Quantitative aspects of linear and affine closed lambda terms

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Abstract

Affine λ-terms are λ-terms in which each bound variable occurs at most once and linear λ-terms are λ-terms in which each bound variable occurs once. and only once. In this paper we count the number of closed affine λ-terms of size n, closed linear λ-terms of size n, affine β-normal forms of size n and linear β-normal forms of size n, for different ways of measuring the size of λ-terms. From these formulas, we show how we can derive programs for generating all the terms of size n for each class. The foundation of all of this is specific data structures, which are contexts in which one counts all the holes at each level of abstractions by λ's.

Keywords: Lambda calculus, combinatorics, functional programming

1 Introduction

The λ-calculus [1] is a well known formal system designed by Alonzo Church [8] for studying the concept of function. It has three kinds of basic operations: variables, application and abstraction (with an operator λ which is a binder of variables). We assume the reader familiar with the λ-calculus and with de Bruijn indices.[1]

In this paper we are interested in terms in which bound variables occur once. A *closed λ-term* is a λ-term in which there is no free variable, i.e., only free variables. An *affine λ-term* is a $\lambda-$term in which bound variables occur at most once. A *linear* λ-term is a λ-term in which bound variables occur once and only once

In this paper we propose a method for counting and generating (including random generation) linear and affine closed λ-terms based on a data structure which we call *SwissCheese* because of its holes. Actually we count those λ-terms up-to α-conversion. Therefore it is adequate to use de Bruijn indices [10], because a term with de Bruijn indices represents an α-equivalence class. An interesting aspect of these terms is the fact that they are simply typed [16, 15]. For instance, generated by the program of Section 5 6, there are 16 linear terms of natural size 8:

$$(\lambda 0\ (\lambda 0\ \lambda 0))\quad (\lambda 0\ \lambda(\lambda 0\ 0))\quad (\lambda 0\ \lambda(0\ \lambda 0))\quad ((\lambda 0\ \lambda 0)\ \lambda 0)\quad (\lambda(\lambda 0\ 0)\ \lambda 0)\quad (\lambda(0\ \lambda 0)\ \lambda 0)\quad \lambda(\lambda 0\ (\lambda 0\ 0))\quad \lambda(\lambda 0\ (0\ \lambda 0))$$

$$\lambda((\lambda 0\ \lambda 0)\ 0)\quad \lambda(\lambda(\lambda 0\ 0)\ 0)\quad \lambda(\lambda(0\ \lambda 0)\ 0)\quad \lambda(0\ (\lambda 0\ \lambda 0))\quad \lambda(0\ \lambda(\lambda 0\ 0))\quad \lambda(0\ \lambda(0\ \lambda 0))\quad \lambda((\lambda 0\ 0)\ \lambda 0)\quad \lambda((0\ \lambda 0)\ \lambda 0)$$

written with explicit variables

$$\lambda x.x\ (\lambda x.x\ \lambda x.x)\quad \lambda x.x\ \lambda y.(\lambda x.x\ y)\quad \lambda x.x\ \lambda y.(y\ \lambda x.x)\quad (\lambda x.x\ \lambda x.x)\ \lambda x.x$$

[1] If the reader is not familiar with the λ-calculus, we advise him to read the introduction of [14], for instance.

$\lambda y.(\lambda x.x\ y)\ \lambda x.x$ $\lambda y.(y\ \lambda x.x)\ \lambda x.x$ $\lambda y.(\lambda x.x\ (\lambda x.x\ y))$ $\lambda y.(\lambda x.x\ (y\ \lambda x.x))$
$\lambda y.((\lambda x.x\ \lambda x.x)\ y)$ $\lambda y.(\lambda z.(\lambda x.x\ z)\ y)$ $\lambda y.(\lambda z.(z\ \lambda x.x)\ y)$ $\lambda y.(y\ (\lambda x.x\ \lambda x.x))$
$\lambda y.(y\ \lambda z.(\lambda x.x\ z))$ $\lambda y.(y\ \lambda z.(z\ \lambda x.x))$ $\lambda y.((\lambda x.x\ y)\ \lambda x.x)$ $\lambda y.((y\ \lambda x.x)\ \lambda x.x)$

and there are 25 affine terms of natural size 7:

$(\lambda 0\ \lambda\lambda 1)$ $(\lambda 0\ \lambda\lambda\lambda 0)$ $(\lambda\lambda 0\ \lambda\lambda 0)$ $(\lambda\lambda 1\ \lambda 0)$ $(\lambda\lambda\lambda 0\ \lambda 0)$ $\lambda(\lambda\lambda 1\ 0)$ $\lambda(\lambda\lambda\lambda 0\ 0)$ $\lambda(0\ \lambda\lambda 1)$
$\lambda(0\ \lambda\lambda\lambda 0)$ $\lambda(\lambda 0\ \lambda 1)$ $\lambda(\lambda 1\ \lambda 0)$ $\lambda\lambda(\lambda 0\ 1)$ $\lambda\lambda(1\ \lambda 0)$ $\lambda(\lambda 0\ \lambda\lambda 0)$ $\lambda(\lambda\lambda 0\ \lambda 0)$ $\lambda\lambda(\lambda\lambda 0\ 0)$
$\lambda\lambda(0\ \lambda\lambda 0)$ $\lambda\lambda\lambda(0\ 1)$ $\lambda\lambda\lambda(1\ 0)$ $\lambda\lambda\lambda\lambda 2$ $\lambda\lambda(\lambda 0\ \lambda 0)$ $\lambda\lambda\lambda(\lambda 0\ 0)$ $\lambda\lambda\lambda(0\ \lambda 0)$ $\lambda\lambda\lambda\lambda\lambda 1$ $\lambda\lambda\lambda\lambda\lambda\lambda 0$

The Haskell programs of this development are on GitHub: `https://github.com/PierreLescanne/CountingGeneratingAfffineLinearClosedLambdaterms`.

Notations

In this paper we use specific notations.

Given a predicate p, the Iverson notation written $[p(x)]$ is the function taking natural values which is 1 if $p(x)$ is true and which is 0 if $p(x)$ is false.

Let $\mathbf{m} \in \mathbb{N}^p$ be the p-tuple $(m_0, ..., m_{p-1})$. In Section 5, we consider infinite tuples. Thus $\mathbf{m} \in \mathbb{N}^\omega$ is the sequence $(m_0, m_1, ...)$.

- p is the *length* of $\mathbf{m}$, which we write also $\mathsf{length}\ \mathbf{m}$
- The p-tuple $(0, ..., 0)$ is written 0^p. 0^ω is the infinite sequence made of 0's.
- The *increment* of a p-tuple at i is:

$$\mathbf{m}^{\uparrow i} = \mathbf{n} \in \mathbb{N}^p \text{ where } n_j = m_j \text{ if } j \neq i \text{ and } n_i = m_i + 1$$

- Putting an element x as *head* of a tuple is written

$$x : \mathbf{m} = x : (m_0, ...) = (x, m_0, ...)$$

  tail removes the head of an tuple:

$$\mathsf{tail}(x : \mathbf{m}) = \mathbf{m}.$$

- $\oplus$ is the componentwise addition on tuples.

2 SwissCheese

The basic concept is this of **m-SwissCheese** or **SwissCheese** if there is no ambiguity on $\mathbf{m}$. That is a λ-term with holes at p levels, which are all counted, using $\mathbf{m}$. The p levels of holes are $\Box_0$,... $\Box_{p-1}$. A hole $\Box_i$ is meant to be a location for a variable at level i, that is under i λ's. According to the way bound variables are inserted when creating abstractions (see below), we consider linear or affine SwissCheeses. The holes have size 0. An $\mathbf{m}$-SwissCheese has m_0 holes at level 0, m_1 holes at level 1, ... m_{p-1} holes at level p. Let $l_{n,\mathbf{m}}$ (resp. $a_{n,\mathbf{m}}$) count the linear (resp. the affine) $\mathbf{m}$-SwissCheese of size n. $l_{n,\mathbf{m}} = l_{n,\mathbf{m}'}$ and $a_{n,\mathbf{m}} = a_{n,\mathbf{m}'}$ if $\mathbf{m}$ is finite, $\mathsf{length}\ \mathbf{m} \geq n$, $m_i = m'_i$ for $i \leq \mathsf{length}\ \mathbf{m}$, and $m'_i = 0$ for $i > \mathsf{length}\ \mathbf{m}$. $l_{n,0^n}$ (resp. $a_{n,0^n}$) counts the closed linear (resp. the closed affine) λ-terms.

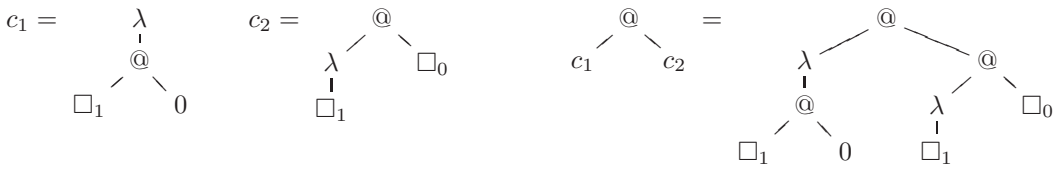

Figure 1: Building a SwissCheese by application

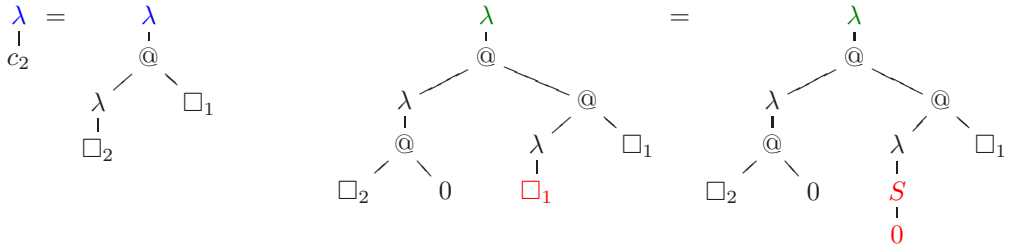

Figure 2: Abstracting SwissCheeses without and with binding

2.1 Growing a SwissCheese

Given two SwissCheeses, we can build a SwissCheese by application like in Fig 1. In Fig. 1, c_1 is a $(0, 1, 0, 0, 0)$-SwissCheese, c_2 is a $(1, 1, 0, 0, 0)$-SwissCheese and $c_1@c_2$ is a $(1, 2, 0, 0, 0)$-SwissCheese.

Given a SwissCheese, there are two ways to grow a SwissCheese to make another SwissCheese by abstraction.

1. We put a λ on the top of a $\mathbf{m}$-SwissCheese c. This increases the levels of the holes: a hole $\square_i$ becomes a hole $\square_{i+1}$. λc is a $(0 : \mathbf{m})$-SwissCheese. See Fig 2 on the left. This way, no index is bound by the top λ, therefore this does not preserve linearity (it preserves affinity however). Therefore this construction is only for building affine SwissCheeses, not for building linear SwissCheeses. In Figure 2 (left), we colour the added λ in blue and we call it *abstraction with no binding*.

2. In the second method for growing a SwissCheese by abstraction, we select first a hole $\square_i$, we top the SwissCheese by a λ, we increment the levels of the other holes and we replace the chosen box by S^i0. In Figure 2 (right), we colour the added λ in green and we call it *abstraction with binding*.

2.2 Measuring SwissCheese

We considers several ways of measuring the size of a SwissCheese derived from what is done on λ-terms. In all these sizes, applications @ and abstractions λ have size 1 and holes have size 0. The differences are in the way variables are measured.

- Variables have size 0, we call this **variable size** 0.
- Variables have size 1, we call this **variable size** 1 .
- Variables (or de Bruijn indices) S^i0 have size $i + 1$, we call this **natural size**.

3 Counting linear closed terms

We start with counting linear terms since they are slightly simpler. We will give recursive formulas first for the numbers $l^\nu_{n,\mathbf{m}}$ of linear SwissCheeses of natural size n with holes set by $\mathbf{m}$, then for the numbers $l^0_{n,\mathbf{m}}$ of linear SwissCheeses of size n, for variable size 0, with holes set by $\mathbf{m}$, eventually for the numbers $l^1_{n,\mathbf{m}}$ of linear SwissCheeses of size n, for variable size 1, with holes set by $\mathbf{m}$. When we do not want to specify a chosen size, we write only $l_{n,\mathbf{m}}$ without superscript.

3.1 Natural size

First let us count linear SwissCheeses with natural size. This is given by the coefficient l^ν which has two arguments: the size n of the SwissCheese and a tuple $\mathbf{m}$ which specifies the number of holes of each level. In other words we are interested by the quantity $l_{n,\mathbf{m}}$. We assume that the length of $\mathbf{m}$ is p, greater than n.

$\mathbf{n = 0}$ whatever size is considered, there is only one SwissCheese of size 0 namely $\Box_0$. This means that the number of SwissCheeses of size 0 is 1 if and only if $\mathbf{m} = (1, 0, 0, ...)$:

$$l_{0,\mathbf{m}} \quad = \quad [m_0 = 1 \wedge \bigwedge_{j=1}^{p} m_j = 0]$$

$\mathbf{n \neq 0}$ and application if a λ-term of size n has holes set by $\mathbf{m}$ and is an application, then it is obtained from a λ term of size k with holes set by $\mathbf{q}$ and a λ term of size $n-k-1$ with holes set by $\mathbf{r}$, with $\mathbf{m} = \mathbf{q} \oplus \mathbf{r}$:

$$\sum_{\mathbf{q}\oplus\mathbf{r}=\mathbf{m}} \sum_{k=0}^{n} l_{k,\mathbf{q}}\, l_{n-1-k,\mathbf{r}}$$

$\mathbf{n \neq 0}$ and abstraction with binding consider a level i, that is a level of hole $\Box_i$. In this hole we put a term $S^{i-1}\,0$ of size i. There are m_i ways to choose a hole $\Box_i$. Therefore there are $m_i\, l^\nu_{n-i-1,\mathbf{m}}$ SwissCheeses which are abstractions with binding in which a $\Box_i$ has been replaced by the de Bruijn index $S^{i-1}\,0$ among $l^\nu_{n,0:\mathbf{m}^{\downarrow\mathbf{i}}}$ SwissCheeses, where $\mathbf{m}^{\downarrow\mathbf{i}}$ is $\mathbf{m}$ in which m_i is decremented. We notice that this refers only to an $\mathbf{m}$ starting with 0. Hence by summing over i and adjusting $\mathbf{m}$, this part contributes as:

$$\sum_{i=0}^{p} (m_i + 1)\; l^\nu_{n-i,\mathbf{m}^{\uparrow i}}$$

to $l^\nu_{n+1,0:\mathbf{m}}$.

We have the following recursive definitions of $l^\nu_{n,\mathbf{m}}$:

$$\begin{aligned} l^\nu_{n+1,0:\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}\,=\,0:\mathbf{m}} \sum_{k=0}^{n} l^\nu_{k,\mathbf{q}}\, l^\nu_{n-k,\mathbf{r}} + \sum_{i=0}^{p} (m_i + 1)\; l^\nu_{n-i,\mathbf{m}^{\uparrow i}} \\ l^\nu_{n+1,(h+1):\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=(h+1):\mathbf{m}} \sum_{k=0}^{n} l^\nu_{k,\mathbf{q}}\, l^\nu_{n-k,\mathbf{r}} \end{aligned}$$

Numbers of closed linear terms with natural size are given in Figure 3.

3.2 Variable size 0

The only difference is that the inserted de Bruijn index has size 0. Therefore we have $m_i\, l^0_{n-1,\mathbf{m}}$ where we had $m_i\, l^\nu_{n-i-1,\mathbf{m}}$ for natural size. Hence the formulas:

$$\begin{aligned} l^0_{n+1,0:\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=0:\mathbf{m}} \sum_{k=0}^{n} l^0_{k,\mathbf{q}}\, l^0_{n-k,\mathbf{r}} + \sum_{i=0}^{n}(m_i+1)\; l^0_{n,\mathbf{m}^{\uparrow i}} \\ l^0_{n+1,(h+1):\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=(h+1):\mathbf{m}} \sum_{k=0}^{n} l^0_{k,\mathbf{q}}\, l^0_{n-k,\mathbf{r}} \end{aligned}$$

The sequence $l^0_{n,0^n}$ of the numbers of closed linear terms is $0, 1, 0, 5, 0, 60, 0, 1105, 0, 27120, 0, 828250$, which is sequence A062980 in the *On-line Encyclopedia of Integer Sequences* with 0's at even indices.

3.3 Variable size 1

The inserted de Bruijn index has size 1. We have $m_i\, l^1_{n-2,\mathbf{m}}$ where we had $m_i\, l^\nu_{n-i-1,\mathbf{m}}$ for natural size.

$$\begin{aligned} l^1_{n+1,0:\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=0:\mathbf{m}} \sum_{k=0}^{n} l^1_{k,\mathbf{q}}\, l^1_{n-k,\mathbf{r}} + \sum_{i=0}^{n-1}(m_i+1)\; l^1_{n-1,\mathbf{m}^{\uparrow i}} \\ l^1_{n+1,(h+1):\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=(h+1):\mathbf{m}} \sum_{k=0}^{n} l^1_{k,\mathbf{q}}\, l^1_{n-k,\mathbf{r}} \end{aligned}$$

As noticed by Grygiel et al. [12] (§ 6.1) There are no linear closed λ-terms of size $3k$ and $3k+1$. However for the values $3k+2$ we get the sequence: $1, 5, 60, 1105, 27120, \ldots$ which is again sequence A062980 of the *On-line Encyclopedia of Integer Sequences*.

4 Counting affine closed terms

We have just to add the case $n \neq 0$ *and abstraction without binding*. Since no index is added, the size increases by 1. The numbers are written $a^\nu_{n,\mathbf{m}}$, $a^0_{n,\mathbf{m}}$, $a^1_{n,\mathbf{m}}$, and $a_{n,\mathbf{m}}$ when the size does not matter. There are $(0:\mathbf{m})$-SwissCheeses of size n that are abstraction without binding. We get the recursive formulas:

4.1 Natural size

$$\begin{aligned} a^\nu_{n+1,0:\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=0:\mathbf{m}} \sum_{k=0}^{n} a^\nu_{k,\mathbf{q}}\, a^\nu_{n-k,\mathbf{r}} + \sum_{i=0}^{n}(m_i+1)\; a^\nu_{n-i,\mathbf{m}^{\uparrow i}} + a^\nu_{n,\mathbf{m}} \\ a^\nu_{n+1,(h+1):\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=(h+1):\mathbf{m}} \sum_{k=0}^{n} a^\nu_{k,\mathbf{q}}\, a^\nu_{n-k,\mathbf{r}} \end{aligned}$$

The numbers of closed affine size with natural size are given in Figure 4.

4.2 Variable size 0

$$\begin{aligned} a^0_{n+1,0:\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=0:\mathbf{m}} \sum_{k=0}^{n} a^0_{k,\mathbf{q}}\, a^0_{n-k,\mathbf{r}} + \sum_{i=0}^{n}(m_i+1)\; a^0_{n,\mathbf{m}^{\uparrow i}} + a^0_{n,\mathbf{m}} \\ a^0_{n+1,(h+1):\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=(h+1):\mathbf{m}} \sum_{k=0}^{n} a^0_{k,\mathbf{q}}\, a^0_{n-k,\mathbf{r}} \end{aligned}$$

The sequence $a^0_{n,0^\omega}$ of the numbers of affine closed terms for variable size 0 is

$$0, 1, 2, 8, 29, 140, 661, 3622, 19993, 120909, 744890, 4887401, 32795272, ...$$

It does not appear in the *On-line Encyclopedia of Integer Sequences*. It corresponds to the coefficients of the generating function $\mathcal{A}(z, 0)$ where

$$\mathcal{A}(z,u) = u + z(\mathcal{A}(z,u))^2 + z\frac{\partial \mathcal{A}(z,u)}{\partial u} + z\mathcal{A}(z,u).$$

4.3 Variable size 1

$$\begin{aligned} a^1_{n+1,0:\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=0:\mathbf{m}} \sum_{k=0}^{n-1} a^1_{k,\mathbf{q}}\, a^1_{n-k,\mathbf{r}} + \sum_{i=0}^{n-1}(m_i+1)\; a^1_{n-1,\mathbf{m}^{\uparrow i}} + a^1_{n,\mathbf{m}} \\ a^1_{n+1,(h+1):\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=(h+1):\mathbf{m}} \sum_{k=0}^{n} a^1_{k,\mathbf{q}}\, a^1_{n-k,\mathbf{r}} \end{aligned}$$

The sequence $a^1_{n,0^\omega}$ of the numbers of affine closed terms for variable size 1 is

$$0, 0, 1, 2, 3, 9, 30, 81, 242, 838, 2799, 9365, 33616, 122937, 449698, 1696724, 6558855, ...$$

It does not appear in the *On-line Encyclopedia of Integer Sequences*. However it corresponds to the coefficient of generating function $\hat{\mathcal{A}}(z, 0)$ where $\hat{\mathcal{A}}(z, u)$ is the solution of the functional equation:

$$\hat{\mathcal{A}}(z,u) = zu + z(\hat{\mathcal{A}}(z,u))^2 + z\frac{\partial \hat{\mathcal{A}}(z,u)}{\partial u} + z\hat{\mathcal{A}}(z,u).$$

Notice that this corrects the wrong assumptions of [12] (Section 6.2).

5 Generating functions

Consider families $F_{\mathbf{m}}(z)$ of generating functions indexed by $\mathbf{m}$, where $\mathbf{m}$ is an infinite tuple of naturals. In fact, we are interested in the infinite tuples $\mathbf{m}$ that are always 0, except a finite number of indices, in order to compute $F_{0^\omega}(z)$, which corresponds to closed λ-terms. Let $\mathbf{u}$ stands for the infinite sequences of variables $(u_0, u_1, ...)$ and $\mathbf{u}^{\mathbf{m}}$ stands for $(u_0^{m_0}, u_1^{m_1}, ..., u_n^{m_n}, ...)$ and $\mathsf{tail}\,(\mathbf{u})$ stand for $(u_1, ...)$. We consider the series of two variables z and $\mathbf{u}$ or double series associated with $F_{\mathbf{m}}(z)$:

$$\mathcal{F}(z,\mathbf{u}) = \sum_{\mathbf{m}\in\mathbb{N}^\omega} F_{\mathbf{m}}(z)\,\mathbf{u}^{\mathbf{m}}.$$

Natural size

$L^\nu_{\mathbf{m}}(z)$ is associated with the numbers of *closed linear SwissCheeses* for natural size:

$$\begin{aligned} L^\nu_{0:\mathbf{m}}(z) &= z \sum_{\mathbf{m}'\oplus\mathbf{m}''=0:\mathbf{m}} L^\nu_{\mathbf{m}'}(z)L^\nu_{\mathbf{m}''}(z) + z\sum_{i=0}^{\infty}(m_i+1)z^i L^\nu_{\mathbf{m}^{\uparrow i}}(z) \\ L^\nu_{(h+1):\mathbf{m}}(z) &= [h = 0 + \bigwedge_{i=0}^{\infty} m_i = 0] + z \sum_{\mathbf{m}'\oplus\mathbf{m}''=(h+1):\mathbf{m}} L^\nu_{\mathbf{m}'}(z)L^\nu_{\mathbf{m}''}(z) \end{aligned}$$

$L^\nu_{0^\omega}$ is the generating function for the closed linear λ-terms. $\mathcal{L}^\nu(z,\mathbf{u})$ is the double series associated with $L^\nu_{\mathbf{m}}(z)$ and is solution of the equation:

$$\mathcal{L}^\nu(z,\mathbf{u}) \quad = \quad u_0 + z(\mathcal{L}^\nu(z,\mathbf{u}))^2 + u_0 \sum_{i=1}^{\infty} z^i \frac{\partial \mathcal{L}^\nu(z,\mathsf{tail}\,(\mathbf{u}))}{\partial u^i}$$

$\mathcal{L}^\nu(z,0^\omega)$ is the generating function of closed linear λ-terms.

For *closed affine SwissCheeses* we get:

$$\begin{aligned} A^\nu_{0:\mathbf{m}}(z) &= z \sum_{\mathbf{m}'\oplus\mathbf{m}''=0:\mathbf{m}} A^\nu_{\mathbf{m}'}(z)A^\nu_{\mathbf{m}''}(z) + z\sum_{i=0}^{\infty}(m_i+1)z^i A^\nu_{\mathbf{m}^{\uparrow i}}(z) + z\,A^\nu_{\mathbf{m}}(z) \\ A^\nu_{(h+1):\mathbf{m}}(z) &= [h=0+\bigwedge_{i=0}^{\infty} m_i=0] + z \sum_{\mathbf{m}'\oplus\mathbf{m}''=(h+1):\mathbf{m}} A^\nu_{\mathbf{m}'}(z)A^\nu_{\mathbf{m}''}(z) \end{aligned}$$

$A^\nu_{0^\omega}$ is the generating function for the affine linear λ-terms. $\mathcal{A}^\nu(z,\mathbf{u})$ is the double series associated with $A^\nu_{\mathbf{m}}(z)$ and is solution of the equation:

$$\mathcal{A}^\nu(z,\mathbf{u}) \quad = \quad u_0 + z(\mathcal{A}^\nu(z,\mathbf{u}))^2 + u_0 \sum_{i=1}^{\infty} z^i \frac{\partial \mathcal{A}^\nu(z,\mathsf{tail}\,(\mathbf{u}))}{\partial u^i} + z\mathcal{A}^\nu(z,\mathsf{tail}\,(\mathbf{u}))$$

$\mathcal{A}^\nu(z,0^\omega)$ is the generating function of closed linear λ-terms.

Variable size 0

$L^0_{\mathbf{m}}$ is associated with the numbers of *closed linear SwissCheeses* for variable size 0:

$$\begin{aligned} L^0_{0:\mathbf{m}}(z) &= z \sum_{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}} L^0_{\mathbf{m}'}(z)L^0_{\mathbf{m}''}(z) + z\sum_{i=0}^{\infty}(m_i+1)L^0_{\mathbf{m}^{\uparrow i}}(z) \\ L^0_{(h+1):\mathbf{m}}(z) &= [h=0+\bigwedge_{i=0}^{\infty} m_i=0] + \sum_{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}} zL^0_{\mathbf{m}'}(z)L^0_{\mathbf{m}''}(z) \end{aligned}$$

$L^0_{0^\omega}$ is the generating function for the closed linear λ-terms. $\mathcal{L}^0(z,\mathbf{u})$ is the double series associated with $L^0_{\mathbf{m}}(z)$ and is solution of the equation:

$$\mathcal{L}^0(z,\mathbf{u}) \quad = \quad u_0 + z(\mathcal{L}^0(z,\mathbf{u}))^2 + u_0 \sum_{i=1}^{\infty} \frac{\partial \mathcal{L}^0(z,\mathsf{tail}\,(\mathbf{u})}{\partial u^i}$$

$\mathcal{L}^0(z,0^\omega)$ is the generating function of closed linear λ-terms.

For *closed affine SwissCheeses* we get:

$$\begin{aligned} A^0_{0:\mathbf{m}}(z) &= z \sum_{\mathbf{m}'\oplus\mathbf{m}''=0:\mathbf{m}} A^0_{\mathbf{m}'}(z)A^0_{\mathbf{m}''}(z) + z\sum_{i=0}^{\infty}(m_i+1)A^0_{\mathbf{m}^{\uparrow i}}(z) + z\,A^0_{\mathbf{m}}(z) \\ A^0_{(h+1):\mathbf{m}}(z) &= [h=0+\bigwedge_{i=0}^{\infty} m_i=0] + \sum_{\mathbf{m}'\oplus\mathbf{m}''=(h+1):\mathbf{m}} zA^0_{\mathbf{m}'}(z)A^0_{\mathbf{m}''}(z) \end{aligned}$$

$A^0_{0^\omega}$ is the generating function for the affine linear λ-terms. $\mathcal{A}^0(z,\mathbf{u})$ is the double series associated with $A^0_{\mathbf{m}}(z)$ and is solution of the equation:

$$\mathcal{A}^0(z,\mathbf{u}) \quad = \quad u_0 + z(\mathcal{A}^0(z,\mathbf{u}))^2 + u_0 \sum_{i=1}^{\infty} \frac{\partial \mathcal{A}^0(z,\mathsf{tail}\,(\mathbf{u}))}{\partial u^i} + z\mathcal{A}^0(z,\mathsf{tail}\,(\mathbf{u}))$$

$\mathcal{A}^0(z,0^\omega)$ is the generating function of closed linear λ-terms. We do not present variable size 1, since it goes exactly the same way.

Variable size 1

The generating functions for $l^1_{n,\mathbf{m}}$ are:

$$\begin{aligned} L^1_{0:\mathbf{m}}(z) &= z \sum_{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}} L^1_{\mathbf{m}'}(z)L^1_{\mathbf{m}''}(z) + z^2 \sum_{i=0}^{\infty}(m_i+1)L^1_{\mathbf{m}^{\uparrow i}}(z) \\ L^1_{(h+1):\mathbf{m}}(z) &= [h = 0 + \bigwedge_{i=0}^{\infty} m_i = 0] + \sum_{\mathbf{m}'\oplus\mathbf{m}''=\mathbf{m}} zL^1_{\mathbf{m}'}(z)L^1_{\mathbf{m}''}(z) \end{aligned}$$

Then we get as associated double series :

$$\mathcal{L}^1(z,\mathbf{u}) = u_0 + z(\mathcal{L}^1(z,\mathbf{u}))^2 + z^2 \sum_{i=1}^{\infty} \frac{\partial \mathcal{L}^1(z,\mathsf{tail}\,(\mathbf{u}))}{\partial u^i}$$

6 Effective computations

The definition of the coefficients $a^{\nu}_{\mathbf{m}}$ and others is highly recursive and requires a mechanism of memoization. In Haskell, this can be done by using the call by need which is at the core of this language. Assume we want to compute the values of $a^{\nu}_{\mathbf{m}}$ until a value `bound` for n. We use a recursive data structure:

```
data Mem = Mem [Mem] | Load [Integer]
```

in which we store the computed values of a function

```
a :: Int -> [Int] -> Integer
```

In our implementation the depth of the recursion of `Mem` is limited by `bound`, which is also the longest tuple $\mathbf{m}$ for which we will compute $a^{\nu}_{\mathbf{m}}$. Associated with `Mem` there is a function

```
access :: Mem -> Int -> [Int] -> Integer access (Load l) n [] = l !! n
access (Mem listM) n (k:m) = access (listM !! k) n m
```

The leaves of the tree memory, corresponding to `Load`, contains the values of the function:

```
memory :: Int -> [Int] -> Mem
memory 0 m = Load [a n (reverse m) | n<-[0..]]
memory k m = Mem [memory (k-1) (j:m) | j<-[0..]]
```

The memory relative to the problem we are interested in is

```
theMemory = memory (bound) []
```

and the access to `theMemory` is given by a specific function:

```
acc :: Int -> [Int] -> Integer acc n m = access theMemory n m
```

Notice that `a` and `acc` have the same signature. This is not a coincidence, since `acc` accesses values of `a` already computed. Now we are ready to express `a`:

```
a 0 m = iv (head m == 1 && all ((==) 0) (tail m))
a n m = aAPP n m + aABSwB n m + aABSnB n m
```

`aAPP` *counts affine terms that are applications:*

```
aAPP n m = sum (map (\((q,r),(k,nk))->(acc k q)*(acc nk r)) (allCombinations m (n-1)))
```

where `allCombinations` returns a list of all the pairs of pairs $(\mathbf{m}', \mathbf{m}'')$ such $\mathbf{m} = \mathbf{m}' \oplus \mathbf{m}''$ and of pairs (k, nk) such that $k + nk = n$. `aABSwB` *counts affine terms that are abstractions with binding.*

```
aABSwB n m
   | head m == 0 = sum [aABSAtD n m i |i<-[1..(n-1)]]
   | otherwise = 0
```

`aABSAtD` *counts affine terms that are abstractions with binding at level* i:

```
aABSAtD n m i = (fromIntegral (1 + m!!i))*(acc (n-i-1) (tail (inc i m) ++ [0]))
```

`aABSnB` *counts affine terms that are abstractions with no binding*:

```
aABSnB n m
     | head m == 0 = (acc (n-1) (tail m ++ [0]))
     | otherwise = 0
```

Anyway the efficiency of this program is limited by the size of the memory, since for computing $a^{\nu}_{n,0^n}$, for instance, we need to compute $a^{\nu}_{\mathbf{r}}$ for about $n!$ values.

7 Generating affine and linear terms

By relatively small changes it is possible to build programs which generate linear and affine terms. For instance for generating affine terms we get.

```
amg :: Int -> [Int] -> [SwissCheese]
amg 0 m =  if (head m == 1 && all ((==) 0) (tail m)) then [Box 0] else []
amg n m = allAPP n m ++ allABSwB n m ++ allABSnB n m

allAPP  :: Int -> [Int] -> [SwissCheese]
allAPP n m = foldr (++) [] (map (\((q,r),(k,nk))-> appSC (cartesian (accAG k q)
                                                                   (accAG nk r))
                            (allCombinations m (n-1)))

allABSAtD :: Int -> [Int] -> Int -> [SwissCheese]
allABSAtD n m i = foldr (++) [] (map (abstract (i-1)) (accAG (n - i - 1)
                                                      (tail (inc i m) ++ [0])))

allABSwB  :: Int -> [Int] -> [SwissCheese]
allABSwB n m
  | head m == 0 = foldr (++) [] [allABSAtD n m i |i<-[1..(n-1)]]
  | otherwise = []

allABSnB  :: Int -> [Int] -> [SwissCheese]
allABSnB n m
  | head m == 0 = map (AbsSC . raise) (accAG (n-1) (tail m ++ [0]))
  | otherwise = []

memoryAG :: Int -> [Int] -> MemSC
memoryAG 0 m = LoadSC [amg n (reverse m) | n<-[0..]]
memoryAG k m = MemSC [memoryAG (k-1) (j:m) | j<-[0..]]

theMemoryAG = memoryAG (upBound) []

accAG :: Int -> [Int] -> [SwissCheese]
accAG n m = accessSC theMemoryAG n m
```

There is similar programs for generating all the terms of size n for variable size 0 and variable size 1. From this, we get programs for generating random affine terms or random linear terms.

8 Normal forms

From the method used for counting affine and linear closed terms, it is easy to deduce method for counting affine and linear closed normal forms. Like before, we use SwissCheeses. In this section we consider only natural size.

8.1 Natural size

Affine closed normal forms

Let us call $anf^{\nu}_{n,\mathbf{m}}$ the numbers of affine SwissCheeses with no β-redex and $ane^{\nu}_{n,\mathbf{m}}$ the numbers of neutral affine SwissCheeses, i.e., affine SwissCheeses with no β-redexes that are sequences of applications starting with a de Bruijn index. In addition we count:

- $anf^\nu\lambda w_{n,m}$ the number of affine SwissCheeses with no β-redex which are abstraction with a binding of a de Bruijn index,
- $anf^\nu\lambda n_{n,m}$ the number of affine SwissCheeses with no β-redex which are abstraction with no binding.

$$\begin{aligned} anf^\nu_{0,\mathbf{m}} &= ane^\nu_{0,\mathbf{m}} \\ anf^\nu_{n+1,\mathbf{m}} &= ane^\nu_{n+1,\mathbf{m}} + anf^\nu\lambda w_{n+1,m} + anf^\nu\lambda n_{n+1,m} \end{aligned}$$

where

$$\begin{aligned} ane^\nu_{0,\mathbf{m}} &= m_0 = 1 \wedge \bigwedge_{j=1}^{p} m_j = 0 \\ ane^\nu_{n+1,\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=0:\mathbf{m}} \sum_{k=0}^{n} ane^\nu_{k,\mathbf{q}}\, anf^\nu_{n-k,\mathbf{r}} \end{aligned}$$

and

$$anf^\nu\lambda w_{n+1,m} = \sum_{i=0}^{n}(m_i+1)\; anf^\nu_{n-i,\mathbf{m}^{\uparrow i}}$$

and

$$anf^\nu\lambda n_{n+1,m} = anf^\nu_{n,m}$$

There are two generating functions, $\mathcal{A}^{nf}$ and $\mathcal{A}^{ne}$, which are associated to $anf^\nu_{n,\mathbf{m}}$ and $anf^\nu_{n,\mathbf{m}}$:

$$\begin{aligned} \mathcal{A}^{nf}(z,\mathbf{u}) &= \mathcal{A}^{ne}(z,\mathbf{u}) + z\sum_{i=1}^{\infty} z^i \frac{\partial\mathcal{A}^{nf}(z,\mathsf{tail}\,(\mathbf{u}))}{\partial u^i} + z\mathcal{A}^{nf}(z,\mathsf{tail}\,(\mathbf{u})) \\ \mathcal{A}^{ne}(z,\mathbf{u}) &= u_0 + z\mathcal{A}^{ne}(z,\mathbf{u})\mathcal{A}^{nf}(z,\mathbf{u}) \end{aligned}$$

Linear closed normal forms

Let us call $lnf^\nu_{n,\mathbf{m}}$ the numbers of linear SwissCheeses with no β-redex and $lne^\nu_{n,\mathbf{m}}$ the numbers of neutral linear SwissCheeses, linear SwissCheeses with no β-redexes that are sequences of applications starting with a de Bruijn index. In addition we count $lnf^\nu\lambda w_{n,m}$ the number of linear SwissCheeses with no β-redex which are abstraction with a binding of a de Bruijn index.

$$\begin{aligned} lnf^\nu_{0,\mathbf{m}} &= lne^\nu_{0,\mathbf{m}} \\ lnf^\nu_{n+1,\mathbf{m}} &= lne^\nu_{n+1,\mathbf{m}} + lnf^\nu\lambda w_{n+1,m} \end{aligned}$$

where

$$\begin{aligned} lne^\nu_{0,\mathbf{m}} &= m_0 = 1 \wedge \bigwedge_{j=1}^{p} m_j = 0 \\ lne^\nu_{n+1,\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=0:\mathbf{m}} \sum_{k=0}^{n} lne^\nu_{k,\mathbf{q}}\, lnf^\nu_{n-k,\mathbf{r}} \end{aligned}$$

and

$$lnf^\nu\lambda w_{n+1,m} = \sum_{i=0}^{n}(m_i+1)\; lnf^\nu_{n-i,\mathbf{m}^{\uparrow i}}$$

with the two generating functions:

$$\begin{aligned}\mathcal{L}^{nf,\nu}(z,\mathbf{u}) &= \mathcal{L}^{ne,\nu}(z,\mathbf{u}) + u_0\sum_{i=1}^{\infty} z^i \frac{\partial \mathcal{L}^{nf,\nu}(z,\mathsf{tail}\,(\mathbf{u}))}{\partial u^i}\\ \mathcal{L}^{ne,\nu}(z,\mathbf{u}) &= u_0 + z\mathcal{L}^{ne,\nu}(z,\mathbf{u})\mathcal{L}^{nf,\nu}(z,\mathbf{u})\end{aligned}$$

We also deduce programs for generating all the closed affine or linear normal forms of a given size from which we deduce programs for generating random closed affine or linear normal forms of a given size. For instance, here are three randoms linear closed normal forms (using de Bruijn indices) of natural size 28:

$$\lambda\lambda\lambda\lambda(2\ \lambda((1\ 2)\ \lambda(0\ (5\ 1))))\quad \lambda(0\ \lambda\lambda(1\ \lambda\lambda((0\ (2\ \lambda\lambda((1\ \lambda 0)\ 0)))\ 1)))\quad \lambda((0\ \lambda 0)\ \lambda\lambda((0((1\ \lambda 0)\lambda\lambda(1\ (0\ \lambda 0))))\lambda 0))$$

8.2 Variable size 0

Linear closed normal forms

A little like previously, let us call $lnf^0_{n,\mathbf{m}}$ the numbers of linear SwissCheeses with no β-redex and $lne^0_{n,\mathbf{m}}$ the numbers of neutral linear SwissCheeses, linear SwissCheeses with no β-redexes that are sequences of applications starting with a de Bruijn index. In addition we count $lnf^0\lambda w_{n,m}$ the number of linear SwissCheeses with no β-redex which are abstraction with a binding of a de Bruijn index. We assume that the reader knows now how to proceed.

$$\begin{aligned}lnf^0_{0,\mathbf{m}} &= lne^0_{0,\mathbf{m}}\\ lnf^0_{n+1,\mathbf{m}} &= lne^0_{n+1,\mathbf{m}} + lnf^0\lambda w_{n+1,m}\end{aligned}$$

where

$$\begin{aligned}lne^0_{0,\mathbf{m}} &= m_0 = 1 \wedge \bigwedge_{j=1}^{p} m_j = 0\\ lne^0_{n+1,\mathbf{m}} &= \sum_{\mathbf{q}\oplus\mathbf{r}=0:\mathbf{m}}\ \sum_{k=0}^{n} lne^0_{k,\mathbf{q}}\, lnf^0_{n-k,\mathbf{r}}\end{aligned}$$

$$lnf^0\lambda w_{n+1,m} = \sum_{i=0}^{n}(m_i+1)\ lnf^0_{n,\mathbf{m}^{\uparrow i}}$$

and the two generating functions:

$$\begin{aligned}\mathcal{L}^{nf,0}(z,\mathbf{u}) &= \mathcal{L}^{ne,0}(z,\mathbf{u}) + u_0\sum_{i=1}^{\infty} \frac{\partial \mathcal{L}^{nf,0}(z,\mathsf{tail}\,(\mathbf{u}))}{\partial u^i}\\ \mathcal{L}^{ne,0}(z,\mathbf{u}) &= u_0 + z\mathcal{L}^{ne,0}(z,\mathbf{u})\mathcal{L}^{nf,0}(z,\mathbf{u})\end{aligned}$$

With no surprise we get for $lnf^0_{n,0^n}$ the sequence:

$$0,1,0,3,0,26,0,367,0,7142,0,176766,0,5304356,...$$

mentioned by Zeilberger in [18] and listing the coefficients of the generating function $\mathcal{L}^{nf,0}(z,0^\omega)$.

We let the reader deduce how to count closed affine normal forms for variable size 0 and closed linear and affine normal forms for variable size 1 alike. Notice that the Haskell programs are on the GitHub site.

9 Related works and Acknowledgement

There are several works on counting λ-terms, for instance on natural size [3, 2], on variable size 1 [5, 9, 17], on variable size 0 [13], on affine terms with variable size 1 [7, 6], on linear λ-terms [20, 18, 19], also on a size based binary representation of the λ-calculus [14] (see [11] for a synthetic view of both natural size and binary size).

We would like to thank Olivier Bodini, Maciej Bendkowski, Katarzyna Grygiel and Noam Zeilberger for stimulating discussions.

10 Conclusion

This work on counting closed terms opens new perspective on the generation, for instance, the random generation of closed lambda terms in the line of [14, 4].

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Data

In the appendix, we give the first values of $l^{\nu}_{n,0^n}$, $a^{\nu}_{n,0^n}$, and $anf^{\nu}_{n,0^n}$.

0	0
1	0
2	1
3	0
4	0
5	3
6	2
7	0
8	16
9	24
10	8
11	117
12	252
13	180
14	1024
15	2680
16	2952
17	10350
18	29420
19	42776
20	116768
21	335520
22	587424
23	1420053
24	3976424
25	7880376
26	18103936
27	48816576
28	104890704
29	237500826
30	617733708
31	1396750576
32	3171222464
33	8014199360
34	18688490336
35	42840683418
36	106063081288
37	251769197688
38	583690110208
39	1425834260080
40	3417671496432
41	8007221710652
42	19404994897976
43	46747189542384
44	110498345360800
45	266679286291872
46	644021392071840
47	1533054190557133
48	3693823999533360
49	8931109667692464
50	21375091547312128
51	51496022711337536
52	124591137939086496
53	299402908258405410
54	721839933329222924
55	1747307145272084192
56	4211741383777966592
57	10165998012602469888
58	24620618729658655936
59	59482734150603634286
60	143764591607556354344
61	348379929166234350008
62	843169238563254723200
63	2040572920613086128400
64	4948102905207104837424
65	11992521016286173712196
66	29059897435554891991144
67	70516464312280927105392
68	171105110698292441423968
69	415095704639682396539232
70	1008016383720573882885792
71	2448305474519849567597826
72	5945721872300885649415632
73	14449388516068567845838736
74	35125352062243788817753856
75	85382289240293493116120064
76	207650379931166057815603296
77	505172267243918348155299780
78	1229005880128485245247395000
79	2991079243470267667831893408
80	7281852742753184123608419712
81	17729171587798767750815341440
82	431774546203254451229443059 84
83	1051854527871170352663154468 68
84	2562738624654251582119480200 48
85	6245274132922529045841219802 08
86	1522355057007327280427270436480
87	3711429775030704772089070886624
88	9050041253711022076275958636128
89	2207315030175885711007204291980 0
90	5384491090939892899064110135166 4
91	1313711355441739145370767749325 76
92	3205886772380856428209209105559 68
93	7824652188858698131838632132314 24
94	1910077425906069707804966102543936
95	4663586586924802791117231052636349
96	1138825956594245283771768874395 3504
97	2781375436189798454346747891722 3008
98	6794178128411320199864569950174 6176
99	1659894857240489642720236007732 71424
100	4055888093051684539631373774423 21728

Figure 3: *Natural size:* numbers of closed linear terms of size n from 0 to 100

0	0
1	0
2	1
3	1
4	2
5	5
6	12
7	25
8	64
9	166
10	405
11	1050
12	2763
13	7239
14	19190
15	51457
16	138538
17	374972
18	1020943
19	2792183
20	7666358
21	21126905
22	58422650
23	162052566
24	450742451
25	1256974690
26	3513731861
27	9843728012
28	27633400879
29	77721141911
30	218984204904
31	618021576627
32	1746906189740
33	4945026080426
34	14017220713131
35	39784695610433
36	113057573020242
37	321649935953313
38	916096006168770
39	2611847503880831
40	7453859187221508
41	21292177500898858
42	60875851617670699
43	174195916730975850
44	498863759031591507
45	1429753835635525063
46	4100730353324163138
47	11769771167532816128
48	33804054749367200891
49	97151933333668422006
50	279385977720772581435
51	803928779462727941247
52	2314623127904669382002
53	6667810436356967142481
54	19218411059885449257096
55	55421020161661024650870
56	159899218321197381984561
57	461557020400062903560120
58	1332920908954281811200519
59	385102706833658369341291O
60	111310324445031365717895 27
61	32186581221116996967632029
62	93108410048006285466998584
63	26944619170241142079040 2033
64	78004372618640316739245 3886
65	22590431899955153159303496 50
66	65446129553902523361872668 73
67	18966737218108971681014445025
68	54985236298270057405776629352
69	1594557373503846378477830553 11
70	46256284862443572496418132348 4
71	13422518844516647330642832516 27
72	38960656221272006256531341005 38
73	11312117748805772104795220337816
74	32853646116456632492645965741531
75	95442534633482460553801961967438
76	27734219154733083964028997881 3667
77	80612518945729190286384826746 3755
78	23436821309112322792857072906041 56
79	68155640237365342080793678163403 59
80	19824812322145727566417303371819466
81	57679033022808238913186144092831856
82	16785178708256139238464824884639 0041
83	48857436867083209324380279046479 6207
84	14224263423808832544597834108453650 06
85	41421045640890442039011908172758646 65
86	12064305885705003967881526911560653106
87	35145647815239737143373764367447378676
88	10240630305212309706205356481810946 8705
89	29844602959866120521617089785033655 0644
90	86993545270502330218903164493280399 0417
91	25362294927043545133096962285927841811 58
92	73955181434251600735379676062987559473 91
93	21568776408467701927134211542478146593789
94	62915493935623036562559989770249004382816
95	18355377588886211325916815013026636241 6356
96	53560066162155696915545354469282662553 2079
97	15631097206725269198996893666262408675151 44
98	45625428188011384523100241312233041869092 33
99	13319630286623965617386598746472280781972745
100	38890520391341859449843201188612375394153776

Figure 4: *Natural size:* numbers of closed affine terms of size n from 0 to 100

0	0
1	0
2	1
3	1
4	2
5	3
6	7
7	10
8	20
9	40
10	77
11	160
12	318
13	671
14	1405
15	2981
16	6312
17	13672
18	29399
19	63697
20	139104
21	304153
22	667219
23	1469241
24	3247176
25	7184288
26	15949179
27	35480426
28	79083472
29	176607519
30	395119875
31	885450388
32	1987289740
33	4466760570
34	10053371987
35	22656801617
36	51121124910
37	115478296639
38	261139629999
39	591138386440
40	1339447594768
41	3037843646560
42	6895841598615
43	15666498585568
44	35620848278448
45	81052838239593
46	184564847153821
47	420564871255118
48	958975854646984
49	2188068392529104
50	4995528560788451
51	11411921511827547
52	26084524952754538
53	59654682828889245
54	136500653558490261
55	312496493161999851
56	715760763686417314
57	1640194881084692664
58	3760284787917366081
59	8624561382605096780
60	19789639944299656346
61	45427337308377290201
62	104320438668034814453
63	239656248361374562433
64	550769764273325683828
65	1266217774600330829940
66	2912050679107531357883
67	6699418399886008666265
68	15417663698156810292010
69	35492710197462925262295
70	81732521943462960197057
71	188270363628099910161436
72	433807135012774797924026
73	999851681931974600766994
74	2305129188866501774481545
75	5315847675735178072941600
76	12262083079763320881047944
77	28292248892584567512609357
78	65294907440089718078048829
79	150729070403767032817820543
80	348031015577337732605480908

Figure 5: *Natural size:* numbers of closed affine normal forms of size n from 0 to 80