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ON THE BROWNIAN SEPARABLE PERMUTON

MICKAËL MAAZOUN

Abstract. The Brownian separable permuton is a random probability measure on the unit square, which was introduced by Bassino, Bouvel, Féray, Gerin, Pierrot (2016) as the scaling limit of the diagram of the uniform separable permutation as size grows to infinity. We show that, almost surely, the permuton is the pushforward of the Lebesgue measure on the graph of a random measure-preserving function associated to a Brownian excursion whose strict local minima are decorated with i.i.d. signs. As a consequence, its support is almost surely totally disconnected, has Hausdorff dimension one, and enjoys self-similarity properties inherited from those of the Brownian excursion. The density function of the averaged permuton is computed and a connection with the shuffling of the Brownian continuum random tree is explored.

1. Introduction

For $n \geq 1$, let $\mathfrak{S}_n$ be the set of permutations of $[\![1, n]\!]$, and $\mathfrak{S} = \sqcup_{n\geq 1}\mathfrak{S}_n$. We use the one line notation $\sigma = (\sigma(1)\,\sigma(2)\,\cdots\,\sigma(n))$ for $\sigma \in \mathfrak{S}_n$. A *pattern* in a permutation $\sigma \in \mathfrak{S}_n$ induced by the indices $1 \leq i_1 < \ldots i_k \leq n$ is the permutation $\pi \in \mathfrak{S}_k$ that is order-isomorphic to the word $(\sigma(i_1), \ldots, \sigma(i_k))$. The *density* of the pattern $\pi \in \mathfrak{S}_k$ in $\sigma \in \mathfrak{S}_n$ is the proportion of increasing k-uples in $[\![1, n]\!]$ that induce π in σ. A *class* of permutations is a subset of $\mathfrak{S}$ that is stable by pattern extraction, and is characterized by the pattern avoidance of some minimal family of permutations called its *basis* [8, 5.1.2]. There is a large literature on the asymptotics of the pattern densities and diagram shape of a large typical permutation in several classes. This type of results can, to some extent, be encoded as convergence to a *permuton*. In [6] (to which we refer the reader for an extensive review of literature), Bassino *et. al.* studied the class of *separable permutations* and showed the convergence, of a uniform large separable permutation to a *Brownian separable permuton*, and the present paper is a detailed study of this object. Let us start with a few definitions.

1.1. **Limits of permutations.** A probability measure on the unit square $[0, 1]^2$ is called a *permuton* if both its marginals on $[0, 1]$ are uniform. With every permutation $\sigma \in \mathfrak{S}_n$ we associate a permuton μ_σ by setting $\mu_\sigma(dxdy) = n\,\mathbb{1}\,[\sigma(\lfloor xn \rfloor) = \lfloor yn \rfloor]\,dxdy$. The set of permutons is equipped with the weak convergence of probability measures, which makes it compact. A sequence of permutations $(\sigma_n)_n$ is said to converge to a permuton μ if and only if μ_{σ_n} converges weakly to μ. This theory was introduced by Hoppen *et. al.* in [14], where it is shown that convergence of a sequence of permutations to a permuton is equivalent to convergence of all pattern densities. As a result, permutons can be alternatively constructed as the completion of the space of permutations w.r.t. convergence of all pattern

densities. This theory is similar to graphons as limits of dense graphs, and unifies the study of the limit shape of the permutation diagram with that of the limit of pattern densities.

1.2. **The case of separable permutations.** A permutation is separable if it does not have (2413) and (3142) as an induced pattern. Separable permutations were introduced in [10], but appeared earlier in the literature [4, 20]. They are counted by the large Schröder numbers: $1, 2, 6, 22, 90, 394, \ldots$ and enjoy many simple characterizations [10, 4, 20, 13]

The one most relevant to this paper is in terms of trees. A signed tree t is an rooted plane tree whose internal nodes are decorated with signs in $\{\oplus, \ominus\}$. We label its leaves $1, \ldots, k$ according to the natural ordering of t. The signs can be interpreted as coding a different ordering of the rooted tree t: we call $\tilde{t}$ the tree obtained from t by reversing the order of the children of each node with a minus sign. The order of the leaves is changed by this procedure, and we set $\sigma(i)$ to be the position in $\tilde{t}$ of the leaf i. We call $\mathrm{perm}(t)$ this permutation $\sigma \in \mathfrak{S}_k$. It turns out [10, Lemma 3.1] that separable permutations are exactly the ones that can be obtained this way.

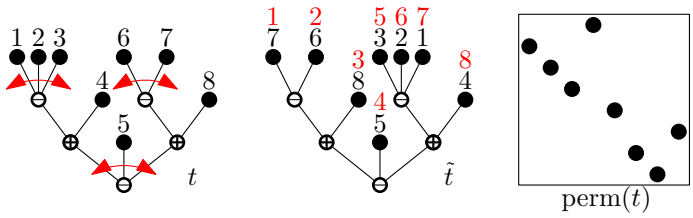

FIGURE 1. The permutation associated to a signed tree.

The article [6] shows that separable permutations have a permuton limit in distribution, yielding the first example of a nondeterministic permuton limit of a permutation class. The representation by signed trees is fundamental in their proof.

Theorem 1.1 (theorem 1.6 of [6])**.** *If σ_n is a uniform separable permutation of size n, then μ_{σ_n} converges in distribution, in the weak topology, to a non-deterministic permuton $\mu^{1/2}$ called the Brownian Separable Permuton of parameter* $1/2$.

This result comes with a characterization of $\mu^{1/2}$ (which we recall in section 2) which suggests that it can be realized as a measurable functional of a *signed Brownian excursion* (see remark 2.7). The authors of [6] left this, along with the study of the support of $\mu^{1/2}$, as open questions that the present paper aims at addressing.

Let us mention that theorem 1.1 was generalized in [5] by the same authors along with the present author to a rather wide range of permutation classes called substitution-closed classes. It yields, among others, a one-parameter family $(\mu^p)_{p\in(0,1)}$ of possible limits, called the *biased Brownian separable permutons*. We set our paper in this generality and fix once and for all $p \in (0, 1)$. We postpone a precise definition of μ^p to section 2.

1.3. **The signed Brownian excursion.** We call *continuous excursion* a nonnegative function $g : [0, 1] \to \mathbb{R}_+$ that is positive on $(0, 1)$. The *inner local minima* of g are the

points of $(0,1)$ in which g is locally minimal, and we say that $x \in (0,1)$ is *not a one-sided minimum* of g if

$$\forall \epsilon > 0, \exists x_1 \in (x-\epsilon, x), x_2 \in (x, x+\epsilon) \text{ s.t. } g(x_1) < g(x) \text{ and } g(x_2) < g(x).$$

A *CRT excursion* is a continuous function $g : [0,1] \to \mathbb{R}_+$ such that:

(CRT1) the inner local minima of g are dense in $[0,1]$,
(CRT2) the values at the inner local minima are all different,
(CRT3) the set of times that are not one-sided minima has Lebesgue measure 1.

In a CRT excursion, all inner local minima are necessarily strict local minima, and hence countable. It will be useful for our purposes to enumerate them in a well-defined manner.

Definition 1.2. A *measurable enumeration* is a sequence $(b_i)_{i\in\mathbb{N}}$ of functions from the set $\mathcal{E}_{\mathrm{CRT}}$ of CRT excursions to $[0,1]$ such that

(ME1) for every $g \in \mathcal{E}_{\mathrm{CRT}}$, $i \mapsto b_i(g)$ is a bijection between $\mathbb{N}$ and the inner local minima of g,
(ME2) for every $i \in \mathbb{N}$, $g \mapsto b_i(g)$ is measurable,
(ME3) the function which maps $(g,u,v) \in \mathcal{E}_{\mathrm{CRT}} \times [0,1]^2$ to $i \in \mathbb{N}$ if $b_i \in (u,v)$ is the unique point in $[u,v]$ in which the minimum of g on $[u,v]$ is reached, and ∞ otherwise, is measurable.

We fix once and for all a measurable enumeration (see section 2 for an explicit construction of one, which comes from [6]). We call *signed excursion* a couple (g,s), where g is a CRT excursion and s is a sequence in $\{\oplus, \ominus\}^{\mathbb{N}}$. The sign s_i is to be considered as attached to the inner local minimum b_i.

In what follows, we consider the signed excursion (e,S), where e is a the normalized Brownian excursion, and S is an independent sequence of independent signs with bias p, that is probability p of being $\oplus$ and $1-p$ of being $\ominus$.

The signed excursion (e,S) is the main ingredient in building μ^p. To that end, if $x < y \in [0,1]$, we say that x and y are g-comparable if and only if the minimum of g on $[x,y]$ is reached at a unique point which is a strict local minimum $b_i \in (x,y)$. In this case, if $s_i = \oplus$, we say $x \lhd_g^s y$, otherwise $y \lhd_g^s x$. The relation $\lhd_g^s$ is not a strict partial order, because it lacks the property of transivity. However we will see later that it can be extended to a total preorder. Moreover, almost surely, the pairs of points that are g-comparable have full Lebesgue measure, since two distinct points which are not one-sided minima are always g-comparable.

1.4. **Main results.** If (g,s) is a signed excursion, we define

$$\varphi_{g,s}(t) = \mathrm{Leb}\{u \in [0,1], u \lhd_g^s t\}, \quad t \in [0,1] \tag{1}$$

and $\mu_{g,s} = (\mathrm{Id}, \varphi_{g,s})_* \mathrm{Leb}$. Here $H_*\nu$ denotes the pushforward measure $\nu(H^{-1}(\cdot))$, whenever H and ν are respectively a measurable function and a measure defined on the same space. Our main theorem is the following:

Theorem 1.3. *The maps $(t, g, s) \mapsto \varphi_{g,s}(t)$ and $(g, s) \mapsto \mu_{g,s}$ are measurable, and the random measure $\mu_{e,S}$ is distributed like μ^p, the biased Brownian separable permuton of parameter p.*

This theorem is proved in section 3, along with a corollary which shows that the convergence of theorem 1.1 can be rewritten without permutons, only in terms of functional convergence. To any permutation $\sigma \in \mathfrak{S}_n$, we associate a càdlàg, piecewise affine, measure-preserving function $\varphi_\sigma : [0,1] \to [0,1]$ with $\varphi_\sigma(x) = \frac{1}{n}(\sigma(\lfloor nt \rfloor + 1) - 1) + \frac{1}{n}\{nt\}$.

Corollary 1.4. *Let σ_n be a random permutation in $\mathfrak{S}_n$ for every $n \in \mathbb{N}$. If μ_{σ_n} converges in distribution to μ^p, then for every $q \in [1, \infty)$, we have the convergence in distribution in the space $L^q([0,1])$:*

$$\varphi_{\sigma_n} \xrightarrow[n\to\infty]{d} \varphi_{e,S}$$

This function $\varphi_{e,S}$ is well-defined as a random element of $L^\infty([0,1])$, and although it has a dense set of discontinuity points, it is continuous at every point which is not a one-sided minimum of e (i.e. on a set which has almost surely measure 1). In section 4, we prove the following result.

Theorem 1.5. *Almost surely, the support of μ^p is totally disconnected, and its Hausdorff dimension is 1 (with one-dimensional Hausdorff measure bounded above by $\sqrt{2}$).*

The claim that the Hausdorff dimension is 1 also comes as a special case of a result of Riera [19]: any permuton limit in distribution of random permutation in a proper class, if it exists, almost surely has a support of Hausdorff dimension 1.

In section 5, we show that μ^p inherits the self-similarity properties of e, in that μ^p contains a lot of rescaled distributional copies of itself. In particular, we get the following theorem, illustrated in fig. 2, which states that μ^p can be obtained by cut-and-pasting three independent Brownian separable permutons.

Theorem 1.6. *Let $(\Delta_0, \Delta_1, \Delta_2)$ be a random variable of* $\mathrm{Dirichlet}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ *distribution. Let μ_0, μ_1, μ_2 be independent and distributed like μ^p, and conditionally on μ_0, let (X_0, Y_0) be a random point of distribution μ_0. Let β be an independent Bernouilli r.v. of parameter p. We define the piecewise affine maps of the unit square into itself:*

$$\begin{aligned} \theta_0(x,y) &= (\eta_0(x), \zeta_0(y)) = \Delta_0(x,y) + (1-\Delta_0)(\mathbb{1}_{[x>X_0]}, \mathbb{1}_{[y>Y_0]}) \\ \theta_1(x,y) &= (\eta_1(x), \zeta_1(y)) = \Delta_1(x,y) + \Delta_0(X_0,Y_0) + \Delta_2(0,\beta) \\ \theta_2(x,y) &= (\eta_2(x), \zeta_2(y)) = \Delta_2(x,y) + \Delta_0(X_0,Y_0) + \Delta_1(1,1-\beta) \end{aligned} \tag{2}$$

Then

$$\Delta_0\theta_{0*}\mu_0 + \Delta_1\theta_{1*}\mu_1 + \Delta_2\theta_{2*}\mu_2 \overset{d}{=} \mu^p, \tag{3}$$

We believe that a result by Albenque and Goldschmidt [1] about the Brownian CRT can be adapted to show that the *distributional identity* (3) characterizes μ^p (see remark 5.5.)

Finally, our construction allows us to compute the averaged permuton $\mathbb{E}\,\mu^p$, obtained by taking $\mathbb{E}\,\mu^p(A) = \mathbb{E}[\mu^p(A)]$. We get the following result.

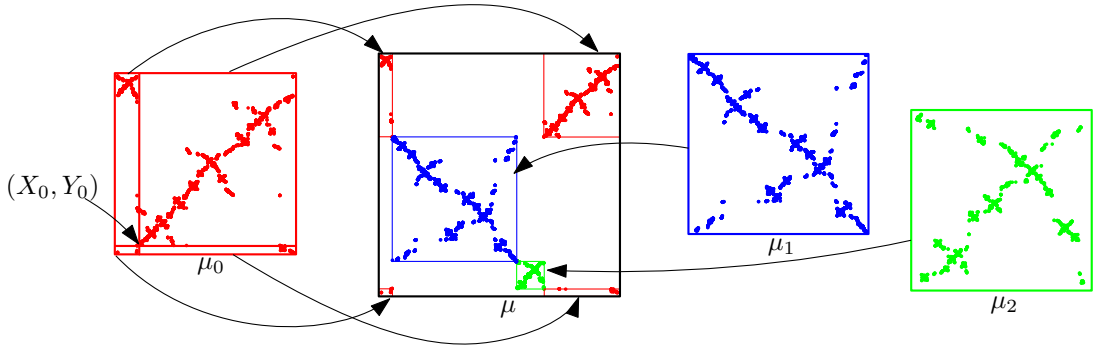

FIGURE 2. The construction of μ from three independent permutons distributed like μ. Here $\beta = 0$ and $(\Delta_0, \Delta_1, \Delta_2) \approx (0.4, 0.5, 0.1)$.

Theorem 1.7. *The permuton $\mathbb{E}\,\mu^p$ is absolutely continuous w.r.t. the Lebesgue measure on the unit square, with density function at (x, y) equal to*

$$\int_{\max(0,x+y-1)}^{\min(x,y)} \frac{3p^2(1-p)^2 da}{2\pi(a(x-a)(1-x-y+a)(y-a))^{3/2}\left(\frac{p^2}{a} + \frac{(1-p)^2}{(x-a)} + \frac{p^2}{(1-x-y+a)} + \frac{(1-p)^2}{(y-a)}\right)^{5/2}}.$$

We discuss a relationship with an existing result by Dokos and Pak on doubly-alternating Baxter permutations at the end of this section.

1.5. **Shuffling of continuous trees.** Through a classical construction (see [16]), a continuous excursion g encodes a continuous rooted tree equipped with a total ordering (analoguous to the depth-first search order of a discrete tree) and a probability measure. This is done by setting $d_g(x, y) = g(x) + g(y) - 2\min_{[x,y]} g$ on $[0, 1]$ and identifying points $x, y \in [0, 1]$ such that $d_g(x, y) = 0$. This yields a quotient metric space $(\mathcal{T}_g, d_g)$ with a continuous canonical surjection $p_g : [0, 1] \mapsto \mathcal{T}_g$. The root is $\rho = p_g(0)$, the order is defined by $x \leq_g y \iff \inf p_g^{-1}(x) \leq \inf p_g^{-1}(y)$, and the measure is $\lambda_g = p_{g*}\,\mathrm{Leb}_{[0,1]}$. When $g = e$, we get the well-known Brownian CRT. Section 7 is devoted to the proof of the following theorem, illustrated in fig. 3.

Theorem 1.8. *There exists a random CRT excursion $\tilde{e}$, defined on the same probability space as (e, S), with the following properties:*

(1) The excursion $\tilde{e}$ has the distribution of a normalized Brownian excursion, with the same field of local times at time 1 as e.
(2) Almost surely, the function $\varphi_{e,S}$ is an isometry between the pseudo-distances d_e and $d_{\tilde{e}}$. In particular, $\tilde{e} \circ \varphi_{e,S} = e$.

This result has an interpretation in terms of shuffling of continuous trees, mirroring the construction of separable permutations described in section 1.2.

When g is a CRT excursion, the construction of $\mathcal{T}_g$ puts the strict local minima of g in bijection with the branching points of $\mathcal{T}_g$. Hence, when (g, s) is a signed excursion, the order $\leq_g^s$ can be defined on the tree $\mathcal{T}_g$ by inverting at all branching points with a minus sign,

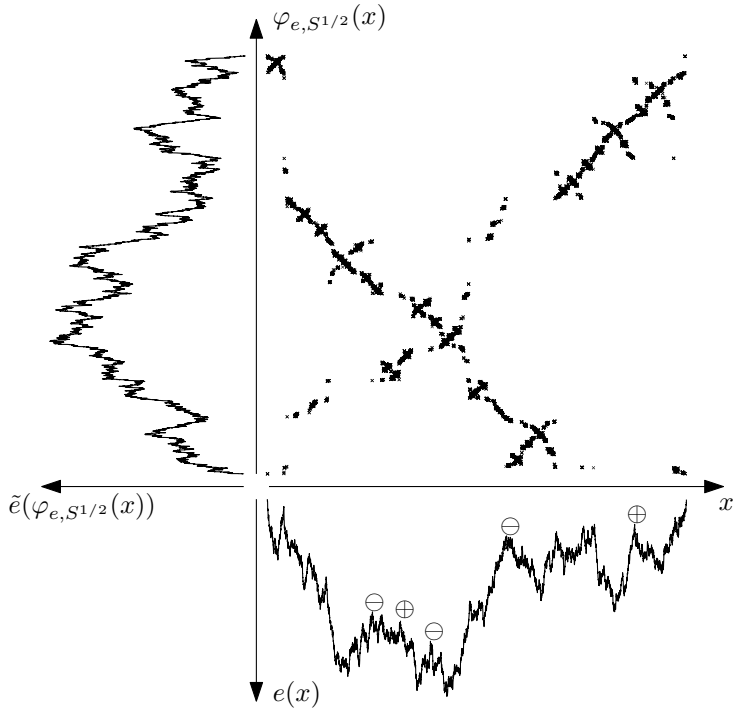

FIGURE 3. A realization of (e, S) (here $p = 1/2$), and the associated functions $\varphi_{e,S}$ and $\tilde{e}$, highlighting the property $\tilde{e} \circ \varphi_{e,S} = e$.

as follows. Let $x, y \in \mathcal{T}_g$ such that $x \leq_g y$. If there exists a strict local minimum b_i such that $\sup p_g^{-1}(x) < b_i < \inf p_g^{-1}(y)$, with $g(b_i) = \inf\{g(t), \sup p_g^{-1}(x) \leq t \leq \inf p_g^{-1}(y)\}$, and $s(b_i) = \ominus$, then set $x \geq_g^s y$. Otherwise, set $x \leq_g^s y$. This defines a total order compatible with the relation on $[0, 1]$ defined in the previous section: when x and y are g-comparable, then $x \lhd_g^s y \iff p_g(x) <_g^s p_g(y)$.

This allows us to give an interpretation of theorem 1.8 in terms of trees. If we consider the tree $(T_{\tilde{e}}, d_{\tilde{e}}, \rho_{\tilde{e}}, \leq_{\tilde{e}}, \lambda_{\tilde{e}})$, from theorem 1.8(2) we deduce, for $x, y \in [0, 1]$, that

$$p_e(x) = p_e(y) \iff d_e(x, y) = 0 \iff d_{\tilde{e}}(\varphi_{e,S}(x), \varphi_{e,S}(y)) = 0 \iff p_{\tilde{e}}(\varphi_{e,S}(x)) = p_{\tilde{e}}(\varphi_{e,S}(y)).$$

So there is a unique map $\jmath : \mathcal{T}_e \to \mathcal{T}_{\tilde{e}}$ such that $\jmath \circ p_e = p_{\tilde{e}} \circ \varphi_{e,S}$. It is immediate than $\jmath$ is an isometry $(\mathcal{T}_e, d_e) \leftrightarrow (\mathcal{T}_{\tilde{e}}, d_{\tilde{e}})$. Moreover, $\jmath$ maps the root of $\mathcal{T}_e$ to the root of $\mathcal{T}_{\tilde{e}}$, is measure preserving and increasing w.r.t. $(\leq_e^S, \leq_{\tilde{e}})$. This discussion can be summarized in the following corollary of theorem 1.8.

Proposition 1.9. *The map $\jmath : \mathcal{T}_e \leftrightarrow \mathcal{T}_{\tilde{e}}$ provides an isomorphism (of pointed, ordered, measured metric spaces) between the tree $(T_e, d_e, \rho_e, \leq_e^S, \lambda_e)$ and the Brownian CRT $(T_{\tilde{e}}, d_{\tilde{e}}, \rho_{\tilde{e}}, \leq_{\tilde{e}}, \lambda_{\tilde{e}})$ constructed from the Brownian excursion $\tilde{e}$.*

Combining this with the result of Duquesne on the uniqueness of coding functions of trees [12, Thm 1.1], we directly get an abstract construction of $\mu_{e,S}$.

Proposition 1.10. *Almost surely, the functions $\tilde{e}$ and $\jmath$ are uniquely determined by the fact that $\tilde{e}$ is continuous and $\jmath$ is an isomorphism between $(T_e, d_e, \rho_e, \leq_e^S, \lambda_e)$ and $(T_{\tilde{e}}, d_{\tilde{e}}, \rho_{\tilde{e}}, \leq_{\tilde{e}}, \lambda_{\tilde{e}})$. Any function ϕ which verifies $p_{\tilde{e}} \circ \phi = \jmath \circ p_e$ must coincide with $\varphi_{e,S}$ on a set of measure 1, hence still verifies $\mu_{e,S} = (\mathrm{Id}, \phi)_* \mathrm{Leb}$.*

1.6. **Comments and perspectives.** Let us mention another natural family of permutations: the doubly-alternating Baxter permutations, which are also the doubly- alternating separable permutations [17], and are counted by the Catalan numbers. The fact that they enjoy a tree decomposition similar to separable permutations, along with simulations [11], allows to boldly conjecture that they converge to the Brownian separable permuton of parameter 1/2. Under the conjecture just stated, the main result of Dokos and Pak [11, Thm 1.1] would provide another expression for the density function α of $\mathbb{E}\,\mu^{1/2}$: for $0 \leq x \leq \min(y, 1-y)$,

$$\alpha(x,y) = \frac{dxdy}{4\pi} \int_0^x du \int_0^{x-u} dv \frac{1}{[(u+v)(y-v)(1-y-u)]^{3/2}},$$

the values on whole unit square being recovered through the invariance under isometries of the square. We were unable to find a direct analytical proof of the equality with the expression of theorem 1.7 for $p = 1/2$.

As already mentioned, the article [5] considers *substitution-closed classes*, which are natural generalizations of the class of separable permutations. Depending on the class, several possible limits appear, among which are the μ^p for p possibly different from 1/2. Another family of possible limits is *the α-stable permuton driven by ν*, for $\alpha \in (1,2)$ and ν itself a random permuton. We believe a continuum construction similar to the one presented here is possible, by considering a α-stable tree, with an independent copy of ν at each branching point, driving the reordering of the (countably infinite number of) branches stemming from that point.

The structure of the paper is as follows. Section 2 contains various definitions that will be needed in the rest of the paper, notably a characterization of μ^p through its marginals, that highlight the link with the signed excursion. Section 3 contains the proof of theorem 1.3, along with some facts about the random function $\varphi_{e,S}$ that are reused later. Sections 4 to 7 are respectively devoted to the proofs of theorems 1.5 to 1.8.

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2. Definitions

First we set a few notations : if $x_1, \ldots, x_k$ are strictly comparable elements of an ordered set $(E, \leq)$, then $\mathrm{rank}_{\leq}(x_1, \ldots, x_k)$ is the permutation α such that $\alpha(i) < \alpha(j) \iff x_i < x_j$ for every $1 \leq i, j \leq k$. The sequence $(x_{\alpha^{-1}(1)} < \ldots < x_{\alpha^{-1}(k)})$ is called the order statistic of $(x_1, \ldots, x_k)$ and denoted $(x_{(1)} < \ldots < x_{(k)})$.

2.1. **Marginals of a permuton.** In this section we want to give a tractable definition of the random permuton μ^p. This will take the form of a characterization through its finite-dimensional marginals, which we define now. If $k \geq 1$ and μ is a random permuton, let $\mathrm{subperm}_k(\mu) = \mathrm{rank}(Y_1, \dots Y_k) \circ \mathrm{rank}(X_1, \dots X_k)^{-1} \in \mathfrak{S}_k$, where conditionally on μ, the (X_i, Y_i) for $i \in [\![1, k]\!]$ are independent and distributed according to μ. Then the distribution of $\mathrm{subperm}_k(\mu)$ is called the *k-dimensional marginal* of μ. The interest of this definition lies in the following result, which is an extension of the main theorem of [14] to random permutations. It

Proposition 2.1 (theorem 2.2 of [5])**.** *If σ_n is a sequence of (possibly random) permutation whose size goes to infinity, then σ_n converges to some random permuton μ in distribution if and only if for every permutation π, $\mathbb{E}[\mathrm{occ}(\pi, \mu)]$ converges to some number Δ_π.*

In this case, the law of μ is characterized by the relations $\mathbb{P}(\mathrm{subperm}_k(\mu) = \pi) = \Delta_\pi$, for $k \geq 1$ and $\pi \in \mathfrak{S}_k$.

This is indeed the result used by [6] and [5] to prove permuton convergence. As a result, the distribution of $\mathrm{subperm}_k(\mu^p)$ for every k is obtained as follows (see [6, prop. 9.1] and [5, def. 5.1])

Definition 2.2. The permuton μ^p is determined by the relations

$$\forall k \geq 1, \quad \mathrm{subperm}_k(\mu^p) \stackrel{d}{=} \mathrm{perm}(t_{k,p}), \tag{4}$$

where $t_{k,p}$ is a uniform binary tree with k leaves, whose internal vertices are decorated with i.i.d. signs that are $\oplus$ with probability p.

In the rest of the section, we make apparent a connection with the signed Brownian excursion.

2.2. **A few facts about excursions.** We start by constructing a measurable enumeration as defined in definition 1.2. Let $(p_i, q_i)_{i \in \mathbb{N}}$ be a fixed enumeration of $\mathbb{Q}^2 \cap [0, 1]$. Let g be a CRT excursion. For $i \geq 1$, define $w_i = \min\{t \in [p_i, q_i] : g(t) = \min_{[p_i, q_i]} g\}$, $i_0 = 0$, and for $k \geq 1$, set recursively

$$i_k = \inf\{i > i_{k-1}, w_i \in (p_i, q_i)\ ,\ w_i \notin \{w_1, \dots, w_{i_{k-1}}\}\}.$$

Finally, for $k \in \mathbb{N}$, set $b_k(g) = w_{i_k}$.

Lemma 2.3. *This construction defines a measurable enumeration.*

Proof. It is immediate that all inner local minima will appear in the sequence $(w_i)_i$. The way the subsequence $(b_i)_i$ of $(w_i)_i$ is chosen guarantees that only inner local minima appear, and only once, in $(b_i)_i$.

Measurability of $g \mapsto b_i(g)$ for every i follows from that of $g \mapsto w_i(g)$ and $k \mapsto i_k$.

To prove (ME3) we see that thanks to item (CRT2), the function $\mathcal{E}_{\mathrm{CRT}} \times [0, 1]^2 \to \mathbb{N} \cup \{\infty\}$

$$(g, x, y) \mapsto \min\left\{ i \in \mathbb{N}, g(b_i(g)) = \min_{[x,y]} g \text{ and } b_i(g) \in (x, y) \text{ and } \min_{[x,y]} g < \min(g(x), g(y)) \right\}$$

is a measurable functional that maps (g, x, y) to $i \in \mathbb{N}$ whenever b_i is the point in (x, y) that is the only global minimum of g on $[x, y]$, and ∞ if no such $i \in \mathbb{N}$ exists. $\square$

We now collect a few facts about CRT excursions. In section 1.5 we saw a that such functions encode continuous trees. So we borrow the vocabulary of trees in a way that is coherent with this encoding: the $x \in [0,1]$ which are not one-sided local minima are called *leaves of* g. The b_i for $i \in \mathbb{N}$ are called *branching points of* g and are identified with $\mathbb{N}$. Set

$$\begin{aligned} a_i &= \sup\{t < b_i : g(t) = g(b_i)\}, \\ c_i &= \inf\{t > b_i : g(t) = g(b_i)\}; \\ h_i &= g(b_i) = g(c_i) = g(a_i). \end{aligned}$$

By definition, for $x \in (a_i, b_i) \cup (b_i, c_i)$, $g(x) \geq h_i$, defining two subexcursions at respectively the left and the right of b_i. We collect an immediate consequence of (CRT2), which states that these subexcursions are nested, with a binary tree structure (which comes from that of $\mathcal{T}_g$).

Lemma 2.4. *For every i, j either $[a_i, c_i] \subset [a_j, c_j]$ or $[a_j, c_j] \subset [a_i, c_i]$ or $[a_i, c_i] \cap [a_j, c_j] = \emptyset$. Furthermore, if $[a_j, c_j] \subset [a_i, c_i]$, then either $j = i$, $[a_j, c_j] \subset (a_i, b_i)$ or $[a_j, c_j] \subset (b_i, c_i)$.*

If $x < y$ are g-comparable, the b_i in which g reaches its minimum between x and y at b_i is called the *most recent common ancestor* of x and y. We extend this notion to branching points: if $[a_i, c_i] \cap [a_j, c_j] = \emptyset$, then b_i and b_j are g-comparable. We can always assume by symmetry that $b_i < b_j$ and call *most recent common ancestor* of i and j the $k \in \mathbb{N}$ such that $[a_i, c_i] \subset (a_k, b_k)$ and $[a_j, c_j] \subset (b_k, c_k)$.

2.3. **Extraction of permutations and trees from a signed excursion.** Let (g, s) be a signed excursion. Recall that x and y are g-comparable if the minimum of g on $[x, y]$ is reached at a unique point, and that point b is a strict local minimum with $b \in (x, y)$. If $x_1, \dots x_n$ are points of $[0, 1]$, pairwise g-comparable, then we define

$$\mathrm{Perm}_{g,s}(x_1, \dots x_n) = \mathrm{rank}_{\lhd_g^s}(x_{(1)}, \dots, x_{(n)}).$$

To understand the structure of these permutations, let us define the (signed) trees extracted from a (signed) excursion. Following Le Gall [16], when g is a CRT excursion and $t_1 < \dots < t_k$ are pairwise g-comparable[1], the discrete plane tree with edge-lengths $\tau(g, t_1, \dots, t_k)$ is constructed recursively as follows:

- If $k = 1$, then $\tau(g, t_1)$ is a leaf labeled t_1.
- If $k \geq 2$, then the minimum of g on $[t_1, t_k]$ is reached at a strict local minimum b_i for some i, and there is $j \in [\![2, k]\!]$ such that $\{t_1, \dots t_{j-1}\} \subset (a_i, b_i)$, and $\{t_j, \dots t_k\} \subset (b_i, c_i)$. Then $\tau(g, t_1, \dots, t_k)$ is a root labeled i, spanning two subtrees $\tau(g, t_1, \dots, t_{j-1})$ and $\tau(g, t_j, \dots, t_k)$.

This yields a binary tree whose internal vertices are put in correspondence with branching points of g. Then, if (g, s) is a signed excursion, we set $\tau^{\pm}(g, s, t_1, \dots t_k)$ to be the tree $\tau(g, t_1, \dots t_k)$, to which we add, at each internal node labeled i, the sign s_i. The following observation is capital:

[1]The definition there is stated differently and covers any continuous function g and choice of points $t_1, \dots t_k$

Observation 2.5. For any signed excursion (g, s) and g-comparable $x_1, \dots, x_n$,

$$\mathrm{Perm}_{g,s}(x_1, \dots x_n) = \mathrm{perm}(\tau^{\pm}(g, s, x_{(1)}, \dots, x_{(n)})).$$

If $U_1, \dots, U_k$ are independent uniform random variables in $[0, 1]$, then they are almost surely leaves of g thanks to (CRT3), hence g-comparable.

We recall that the signed Brownian excursion (e, S) is built by taking e to be a normalized Brownian excursion, and S an independent i.i.d. sequence in $\{\oplus, \ominus\}$, where $\mathbb{P}(S_1 = \oplus) = p$. Then a consequence of [16, Theorem 2.11] is that the tree $\tau^{\pm}(e, S, U_{(1)}, \dots U_{(k)})$ is a uniform binary tree with k leaves, independently decorated with i.i.d. signs of bias p. From definition 2.2 and observation 2.5 follows a new characterization of μ^p, which we use in this paper.

Proposition 2.6. *The permuton μ^p is determined by the relations*

$$\forall k \geq 1, \quad \mathrm{subperm}_k(\mu^p) \stackrel{d}{=} \mathrm{Perm}_{e,S}(U_1, \dots U_k). \tag{5}$$

Remark 2.7. This connection with the Brownian excursion was present in [6] for $p = 1/2$. The main result of that paper actually goes further: the conditional distribution of the l.h.s. given $\mu^{1/2}$ equals (in distribution) the conditional distribution of the r.h.s given (e, S), jointly for all k (see [6, thm. 1.6] and its proof). This indeed strongly hinted at the existence of a direct construction of $\mu^{1/2}$ from (e, S), made explicit in the present paper.

3. The function φ

Theorem 1.3 follows from the next two propositions.

Proposition 3.1. *If g is a CRT excursion and s a sequence of signs, then $(g, s, t) \mapsto \varphi_{g,s}(t)$ and $(g, s) \mapsto \mu_{g,s}$ are measurable. Furthermore, $\varphi_{g,s*}\,\mathrm{Leb} = \mathrm{Leb}$, hence $\mu_{g,s}$ is a permuton.*

Proof. For the measurability, remark that $((g, s, t), u) \mapsto \mathbb{1}[u \lhd_g^s t]$ is a measurable function, as a result of item (ME3). Then Fubini's theorem implies that its partial integral over u is a measurable function of (g, s, t).

Now we only have to prove that $\varphi_* \,\mathrm{Leb} = \mathrm{Leb}$. Let $(Z_i)_{i\geq 1}$ be independent uniformly distributed random variables in $[0, 1]$. For $k \geq 2$, let $U_{1,k} = \frac{1}{k-1}\#\{i \in [\![2, k]\!] : Z_i \lhd_g^s Z_1\}$ and $U_1 = \lim_{k\to\infty} U_{1,k}$. We can apply the law of large numbers conditionally on Z_1 to the sequence $\mathbb{1}_{Z_2 \lhd_g^s Z_1}, \mathbb{1}_{Z_3 \lhd_g^s Z_1}, \dots$ (which is i.i.d given Z_1) to show that this limit is well defined and equal almost surely to $\mathrm{Leb}\{t : t \lhd_g^s Z_1\} = \varphi(Z_1)$. This means that U_1 has distribution $\varphi_* \,\mathrm{Leb}$. On the other hand, by exchangeability of the Z_i, the $U_{1,k}$ are uniform over $\{\frac{1}{k-1}, \dots, \frac{k-1}{k-1}\}$ so the distribution of the limit U_1 must be uniform. This means precisely that $\varphi_* \,\mathrm{Leb} = \mathrm{Leb}$. □

Proposition 3.2. *The Brownian separable permuton μ^p is distributed like $\mu_{e,S}$.*

Proof. By definition of $\mu_{g,s}$, $\mathrm{subperm}_k(\mu_{e,S})$ can be realized as $\mathrm{rank}(\mathbf{Y}) \circ \mathrm{rank}(\mathbf{X})^{-1}$ where $X_1, \dots X_k$ are independent uniform in $[0, 1]$ and $Y_i = \varphi_{e,S}(X_i)$ for $i \in [\![1, k]\!]$ Since $x \lhd_e^S y$ implies $\varphi_{e,S}(x) \leq \varphi_{e,S}(y)$, and moreover since the Y_i are almost surely distinct, then almost surely $\mathrm{subperm}_k(\mu_{e,S}) = \mathrm{Perm}_{e,S}(X_1, \dots X_k)$. According to proposition 2.6, this property characterizes μ^p among permutons. □

We now collect a few results about the excursion and the function φ. The first one states that $[0,1]$ can almost be covered by a union of small subexcursions.

Lemma 3.3. *Let g be a CRT excursion, and $\delta > 0$, $\epsilon > 0$. There exists a finite $I \subset \mathbb{N}$ such that the $([a_i, c_i])_{i\in I}$ are disjoint, $c_i - a_i \le \epsilon$ for every i, and* $\mathrm{Leb}(\bigsqcup_{i\in I}[a_i, c_i]) = \sum(c_i - a_i) > 1 - \delta$.

Proof. Let x be a leaf of the excursion g. Let $x_0 < x$ be another leaf. Define recursively b_{k_n} to be the most recent common ancestor of x_n and x, and x_{n+1} to be a leaf in $(\max\{b_{k_n}, x - \frac{1}{n}\}, x)$. This is possible by density of the leaves. Then necessarily $x \in [a_{k_n}, c_{k_n}]$ and a_{k_n} converges to x. Hence $g(c_{k_n}) = g(a_{k_n})$ converges to $g(x)$, which implies that $c_{k_n} - a_{k_n}$ converges to 0 (otherwise x couldn't be a leaf). Hence there must be a i such that $|c_i - a_i| \le \epsilon$ and $x \in [a_i, c_i]$.

We deduce that $\bigcup_{i:c_i-a_i\le\epsilon}[a_i, c_i]$ has measure 1. So a finite union can be found with measure $\ge 1 - \delta$. Now thanks to lemma 2.4, this union can be readily rewritten as a disjoint union. □

Now we want to characterize how the function $\varphi_{g,s}$ behaves on a pair of sibling subexcursions defined by an interval of the form $[a_i, c_i]$. Set $a'_i = \varphi_{g,s}(a_i)$, $c'_i = a'_i + c_i - a_i$, $b'_i = a'_i + (b_i - a_i)\,\mathbb{1}[s_i = \oplus] + (c_i - b_i)\,\mathbb{1}[s_i = \ominus]$. The numbers $a'_i, b'_i, c'_i \in [0,1]$ can be interpreted as the equivalent of a_i, b_i, c_i for the shuffled order.

Lemma 3.4. *For $i \in \mathbb{N}$, we have*

if $t \in [a_i, b_i]$ and $s_i = \oplus$, then $\varphi_{g,s}(t) = a'_i + \mathrm{Leb}\{x \in [a_i, b_i] : x \lhd^s_g t\} \in [a'_i, b'_i]$.

if $t \in [b_i, c_i]$ and $s_i = \oplus$, then $\varphi_{g,s}(t) = b'_i + \mathrm{Leb}\{x \in [b_i, c_i] : x \lhd^s_g t\} \in [b'_i, c'_i]$.

if $t \in [a_i, b_i]$ and $s_i = \ominus$, then $\varphi_{g,s}(t) = b'_i + \mathrm{Leb}\{x \in [a_i, b_i] : x \lhd^s_g t\} \in [b'_i, c'_i]$.

if $t \in [b_i, c_i]$ and $s_i = \ominus$, then $\varphi_{g,s}(t) = a'_i + \mathrm{Leb}\{x \in [b_i, c_i] : x \lhd^s_g t\} \in [a'_i, b'_i]$.

If $t \in [0, a_i) \cup (c_i, 1]$, then

$$\varphi_{g,s}(t) = \mathrm{Leb}\{x \in [0, a_i) \cup (c_i, 1] : x \lhd^s_g t\} + \mathbb{1}[a_i \lhd^s_g t](c_i - a_i) \in [0, a'_i] \cup [c'_i, 1]$$

Proof. We prove the first and last equalities, as the others have a symmetric proof. If $s_i = \oplus$, $t \in [a_i, b_i]$ and u is a leaf, then $u \lhd^s_g t$ if and only if $u \in [0, a_i) \cup (c_i, 1]$ and $u \lhd^s_g a_i$, or $u \in [a_i, b_i]$ and $u \lhd^s_g t$. The first claim follows by taking the measure of such u.

For the last equality, we see that if $t \in [0, a_i) \cup (c_i, 1]$ and $u \in [a_i, c_i]$, then $u \lhd^s_g t$ if and only if $a_i \lhd^s_g t$. □

Lemma 3.5. *If $[a_j, c_j] \subset (a_i, b_i)$, then either $s_i = \oplus$ and $[a'_j, c'_j] \subset [a'_i, b'_i]$, or $s_i = \ominus$ and $[a'_j, c'_j] \subset [b'_i, c'_i]$.*

If $[a_j, c_j] \subset (b_i, c_i)$, then either $s_i = \oplus$ and $[a'_j, c'_j] \subset [b'_i, c'_i]$, or $s_i = \ominus$ and $[a'_j, c'_j] \subset [a'_i, b'_i]$.

Proof. The four claims have a symmetrical proof, hence we only prove the first. If $s_i = \oplus$ and $[a_j, c_j] \subset (a_i, b_i)$, then the previous lemma implies readily $a'_i \le a'_j$. We need to prove $c'_j \le b'_i$, that is $a'_j + c_j - a_j \le a'_i + b_i - a_i$, which is equivalent to $a'_j - a'_i \le a_j - a_i + b_i - c_j$.

This is exactly the inequality of measures derived from the inclusion $\{x, a_i \lhd_g^s x \lhd_g^s a_j\} \subset [a_i, a_j] \sqcup [c_j, b_i]$ □

Now we can prove corollary 1.4.

Proof of corollary 1.4. We consider the Kolmogorov distance between probability measures, which is the uniform distance on the bivariate CDFs ($d_K(\nu, \pi) = \sup_{0 \le x,y \le 1} |\nu - \pi|([0,x] \times [0,y])$). We use the fact that convergence of permutons is metrized by d_K [14, lemma 5.3], and the following result:

Lemma 3.6. *If $\sigma \in \mathfrak{S}_n$, $d_K(\mu_\sigma, (\mathrm{Id}, \varphi_\sigma)_* \mathrm{Leb}) \le \frac{2}{n}$*

Proof. It is enough to notice that both CDFs coincide on points whose coordinates are entire multiples of $1/n$ and use the fact that CDFs of permutons are 1-Lipschitz [14, eq. 7] □

All together, this implies $(\mathrm{Id}, \varphi_{\sigma_n})_* \mathrm{Leb} \xrightarrow{d} (\mathrm{Id}, \varphi_{e,S})_* \mathrm{Leb}$. With the Skorokhod coupling we can assume without loss of generality, that the convergence is in fact almost sure. Let ϵ and δ be positive real numbers, and apply lemma 3.3. Then

$$\begin{aligned}\mathrm{Leb}(x : |\varphi_{\sigma_n}(x) - \varphi_{e,S}(x)| > \epsilon) \le \mathrm{Leb}(x : x \notin \bigsqcup_{i \in I} [a_i, c_i]) \\ + \mathrm{Leb}(x : \exists i \text{ s.t. } x \in [a_i, c_i], \varphi_{\sigma_n}(x) \notin [a_i', c_i'])\end{aligned}$$

The first term is smaller than δ by construction, and the second term converges to $\mathrm{Leb}(x : \exists i \text{ s.t. } x \in [a_i, c_i], \varphi_{e,S}(x) \notin [a_i', c_i']) = 0$ because of the narrow convergence of $(\mathrm{Id}, \varphi_{\sigma_n})$ to $(\mathrm{Id}, \varphi_{e,S})$ and the Portmanteau theorem (indeed permutons put no mass on the boundary of rectangles, because they have uniform marginals). So for $q \ge 1$, $||\varphi_{\sigma_n} - \varphi_{e,S}||_{L^q}^q \le \epsilon^q + \delta + o(1)$. This last quantity can be made arbitrary small by choosing first ϵ and δ small enough and then n large enough. We have proven almost sure convergence of $\varphi_{\sigma_n} \xrightarrow{L^p} \varphi_{e,S}$ in some coupling, hence the corollary. □

We end this section by considering the following property of signed excursions (g, s):

$$\text{(A)} \quad \forall i \neq j, \quad [a_j', c_j'] \subset [a_i', c_i'] \implies \{h_l : l \ge 1, [a_l', c_l'] \subset [a_i', c_i'] \text{ and } [a_j', c_j'] \subset [b_l', c_l']\} \\ \text{and } \{h_l : l \ge 1, [a_l', c_l'] \subset [a_i', c_i'] \text{ and } [a_j', c_j'] \subset [a_l', b_l']\} \text{ are dense in } [h_i, h_j]$$

It is very similar to the "order-leaf-tight" property of continuum trees defined in [2]. Loosely said, it means that it is impossible to find a nontrivial ancestral path in the tree $\mathcal{T}_g$ without a density of points both on the right and on the left where a subtree is grafted. "left" and "right" are understood with regard to the shuffled order $\le_g^s$. This is crucial to the proof of theorem 1.8. We show that it holds almost surely in our setting.

Proposition 3.7. *Let g be a CRT excursion, $p \in (0,1)$ and S be a random i.i.d. sequence of signs with bias p. Then with probability one, (g, S) verifies property* (A)

Proof. By symmetry we prove only the first claim and by countable union we fix i and j. Let $K = \{l \geq 1 : [a_l, c_l] \subset [a_i, c_i] \text{ and } [a_j, c_j] \subset [b_l, c_l]\}$, and $\widetilde{K} = \{l : l \geq 1, [a'_l, c'_l] \subset [a'_i, c'_i] \text{ and } [a'_j, c'_j] \subset [b'_l, c'_l]\}$. For $y \in (h_i, h_j) \cap \mathbb{Q}$, consider $x = \sup\{t \in [a_i, a_j] : g(t) = y\}$. Then by definition $g(x) = y$ and $g(t) > y$ for $t > x$. Consider a sequence of leaves $x_n \nearrow x$ and the minimum b_{k_n} of g between x_n and a_i. Then necessarily $k_n \in K$ and $x_n < b_{k_n} < x$. So $h_{k_n} \to y$.

Now with probability one a subsequence $(k'_n)_n$ of $(k_n)_n$ can be found with $s_{k'_n} = \oplus$ for every n. Then lemma 3.5 implies that $k'_n \in \widetilde{K}$, and $h_{k'_n} \to y$. By countable union over y we have shown that $\overline{\{h_l, l \in \widetilde{K}\}}$ countains $(h_i, h_j) \cap \mathbb{Q}$. So it contains $[h_i, h_j]$ from which the proposition follows. □

An immediate consequence of property (A) is the following improvement on lemma 3.5, with strict inclusions.

Lemma 3.8. *Suppose* (g, s) *verifies* (A). *Let* $i \neq j$.

If $[a_j, c_j] \subset (a_i, b_i)$, *then either* $s_i = \oplus$ *and* $[a'_j, c'_j] \subset (a'_i, b'_i)$, *or* $s_i = \ominus$ *and* $[a'_j, c'_j] \subset (b'_i, c'_i)$.

If $[a_j, c_j] \subset (b_i, c_i)$, *then either* $s_i = \oplus$ *and* $[a'_j, c'_j] \subset (b'_i, c'_i)$, *or* $s_i = \ominus$ *and* $[a'_j, c'_j] \subset (a'_i, b'_i)$.

If $[a_i, c_i] \cap [a_j, c_j] = \emptyset$, *then* $[a'_i, c'_i] \cap [a'_j, c'_j] = \emptyset$.

4. The support of the permuton

Theorem 1.5 follows readily from the two propositions of this section.

Proposition 4.1. *For every signed excursion* (g, s), $\mu_{g,s}$ *has Hausdorff dimension* 1 *and its* 1*-dimensional Hausdorff measure is* $\leq \sqrt{2}$.

Proof. We start by showing that $\dim_H(\text{supp}(\mu)) \geq 1$. If π_1 is the projection of the unit square to its first coordinate, then $\pi_1(\text{supp}(\mu)) = [0, 1]$, otherwise μ couldn't have a uniform marginal. We conclude with the following lemma, which is immediate from the definition of Hausdorff dimension:

Lemma 4.2. *If* $\theta : (E, d_E) \to (F, d_F)$ *is a contraction, then for* $X \subset E$, $\dim_H(X) > \dim_H(\theta(X))$

To prove the upper bound, we apply lemma 3.3 for some choice of $\epsilon > \delta > 0$. Let I be the set of indices provided by the lemma. Let $J = \{k : \exists i, j \in I, [a_i, c_i] \subset (a_k, b_k), [a_j, c_j] \subset (b_k, c_k)\}$. Let $K = I \sqcup J$ We have the following fact, which is a direct consequence of the nested structure of the $[a_i, c_i]$.

Fact 4.3. *For every* $i \in J$, *there exists an* $i_l \in K$ *such that for every* $j \in K$, $[a_j, c_j] \subset [a_i, b_i]$ *implies* $[a_j, c_j] \subset [a_{i_l}, c_{i_l}] \subset [a_i, b_i]$. *Similarly for every* $i \in J$, *there exists be an* $i_r \in K$ *such that for every* $j \in K$, $[a_j, c_j] \subset [b_i, c_i]$ *implies* $[a_j, c_j] \subset [a_{i_r}, c_{i_r}] \subset [b_i, c_i]$. *Also there exists* $\star \in J$ *such that for every* $k \in K$, $[a_k, c_k] \subset [a_\star, c_\star]$.

We can define the following subsets of the unit square, which we use to cover $\mathrm{supp}(\mu_{g,s})$:

$$\begin{aligned}
A_i &= ([a_i, a_{i_l}] \cup [c_{i_l}, b_i]) \times ([a'_i, a'_{i_l}] \cup [c'_{i_l}, b'_i]) \\
&\quad \cup ([b_i, a_{i_r}] \cup [c_{i_r}, c_i]) \times ([b'_i, a'_{i_r}] \cup [c'_{i_r}, c'_i]) && \text{if } i \in J \text{ and } s_i = \oplus \\
A_i &= ([a_i, a_{i_l}] \cup [c_{i_l}, b_i]) \times ([b'_i, a'_{i_l}] \cup [c'_{i_l}, c'_i]) \\
&\quad \cup ([b_i, a_{i_r}] \cup [c_{i_r}, c_i]) \times ([a'_i, a'_{i_r}] \cup [c'_{i_r}, b'_i]) && \text{if } i \in J \text{ and } s_i = \ominus \\
A_i &= [a_i, c_i] \times [a'_i, c'_i] && \text{if } i \in I \\
A_0 &= ([0, a_\star] \cup [c_\star, 1]) \times ([0, a'_\star] \cup [c'_\star, 1])
\end{aligned}$$

By construction and fact 4.3, $\bigcup_{i \in K \cup \{0\}} \pi_1(A_i) = [0,1]$, and lemma 3.4 implies that for $x \in \pi_1(A_i)$, $(x, \varphi_{g,s}(x)) \in A_i$. This one has:

$$(\mathrm{Id}, \varphi_{g,s})[0,1] \subset \bigcup_{i \in K \cup \{0\}} A_i. \tag{6}$$

The rest of the proof is devoted to rewriting the right-hand side of (6) as an union of sets in

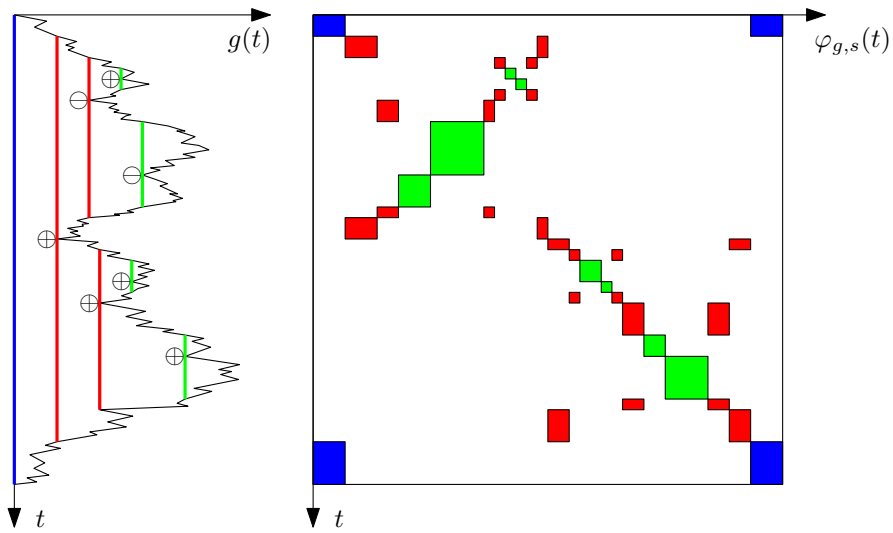


FIGURE 4. A_0 in blue, A_i for $i \in I$ in green, and A_i for $i \in J$ in red.

which we control the sum of diameters. Now, for $i \in I$, $\mathrm{diam}(A_i) = \mathrm{diam}([a_i, c_i] \times [a'_i, c'_i]) = \sqrt{2}(c_i - a_i)$. We deduce that

$$\sum_{i \in I} \mathrm{diam}(A_i) \leq \sqrt{2}. \tag{7}$$

For $i \in J$, A_i is the union of 8 rectangles $A_i^1, \dots A_i^8$. We have that

$$\sum_{j=1}^{8} \text{width}(A_i^j) = 2[(c_i - a_i) - (c_{i_l} - a_{i_l}) - (c_{i_r} - a_{i_r})]$$
$$\sum_{j=1}^{8} \text{height}(A_i^j) = 2[(c'_i - a'_i) - (c'_{i_l} - a'_{i_l}) - (c'_{i_r} - a'_{i_r})].$$

And both these quantities are equal and their value is 2 $\text{Leb}(\pi_1(A_i))$. Similarly, A_0 is the union of 4 rectangles $A_0^1, \dots, A_0^4$ whose widths and heights both sum to $2\,\text{Leb}(\pi_1(A_0))$. Hence

$$\begin{aligned}\sum_{j=1}^{4} \text{diam}(A_0^j) + \sum_{i\in J}\sum_{j=1}^{8} \text{diam}(A_i^j) &\leq \sum_{j=1}^{4}(\text{width}+\text{height})(A_0^j) + \sum_{i\in J}\sum_{j=1}^{8}(\text{width}+\text{height})(A_i^j)\\ &= 4\,\text{Leb}(\pi_1(A_0)) + 4\sum_{i\in J}\text{Leb}(\pi_1(A_i))\\ &= 4\,\text{Leb}([0,1]\setminus\bigcup_{i\in I}[a_i,c_i]) \leq 4\delta \qquad (8)\end{aligned}$$

By taking the closure and rewriting the right-hand side in eq. (6), we get

$$\text{supp}(\mu_{g,s}) \subset \overline{(\text{Id},\varphi_{g,s})[0,1]} \subset \left(\bigcup_{i\in I} A_i\right) \cup \left(\bigcup_{j=0}^{4} A_0^j\right) \cup \left(\bigcup_{i\in J}\bigcup_{j=1}^{8} A_i^j\right) \qquad (9)$$

Summing (7) and (8) shows that the sum of diameters in the cover (9) can't exceed $4\delta+\sqrt{2}$. Moreover, each square and rectangle in the cover has diameter bounded by $\sqrt{2}\epsilon$. This implies that $\text{supp}(\mu)$ has 1-dimensional Hausdorff measure bounded above by $\sqrt{2}$. □

Proposition 4.4. *If S is an i.i.d sequence of nondeterministic signs, then* $\text{supp}(\mu_{g,S})$ *is almost surely totally disconnected.*

Proof. We re-use the notations of the last proof, with $\epsilon > \delta > 0$. We now show that almost surely, we can build sets $\bar{I} \supset I$ and $\bar{J} \supset J$ such that

(1) the statement of fact 4.3 is still true when J is replaced by $\bar{J}$ and K by $\bar{K} = \bar{I} \sqcup \bar{J}$,
(2) for all $i \in \bar{I}$, $c_i - a_i \leq \epsilon$,
(3) $\text{Leb}([0,1] \setminus \bigsqcup_{i\in\bar{I}}[a_i,c_i]) < \delta$,

with the following added constraint:

$$\forall i \in J, \quad s(b_{i_r}) = s(b_{i_l}) \neq s(b_i). \qquad (10)$$

This is done by adding successively indices to I in order to create new branching points in between two branching points of the same sign. Condsider $i \in J$ and its left child i_l, with $s_i = s_{i_l} = \epsilon$. We can build, as in the proof of lemma 3.3, an infinite sequence $(b_{r_n})_n$ such that $[a_{r_n}, c_{r_n}] \subset [a_i, b_i]$ and $[b_{r_k}, c_{r_k}] \supset [a_{i_l}, c_{i_l}]$. Almost surely, one of the r_n, which we denote $j = j(i, i_l)$, is such that $s_j \neq \epsilon$. We can then find, by the same reasoning, a

$k = k(j(i, i_l))$ such that $[a_k, c_k] \subset [a_j, c_j]$ and $s_k = \epsilon$. We proceed similarly for $i \in J$ such that $s_i = s_{i_r}$. We can now set

$$\bar{I} = I \cup \{k(i, i_l) : i \in J, s_i \neq s_{i_l}\} \cup \{k(i, i_r) : i \in J, s_i \neq s_{i_r}\}$$
$$\bar{J} = J \cup \{j(i, i_l) : i \in J, s_i \neq s_{i_l}\} \cup \{j(i, i_r) : i \in J, s_i \neq s_{i_r}\}.$$

By construction, fact 4.3 applies to $\bar{I}$ and $\bar{J}$, and (10) is verified.

Now we can define the sets $(A_i)_{i \in \bar{K} \cup \{0\}}$ as in the previous proof, and we still have

$$\operatorname{supp} \mu_{e,S} \subset C = \bigcup_{i \in \bar{K} \cup \{0\}} A_i.$$

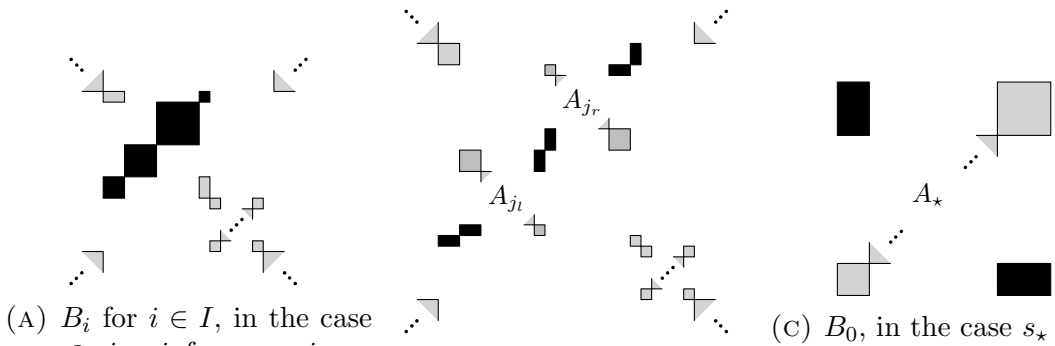
We will show that the diameter of any connected component of C is almost surely bounded by $4\epsilon + 2\delta$. This is enough to show that $\operatorname{supp}(\mu_{g,S})$ is totally disconnected.

For $x \in C$, let us denote by $\mathcal{C}(x)$ the connected component of C containing x, and for $X \subset C$, set $\mathcal{C}(X) = \cup_{x \in X} \mathcal{C}(x)$. We now set, for $i \in \bar{I}$, $B_i = \mathcal{C}(A_i)$, for $i \in \bar{J}$ $B_i = \mathcal{C}(A_i) \setminus \mathcal{C}(A_{i_l}) \setminus \mathcal{C}(A_{i_r})$, and $B_0 = \mathcal{C}(A_0) \setminus \mathcal{C}(A_*)$. Then, immediate induction yields

$$C = \bigsqcup_{i \in \bar{K} \cup \{0\}} B_i.$$

Now remark that the sets B_i were obtained by inclusion and exclusion of full connected components of C. Hence each connected component of C appears as a connected component of one of the B_i, that we now consider.

It turns out (see fig. 5) that for $i \in \bar{I}$, B_i has only one connected component, and its diameter is bounded above by $4\epsilon + 2\delta$. For $i \in \bar{J}$, B_i has three connected components, whose diameter is bounded above by 2δ. For $i = 0$, B_0 has two connected components, and their diameter is also bounded above by 2δ. □



(A) B_i for $i \in I$, in the case $s_i = \oplus$, $i = j_l$ for some j.

(B) B_j for $j \in J$, in the case $s_j = \oplus$, $j = j'_l$ for some j'.

(C) B_0, in the case $s_\star = \oplus$.

FIGURE 5. The proof of total disconnection.

5. Self-similarity

Given a CRT excursion g and one of its branching points b , one can build three subexcursions by cut-and-pasting, which encode the three connected components of $\mathcal{T}_g \setminus \{p_g(b)\}$. The goal of this section is do the same procedure on signed excursions, and observe the consequences on the associated permutons. This will allow us to prove theorem 1.6 in a "reversed" fashion: we start from μ, build μ_1, μ_2 and μ_3 by cutting along a suitably chosen branching point, as to be able to use a result of Aldous [3] and identify the distribution and relative sizes of the subexcursions.

Let (g, s) be a signed excursion. Given $\bar{\imath} \in \mathbb{N}$, we can obtain 3 excursions by looking at the values of g on $[a_{\bar{\imath}}, b_{\bar{\imath}}]$, $[b_{\bar{\imath}}, c_{\bar{\imath}}]$ and $[0, a_{\bar{\imath}}] \sqcup [c_{\bar{\imath}}, 1]$. More precisely, following [3], we define

$$\Delta_0 = 1 - c_{\bar{\imath}} + a_{\bar{\imath}}, \Delta_1 = b_{\bar{\imath}} - a_{\bar{\imath}}, \Delta_2 = c_{\bar{\imath}} - b_{\bar{\imath}}, X_0 = \frac{a_{\bar{\imath}}}{\Delta_0}, Y_0 = \frac{a'_{\bar{\imath}}}{\Delta_0}, \beta = s_{\bar{\imath}}. \tag{11}$$

Given these constants, we can define the contractions $\theta_k, \eta_k, \zeta_k$ for $k \in \{0, 1, 2\}$, as in (2), and

$$g_k = \frac{1}{\sqrt{\Delta_k}} g \circ \eta_k, \quad k \in \{0, 1, 2\}. \tag{12}$$

Because each η_k is a piecewise affine function, it pulls back the strict local minima of g that are in the interior of $\mathrm{Im}(\eta_k)$ onto strict local minima of g_k. This is made explicit in the following result:

Proposition 5.1. *For $k \in \{0, 1, 2\}$, there is an injective map $\vartheta_k : \mathbb{N} \to \mathbb{N}$, such that*

$$\forall i \in \mathbb{N}, \quad \eta_k(b_i(g_k)) = b_{\vartheta_k(i)}(g).$$

Moreover, the $\vartheta_k(\mathbb{N})$, for $k \in \{0, 1, 2\}$, form a partition of $\mathbb{N} \setminus \{\bar{\imath}\}$. Finally, for $k \in \{0, 1, 2\}$, the map $(g, \bar{\imath}, i) \mapsto \vartheta_k(i)$ is measurable.

Proof. We set $\vartheta_k(i) = \min\{j \in \mathbb{N} : \eta_k(b_i(g_k)) = b_j(g)\}$, and the measurability claim follows from measurability of $(i, g) \mapsto b_i(g)$, $(\bar{\imath}, g) \mapsto \eta_k$ and $(\bar{\imath}, g) \mapsto g_k$. The other claims are immediate by construction and from the definition of a measurable enumeration. □

We can now transport the signs of g onto signs of the g_k by setting $s_i^k = s_{\vartheta_k(i)}$ for $k \in \{0, 1, 2\}$ and $i \in \mathbb{N}$. A result of this construction is the following crucial observations:

Observation 5.2. For $x < y \in [0, 1]$, and $k \in \{0, 1, 2\}$, $x \lhd^s_{g_k} y$ if and only if $\eta_k(x) \lhd^s_g \eta_k(y)$.

Observation 5.3. The map $(g, \bar{\imath}, (s_i)_{i \in \mathbb{N}}) \mapsto s_i^k$ is measurable for every $i \in \mathbb{N}$ and $k \in \{0, 1, 2\}$

Now we want to use lemma 3.4 to show that our function $\varphi_{g,s}$ can be cut out into rescaled copies of φ_{g_k, s^k}, which translates immediately in termes of measures.

Proposition 5.4. *For $\bar{\imath} \in \mathbb{N}$, $k \in \{0, 1, 2\}$ and $t \in [0, 1]$,*

$$\varphi_{g,s} \circ \eta_k(t) = \zeta_k \circ \varphi_{g_k, s^k}(t). \tag{13}$$

As a consequence,

$$\mu_{g,s} = \sum_{k=0}^{2} \Delta_k \cdot (\theta_{k*} \mu_{g_k, s^k}).$$

Proof. Let us prove (13) for $k = 0$.

$$\begin{aligned}\varphi_{g,s}(\eta_0(t)) &= \mathrm{Leb}\{x \in [0, a_{\bar\imath}) \cup (c_{\bar\imath}, 1] : x \lhd_g^s \eta_0(t)\} + \mathbb{1}[a_{\bar\imath} \lhd_g^s t](c_{\bar\imath} - a_{\bar\imath}) \\ &= \mathrm{Leb}\{x \in [0, a_{\bar\imath}) \cup (c_{\bar\imath}, 1] : x \lhd_g^s \eta_0(t)\} \\ &\quad + (c_{\bar\imath} - a_{\bar\imath})\, \mathbb{1}\left[\mathrm{Leb}\{x \in [0, a_{\bar\imath}) \cup (c_{\bar\imath}, 1] : x \lhd_g^s \eta_0(t)\} > a'_{\bar\imath}\right] \\ &= \Delta_0 \,\mathrm{Leb}\{y \in [0,1] : y \lhd_{e_0}^{s^0} t\} + (1 - \Delta_0)\, \mathbb{1}\left[\Delta_0 \,\mathrm{Leb}\{y \in [0,1] : y \lhd_{e_0}^{s^0} t\} > \Delta_0 Y_0\right] \\ &= \zeta_0(\varphi_{e_0,s^0}(t))\end{aligned}$$

Where the first two equalities come from lemma 3.4 and the third is the result of the change of variable $x = \eta_0(y)$. Now, for $k = 1$,

$$\begin{aligned}\varphi_{g,s}(\eta_1(t)) &= a'_{\bar\imath} + (b'_{\bar\imath} - a'_{\bar\imath})\, \mathbb{1}[s_{\bar\imath} = \ominus] + \mathrm{Leb}\{x \in [a_{\bar\imath}, b_{\bar\imath}] : x \lhd_g^s \eta_1(t)\} \\ &= \Delta_0 Y_0 + \Delta_2 \beta + (b_{\bar\imath} - a_{\bar\imath}) \,\mathrm{Leb}\{y \in [0,1] : y \lhd_{g_1}^{s^1} t\} \\ &= \zeta_1(\varphi_{g_1,s^1}(t))\end{aligned}$$

where the first equality comes from lemma 3.4 and the second is the result of the change of variable $x = \eta_1(y)$. The case $k = 2$ is similar. □

This is all we need to show theorem 1.6.

Proof of theorem 1.6. If e is an Brownian excursion, and $X_l < X_r$ are reordered uniform independent random variables in $[0, 1]$, independent of e, then almost surely there is a $\bar\imath$ such that $b_{\bar\imath} = \mathrm{argmin}_{[X_l,X_r]} e$. Define $\Delta_0, \Delta_1, \Delta_2, X_0, Y_0, \beta$ as in (11). This allows us to define the θ_k as in (2) and the e_k, s^k as before.

A result of Aldous [3, cor. 5] states that e_0, e_1, e_2 are Brownian excursions, $(\Delta_0, \Delta_1, \Delta_2)$ is a Dirichlet$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ partition of 1, and X_0 is uniform in $[0, 1]$, all these random variables being independent.

Now, as a consequence of observation 5.3, for $k \in [0, 1]$ and $i \in \mathbb{N}$, S_i^k is a random variable. Given e and $\bar\imath$, the S^k for $k \in [0, 1]$ and β are permutations of disjoint subsequences of S. As a result, the S^k and β are independent (and independent of (e, X_l, X_r)), and distributed as i.i.d. sequences of signs of bias p.

We finally set $\mu_k = \mu_{e_k,S^k}$ for $k \in \{0, 1, 2\}$ and need only prove

$$Y_0 = \varphi_{e_0,S^0}(X_0) \text{ a.s.} \tag{14}$$

to show that the collection of random variables $((\Delta_k)_{k\in\{0,1,2\}}, (\mu_k)_{k\in\{0,1,2\}}, (X_0, Y_0), \beta)$ has the joint distribution assumed in theorem 1.6. Proposition 5.4 then yields the theorem. Let us now prove (14).

$$\begin{aligned}\Delta_0 Y_0 = a'_i = \mathrm{Leb}\{x \in [0, a_i) \cup (c_i, 1) : x \le_e^S a_i\} &= \Delta_0 \,\mathrm{Leb}\{y \in [0,1] : y \le_{e_0}^{S_0} \eta_0^{-1}(a_i)\} \\ &= \Delta_0 \,\mathrm{Leb}\{y \in [0,1] : y \le_{e_0}^{S_0} X_0\} = \Delta_0 \varphi_{e_0,S^0}(X_0). \quad \square\end{aligned}$$

Remark 5.5. As seen in the proof, theorem 1.6 is a direct consequence of the self-similarity property of the Brownian CRT [3, thm. 2]. It was shown [1] that this property actually characterizes the Brownian CRT in the space of measured $\mathbb{R}$-trees. We believe that the

arguments of Albenque and Goldschmidt can be transposed in our setting, to show that the law of μ^p is the only distribution on permutons which verifies (3). The main reason backing that claim is the following: permutons are characterized by their finite-dimensional marginals, just like measured $\mathbb{R}$-trees are determined by their reduced trees (see section 3 in [1]).

6. Expectation of the permuton

In this section we shall compute the density function of the averaged permuton $\mathbb{E}\,\mu^p$ for $p \in (0,1)$. We know that $\mu^p = \mu_{e,S}$, where e is a normalized Brownian excursion and S is an independent sequence of i.i.d. signs with bias p. Since for fixed (g,s), the measure $\mu_{g,s}$ is the distribution of the random pair $(U, \varphi_{g,s}(U))$ with U uniform in $[0,1]$, then by Fubini's theorem, we get the following:

Lemma 6.1. *$\mathbb{E}\,\mu^p$ is the distribution of the random pair $(U, \varphi_{e,S}(U))$, where e is a normalized Brownian excursion, S is an independent sequence of i.i.d. signs with bias p, and U is uniform, those three random variables being independent.*

Let $(B_t)_{0\le t\le 1}$ be a normalized Brownian bridge between 0 and 0. Define its local time at 0 as follows: for $t \in [0,1]$, set $L_t = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{0\le |B_s|\le \varepsilon}\, ds$ in probability. Define also its right-continuous inverse $(T_l)_{l\ge 0}$.

We set $\Delta T_l = T_l - T_{l^-}$ for $l \ge 0$. We suppose that each $l \ge 0$ such that $\Delta T_l > 0$ is equipped with an independent sign ϵ_l with bias p. We will use a result of Bertoin and Pitman [7] to rewrite the measure $\mathbb{E}\,\mu^p$ as the distribution of some functional of B.

Lemma 6.2. *The measure $\mathbb{E}\,\mu^p$ is the distribution of $\left(\frac{P_1+P_2}{P_1+P_2+P_3+P_4}, \frac{P_1+P_4}{P_1+P_2+P_3+P_4}\right)$, where*

$$\begin{aligned} P_1 &= \textstyle\sum_{l<L_1/2,\epsilon_l=\oplus} \Delta T_l, \quad P_2 = \sum_{l<L_1/2,\epsilon_l=\ominus} \Delta T_l \\ P_3 &= \textstyle\sum_{l>L_1/2,\epsilon_l=\oplus} \Delta T_l, \quad P_4 = \sum_{l>L_1/2,\epsilon_l=\ominus} \Delta T_l \end{aligned} \tag{15}$$

Proof. We will build a suitable coupling of (e,S,U) on one hand, and (B,ϵ) on the other hand. Start with the bridge B, and set $U = T_{L_1/2}$. Define $(K_t)_{0\le t\le 1}$ as follows: $K_t = L_t$ for $0 \le t \le U$ and $K_t = L_1 - L_t$ when $U \le t \le 1$. Theorem 3.2 of [7] tells us that if we set $e = K + |B|$, then (e,U) is distributed as a Brownian excursion with an independent uniform variable in $[0,1]$. Moreover, the following holds almost surely: for $0 \le t \le U$, $K_t = \inf_{t\le s\le U} e_s$ and for $U \le t \le 1$, $K_t = \inf_{U\le s\le t} e_s$. Finally let S be a sequence of i.i.d. signs with bias p, independent of (B,e,U). The triple (e,S,U) has the desired distribution. We can transfer some of the signs of S to form the marking process $(\epsilon_l)_{l\ge 0,\Delta_l>0}$. First remark that almost surely, U is not a one-sided local minimum of e. For $l \ge 0$ such that $\Delta T_l > 0$,

- either $l < L_1/2$ and then $T_{l^-} < T_l < U$, in which case T_l is an inner local minimum $b_{\imath_l}$ of e for some $\imath_l \in \mathbb{N}$. We then set $\epsilon_l = S_{\imath_l}$.
- either $l > L_1/2$ and then $T_{l^-} < T_l < U$, in which case T_{l^-} is an inner local minimum $b_{\imath_l}$ of e for some $\imath_l \in \mathbb{N}$. We then set $\epsilon_l = S_{\imath_l}$.

The sequence $(\imath_l)_{l:\Delta T_l>0}$ is a random injection into $\mathbb{N}$ that solely depends on B. So conditional on B, the signs in $(\epsilon_l)_{l:\Delta T_l>0}$ are i.i.d. and of bias p. Then (B,ϵ) has the desired distribution.

We now show that in this coupling we have the almost sure equality $(U,\varphi_{e,S}(U)) = \left(\frac{P_1+P_2}{P_1+P_2+P_3+P_4}, \frac{P_1+P_4}{P_1+P_2+P_3+P_4}\right)$. Then lemma 6.1 implies the present lemma. If we define

$$\hat{P}_1 = \mathrm{Leb}\{t : 0 \le t \le U, t \lhd_e^S U\}, \quad \hat{P}_2 = \mathrm{Leb}\{t : 0 \le t \le U, t \rhd_e^S U\},$$
$$\hat{P}_3 = \mathrm{Leb}\{t : U \le t \le 1, t \rhd_e^S U\}, \quad \hat{P}_4 = \mathrm{Leb}\{t : U \le t \le 1, t \lhd_e^S U\},$$

then it is immediate that $\hat{P}_1+\hat{P}_2+\hat{P}_3+\hat{P}_4 = 1$ almost surely, $\hat{P}_1+\hat{P}_2 = U$ and $\hat{P}_1+\hat{P}_4 = \varphi_{e,S}(U)$. Now we need only show that the $P_i = \hat{P}_i$ for $1 \le i \le 4$. For instance for $i=1$, we need to observe that $t \in [0,1]$ is such that $t < U$ and $t \lhd_e^S U$ if and only if there is a $b_i \in (t,U)$ such that b_i is the unique minimum of e on $[t,U]$ and $S_i = \oplus$. Such b_i is necessarily equal to T_l for some $l < L_1/2$ such that $T_{l^-} < t < T_l$, and then $S_i = \epsilon_l$. We have shown the following logical equivalence for $t \in [0,1]$:

$$t \le U \text{ and } t \lhd_e^S U \iff \exists\, l < L_1/2 \text{ s.t. } T_{l^-} < t < T_l \text{ and } \epsilon_l = \oplus.$$

Taking the Lebesgue measure on both sides yields $\hat{P}_1 = P_1$. For $i = 2,3,4$, the proof is symmetric. □

Let $\mathcal{U}$ be the set of continuous excursions of variable length, with $R : \mathcal{U} \to \mathbb{R}^+$ denoting the length statistic. Let N be the Itô excursion measure of Brownian motion. For $\theta \ge 0$, define the measure $\Lambda^\theta(dr) = e^{-\theta r} N(R \in dr)$. Denote by $(X_l^\theta)_{l\ge 0}$ the process of sums up to time l of a Poisson point process of intensity $dt\Lambda^\theta$. This is a well-defined process because $\int \Lambda^\theta(dr)(r \wedge 1)$ is finite. We can state the following rewriting of the distribution $\mathbb{E}\,\mu^p$.

Lemma 6.3. *For any $\theta > 0$, $\mathbb{E}\,\mu^p$ is the distribution of $\left(\frac{\mathcal{P}_1+\mathcal{P}_2}{\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3+\mathcal{P}_4}, \frac{\mathcal{P}_1+\mathcal{P}_4}{\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3+\mathcal{P}_4}\right)$, where conditional on a random variable λ_Y with exponential distribution of parameter $\sqrt{2\theta}$, we define the variables $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ to be independent with $\mathcal{P}_1 \overset{d}{=} \mathcal{P}_3 \overset{d}{=} X^\theta_{p\lambda_Y/2}$ and $\mathcal{P}_2 \overset{d}{=} \mathcal{P}_4 \overset{d}{=} X^\theta_{(1-p)\lambda_Y/2}$.*

Proof. Let us reuse the notations of lemma 6.2. We make use of the results of Perman and Wellner [18], which show that the most tractable object in terms of its excursions is not the normalized Brownian bridge, but the random-length bridge $(\beta_t)_{t\ge 0}$ defined as follows: $\beta_t = \mathbb{1}_{0\le t\le Y}\sqrt{Y}B_{t/Y}$ where Y is a random variable of distribution $\Gamma(1/2,\theta)$ independent of B. Its local time λ, inverse local time τ and jump process $\Delta\tau$ are related to those of B by $\lambda_t = \sqrt{Y}L_{t/Y}$, $\tau_l = YT_{l/\sqrt{Y}}$ and $\Delta\tau_l = Y\Delta T_{l/\sqrt{Y}}$. The marking process ϵ can be modified accordingly by setting $\varepsilon_l = \epsilon_{l/\sqrt{Y}}$ for $l \ge 0$ such that $\Delta\tau_l > 0$.

Now if we set

$$\mathcal{P}_1 = \textstyle\sum_{l<\lambda_1/2,\epsilon_l=\oplus} \Delta\tau_l, \quad \mathcal{P}_2 = \sum_{l<\lambda_1/2,\epsilon_l=\ominus} \Delta\tau_l$$
$$\mathcal{P}_3 = \textstyle\sum_{l>\lambda_1/2,\epsilon_l=\oplus} \Delta\tau_l, \quad \mathcal{P}_4 = \sum_{l>\lambda_1/2,\epsilon_l=\ominus} \Delta\tau_l$$

then by construction, $\left(\frac{P_1+P_2}{P_1+P_2+P_3+P_4}, \frac{P_1+P_4}{P_1+P_2+P_3+P_4}\right) = \left(\frac{\mathcal{P}_1+\mathcal{P}_2}{\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3+\mathcal{P}_4}, \frac{\mathcal{P}_1+\mathcal{P}_4}{\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3+\mathcal{P}_4}\right)$.

We now have to identify the joint distribution of the $\mathcal{P}_i$. It results from [18, thm 1 and 4] that λ_Y is distributed as an exponential random variable of parameter $\sqrt{2\theta}$, and that, conditional on λ_Y, the excursions of β away from 0, parametrized by the local time, form a Poisson point process of intensity $dle^{-\theta R(w)}N(dw)$ over $[0,\lambda_Y]\times\mathcal{U}$. The random set $\{(l,\Delta\tau_l), l\geq 0, \Delta_l>0\}$, which is just the point process of excursion lengths, is then also Poisson with intensity $dl\Lambda^\theta(dt)$ over $[0,\lambda_Y]\times\mathbb{R}_+$. This results from the mapping property of Poisson processes. Now, since the marking process $(\varepsilon_l)_{l\geq 0}$ is a choice of i.i.d. marks, chosen independent of B, the marking property of point processes [15, sect. 2.3] tells us that $\{(l,\Delta\tau_l,\varepsilon_l), l\geq 0, \Delta_l>0\}$ is itself a Poisson process of intensity $dl\Lambda^\theta(dt)(p\delta_\oplus+(1-p)\delta_\ominus)(d\varepsilon)$ over $[0,\lambda_Y]\times\mathbb{R}_+\times\{\oplus,\ominus\}$.

Since they are functionals of the same Poisson process restricted to disjoint subsets, the processes $\{\Delta\tau_l, 0\leq l\leq\lambda_Y/2, \Delta_l>0, \varepsilon_l=\oplus\}$, $\{\Delta\tau_l, 0\leq l\leq\lambda_Y/2, \Delta_l>0, \varepsilon_l=\ominus\}$, $\{\Delta\tau_l, \lambda_Y/2\leq l\leq\lambda_Y, \Delta_l>0, \varepsilon_l=\oplus\}$ and $\{\Delta\tau_l, \lambda_Y/2\leq l\leq\lambda_Y, \Delta_l>0, \varepsilon_l=\ominus\}$, are independent. Moreover, by the mapping property, they are themselves Poisson, with respective intensity measures $\frac{p\lambda_Y}{2}\Lambda_\theta(dr)$, $\frac{(1-p)\lambda_Y}{2}\Lambda_\theta(dr)$, $\frac{p\lambda_Y}{2}\Lambda_\theta(dr)$ and $\frac{(1-p)\lambda_Y}{2}\Lambda_\theta(dr)$. The lemma follows. □

Proof of theorem 1.7. By a classical argument using Girsanov's theorem[2], X_l^θ is distributed as the hitting time of level l by a Brownian motion with positive drift θ, hence its density is $\frac{d}{dt}\mathbb{P}(X_l^\theta\in dt)=y_l^\theta(t)=\mathbb{1}_{t\geq 0}\frac{e^{-\theta t}\,l\,e^{-l^2/(2t)}}{e^{-\sqrt{2\theta}l}\sqrt{2\pi t^3}}$(see [9, ch. II.1, eq. 2.0.2]).

Then, going back to the notations of lemma 6.3, the joint density of $(\mathcal{P}_1,\mathcal{P}_2,\mathcal{P}_3,\mathcal{P}_4)$ at $(t_1,t_2,t_3,t_4)\in(\mathbb{R}_+)^4$ equals

$$\begin{aligned}
&\int_0^\infty d\lambda\sqrt{2\theta}e^{-\sqrt{2\theta}\lambda}y^\theta_{p\lambda/2}(dt_1)y^\theta_{(1-p)\lambda/2}(dt_2)y^\theta_{(1-p)\lambda/2}(dt_3)y^\theta_{p\lambda/2}(dt_4)\\
&=\frac{\sqrt{2\theta}p^2(1-p)^2}{2^4(\sqrt{2\pi})^4}\frac{e^{-\theta(t_1+t_2+t_3+t_4)}}{(t_1t_2t_3t_4)^{3/2}}\int_0^\infty\lambda^4e^{-\lambda^2/2\left(\frac{p^2}{4t_1}+\frac{(1-p)^2}{4t_2}+\frac{p^2}{4t_3}+\frac{(1-p)^2}{4t_4}\right)}d\lambda\\
&=\frac{\sqrt{2\theta}p^2(1-p)^2}{2^4(\sqrt{2\pi})^4}\frac{e^{-\theta(t_1+t_2+t_3+t_4)}}{(t_1t_2t_3t_4)^{3/2}}\frac{3\sqrt{2\pi}}{2\left(\frac{p^2}{4t_1}+\frac{(1-p)^2}{4t_2}+\frac{p^2}{4t_3}+\frac{(1-p)^2}{4t_4}\right)^{5/2}}.
\end{aligned}$$

Now we define the random variables $S=\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3+\mathcal{P}_4$, $Q=\mathcal{P}_1/S$, $U=(\mathcal{P}_1+\mathcal{P}_2)/S$ and $V=(\mathcal{P}_1+\mathcal{P}_4)/S$. According to lemma 6.3, $\mathbb{E}\,\mu^p$ is the distribution of the couple (U,V). It follows from the Lebesgue change of variables theorem that the joint density of

[2]It also follows from Campbell's formula [15, sect. 3.2] and [9, ch. II.1, eq. 2.0.1]

(S,Q,U,V) at $(s,q,u,v) \in (\mathbb{R}_+ \times \mathbb{R}^+ \times [0,1] \times [0,1])$ is equal to

$$\frac{s^3\, \mathbb{1}_{\max(0,u+v-1)\le q\le \min(u,v)}\, \frac{3\sqrt{2\theta}p^2(1-p)^2}{2^5(\sqrt{2\pi})^3} e^{-\theta s}}{(sq\, s(u-q)\, s(1-u-v+q)\, s(v-q))^{3/2}\left(\frac{p^2}{4sq}+\frac{(1-p)^2}{4s(u-q)}+\frac{p^2}{4s(1-u-v+q)}+\frac{(1-p)^2}{4s(v-q)}\right)^{5/2}}$$
$$= \left(\frac{\sqrt{\theta}e^{-\theta s}}{\sqrt{\pi}\sqrt{s}}\right) \frac{\frac{3p^2(1-p)^2}{2\pi}\, \mathbb{1}_{\max(0,u+v-1)\le q\le \min(u,v)}}{(q(u-q)(1-u-v+q)(v-q))^{3/2}\left(\frac{p^2}{q}+\frac{(1-p)^2}{(u-q)}+\frac{p^2}{(1-u-v+q)}+\frac{(1-p)^2}{(v-q)}\right)^{5/2}}.$$

Now we get the joint distribution of (U,V) by integrating with respect to s and q, which immediately yields theorem 1.7. □

7. Shuffling of continuous trees

The goal of this section is to build, from a signed excursion (g,s), a shuffled excursion $f_{g,s}$, that verifies the conclusions of theorem 1.8 after setting $\tilde{e} = f_{e,S}$. This will not be possible for every choice of deterministic signed excursion, but we will show that it is possible for signed excursions with property (A), which is the case of (e,S) with probability 1.

We start from the following observation: for every CRT excursion g, if we define the a_i, b_i, c_i, h_i as before, then by density of the branching points it is easy to see that

$$g(t) = \sup_i h_i\, \mathbb{1}_{[a_i,c_i]}(t).$$

Hence, given a sequence of signs s, which provides us the numbers a'_i, b'_i, c'_i, it is natural to define a shuffled version as such:

$$f_{g,s}(t) = \sup_i h_i\, \mathbb{1}_{[a'_i,c'_i]}(t)$$

The map $(g,s,t) \mapsto f_{g,s}(t)$ is measurable because the $g(a_i)$, a'_i and c'_i are measurable functions of g and s.

From now on, we will drop the dependency in (g,s) in the proofs. So we set $f = f_{g,s}$ and $\varphi = \varphi_{g,s}$. The first step is to show that f is continuous whenever (g,s) verifies (A). We start with two lemmas. Let $\omega(g,\delta)$ stand for the modulus of continuity of g at radius δ.

Lemma 7.1. *For $a'_k \le u \le b'_k$, $h_k \le f(u) \le h_k + \omega(g, b'_k - a'_k)$.*
For $b'_k \le u \le c'_k$, $h_k \le f(u) \le h_k + \omega(g, c'_k - b'_k)$.

Proof. The two claims are symmetric, thus only the first is proved. Recall that $f(u) = \sup_{[a'_i,c'_i]\ni u} h_i$ and suppose $u \in [a'_k, b'_k]$. For i such that $[a'_i, c'_i] \ni u$, either $h_i \le h_k$, or $h_i > h_k$. In the latter case, $[a'_i, c'_i] \subset [a'_k, b'_k]$. Hence $|a_i - b_k| < |b'_k - a'_k|$, and $h_i - h_k = g(a_i) - g(a_k) \le \omega(g, b_k - a_k) = \omega(g, b'_k - a'_k)$.

This shows that for every i such that $[a'_i, c'_i] \ni u$, $h_i < h_k + \omega(g, b'_k - a'_k)$ Taking the supremum gives the claim of the lemma. □

Lemma 7.2. *The b'_i, for $i \in \mathbb{N}$, are dense in $[0,1]$.*

Proof. The leaves of g are of full Lebesgue measure. If x and y are leaves, there is a i such that $a_i < x < b_i < y < c_i$. As a result of lemma 3.4, b'_i must lie between $\varphi(x)$ and $\varphi(y)$. Since φ is measure-preserving, the images of leaves of g by φ are of full measure, and hence dense in $[0,1]$. So the b'_i are dense. □

Proposition 7.3. *Under* (A)*, the function f is continuous.*

Proof. Let t be in $[0,1]$ and $\delta > 0$. By lemma 7.2, we can find $b'_i < t < b'_j$ with $(b'_j - b'_i) \leq \delta$. Let k be the most recent common ancestor of i and j, so that $b'_i < b'_k < b'_j$. We shall show that there is a continuous function $\underline{f}$ such that for $u \in [b'_i, b'_j]$,

$$\underline{f}(u) \leq f(u) \leq \underline{f}(u) + \omega(g, \delta) \tag{16}$$

Which is enough, since δ was arbitrary, to show continuity in t. We build $\underline{f}$ and show (16) on $[b'_k, b'_j]$ only. The interval $[b'_i, b'_k]$ can be treated with a symmetric proof.

Set $\underline{f} : [b'_k, b'_j] \to \mathbb{R}_+$, with

$$\underline{f} = \sup\{h_l \mathbb{1}_{[a'_l, c'_l]} \mid l \colon [a'_k, c'_k] \supset [a'_l, c'_l] \supset [a'_j, c'_j]\}.$$

Clearly, $\underline{f} \leq f$. It is also clear that $\underline{f}$ is increasing from h_k to h_j, because the indicator functions are nested and h_l increases as a'_l decreases. Lemma 3.8 implies that the a'_l are all distinct, while property (A) implies that the h_l are dense in $[h_k, h_j]$. This implies continuity of $\underline{f}$.

Now we shall show (16) for u in $[b'_k, b'_j]$.

Case 1: for every l s.t. $u \in [a'_l, c'_l]$, we have $[a'_l, c'_l] \supset [a'_j, c'_j]$. Then $f(u) = \underline{f}(u)$.

Case 2: there exists l s.t. $x \in [a'_l, c'_l]$ and $[a'_l, c'_l] \not\supset [a'_j, c'_j]$. Then consider the most recent common ancestor m of l and j. Necessarily,

$$b'_k < a'_m < a'_l < u < c'_l < b'_m < a'_j < c'_j < c'_m.$$

Then lemma 7.1 gives $h_m \leq g(u) \leq h_m + \omega(g, \delta)$. It is clear that $h_m = \underline{f}(u)$, proving (16). □

Now that we have shown that f is continuous, it becomes possible to define the distance d_f on $[0,1]$ and the structured real tree $\mathcal{T}_f$.

Proposition 7.4. *Under* (A)*, we have $g = f \circ \varphi$, and furthermore, φ is a $([0,1], d_g) \to ([0,1], d_f)$ isometry.*

Proof. Let $t \in [0,1]$. To show $g(t) = f(\varphi(t))$ it is enough to see that

$$\{k : t \in [a_k, c_k]\} = \{k : \varphi(t) \in [a'_k, c'_k]\}. \tag{17}$$

because $e(t)$ and $f(\varphi(t))$ are just the sup of $i \mapsto h_i$ over these two respective sets. If k is such that $t \in [a_k, c_k]$, then by lemma 3.4, $\varphi(t) \in [a'_k, c'_k]$. If on the other hand k is such that $t \notin [a_k, c_k]$, then by symmetry suppose $t < a_k$, it is then possible to find i such that $t < a_i < a_k < c_k \leq c_i$. Then lemmas 3.4 and 3.8 imply that $\varphi(t) \notin [a'_k, c'_k]$.

Now to show that φ is a (d_g, d_f) isometry, we need only show that for $x < y$,

$$\min_{[x,y]} g = \min_{[\varphi(x), \varphi(y)]} f.$$

Case 1: $\min_{[x,y]} g = g(x)$. Then for every i, $x \in [a_i, c_i]$ implies $y \in [a_i, c_i]$. So $\varphi(x) \in [a'_i, c'_i]$ implies $\varphi(y) \in [a'_i, c'_i]$ and then $[\varphi(x), \varphi(y)] \subset [a'_i, c'_i]$. The definition of f then yields $f(t) \geq f(\varphi(x))$ for every $t \in [\varphi(x), \varphi(y)]$. Hence

$$\min_{[\varphi(x),\varphi(y)]} f = f(\varphi(x)) = g(x) = \min_{[x,y]} g.$$

Case 2: $\min_{[x,y]} g = g(y)$. This case is similar by symmetry.
Case 3: $\min_{[x,y]} g = b_i$ for some $b_i \in (x, y)$. Then we conclude immediately by applying case 2 on $[x, b_i]$ and case 1 on $[b_i, y]$. □

Proposition 7.5. *The random continuous function $f_{e,S}$ has the distribution of a Brownian excursion with the same local time at 1 as e.*

Proof. The claim on the local time is an immediate consequence of the fact that for every $y \geq 0$, $\mathrm{Leb}\{t, f_{g,s}(t) \leq y\} = \mathrm{Leb}\{t, f_{g,s}(\varphi_{g,s}(t)) \leq y\} = \mathrm{Leb}\{t, g(t) \leq y\}$.

To show that the random continuous functions e and $f = f_{e,S}$ have the same distribution, we shall show that for every $k \geq 1$, if $U_{(1)} < \ldots < U_{(k)}$ are reordered uniform variables in $[0, 1]$, independent of e, S, then

$$(e(U_{(1)}), \ldots, e(U_{(k)})) \stackrel{d}{=} (f(U_{(1)}), \ldots, f(U_{(k)})). \tag{18}$$

Deriving $e \stackrel{d}{=} f$ from there is classical, see for instance the end of the proof of the direct implication of [2, thm. 20].

Let us consider $U_{(1)} < \ldots < U_{(k)}$ the order statistics of p uniform random variables in $[0, 1]$, independent of e, S. Set $V_i = \varphi(U_{(i)})$ for every $1 \leq i \leq k$. Then there exists $\alpha \in \mathfrak{S}_k$ such that $W_1 = V_{\alpha(1)} < \ldots < V_{\alpha(k)} = W_k$. Since φ preserves the Lebesgue measure, $(W_1, \ldots, W_k)$ has the distribution of the order statistic of p uniform variables.

We consider the *marked trees*, as per the definition of [16, sect. 2.5], associated to a CRT excursion and a finite number of points. For any set $\mathbf{t} = (t_1 < \ldots < t_k)$ of leaves of g, $\theta(g; \mathbf{t})$ is built from the tree $\tau(g; \mathbf{t})$ by adding edge-lengths compatible with the distances in the tree $\mathcal{T}_g$. Since the root of $\tau(g; \mathbf{t})$ has a positive height, a new root $\emptyset$ is added below it. It is characterized (among plane trees with edge-lengths up to isomorphism) by the following fact:

$$d_{\theta(g;\mathbf{t})}(\ell_i, \ell_j) = d_g(t_i, t_j), \quad d_{\theta(g;\mathbf{t})}(\emptyset, \ell_i) = g(t_i), \tag{19}$$

where $d_{\theta(g;\mathbf{t})}$ denotes the graph distance, taking edge-lengths into account, and in any plane tree $\ell_1, \ldots, \ell_k$ is an enumeration of the leaves in the natural ordering.

Let $T = \theta(e; \mathbf{U})$, and let $\widetilde{T}$ be obtained from T by inverting the order of the children at each branching point corresponding to a b_i where the sign s_i is a $\ominus$. By definition there is an isomorphism of rooted trees with edge-lengths $\widetilde{T} \leftrightarrow T$. This isomorphism necessarily permutes the leaves: set $\beta \in \mathfrak{S}_k$ such that $\ell_i(\widetilde{T}) \leftrightarrow \ell_{\beta(i)}(T)$. Then by construction β is such that $\varphi_{e,S}(U_{\beta(1)}) < \ldots < \varphi_{e,S}(U_{\beta(k)})$. We deduce $\beta = \alpha$, and hence

$$d_{\widetilde{T}}(\ell_i, \ell_j) = d_T(\ell_{\alpha(i)}, \ell_{\alpha(j)}) = d_e(U_{\alpha(i)}, U_{\alpha(j)}) = d_f(\varphi(U_{\alpha(i)}), \varphi(U_{\alpha(j)})) = d_{\theta(f;\mathbf{W})}(\ell_i, \ell_j)$$
$$d_{\widetilde{T}}(\ell_i, \emptyset) = d_T(\ell_{\alpha(i)}, \emptyset) = g(U_{\alpha(i)}) = g(W_i) = d_{\theta(f;\mathbf{W})}(\ell_i, \emptyset).$$

So $\widetilde{T} = \theta(f, \mathbf{W})$.

Finally we consider the distribution of $\widetilde{T}$. Theorem 2.11 of [16] tells us that the structure of T is that of a uniform planted binary tree with k leaves, and the edge-lengths are exchangeable. So an independent shuffling of T is still distributed like T, and this is the case of $\widetilde{T}$. We deduce $\theta(e; \mathbf{U}) = T \stackrel{d}{=} \widetilde{T} = \theta(f; \mathbf{W})$. From there, (19) implies that we can recover (18). □

Now theorem 1.8 follows from propositions 3.7 and 7.3 to 7.5, after setting $\widetilde{e} = f_{e,S}$.

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