \mathrm{Spec}\, A^h,$$

can be described as the composition of the henselian version of the map $\tilde{\alpha}^{\mathrm{FM}}_{r,n}$ with the natural map $C(G_R, \mathbf{Z}/p^n(r)') \to \mathrm{R}\Gamma(Y_{\mathrm{tr},\text{ét}}, \mathbf{Z}/p^n(r)')$ for $R := A^h$ and G_R - the Galois group of the maximal extension of R unramified in characteristic zero outside the divisor D_K. The henselian version of the period map $\tilde{\alpha}^{\mathrm{FM}}_{r,n}$ is obtained by replacing $\overline{\widehat{R}}$ with $\overline{R}$ and $G_{\widehat{R}}$ with G_R. We set $\mathrm{Syn}(A,r)_n = \mathrm{Syn}(\widehat{A},r)_n$.

Let $i \leq r$. We need to show that the map

$$\alpha^{\mathrm{FM}}_{r,n}: \quad H^i\mathrm{Syn}(A,r)_n \xrightarrow{\tilde{\alpha}^{\mathrm{FM}}_{r,n}} H^iC(G_R, \mathbf{Z}/p^n(r)') \to H^i\mathrm{R}\Gamma(Y_{\mathrm{tr},\text{ét}}, \mathbf{Z}/p^n(r)'), \tag{5.5}$$

is an isomorphism (up to the wanted constants). To do that we will pass to the completion of A. Consider the following commutative diagram.

$$\begin{tikzcd}
H^i(\mathrm{Syn}(A,r)_n) \arrow[r, "\tilde{\alpha}^{\mathrm{FM}}_{r,n,A}"] \arrow[d, equal] & H^i(G_R, \mathbf{Z}/p^n(r)') \arrow[r] \arrow[d, "\wr"'] & H^i(U_{\text{ét}}, \mathbf{Z}/p^n(r)') \arrow[d, "\wr"', "g"] \\
H^i(\mathrm{Syn}(\widehat{A},r)_n) \arrow[r, "\tilde{\alpha}^{\mathrm{FM}}_{r,n,\widehat{A}}", "\sim"'] & H^i(G_{\widehat{R}}, \mathbf{Z}/p^n(r)') \arrow[r, "f", "\sim"'] & H^i(\mathscr{U}_{\text{ét}}, \mathbf{Z}/p^n(r)'),
\end{tikzcd} \tag{5.6}$$

where $U = (\mathrm{Spec}\, R)_{\mathrm{tr}}, \mathscr{U} := (\mathrm{Sp}\, \widehat{R}[1/p])_{\mathrm{tr}}$. Now, the middle vertical arrow is an isomorphism because the two Galois groups are equal by Lemma 5.8 (combination of Abhyankar's Lemma and Elkik's approximation techniques), the map f is an isomorphism by a $K(\pi,1)$-Lemma (see 5.1.4), and the map g is an isomorphism by a rigid GAGA argument (see 5.1.5). All of which we prove below. It thus remains to prove that the map

$$\tilde{\alpha}^{\mathrm{FM}}_{r,n} = \tilde{\alpha}^{\mathrm{FM}}_{r,n,\widehat{A}}: \quad H^i(\mathrm{Syn}(\widehat{A},r)_n) \to H^i(G_{\widehat{R}}, \mathbf{Z}/p^n(r)'), \quad i \leq r,$$

is an isomorphism (up to the wanted constants). In the case that K has enough roots of unity this follows from Theorem 4.16 and Theorem 4.14, which proves the first claim of the theorem.

To prove the second claim, we pass from $R = A$ to $R^i := R(\zeta_{p^i})$, $i \geq c(K)+3$, so that R^i has enough roots of unity. There the Fontaine-Messing period map is a p^{Nr}-quasi-isomorphism, for a universal constant N (as we have just shown). To descend note that this period map is $G_i = \mathrm{Gal}(K_i/K)$-equivariant, i.e., that the following diagram commutes (cf. Remark 5.10 below).

$$\begin{tikzcd}
\mathrm{R}\Gamma(G_i, \mathrm{Syn}(R^i,r)_n) \arrow[r, "\tilde{\alpha}^{\mathrm{FM}}_{r,n}"] & \mathrm{R}\Gamma(G_i, C(G_{R^i}, \mathbf{Z}/p^n(r)')) \\
\mathrm{Syn}(R,r)_n \arrow[u] \arrow[r, "\tilde{\alpha}^{\mathrm{FM}}_{r,n}"] & C(G_R, \mathbf{Z}/p^n(r)') \arrow[u, "\wr"']
\end{tikzcd}$$

Hence to finish the proof of our theorem it suffices to quote Lemma 5.9 below. □

5.1.3. *Comparison of Galois groups.* Recall the following theorem of Lütkebohmert [43].

Theorem 5.7. (Riemann's Existence) *Let $\mathscr{X}$ be a smooth quasi compact rigid space. Let $\mathscr{Z} \subset \mathscr{X}$ be a normal crossings divisor and set $\mathscr{U} = \mathscr{X} \setminus \mathscr{Z}$. Then any finite étale covering of $\mathscr{U}$ extends (uniquely) to a finite flat normal covering of $\mathscr{X}$.*

Proof. This is the main theorem of [43]. One uses the description of the tubular neighborhoods of the divisor $\mathscr{Z}$ from [38, Theorem 1.18] to pass to pointed disks (of varied dimensions), which then can be treated by the extension lemma [43, Lemma 3.3]. □

Let R be a ring as in (5.3) or a henselization at (p) of such a ring or a ring as in 2.2.2. Let $R_m := R(\zeta_m)[X_{a+b+1}^{m^{-1}}, \cdots, X_d^{m^{-1}}]$, for a choice of a primitive m'th root of unity ζ_m, and $R'_\infty := \cup_m R_m$. Let $\overline{R}'_\infty$ be the maximal extension of R'_∞ which is étale in characteristic zero. We define similarly $\widehat{R}'_\infty$ and $\overline{\widehat{R}}'_\infty$.

Lemma 5.8. (i) (Abhyankar's Lemma) *The natural inclusion $\overline{R}'_\infty \subset \overline{R}$ is an equality. In particular, we have the natural isomorphism*

$$G_R = \mathrm{Gal}(\overline{R}[1/p]/R[1/p]) \xrightarrow{\sim} \mathrm{Gal}(\overline{R}'_\infty[1/p]/R[1/p]).$$

(ii) (Approximation) *Let R be a henselization of a ring of the form (5.3). The natural map $G_{\widehat{R}} \to G_R$ is an isomorphism.*

Proof. For the first statement, we first assume that R is of the form (5.3). Recall that $\overline{R}$ is the direct limit of a maximal chain of normal R-algebras, which are domains and, after inverting p, are finite and étale extensions of $R[1/p][X_{a+b+1}^{-1}, \cdots, X_d^{-1}]$. Similarly, $\overline{R}'_\infty$ is the direct limit of a maximal chain of normal R-algebras, which are domains and, after inverting p, are finite and étale extensions of $R'_\infty[1/p]$. Let $Y \to \mathrm{Spec}(R[1/p])$ be a normal, connected, finite scheme such that the base change $Y_{(\mathrm{Spec}\, R)_{\mathrm{tr}}} \to (\mathrm{Spec}\, R)_{\mathrm{tr}}$, $(\mathrm{Spec}\, R)_{\mathrm{tr}} := \mathrm{Spec}(R[1/p]) \setminus \{X_{a+b+1} \cdots X_d = 0\}$, is a finite étale extension. Since $R[1/p]$ is regular, by Abhyankar's Lemma [69, XIII, Proposition 5.2], there exists a number $m \geq 1$ such that the finite extension $Y_m := (Y \times_{\mathrm{Spec}(R[1/p])} \mathrm{Spec}(R(\zeta_m)[1/p][X_{a+b+1}^{m^{-1}}, \cdots, X_d^{m^{-1}}]))^{\mathrm{norm}}$ of $\mathrm{Spec}(R[1/p][X_{a+b+1}^{m^{-1}}, \cdots, X_d^{m^{-1}}])$ is étale. The first statement of the lemma for follows for R and for its henselization at (p). For R as in 2.2.2, we argue in the same way noting that the rigid space Y above exists by Theorem 5.7.

For the second statement, since clearly $\mathrm{Gal}(\widehat{R}'_\infty[1/p]/\widehat{R}[1/p])) \xrightarrow{\sim} \mathrm{Gal}(R'_\infty[1/p]/R[1/p]))$, by (i), it suffices to show that the natural map

$$\mathrm{Gal}(\overline{\widehat{R}}'_\infty[1/p]/\widehat{R}_\infty[1/p])) \to \mathrm{Gal}(\overline{R}'_\infty[1/p]/R_\infty[1/p]))$$

is an isomorphism. But

$$\mathrm{Gal}(\overline{\widehat{R}}'_\infty[1/p]/\widehat{R}_\infty[1/p])) = \lim_m \mathrm{Gal}(\overline{\widehat{R}}_m[1/p]/\widehat{R}_m[1/p])),$$
$$\mathrm{Gal}(\overline{R}'_\infty[1/p]/R_\infty[1/p])) = \lim_m \mathrm{Gal}(\overline{R}_m[1/p]/R_m[1/p])),$$

where $\overline{\widehat{R}}_m[1/p]$ (resp. $\overline{R}_m[1/p]$) denotes the maximal étale extension of $\widehat{R}_m[1/p]$ (resp. $R_m[1/p]$). Since R_m is henselian at (p) and $(R_m)^\wedge = \widehat{R}_m$, we have, by Elkik's theorem [23, Corollary p. 579], that, for $m \geq 0$,

$$\mathrm{Gal}(\overline{\widehat{R}}_m[1/p]/\widehat{R}_m[1/p])) \xrightarrow{\sim} \mathrm{Gal}(\overline{R}_m[1/p]/R_m[1/p])).$$

To conclude we pass to the limit over m. □

5.1.4. *$K(\pi, 1)$-Lemmas.* To show that the map f in diagram 5.6 is an isomorphism, note that this is just the $K(\pi, 1)$-Lemma for $\mathscr{U}$ and p-coefficients. In the case of the divisor at infinity D being trivial this is proved by Scholze [57, Theorem 1.2]. The general case we will treat the way Faltings treated the algebraic case. We have to show that, for any locally constant constructible p^n-torsion sheaf $\mathscr{M}$, any class $\beta \in H^i(\mathscr{U}_{\text{ét}}, \mathscr{M})$, $i > 0$, dies in a finite étale extension of $\mathscr{U}$. During the proof we will often pass to the extension of $\mathscr{U}$ generated by taking k'th roots of all the U_i's, $k \geq 0$. This is possible since the normalization of $\mathscr{U}$ in this extension satisfies the same hypothesis as $\mathscr{U}$.

By Abhyankar's Lemma 5.8 we have $G_{\widehat{R}} = \mathrm{Gal}(\overline{\widehat{R}}'_\infty[1/p]/\widehat{R}[1/p])$. Hence, by adjoining k'th roots of U_i's, for some $k \geq 0$, and adding some roots of unity if necessary, we can assume that the sheaf $\mathscr{M}$ is unramified, i.e., that it is a module for the fundamental group of $\mathrm{Spec}(\widehat{R}[1/p])$. Let $j : \mathscr{U} \hookrightarrow \mathrm{Sp}(\widehat{R}[1/p]), f = U_1 \cdots U_b$. There is a spectral sequence

$$E_2^{s,t} = H^s(\mathrm{Sp}(\widehat{R}[1/p])_{\text{ét}}, \mathrm{R}^t j_* \mathscr{M}) \Rightarrow H^{s+t}(\mathscr{U}_{\text{ét}}, \mathscr{M}).$$

Taking p^k'th roots of f induces multiplication by p^{tk} on $E_2^{s,t}$. This can be checked on the finite extension of $\mathrm{Spec}(R[1/p])$ that trivializes $\mathscr{M}$ and there it follows from the purity statement [21, Thm 7.2.2], [45,

Proposition 2.0.2], i.e., the fact that

$$R^t j_* \mathbf{Z}/p^n \simeq \wedge^t \overline{M}_Y^{\mathrm{gp}} \otimes \mathbf{Z}/p^n(-1), \quad Y := \mathrm{Sp}(\widehat{R}[1/p]),$$

where M_Y is the log-structure on Y associated to D_K: $M_Y = j_* \mathscr{O}_{\mathscr{U}}^* \cap \mathscr{O}_Y$, M_Y^{gp} is the induced sheaf of abelian groups, and $\overline{M}_Y := M_Y/\mathscr{O}_Y^*$.

Repeating this procedure several times we reduce to the case when β is a restriction of some cohomology class on $\mathrm{Sp}(\widehat{R}[1/p])$. Then we can use Scholze's $K(\pi, 1)$-Lemma again.

5.1.5. *Rigid GAGA.* To show that the map g in diagram (5.6) is an isomorphism, we will argue by induction on the number of irreducible components of the divisor D_K, i.e., on s. If $s = 0$ this is a result of Gabber [29, Thm 1]. Assume that the isomorphism is true for $s - 1$. The case of s follows now easily from the following commutative diagram of localization sequences in the étale and rigid étale cohomology [21, Thm 7.2.2](we assumed $s = 1$ for simplicity).

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(D_{K,\text{ét}}, \mathbf{Z}/p^n) \arrow[r, "i_*"] \arrow[d, "\wr"] & H^i(T_{\text{ét}}, \mathbf{Z}/p^n) \arrow[r, "j^*"] \arrow[d, "\wr"] & H^i(U_{\text{ét}}, \mathbf{Z}/p^n) \arrow[r] \arrow[d] & H^{i-1}(D_{K,\text{ét}}, \mathbf{Z}/p^n) \arrow[r] \arrow[d, "\wr"] & \cdots \\
\cdots \arrow[r] & H^i(\mathscr{D}_{\text{ét}}, \mathbf{Z}/p^n) \arrow[r, "i_*"] & H^i(\mathscr{T}_{\text{ét}}, \mathbf{Z}/p^n) \arrow[r, "j^*"] & H^i(\mathscr{U}_{\text{ét}}, \mathbf{Z}/p^n) \arrow[r] & H^{i-1}(\mathscr{D}_{\text{ét}}, \mathbf{Z}/p^n) \arrow[r] & \cdots
\end{tikzcd}$$

Here we set $T := \mathrm{Spec}(R[1/p])$, $\mathscr{T} := \mathrm{Sp}(\widehat{R}[1/p])$, and $\mathscr{D}$ is the rigid analytic spaces associated to D_K. The vertical isomorphisms follow from the inductive assumption.

5.1.6. *Galois descent.*

Lemma 5.9. *Let K_1 be a Galois extension of K, with relative ramification index e_1 and Galois group G. Set $R_1 := R \otimes_{\mathscr{O}_K} \mathscr{O}_{K_1}$. The natural map*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(R, r) \to \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{syn}}(R_1, r))$$

is a p^{cre_1}-quasi-isomorphism for some universal constant c.

Remark 5.10. One can see the G-action on $\mathrm{R}\Gamma_{\mathrm{syn}}(R_1, r)$ and the map $\mathrm{R}\Gamma_{\mathrm{syn}}(R, r) \to \mathrm{R}\Gamma_{\mathrm{syn}}(R_1, r)$ very explicitly. Namely, instead of the coordinate ring $\mathscr{O}_F[T_1]$ of $\mathscr{O}_{K_1}$, $T_1 \mapsto \varpi_1$, one can take the coordinate ring $\mathscr{O}_F[T, X_\sigma, \sigma \in G]$, $T \mapsto \varpi, X_\sigma \mapsto \sigma(\varpi_1)$. The action of G on this coordinate ring is defined by $g(T) = T, g(X_\sigma) = X_{g\sigma}, g \in G$.

Proof. Since crystalline cohomology satisfies étale descent we may assume that our extension is totally ramified. Write

$$\mathrm{R}\Gamma_{\mathrm{syn}}(R, r) = [\mathrm{R}\Gamma_{\mathrm{cr}}(R)^{\varphi = p^r} \xrightarrow{\mathrm{can}} \mathrm{R}\Gamma_{\mathrm{cr}}(R)/F^r],$$
$$\mathrm{R}\Gamma_{\mathrm{cr}}(R)^{\varphi = p^r} := [\mathrm{R}\Gamma_{\mathrm{cr}}(R) \xrightarrow{p^r - \varphi} \mathrm{R}\Gamma_{\mathrm{cr}}(R)]$$

It suffices to show that the maps

$$\mathrm{R}\Gamma_{\mathrm{cr}}(R)^{\varphi = p^r} \xrightarrow{\sim} \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{cr}}(R_1)^{\varphi = p^r}), \quad \mathrm{R}\Gamma_{\mathrm{cr}}(R)/F^r \xrightarrow{\sim} \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{cr}}(R_1)/F^r)$$

are p^{cre_1}-quasi-isomorphisms.

To treat the first map consider the following diagram of maps of schemes

$$\begin{tikzcd} R_0 \arrow[r, hook, "i_{R_1}"] \arrow[dr, hook, "i_R"] \arrow[dd] & R_1 \arrow[d] \\ & R \arrow[d] \\ k \arrow[r, hook] & \mathscr{O}_K \end{tikzcd}$$

It yields the commutative diagram

$$\begin{tikzcd} \mathrm{R}\Gamma_{\mathrm{cr}}(R) \arrow[r, "i_R^*"] \arrow[d] & \mathrm{R}\Gamma_{\mathrm{cr}}(R_0) \arrow[d] \\ \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{cr}}(R_1)) \arrow[r, "i_{R_1}^*"] & \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{cr}}(R_0)) \end{tikzcd}$$

Since the right vertical map is a e_1-quasi-isomorphism, it follows that it suffices to show that the morphism

$$\mathrm{R}\Gamma_{\mathrm{cr}}(R)^{\varphi=p^r} \xrightarrow{i_R^*} \mathrm{R}\Gamma_{\mathrm{cr}}(R_0)^{\varphi=p^r}$$

and its analog for R_1 are $p^{cr}e_1$-quasi-isomorphisms. To see this for R consider the following factorization

$$\varphi^m : \quad \mathrm{R}\Gamma_{\mathrm{cr}}(R/\mathscr{O}_F)^{\varphi=p^r} \xrightarrow{i_R^*} \mathrm{R}\Gamma_{\mathrm{cr}}(R_0/\mathscr{O}_F)^{\varphi=p^r} \xrightarrow{j_m} \mathrm{R}\Gamma_{\mathrm{cr}}(R/\mathscr{O}_F)^{\varphi=p^r}$$

of the m'th power of the Frobenius, where $p^m \geq e$. We also have $i_R^* j_m = \varphi^m$. Since φ^m is a p^{2rm}-quasi-isomorphism on $\mathrm{R}\Gamma_{\mathrm{cr}}(R)^{\varphi=p^r}$ and on $\mathrm{R}\Gamma_{\mathrm{cr}}(R_0)^{\varphi=p^r}$ both i_R^* and j_m are p^{4rm}-quasi-isomorphisms as well. The reasoning in the case of R_1 is analogous.

It remains to show that the map

$$\mathrm{R}\Gamma_{\mathrm{cr}}(R)/F^r \xrightarrow{\sim} \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{cr}}(R_1)/F^r)$$

is a $rp^{v_p(\delta_{K_1/K_0})} + e_1$-quasi-isomorphism. For that recall that Beilinson's identification of filtered log-crystalline cohomology with filtered derived log de Rham complex [8, 1.9.2] allows to prove that, for $n \geq 1$, the natural map

$$\mathrm{R}\Gamma_{\mathrm{cr}}(R)_n/F^r \to \mathrm{R}\Gamma_{\mathrm{cr}}(R/\mathscr{O}_K^{\times})_n/F^r \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(R/\mathscr{O}_K)_n/F^r$$

is a $rp^{v_p(\delta_{K/K_0})}$-quasi-isomorphism [46, proof of Corollary 2.4]. Same is true for R_1 with a constant $rp^{v_p(\delta_{K_1/K_0})}$. It follows that it suffices to show that the map

$$\mathrm{R}\Gamma_{\mathrm{dR}}(R/\mathscr{O}_K)_n/F^r \to \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{dR}}(R_1/\mathscr{O}_{K_1})_n/F^r)$$

is a e_1-quasi-isomorphism. But, since, $\Omega^{\bullet}_{R_1/\mathscr{O}_{K_1}} = \Omega^{\bullet}_{R/\mathscr{O}_K} \otimes_{\mathscr{O}_K} \mathscr{O}_{K_1}$, this is clear. □

Using Corollary 3.16 and the proof of Theorem 5.4, we can relate de Rham and p-adic rigid étale cohomology. More specifically, let $\mathscr{X}$ be a quasi-compact formal, semistable scheme over $\mathscr{O}_K$ (i.e., locally of the form described in section 2.1.2 with $h = 1$). For $i \geq 0$, consider the composition

$$\alpha_{r,i} : H^{i-1}_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}) \to H^i_{\mathrm{syn}}(\mathscr{X}, r)_{\mathbf{Q}} \xrightarrow{\alpha_r^{FM}} H^i_{\text{ét}}(\mathscr{X}_{K,\mathrm{tr}}, \mathbf{Q}_p(r)).$$

Corollary 5.11. *For $i \leq r-1$, the map*

$$\alpha_{r,i} : H^{i-1}_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}) \to H^i_{\text{ét}}(\mathscr{X}_{K,\mathrm{tr}}, \mathbf{Q}_p(r))$$

is an isomorphism. Moreover, the map $\alpha_{r,r} : H^{r-1}_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}) \to H^r_{\text{ét}}(\mathscr{X}_{K,\mathrm{tr}}, \mathbf{Q}_p(r))$ is injective.

5.1.7. *Geometric syntomic cohomology.* Recall the definition of the geometric syntomic cohomology, i.e., syntomic cohomology over $\overline{K}$. For a log-scheme X, log-smooth over $\mathscr{O}_K^\times$ and universally saturated, we have the absolute log-crystalline cohomology complexes and their completions

$$\begin{aligned}
\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K}}}, \mathscr{J}^{[r]})_n &:= \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K}},\text{ét}}, \mathrm{R}u_{X_{\mathscr{O}_{\overline{K}},n}/W_n(k)*}\mathscr{J}^{[r]}_{X_{\mathscr{O}_{\overline{K}},n}/W_n(k)}),\\
\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K}}}, \mathscr{J}^{[r]}) &:= \mathrm{holim}_n\, \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K}}}, \mathscr{J}^{[r]})_n,\\
\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K}}}, \mathscr{J}^{[r]})_{\mathbf{Q}_p} &:= \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K}}}, \mathscr{J}^{[r]}) \otimes \mathbf{Q}_p
\end{aligned}$$

By [7, Theorem 1.18], if X is proper, the complex $\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K}}})$ is a perfect $\mathbf{A}_{\mathrm{cr}}$-complex and

$$\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K}}})_n \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K}}}) \otimes^L_{\mathbf{A}_{\mathrm{cr}}} \mathbf{A}_{\mathrm{cr}}/p^n \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K}}}) \otimes^L \mathbf{Z}/p^n.$$

In general, we have:

$$\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K}}}, \mathscr{J}^{[r]})_n \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\mathscr{O}_{\overline{K}}}, \mathscr{J}^{[r]}) \otimes^L \mathbf{Z}/p^n.$$

Moreover, $F^r\mathbf{A}_{\mathrm{cr}} = \mathrm{R}\Gamma_{\mathrm{cr}}(\mathrm{Spec}(\mathscr{O}_{\overline{K}}), \mathscr{J}^{[r]})$ [63, 1.6.3,1.6.4].

For $r \geq 0$, the mod-p^n, completed, and rational syntomic complexes $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\mathscr{O}_{\overline{K}}}, r)_n$, $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\mathscr{O}_{\overline{K}}}, r)$, and $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\mathscr{O}_{\overline{K}}}, r)_{\mathbf{Q}}$ are defined by the analogs of formulas we have used in the arithmetic case. We have $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\mathscr{O}_{\overline{K}}}, r)_n \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(X_{\mathscr{O}_{\overline{K}}}, r) \otimes^L \mathbf{Z}/p^n$. Let $\mathscr{S}(r)$ be the étale-sheafification of the presheaf sending étale map $U \to X_{\mathscr{O}_{\overline{K}}}$ to $\mathrm{R}\Gamma_{\mathrm{syn}}(U, r)$. Let $\mathscr{S}_n(r)$ denote the étale-sheafification of the mod-p^n version of this presheaf.

Corollary 5.12. (i) *Let X be a semistable scheme over $\mathscr{O}_K$ or a base change of a semistable scheme over the ring of integers of a subfield of K. Then for $0 \leq j \leq r$, the kernel and cokernel of the map*

$$\alpha^{\mathrm{FM}}_{r,n}: \quad \mathscr{H}^j(\mathscr{S}_n(r)_{X_{\mathscr{O}_{\overline{K}}}}) \to \bar{i}^* R^j \bar{j}_* \mathbf{Z}/p^n(r)'_{X_K}$$

is annihilated by p^{Nr}, for a universal constant N. Here $\bar{i}: X_0 \hookrightarrow X_{\mathscr{O}_{\overline{K}}}, \bar{j}: X_{\mathrm{tr},\overline{K}} \hookrightarrow X_{\mathscr{O}_{\overline{K}}}$.

(ii) *Let X be a fine and saturated log-scheme log-smooth over $\mathscr{O}_K^\times$. If X is proper then the map*

$$\alpha^{\mathrm{FM}}_r: \quad H^j_{\mathrm{syn}}(X_{\mathscr{O}_{\overline{K}}}, r)_{\mathbf{Q}} \xrightarrow{\sim} H^j(X_{\mathrm{tr},\overline{K}}, \mathbf{Q}_p(r)), \quad j \leq r,$$

is an isomorphism.

Proof. The first claim follows from Theorem 5.4 by going to the limit over finite extensions of the base field.

It implies the second statement of the corollary for semistable schemes and with the group $HN^j_{\mathrm{syn}}(X_{\mathscr{O}_{\overline{K}}}, r)$ as the domain. Here $HN^j_{\mathrm{syn}}(X_{\mathscr{O}_{\overline{K}}}, r) := \varprojlim_n H^j\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\mathscr{O}_{\overline{K}}}, r)_n$, $j \geq 0$, denotes the naive syntomic cohomology. We have a natural map $H^j_{\mathrm{syn}}(X_{\mathscr{O}_{\overline{K}}}, r) \to HN^j_{\mathrm{syn}}(X_{\mathscr{O}_{\overline{K}}}, r)$. Since the groups $H^*(X_{\mathrm{tr},\overline{K}}, \mathbf{Z}/p^n(r))$ are finite all the relevant higher derived projective limits vanish and we can replace the naive syntomic groups with the "real" ones.

To pass to general log-smooth schemes we use the following observation. We can assume that K has enough roots of unity. By [55, Theorem 2.9], there exists a ramified extension K_1 of K such that the base change $X_{\mathscr{O}_{K_1}}$ has a semistable model. That is, there exists a log-blow-up $\pi: Y \to X_{\mathscr{O}_{K_1}}$ such that Y is a semistable scheme (with no multiplicities in the special fiber). We have the following commutative diagram

$$\begin{tikzcd}
H^j_{\mathrm{syn}}(Y, r)_n \arrow[r, "\alpha^{\mathrm{FM}}_{r,n}"] & H^j(Y_{\mathrm{tr}}, \mathbf{Z}/p^n(r)') \arrow[d, equal, "\pi^*"'] \\
H^j_{\mathrm{syn}}(X_{\mathscr{O}_{K_1}}, r)_n \arrow[r, "\alpha^{\mathrm{FM}}_{r,n}"] \arrow[u, "\pi^*"] & H^j(X_{\mathrm{tr},K_1}, \mathbf{Z}/p^n(r)')
\end{tikzcd}$$

Since log-blow-ups do not change syntomic cohomology [48, Proposition 2.3] and the top horizontal map is a p^{Nr}-quasi-isomorphism, for a constant N independent of n, so is the bottom one.

The final claim in the corollary is now clear. □

5.2. **Semistable comparison theorems.** We show in this section how the comparison theorem for p-adic nearby cycles (cf. Corollary 5.12) combined with the theory of finite dimensional Banach Spaces allows us to reprove the semistable comparison theorem for schemes and prove a semistable comparison theorem for formal schemes.

5.2.1. *Semistable conjecture for schemes.* For X proper and semistable over $\mathscr{O}_K$ (with no multiplicities in the special fiber), we have [46, 3.2]

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\mathscr{O}_{\overline{K}}}, r)_{\mathbf{Q}} = [[\mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_F \mathbf{B}_{\mathrm{st}}^+]^{\varphi=p^r, N=0} \xrightarrow{\iota_{\mathrm{dR}}} (\mathrm{R}\Gamma_{\mathrm{dR}}(X_K) \otimes_K \mathbf{B}_{\mathrm{dR}}^+)/F^r] \tag{5.13}$$

Here $\mathrm{R}\Gamma_{\mathrm{HK}}(X) \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(X_0/\mathscr{O}_F^0)_{\mathbf{Q}}$ is the Hyodo-Kato cohomology of X defined by Beilinson [7] and $\iota_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)$ is the Hyodo-Kato map (associated to ϖ). The Hyodo-Kato complexes are built from finite rank (φ, N)-modules. We set

$$[\mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_F \mathbf{B}_{\mathrm{st}}^+]^{\varphi=p^r, N=0} := \left[\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_F \mathbf{B}_{\mathrm{st}}^+ & \xrightarrow{1-\varphi/p^r} & \mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_F \mathbf{B}_{\mathrm{st}}^+ \\ \downarrow N & & \downarrow N \\ \mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_F \mathbf{B}_{\mathrm{st}}^+ & \xrightarrow{1-\varphi/p^{r-1}} & \mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_F \mathbf{B}_{\mathrm{st}}^+ \end{array}\right]$$

Recall that $H^j[\mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes_F \mathbf{B}_{\mathrm{st}}^+]^{\varphi=p^r, N=0} = (H^j_{\mathrm{HK}}(X) \otimes_F \mathbf{B}_{\mathrm{st}}^+)^{\varphi=p^r, N=0}$ [46, Corollary 3.25]. By the degeneration of the Hodge-de Rham spectral sequence [22, Corollary 4.2.4] we also have $H^j((\mathrm{R}\Gamma_{\mathrm{dR}}(X_K) \otimes_K \mathbf{B}_{\mathrm{dR}}^+)/F^r) = (H^j_{\mathrm{dR}}(X_K) \otimes_K \mathbf{B}_{\mathrm{dR}}^+)/F^r$. It follows that we have the long exact sequence

$$\begin{aligned} \to (H^{j-1}_{\mathrm{dR}}(X_K) \otimes_K \mathbf{B}_{\mathrm{dR}}^+)/F^r \to H^j_{\mathrm{syn}}(X_{\mathscr{O}_{\overline{K}}}, r)_{\mathbf{Q}} \to (H^j_{\mathrm{HK}}(X) \otimes_F \mathbf{B}_{\mathrm{st}}^+)^{\varphi=p^r, N=0} \\ \to (H^j_{\mathrm{dR}}(X_K) \otimes_K \mathbf{B}_{\mathrm{dR}}^+)/F^r \to \end{aligned} \tag{5.14}$$

Corollary 5.15. (Semistable conjecture) *Let X be a proper, fine and saturated log-scheme, log-smooth over $\mathscr{O}_K^\times$, with Cartier type reduction. There exists a natural $\mathbf{B}_{\mathrm{st}}$-linear Galois equivariant period isomorphism*

$$\alpha : \quad H^i(X_{\mathrm{tr},\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}} \simeq H^i_{\mathrm{HK}}(X) \otimes_F \mathbf{B}_{\mathrm{st}}$$

that preserves the Frobenius and the monodromy operators, and, after extension to $\mathbf{B}_{\mathrm{dR}}$, induces a filtered isomorphism

$$\alpha : \quad H^i(X_{\mathrm{tr},\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}} \simeq H^i_{\mathrm{dR}}(X_K) \otimes_K \mathbf{B}_{\mathrm{dR}}$$

Proof. Take $r \geq i$. The period map is induced by the following composition

$$H^i(X_{tr,\overline{K}}, \mathbf{Q}_p(r)) \overset{\alpha_r^{\mathrm{FM}}}{\leftarrow} H^i_{\mathrm{syn}}(X_{\mathscr{O}_{\overline{K}}}, r)_{\mathbf{Q}} \to (H^i_{\mathrm{HK}}(X) \otimes_F \mathbf{B}_{\mathrm{st}})^{\varphi=p^r, N=0} \to H^i_{\mathrm{HK}}(X) \otimes_F \mathbf{B}_{\mathrm{st}}.$$

The first map is an isomorphism by Corollary 5.12. The period map is clearly compatible with Galois action, Frobenius, monodromy, as well as with filtration (after extending to $\mathbf{B}_{\mathrm{dR}}$). To prove our corollary, it suffices to show that $H^i_{\mathrm{dR}}(X_K)$ is admissible and that the above map induces an isomorphism $H^i(X_{\mathrm{tr},\overline{K}}, \mathbf{Q}_p) \simeq \mathbf{V}_{\mathrm{st}}(H^i_{\mathrm{dR}}(X_K))$. To do that we will use Corollary 5.12, exact sequence (5.14), and the theory of finite dimensional Banach Spaces from [19] to prove Proposition 5.20 below that will imply our corollary (take $D^i := (H^i_{\mathrm{HK}}(X), H^i_{\mathrm{dR}}(X_K), \iota_{\mathrm{dR}} : H^i_{\mathrm{HK}}(X) \to H^i_{\mathrm{dR}}(X_K))$. □

5.2.2. *Finite dimensional Banach Spaces.* Recall [19] that a finite dimensional Banach Space W is, morally, a finite dimensional C-vector space ($C := \overline{K}^{\wedge}$) up to a finite dimensional $\mathbf{Q}_p$-vector space. It has a Dimension[11] $\operatorname{Dim} W = (a, b)$, where $a = \dim W \in \mathbf{N}$, the *dimension of* W, is the dimension of the C-vector space and $b = \operatorname{ht} W \in \mathbf{Z}$, the *height of* W, is the dimension of the $\mathbf{Q}_p$-vector space. More precisely, a *Banach Space* $\mathbb{W}$ is a functor $\Lambda \mapsto \mathbb{W}(\Lambda)$, from the category of sympathetic algebras (spectral Banach algebras, such that $x \mapsto x^p$ is surjective on $\{x,\ |x-1| < 1\}$; such an algebra is, in particular, perfectoid) to the category of $\mathbf{Q}_p$-Banach spaces. Trivial examples of such objects are:

- finite dimensional $\mathbf{Q}_p$-vector spaces V, with associated functor $\Lambda \mapsto V$ for all Λ,
- $\mathbb{V}^d$, for $d \in \mathbf{N}$, with $\mathbb{V}^d(\Lambda) = \Lambda^d$, for all Λ.

A Banach Space $\mathbb{W}$ is *finite dimensional* if it "is equal to $\mathbb{V}^d$, for some $d \in \mathbf{N}$, up to finite dimensional $\mathbf{Q}_p$-vector spaces". More precisely, we ask that there exists finite dimensional $\mathbf{Q}_p$-vector spaces V_1, V_2 and exact sequences[12]

$$0 \to V_1 \to \mathbb{Y} \to \mathbb{V}^d \to 0, \quad 0 \to V_2 \to \mathbb{Y} \to \mathbb{W} \to 0,$$

so that $\mathbb{W}$ is obtained from $\mathbb{V}^d$ by "adding V_1 and moding out by V_2". Then $\dim \mathbb{W} = d$ and $\operatorname{ht} \mathbb{W} = \dim_{\mathbf{Q}_p} V_1 - \dim_{\mathbf{Q}_p} V_2$. (We are, in general, only interested in $W = \mathbb{W}(C)$ but, without the extra structure, it would be impossible to speak of its Dimension.)

Proposition 5.16. (i) *The Dimension of a finite dimensional Banach Space is independant of the choices made in its definition.*

(ii) *If $f : \mathbb{W}_1 \to \mathbb{W}_2$ is a morphism of finite dimensional Banach Spaces, then* $\operatorname{Ker} f$, $\operatorname{Coker} f$ *and* $\operatorname{Im} f$ *are finite dimensional Banach Spaces, and we have*

$$\operatorname{Dim} \mathbb{W}_1 = \operatorname{Dim} \operatorname{Ker} f + \operatorname{Dim} \operatorname{Im} f \quad \text{and} \quad \operatorname{Dim} \mathbb{W}_2 = \operatorname{Dim} \operatorname{Coker} f + \operatorname{Dim} \operatorname{Im} f.$$

(iii) *If* $\dim \mathbb{W} = 0$, *then* $\operatorname{ht} \mathbb{W} \geq 0$.

(iv) *If $\mathbb{W}$ has an increasing filtration such that the successive quotients are $\mathbb{V}^1$, and if $\mathbb{W}'$ is a sub-Banach Space of $\mathbb{W}$, then* $\operatorname{ht} \mathbb{W}' \geq 0$.

Proof. The first two points are the core of the theory [19, Th. 0.4]. The third point is obvious and the fourth is [20, Lemme 2.6]. □

Corollary 5.17. (i) *If $\mathbb{W}_1$ is a successive extension of $\mathbb{V}^1$'s, and if $\mathbb{W}_2$ is of dimension* 0, *then any morphism $\mathbb{W}_1 \to \mathbb{W}_2$ is the* 0*-map.*

(ii) *Let $\mathbb{W}_1, \mathbb{W}_2$ be finite dimensional Banach Spaces, $W_1 = \mathbb{W}_1(C)$ and $W_2 = \mathbb{W}_2(C)$. Suppose that $\mathbb{W}_2$ is a successive extension of $\mathbb{V}^1$'s, and that we are given a $\mathbf{Q}_p$-linear map $f : W_1 \to W_2$ which lifts to a morphism $f : \mathbb{W}_1 \to \mathbb{W}_2$ of finite dimensional Banach Spaces. Then, if we are in one of these situations:*

- $\dim_{\mathbf{Q}_p} \operatorname{Coker}\big(f : W_1 \to W_2\big) < \infty$,
- $\dim_{\mathbf{Q}_p} \operatorname{Ker}\big(f : W_1 \to W_2\big) < \infty$ *and* $\dim \mathbb{W}_1 = \dim \mathbb{W}_2$,

the map $f : W_1 \to W_2$ is surjective.

(iii) *If $f : \mathbb{W}_1 \to \mathbb{W}_2$ is a morphism of finite dimensional Banach Spaces, and if the kernel and cokernel of $f_C : \mathbb{W}_1(C) \to \mathbb{W}_2(C)$ are finite dimensional over $\mathbf{Q}_p$, then* $\dim \mathbb{W}_1 = \dim \mathbb{W}_2$.

Proof. Let $f : \mathbb{W}_1 \to \mathbb{W}_2$ be a morphism. We have $\dim \operatorname{Im} f = 0$ since $\dim \mathbb{W}_2 = 0$. Now, $\operatorname{ht} \mathbb{W}_1 = 0$; hence $\operatorname{ht} \operatorname{Im} f = -\operatorname{ht} \operatorname{Ker} f$, and $\operatorname{ht} \operatorname{Ker} f \geq 0$ by (iv) of Proposition 5.16 and our assumption on $\mathbb{W}_1$. So $\operatorname{ht} \operatorname{Im} f \leq 0$, and since $\dim \operatorname{Im} f = 0$, we obtain $\operatorname{Im} f = 0$, as wanted.

To prove (ii) in the first situation, note that the Banach Space $\operatorname{Coker}(f : \mathbb{W}_1 \to \mathbb{W}_2)$ is of dimension 0 since its C-points are finite dimensional over $\mathbf{Q}_p$ by assumption. Hence we can apply (i) to deduce that

[11] In [19], the dimension is called the "dimension principale", noted $\dim_{\mathrm{pr}}$, and the height is called the "dimension résiduelle", noted $\dim_{\mathrm{res}}$, and the Dimension is called simply the "dimension".

[12] A sequence $0 \to \mathbb{W}_1 \to \mathbb{W}_2 \to \mathbb{W}_3 \to 0$ is exact if and only if $0 \to \mathbb{W}_1(\Lambda) \to \mathbb{W}_2(\Lambda) \to \mathbb{W}_3(\Lambda) \to 0$ is exact for all Λ.

this cokernel is 0, hence also the cokernel of $f : W_1 \to W_2$. In the second situation, the assumption $\dim_{\mathbf{Q}_p} \mathrm{Ker}\big(f : W_1 \to W_2\big) < \infty$ imply that $\dim \mathrm{Ker}\big(f : \mathbb{W}_1 \to \mathbb{W}_2\big) = 0$ and, since $\dim \mathbb{W}_1 = \dim \mathbb{W}_2$, this implies that $\dim \mathrm{Coker}\big(f : \mathbb{W}_1 \to \mathbb{W}_2\big) = 0$, and we conclude as before.

Finally, (iii) is a consequence of (ii) of Proposition 5.16, which implies (forgetting the heights),

$$\dim \mathbb{W}_1 - \dim \mathbb{W}_2 = \dim \mathrm{Ker}\, f - \dim \mathrm{Coker}\, f,$$

and the fact that $\dim \mathrm{Ker}\, f = \dim \mathrm{Coker}\, f = 0$ since their C points are of finite dimension over $\mathbf{Q}_p$ by assumption. □

Example 5.18. Let $D = (D_{\mathrm{st}}, D_{\mathrm{dR}}, \lambda)$, where:

- D_{st} is a finite dimensional F-vector space with a bijective semi-linear Frobenius φ, and a linear monodromy operator N such that $N\varphi = p\varphi N$,
- D_{dR} is a finite dimensional K-vector space with a decreasing, separated, exhausting filtration indexed by $\mathbf{N}$,
- $\lambda : D_{\mathrm{st}} \to D_{\mathrm{dR}}$ is a F-linear map.

If $r \in \mathbf{N}$, define

$$X^r_{\mathrm{st}}(D) = (t^{-r}\mathbf{B}^+_{\mathrm{st}} \otimes_F D_{\mathrm{st}})^{\varphi=1,N=0} \quad \text{and} \quad X^r_{\mathrm{dR}}(D) = (t^{-r}\mathbf{B}^+_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}})/F^0.$$

These are the C-points of finite dimensional Banach Spaces $\mathbb{X}^r_{\mathrm{st}}(D)$ and $\mathbb{X}^r_{\mathrm{dR}}(D)$, and we have (cf. [19, Proposition 0.8]),

$$\dim \mathbb{X}^r_{\mathrm{dR}}(D) = (r \dim_K D_{\mathrm{dR}} - \sum_{i=1}^{r} \dim F^i D_{\mathrm{dR}}, 0)$$

$$\dim \mathbb{X}^r_{\mathrm{st}}(D) = \sum_{r_i \le r} (r - r_i, 1), \quad \text{where the } r_i\text{'s are the slopes of } \varphi\text{, repeated with multiplicity.}$$

In particular, if $F^{r+1}D_{\mathrm{dR}} = 0$ and if all r_i's are $\le r$ (we let $r(D)$ be the smallest r with these properties), then

$$\dim \mathbb{X}^r_{\mathrm{st}}(D) = (r \dim_F D_{\mathrm{st}} - t_N(D_{\mathrm{st}}), \dim_F D_{\mathrm{st}}) \quad \text{and} \quad \dim \mathbb{X}^r_{\mathrm{dR}}(D) = (r \dim_K D_{\mathrm{dR}} - t_H(D_{\mathrm{dR}}), 0).$$

Here $t_N(D) = v_p(\det \varphi)$ and $t_H(D) = \sum_{i \ge 0} i \dim_K (F^i D_{\mathrm{dR}}/F^{i+1}D_{\mathrm{dR}})$.

Now, the map λ extends (using the natural map $\mathbb{B}^+_{\mathrm{st}} \to \mathbb{B}^+_{\mathrm{dR}}$) to a map $\mathbb{X}^r_{\mathrm{st}}(D) \to \mathbb{X}^r_{\mathrm{dR}}(D)$ of finite dimensional Banach Spaces and, if $\lambda : K \otimes_F D_{\mathrm{st}} \to D_{\mathrm{dR}}$ is bijective (in which case we set $\mathrm{rk}\, D = \dim_F D_{\mathrm{st}} = \dim_K D_{\mathrm{dR}}$), then the kernel of the C points of this map is $\mathbf{V}_{\mathrm{st}}(D)$ if $r \ge r(D)$ ([19, Proposition 10.14]).

The following result is a variant of the theorem "weakly admissible implies admissible".

Lemma 5.19. *If $\lambda : K \otimes_F D_{\mathrm{st}} \to D_{\mathrm{dR}}$ is an isomorphism, and if $t_H(D_{\mathrm{dR}}) = t_N(D_{\mathrm{st}})$, the following conditions are equivalent:*

(i) *$\mathbf{V}_{\mathrm{st}}(D)$ is finite dimensional over $\mathbf{Q}_p$,*

(ii) *the map $X^r_{\mathrm{st}}(D) \to X^r_{\mathrm{dR}}(D)$ is surjective for $r = r(D)$.*

(ii′) *the map $X^r_{\mathrm{st}}(D) \to X^r_{\mathrm{dR}}(D)$ is surjective for all $r \ge r(D)$.*

Moreover, they imply:

(iii) *$t_N(D') \ge t_H(D')$ for all $D' \subset D$ stable by N and φ (i.e. D is weakly admissible).*

(iv) *$\dim_{\mathbf{Q}_p} \mathbf{V}_{\mathrm{st}}(D) = \mathrm{rk}\, D$.*

Proof. The hypothesis $t_H(D_{\mathrm{dR}}) = t_N(D_{\mathrm{st}})$ implies that $\dim(\mathbb{X}^r_{\mathrm{dR}}(D)) = \dim(\mathbb{X}^r_{\mathrm{st}}(D))$ for all $r \ge r(D)$. Now, $\mathbb{X}^r_{\mathrm{dR}}(D)$ is a sucessive extension of $\mathbb{V}^1$'s; hence, if $\mathbf{V}_{\mathrm{st}}(D)$ is finite dimensional over $\mathbf{Q}_p$, we are in the second situation considered in Lemma 5.17 (ii), which proves the implications (i)⇒(ii) and (i)⇒(ii′).

The converse implications and (iv) are just Dimension arguments.

Now, if there exists D' with $t_N(D') < t_H(D')$, then $\dim(\mathbb{X}^r_{\mathrm{st}}(D')) > \dim(\mathbb{X}^r_{\mathrm{dR}}(D'))$ if r is big enough, so the dimension of $\mathrm{Ker}\big(\mathbb{X}^r_{\mathrm{st}}(D') \to \mathbb{X}^r_{\mathrm{dR}}(D')\big)$ is ≥ 1 and $\mathbf{V}_{\mathrm{st}}(D')$ is infinite dimensional over $\mathbf{Q}_p$. As this contradicts our hypothesis that $\mathbf{V}_{\mathrm{st}}(D)$ is finite dimensional over $\mathbf{Q}_p$, this proves (iii). □

Proposition 5.20. *Assume that, for $r \in \mathbf{N}$, we have a set of $D^i = (D^i_{\mathrm{st}}, D^i_{\mathrm{dR}}, \lambda^i)$, $i \in \mathbf{N}$, as above but with the additional condition $F^{i+1}D^i_{\mathrm{dR}} = 0$, and long exact sequences*

$$\cdots \to H^i(r) \to X^r_{\mathrm{st}}(D^i) \to X^r_{\mathrm{dR}}(D^i) \to H^{i+1}(r) \to \cdots$$

Assume that $\dim_{\mathbf{Q}_p} H^i(r)$ is finite if $r \geq i$. Then:
(i) *the map $\lambda^i : K \otimes_F D^i_{\mathrm{st}} \to D^i_{\mathrm{dR}}$ is bijective, if $i \geq 0$,*
(ii) *D^i is weakly admissible, if $i \geq 0$,*
(iii) *the map $X^r_{\mathrm{st}}(D^i) \to X^r_{\mathrm{dR}}(D^i)$ is surjective, if $r \geq i$,*
(iv) *$H^i(r) = \mathbf{V}_{\mathrm{st}}(D^i)$, if $r \geq i$.*

Proof. To see (i) note that the fact that $H^i(r)$ and $H^{i+1}(r)$ are finite dimensional over $\mathbf{Q}_p$ for r big enough implies that $\dim(\mathbb{X}^r_{\mathrm{dR}}(D^i)) = \dim(\mathbb{X}^r_{\mathrm{st}}(D^i))$ for all such r by (iii) of Corollary 5.17. This, in turn, implies that $\dim_F D^i_{\mathrm{st}} = \dim_K D^i_{\mathrm{dR}}$ and $t_H(D^i) = t_N(D^i)$.

Now, let $D' := \mathrm{Im}(\lambda^i) \subset D^i_{\mathrm{dR}}$. Then the image of $\mathbb{X}^r_{\mathrm{st}}(D^i)$ in $\mathbb{X}^r_{\mathrm{dR}}(D^i)$ is included in $\mathbb{X}^r_{\mathrm{dR}}(D')$ and so the cokernel surjects onto $\mathbb{X}^r_{\mathrm{dR}}(D^i/D') = 0$. Since its C-points inject into $H^{i+1}(r)$ which is finite dimensional over $\mathbf{Q}_p$, it follows that $\dim(\mathbb{X}^r_{\mathrm{dR}}(D^i/D')) = 0$ if $r \geq i+1$. This implies that $D^i/D' = 0$, and that λ^i is surjective, hence bijective.

Claim (ii) follows from claim (i) and Lemma 5.19. Note that it implies that $r(D^i) \leq i$ since $F^{i+1}D^i_{\mathrm{dR}} = 0$ by assumption.

If $i \leq r-1$, the map $X^r_{\mathrm{dR}}(D^i) \to H^{i+1}(r)$ is the zero map, since the source is the C-points of a successive extension of $\mathbb{V}^1$'s and the target is a finite dimensional $\mathbf{Q}_p$-vector space by assumption. This allows to split the long exact sequence into short exact sequences $0 \to H^i(r) \to X^r_{\mathrm{st}}(D^i) \to X^r_{\mathrm{dR}}(D^i) \to 0$ for $i \leq r-1$ and $0 \to H^r(r) \to X^r_{\mathrm{st}}(D^r) \to X^r_{\mathrm{dR}}(D^r)$. We conclude that $H^i(r) = \mathbf{V}_{\mathrm{st}}(D^i)$, if $i \leq r$, using inequality $r(D^i) \leq i$ and Lemma 5.19. □

5.2.3. *Semistable conjecture for formal schemes.* Let $\mathscr{X}$ be a fine log (p-adic) formal scheme, log-smooth over $\mathscr{O}_K^\times$, and universally saturated. We define the arithmetic and geometric syntomic cohomology of $\mathscr{X}$ by the same formulas as in the algebraic setting.

For $\mathscr{X}$ semistable over $\mathscr{O}_K$, we have a rigid analytic version of the period map of Fontaine-Messing

$$\alpha^{\mathrm{FM}}_{r,n} : \quad H^j_{\mathrm{syn}}(\mathscr{X}, r)_n \to H^j(\mathscr{X}_{K,\mathrm{tr}}, \mathbf{Z}/p^n(r)').$$

The definition is a straightforward analog of the algebraic definition [63, p. 321]. Namely, we cover $\mathscr{X}$ with open affine formal schemes of the right presentation, use the Fontaine-Messing map on each open set almost verbatim to end up with a Čech complex of Galois cochains. Then we use the $K(\pi,1)$-Lemma for p-coefficients to pass to a Čech complex of étale cohomology. That is, in the local version of this map one combines the map $\tilde{\alpha}^{\mathrm{FM}}_{r,n} : \mathrm{Syn}(R,r)_n \to C(G_R, \mathbf{Z}/p^n(r)')$ with the quasi-isomorphism $C(G_R, \mathbf{Z}/p^n(r)') \xrightarrow{\sim} \mathrm{R}\Gamma((\mathrm{Sp}(R[1/p]))_{\mathrm{tr,\acute{e}t}}, \mathbf{Z}/p^n(r)')$. This quasi-isomorphism is obtained from the Riemann's Existence Theorem 5.7 by Abhyankar's Lemma argument as in Lemma 5.8. Finally, we descend.

Corollary 5.21. *Let $\mathscr{X}$ be a proper semistable formal scheme over $\mathscr{O}_K$. Then the period map*

$$\alpha^{\mathrm{FM}}_{r} : \quad H^j_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}}, r)_{\mathbf{Q}} \xrightarrow{\sim} H^j(\mathscr{X}_{\overline{K},\mathrm{tr}}, \mathbf{Q}_p(r)), \quad j \leq r,$$

is an isomorphism.

Proof. The first claim follows from Theorem 5.4 by going to the limit over finite extensions of the base field. It implies the second statement with the naive syntomic cohomology groups $HN^j_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}}, r) :=$

$\varprojlim_n H^j \mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}}, r)_n$ in place of the syntomic cohomology groups. Since the groups $H^*(\mathscr{X}_{\overline{K},\mathrm{tr}}, \mathbf{Z}/p^n(r))$ are finite [57, Theorem 1.1] all the relevant higher derived projective limits vanish and we can replace the naive syntomic groups with the "real" ones. The final claim in the corollary is now clear. □

To proceed, we need to derive a presentation of the geometric syntomic cohomology using Hyodo-Kato and de Rham cohomologies analogous to the one we have used in the algebraic setting. Assume that the log-scheme $\mathscr{X}_1$ is of Cartier type over $\mathrm{Spec}(\mathscr{O}_{K,1})^\times$. We have the Hyodo-Kato cohomology complex $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}) \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}_F^0)_{\mathbf{Q}}$ and the quasi-isomorphisms [7, 1.16.2]

$$(\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}_{\mathrm{st}}^+)^{N=0} \simeq \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}_{\mathrm{cr}}^+, \quad \iota_{\mathrm{cr}} : \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}_{\mathrm{cr}}^+ \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})_{\mathbf{Q}},$$

that are compatible with the action of N and φ. Recall that the complex $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X})$ is built from finite rank (φ, N)-modules over F. We also have the Hyodo-Kato isomorphism [7, 1.16.2]

$$\iota_{\mathrm{dR}} : \quad \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}) \otimes_F K \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K).$$

Since, for $\mathscr{X}$ semistable over $\mathscr{O}_K$, we have $\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}})$, in that case the Hyodo-Kato isomorphism has the following form

$$\iota_{\mathrm{dR}} : \quad \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}) \otimes_F K \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}).$$

Proposition 5.22. *For $\mathscr{X}$ proper and semistable over $\mathscr{O}_K$ with Cartier type reduction, we have*

(i) *the quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}}, r)_{\mathbf{Q}} \simeq [[\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}_{\mathrm{st}}^+]^{\varphi=p^r, N=0} \xrightarrow{\iota_{\mathrm{dR}}} (\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}_{\mathrm{dR}}^+)/F^r] \tag{5.23}$$

(ii) *for $j < r$, the following short exact sequences*

$$0 \to H^j_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}}, r)_{\mathbf{Q}} \to (H^j_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}_{\mathrm{st}}^+)^{\varphi=p^r, N=0} \to (H^j_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}_{\mathrm{dR}}^+)/F^r \to 0 \tag{5.24}$$

Proof. Recall [7] that we have the complexes

$$\mathrm{R}\Gamma^\natural_{\mathrm{dR}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})_n := \mathrm{R}\Gamma(\mathscr{X}_{\mathscr{O}_{\overline{K}},\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{\mathscr{X}_{\mathscr{O}_{\overline{K},n}}/\mathscr{O}_{F,n}}), \quad \mathrm{R}\Gamma^\natural_{\mathrm{dR}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})\widehat{\otimes}\mathbf{Z}_p := \mathrm{holim}_n\, \mathrm{R}\Gamma^\natural_{\mathrm{dR}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})_n,$$

$$\mathrm{R}\Gamma^\natural_{\mathrm{dR}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})\widehat{\otimes}\mathbf{Q}_p := (\mathrm{R}\Gamma^\natural_{\mathrm{dR}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})\widehat{\otimes}\mathbf{Z}_p) \otimes \mathbf{Q}.$$

These are objects in the filtered ∞-derived category. Here $\mathrm{L}\Omega^{\bullet,\wedge}_{\mathscr{X}_{\mathscr{O}_{\overline{K},n}}/\mathscr{O}_{F,n}}$ denotes the derived log de Rham complex (see [6] for a review). The hat refers to the completion with respect to the Hodge filtration (in the sense of prosystems). Set $\mathbf{A}_{\mathrm{dR}} := \mathrm{L}\Omega^{\bullet,\wedge}_{\mathscr{O}_{\overline{K}}/\mathscr{O}_K}$. By [7, Lemma 3.2] we have $\mathbf{A}_{\mathrm{dR}} = \mathrm{R}\Gamma^\natural_{\mathrm{dR}}(\mathscr{O}_{\overline{K}}^\times)$. The corresponding F-filtered algebras $\mathbf{A}_{\mathrm{dR},n}$, $\mathbf{A}_{\mathrm{dR}}\widehat{\otimes}\mathbf{Z}_p$, $\mathbf{A}_{\mathrm{dR}}\widehat{\otimes}\mathbf{Q}_p$ are acyclic in nonzero degrees and the projections $\cdot/F^{m+1} \to \cdot/F^m$ are surjective. Thus (we set $\lim_F := \mathrm{holim}_F$)

$$\lim_F \mathbf{A}_{\mathrm{dR}}\widehat{\otimes}\mathbf{Q}_p = \varprojlim_m H^0(\mathbf{A}_{\mathrm{dR}}\widehat{\otimes}\mathbf{Q}_p/F^m) = \mathbf{B}_{\mathrm{dR}}^+, \quad \mathbf{A}_{\mathrm{dR}}\widehat{\otimes}\mathbf{Q}_p/F^m = \mathbf{B}_{\mathrm{dR}}^+/F^m.$$

Recall that we have the quasi-isomorphism [46, Thm 2.1]

$$\kappa_r^{-1} : \quad \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})_{\mathbf{Q}}/F^r \xrightarrow{\sim} \mathrm{R}\Gamma^\natural_{\mathrm{dR}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})\widehat{\otimes}\mathbf{Q}_p/F^r.$$

This yields the first quasi-isomorphism in the following diagram.

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}}, r)_{\mathbf{Q}} &\xrightarrow{\sim} [\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})_{\mathbf{Q}} \xrightarrow{(1-\varphi_r, \kappa_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})_{\mathbf{Q}} \oplus (\mathrm{R}\Gamma^\natural_{\mathrm{dR}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})\widehat{\otimes}\mathbf{Q}_p)/F^r] \\ &\xrightarrow{\sim} [\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})_{\mathbf{Q}} \xrightarrow{(1-\varphi_r, \gamma_r^{-1}\kappa_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})_{\mathbf{Q}} \oplus (\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}_{\mathrm{dR}}^+)/F^r]\end{aligned}$$

To define the second quasi-isomorphism note that the natural map $\mathrm{R}\Gamma^\natural_{\mathrm{dR}}(\mathscr{X})\widehat{\otimes}\mathbf{Q}_p \to \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K)$ is a filtered quasi-isomorphism: it suffices to show that the natural map

$$\mathrm{gr}^i_F\, \mathrm{R}\Gamma(\mathscr{X}_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{\mathscr{X}/\mathscr{O}_F})\widehat{\otimes}\mathbf{Q}_p \to \mathrm{gr}^i_F\, \mathrm{R}\Gamma(\mathscr{X}_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{\mathscr{X}/\mathscr{O}_K^\times})\widehat{\otimes}\mathbf{Q}_p$$

is a quasi-isomorphism for all $i \geq 0$ and this was done in [46, proof of Corollary 2.4]. It yields, by extension to $\mathbf{A}_{\mathrm{dR}}\widehat{\otimes}\mathbf{Q}_p$, a quasi-isomorphism of F-filtered $\overline{K}$-algebras

$$\gamma : \quad \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K (\mathbf{A}_{\mathrm{dR}}\widehat{\otimes}\mathbf{Q}_p) \xrightarrow{\sim} \mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})\widehat{\otimes}\mathbf{Q}_p :$$

one can use here the arguments of [7, 3.5] almost verbatim. Its mod F^r-version γ_r is the quasi-isomorphism

$$\gamma_r : \quad (\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r \xrightarrow{\sim} \mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})/F^r$$

The quasi-isomorphism (5.23) is now defined by the following composition of morphisms

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}}, r) \xrightarrow{\sim} \left[\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})_{\mathbf{Q}} \xrightarrow{(1-\varphi_r, \gamma_r^{-1}\kappa_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})_{\mathbf{Q}} \oplus (\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r \right]$$

$$\xleftarrow{\sim} \left[\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}^+_{\mathrm{st}} & \xrightarrow{(1-\varphi_r, \iota_{\mathrm{dR}}\otimes\iota)} & \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}^+_{\mathrm{st}} \oplus (\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r \\ \downarrow N & & \downarrow (N,0) \\ \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}^+_{\mathrm{st}} & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}^+_{\mathrm{st}} \end{array} \right]$$

Here second quasi-isomorphism uses the fact that the following diagram commutes (proof of [46, Lemma 3.23] goes through)

$$\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}})_{\mathbf{Q}} \otimes_{\mathbf{B}^+_{\mathrm{cr}}} \mathbf{B}^+_{\mathrm{st}} & \xrightarrow{\gamma_r^{-1}\kappa_r^{-1}\otimes\iota} & (\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r \\ \uparrow \iota_{cr} \wr & \nearrow_{\iota_{\mathrm{dR}}\otimes\iota} & \\ \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}^+_{\mathrm{st}} & & \end{array}$$

This proves the first claim of our proposition.

For the second claim, taking cohomology of the homotopy fiber (5.23) we get the long exact sequence

$$\begin{aligned} \to H^{j-1}((\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r) \to H^j_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}}, r)_{\mathbf{Q}} \to H^j[\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}^+_{\mathrm{st}}]^{\varphi=p^r, N=0} \\ \to H^j((\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r) \to \end{aligned}$$

But, arguing as in the algebraic case [46, Corollary 3.25], we obtain the isomorphism

$$H^j[\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}^+_{\mathrm{st}}]^{\varphi=p^r, N=0} \simeq (H^j_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}^+_{\mathrm{st}})^{\varphi=p^r, N=0}.$$

Hence we have the long exact sequence

$$\begin{aligned} \to H^{j-1}((\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r) \to H^j_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}}, r)_{\mathbf{Q}} \to (H^j_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}^+_{\mathrm{st}})^{\varphi=p^r, N=0} \\ \to H^j((\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r) \to \end{aligned}$$

To show that it splits into short exact sequences for $j \leq r$, we start with the observation that the cohomology groups $H^j((\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r)$ are finite dimensional Banach Spaces which are successive extensions of $\mathbb{V}^1$. This is because they are finite length $\mathbf{B}^+_{\mathrm{dR}}$-modules: we have

$$H^j(\mathrm{gr}^i_F(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}})) \simeq \bigoplus_{k\geq 0} H^{j-k}(\Omega^k_{\mathscr{X}_K}) \otimes_K \mathrm{gr}^{i-k}_F \mathbf{B}^+_{\mathrm{dR}}.$$

We get the required short exact sequences by the same argument that we have used in the proof of Proposition 5.20 (the key point being that, by Corollary 5.21, the groups $H^j_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}}, r)_{\mathbf{Q}}$, $j \leq r$, are finite dimensional vector spaces over $\mathbf{Q}_p$).

It remains to show that, for $j < r$, we have an isomorphism

$$H^j((\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r) \simeq (H^j_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r.$$

Just as in the algebraic case this statement (for all j) is an immediate result of the degeneration of the Hodge-de Rham spectral sequence which follows from the de Rham comparison theorem for proper rigid analytic spaces proved by Scholze [57]. It also can be proved directly via the following arguments. By (5.24), we have a surjection

$$(H^j_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}^+_{\mathrm{st}})^{\varphi=p^r,N=0} \to H^j((\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r), \quad j < r.$$

Since the above map factors through the natural map

$$(H^j_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r \to H^j((\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r), \tag{5.25}$$

that latter is surjective as well. But it is also injective. Indeed, we have the distinguished triangle

$$F^r(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}}) \to \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}} \to (\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r$$

It yields the long exact sequence of cohomology groups

$$\to H^jF^r(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}}) \xrightarrow{f_j} H^j_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}} \to H^j(\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}})/F^r) \to$$

Since $F^r(H^j_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K \mathbf{B}^+_{\mathrm{dR}}) = \mathrm{Im}\, f_j$, the map in (5.25) is injective. We are done. □

Corollary 5.26. (Semistable conjecture) *Let $\mathscr{X}$ be a proper semistable formal scheme over $\mathscr{O}_K$. There exists a natural $\mathbf{B}_{\mathrm{st}}$-linear Galois equivariant period isomorphism*

$$\alpha: \quad H^j(\mathscr{X}_{\overline{K},\mathrm{tr}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}} \simeq H^j_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}_{\mathrm{st}}$$

that preserves the Frobenius and the monodromy operators, and induces a filtered isomorphism

$$\alpha: \quad H^j(\mathscr{X}_{\overline{K},\mathrm{tr}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}} \simeq H^j_{\mathrm{dR}}(\mathscr{X}_{K,\mathrm{tr}}) \otimes_K \mathbf{B}_{\mathrm{dR}}.$$

Proof. Take $r > j$. The period map is induced by the following composition

$$H^j(\mathscr{X}_{\overline{K},\mathrm{tr}}, \mathbf{Q}_p(r)) \overset{\alpha^{\mathrm{FM}}_r}{\leftarrow} H^j_{\mathrm{syn}}(\mathscr{X}_{\mathscr{O}_{\overline{K}}}, r)_{\mathbf{Q}} \to (H^j_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}_{\mathrm{st}})^{\varphi=p^r,N=0} \to H^j_{\mathrm{HK}}(\mathscr{X}) \otimes_F \mathbf{B}_{\mathrm{st}}.$$

The first morphism is an isomorphism by Corollary 5.21. The proof of our corollary proceeds now as the proof of Corollary 5.15 using the short exact sequence (5.24). □

Remark 5.27. It is shown in [49] that the distinguished triangles (5.13) and (5.23) lift canonically to the category of finite dimensional Banach Spaces.

Remark 5.28. It is likely that the de Rham conjecture for smooth and proper rigid analytic spaces proved by Scholze [57] can be derived from the semistable comparison in Corollary 5.26 using the existence of local (in the étale topology) semistable formal models of smooth rigid analytic spaces proved by Hartl [31] and the "gluing on the generic fiber" techniques of [7], [46].

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