Divergence-free positive symmetric tensors and fluid dynamics

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Abstract

We consider $d \times d$ tensors $A(x)$ that are symmetric, positive semi-definite, and whose row-divergence vanishes identically. We establish sharp inequalities for the integral of $(\det A)^{\frac{1}{d-1}}$. We apply them to models of compressible inviscid fluids: Euler equations, Euler–Fourier, relativistic Euler, Boltzman, BGK, etc... We deduce an *a priori* estimate for a new quantity, namely the space-time integral of $\rho^{\frac{1}{n}} p$, where ρ is the mass density, p the pressure and n the space dimension. For kinetic models, the corresponding quantity generalizes Bony's functional.

Notations. The integer $d \geq 2$ is the number of independent variables, which are often space-time coordinates. It serves also for the size of square matrices. If $1 \leq j \leq d$ and $x \in \mathbb{R}^d$ are given, we set $\widehat{x_j} = (\dots, x_{j-1}, x_{j+1}, \dots) \in \mathbb{R}^{d-1}$; the projection $x \mapsto \widehat{x_j}$ ignores the j-th coordinate. The transpose of a matrix M is M^T. If $A \in \mathbf{M}_d(\mathbb{R})$, its cofactor matrix $\widehat{A}$ satisfies

$$\widehat{A}^T A = A \widehat{A}^T = (\det A) I_d, \qquad \det \widehat{A} = (\det A)^{d-1}.$$

Because we shall deal only with symmetric matrices, we have simply $\widehat{A} A = A \widehat{A} = (\det A) I_d$. The space of $d \times d$ symmetric matrices with real entries is $\mathbf{Sym}_d$. The cones of positive definite, respectively positive semi-definite, matrices are $\mathbf{SPD}_d$ and $\mathbf{Sym}_d^+$. If $u \in \mathbb{R}^d$, $u \otimes u \in \mathbf{Sym}_d^+$ denotes the rank-one matrix of entries $u_i u_j$.

The unit sphere of $\mathbb{R}^d$ is S^{d-1}. The Euclidian volume of an open subset Ω of $\mathbb{R}^d$ is denoted $|\Omega|$. If the boundary $\partial\Omega$ is rectifiable, we denote the same way $|\partial\Omega|$ its area, and $ds(x)$ the area element. For instance, the ball B_r of radius r and its boundary, the sphere S_r, satisfy $|B_r| = \frac{r}{d} |S_r|$. If Ω has a Lipschitz boundary, its outer unit normal $\vec{n}$ is defined almost everywhere.

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If $f:\Omega\to\mathbb{R}$ is integrable, its average over Ω is the number

$$\fint_\Omega f(x)\,dx := \frac{1}{|\Omega|}\int_\Omega f(x)\,dx.$$

Given a lattice Γ of $\mathbb{R}^d$, and $f:\mathbb{R}^d\to\mathbb{R}$ a Γ-periodic, locally integrable function, we denote

$$\int_{\mathbb{R}^d/\Gamma} f(x)\,dx$$

the value of the integral of f over any fundamental domain. We define as above the average value

$$\fint_{\mathbb{R}^d/\Gamma} f(x)\,dx.$$

For our purpose, a *tensor* is a matrix-valued function $x\mapsto T(x)\in\mathbf{M}_{p\times q}(\mathbb{R})$. If $q=d$ and if the derivatives make sense (say as distributions), we form

$$\mathrm{Div}T=\left(\sum_{j=1}^d\partial_j t_{ij}\right)_{1\le i\le p},$$

which is vector-valued. We emphasize the uppercase letter D in this context. We reserve the lower case operator `div` for vector fields.

If $1\le p\le\infty$, its conjugate exponent is p'.

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1 Motivations

We first define the mathematical object under consideration.

Definition 1.1 *Let Ω be an open subset of $\mathbb{R}^d$. A* divergence-free positive symmetric tensor *(in short, a* DPT*) is a locally integrable tensor $x\mapsto A(x)$ over Ω with the properties that $A(x)\in\mathbf{Sym}_d^+$ almost everywhere, and* $\mathrm{Div}A=0$.

The following fact is obvious.

Lemma 1.1 (Congruence.) *If A is a DPT and $P\in\mathbf{GL}_d(\mathbb{R})$ is given, then the tensor*

$$B(y):=PA(P^{-1}y)P^T$$

is also a DPT.

1.1 Where do the divergence-free positive symmetric tensors occur ?

Most of our examples, though not all of them, come from fluid dynamics, where a DPT contains a stress tensor.

Compressible gas. In space dimension $n \geq 1$, a gas is described by a mass density $\rho \geq 0$, a velocity u and a pressure $p \geq 0$. These fields obey the Euler equations (conservation of mass and momentum)

$$\partial_t \rho + \mathrm{div}_y(\rho u) = 0, \qquad \partial_t(\rho u) + \mathrm{Div}_y(\rho u \otimes u) + \nabla_y p = 0.$$

Here $x = (t, y)$ and $d = 1 + n$. The tensor

$$A(t,y) = \begin{pmatrix} \rho & \rho u^T \\ \rho u & \rho u \otimes u + pI_n \end{pmatrix}$$

is a DPT.

Rarefied gas. It is described by a density function $f(t,y,v) \geq 0$ where $v \in \mathbb{R}^n$ is the particle velocity. The evolution is governed by a kinetic equation

$$(\partial_t + v \cdot \nabla_y) f = Q[f(t,y,\cdot)].$$

The left-hand side is the transport operator, while the right-hand side, a non-local operator acting on the velocity variable, accounts for the interaction between particles. This class contains the Boltzman equation, as well as the discrete kinetic models or the BGK model. When the collisions are elastic, the mass, momentum and energy are conserved. This is reflected by the properties

$$\int_{\mathbb{R}^n} Q[g](v)\,dv = 0, \qquad \int_{\mathbb{R}^n} Q[g](v) v\,dv = 0, \qquad \int_{\mathbb{R}^n} Q[g](v)|v|^2\,dv = 0$$

for every reasonable function $g(v)$. Integrating the kinetic equation against dv, $v\,dv$ and $\frac{1}{2}|v|^2 dv$, we obtain again, at least formally, the conservation laws

$$\partial_t \rho + \mathrm{div}_y m = 0, \qquad \partial_t m + \mathrm{Div}_y T = 0, \qquad \partial_t E + \mathrm{div}_y Q = 0,$$

where

$$\rho(t,y) := \int_{\mathbb{R}^n} f(t,x,v)\,dv, \qquad m(t,y) := \int_{\mathbb{R}^n} f(t,x,v)\,v\,dv, \qquad E := \int_{\mathbb{R}^n} f(t,y,v)\,\frac{1}{2}|v|^2 dv$$

are the mass density, linear momentum and energy, while

$$T := \int_{\mathbb{R}^n} f(t,y,v)\,v \otimes v\,dv, \qquad Q := \int_{\mathbb{R}^n} f(t,y,v)\,\frac{1}{2}|v|^2 v\,dv$$

are fluxes. The tensor

$$A(t,y) = \begin{pmatrix} \rho & m^T \\ m & T \end{pmatrix}$$

is again a DPT.

Steady / self-similar flows. Let us go back to gas dynamics. If the flow is steady, then on the one hand $\mathrm{div}(\rho u) = 0$, and on the other hand $\mathrm{Div}(\rho u \otimes u) + \nabla p = 0$. Therefore the tensor $A = \rho u \otimes u + pI_n$ is a DPT in the physical domain $\Omega \subset \mathbb{R}^n$.

If instead the flow is self-similar, in the sense that ρ, u and p depend only upon $\xi = \frac{y}{t}$ (this is reminiscent to the multi-D Riemann Problem), then it obeys to the reduced system

$$\mathrm{div}_\xi(\rho v) + n\rho = 0, \qquad \mathrm{Div}_\xi(\rho v \otimes v) + \nabla_\xi p + (n+1)\rho v = 0, \tag{1}$$

where $v := u(\xi) - \xi$ is the *pseudo-velocity*. The tensor $A := \rho v \otimes v + pI_n$ is not a DPT, because of the source term $(n+1)\rho v$. However it is positive semi-definite, and we shall be able to handle such a situation.

Relastivistic gas dynamics. In the Minkowski space, the Euler equations write $\mathrm{Div}\, T = 0$ where T is the stress-energy tensor. This is another instance of a DPT.

Periodic homogenization of elliptic operators. This is completely different context, for which we refer to [1, 18]. A Γ-periodic symmetric tensor $A(x)$ is given, which satisfies the bounds

$$\alpha|\xi|^2 \le \xi^T A(x)\xi \le \beta|\xi|^2, \qquad \forall \xi \in \mathbb{R}^d,$$

where $0 < \alpha \le \beta < +\infty$ are constants. The differential operator $Lu = \mathrm{div}(A\nabla u)$ is uniformly elliptic. Given a vector ξ, the problem

$$\mathrm{div}(A(\xi + \nabla u)) = 0$$

admits a unique Γ-periodic solution $u_\xi \in H^1_{loc}$, up to an additive constant. Such a PDE governs the temperature or the electric potential at equilibrium in a periodic non-homogeneous medium. The macroscopic behaviour of the medium is well described by the so-called *effective tensor* A_{eff}, whose definition is

$$A_{\mathrm{eff}}\xi = \fint_{\mathbb{R}^d/\Gamma} A(x)(\xi + \nabla u_\xi)\, dx.$$

An equivalent formulation is

$$\xi^T A_{\mathrm{eff}}\xi = \fint_{\mathbb{R}^d/\Gamma} (\xi + \nabla u_\xi)^T A(x)(\xi + \nabla u_\xi)\, dx = \inf_{w \in H^1_{per}} \fint_{\mathbb{R}^d/\Gamma} (\xi + \nabla w)^T A(x)(\xi + \nabla w)\, dx. \tag{2}$$

In particular, $A_{\mathrm{eff}} \in \mathbf{SPD}_d$. The effective tensor is known to obey the sharp bounds

$$A_- \le A_{\mathrm{eff}} \le A_+ \tag{3}$$

where $A_\pm$ are the harmonic and arithmetic means of $A(x)$:

$$A_+ = \fint_{\mathbb{R}^d/\Gamma} A(x)\, dx, \qquad A_- = \left(\fint_{\mathbb{R}^d/\Gamma} A(x)^{-1}\, dx\right).$$

Proposition 1.1 *The effective tensor* $A_{\rm eff}$ *equals the upper bound* A_+ *if, and only if,* A *is a DPT.*

Although this is a classical and simple fact, we recall the proof. Taking $w \equiv 0$ in (2), we obtain the upper bound $\xi^T A_{\rm eff}\xi \le \xi^T A_+\xi$. If $A_{\rm eff} = A_+$, this implies that the infimum is attained precisely at constants ; in other words $\nabla u_\xi \equiv 0$. But then ${\rm div}(A(\xi + \nabla u_\xi)) = 0$ writes ${\rm div}(A\xi) = 0$. This being true for every ξ, we have ${\rm Div} A = 0$. The converse is immediate: if ${\rm Div} A = 0$, then u_ξ is just a constant, and therefore $\xi^T A_{\rm eff}\xi = \xi^T A_+\xi$.

The role of the effective tensor is the following. Given $f \in H^{-1}(\Omega)$ and a small scale $\varepsilon > 0$, the solution u^ε of the Dirichlet boundary-value problem

$$
{\rm div}\left(A(\frac{x}{\varepsilon})\nabla u^\varepsilon\right) = f(x), \qquad u^\varepsilon|_{\partial\Omega} = 0
$$

remains bounded in $H^1(\Omega)$ and converges weakly as $\varepsilon \to 0$ towards the solution $\bar u$ of the same problem with the effective matrix:

$$
{\rm div}\,(A_{\rm eff}\nabla \bar u) = f(x), \qquad \bar u|_{\partial\Omega} = 0.
$$

When $f \in L^2(\Omega)$ instead, the sequence u^ε remains bounded in $H^2(\Omega)$ only if $A_{\rm eff}$ coincides with A_+, see [6]. This is due to the fact that the first corrector in the expansion of u^ε in terms of ε becomes trivial.

1.2 Λ-concave functions

Let K be a convex subset of some space $\mathbb{R}^N$ and $F : K \to \mathbb{R}$ be a continuous function. We consider measurable functions $u : \Omega \to K$ (say, bounded ones). Let us recall that F is concave if, and only if the inequality

$$
\tag{4} \fint_\Omega F(u)\,dx \le F\left(\fint_\Omega u\,dx\right)
$$

for every such u. This is just a reformulation of Jensen's inequality. In particular, the equality holds true for every u if, and only if F is affine.

A general question, first addressed by F. Murat and L. Tartar [13, 17] is whether a differential constraint imposed to u allows some non-concave functions F to satisfy (4). For instance, the following is known [3]. If $\Omega = \mathbb{R}^d/\Gamma$, and $u = \nabla\phi$ (hence F applies to $d \times m$ matrices, and ${\rm curl}\, u = 0$) is Γ-periodic, then the equality holds true in (4) whenever F is a linear combination of minors. And the inequality is valid for every *polyconcave* function, that is a concave function of all the minors.

The same question is addressed here, when $\mathbb{R}^N = \mathbf{Sym}_d$, the cone K is $\mathbf{Sym}_d^+$ and the differential constraint is ${\rm Div} A = 0$. Every concave function satisfies it, in a trivial manner because the inequality does not involve the differential constraint. A fundamental example of that situation is the function

$$
A \mapsto ({\rm det} A)^{\frac{1}{d}},
$$

which is concave over $\mathbf{Sym}_d^+$ (see [14] Section 6.6).

A necessary condition. Let us recall a construction due to Tartar [17]. Let $B,C\in\mathbf{Sym}_d^+$ be given, such that $C-B$ is singular (that is $\det(C-B)=0$). Then there exists a non-zero vector ξ such that $(C-B)\xi=0$. This ensures that for every function $g:\mathbb{R}\to\{0,1\}$, the tensor

$$A(x):=g(x\cdot\xi)B+(1-g(x\cdot\xi))C$$

is a DPT. If F satisfies (4) then in particular we have

$$⨏_{\mathbb{R}^d/\mathbb{Z}^d}F(g(x\cdot\xi)B+(1-g(x\cdot\xi))C)\,dx\le F\left(⨏_{\mathbb{R}^d/\mathbb{Z}^d}(g(x\cdot\xi)B+(1-g(x\cdot\xi))C)\,dx\right).$$

With θ the mean value of g, this is

$$\theta F(B)+(1-\theta)F(C)\le F(\theta B+(1-\theta)C).$$

The restriction of F to the segment $[B,C]$ must therefore be concave. We say that F is Λ-*concave*, where Λ is the cone of singular symmetric matrices.

Let us go back to the trivial example of $A\mapsto(\det A)^{\frac{1}{d}}$. Is it possible to improve the exponent $\frac{1}{d}$ while keeping the Λ-concavity ? The answer is positive:

Proposition 1.2 *For an exponent $\alpha>0$, the map*

$$\begin{array}{ccc} A & \longmapsto & (\det A)^{\alpha}\\ \mathbf{Sym}_d^+ & \to & \mathbb{R}^+\end{array}$$

is Λ-concave if, and only if $\alpha\le\frac{1}{d-1}$.

Proof

Let $A,A+B\in\mathbf{Sym}_d^+$ be such that $\det B=0$ and denote $f(t)=(\det(A+tB))^{\frac{1}{d-1}}$. To prove that f is concave over $[0,1]$, it is enough to prove that $f(t)\le f(0)+tf'(0)$. Using a congruence, we may assume that $A=I_d$. Another congruence, by an orthogonal matrix P, allows us to assume that in addition, B is diagonal: $B=\mathrm{diag}(b_1,\dots,b_{d-1},0)$. Then, using the arithmetico-geometric inequality,

$$f(t)=\prod_{j=1}^{d-1}(1+tb_j)^{\frac{1}{d-1}}\le\frac{1}{d-1}\sum_{j=1}^{d-1}(1+tb_j)=f(0)+tf'(0).$$

If $\alpha<\frac{1}{d-1}$, then the function F_α under consideration is a composition $\phi_\alpha\circ F_{\frac{1}{d-1}}$ where $\phi_\alpha(s)=s^{\frac{\alpha}{d-1}}$. Since ϕ_α is concave increasing and $F_{\frac{1}{d-1}}$ is concave, F_α is concave.

Conversely, if F_α is Λ-concave and $B=\mathrm{diag}(b_1,\dots,b_{d-1},0)$ is singular, diagonal with all $b_j>0$, then

$$t\mapsto\prod_{j=1}^{d-1}(1+tb_j)^{\alpha}$$

must be concave. In particular it must be sub-linear, which implies $\alpha \le \frac{1}{d-1}$.

■

Once we know that F_α passes the test of Λ-concavity, it becomes natural to ask whether it satisfies a functional inequality, such as (4) when $\Omega = \mathbb{R}^d/\Gamma$, or something similar when Ω is a bounded domain.

Clues are provided by two particular cases:

Diagonal case. A diagonal DPT is a map $x \mapsto \mathrm{diag}(g_1(\widehat{x_1}), \dots, g_d(\widehat{x_d}))$, where the j-th function (non-negative) does not depend upon x_j. Such a tensor is periodic whenever the g_j's are so, and the lattice is parallel to the axes. This situation enjoys an inequality due to Gagliardo [9]:

$$\fint_{\mathbb{R}^d/\Gamma} \left(\prod_{j=1}^{d} g_j(\widehat{x_j}) \right)^{\frac{1}{d-1}} dx \le \prod_{j=1}^{d} \left(\fint_{\mathbb{R}^{d-1}/\Gamma_j} g_j(\widehat{x_j})\, d\widehat{x_j} \right)^{\frac{1}{d-1}}, \tag{5}$$

where the lattice Γ_j is the projection of Γ on the hyperplane $x_j = 0$. The right-hand side can be viewed as the average of a power of $\det A$, while the left-hand side is the power of the determinant of the average matrix.

Cofactors of Hessian. Let $\phi \in W^{2,d-1}(\Omega)$ be a convex function over a convex domain Ω. Let us form its Hessian matrix $\nabla^2\phi$, and then the positive symmetric tensor $A = \widehat{\nabla^2\phi}$.

Lemma 1.2 *The tensor defined above is a DPT.*

The proof consists in remarking that the differential form $\omega_j := \sum_i a_{ij} dx_j$ is nothing but the exterior product $\cdots \wedge d\phi_{j-1} \wedge d\phi_{j+1} \wedge \cdots$, where only the factor $d\phi_j$ has been omited. This $(d-1)$-form is obviously closed, and this translates into the identity $\sum_i \partial_i a_{ij} = 0$.

It turns out that $(\det A)^{\frac{1}{d-1}} = \det \nabla^2\phi$ is itself an exterior derivative, for instance that of $\phi_j\omega_j$. Therefore

$$\int_\Omega (\det A)^{\frac{1}{d-1}} dx$$

is actually a boundary integral.

In the periodic case, we assume that only $\nabla^2\phi$ is Γ-periodic, and we write $\phi(x) = \frac{1}{2}x^T Sx + \text{linear} + \psi(x)$ where ψ is Γ-periodic. Then we have

$$\fint_{\mathbb{R}^d/\Gamma} (\det A)^{\frac{1}{d-1}} dx = \fint_{\mathbb{R}^d/\Gamma} \det(S + \nabla^2\psi)\, dx = \det S,$$

because the determinant of $S + \nabla\phi^2$ is the sum of $\det S$ and a linear combination of minors of $\nabla^2\phi$, each one being a divergence, thus integrating to zero. On the other hand we have

$$\fint_{\mathbb{R}^d/\Gamma} A(x)\, dx = \fint_{\mathbb{R}^d/\Gamma} \widehat{S + \nabla^2\psi} = \widehat{S}$$

for the same reason. We infer a remarkable identity:

Proposition 1.3 *The formula $A=\widehat{S+\nabla^2\psi}$, where ψ is Γ-periodic and $x\mapsto\frac{1}{2}x^TSx+\psi(x)$ is convex, provides a DPT, which satisfies*

$$\fint_{\mathbb{R}^d/\Gamma}(\det A)^{\frac{1}{d-1}}dx=\left(\det\fint_{\mathbb{R}^d/\Gamma}A(x)\,dx\right)^{\frac{1}{d-1}}.$$

Both particular cases above are given in a periodic context but have counterparts in bounded convex domains. We shall explain below how they embed into results that are valid for every DPT. The version in a bounded convex domain will involve the trace $A\vec{n}$, an object that makes sense just because of the divergence-free property.

The next two sections contain our results. Up to our knowledge, they have not been uncovered so far, perhaps because the DPT structure has been overlooked, or has been examined only at the linear level. Our results are two-fold. On the one hand we make general statements about DPTs, which we prove in Sections 4 and 5. On the other hand, we give several applications to gas dynamics. They concern either the Euler system of a compressible fluid, or the kinetic models, for instance that of Boltzmann. Their proofs are given in Section 6.

2 General statements

We present two abstract results about DPT, which cover the periodic case, and that of a convex bounded domain. The central object here is the application $F_{\frac{1}{d-1}}$:

$$\begin{array}{rcl} A & \longmapsto & (\det A)^{\frac{1}{d-1}}, \\ \mathbf{Sym}_d^+ & \longrightarrow & \mathbb{R}^+ \end{array}$$

2.1 Periodic case

Theorem 2.1 *Let the DPT $x\longmapsto A(x)$ be Γ-periodic, with $A\in L^{\frac{d}{d-1}}(\mathbb{R}^d)$. Then there holds*

$$\fint_{\mathbb{R}^d/\Gamma}(\det A(x))^{\frac{1}{d-1}}dx\le\left(\det\fint_{\mathbb{R}^d/\Gamma}A(x)\,dx\right)^{\frac{1}{d-1}}. \tag{6}$$

An easy consequence is the following.

Corollary 2.1 *Let Ω be an open set of $\mathbb{R}^d$. Let $\bar{A}\in\mathbf{SPD}_d$ be given, and A be a DPT over Ω, such that $A-\bar{A}$ is compactly supported. Then*

$$\fint_\Omega A(x)\,dx=\bar{A}\quad\textit{and}\quad\fint_\Omega(\det A(x))^{\frac{1}{d-1}}dx\le\left(\det\fint_\Omega A(x)\,dx\right)^{\frac{1}{d-1}}.$$

The inequality (6) of Theorem 2.1 is actually sharp:

Proposition 2.1 *In the situation of Theorem 2.1, suppose that $x \mapsto \det A$ is a smooth function, bounded by below and by above. Then the equality case in (6) is achieved if, and only if $A = \widehat{\nabla^2 \theta}$, where θ is a convex function whose Hessian is periodic.*

We expect that the assumptions that $\det A$ is smooth and bounded below by a positive constant can be removed, though we do not dwell into more details here.

Within the context of periodic homogenization, (6) applies to the case where $A_{\rm eff} = A_+$. One might wonder whether it is a particular case of a more general inequality, once $A_{\rm eff}$ differs from A_+. We leave this question open, but it is easy to rule out the tempting inequality

$$\text{(7)} \qquad \fint_{\mathbb{R}^d/\Gamma} (\det A(x))^{\frac{1}{d-1}} dx \quad \stackrel{?}{\leq} \quad (\det A_{\rm eff})^{\frac{1}{d-1}} .$$

As a matter of fact, the upper bound in (3) and the monotonicity of the determinant tell us that

$$\det A_{\rm eff} \leq \det \fint_{\mathbb{R}^d/\Gamma} A(x)\, dx.$$

If the inequality (7) was true, then the average of $F(A) := (\det A)^{\frac{1}{d-1}}$ would be bounded above by F of the average, for every $x \mapsto A(x)$ taking values in $\mathbf{SPD}_d$. This would imply the concavity of $F = F_{\frac{1}{d-1}}$ over $\mathbf{SPD}_d$, which we know is false (Proposition 1.2).

2.2 Bounded domain

We assume now that the domain Ω is convex. We recall that if a divergence-free vector field $\vec{q}$ belongs to $L^p(\Omega)$, then it admits a normal trace $\gamma_\nu \vec{q}$ which belongs to the Sobolev space $W^{-\frac{1}{p},p}(\partial\Omega)$. It is defined by duality, by the formula

$$\langle \gamma_\nu \vec{q}, \gamma_0 w \rangle = \int_\Omega \vec{q} \cdot \nabla w \, dx, \qquad \forall w \in W^{1,p'}(\Omega),$$

where γ_0 is the standard trace operator from $W^{1,p'}(\Omega)$ into $W^{\frac{1}{p},p'}(\partial\Omega)$.

When $\vec{q}$ is a smooth field, $\gamma_\nu \vec{q}$ coincides with the pointwise normal trace $\vec{q}|_{\partial\Omega} \cdot \vec{n}$. We say that $\vec{q}$ has an integrable normal trace if the distribution $\gamma_\nu \vec{q}$ coincides with an integrable function ; then we write $\vec{q} \cdot \vec{n}$ instead. For instance, and this is the case below, the row-wise trace $\gamma_\nu A$ of a DPT of class $L^d(\Omega)$ makes sense in $W^{-\frac{1}{d},d}(\partial\Omega)$, and we denote this trace $A\vec{n}$ when it is integrable.

Theorem 2.2 *Let Ω be a bounded convex open subset in $\mathbb{R}^d$. Let A be a DPT over Ω that belongs to $L^{\frac{d}{d-1}}_{loc}(\mathbb{R}^d)$ and has an integrable normal trace. Then there holds*

$$\int_\Omega (\det A(x))^{\frac{1}{d-1}} dx \le \frac{1}{d|S^{d-1}|^{\frac{1}{d-1}}} \|A\vec{n}\|_{L^1(\partial\Omega)}^{\frac{d}{d-1}}. \tag{8}$$

If A is only symmetric non-negative, but $\mathrm{Div} A$ *is a bounded measure (therefore A is not a DPT), then we have*

$$\int_\Omega (\det A(x))^{\frac{1}{d-1}} dx \le \frac{1}{d|S^{d-1}|^{\frac{1}{d-1}}} \left(\|A\vec{n}\|_{L^1(\partial\Omega)} + \|\mathrm{Div} A\|_{\mathcal{M}(\Omega)}\right)^{\frac{d}{d-1}}, \tag{9}$$

where the second norm is the total mass of the measure $|\mathrm{Div} A|$.

A somehow more elegant form of (8) happens when Ω is a ball:

$$⨍_{B_r} (\det A(x))^{\frac{1}{d-1}} dx \le \left(⨍_{S_r} |A\vec{n}|\, ds(x)\right)^{\frac{d}{d-1}}. \tag{10}$$

The inequalities (6) and (10) can be viewed as *non-commutative* analogues of the Gagliardo inequality (5).

Once again, the inequality (8) is sharp, and we have

Proposition 2.2 *In the situation of Theorem 2.2, suppose that $x \mapsto \det A$ is a smooth function, bounded by below and by above. Then the equality case in (8) is achieved if, and only if $A = \widehat{\nabla^2\theta}$, where θ is a convex function such that $\nabla\theta(\Omega)$ is a ball centered at the origin.*

On a qualitative side, we have the following result.

Proposition 2.3 *Let Ω be a bounded open subset of $\mathbb{R}^d$ with a Lipschitz boundary. Let A be a DPT over Ω. If $\vec{n}^T A\vec{n} \equiv 0$ over $\partial\Omega$, then A vanishes identically over Ω.*

3 Applications to gas dynamics

We intend to apply or adapt Theorem 2.2 in a situation where the first independent variable is a time variable, and the other ones represent spatial coordinates. We therefore set $d = 1+n$ and $x = (t,y)$ where $t \in \mathbb{R}$ and $y \in \mathbb{R}^n$. We write a DPT blockwise

$$A(t,y) = \begin{pmatrix} \rho & m^T \\ m & S \end{pmatrix},$$

where $\rho \ge 0$ and m can be interpreted as the densities of mass and linear momentum. We begin with an abstract result.

Theorem 3.1 *Let A be a DPT over a slab $(0,T)\times\mathbb{R}^n$. We assume*

$$A\in L^1((0,T)\times\mathbb{R}^n)\cap L^{\frac{d}{d-1}}_{loc}((0,T)\times\mathbb{R}^n).$$

There exists a constant c_n, depending only upon the space dimension (but neither on T, nor on A) such that, with the notations above

$$\int_0^T dt\int_{\mathbb{R}^n}(\det A)^{\frac{1}{n}}dy\le c_n\left(\|m(0,\cdot)\|_{L^1(\mathbb{R}^n)}+\|m(T,\cdot)\|_{L^1(\mathbb{R}^n)}\right)\left(\int_{\mathbb{R}^n}\rho(0,y)\,dy\right)^{\frac{1}{n}}.$$

3.1 Euler equations

For a compressible, inviscid gas, the flux of momentum is given by

$$S=\frac{m\otimes m}{\rho}+pI_n,$$

where the pressure $p\ge 0$ is given by an equation of state. The latter is given in terms of the density ρ (if the gas is barotropic or isentropic) or of the density and the temperature ϑ (adiabatic gas). In both cases, the Euler system $\mathrm{Div}A=0$ expresses the conservation of mass and momentum, and is supplemented by an energy balance law

$$\partial_t E+\mathrm{div}_y\left[(E+p)\frac{m}{\rho}\right]\le 0,\qquad E:=\frac{|m|^2}{2\rho}+\rho e,$$

where $e\ge 0$ is the internal energy per unit mass. This inequality is an equality in the adiabatic case. Its main role is to provide an *a priori* energy estimate

$$\sup_{t\ge 0}\int_{\mathbb{R}^n}E(t,y)\,dy\le E_0:=\int_{\mathbb{R}^n}E(0,y)\,dy,$$

whenever the total energy E_0 at initial time is finite.

For reasonable equations of state, like those of a polytropic gas ($p=a\rho^\gamma$ for a constant $\gamma>1$) or a perfect gas ($p=(\gamma-1)\rho e$), the internal energy per unit volume ρe dominates the pressure:

$$p\le C\rho e \tag{11}$$

for some finite constant C. This, together with the assumption $E_0<\infty$, ensures that $S\in L^1((0,T)\times\mathbb{R}^n)$. If in addition the total mass

$$M_0:=\int_{\mathbb{R}^n}\rho(0,y)\,dy$$

is finite, then $A\in L^1((0,T)\times\mathbb{R}^n)$ (remark that the total mass remains constant in time).

Applying Theorem 3.1 to the Euler system, we infer the estimate

$$\int_0^T dt\int_{\mathbb{R}^n}\rho^{\frac{1}{n}}p\,dy\le 2c_nM_0^{\frac{1}{2}+\frac{1}{n}}(2E_0)^{1/2}.$$

This inequality can be sharpened after remarking that the left-hand side does not depend upon the Galilean frame, while the right-hand side, more precisely E_0, does. We may replace in the inequality above the initial velocity $u_0 = \frac{m}{\rho}(0,\cdot)$ by $u_0 - \vec{c}$ where $\vec{c}$ is an arbitrary constant (this constant represents the velocity of a Galilean frame with respect to a reference frame). Eventually, we may choose the vector $\vec{c}$ which minimizes the resulting quantity

$$\int_{\mathbb{R}^n} \left(\frac{1}{2}\rho_0|u_0-\vec{c}|^2+\rho_0 e_0\right) dx.$$

This yields the following result.

Theorem 3.2 *We assume that the equation of state implies (11).*

Consider an admissible (in the sense above) flow, solution of the Euler equations of a compressible fluid in $(0,T)\times\mathbb{R}^n$. We assume a finite mass M_0 and energy E_0, and that the tensor A belongs to $L_{loc}^{\frac{d}{d-1}}((0,T)\times\mathbb{R}^n)$. Then the following estimate holds true:

$$\int_0^T dt \int_{\mathbb{R}^n} \rho^{\frac{1}{n}} p\, dy \leq 2c_n M_0^{\frac{1}{n}} D_0^{\frac{1}{2}}, \tag{12}$$

where

$$D_0 := \int_{\mathbb{R}^n} \rho_0 dy \int_{\mathbb{R}^n} (\rho_0|u_0|^2+2\rho_0 e_0)\, dy - \left|\int_{\mathbb{R}^n} \rho_0 u_0 dy\right|^2.$$

Remarks.

- For full gas dynamics, the quantity D_0 is an invariant of the flow. For a barotropic flow, the energy may decay, but the mass and linear momentum are preserved ; the corresponding quantity $D(t)$ is therefore non-increasing.
- We did not make any local hypothesis about the equation of state. We did not even ask for hyperbolicity. Thus (12) could be used to control of blow-up for models with phase transition (Van der Waals gas). Our assumption (11) is more of a global nature. For instance, if the gas is barotropic, then $\rho\mapsto p,e$ are linked by $p=\rho^2 e'$ and our assumption is just that

  $$\rho\frac{de}{d\rho}\leq Ce$$

  for some finite constant C.
- When the solution is globally defined, we even have

  $$\int_0^\infty dt \int_{\mathbb{R}^n} \rho^{\frac{1}{n}} p\, dy \leq 2c_n M_0^{\frac{1}{n}} D_0^{\frac{1}{2}}. \tag{13}$$

- Our estimate shows that the fluid cannot concentrate, unless $\rho^{\frac{1}{n}}p = O(\rho)$ as $\rho\to+\infty$. This rules out the so-called *delta-shocks* for most of the reasonable equations of state.

Polytropic gas. When $p(\rho)=\mathrm{cst}\cdot\rho^\gamma$ with adiabatic constant $\gamma>1$, (12) is an estimate of ρ in $L^{\gamma+\frac{1}{n}}_{t,y}$, which up to our knowledge is new. Combining this with the estimates of ρ in $L^\infty_t(L^1_y)$ (conservation of total mass) and in $L^\infty_t(L^\gamma_y)$ (decay of total energy), and using the Hölder inequality, we infer estimates of ρ in $L^q_t(L^r_y)$ for every (q,r) such that the point $\left(\frac{1}{q},\frac{1}{r}\right)$ lies within the triangle whose vertices are

$$(0,1),\qquad \left(0,\frac{1}{\gamma}\right)\quad\text{and}\quad \left(\frac{n}{n\gamma+1},\frac{n}{n\gamma+1}\right).$$

A similar interpolation argument, which involves the decay of energy, ensures that

$$\int_0^T\left(\int_{\mathbb{R}^n}\rho^\alpha|m|\,dy\right)^2dt<\infty,\qquad \alpha:=\frac{1}{2}\left(\frac{1}{n}+\gamma-1\right).$$

When $T=+\infty$, (13) can be compared with other dispersion estimates, for instance (see [16])

$$\int_{\mathbb{R}^n}\rho^\gamma dy=O\left((1+t)^{-n(\gamma-1)}\right),$$

when the gas has finite inertia

$$I_0:=\int_{\mathbb{R}^n}\rho(0,y)\frac{|y|^2}{2}\,dy.$$

Perfect non-isentropic gas. When $p=(\gamma-1)\rho e$, a similar argument yields an estimate of $\rho^{1+\frac{1}{nq}}e^r$ in $L^q_t(L^1_x)$, whenever $1\le q\le\infty$ and $r-1\le\frac{1}{q}\le r$.

Euler–Fourier system. The Euler–Fourier system governs the motion of an inviscid but heat-conducting gas. The only difference with the Euler system is that the conservation law of energy incorporates a dissipative term $\mathrm{div}_y(\kappa\nabla_y\vartheta)$, where ϑ is the temperature and $\kappa>0$ the thermal conductivity. Because the conservation of mass and momentum still writes $\mathrm{Div}A$ with the same A as before, and because the total energy is conserved, Theorem 3.2 applies to this case.

On the contrary, our theorem does not apply to the Navier–Stokes system for a compressible fluid, because then the divergence-free tensor is not positive semi-definite.

The role of Estimate (12). Theorem 3.2 is an *a priori* estimate which suggests a functional space where to search for admissible solutions of the Euler equation. For finite initial mass and energy, one should look for a flow satisfying the following three requirements: – the total mass is conserved, – the total energy is a non-increasing function of time (a constant in the adiabatic case), – and $\rho^{\frac{1}{n}}p\in L^1_{t,y}$.

To this end, the construction of a solution to the Cauchy problem should involve an approximation process which is consistent with these estimates. For this purpose, a vanishing viscosity approach (say, the compressible Navier–Stokes equation) does not seem suitable. As we shall see below, the Boltzmann equation is more appropriate, but this observation just shifts the consistency problem to an other level. An other approach is to design numerical schemes, which are consistent with the Euler equations and meanwhile with the above requirements. There exist several schemes that preserve the symmetric positive structure, for instance the Lax–Friedrichs and Godunov schemes in space dimension one, or their muti-dimensional variants. However they provide approximations for which the mass of the Radon measure $\mathrm{Div} A^{\Delta t,\Delta y}$ tends to $+\infty$ as $\Delta t,\Delta y\to 0$. The second part of Theorem 2.2 yields

$$\int_\Omega (\det A^{\Delta t,\Delta})^{\frac{1}{d-1}}dy\,dt \le \frac{1}{d|S^{d-1}|^{\frac{1}{d-1}}}\left(\|A^{\Delta t,\Delta x}\vec n\|_{L^1(\partial\Omega)}+\|\mathrm{Div} A^{\Delta t,\Delta x}\|_{\mathcal{M}(\Omega)}\right)^{\frac{d}{d-1}},$$

where the right-hand side tends to $+\infty$ when $\Delta t,\Delta y\to 0$. Thus it is unclear whether the limit of such schemes satisfies the estimate (12).

Notice that we must not require the integrability $A\in L^{\frac{d}{d-1}}$, which is only a technical need for our proof. As a matter of fact, the various entries a_{ij} have distinct physical dimensions, so that such an integrability hardly makes sense. On the contrary, $\det A$ is a well-defined quantity from the physical point of view.

We also point out that, although our new estimate is a genuine improvement, it is still not sufficient to ensure the local integrability of the energy flux

$$\left(\frac{1}{2}\rho|u|^2+\rho e+p\right)u,$$

and therefore to give sense to the conservation law of energy.

3.2 Self-similar flows

We now consider the problem (1) in space dimension n. The tensor $A=\rho v\otimes v+pI_n$ (recall that v is the pseudo-velocity), though positive semi-definite, is not a DPT. The second part of Theorem 2.2, plus the formula $\det A=p^{n-1}(p+\rho|v|^2)$, yield

$$(14)\quad \begin{aligned}\int_\Omega p(p+\rho|v|^2)^{\frac{1}{n-1}}d\xi &\le \frac{1}{n|S^{n-1}|^{\frac{1}{n-1}}}\left(\|p\vec n+\rho(v\cdot\vec n)v\|_{L^1(\partial\Omega)}+(n+1)\int_\Omega\rho|v|\,d\xi\right)^{\frac{n}{n-1}}\\ &\le \frac{1}{n|S^{n-1}|^{\frac{1}{n-1}}}\left(\int_{\partial\Omega}(p+\rho|v|^2)\,ds(\xi)+(n+1)\int_\Omega\rho|v|\,d\xi\right)^{\frac{n}{n-1}}\end{aligned}$$

for every convex subdomain Ω. For a ball B_r of radius r and arbitrary center, this writes

$$(15)\qquad \fint_{B_r} p(p+\rho|v|^2)^{\frac{1}{n-1}}d\xi \le \left(\fint_{\partial B_r}(p+\rho|v|^2)\,ds(\xi)+\frac{n+1}{n}r\fint_{B_r}\rho|v|\,d\xi\right)^{\frac{n}{n-1}}$$

Remark that, contrary to the situation of the Cauchy problem, we do not have the freedom to choose among equivalent coordinate frames. There is no improvement of (14) or (15) similar to (12).

Riemann problem. The Riemann problem is a special form of the Cauchy problem, where the initial data (density, momentum, energy) is positively homogeneous of degree zero ; for instance, the initial density has the form $\rho_0(\frac{y}{|y|})$. In practice, we suppose that the physical space $\mathbb{R}^n$ is partitioned into conical cells with polygonal sections, and that the data is constant in each cell. Such a data depends on finitely many parameters.

Because the Euler equations are PDEs of homogeneous degree one, the admissible solution, whether there exists a unique one, must be self-similar too. For instance $\rho(t,x) = \bar{\rho}(\frac{x}{t})$. Denoting $\xi = \frac{x}{t}$ the self-similar variable, every conservation law $\partial_t f + \mathrm{div}_y q = 0$ becomes $\mathrm{div}_\xi q = \xi \cdot \nabla_\xi f$. For instance, droping the bars, we have $\mathrm{div}_\xi(\rho u) = \xi \cdot \nabla \rho$. These new equations involve explicitly the independent variable ξ, but the introduction of the *pseudo-velocity* $v(\xi) := u(\xi) - \xi$ allows us to get rid of it. In terms of ρ, v, p, e and ξ-derivatives, the reduced Euler system becomes

$$\mathrm{div}(\rho v) + n\rho = 0, \qquad \mathrm{Div}(\rho v \otimes v) + \nabla p + (n+1)\rho v = 0 \tag{16}$$

and

$$\mathrm{div}\left((\frac{1}{2}\rho |v|^2 + \rho e + p) v \right) + \left(\frac{n}{2} + 1 \right) \rho |v|^2 + n(\rho e + p) = 0. \tag{17}$$

The initial data to the Riemann problem becomes a data at infinity for the reduced system. Let us mention that for an isentropic flow, (17) is not an equation but merely an inequation, which plays the role of an entropy inequality.

The 3-dimensional RP is still widely open. We therefore limit ourselves to the 2-dimensional case ($n = 2$). The tools and strategy for the analysis of the Riemann problem are described in the review paper [15]. The plane splits into a compact subsonic region Ω_{sub} and its complement the supersonic domain Ω_{sup}. *Subsonic* means that $|v| \leq \mathbf{c}$ where $\mathbf{c}$ is the sound speed, a function of the internal variables ρ and p. In the supersonic region, the system is of hyperbolic type and one can solve a kind of Cauchy problem, starting from the data at infinity. This Cauchy problem has an explicit solution outside of some ball $B_R(0)$. It is made of constant states separated by simple waves depending only on one coordinate ; these waves are shocks, rarefaction waves and/or contact discontinuities. An *a priori* estimate of the radius R is available. The situation in the rest of the supersonic region may be more involved, with genuinely 2-D interactions of simple waves ; even the interface between Ω_{sup} and Ω_{sub} is not fully explicit, a part of it being a free boundary. But these facts do not raise obstacles for the following calculations.

The conservation laws of mass and energy allow us to establish two *a priori* estimates. On the one hand, we have (recall that $n = 2$)

$$2\int_{B_R(0)} \rho \, d\xi = -\int_{B_R(0)} \mathrm{div}(\rho v) \, d\xi = -\int_{S_R(0)} \rho v \cdot \vec{n} \, d\xi,$$

where the last integral is computed explicitly because we know explicitly the solution over S_R. On the other hand, we have

$$2\int_{B_R(0)} (\rho |v|^2 + \rho e + p) \, d\xi \leq -\int_{S_R(0)} (\frac{1}{2}\rho |v|^2 + \rho e + p) v \cdot \vec{n} \, d\xi,$$

where again the right-hand side is known explicitly. In the non-isentropic case, we also have a minimum principle for the physical entropy s, which is nothing but the second principle of thermodynamics: $s \geq s_{\min}$ where $s_{\min}$ is the minimum value of s in the data. Let us assume a polytropic gas ($p = \mathrm{cst} \cdot \rho^\gamma$) or a perfect gas ($p = (\gamma - 1)\rho e$). In the latter case, we have $p \geq (\gamma - 1)e^{s_{\min}}\rho^\gamma$. Therefore the energy estimate yields an upper bound for

$$\int_{B_R(0)} \rho^\gamma d\xi \qquad \text{and} \qquad \int_{B_R(0)} \rho|v|^2 d\xi. \tag{18}$$

In particular, a so-called *Delta-shock* cannot take place in this situation.

These estimates can be completed by applying (14) to the tensor $A = \rho v \otimes v + pI_2$ in the ball $B_R(0)$. To this end, we show that the right-hand side is fully controled. On the one hand, the boundary integral

$$\int_{S_R(0)} (p + \rho|v|^2)\, ds(\xi)$$

is estimated explicitly as before. On the other hand, the last integral is bounded by

$$\left(\int_{B_R(0)} \rho\, d\xi\right)^{\frac{1}{2}} \left(\int_{B_R(0)} \rho|v|^2\, d\xi\right)^{\frac{1}{2}},$$

where both factors have been estimated previously. We therefore obtain an estimate of

$$\int_{B_R(0)} \rho^{2\gamma} d\xi \qquad \text{and} \qquad \int_{B_R(0)} \rho^{\gamma+1}|v|^2 d\xi. \tag{19}$$

This integrability is significantly better than that in (18).

3.3 Relativistic gas dynamics

In the Minkowski space-time $\mathbb{R}^{1+n}$ of special relativity, an isentropic gas is governed by the Euler system (see [12])

$$\begin{aligned}
\partial_t \left(\frac{\rho c^2 + p}{c^2 - |v|^2} - \frac{p}{c^2} \right) + \mathrm{div}_y \left(\frac{\rho c^2 + p}{c^2 - |v|^2} v \right) &= 0, \\
\partial_t \left(\frac{\rho c^2 + p}{c^2 - |v|^2} v \right) + \mathrm{Div}_y \left(\frac{\rho c^2 + p}{c^2 - |v|^2} v \otimes v \right) + \nabla p &= 0,
\end{aligned}$$

where the constant $c > 0$ is the speed of light. Here ρ is the mass density at rest, and p is the pressure. The fluid velocity is constrained by $|v| < c$.

It is a simple exercise to verify that the stress-energy tensor

$$A = \begin{pmatrix} \frac{\rho c^2+p}{c^2-|v|^2} - \frac{p}{c^2} & \frac{\rho c^2+p}{c^2-|v|^2} v^T \\ \frac{\rho c^2+p}{c^2-|v|^2} v & \frac{\rho c^2+p}{c^2-|v|^2} v \otimes v + pI_n \end{pmatrix}$$

is positive semi-definite. Our Theorems 2.2 and 3.1 therefore apply. What is perhaps surprising is that the determinant of A is unchanged ! Its value is still ρp^n. We infer

$$\int_0^T dt \int_{\mathbb{R}^n} \rho^{\frac{1}{n}} p\, dy \le c_n \left(\int_{\mathbb{R}^n} \frac{\rho c^2 + p}{c^2 - |v|^2} |v|\, dy \bigg|_{t=0} + (\text{same}) \bigg|_{t=T} \right) \left(\int_{\mathbb{R}^n} \frac{\rho c^4 + p|v|^2}{c^2(c^2 - |v|^2)}\, dy \right)_{t=0}^{\frac{1}{n}}.$$

We warn the reader that mass and energy are related to each other in relativity theory. The last integral in the inequality above accounts for both. We denote below its value μ_0.

Suppose an equation of state of the form $p = a^2\rho$, where $a > 0$ is a constant. When $a^2 = \frac{c^2}{3}$, this follows directly from the Stefan–Boltzmann law for a gas in thermodynamical equilibrium, as discussed page 12 of A. M. Anile's book [2]. Then the contribution of the momentum can be estimated after using $|v| \le \frac{1}{2ca}(c^2 + a^2|v|^2)$:

$$\int_{\mathbb{R}^n} \frac{\rho c^2 + p}{c^2 - |v|^2} |v|\, dy \le \frac{c^2 + a^2}{2a} c\mu_0.$$

We deduce the *a priori* estimate

$$\int_0^T dt \int_{\mathbb{R}^n} \rho^{\frac{1}{n}} p\, dy \le c_n \frac{c^2 + a^2}{a} c\mu_0^{1+\frac{1}{n}}. \tag{20}$$

3.4 Kinetic equations

We now turn to the class of kinetic equations

$$(\partial_t + v \cdot \nabla_y) f(t,y,v) = Q[f(t,y,\cdot)] \tag{21}$$

where Q is compatible with the minimum principle $f \ge 0$ and with the conservation of mass, momentum and energy. This includes the Boltzman equation, the BGK model and most of the discrete velocity models. Then we apply Theorem 2.2 to the non-negative tensor

$$A(t,y) := \int_{\mathbb{R}^n} f(t,y,v) \begin{pmatrix} 1 \\ v \end{pmatrix} \otimes \begin{pmatrix} 1 \\ v \end{pmatrix} dv.$$

The following result is a far-reaching extension of an estimate that J.-M. Bony [4] obtained for a one-dimensional discrete velocity model.

Theorem 3.3 *Consider an admissible flow of a kinetic equation of the form (21). Assume a finite mass an energy*

$$M_0 = \int_{\mathbb{R}^n} dy \int_{\mathbb{R}^n} f_0(y,v)\, dv, \qquad E_0 = \int_{\mathbb{R}^n} dy \int_{\mathbb{R}^n} f_0(y,v) \frac{|v|^2}{2}\, dv,$$

and that the moments

$$\rho(t,y) = \int_{\mathbb{R}^n} f(t,y,v)\, dv, \qquad Tr\, S(t,y) = \int_{\mathbb{R}^n} f(t,y,v)|v|^2 dv$$

belong to $L^{\frac{d}{d-1}}_{loc}((0,T)\times\mathbb{R}^n)$*. Then the following estimate holds true:*

(22) $$\int_0^T dt\int_{\mathbb{R}^n}dy\left(\frac{1}{d!}\int_{\mathbb{R}^n}^{\otimes(n+1)} f(t,y,v^0)\cdots f(t,y,v^n)(\Delta(v^0,\dots,v^n))^2dv^0\cdots dv^n\right)^{\frac{1}{n}}\le 2c_nM_0^{\frac{1}{n}}D_0^{1/2},$$

where

$$\Delta(v_0,\dots,v_n):=\begin{vmatrix}1 & \cdots & 1\\ v^0 & \cdots & v^n\end{vmatrix}$$

is the volume of the simplex spanned by $(v^0,\dots,v^n)$*, and*

$$D_0=\frac{1}{2}\int_{\mathbb{R}^n}^{\otimes 4} f_0(y,v)f_0(y',v')|v'-v|^2dydy'dvdv'.$$

Again, this estimate suggests to narrow the functional space where to search for a solution. Besides the usual constraints

$$\sup_t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(1+|x|^2+|v|^2+\log^+ f)\,f\,dv\,dx<\infty,$$

we should impose that the expression

$$I_T:=\int_0^T dt\int_{\mathbb{R}^n}dy\left(\frac{1}{d!}\int\cdots\int_{\mathbb{R}^n}^{\otimes(n+1)} f(t,y,v^0)\cdots f(t,y,v^n)(\Delta(v^0,\dots,v^n))^2dv^0\cdots dv^n\right)^{\frac{1}{n}}$$

be finite. An open problem is to understand the physical meaning of I_T.

Comments.

- The $d\times d$ determinant $\Delta(v_0,\dots,v_n)$ vanishes precisely when the points $v^0,\dots,v^n$ are affinely dependent in the space $\mathbb{R}^n$, therefore are non generic. The estimate (22) tends to force the support of $f(t,y,\cdot)$ to keep close to some affine hyperplane $\Pi(t,y)$.

- Of course, Boltzman's H-theorem, which tends to force $f(t,y,\cdot)$ to be close to a Maxwellian distribution, has the opposite effect. The combination of both estimates is expected to produce a nice control of the density f.

- This estimate controls the (t,y)-integrability of an expression homogeneous in f of degree $1+\frac{1}{n}$. This is slightly but strictly better than the controls given by the mass and energy (both linear in f) or by the H-Theorem (control in $f\log f$). The price to pay is an integration in the time variable ; this looks like what happens in Strichartz estimates for dispersive PDEs.

  The little gain in integrability raises the question whether the Boltzmann admits weak solutions for large data, and not only renormalized ones. Using this gain, C. Cercignani [5] proved the existence of weak solutions to the Cauchy problem in dimension $n=1$.

- If we had just applied the Jensen inequality, the exponent in (22) would have been $\frac{1}{n+1}$, and the (t,y)-integrand should be homogeneous of degree 1, conveying an information already contained in the mass and energy.

3.4.1 Renormalized solutions

The existence of distributional solutions to the Cauchy problem for the Boltzmann equation has not yet been proved, except in space dimension $n = 1$. Instead, we know the existence of *renormalized solutions*, whose existence has been proved by R. DiPerna & P.-L. Lions [8]. We shall not even give a precise definition of this notion, but we content ourselves to recall that it implies the conservation of mass and momentum in the sense that we have

$$\partial_t \int_{\mathbb{R}^n} f\,dv + \mathrm{div}_y \int_{\mathbb{R}^n} fv\,dv = 0, \qquad \partial_t \int_{\mathbb{R}^n} fv\,dv + \mathrm{Div}_y \left(\int_{\mathbb{R}^n} fv \otimes v\,dv + \Sigma \right) = 0.$$

The so-called *defect measure* Σ takes values in $\mathbf{Sym}_n^+$; see [11]. In addition, the total energy

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(v)|v|^2 dv\,dy + \frac{1}{2} \int \mathrm{Tr}\,\Sigma(t,\cdot)$$

is a non-increasing function of time. Our DPT is

$$A = \begin{pmatrix} \int_{\mathbb{R}^n} f\,dv & \int_{\mathbb{R}^n} fv\,dv \\ \int_{\mathbb{R}^n} fv^T\,dv & \int_{\mathbb{R}^n} fv \otimes v\,dv + \Sigma \end{pmatrix}.$$

The components ρ, m of $A\vec{e}_t$ are still the mass density and linear momentum. Theorem 3.1 yields an inequality

$$\int_0^T dt \int_{\mathbb{R}^n} (\det A)^{\frac{1}{n}} dy \le c_n M_0^{\frac{1}{n}} (\|m(0)\|_{L^1(\mathbb{R}^n)} + \|m(T)\|_{L^1(\mathbb{R}^n)}),$$

from which we may extract two informations, using the monotonicity of the determinant. On the one hand, we have

$$\begin{pmatrix} \int_{\mathbb{R}^n} f\,dv & \int_{\mathbb{R}^n} fv\,dv \\ \int_{\mathbb{R}^n} fv^T\,dv & \int_{\mathbb{R}^n} fv \otimes v\,dv \end{pmatrix} \le A,$$

from which we infer the same estimate (22) as in the case of distributional solutions. On the other hand, the Schur complement formula (see [14] Proposition 3.9) gives

$$\det A = \rho \det \left(\int_{\mathbb{R}^n} fv \otimes v\,dv - \frac{1}{\int_{\mathbb{R}^n} f\,dv} \int_{\mathbb{R}^n} fv\,dv \otimes \int_{\mathbb{R}^n} fv\,dv + \Sigma \right) \ge \rho \det \Sigma.$$

We infer an estimate of the defect measure against the mass density:

$$\int_0^T dt \int_{\mathbb{R}^n} (\rho \det \Sigma)^{\frac{1}{n}} dy \le c_n' M_0^{\frac{1}{n}} D_1^{\frac{1}{2}}, \tag{23}$$

where

$$D_1 = D_0 + M_0 \int \mathrm{Tr}\,\Sigma(t,\cdot).$$

Notice that, because Σ is a Radon measure taking values in $\mathbf{Sym}_n^+$ and $\det^{\frac{1}{n}}$ is homogeneous of degree over this cone, the expression $(\det \Sigma)^{\frac{1}{n}}$ makes sense as a bounded measure.

4 Convex domain

4.1 Proof of Theorem 2.2

In this paragraph, we consider a DPT over a bounded convex domain Ω. To prove Theorem 2.2, it is enough to consider the case where A is uniformly positive definite: just replace $A(x)$ by $A(x)+\delta I_d$ with $\delta>0$ (such a tensor is still a DPT) and then pass to the limit as $\delta\to 0+$.

From now on, we therefore assume that Ω has a smooth boundary and that $A(x)\geq\delta I_d$ for some $\delta>0$ independent of x.

Let f denote the function $(\det A)^{\frac{1}{d-1}}$. One has $f=(f\det A)^{\frac{1}{d}}$. The density of $\mathcal{C}^\infty(\overline{\Omega})$ in $L^1(\Omega)$ provides a sequence of smooth functions $f_\varepsilon:\Omega\to\mathbb{R}$ that satisfies the following requirements. To begin with, $\frac{\delta}{2}\leq f_\varepsilon(x)\leq C_\varepsilon$ for every x, where C_ε is a finite constant depending on ε. Then

$$\int_\Omega f_\varepsilon(x)\,dx=\int_\Omega f(x)\,dx,$$

and finally

$$\|f_\varepsilon-f\|_{L^1(\Omega)}\stackrel{\varepsilon\to 0}{\longrightarrow}0.$$

From the latter, we deduce that $f_\varepsilon^{1/d}\to f^{1/d}$ in $L^d(\Omega)$ and therefore $f_\varepsilon^{1/d}f^{1-1/d}\to f$ in $L^1(\Omega)$. It will thus be enough to estimate

$$\int_\Omega f_\varepsilon^{1/d}f^{1-1/d}\,dx=\int_\Omega (f_\varepsilon\det A)^{1/d}dx.$$

To do so, we consider the ball $B_r=B_r(0)$, centered at the origin, whose volume equals the integral of f (that is, that of f_ε) over Ω. A theorem due to Y. Brenier (see Theorem 2.12 in [19], or Theorem 3.1 in [7]) ensures the existence and uniqueness of an optimal transport from the measure $f_\varepsilon(x)\,dx$ to the Lebesgue measure over B_r. This transport is given by a gradient map $\nabla\psi_\varepsilon$, which is the solution of the Monge–Ampère equation

$$\det\nabla^2\psi_\varepsilon=f_\varepsilon\qquad\text{in }\Omega$$

such that ψ_ε is convex and $\nabla\psi_\varepsilon(\Omega)=B_r$; see Theorem 4.10 of [19] or Theorem 3.3 of [7]. Finally, ψ_ε is a smooth function (Theorem 4.13 of [19]). In particular, the image of the boundary $\partial\Omega$ under $\nabla\psi_\varepsilon$ is the sphere S_r.

We therefore have

$$(f_\varepsilon\det A)^{\frac{1}{d}}=(\det A\cdot\det\nabla^2\psi_\varepsilon)^{\frac{1}{d}}=(\det(A\nabla^2\psi_\varepsilon))^{\frac{1}{d}}.$$

Let $\lambda_1(x),\dots,\lambda_d(x)$ be the spectrum of the matrix $A\nabla^2\psi_\varepsilon$. This is not a symmetric matrix, but because it is the product of a positive definite matrix and a positive semi-definite one, it is diagonalisable with non-negative real eigenvalues: the λ_j's are real and ≥ 0 (Proposition 6.1 in [14]). Applying the arithmetico-geometric inequality (AGI), we have

$$(\det(A\nabla^2\psi_\varepsilon))^{\frac{1}{d}} = \left(\prod_{j=1}^{d}\lambda_j\right)^{\frac{1}{d}} \leq \frac{1}{d}\sum_{j=1}^{d}\lambda_j = \frac{1}{d}\mathrm{Tr}\,(A\nabla^2\psi_\varepsilon).$$

Because A is divergence-free, one has $\mathrm{Tr}\,(A\nabla^2\psi_\varepsilon) = \mathrm{div}(A\nabla\psi_\varepsilon)$. We infer

$$\int_\Omega f_\varepsilon(x)^{\frac{1}{d}} f(x)^{1-\frac{1}{d}}\,dx \leq \frac{1}{d}\int_\Omega \mathrm{div}(A\nabla\psi_\varepsilon)\,dx = \frac{1}{d}\int_{\partial\Omega}(A\nabla\psi_\varepsilon)\cdot\vec{n}\,ds(x) = \frac{1}{d}\int_{\partial\Omega}(A\vec{n})\cdot\nabla\psi_\varepsilon\,ds(x).$$

Because $\nabla\psi_\varepsilon$ takes values in B_r, there comes

$$\int_\Omega f(x)\,dx = \lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon(x)^{\frac{1}{d}} f(x)^{1-\frac{1}{d}}\,dx \leq \frac{r}{d}\,\|A\vec{n}\|_{L^1(\partial\Omega)}.$$

We complete the proof of the theorem by the calculation of the radius r :

$$\frac{r^d}{d}\,|S^{d-1}| = |B_r| = \int_\Omega f(x)\,dx.$$

If instead $\mathrm{Div}A$ is a bounded measure, then we have $\mathrm{Tr}\,(A\nabla^2\psi_\varepsilon) = \mathrm{div}(A\nabla\psi_\varepsilon) - (\mathrm{Div}A)\nabla\psi_\varepsilon$. The same calculation yields the bound

$$\int_\Omega f(x)\,dx \leq \frac{r}{d}\left(\|A\vec{n}\|_{L^1(\partial\Omega)} + \|\mathrm{Div}A\|_{\mathcal{M}(\Omega)}\right)$$

and the conclusion follows.

4.2 The equality case: proof of Proposition 2.2

Since we assume that f is smooth and bounded below and above, we may take $f_\varepsilon = f$. Let us examine the proof above. In order to have equalities everywhere, we need in particular that the AGI be an equality, that is the λ_j's be equal to each other. Then the diagonalisable matrix $A\nabla^2\psi$, with only one eigenvalue $\lambda(x)$, must equal $\lambda(x)I_d$. In other words, there is a scalar field $a > 0$ such that $A(x) = a(x)\widehat{\nabla^2\psi}$. In particular $\widehat{\nabla^2\psi}$ is positive definite. Because both A and $\widehat{\nabla^2\psi}$ are divergence-free (Lemma 1.2), we find that $(\widehat{\nabla^2\psi})\nabla a = 0$, that is $\nabla a = 0$. Thus a is a constant. Up to replacing ψ by $a^{-1/(d-1)}\psi$, we infer that $A = \widehat{\nabla^2\psi}$. By construction the image of Ω by $\nabla\psi$ is a ball centered at the origin.

Conversely, if ψ is such a convex function, and $A(x) := \widehat{\nabla^2\psi}$, then we know that A is a DPT. Let us examine the calculations of the previous paragraph. There is no need of an f_ε, we can just keep

f itself. Likewise, we take ψ instead of ψ_ε. Because $A\nabla^2\psi = (\det\nabla^2\psi)I_d$, the AGI is actually an equality and we have

$$\int_\Omega f(x)\,dx = \frac{1}{d}\int_\Omega \mathrm{div}(A\nabla\psi)\,dx = \frac{1}{d}\int_{\partial\Omega}(A\nabla\psi)\cdot\vec{n}\,dx = \frac{1}{d}\int_{\partial\Omega}(A\vec{n})\cdot\nabla\psi\,dx.$$

We claim that $A\vec{n}$ and $\nabla\psi$ are positively colinear along the boundary. It amounts to proving that $\vec{n}$ and $A^{-1}\nabla\psi$ are so. But the latter vector equals

$$\frac{1}{\det\nabla^2\psi}\nabla^2\psi\nabla\psi = \frac{1}{\det\nabla^2\psi}\nabla\left(|\nabla\psi|^2\right).$$

Because $|\nabla\psi|^2$ is $\le r$ everywhere, but equals r on $\partial\Omega$, its gradient is normal to the boundary and points outward. This proves the claim.

We therefore have $(A\vec{n})\cdot\nabla\psi = |A\vec{n}|\cdot|\nabla\psi| = r|A\vec{n}|$ over $\partial\Omega$, and we infer

$$\int_\Omega f(x)\,dx = \frac{r}{d}\,\|A\vec{n}\|_{1,\partial\Omega}.$$

This ends the proof of the proposition.

4.3 Compactly supported case. Proof of Proposition 2.3

Because $A(x)$ is positive semi-definite, $\vec{n}^T A\vec{n} = 0$ implies $A\vec{n} = 0$. This ensures that the extension of A to $\mathbb{R}^d$ by $A \equiv 0$ over $\mathbb{R}^d \setminus \Omega$, is still a DPT over $\mathbb{R}^d$. Let us denote it A, which is compactly supported. Let ϕ_ε be a non-negative mollifier and set $A^\varepsilon = \phi_\varepsilon * A$. This is a compactly supported DPT, of class C^∞. Its Fourier transform is therefore in the Schwartz class. The divergence-free constraint translates into $\mathcal{F}A^\varepsilon(\xi)\xi \equiv 0$. Taking $\eta \in S^{d-1}$ and $r > 0$, we have $\mathcal{F}A^\varepsilon(r\eta)\eta = 0$. Letting $r \to 0+$, we obtain $\mathcal{F}A^\varepsilon(0)\eta = 0$. In other words $\mathcal{F}A^\varepsilon(0) = 0_d$, that is

$$\int_{\mathbb{R}^d} A^\varepsilon(x)\,dx = 0_d.$$

From there, the non-negativity of $A^\varepsilon(x)$ for all x implies $A^\varepsilon \equiv 0_d$. Passing to the limit as $\varepsilon \to 0+$, we infer $A \equiv 0_d$.

5 Periodic tensors: proof of Theorem 2.1

Reduction. As in the previous section, we may assume that A is uniformly positive definite: $A(x) \ge \delta I_d$ for almost every x. Also, we may approximate A by a smooth DPT $A_\varepsilon = \phi_\varepsilon * A$, where

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^d}\phi\left(\frac{x}{\varepsilon}\right), \qquad \phi \in \mathcal{D}^+(\mathbb{R}^d) \quad \text{and} \quad \int_{\mathbb{R}^d}\phi(x)\,dx = 1.$$

This A_ε is smooth and still satisfies $A_\varepsilon(x) \ge \delta I_d$. Since it converges to A in $L^{\frac{d}{d-1}}_{loc}$, the inequality (6) for A_ε will pass to the limit and give the same for A. We therefore may restrict to the case where A is smooth and uniformly positive definite.

The proof. We start as in the previous section, by writing

$$f = (f \det A)^{\frac{1}{d}}.$$

We apply Theorem 2.2 of [10] : given a matrix $S \in \mathbf{SDP}_d$ such that

$$\det S = \fint_{\mathbb{R}^d/\Gamma} f(x)\,dx, \tag{24}$$

there exists a Γ-periodic function $\phi_S \in \mathcal{C}^\infty$ such that $\det(S + \nabla^2\phi_S) = f$. In other words, the function $\psi_S(x) = \frac{1}{2}x^T Sx + \phi_S(x)$ solves the Monge–Ampère equation $\det \nabla^2\psi_S = f$.

Proceeding as in the bounded case, we have the inequality

$$f \leq \frac{1}{d}\mathrm{Tr}\,(A\nabla^2\psi_S) = \frac{1}{d}\mathrm{div}(A\nabla\psi_S) = \frac{1}{d}\mathrm{div}(A(Sx + \nabla\phi_S)).$$

Integrating over a fundamental domain, we obtain

$$\fint_{\mathbb{R}^d/\Gamma} f(x)\,dx \leq \frac{1}{d}\fint_{\mathbb{R}^d/\Gamma}(\mathrm{Tr}\,(AS) + \mathrm{div}(A\nabla\phi_S))\,dx = \frac{1}{d}\mathrm{Tr}\,(A_+S). \tag{25}$$

There remains to minimize $\mathrm{Tr}\,(A_+S)$ under the constraint (24). The minimum is achieved with $S = \lambda\widehat{A_+}$, where λ is determined by

$$\lambda^d(\det A_+)^{d-1} = \fint_{\mathbb{R}^d/\Gamma} f(x)\,dx.$$

With this choice, (25) becomes

$$\fint_{\mathbb{R}^d/\Gamma} f(x)\,dx \leq \lambda \det A_+,$$

which is nothing but (6).

The proof of Proposition 2.1 (the case of equality) is the same as that of Proposition 2.2.

Proof of Corollary 2.1. Let $B(x) := A(x) - \bar{A}$, which satisfies $\mathrm{Div}\,B \equiv 0$ and is compactly supported. Integrating by parts twice and using the assumption, we have

$$\int_\Omega b_{ij}dx = \int_\Omega b_{ij}\partial_i x_i dx = -\int_\Omega x_i\partial_i b_{ij}dx = \sum_{k\neq i}\int_\Omega x_i\partial_k b_{kj}dx = -\sum_{k\neq i}\int_\Omega b_{kj}\partial_k x_i dx = 0,$$

whence the equality

$$\fint_\Omega A(x)\,dx = \bar{A}.$$

Let K be a cube containing Ω. We extend A to K by setting $\hat{A}(x) = \bar{A}$ whenever $x \in K \setminus \Omega$. Next we extend $\hat{A}$ by periodicity to $\mathbb{R}^d$, K being a fundamental domain. This $\hat{A}$ is a periodic DPT and has mean $\bar{A}$. Applying (6) to $\hat{A}$, we have

$$\int_\Omega (\det A)^{\frac{1}{d-1}}\,dx = \int_K (\det A)^{\frac{1}{d-1}}\,dx - (|K| - |\Omega|)(\det\bar{A})^{\frac{1}{d-1}} \leq (|K| - (|K| - |\Omega|))(\det\bar{A})^{\frac{1}{d-1}},$$

from which the obtain the desired inequality.

6 Gas dynamics with finite mass and energy

6.1 Proof of Theorem 3.1

Let us apply Theorem 2.2 in the bounded convex domain $\Omega = (0,T)\times B_R$ for some $R>0$. We have

$$\int_0^T dt\int_{B_R}(\det A)^{\frac{1}{n}}dy \le \frac{1}{(n+1)|S^n|^{\frac{1}{n}}}\,\|A\vec{n}\|_{1,\partial\Omega}^{1+\frac{1}{n}}. \tag{26}$$

The boundary consists in three parts: an initial ball $\{0\}\times B_R$, a final ball $\{T\}\times B_R$, and a lateral boundary $(0,T)\times S_R$. The latter contributes to

$$g(R) := \int_0^T dt\int_{S_R}\left|A\frac{y}{|y|}\right| dy.$$

Because A is integrable, we have $g\in L^1(0,+\infty)$ and therefore there is a subsequence $R_m\to+\infty$ such that $g(R_m)\to 0$. Passing to the limit in (26), we obtain

$$\int_0^T dt\int_{\mathbb{R}^n}(\det A)^{\frac{1}{n}}dy \le \frac{1}{(n+1)|S^n|^{\frac{1}{n}}}\left(\|(\rho,m)(0)\|_{L^1(\mathbb{R}^n)}+\|(\rho,m)(T)\|_{L^1(\mathbb{R}^n)}\right)^{1+\frac{1}{n}}. \tag{27}$$

The latter estimate has the drawback that it is not homogeneous from a physical point of view. The density ρ and the momentum m have different dimensions and the norm

$$\|(\rho,m)\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\sqrt{\rho^2+|m|^2}\,dy$$

is not physically meaningful.

To recover the homogeneity, we introduce a scaling

$$t'=\lambda t,\quad y'=y,\quad \rho'=\lambda^2\rho,\quad m'=\lambda m,\quad T'=T.$$

The corresponding A' is a DPT over the slab $(0,T')\times\mathbb{R}^n$ where $T'=\lambda T$. Applying (27) to A', we infer

$$\lambda^{1+\frac{2}{n}}\int_0^T dt\int_{\mathbb{R}^n}(\det A)^{\frac{1}{n}}dy \le \frac{1}{(n+1)|S^n|^{\frac{1}{n}}}\left(\|(\lambda^2\rho,\lambda m)(0)\|_{L^1(\mathbb{R}^n)}+\|(\lambda^2\rho,\lambda m)(T)\|_{L^1(\mathbb{R}^n)}\right)^{1+\frac{1}{n}}.$$

Simplifying by λ and then defining $\lambda =: \mu^{n+1}$, this becomes

$$\begin{aligned}\int_0^T dt\int_{\mathbb{R}^n}(\det A)^{\frac{1}{n}}dy &\le \frac{1}{(n+1)|S^n|^{\frac{1}{n}}}\left(\|(\mu^n\rho,\mu^{-1}m)(0)\|_{L^1(\mathbb{R}^n)}+\|(\mu^n\rho,\mu^{-1}m)(T)\|_{L^1(\mathbb{R}^n)}\right)^{1+\frac{1}{n}}\\ &\le \frac{1}{(n+1)|S^n|^{\frac{1}{n}}}\left(2\mu^nM_0+\mu^{-1}(\|m(0)\|_{L^1(\mathbb{R}^n)}+\|m(T)\|_{L^1(\mathbb{R}^n))}\right)^{1+\frac{1}{n}}.\end{aligned}$$

We are free to choose the parameter $\mu>0$, and we make the choice

$$\lambda=\mu^{n+1}=\frac{\|m(0)\|_{L^1(\mathbb{R}^n)}+\|m(T)\|_{L^1(\mathbb{R}^n))}}{M_0}.$$

This yields the estimate in Theorem 3.1.

6.2 The Euler and kinetic equations

For the Euler equation, we only have to remark that A is positive semi-definite and $\det A = \rho p^n$.

Likewise, for a kinetic equation, we only have to calculate the determinant of

$$A(t,y) = \begin{pmatrix} \int_{\mathbb{R}^n} f(t,y,v)\,dv & \int_{\mathbb{R}^n} f(t,y,v)v^T\,dv \\ \int_{\mathbb{R}^n} f(t,y,v)v\,dv & \int_{\mathbb{R}^n} f(t,y,v)v\otimes v\,dv \end{pmatrix}.$$

The formula

$$\begin{vmatrix} \int_{\mathbb{R}^n} f(v)\,dv & \int_{\mathbb{R}^n} f(v)v^T\,dv \\ \int_{\mathbb{R}^n} f(v)v\,dv & \int_{\mathbb{R}^n} f(v)v\otimes v\,dv \end{vmatrix} = \frac{1}{d!}\int\cdots\int_{(\mathbb{R}^n)^{n+1}} f(v^0)\cdots f(v^n)(\Delta(v^0,\dots,v^n))^2 dv^0\cdots dv^n$$

is a particular case of the more general one

$$\det\left(\left(\int \phi_i\phi_j d\mu(v)\right)\right)_{1\le i,j\le N} = \frac{1}{N!}\int^{\otimes N}\left(\det((\phi_i(v_j)))_{1\le i,j\le N}\right)^2 d\mu(v_1)\cdots d\mu(v_N). \tag{28}$$

To prove (28), we develop the left-hand side as

$$\sum_{\sigma\in\mathfrak{S}_N}\varepsilon(\sigma)\prod_i\int\phi_i(v)\phi_{\sigma(i)}(v)\,dv$$

and write

$$\prod_i\int\phi_i(v)\phi_{\sigma(i)}(v)\,dv = \frac{1}{N!}\int^{\otimes N}\sum_{\rho\in\mathfrak{S}_N}\prod_i\phi_i(v^{\rho(i)})\phi_{\sigma(i)}(v^{\rho(i)})\,d\mu(v_1)\cdots d\mu(v_N).$$

There remains to verify

$$\sum_{\sigma\in\mathfrak{S}_N}\varepsilon(\sigma)\sum_{\rho\in\mathfrak{S}_N}\prod_i\phi_i(v^{\rho(i)})\phi_{\sigma(i)}(v^{\rho(i)}) = \left(\sum_{\lambda\in\mathfrak{S}_N}\varepsilon(\lambda)\prod_i\phi_i(v^{\lambda(i)})\right)^2,$$

which is immediate.

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