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Delocalization at small energy for heavy-tailed random matrices

Charles Bordenave* and Alice Guionnet†

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Abstract

We prove that the eigenvectors associated to small enough eigenvalues of an heavy-tailed symmetric random matrix are delocalized with probability tending to one as the size of the matrix grows to infinity. The delocalization is measured thanks to a simple criterion related to the inverse participation ratio which computes an average ratio of L^4 and L^2 -norms of vectors. In contrast, as a consequence of a previous result, for random matrices with sufficiently heavy tails, the eigenvectors associated to large enough eigenvalues are localized according to the same criterion. The proof is based on a new analysis of the fixed point equation satisfied asymptotically by the law of a diagonal entry of the resolvent of this matrix.

1 Introduction

Anderson localization has attracted a lot of interest in both mathematical and physical communities over the past decades (we refer to the book [29] and the recent monograph [8]). One of the most tractable models to study this phenomenon is given by random Schrödinger operators on trees, see notably [1, 2] for the derivation of the phase diagram at the physical level of rigor, [5, 3, 4] for proofs of localization at strong disorder and high energy, and [27, 6, 7, 26] for proofs in the regime of delocalization. It was shown that at small energy the system displays delocalized waves whereas at large energy waves are localized. This phenomenon is related to the transition between a continuous spectrum and a discrete spectrum. Even, a transition between these two phases at a given energy, the so-called mobility edge, could be proved [7]. Such a transition is expected to happen in much more general settings, see e.g. [24]. In this article, we prove the existence of a similar phenomenon for random matrices with heavy tails, as conjectured in [17, 31, 33]. This is in contrast with the full delocalization observed for light tails Wigner matrices [22, 21, 23]. Indeed, we shall prove that for Lévy matrices with heavy enough tail, eigenvectors are delocalized for small

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enough energy whereas they are localized for large enough energy. We are not able to prove a sharp transition but the mobility edge value is predicted in [17, 33] based on the replica trick. In fact, we already proved in [16] that eigenvectors are delocalized provided the entries have roughly speaking finite L^1-norm, whereas a localization phenomenon appears for sufficiently heavy tail and large energy. However, we left open the question of proving delocalization at small energy and very heavy tails, or at least to exhibit a single criterion which would allow to distinguish these two phases. In this article we remedy this point.

Let us first describe roughly our main results. Consider a symmetric matrix A of size $n \times n$ with independent equidistributed real entries above the diagonal. Assume that the tail of A_{ij} is such that, for some $0 < \alpha < 2$,

$$n\mathbb{P}(|A_{ij}| \geq t) \simeq_{t\to\infty} t^{-\alpha},$$

(in a sense which will be made precise later on). Then, for $z \in \mathbb{C}\backslash\mathbb{R}$, consider the following fractional moment of the resolvent:

$$y_z(\beta) = \frac{1}{n}\sum_{k=1}^{n} (\Im(A - zI)^{-1}_{kk})^{\beta}, \tag{1}$$

(y_z, as other variables in this text, may depend implicitly in n). For $\beta = 1$, as n goes to infinity and then z goes to $E \in \mathbb{R}$ on the real line, $y_z(1)/\pi$ converges towards the spectral density which turns out to be positive [12, 11, 15]. However, we proved in [16] that for $\beta = \alpha/2$, and for sufficiently heavy tails ($0 < \alpha < 2/3$), as n goes to infinity and then z goes to $E \in \mathbb{R}$ large enough, $y_z(\alpha/2)$ goes to zero. This can be shown to imply a localization of eigenvectors with large enough eigenvalue or energy. On the other end, we prove in the present article that for $0 < \alpha < 2$ (outside a countable subset of $(0, 2)$), as n goes to infinity and then z goes to $E \in \mathbb{R}$ small enough, $y_z(\alpha/2)$ is bounded below by a positive constant. Back to eigenvectors, this in turn allows to prove delocalization at small energies versus localization at high energies according to the following criterion. Consider an orthonormal basis of $\mathbb{R}^n$ of eigenvectors of A. Let $I = [E - \eta_n, E + \eta_n]$ be an interval of the real line so that η_n goes to zero as n goes to infinity. Let Λ_I denote the set of eigenvectors of A with eigenvalues in I and set, if Λ_I is not empty,

$$Q_I = n\sum_{k=1}^{n} \left(\frac{1}{|\Lambda_I|} \sum_{u\in\Lambda_I} \langle u, e_k\rangle^2 \right)^2 .$$

We will explain below why Q_I is related to the nature of eigenvectors in Λ_I. Then, the main result of this article is that for η_n going to zero more slowly than $n^{-\rho}$ for some $\rho > 0$, for E large enough, Q_I goes to infinity (Theorem 1.1), whereas for E small enough, it remains finite (Theorem 1.2). See Figure 1 for an illustration of known and predicted results on eigenvectors.

Let us now describe our results more precisely. For integer $n \geq 1$, we consider an array $(A_{ij})_{1\leq i\leq j\leq n}$ of i.i.d. real random variables and set, for $i > j$, $A_{ij} = A_{ji}$. Then, we define the random symmetric matrix:

$$A = (A_{ij})_{1\leq i,j\leq n}.$$

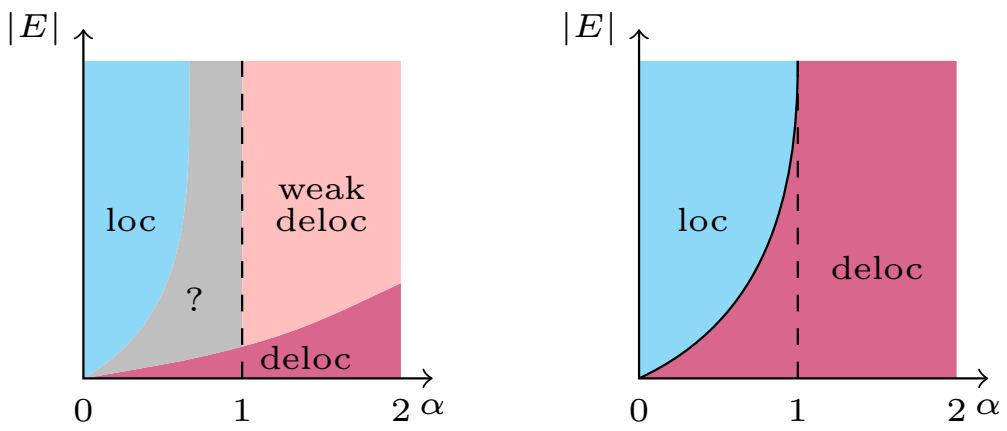


Figure 1: Left: illustration of known results on eigenvectors. The blue 'loc' zone represents the localization regime in Theorem 1.1, the red 'deloc' zone is Theorem 1.2. The pink 'weak deloc' zone is the L^∞-bound on eigenvectors (5), [16, Theorem 1.2]. Right: the phase diagram of Bouchaud-Cizeau [17] as corrected by Tarquini-Biroli-Tarzia [33]. The mobility edge is the boundary of the localization and delocalization regimes.

The eigenvalues of the matrix A are real and are denoted by $\lambda_n \leq \cdots \leq \lambda_1$. We also consider an orthogonal basis $(u_1, \ldots, u_n)$ of $\mathbb{R}^n$ of eigenvectors of A, for $1 \leq k \leq n$, $Au_k = \lambda_k u_k$.

If $X_{11} = \sqrt{n}A_{11}$ is a random variable independent of n and with variance equal to 1, then A is a normalized Wigner matrix. In the large n limit, the spectral properties of this matrix are now well understood, see e.g. [10, 14, 9, 20, 32]. The starting point of this analysis is the Wigner's semicircle law, which asserts that for any interval $I \subset \mathbb{R}$, the proportion of eigenvalues in I is asymptotically close to $\mu_{sc}(I)$ where μ_{sc} is the distribution with support $[-2, 2]$ and density $f_{sc}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$. Many more properties of the spectrum are known. For example, if X_{11} is centered and has a subexponential tail, then, from [22, 23], for any $p \in (2, \infty]$ and $\varepsilon > 0$, with high probability,

$$\max_{1 \leq k \leq n} \|u_k\|_p \leq n^{1/p - 1/2 + \varepsilon}, \tag{2}$$

where for $u \in \mathbb{R}^n$, $\|u\|_p = (\sum_{i=1}^n |u_i|^p)^{1/p}$ and $\|u\|_\infty = \max |u_i|$. This implies that the eigenvectors are strongly delocalized with respect to the canonical basis.

In this paper, we are interested in heavy-tailed matrices, it corresponds to the assumption that the measure defined on $\mathbb{R}_+ = (0, \infty)$,

$$L_n(\cdot) = n\mathbb{P}(|A_{11}|^2 \in \cdot) \tag{3}$$

converges vaguely as n goes to infinity to a (non trivial) Radon measure L on $\mathbb{R}_+$. For example, if A_{11} is a Bernoulli 0-1 variable with mean c/n, then L is equal $c\delta_1$. In this case, (up to the irrelevant diagonal terms) A is the adjacency matrix of an Erdős-Rényi graph, where each edge is present

independently with probability c/n. In this paper, we will focus on Lévy matrices introduced by Bouchaud and Cizeau [17]. They can be defined as follows. We fix $0 < \alpha < 2$ and assume that $X_{11} = n^{1/\alpha} A_{11}$ is a random variable independent of n such that

$$\mathbb{P}(|X_{11}| \geq t) \sim_{t\to\infty} t^{-\alpha}. \tag{4}$$

In the above setting, they correspond to the case $L = (\alpha/2)x^{-\alpha/2-1}dx$, where dx is the Lebesgue measure on $\mathbb{R}_+$. For technical simplicity, we will further restrict to X_{11} a symmetric α-stable random variable such that for all $t \in \mathbb{R}$,

$$\mathbb{E}\exp(itX_{11}) = \exp(-\sigma^\alpha |t|^\alpha),$$

with $\sigma^\alpha = \pi/(2\sin(\pi\alpha/2)\Gamma(\alpha))$. With this choice, the random variable X_{11} is normalized in the sense that (4) holds.

The spectrum of heavy-tailed matrices is far from being perfectly understood. It differs significantly from classical Wigner matrices. This model of random matrix admits, as scaling limit, a self-adjoint operator on a random tree (see [15]). In the Lévy case (4), for any interval $I \subset \mathbb{R}$, the proportion of eigenvalues in I is asymptotically close to $\mu_\star(I)$ where the probability measure $\mu_\star$ depends on $0 < \alpha < 2$, it is symmetric, has support $\mathbb{R}$, a bounded density $f_\star$ which is analytic outside a finite set of points. Moreover, $f_\star(0)$ has an explicit expression and as x goes to $\pm\infty$, $f_\star(x) \sim (\alpha/2)|x|^{-\alpha-1}$, see [19, 13, 11, 15].

The eigenvectors of Lévy matrices have been rigorously studied in [16]. It is shown there that if $1 < \alpha < 2$, there is a finite set $\mathcal{E}_\alpha$ such if $K \subset \mathbb{R}\backslash\mathcal{E}_\alpha$ is a compact set, for any $\varepsilon > 0$, with high probability,

$$\max\{\|u_k\|_\infty : \lambda_k \in K\} \leq n^{-\rho+\varepsilon}, \tag{5}$$

where $\rho = (\alpha-1)/((2\alpha) \vee (8-3\alpha))$. Since $\|u\|_p \leq \|u\|_\infty^{1-2/p}\|u\|_2^{2/p}$, it implies that the L^p-norm of most eigenvectors is $O(n^{2\rho/p-\rho+o(1)})$. Notice that when $\alpha \to 2$, then $\rho \to 1/4$ and it does not match with (2). We do not expect the upper bound (5) to be optimal.

When $0 < \alpha < 1$, the situation turns out to be very different. In [17], Bouchaud and Cizeau have conjectured the existence of a mobility edge, $E_\alpha > 0$, such that all eigenvectors u_k with $|\lambda_k| < E_\alpha - o(1)$ are delocalized in a sense similar to (5) while eigenvectors u_k with $|\lambda_k| > E_\alpha + o(1)$ are localized, that is they have a sparse representation in the canonical basis. A nearly-explicit formula for E_α is computed in [33]. In [16], the existence of this localized phase was established when $0 < \alpha < 2/3$. More precisely, for I an interval of $\mathbb{R}$, as above Λ_I is the set of eigenvectors whose eigenvalues are in I. Then, if Λ_I is not empty, for $1 \leq k \leq n$, we set

$$P_I(k) = \frac{1}{|\Lambda_I|}\sum_{u\in\Lambda_I} \langle u, e_k\rangle^2,$$

where $|\Lambda_I|$ is the cardinal of Λ_I. In words, $P_I(k)$ is the average amplitude of the k-th coordinate of eigenvectors in Λ_I. Theorem 1.1 in [16] asserts that $|\Lambda_I|$ is of order $n|I|$ for intervals I of length

$|I| \geq n^{-\rho}$ for some $\rho > 0$ (depending on α). By construction, P_I is a probability vector:

$$\sum_{k=1}^{n} P_I(k) = 1.$$

Observe also that $P_{\mathbb{R}}(k) = 1/n$. If the eigenvectors in Λ_I are localized and I contains few eigenvalues, then we might expect that for some k, $P_I(k) \gg 1/n$, while for most of the others $P_I(k) = o(1/n)$. Alternatively, if the eigenvectors in Λ_I are well delocalized, then $P_I(k) = O(1/n)$ for all k. More quantitatively, we will measure the (de)localization of eigenvectors through

$$Q_I = n \sum_{k=1}^{n} P_I(k)^2 \ \in [1, n]. \tag{6}$$

The scalar $\log(Q_I)$ is proportional to the Rényi divergence of order 2 of P_I with respect to the uniform measure $(1/n, \ldots, 1/n)$. If $Q_I \leq C$ then for any $t > 0$, the number of k such that $P_I(k) \geq t\sqrt{C}/n$ is at most n/t^2. The scalar Q_I is also closely related to the *inverse participation ratio* which can be defined as (the inverse of)

$$\Pi_I = \frac{n}{|\Lambda_I|} \sum_{u \in \Lambda_I} \sum_{k=1}^{n} \langle u, e_k \rangle^4 = \frac{n}{|\Lambda_I|} \sum_{u \in \Lambda_I} \|u\|_4^4 \ \in [1, n].$$

Using $\sum_{u \in S} x_u^4 \leq (\sum_{u \in S} x_u^2)^2 \leq |S| \sum_{u \in S} x_u^4$, we find

$$Q_I \leq \Pi_I \leq Q_I |\Lambda_I|.$$

We will write that a sequence of events E_n defined on our underlying probability space holds with overwhelming probability, if for any $t > 0$, $n^t \mathbb{P}(E_n^c)$ goes to 0 as n goes to infinity. As we shall check, [16, Theorem 1.3] implies the following localization statement.

Theorem 1.1 (Localization of eigenvectors of large eigenvalues [16])**.** *Let $0 < \alpha < 2/3$, $0 < \kappa < \alpha/2$ and $\rho = \alpha/(2 + 3\alpha)$. There exists $E_{\alpha,\kappa}$ such that for any compact $K \subset [-E_{\alpha,\kappa}, E_{\alpha,\kappa}]^c$, there is a constant $c_1 > 0$ and, if $I \subset K$ is an interval of length $|I| \geq n^{-\rho}(\log n)^2$, then*

$$Q_I \geq c_1 |I|^{-\frac{2\kappa}{2-\alpha}},$$

with overwhelming probability.

In this work, we shall prove the converse of this statement and prove notably that there exists a neighborhood of 0 where eigenvectors are delocalized.

Theorem 1.2 (Delocalization of eigenvectors of small eigenvalues)**.** *There exists a countable set $\mathcal{A} \subset (0, 2)$ with no accumulation point on $(0, 2) \backslash \{1/2, 1\}$ such that the following holds. Let $\alpha \in (0, 2) \backslash \mathcal{A}$ and $\rho' = \alpha/(4 + \alpha) \wedge 1/4$. There are $E'_\alpha > 0$ and constants $c_0, c_1 > 0$ such that, if $I \subset [-E'_\alpha, E'_\alpha]$ is an interval of length $|I| \geq n^{-\rho'}(\log n)^{c_0}$, then*

$$Q_I \leq c_1,$$

with overwhelming probability.

As we shall see in the course of the proof, Q_I is finite for $I = [E-\eta, E+\eta]$ with η going to zero iff the fractional moment of the resolvent $y_z(\alpha/2)$, defined in (1), is bounded below by a positive constant as n goes to infinity. Our point will therefore be to provide such a bound.

We note that the parameter Q_I could be replaced for any $p > 1$ by

$$n^{p-1} \sum_{k=1}^{n} P_I(k)^p.$$

Then, the statements of Theorem 1.1 and Theorem 1.2 are essentially unchanged (up to modifying the value of ρ' in Theorem 1.2 and the exponent $\delta > 0$ in the lower bound $Q_I \geq c_1 |I|^{-\delta}$ in Theorem 1.1). We have chosen to treat the case $p = 2$ for its connection with the inverse participation ratio.

There are still many open problems concerning Lévy matrices. In the forthcoming Corollary 5.13, we prove a local law (i.e. a sharp quantitative estimate of $|\Lambda_I|$ for intervals I of vanishing size) which improves for small value of α on Theorem 1.1 in [16]. We conjecture that such local law holds for all $\alpha \in (0, 2)$ and for all intervals I of length much larger that $1/n$.

There is no rigorous results on the local eigenvalue statistics for Lévy matrices, see [33] for a recent account of the predictions in the physics literature. It is expected that for $1 < \alpha < 2$ the local eigenvalue statistics are similar to those of the Gaussian Orthogonal Ensemble (GOE). For $0 < \alpha < 1$ and energies larger than some E_α, we expect that the local eigenvalue statistics are asymptotically described by a Poisson point process. In the regime $0 < \alpha < 1$ and energies smaller than E_α, [33] also predicts a GOE statistics.

For $0 < \alpha < 1$, proving the existence of such mobility edge E_α is already a very interesting open problem. The core of the difficulty is to better understand a fixed point equation which is satisfied by the weak limit of $(A - zI)^{-1}_{11}$ as n goes to infinity. More precisely, in [15] it is shown that the analytic function $z \mapsto (A - zI)^{-1}_{11}$ on $\mathbb{C}_+$ converges weakly for the finite dimensional convergence to the random analytic function $z \mapsto R_\star(z)$ on $\mathbb{C}_+$ which is the unique solution of the recursive distributional equation for all $z \in \mathbb{C}_+$,

$$R_\star(z) \stackrel{d}{=} - \left(z + \sum_{k \geq 1} \xi_k R_k(z) \right)^{-1}, \tag{7}$$

where $\{\xi_k\}_{k \geq 1}$ is a Poisson process on $\mathbb{R}_+$ of intensity measure $(\alpha/2) x^{-\alpha/2-1} dx$, independent of $(R_k)_{k \geq 1}$, a sequence of independent copies of $R_\star$. In [15], $R_\star(z)$ is shown to be the resolvent at a vector of a random self-adjoint operator defined on Aldous' Poisson Weighted Infinite Tree (for its precise definition, see [15]). Equation (7) is closely related to the equation satisfied by the resolvent of random Schrödinger operator on a tree (see the monograph [8]). The random diagonal potential in random Schrödinger operators is replaced here by a random hopping factor between the vertices of the tree. The mobility edge could be alternately described as the transition between absolutely continuous and singular spectrum for this random self-adjoint operator defined on Aldous' Poisson Weighted Infinite Tree. Without entering into more details, we may mention

that Theorem 1.1 and Theorem 1.2 have a consequence for the spectral decomposition of this random operator (respectively, almost surely, no absolutely continuous spectrum at high energies for $0 < \alpha < 2/3$ and purely absolutely continuous spectrum at low energies). Also, it would be interesting to understand to which extent the methods of [5, 7] are applicable for this random operator.

More generally, the Lévy matrix is an example of a broader class of random matrices with heavy tails. The qualitative behavior of the spectrum depends on the Radon measure L_n in (3) and its vague limit which we denoted by L. It is a challenging question to understand how L influences the nature of the spectrum around a given energy (regularity of the limiting spectral measure, localization of eigenvectors, local eigenvalue statistics).

The paper is organized as follows. In Section 2, we prove that Theorem 1.1 is a direct consequence of a result in [16]. Section 3 gives an outline of the proof of Theorem 1.2. The actual proof is contained in Section 4 and Section 5.

2 Proof of Theorem 1.1

We use a localization estimate proved in [16]. For $0 < \alpha < 2/3$, $0 < \kappa < \alpha/2$ and $\rho = \alpha/(2+3\alpha)$, it follows from [16, Theorem 1.3] that there exists $E_{\alpha,\kappa}$ such that for any compact $K \subset [-E_{\alpha,\kappa}, E_{\alpha,\kappa}]^c$, there are constants $c_0, c_1 > 0$ and for all integers $n \geq 1$, if $I \subset K$ is an interval of length $|I| \geq n^{-\rho}(\log n)^2$, then

$$n^{\alpha/2-1} \sum_{k=1}^{n} P_I(k)^{\alpha/2} \leq c_1 |I|^{\kappa}, \tag{8}$$

with overwhelming probability.

We may use duality to obtain from (8) a lower bound on Q_I. From Hölder inequality, we write for $0 < \varepsilon < 1$ and $1/p + 1/q = 1$,

$$\begin{aligned} 1 = \sum_{k=1}^{n} P_I(k) &= \frac{1}{n} \sum_{k=1}^{n} (nP_I(k))^{\varepsilon} (nP_I(k))^{1-\varepsilon} \\ &\leq \left(n^{\varepsilon p - 1} \sum_{k=1}^{n} P_I^{\varepsilon p}(k) \right)^{1/p} \left(n^{(1-\varepsilon)q - 1} \sum_{k=1}^{n} P_I^{(1-\varepsilon)q}(k) \right)^{1/q}. \end{aligned}$$

We choose $\varepsilon = \alpha/(4-\alpha)$ and $p = 2 - \alpha/2$. We have $\varepsilon p = \alpha/2$, $(1-\varepsilon)q = 2$ and $p/q = 1 - \alpha/2$. Hence, if the event (8) holds, we deduce that

$$(c_1 |I|^{\kappa})^{-\frac{q}{p}} = c_1' |I|^{-\frac{2\kappa}{2-\alpha}} \leq Q_I.$$

It completes the proof of Theorem 1.1.

3 Outline of proof of Theorem 1.2

3.1 Connection with the resolvent

For $z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$, the resolvent matrix of A is defined as

$$R(z) = (A - zI)^{-1}.$$

The next lemma shows that the quadratic mean of the diagonal coefficients of the resolvent upper bounds Q_I.

Lemma 3.1. *Let $\eta > 0$, $I = [\lambda - \eta, \lambda + \eta]$ and $z = \lambda + i\eta \in \mathbb{C}_+$. If $|\Lambda_I| \neq 0$, we have*

$$Q_I \leq \left(\frac{n|I|}{|\Lambda_I|}\right)^2 \left(\frac{1}{n}\sum_{k=1}^{n} (\Im R_{kk}(z))^2\right).$$

Proof. We use the classical bound for $1 \leq k \leq n$,

$$\sum_{u \in \Lambda_I} \langle u, e_k\rangle^2 \leq \sum_{j=1}^{n} \frac{2\eta^2 \langle u_j, e_k\rangle^2}{\eta^2 + (\lambda_j - \lambda)^2} = 2\eta \Im R_{kk}(z)$$

We get,

$$Q_I \leq \frac{4n\eta^2}{|\Lambda_I|^2} \sum_{k=1}^{n} (\Im R_{kk}(z))^2,$$

as requested. □

Incidentally, we remark from [16, Lemma 5.9] and the above proof of Theorem 1.1 that there is a converse lower bound of Q_I involving the average of $(\Im G_{kk}(z))^\beta$ for any $0 < \beta < 1$.

We may now briefly describe the strategy behind Theorem 1.2. Take $I = [\lambda - \eta, \lambda + \eta] \subset K$ be an interval and $z = \lambda + i\eta$. First, from [16, Theorem 1.1], there exists a constant $c > 0$, such that, with overwhelming probability, $|\Lambda_I| \geq cn|I|$. Thanks to Lemma 3.1, it is thus sufficient to prove that

$$\frac{1}{n}\sum_{k=1}^{n} (\Im R_{kk}(z))^2 = O(1).$$

From general concentration inequalities, it turns out that the above quantity is self averaging for $\eta \geq n^{-\rho}$. Using the exchangeability of the coordinates, it remains to prove that

$$\mathbb{E}(\Im R_{11}(z))^2 = O(1).$$

Now, the law of $R_{11}(z)$ converges as n goes to infinity to a limit random variable, say $R_\star(z)$, whose law satisfies a fixed point equation. In Section 4, in the spirit of [27], we will study this fixed point and prove, by an implicit function theorem, that $\mathbb{E}(\Im R_\star(z))^2 = O(1)$. In Section 5, it will remain to establish an explicit convergence rate of $R_{11}(z)$ to $R_\star(z)$ to conclude the proof of Theorem 1.2. A careful choice of the norm for this convergence will be very important. We outline the content of these two sections in the next subsection.

3.2 The fixed point equations

The starting point in our approach is Schur's complement formula,

$$\begin{aligned} R_{11}(z) &= -\left(z - n^{-\frac{1}{\alpha}} X_{11} + n^{-\frac{2}{\alpha}} \sum_{2\le k,\ell\le n} X_{1k} X_{1\ell} R^{(1)}_{k\ell}(z) \right)^{-1} \\ &= -\left(z + n^{-\frac{2}{\alpha}} \sum_{2\le k\le n} X_{1k}^2 R^{(1)}_{kk}(z) + T_z \right)^{-1}, \end{aligned} \tag{9}$$

where $R^{(1)}$ is the resolvent of the $(n-1)\times(n-1)$ matrix $(X_{k\ell})_{2\le k,\ell\le n}$ and we have set

$$T_z = -n^{-\frac{1}{\alpha}} X_{11} + n^{-\frac{2}{\alpha}} \sum_{2\le k\ne\ell\le n} X_{1k} X_{1\ell} R^{(1)}_{k\ell}(z).$$

It turns out that T_z is negligible, at least for $\Im z$ large enough. Assuming that, we observe that the moments of $\Im R_{11}(z)$ are governed by the order parameter

$$y_z = \frac{1}{n} \sum_{k=1}^{n} (\Im R_{kk}(z))^{\frac{\alpha}{2}}.$$

Indeed,

$$\begin{aligned} \Im R_{11}(z) &\simeq -\Im \left(z + n^{-\frac{2}{\alpha}} \sum_{2\le k\le n} X_{1k}^2 R^{(1)}_{kk}(z) \right)^{-1} \\ &\le \left(n^{-\frac{2}{\alpha}} \sum_{2\le k\le n} X_{1k}^2 \Im R^{(1)}_{kk}(z) \right)^{-1}. \end{aligned}$$

The resolvent R and $R^{(1)}$ being close, we can justify that $y_z \simeq \frac{1}{n}\sum_{k=2}^{n} (\Im R^{(1)}_{kk}(z))^{\frac{\alpha}{2}}$. Then, taking moments and using the formula, for $p>0$, $\Re(x)>0$,

$$\frac{1}{x^p} = \frac{1}{\Gamma(p)} \int_0^\infty t^{p-1} e^{-xt} dt, \tag{10}$$

we deduce that

$$\begin{aligned} \mathbb{E}[(\Im R_{11}(z))^p] &\le \frac{1}{\Gamma(p)} \int_0^\infty t^{p-1} \mathbb{E}[e^{-t(\Im R_{11}(z))^{-1}}] dt \\ &\lesssim \frac{1}{\Gamma(p)} \int_0^\infty t^{p-1} e^{-\Gamma(1-\alpha/2) t^{\alpha/2} n^{-1} \sum_{k=2}^{n} (\Im R^{(1)}_{kk}(z))^{\frac{\alpha}{2}}} dt \\ &\lesssim \frac{1}{\Gamma(p)} \int_0^\infty t^{p-1} e^{-\Gamma(1-\alpha/2) t^{\alpha/2} y_z} dt, \end{aligned} \tag{11}$$

where, in the second step, we used that the variable X_{1k}^2 are in the domain of attraction of the non-negative $\alpha/2$-stable law (this approximation will be made more precise notably thanks to Lemma

5.7). The main point becomes to lower bound y_z. To this end, we shall extend it as a function on $\mathcal{K}_1 = \{u \in \mathbb{C} : \arg(u) \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$ and set

$$\gamma_z(u) = \Gamma\Big(1 - \frac{\alpha}{2}\Big) \times \frac{1}{n} \sum_{k=1}^{n} (-iR_{kk}(z).u)^{\frac{\alpha}{2}}$$

where

$$h.u = \Re(u)h + \Im(u)\bar{h} = (\Re(u) + \Im(u))\Re(h) + i(\Re(u) - \Im(u))\Im(h).$$

Observe that we wish to lower bound

$$\gamma_z(e^{i\pi/4}) = 2^{\frac{\alpha}{4}}\Gamma\Big(1 - \frac{\alpha}{2}\Big) \times \frac{1}{n} \sum_{k=1}^{n} (\Im(R_{kk}(z)))^{\frac{\alpha}{2}} = 2^{\frac{\alpha}{4}}\Gamma\Big(1 - \frac{\alpha}{2}\Big) y_z \,.$$

We shall study the function γ_z thanks to a fixed point argument. We shall use that γ_z is homogeneous. Also, we shall further restrict ourselves to u in the first quadrant $\mathcal{K}_1^+ = \{z \in \mathbb{C} : \arg(z) \in [0, \frac{\pi}{2}]\}$. Here and after, for $z \in \mathbb{C}$, we take the argument $\arg(z)$ in $(-\pi, \pi]$. We can see that γ_z is approximately solution of a fixed point equation by using (9). To state this result, let us first define the space $\mathcal{H}_{\alpha/2,\kappa}$, $\kappa \in [0, 1)$, in which we will consider γ_z. For any $\beta \in \mathbb{C}$, we let $\mathcal{H}_\beta$ denote the space of C^1 functions g from $\mathcal{K}_1^+$ to $\mathbb{C}$ such that $g(\lambda u) = \lambda^\beta g(u)$, for all $\lambda \geq 0$. For $\kappa \in [0, 1)$, we endow $\mathcal{H}_\beta$ with the norms

$$\begin{aligned} \|g\|_\infty &= \sup_{u \in S_1^+} |g(u)|, \\ \|g\|_\kappa &= \|g\|_\infty + \sup_{u \in S_1^+} \sqrt{|(i.u)^\kappa \partial_1 g(u)|^2 + |(i.u)^\kappa \partial_i g(u)|^2}, \end{aligned} \tag{12}$$

where $S_1^+ = \{u \in \mathcal{K}_1^+, |u| = 1\}$ and $\partial_\varepsilon g(u) \in \mathbb{C}$ is the partial derivative of g at u with respect to the real ($\varepsilon = 1$) or imaginary part ($\varepsilon = i$) of u. We denote $\mathcal{H}_{\beta,\kappa}$ the closure of $\mathcal{H}_\beta$ for $\|.\|_\kappa$. The space $\mathcal{H}_{\beta,\kappa}$ is a Banach space. Notice also that $\mathcal{H}_{\beta,0}$ and $\mathcal{H}_\beta$ coincide. The norm $\|.\|_\kappa$ will turn out to be useful with $\kappa > 0$ to obtain concentration estimates for γ_z as well as to establish existence and good properties for its limit $\gamma_z^\star$ (there $\kappa \geq 0$ is sufficient).

We define formally the function F given for $h \in \mathcal{K}_1$, $u \in S_+^1$ and $g \in \mathcal{H}_{\alpha/2}$ by

$$F_h(g)(u) = \int_0^{\frac{\pi}{2}} d\theta (\sin 2\theta)^{\frac{\alpha}{2}-1} \int_0^\infty dy\, y^{-\frac{\alpha}{2}-1} \int_0^\infty dr\, r^{\frac{\alpha}{2}-1} e^{-rh.e^{i\theta}} \left(e^{-r^{\frac{\alpha}{2}} g(e^{i\theta})} - e^{-yrh.u} e^{-r^{\frac{\alpha}{2}} g(e^{i\theta}+yu)} \right). \tag{13}$$

F_{-iz} is related to a fixed point equation satisfied by γ_z. Namely let

$$c_\alpha = \frac{\alpha}{2^{\frac{\alpha}{2}}\Gamma(\alpha/2)^2} \quad \text{and} \quad \check{u} = i\bar{u} = \Im(u) + i\Re(u). \tag{14}$$

For $z \in \mathbb{C}_+$, we introduce the map G_z on $\mathcal{H}_{\alpha/2}$ given by

$$G_z(f)(u) = c_\alpha F_{-iz}(f)(\check{u}) \,. \tag{15}$$

Finally we let

$$\bar{\gamma}_z(u) = \mathbb{E}\gamma_z(u) = \Gamma\Big(1-\frac{\alpha}{2}\Big)\mathbb{E}(-iR_{11}(z).u)^{\frac{\alpha}{2}}.$$

Then, $\bar{\gamma}_z(u) \in \mathcal{H}_{\alpha/2}$ and we shall prove that

Proposition 3.2. *Let $\alpha \in (0,2)$ and $\delta \in (0,\alpha/2)$. There exists $c>0$ such that if for $z \in \mathbb{C}_+$ and some $\varepsilon > 0$, we have $|z| \le \varepsilon^{-1}$, $\Im z \ge n^{-\delta/\alpha+\varepsilon}$, $\mathbb{E}(\Im R_{11}(z))^{\alpha/2} \ge \varepsilon$ and $\mathbb{E}|R_{11}(z)| \le \varepsilon^{-1}$ then for all $p \ge \alpha/2$ and all n large enough (depending on α, ε, p),*

$$\begin{aligned}
\|\bar{\gamma}_z - G_z(\bar{\gamma}_z)\|_{1-\alpha/2+\delta} &\le (\log n)^c\Big(\eta^{-\alpha/2}\bar{M}_z^{\alpha/2} + \eta^{-\alpha/2}n^{-\delta/2} + \bar{M}_z^{1-\alpha/2}\mathbf{1}_{\alpha>1}\Big)\\
|\mathbb{E}|R_{11}(z)|^p - r_{p,z}(\bar{\gamma}_z)| &\le (\log n)^c\Big(\eta^{-p}\bar{M}_z^{\alpha/2} + \eta^{-\alpha/2}n^{-\delta/2}\Big)\\
|\mathbb{E}(-iR_{11}(z))^p - s_{p,z}(\bar{\gamma}_z(1))| &\le (\log n)^c\Big(\eta^{-p}\bar{M}_z^{\alpha/2} + \eta^{-\alpha/2}n^{-\delta/2}\Big),
\end{aligned}$$

where we have set $\eta = \Im z$, $\bar{M}_z = \mathbb{E}\Im R_{22}^{(1)}(z)/(n\Im z)$ and, for $f \in \mathcal{H}_{\alpha/2}$

$$r_{p,z}(f) = \frac{2^{1-p/2}}{\Gamma(p/2)^2}\int_0^{\frac{\pi}{2}} d\theta \sin(2\theta)^{p/2-1}\int_0^\infty dr r^{p-1}e^{r(iz).e^{i\theta}-r^{\alpha/2}f(e^{i\theta})},$$

and for $x \in \mathcal{K}_1$,

$$s_{p,z}(x) = \frac{1}{\Gamma(p)}\int_0^\infty dr r^{p-1}e^{irz-r^{\alpha/2}x}.$$

Using that $R^{(1)}$ is close to R, we can upper bound $\bar{M}_z$ if we can lower bound $\bar{\gamma}_z$ by (11). Similarly, we can upper bound $\mathbb{E}|R_{11}|^p$ if we can lower bound $\bar{\gamma}_z$ by (11). Assuming for a moment that we can obtain such bounds (by using a bootstrap argument) the above proposition shows that $\bar{\gamma}_z$ is approximately a fixed point for G_z.

It turns out that, for any $z \in \mathbb{C}_+$, $\bar{\gamma}_z$ converges to $\gamma_z^\star \in \mathcal{H}_{\alpha/2}$ as n goes to infinity, where $\gamma_z^\star$ is a solution of the equation $f = G_z(f)$ (even we cannot prove that there is a unique solution of this equation). We will check that this last equation has a unique explicit solution of interest for $z = it$, $t \ge 0$ with $\gamma_0^\star$ also in $\mathcal{H}_{\alpha/2}$. We will study the solutions of the equation $f = G_z(f)$ for z close to 0 and f close to $\gamma_0^\star$ thanks to the Implicit Function Theorem on Banach space. We will show that for most α in $(0,2)$, if $|z|$ is small enough, G_z has a unique fixed point in the neighborhood of $\gamma_0^\star$. Moreover, the real part of this solution is lower bounded by a positive constant. Let us summarize these results in the following statement:

Proposition 3.3. *There exists a countable subset $\mathcal{A} \subset (0,2)$ with no accumulation point on $(0,2)\backslash\{1/2,1\}$ such that the following holds. Let $\kappa \in [0,1)$ and $\alpha \in (0,2)\backslash\mathcal{A}$. There exists $\tau > 0$ such that if $|z| \le \tau$, then $\gamma_z^\star$ is the unique $f \in \mathcal{H}_{\alpha/2,\kappa}$ such that $f = G_z(f)$ and $\|f - \gamma_0^\star\|_\kappa \le \tau$. Moreover, uniformly in $|z| \le \tau$, $\gamma_z^\star(e^{i\pi/4})$ is real and bounded from below and, for any $p > 0$, $r_{p,z}(\gamma_z^\star)$ is bounded from above.*

The possible existence of the set $\mathcal{A}$ should be purely technical. Our proof requires that $\mathcal{A}$ contains $\{1/2, 1\}$, but it could be larger as our argument is based on the fact that some function, analytic in α, does not vanish except possible on a set with no accumulation points.

We will also deduce the following result.

Proposition 3.4. *Let $\mathcal{A}$ be as in Proposition 3.3, $\alpha \in (0,2)\backslash\mathcal{A}$, $\kappa \in [0,1)$ and $\gamma_z^\star$ be as in Proposition 3.3 for $|z|$ small enough. There exist $\tau > 0$ and $c > 0$ such that if $|z| \le \tau$ and $\|\gamma - \gamma_z^\star\|_\kappa \le \tau$ then*

$$\|\gamma - \gamma_z^\star\|_\kappa \le c\,\|\gamma - G_z(\gamma)\|_\kappa.$$

As a corollary of the above three propositions, we will prove that $\gamma_z(e^{i\pi/4})$ is lower bounded for n sufficiently large. In the next section, we study the fixed point equation for G_z and establish Proposition 3.3 and Proposition 3.4. In Section 5, we will prove Proposition 3.2 and complete the details of the proof of Theorem 1.2.

4 Analysis of the limiting fixed point equation

In this section, we study the fixed point equation $f = G_z(f)$ where G_z was defined by (15). We start with general properties of the function G_z in Subsection 4.1. Then, in Subsection 4.2, we will study the regime z in a neighborhood of 0 thanks to an implicit function theorem and provide the proofs of Proposition 3.3 and Proposition 3.4.

4.1 Analysis of the function F_h

In this first part, we show that the function F_h is well defined as a map from $\mathcal{H}^0_{\alpha/2,\kappa} = \cup_{\varepsilon>0}\mathcal{H}^\varepsilon_{\alpha/2,\kappa}$ into $\mathcal{H}_{\alpha/2,\kappa}$ with $\mathcal{H}^\varepsilon_{\alpha/2,\kappa}$ the set of functions in $f \in \mathcal{H}_{\alpha/2,\kappa}$ such that $\Re f(u) > \varepsilon$ for all u in S_1^+. We also check that F_h has good regularity properties. Notice that $\mathcal{H}^0_{\alpha/2,\kappa}$ is an open subset of $\mathcal{H}_{\alpha/2,\kappa}$ for our $\|\cdot\|_\kappa$ norm. We set $\mathcal{H}^\varepsilon_{\alpha/2} = \mathcal{H}^\varepsilon_{\alpha/2,0}$. Finally, the closure $\bar{\mathcal{H}}^0_{\alpha/2,\kappa}$ of $\mathcal{H}^0_{\alpha/2,\kappa}$ is the set of functions in $\mathcal{H}_{\alpha/2,\kappa}$ whose real part is non-negative on $\mathcal{K}_1^+$.

Lemma 4.1. *Let $h \in \mathcal{K}_1$ and $0 < \alpha < 2$. Let $\kappa \in [0,1)$. F_h defines a map from $\mathcal{H}^0_{\alpha/2,\kappa}$ to $\mathcal{H}_{\alpha/2,\kappa}$. Moreover, if $\Re(h) > 0$, F_h defines a map from $\bar{\mathcal{H}}^0_{\alpha/2,\kappa}$ to $\bar{\mathcal{H}}^0_{\alpha/2,\kappa}$ and, for some constant $c = c(\alpha) > 0$,*

$$\|F_h(g)\|_\kappa \le \frac{c}{\Re(h)^{\alpha/2}} + \frac{c}{\Re(h)^\alpha}\|g\|_\kappa\,. \tag{16}$$

Finally, if $g \in \bar{\mathcal{H}}^0_{\alpha/2,\kappa}$, $|r_{p,ih}(g)| \le c/\Re(h)^p$ and $|s_{p,ih}(g(1))| \le c/\Re(h)^p$ for some constant $c = c(\alpha, p)$.

We shall also prove that F_h is Fréchet differentiable and more precisely

Lemma 4.2. *Let $h \in \mathcal{K}_1$, $0 < \alpha < 2$, $\kappa \in [0,1)$, and $g \in \mathcal{H}^0_{\alpha/2,\kappa}$. The Fréchet derivative of F_h at g is the bounded operator given, for any $f \in \mathcal{H}_{\alpha/2,\kappa}$, by*

$$DF_h(g)(f)(u) = \int_0^{\frac{\pi}{2}} d\theta (\sin 2\theta)^{\frac{\alpha}{2}-1} \int_0^\infty dy\, y^{-\frac{\alpha}{2}-1} \times \int_0^\infty dr\, r^{\alpha-1} e^{-rh.e^{i\theta}} \left(f(e^{i\theta}) e^{-r^{\frac{\alpha}{2}} g(e^{i\theta})} - f(e^{i\theta}+yu) e^{-yrh.u} e^{-r^{\frac{\alpha}{2}} g(e^{i\theta}+yu)} \right).$$

Moreover, $(h,g) \mapsto F_h(g)$ is continuously differentiable on $\mathcal{K}_1 \times \mathcal{H}^0_{\alpha/2,\kappa}$ and $(h,g) \mapsto DF_h(g)$ is continuous in $\mathcal{K}_1 \times \mathcal{H}^0_{\alpha/2,\kappa} \to \mathcal{B}(\mathcal{H}_{\alpha/2,\kappa}, \mathcal{H}_{\alpha/2,\kappa})$.

As a corollary we shall see that all the functions defined in Proposition 3.2 are Lipschitz in some appropriate norm.

Lemma 4.3. *For any $\alpha \in (0,2)$, $\kappa \in [0,1)$ and $a > 0$, G_z is Lipschitz on $\mathcal{H}^a_{\frac{\alpha}{2},\kappa}$: there exists $c > 0$ such that for any $z \in \mathbb{C}$ and $f, g \in \mathcal{H}^a_{\frac{\alpha}{2},\kappa}$,*

$$\|G_z(f) - G_z(g)\|_\kappa \le c\, \|f-g\|_\kappa + c(\|f\|_\kappa + \|g\|_\kappa)\|f-g\|_\infty .$$

Similarly, for any $z \in \mathbb{C}$ and $f, g \in \mathcal{H}^a_{\frac{\alpha}{2},\kappa}$, any $x, y \in \mathcal{K}_1$, $\Re(x) \wedge \Re(y) \ge a$, any $p > 0$,

$$|r_{p,z}(f) - r_{p,z}(g)| \le c\, \|f-g\|_\infty \quad \text{and} \quad |s_{p,z}(x) - s_{p,z}(y)| \le c\, |x-y|.$$

Proof of Lemmas 4.1 and 4.2. We treat simultaneously the case where $g \in \mathcal{H}^a_{\alpha/2,\kappa}$ for some $a > 0$ or $g \in \bar{\mathcal{H}}^0_{\alpha/2,\kappa}$ but $\Re(h) \ge b > 0$. We set for $\zeta \ge 0$ (mainly $\zeta = 0, \alpha, \alpha/2, 1$) and $f \in \mathcal{H}_{\zeta,\kappa}$,

$$\varphi^{\zeta,h}_{g,f}(u) = \int_0^{\frac{\pi}{2}} d\theta (\sin 2\theta)^{\frac{\alpha}{2}-1} \int_0^\infty dy\, y^{-\frac{\alpha}{2}-1} \times \int_0^\infty dr\, r^{\frac{\alpha}{2}+\zeta-1} \left(f(e^{i\theta}) e^{-rh.e^{i\theta}} e^{-r^{\frac{\alpha}{2}} g(e^{i\theta})} - f(e^{i\theta}+yu) e^{-rh.(e^{i\theta}+yu)} e^{-r^{\frac{\alpha}{2}} g(e^{i\theta}+yu)} \right).$$

This corresponds to the definition of F_h when $f = 1$ and $\zeta = 0$, and to its derivative in the direction of f when $\zeta = \alpha/2$. We need to check that $\varphi^{\zeta,h}_{g,f}$ has finite norm. To take into account the singularity of the integration in y at the origin, we cut the integral over y in two pieces: one accounts for integration over $[0,1/2]$ and the other for the integration over $[1/2,+\infty)$. We let $\varphi^{\zeta,h}_{g,f}(u) = \varphi^{\zeta,h}_{g,f(1)}(0) - \varphi^{\zeta,h}_{g,f(1)}(u) + \varphi^{\zeta,h}_{g,f(2)}(u)$ with

$$\varphi^{\zeta,h}_{g,f(1)}(u) = \int_0^{\frac{\pi}{2}} d\theta (\sin 2\theta)^{\frac{\alpha}{2}-1} \int_{\frac{1}{2}}^\infty dy\, y^{-1-\frac{\alpha}{2}} \int_0^\infty dr\, r^{\frac{\alpha}{2}+\zeta-1} e^{-rh.(e^{i\theta}+yu)} f(e^{i\theta}+yu) e^{-r^{\frac{\alpha}{2}} g(e^{i\theta}+yu)},$$

$$\varphi^{\zeta,h}_{g,f(2)}(u) = \int_0^{\frac{\pi}{2}} d\theta (\sin 2\theta)^{\frac{\alpha}{2}-1} \int_0^{\frac{1}{2}} dy\, y^{-1-\frac{\alpha}{2}} \times \int_0^\infty dr\, r^{\frac{\alpha}{2}+\zeta-1} \left(f(e^{i\theta}) e^{-rh.e^{i\theta}} e^{-r^{\frac{\alpha}{2}} g(e^{i\theta})} - f(e^{i\theta}+yu) e^{-rh.(e^{i\theta}+yu)} e^{-r^{\frac{\alpha}{2}} g(e^{i\theta}+yu)} \right).$$

For the first expression, we can bound the integral uniformly in θ by using $\Re(h.u) \geq \Re(h)|u|$ for all $u \in \mathcal{K}_1^+$. We get a finite constant C such that

$$\begin{aligned}|\varphi^{\zeta,h}_{g,f(1)}(u)| &\leq C\|f\|_\infty \sup_{\theta\in[0,\frac{\pi}{2}]} \int_{\frac{1}{2}}^\infty dy\, y^{-1-\frac{\alpha}{2}} \int_0^\infty dr\, r^{\frac{\alpha}{2}-1+\zeta} |e^{i\theta}+yu|^\zeta e^{-ar^{\frac{\alpha}{2}}|e^{i\theta}+yu|^{\frac{\alpha}{2}}} e^{-br|e^{i\theta}+yu|} \\ &\leq C\|f\|_\infty \sup_{\theta\in[0,\frac{\pi}{2}]} \int_{\frac{1}{2}}^\infty dy\, y^{-1-\frac{\alpha}{2}} |e^{i\theta}+yu|^{-\frac{\alpha}{2}}\end{aligned}$$

which are bounded uniformly in $u \in \mathcal{K}_1^+$ (observe that $|e^{i\theta}+yu| \geq 1$ for $e^{i\theta} \in S_1^+$ and $y \geq 0$). The constant C depends on $a \vee b > 0$. We used (and will use again repeatidly) the following straightforward inequality : for any $h \in \mathcal{K}_1$, $x \in \mathcal{K}_1$ and $\beta > 0$,

$$\int_0^\infty \left| r^{\beta-1} e^{-rh} e^{-r^{\frac{\alpha}{2}}x} \right| dr \leq \min\left(\Gamma\left(\frac{2\beta}{\alpha}\right) \Re(x)^{-\frac{2\beta}{\alpha}}, \Gamma(\beta)\Re(h)^{-\beta} \right). \tag{17}$$

Similarly for $\varphi^{\zeta,h}_{g,f(2)}$, we have to bound, for $0 \leq y \leq 1/2$ and $0 \leq \theta \leq \pi/2$,

$$\begin{aligned}&L(r,y,\theta) := \left| e^{-rh.e^{i\theta}} \left(f(e^{i\theta}) e^{-r^{\frac{\alpha}{2}} g(e^{i\theta})} - f(e^{i\theta}+yu) e^{-yrh.u} e^{-r^{\frac{\alpha}{2}} g(e^{i\theta}+yu)} \right) \right| \qquad (18) \\ &\leq \left| (f(e^{i\theta}) - f(e^{i\theta}+yu) e^{-yrh.u}) e^{-r^{\frac{\alpha}{2}} g(e^{i\theta})} \right| e^{-br} + |f(e^{i\theta}+yu)| \left| e^{-r^{\frac{\alpha}{2}} g(e^{i\theta})} - e^{-r^{\frac{\alpha}{2}} g(e^{i\theta}+yu)} \right| e^{-br}.\end{aligned}$$

To bound increments of functions in terms of the κ-norm, let us use that the linearity of $x \mapsto i.x$ and that if $g \in \mathcal{H}_{\alpha/2}$ then its derivative is homogeneous of order $\alpha/2 - 1$. We get for $z, w \in \mathcal{K}_1^+$,

$$|g(z) - g(w)| \leq \|g\|_\kappa |z-w| \int_0^1 dt \frac{|z + t(w-z)|^{\kappa + \frac{\alpha}{2} - 1}}{|i.(z+t(w-z))|^\kappa}.$$

A similar bound holds for f with ζ in place of $\alpha/2$. Notice that for $z = e^{i\theta}$ and $w = e^{i\theta} + yu$, $z + t(w-z) = e^{i\theta} + tyu$. Using that $|e^{-x} - e^{-y}| \leq |x-y| e^{-\Re(x)\wedge\Re(y)}$ and $1 \leq |e^{i\theta} + tyu| \leq 3/2$ for $t \in [0,1]$ and $y \in [0, 1/2]$, we deduce that

$$L(r,y,\theta) \leq Cy(\|f\|_\kappa + \|f\|_\infty r + \|f\|_\infty \|g\|_\kappa r^{\frac{\alpha}{2}}) e^{-ar^{\frac{\alpha}{2}} - br} \phi(e^{i\theta}, e^{i\theta}+yu),$$

where C is a constant and

$$\phi(z,w) = 1 + \int_0^1 \frac{dt}{|i.(z+t(w-z))|^\kappa}.$$

For $z = e^{i\theta}$ and $w = e^{i\theta} + yu$,

$$|i.(z+t(w-z))| = |\cos(\theta) - \sin(\theta) + ty(\Re(u) - \Im(u))|.$$

Now, the $[0, \pi/2] \to [-1, 1]$ function $w(\theta) = \cos(\theta) - \sin(\theta)$ is decreasing and $|w'(\theta)| = \cos\theta + \sin\theta \in [1, \sqrt{2}]$. Since $|ty(\Re(u) - \Im(u))| \leq 1/2$, it follows that $\theta \mapsto |i.(z+t(w-z))|$ vanishes once at $\theta_0 \in (\delta, \pi/2 - \delta)$ for some $\delta > 0$. As a consequence, we find that since $\kappa < 1$,

$$\sup_{y\in[0,1/2], u\in S_1^+} \int_0^{\frac{\pi}{2}} d\theta |(\sin 2\theta)^{\frac{\alpha}{2}-1} |\phi(e^{i\theta}, e^{i\theta}+yu)| < +\infty.$$

Therefore, we can integrate $L(r,y,\theta)$ under θ and y to find that

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} d\theta(\sin 2\theta)^{\frac{\alpha}{2}-1}\int_0^{1/2} dy\, y^{-\frac{\alpha}{2}-1}\int_0^{\infty} dr\, r^{\frac{\alpha}{2}+\zeta-1}L(r,y,\theta)\\
&\leq C\int_0^{\frac{\pi}{2}} d\theta(\sin 2\theta)^{\frac{\alpha}{2}-1}\int_0^{1/2} dy\, y^{-\frac{\alpha}{2}-1}\int_0^{\infty} dr\, r^{\frac{\alpha}{2}+\zeta-1} y\left(\|f\|_\kappa+\|f\|_\infty r\|g\|_\kappa\|f\|_\infty r^{\frac{\alpha}{2}}\right)e^{-ar^{\frac{\alpha}{2}}-br}\phi(e^{i\theta},e^{i\theta}+yu)\\
&\leq C\left(\|f\|_\kappa+\|g\|_\kappa\|f\|_\infty\right),
\end{aligned}$$

where the constant C changes from line to line (and depends of $a\vee b$). We thus obtain that $\|\varphi^{\zeta,h}_{g,f(k)}\|_\infty<\infty$ for $k\in\{1,2\}$, and collecting all bounds that $\|\varphi^{\zeta,h}_{g,f}\|_\infty<\infty$ is finite. We now check that $\|(i.u)^\kappa\partial_\varepsilon\varphi^{\zeta,h}_{g,f}\|_\infty<\infty$ for $\varepsilon\in\{1,i\}$. To this end, notice that by homogeneity,

$$\partial_\varepsilon e^{-r^{\frac{\alpha}{2}}g(e^{i\theta}+yu)}=yr^{\frac{\alpha}{2}}|e^{i\theta}+yu|^{\frac{\alpha}{2}-1}\partial_\varepsilon g(v)e^{-r^{\frac{\alpha}{2}}g(e^{i\theta}+yu)} \tag{19}$$

where $v=(e^{i\theta}+yu)/|e^{i\theta}+yu|\in S_1^+$. Therefore, we get for $g\in\mathcal{H}_{\alpha/2,\kappa}$ and $f\in\mathcal{H}_{\zeta,\kappa}$,

$$\begin{aligned}
\left|\partial_\varepsilon[f(e^{i\theta}+yu)e^{-rh.(e^{i\theta}+yu)}e^{-r^{\frac{\alpha}{2}}g(e^{i\theta}+yu)}]\right| &\leq yr|h||e^{i\theta}+yu|^\zeta\|f\|_\infty e^{-ar^{\frac{\alpha}{2}}|e^{i\theta}+yu|^{\frac{\alpha}{2}}-br|e^{i\theta}+yu|}\\
&+y|e^{i\theta}+yu|^{\zeta-1+\kappa}\|f\|_\kappa|i.(e^{i\theta}+yu)|^{-\kappa}e^{-ar^{\frac{\alpha}{2}}|e^{i\theta}+yu|^{\frac{\alpha}{2}}-br|e^{i\theta}+yu|}\\
&+yr^{\frac{\alpha}{2}}|i.(e^{i\theta}+yu)|^{-\kappa}|e^{i\theta}+yu|^{\frac{\alpha}{2}+\zeta-1+\kappa}\|f\|_\infty\|g\|_\kappa e^{-ar^{\frac{\alpha}{2}}|e^{i\theta}+yu|^{\frac{\alpha}{2}}-br|e^{i\theta}+yu|}.
\end{aligned}$$

Using (17), we get for some constant C, depending on (α,a,b) such that

$$\begin{aligned}
\int_0^{\infty} dr\, r^{\frac{\alpha}{2}+\zeta-1}\left|\partial_\varepsilon[f(e^{i\theta}+yu)e^{-yrh.u}e^{-r^{\frac{\alpha}{2}}g(e^{i\theta}+yu)}]\right| &\leq Cy\Big(|h||e^{i\theta}+yu|^{-\frac{\alpha}{2}-1}\|f\|_\infty\\
&+|e^{i\theta}+yu|^{-\frac{\alpha}{2}-1+\kappa}\|f\|_\kappa|i.(e^{i\theta}+yu)|^{-\kappa}+|i.(e^{i\theta}+yu)|^{-\kappa}|e^{i\theta}+yu|^{-\frac{\alpha}{2}-1+\kappa}\|f\|_\infty\|g\|_\kappa\Big).
\end{aligned} \tag{20}$$

The above expression is integrable on $[0,\infty)\times[0,\pi/2]$ with respect to $y^{-\frac{\alpha}{2}-1}(\sin 2\theta)^{\frac{\alpha}{2}-1}dyd\theta$ for any $\kappa\in(0,1)$. Indeed, take $B\geq 0$. Then, we claim that

$$J_{B,\alpha,\kappa}(u):=|i.u|^\kappa\int_0^{\frac{\pi}{2}} d\theta(\sin 2\theta)^{\frac{\alpha}{2}-1}\int_0^{\infty} dy\, y^{-\frac{\alpha}{2}}\frac{|e^{i\theta}+yu|^{-B}}{|i.(e^{i\theta}+yu)|^\kappa} \tag{21}$$

is uniformly bounded if $B+\kappa>1-\frac{\alpha}{2}$. To see that, observe that if $i.u=0$ the integral is clearly finite, where as otherwise

$$J_{B,\alpha,\kappa}(u)=\int_0^{\frac{\pi}{2}} d\theta(\sin 2\theta)^{\frac{\alpha}{2}-1}\int_0^{\infty} dy\, y^{-\frac{\alpha}{2}}\frac{|e^{i\theta}+yu|^{-B}}{|\frac{i.e^{i\theta}}{i.u}+y|^\kappa}.$$

Cut this integral into two pieces: either θ is small or close to $\pi/2$, say in $[0,\pi/6]\cup[\pi/3,\pi/2]$, or it is in $(\pi/6,\pi/3)$, at distance greater than $\pi/6$ from 0 and $\pi/2$. In the first case, we integrate in y a function which is uniformly bounded at a positive distance of $y_0=-i.e^{i\theta}/i.u$ which is away from

the origin, blows up at most at y_0 where it behaves like $|y-y_0|^{-\kappa}$ and behaves like $y^{-B-\kappa-\frac{\alpha}{2}}$ at infinity (independently of $i.u$), and hence with uniformly bounded integral. In the second case, we can integrate first on θ, bounding $(\sin 2\theta)^{-1}$ uniformly from above, whereas we can use the bound $|e^{i\theta}+yu| \geq 1 \vee y$ for $\theta \in [0, \frac{\pi}{2}]$, $u \in S_1^+$. Then, observe that

$$\psi(x) = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{d\theta}{|i.e^{i\theta}+x|^{\kappa}}$$

is uniformly bounded, and goes to zero as $|x|^{-\kappa}$ as x goes to infinity. Hence, we bound ψ from above uniformly by a constant times $(1+|x|^{\kappa})^{-1}$. This implies that the second part of the integral is bounded, up to a multiplicative constant, by

$$|i.u|^{\kappa} \int_0^{\infty} dy\, y^{-\frac{\alpha}{2}} \frac{(1\vee y)^{-B}}{1+|i.u|^{\kappa}y^{\kappa}} = |i.u|^{\kappa+\frac{\alpha}{2}-1} \int_0^{|i.u|} dx\, \frac{x^{-\frac{\alpha}{2}}}{1+x^{\kappa}} + |i.u|^{\kappa+\frac{\alpha}{2}+B-1} \int_{|i.u|}^{\infty} dx\, \frac{x^{-\frac{\alpha}{2}-B}}{1+x^{\kappa}},$$

which is uniformly bounded if $B+\kappa > 1-\frac{\alpha}{2}$ (and even vanishes as $|i.u|$ goes to 0 : the first term is of order $|i.u|^{\kappa}$ and the order of the second term depends on whether $\alpha/2+B$ is less, equal or larger than 1).

Applying (21) in (20) (with $B = 1+\alpha/2-\kappa$ or $1+\alpha/2$) shows that $\varphi_{g,f}^{\zeta,h} \in \mathcal{H}_{\alpha/2,\kappa}$ and in fact collecting the previous bounds we get

$$\|\varphi_{g,f}^{\zeta,h}\|_{\kappa} \leq C(\|f\|_{\kappa} + \|f\|_{\infty}\|g\|_{\kappa}) \tag{22}$$

with a finite constant C depending on $a \vee b > 0$.

In particular F_h is bounded from $\mathcal{H}^a_{\alpha/2,\kappa}$ into $\mathcal{H}_{\alpha/2,\kappa}$ if $a > 0$ and F_h is bounded from $\bar{\mathcal{H}}^0_{\alpha/2,\kappa}$ into $\mathcal{H}_{\alpha/2,\kappa}$ if $\Re(h) \geq b$. In this last case, our proof also shows that

$$\|F_h(g) - F_h(0)\|_{\kappa} \leq C_{\Re(h)}\|g\|_{\kappa}.$$

for some constant $C_{\Re(h)}$ depending on $\Re(h)$. However, from the homogeneity relation, for $t > 0$,

$$F_h(t^{\alpha/2}g) = t^{-\alpha/2}F_{h/t}(g).$$

we get, with $t = \Re(h)$,

$$\begin{aligned} \|F_h(g)\|_{\kappa} &= t^{-\alpha/2}\|F_{h/t}(t^{-\alpha/2}g)\|_{\kappa} \\ &\leq t^{-\alpha/2}\|F_{h/t}(0)\|_{\kappa} + t^{-\alpha/2}\|F_{h/t}(t^{-\alpha/2}g) - F_{h/t}(0)\|_{\kappa} \leq Ct^{-\alpha/2} + t^{-\alpha}C_1\|g\|_{\kappa} \end{aligned}$$

where we noticed that $\|F_h(0)\|_{\kappa}$ is uniformly bounded when $\Re(h) = 1$. Finally, the last statement of Lemma 4.1 is an immediate consequence of inequality (17). This completes the proof of Lemma 4.1. The fact that $DF_h(g)(f)(u) = \varphi_{g,f}^{\alpha,h}(u)$ follows easily since $g \in \mathcal{H}^a_{\alpha/2,\kappa}$ implies that $g+f \in \mathcal{H}^{a'}_{\alpha/2,\kappa}$ with $a' = a - \|f\|_{\infty}$. Finally, from (22), $DF_h(g)$ is a bounded operator.

We now check the continuity of $DF_h(g)$ in h and g. It is sufficient to prove that for any $a > 0$, there exists a constant $c = c(\alpha, a)$ such that for all $f \in \mathcal{H}_{\alpha/2,\kappa}$, $g, g' \in \mathcal{H}^a_{\alpha/2,\kappa}$, $h, h' \in \mathcal{K}_1$,

$$\|DF_h(g)(f) - DF_{h'}(g')(f)\|_\kappa \leq c\|f\|_\kappa(|h - h'| + \|g - g'\|_\kappa). \tag{23}$$

To this end, we prove the same bound for $\varphi^{\alpha/2,h}_{g,f}$ instead of $DF_h(g)$. By interpolation, we may write that

$$\varphi^{\alpha/2,h}_{g,f(2)} - \varphi^{\alpha/2,h'}_{g',f(2)} = -\int_0^1 ds[\varphi^{\alpha/2+1,h_s}_{g,(h-h').*f} + \varphi^{\alpha,h'}_{g_s,f(g-g')}]$$

with $h_s = sh + (1-s)h', g_s = sg + (1-s)g'$ and $[(h-h').*f](u) = (h-h').uf(u)$ is in $\mathcal{H}_{\alpha/2+1}$. The bound then again follows from (22). It completes the proof of (23). In particular, we see that $h \to DF_h(g)(f)$ is Lipschitz. Similarly, we can compute the partial derivative in h of $(h, g) \mapsto F_h(g)$ with respect to the real ($\varepsilon = 1$) or imaginary part ($\varepsilon = i$) of h. We find

$$\partial_\varepsilon F_h(g) = \varphi^{1,h}_{g,\varepsilon.*1}$$

and see that it has finite $\|.\|_\kappa$ norm and it is continuous in g.

□

Proof of Lemma 4.3. From

$$G_z(f) - G_z(g) = \int_0^1 d\theta DG_z(\theta f + (1-\theta)g)(f - g)$$

we deduce from (22) that

$$\|G_z(f) - G_z(g)\|_\kappa \leq c\,\|f - g\|_\kappa + c\,(\|f\|_\kappa + \|g\|_\kappa)\|f - g\|_\infty.$$

The proof of the second statement is straightforward. □

4.2 Implicit function theorem : proofs of Proposition 3.3 and Proposition 3.4

In this subsection, we analyze the solutions for the fixed point equation governed by G_z for z small. For $z \in \mathbb{C}_+$, we first connect G_z with the random variable $R_\star(z)$ in $\mathbb{C}_+$ which satisfies the fixed point equation (7) introduced in [15]. To this end, we set

$$\gamma^\star_z(u) := \Gamma(1 - \frac{\alpha}{2})\mathbb{E}[(-iR_\star(z).u)^{\frac{\alpha}{2}}] \in \bar{\mathcal{H}}^0_{\alpha/2}\,.$$

Integrating (7), it turns out that $\gamma^\star_z$ satisfies the following fixed point equation.

Lemma 4.4. *Let $z \in \mathbb{C}_+$ and $0 < \alpha < 2$. Then for all $u \in \mathcal{K}^+_1$,*

$$\gamma^\star_z(u) = G_z(\gamma^\star_z)(u) = c_\alpha F_{-iz}(\gamma^\star_z)(\check{u}),$$

where c_α and $\check{u}$ are defined in (14)*. Moreover, for any $p > 0$, $\mathbb{E}|R_\star(z)|^p = r_{p,z}(\gamma^\star_z)$ and $\mathbb{E}(-iR_\star)^p = s_{p,z}(\gamma^\star_z(1))$.*

Proof. The statements are a consequence of the Lévy-Khintchine formula applied to the Poisson process (ξ_k), see e.g. [15, (4.5)]: if (w_k) are independent of (ξ_k), i.i.d. complex random variables with $\Re(w_k) > 0$ we have

$$\mathbb{E}\exp\left(-\sum_k \xi_k w_k\right) = \exp\left(-\Gamma\Big(1-\frac{\alpha}{2}\Big)\mathbb{E}w_1^{\alpha/2}\right). \tag{24}$$

The first statement of the lemma is [16, Lemma 5.3]. The second statement is a straightforward modification of the proof of the forthcoming Corollary 5.8 and (24). To avoid repetition we skip the details of the proof. □

We first rely on results from [15] to find $\gamma_0^\star$. We know that for $z = i\eta$, $\eta > 0$, $R_\star(i\eta)$ is pure imaginary (see [15]). It follows that $\gamma^\star_{i\eta}(u) = (1.u)^{\frac{\alpha}{2}} a_\eta$, where a_η satisfies an explicit fixed point equation. Also, we have $a_\eta \to a_0$ as $\eta \to 0$ and, by [15, Lemma 4.3],

$$a_0 = \left(\frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}\right)}\right)^{1/2}.$$

In particular, $\gamma_0^\star(u) := (1.u)^{\frac{\alpha}{2}} a_0$ is in $\mathcal{H}^0_{\alpha/2}$, it is the limit of $\gamma^\star_{i\eta}$ as $\eta \to 0$ and, by Lemma 4.2, it satisfies the fixed point equation $\gamma_0^\star = G_0(\gamma_0^\star)$. We are interested in $DG_0(\gamma_0^\star)$. First, by Lemma 4.2,

$$\begin{aligned} DF_0(\gamma_0^\star)(f)(u) &= \int_0^{\frac{\pi}{2}} d\theta (\sin 2\theta)^{\frac{\alpha}{2}-1} \int_0^\infty dy\, y^{-\frac{\alpha}{2}-1} \\ &\qquad \times \int_0^\infty dr\, r^{\alpha-1}\left(f(e^{i\theta})e^{-r^{\frac{\alpha}{2}} a_0 (1.e^{i\theta})^{\frac{\alpha}{2}}} - f(e^{i\theta}+yu)e^{-r^{\frac{\alpha}{2}} a_0 (1.(e^{i\theta}+yu))^{\frac{\alpha}{2}}}\right) \\ &= \frac{2}{\alpha a_0^2}\int_0^{\frac{\pi}{2}} d\theta (\sin 2\theta)^{\frac{\alpha}{2}-1} \int_0^\infty dy\, y^{-\frac{\alpha}{2}-1}\left(\frac{f(e^{i\theta})}{(1.e^{i\theta})^\alpha} - \frac{f(e^{i\theta}+yu)}{(1.(e^{i\theta}+yu))^\alpha}\right). \end{aligned} \tag{25}$$

Observe that, for $z \in \mathcal{K}_1^+$, $(1.z) = \Re(z) + \Im(z)$ and $|z| \le (1.z) \le \sqrt{2}|z|$. Define the unitary operator, $J(f)(u) = f(\check{u})$. and for $\alpha \in (0,2)$, we consider the operator on $\mathcal{H}_{\alpha/2,\kappa}$ given by

$$K_\alpha = -c_\alpha DF_0(\gamma_0^\star) J,$$

where, with an abuse of notation, $DF_0(\gamma_0^\star)$ is the operator defined on the right hand side of (25). Then $-K_\alpha$ is precisely equal to $DG_0(\gamma_0^\star)$. Our goal is to show that $I + K_\alpha$ is invertible to apply the implicit function theorem. The main result of this section is the following result.

Theorem 4.5. *Let $\kappa \in [0,1)$. Let $\mathcal{A}$ be the subset of α in $(0,2)$ such that $I + K_\alpha$ is not an isomorphism of $\mathcal{H}_{\alpha/2,\kappa}$. If F is a closed subset of $(0,2)\backslash\{1/2, 1\}$, then $\mathcal{A} \cap F$ is finite.*

The proof of Theorem 4.5 will require a careful analysis of the operator K_α. We postpone it to the next subsection. We first use it to prove Proposition 3.3 and Proposition 3.4.

Proof of Proposition 3.3. Since $z \mapsto R_\star(z)$ is analytic and $|R_\star(z)| \leq 1/\Im(z)$, it is easy to check that $z \mapsto \gamma_z^\star$ is continuous from $\mathbb{C}_+$ to $\mathcal{H}_{\alpha,\kappa}$ (see the forthcoming Lemma 5.2). Also, as already pointed out, $\gamma_z^\star$ converges to $\gamma_0^\star$ as z goes to 0. Hence, by Lemma 4.2, the definition of $\mathcal{A}$, and the Implicit Function Theorem in Banach spaces (see [28, Theorem 2.7.2]), for any $t > 0$, there exists $\tau > 0$ such that for all $z \in \mathbb{C}_+$, $|z| \leq \tau$,

$$\|\gamma_z^\star - \gamma_0^\star\|_\kappa \leq t.$$

We take $t = 2^{\alpha/4} a_0/2$. We deduce that if $|z| \leq \tau$,

$$\Gamma(1-\alpha/2) 2^{\frac{\alpha}{4}} \mathbb{E}(\Im R_\star(z))^{\frac{\alpha}{2}} = \gamma_z^\star(e^{i\pi/4}) \geq \gamma_0^\star(e^{i\pi/4}) - t = t.$$

Hence, with $c = a_0/2$, if $|z| \leq c_0$, we have

$$\Gamma(1-\alpha/2)\mathbb{E}(\Im R_\star(z))^{\frac{\alpha}{2}} \geq c.$$

Now, from (7), we have the bound, $|R_\star(z)| \leq 1/\left(\sum_{k\geq 1} \xi_k \Im R_k(z)\right)$. Using Lemma 4.4 and the formula (10), we get

$$\begin{aligned} r_{p,z}(\gamma_z^\star) = \mathbb{E}|R_\star(z)|^p &\leq \frac{1}{\Gamma(p)} \int_0^\infty x^{p-1} \mathbb{E} e^{-x\sum_k \xi_k \Im(R_k)} dx \\ &= \frac{1}{\Gamma(p)} \int_0^\infty x^{p-1} e^{-x^{\alpha/2}\Gamma(1-\alpha/2)\mathbb{E}\Im(R_\star)^{\alpha/2}} dx \\ &\leq \frac{1}{\Gamma(p)} \int_0^\infty x^{p-1} e^{-cx^{\alpha/2}} dx, \end{aligned}$$

where at the second line we have used (24). □

Proof of Proposition 3.4. From Theorem 4.5, Lemma 4.1, Lemma 4.2, we may apply the inverse function Theorem to $T : (z, f) \mapsto (T_1(z), T_2(z)) = (z, f - G_z(f))$ on $\mathbb{C} \times \mathcal{H}^0_{\alpha/2,\kappa}$. It follows that there exists an open neighborhood U of $(0, \gamma_0^\star)$, such that T is an homeomorphism from U to $V = T(U)$. Moreover, $T_{|U}^{-1}$ has Fréchet derivative at $y = (z, g) \in V$ given by $(DT(T_{|U}^{-1} y))^{-1}$. If τ is taken small enough, $(z, \gamma_z^\star) \in U$ for all $|z| \leq \tau$. Also, since $\gamma_z^\star = G_z(\gamma_z^\star)$, if τ is taken small enough, we may further assume that $\{z\} \times B(\gamma_z^\star, 2\tau) \subset U$ and $\{z\} \times B(0, 2\tau) \subset V$, where $B(f, \tau)$ is the open ball for the norm $\|\cdot\|_\kappa$. We then apply the fundamental theorem of calculus to $t \mapsto T_{|U}^{-1}(z, f_t)$ where $f_t = t(\gamma - G_z(\gamma))$ to deduce

$$\|\gamma_z^\star - \gamma\|_\kappa = \|\left(T^{-1}(z,0) - T^{-1}(z, \gamma - G_z(\gamma)\right)_2\|_\kappa \leq c\|\gamma - G_z(\gamma)\|_\kappa .$$

It implies the Proposition with $c = \sup \|(DT(x))^{-1}\|$, where $\|\cdot\|$ is the operator norm and the supremum is over all $x = (z, f)$, $|z| \leq \tau$, $f \in B(\gamma_z^\star, \tau)$. □

4.3 Proof of Theorem 4.5

The strategy of the proof of Theorem 4.5 is quite intricate. The road map is the following:

1. Prove that the operator K_α is compact on the Banach space $\mathcal{H}_{\alpha/2,\kappa}$. Hence, $I+K_\alpha$ is not an isomorphism iff -1 is an eigenvalue of K_α.

2. We could not study directly the spectrum of K_α but that of another compact operator H_α, an operator on a larger Hilbert space whose spectrum contains the spectrum of K_α. We prove that H_α^m is trace class for m large enough (depending on α).

3. To show that -1 is not an eigenvalue of H_α, we want to show that for some large even number m, $\det(I-H_\alpha^m)$ does not vanish except possibly on a discrete set of α, where $\det(I+\cdot)$ denotes the Fredholm determinant. To this end, we first show that $\alpha \mapsto \det(I-H_\alpha^m)$ is analytic on $\Re(\alpha) \in (0,2)\backslash\{1/2,1\}$.

4. We finally check that $\alpha \mapsto \det(I-H_\alpha^m)$ does not vanish when the imaginary part of α goes to infinity and conclude the whole argument.

Step 1 : Compactness. We start by the compactness of $K_\alpha = -c_\alpha DF_0(\gamma_0^\star)J$.

Lemma 4.6. *For any $\kappa \in (0,1)$, any $\alpha \in U = \{z \in \mathbb{C} : \Re(z) \in (0,2)\}$, the operator K_α is compact in $\mathcal{H}_{\alpha/2,\kappa}$.*

Proof. We introduce the multiplication operator, $M(f)(u) = f(u)(1.u)^{-\alpha}$. We notice that M is a bounded operator from $\mathcal{H}_{\alpha/2,\kappa}$ to $\mathcal{H}_{-\alpha/2,\kappa}$. We will now prove that the $\mathcal{H}_{-\alpha/2,\kappa}$ to $\mathcal{H}_{\alpha/2,\kappa}$ operator

$$Tf(u) = \int_0^{\frac{\pi}{2}} d\theta (\sin 2\theta)^{\frac{\alpha}{2}-1} \int_0^\infty dy\, y^{-\frac{\alpha}{2}-1} \Big(f(e^{i\theta}) - f(e^{i\theta}+yu) \Big)$$

is compact. It will conclude the proof since $DF_0(\gamma_0^\star) = \big(2/(\alpha a_0^2)\big)\, TM$. Let B be the set of $f \in \mathcal{H}_{-\alpha/2,\kappa}$ such that $\|f\|_\kappa \le 1$. We should prove that TB is a compact set of $\mathcal{H}_{\alpha/2,\kappa}$. Note that M is bounded, invertible and with bounded inverse as $1.u$ is bounded above and below by a positive number. By Lemma 4.2, $T = \big(2/(\alpha a_0^2)\big)^{-1} DF_0(\gamma_0^\star)M^{-1}$ is a bounded operator in $\mathcal{H}_{\alpha/2,\kappa}$ and $|Tf(1)| \le C\|f\|_\kappa$ is bounded. It is sufficient to prove that there exists a compact set K of $C_\kappa(S_1^+)$ (the space of continuous functions on S_1^+ equipped with the norm $\||i.u|^\kappa \cdot\|_\infty$) such that $u \mapsto \partial_\varepsilon(Tf)(u)$ is in K for all $f \in B$ and $\varepsilon \in \{1,i\}$.

To this end, we observe that, if $g(u) = \partial_\varepsilon f(u)$, then $\partial_\varepsilon(Tf)(u) = -Pg(u)$ where

$$Pg(u) = \int_0^{\frac{\pi}{2}} d\theta (\sin 2\theta)^{\frac{\alpha}{2}-1} \int_0^\infty dy\, y^{-\frac{\alpha}{2}} g\left(\frac{e^{i\theta}+yu}{|e^{i\theta}+yu|}\right) |e^{i\theta}+yu|^{-1-\frac{\alpha}{2}}. \tag{26}$$

We should thus prove that P is a compact operator on $C_\kappa(S_1^+)$. This amounts to prove that $P_\kappa(g)(u) = (i.u)^\kappa P((i.*)^{-\kappa}g)(u)$ is compact in $C_0(S_1^+) = C(S_1^+)$. To this end, we write P as a

kernel operator. We assume that $u = e^{i\omega}$. We wish to perform the change of variable from (θ, y) to (θ, ψ) defined, for $\theta \neq \omega$, by

$$e^{i\psi} = \frac{e^{i\theta} + yu}{|e^{i\theta} + yu|}.$$

The set $y \in (0, \infty)$ is mapped bijectively to $\psi \in (\theta, \omega)$ if $\theta < \omega$ or $\psi \in (\omega, \theta)$ if $\theta > \omega$. Since $|e^{i\theta} + yu| = e^{i(\theta-\psi)} + ye^{i(\omega-\psi)}$ is a real number, taking the imaginary part, we arrive at

$$y = \frac{\sin(\theta - \psi)}{\sin(\psi - \omega)}.$$

Taking the real part, we find

$$|e^{i\theta} + yu| = \cos(\theta - \psi) + \frac{\sin(\theta - \psi)\cos(\omega - \psi)}{\sin(\psi - \omega)} = \frac{\sin(\theta - \omega)}{\sin(\psi - \omega)}.$$

Also,

$$\frac{dy}{d\psi} = -\frac{\cos(\theta - \psi)\sin(\psi - \omega) + \cos(\psi - \omega)\sin(\theta - \psi)}{\sin^2(\psi - \omega)} = -\frac{\sin(\theta - \omega)}{\sin^2(\psi - \omega)}.$$

We may then express P as

$$Pg(e^{i\omega}) = \int_0^{\frac{\pi}{2}} d\psi g(e^{i\psi}) k(\omega, \psi), \tag{27}$$

where k is a kernel on $[0, \pi/2]^2$ defined by, for $\psi > \omega$,

$$k(\omega, \psi) = \sin(\psi - \omega)^{\alpha-1} \int_{\psi}^{\frac{\pi}{2}} d\theta (\sin 2\theta)^{\frac{\alpha}{2}-1} \sin(\theta - \psi)^{-\frac{\alpha}{2}} \sin(\theta - \omega)^{-\frac{\alpha}{2}},$$

while if $\psi < \omega$,

$$k(\omega, \psi) = \sin(\omega - \psi)^{\alpha-1} \int_{0}^{\psi} d\theta (\sin 2\theta)^{\frac{\alpha}{2}-1} \sin(\theta - \psi)^{-\frac{\alpha}{2}} \sin(\theta - \omega)^{-\frac{\alpha}{2}}.$$

This implies that P_κ has kernel

$$k_\kappa(\omega, \psi) = (i.e^{i\omega})^\kappa (i.e^{i\psi})^{-\kappa} k(\omega, \psi).$$

With our alternative expression for P_κ, we can readily prove the compactness on $C(S_1^+)$. From [25, Theorem 12.1], it suffices to prove that for any $\omega \in [0, \pi/2]$, if $\omega_n \to \omega$,

$$\int_0^{\pi/2} |k_\kappa(\omega, \psi) - k_\kappa(\omega_n, \psi)| d\psi \to 0. \tag{28}$$

If $\alpha \in (0, 2)$ is real, then $k_\kappa(\omega, \psi) \geq 0$ and we have that

$$\int_0^{\pi/2} |k_\kappa(\omega, \psi)| d\psi = |i.e^{i\omega}|^\kappa \int_0^{\frac{\pi}{2}} d\theta (\sin 2\theta)^{\frac{\alpha}{2}-1} \int_0^\infty dy\, y^{-\frac{\alpha}{2}} \frac{|e^{i\theta} + ye^{i\omega}|^{-1-\frac{\alpha}{2}+\kappa}}{|i.(e^{i\theta} + ye^{i\omega})|^\kappa}$$

is continuous in ω by dominated convergence (recall $\kappa < 1$ and $|e^{i\theta} + ye^{i\omega}|$ is bounded below). Consequently, from Scheffé's Lemma, to prove (28), it is sufficient to check that for almost all $\psi \in [0, \pi/2]$, $k(\omega_n, \psi)$ converges to $k(\omega, \psi)$. It is however immediate that for any $\psi \neq \omega$, the above convergence holds. It completes the proof in the case α real. In the general case, writing explicitly the dependence of k in α, we find $|k^\alpha(\omega, \psi)| \leq k^{\Re(\alpha)}(\omega, \psi)$. We conclude similarly by dominated convergence that $\int_0^{\pi/2} |k_\kappa^\alpha(\omega_n, \psi)| d\psi \to \int_0^{\pi/2} |k_\kappa^\alpha(\omega, \psi)| d\psi$. Almost sure convergence of $k^\alpha(\omega_n, \psi)$ is again clear from the formula. □

Step 2 : Trace class for an affiliated operator. The next step would be to compute the spectrum of K_α. Unfortunately, we have not been able to compute it explicitly. We instead first define an operator H_α whose spectrum contains the spectrum of K_α. For a suitable even power m, we will then prove that the Fredholm determinant of $I - H_\alpha^m$ depends analytically on α. To this end, we set $X = S_1^+ \times \{0, 1, i\}$ and consider the Hilbert spaces $L^2_\kappa(S_1^+)$ and $L^2_\kappa(X)$ with respective norms

$$\|g\|_{L^2_\kappa(S_1^+)} = \sqrt{\int_0^{\frac{\pi}{2}} |g(e^{i\psi})|^2 \frac{d\psi}{|i.e^{i\psi}|^\kappa}} \quad \text{and} \quad \|f\|_{L^2_\kappa(X)} = \sqrt{\|f_0\|^2_{L^2_0(S_1^+)} + \sum_{\varepsilon \in \{1,i\}} \|f_\varepsilon\|^2_{L^2_\kappa(S_1^+)}},$$

where, for shorter notation, we have set $f_\varepsilon : u \mapsto f(u, \varepsilon)$. We extend a function f in $L^2_\kappa(X)$ to a function on $\mathcal{K}_1^+ \times \{0, 1, i\}$, by setting $f(\lambda u, 0) = \lambda^{\alpha/2} f(u, 0)$, $f(\lambda u, \varepsilon) = \lambda^{\alpha/2-1} f(u, \varepsilon)$ for $\varepsilon \in \{1, i\}$. We define a new operator $S : C_\kappa(X) \to C_\kappa(X)$ by the formula, for $u = u_1 + iu_2$ and $\varepsilon \in \{1, i\}$,

$$\begin{aligned}
Sf(u,0) &= \frac{2}{\alpha} \int_0^{\frac{\pi}{2}} d\theta (\sin 2\theta)^{\frac{\alpha}{2}-1} \int_0^\infty dy\, y^{-\frac{\alpha}{2}} \left(\frac{u_1 f_1(e^{i\theta} + yu) + u_2 f_i(e^{i\theta} + yu)}{(1.(e^{i\theta} + yu))^\alpha} - \alpha \frac{(1.u) f_0(e^{i\theta} + yu)}{(1.(e^{i\theta} + yu))^{1+\alpha}} \right) \\
Sf(u,\varepsilon) &= \int_0^{\frac{\pi}{2}} d\theta (\sin 2\theta)^{\frac{\alpha}{2}-1} \int_0^\infty dy\, y^{-\frac{\alpha}{2}} \left(\frac{f_\varepsilon(e^{i\theta} + yu)}{(1.(e^{i\theta} + yu))^\alpha} - \alpha \frac{(1.u) f_0(e^{i\theta} + yu)}{(1.(e^{i\theta} + yu))^{1+\alpha}} \right).
\end{aligned}$$

Let P be defined by (26), we observe that in matrix form

$$S = \begin{pmatrix} M_{00} & M_{01} & M_{0i} \\ M_{10} & I & 0 \\ M_{i0} & 0 & I \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix} \begin{pmatrix} N_0 & 0 & 0 \\ 0 & N_1 & 0 \\ 0 & 0 & N_i \end{pmatrix}, \tag{29}$$

where $M_{\varepsilon\varepsilon'}$ and N_ε are the bounded multiplication operators in $L^2(S_1^+)$, $M_{00} f(u) = -2(1.u) f(u)$, $M_{10} f(u) = M_{i0} f(u) = -\alpha (1.u) f(u)$, $M_{01} f(u) = \frac{2}{\alpha} u_1 f(u)$, $M_{0i} f(u) = \frac{2}{\alpha} u_2 f(u)$, $N_0 f(u) = (1.u)^{-\alpha-1} f(u)$ and $N_1 f(u) = N_i f(u) = (1.u)^{-\alpha} f(u)$. The proof of Lemma 4.6 shows that S is a compact operator on $C_\kappa(X)$ since P is compact and the $M_{\varepsilon\varepsilon'}$ and N_ε are bounded. We set

$$H_\alpha = c'_\alpha S J, \tag{30}$$

where $c'_\alpha = c_\alpha(2/(\alpha a_0^2))$ and the operator J is extended on $L^2_\kappa(X)$ by setting $(Jf)_\varepsilon(u) = f_\varepsilon(\breve{u})$. Now, if $f \in \mathcal{H}_{\alpha/2,\kappa}$, we set $\partial_0 f(u) = f$ and $\bar{f}(u,\varepsilon) = \partial_\varepsilon f(u) \in L^2_\kappa(X)$. We shall use the identity

$$\begin{aligned}\int_0^\infty dy\, y^{-\frac{\alpha}{2}-1}\left(\frac{f(e^{i\theta})}{(1.e^{i\theta})^\alpha} - \frac{f(e^{i\theta}+yu)}{(1.(e^{i\theta}+yu))^\alpha}\right) &= -\int_0^\infty dy\, y^{-\frac{\alpha}{2}-1}\int_0^y dt\,\partial_u\left(\frac{f(v)}{(1.v)^\alpha}\right)_{v=e^{i\theta}+tu}\\ &= -\frac{2}{\alpha}\int_0^\infty dt\, t^{-\frac{\alpha}{2}}\left(\frac{\partial_u f(e^{i\theta}+tu)}{(1.(e^{i\theta}+tu))^\alpha} - \alpha\frac{(1.u)f(e^{i\theta}+tu)}{(1.(e^{i\theta}+tu))^{1+\alpha}}\right).\end{aligned}$$

Hence, we have by (25) for $\varepsilon = 0, 1, i$,

$$\partial_\varepsilon\left(K_\alpha f\right)(u) = (H_\alpha \bar{f})_\varepsilon(u).$$

In particular, if $f \in \mathcal{H}_{\alpha/2,\kappa}$ is an eigenvector of K_α then, we have $\bar{f} \in C_\kappa(X)$ and $\bar{f}$ is an eigenvector of H_α with the same eigenvalue. It follows that the spectrum of K_α is included in the spectrum of H_α. We may summarize this inclusion as

$$\sigma_{\mathcal{H}_{\alpha/2,\kappa}}(K_\alpha) \subseteq \sigma_{C_\kappa(X)}(H_\alpha). \tag{31}$$

Since $C_\kappa(X) \subset L^2_\kappa(X)$, we can see H_α as an operator on $L^2_\kappa(X)$. The next lemma is the main technical ingredient for this part of the proof. It proves that if $\Re(\alpha)$ is not equal to 1 or 1/2, H^m_α is a trace class operator if m is large enough (depending on α).

Proposition 4.7. *Let $\kappa \in (0,1)$. For integer $\ell \geq 0$, let $V_\ell = \{\alpha \in \mathbb{C} : 2^{-\ell} < \Re(\alpha) < 2^{-\ell+1}\}$ and $W_\ell = V_\ell/3 = \{\alpha \in \mathbb{C} : 2^{-\ell} < 3\Re(\alpha) < 2^{-\ell+1}\}$. If $\alpha \in V_\ell$ then $H_\alpha^{2^\ell}$ is a Hilbert-Schmidt operator in $L^2_\kappa(X)$. Finally, if $\alpha \in W_\ell$ with $\ell \geq 1$, then $H_\alpha^{3\cdot 2^\ell}$ is a Hilbert-Schmidt operator in $L^2_\kappa(X)$.*

Proof. Let P be defined by (26) and set $(Qf)(u,\varepsilon) = P(f_\varepsilon)(u)$. From (29), and since the $M_{\varepsilon\varepsilon'}$ and the N_ε are uniformly bounded, the Hilbert Schmidt norm of H^m_α in $L^2_\kappa(X)$ is always bounded by the Hilbert-Schmidt norm of P^m in $L^2_\kappa(S_1^+)$ (up to a constant factor). We deduce that it is sufficient to prove that P^m is Hilbert-Schmidt on $L^2_\kappa(S_1^+)$.

Bound on the kernel of P. It is simpler to work with P in its kernel form (27). We claim that,

$$|k(\omega,\psi)| \leq \begin{cases} C|\psi-\omega|^{\Re(\alpha)-1}\left(\sin(2\psi)\wedge\sin(2\omega)\right)^{-\Re(\alpha)/2} & \text{if } 0<\Re(\alpha)<1,\\ C\left(\sin(2\psi)\wedge\sin(2\omega)\right)^{-1/2}\left(1\vee\ln\left(\frac{\sin(2\psi)}{|\omega-\psi|}\right)\right) & \text{if } \Re(\alpha)=1,\\ C\left(\sin(2\psi)\wedge\sin(2\omega)\right)^{\Re(\alpha)/2-1} & \text{if } 1<\Re(\alpha)<2,\end{cases} \tag{32}$$

where the constant C is uniform over all $\Re(\alpha)$ in a closed set of $(0,2)$. Indeed, let us assume for example that $\psi < \omega$ and prove that (32) holds. If $\beta = \Re(\alpha)$, since $\sin(x) \geq 2x/\pi$ for $x \in [0,\pi/2]$, we get

$$|k(\omega,\psi)| \leq C|\psi-\omega|^{\beta-1}\int_0^\psi \left(\theta\wedge\left(\frac{\pi}{2}-\theta\right)\right)^{\frac{\beta}{2}-1}|\theta-\psi|^{-\frac{\beta}{2}}|\theta-\omega|^{-\frac{\beta}{2}}d\theta. \tag{33}$$

Now, for $0<a<b$, if $0<\beta<1$,

$$I = \int_0^a x^{\frac{\beta}{2}-1}|x-a|^{-\frac{\beta}{2}}|x-b|^{-\frac{\beta}{2}}dx \;=\; a^{-\frac{\beta}{2}}\int_0^1 x^{\frac{\beta}{2}-1}|x-1|^{-\frac{\beta}{2}}|x-b/a|^{-\frac{\beta}{2}}dx \leq Cb^{-\frac{\beta}{2}}.$$

If $1 \leq \beta < 2$, then a singularity appears when b/a is close to 1.

$$\begin{aligned} I &\leq Ca^{-\frac{\beta}{2}} \int_0^1 |x-1|^{-\frac{\beta}{2}} |x-b/a|^{-\frac{\beta}{2}} dx \\ &= Ca^{-\frac{\beta}{2}} \int_0^1 x^{-\frac{\beta}{2}} (x+(b/a-1))^{-\frac{\beta}{2}} dx \\ &\leq Ca^{-\frac{\beta}{2}} (b/a-1)^{-\beta+1} \int_0^{(b/a-1)^{-1}} x^{-\frac{\beta}{2}} (x+1)^{-\frac{\beta}{2}} dx. \end{aligned}$$

Since $\sin(x) \leq x$, we deduce from (33) that (32) holds if $\psi \leq \pi/4, \psi < \omega$. If $\pi/4 < \psi < \omega < \pi/2$ then, we should upper bound for $0 < a < b < 1$,

$$II = \int_0^a |1-x|^{\frac{\beta}{2}-1} |a-x|^{-\frac{\beta}{2}} |b-x|^{-\frac{\beta}{2}} dx = \int_0^a (x+(1-a))^{\frac{\beta}{2}-1} x^{-\frac{\beta}{2}} (x+b-a)^{-\frac{\beta}{2}} dx.$$

Now, with $s = b-a$ and $t = 1-a > s$,

$$\begin{aligned} \int_0^1 x^{-\frac{\beta}{2}} (x+t)^{\frac{\beta}{2}-1} (x+s)^{-\frac{\beta}{2}} dx &= s^{-\beta/2} \int_0^{1/s} x^{-\frac{\beta}{2}} (x+t/s)^{\frac{\beta}{2}-1} (x+1)^{-\frac{\beta}{2}} dx \\ &\leq Cst^{-\beta/2} + s^{-\beta/2} \int_1^\infty x^{-\beta} (x+t/s)^{\frac{\beta}{2}-1} dx \\ &\leq Cst^{-\beta/2} + t^{-\beta/2} \int_{s/t}^\infty x^{-\beta} (x+1)^{\frac{\beta}{2}-1} dx. \end{aligned}$$

If $0 < \beta < 1$, the above expression is $O(t^{-\beta/2})$. If $\beta > 1$, it is of order $O(s^{1-\beta} t^{\beta/2-1})$. Finally, if $\beta = 1$, it is $O(t^{-1/2}(1 \vee \log(t/s)))$. It completes the proof of (32).

Case $\Re(\alpha) > 1$. We have that, if $\delta, \kappa \in [0,1)$,

$$\int_{[0,1]^2} \frac{dx}{|x-1/2|^\kappa} \frac{dy}{|y-1/2|^\kappa} (x \wedge y)^{-\delta} = 2 \int_0^1 \frac{dy}{|x-1/2|^\kappa} \int_0^y \frac{dx}{|x-1/2|^\kappa} x^{-\delta} < \infty. \tag{34}$$

In particular, if $\beta = \Re(\alpha) > 1$, then, we deduce from (32) applied to $\delta = 2-\beta$ and the fact that if $u = e^{i\theta}$, $\theta \in (0, \pi/2)$,

$$|u - e^{i\pi/4}| \leq |\theta - \pi/4| \leq (\pi/2) |\sin(\theta - \pi/4)| = (\pi/(2\sqrt{2})) |i.u|\,, \tag{35}$$

that

$$\int_{[0,\pi/2]^2} |k(\omega,\psi)|^2 \frac{d\omega}{|i.e^{i\omega}|^\kappa} \frac{d\psi}{|i.e^{i\psi}|^\kappa} \leq C \left(\int_{[0,\frac{\pi}{2}]^2} (\psi \wedge \omega)^{\beta-2} \frac{d\omega}{|\omega - \frac{\pi}{4}|^\kappa} \frac{d\psi}{|\psi - \frac{\pi}{4}|^\kappa} \right)^2$$

is finite for $\beta > 1$ and $\kappa < 1$. Hence P and H_α are Hilbert-Schmidt operators in $L^2_\kappa(X)$.

Case $0 < \Re(\alpha) < 1$. If $\beta = \Re(\alpha) < 1$, then, we need to take powers of P, to obtain a kernel operator in the Hilbert-Schmidt class. Let $\ell \geq 1$, we assume that

$$2^{-\ell} < \beta < 2^{-\ell+1}. \tag{36}$$

By recursion, we will upper bound the kernel of P^{2^t} and show that P^{2^ℓ} is Hilbert-Schmidt in $L^2_\kappa(S_1^+)$. To perform that, we will use the following inequality in the recursion. Let $0 < \zeta, \zeta' < 1$ and $0 < a < b$, we set

$$\begin{aligned} I_{\zeta,\zeta'}(a,b) &= \int_0^1 |x-a|^{\zeta-1}|x-b|^{\zeta'-1}(x\wedge a)^{-\zeta/2}(x\wedge b)^{-\zeta'/2}dx \\ &\leq \begin{cases} Ca^{-(\zeta+\zeta')/2}(b-a)^{\zeta+\zeta'-1} & \text{if } 0<\zeta+\zeta'<1, \\ Ca^{-\zeta/2}b^{-\zeta'/2} & \text{if } 1<\zeta+\zeta'. \end{cases} \end{aligned} \tag{37}$$

and $I_\zeta = I_{\zeta,\zeta}$. Indeed, as in (34), we decompose the integration interval over $[0,a]$, $[a,b]$ and $[b,1]$. It follows easily from the scaling arguments below (33) that the integral over $[0,a]$ is dominating if $2\zeta < 1$ and the integral over $[b,1]$ is dominating if $2\zeta > 1$.

From (32), up to a change a variable $x = \sin(\omega)$, it suffices to prove that the kernel operator $\bar{P}$ in $L^2([0,1])$,

$$\bar{k}(x,y) = |x-y|^{\beta-1}(x\wedge y)^{-\beta/2}$$

satisfies that $\bar{P}^{2^\ell}$ is Hilbert-Schmidt in $L^2_\kappa([0,1]) = \left\{ f : \int_0^1 |f(x)|^2 |x-\frac{1}{2}|^{-\kappa}dx < \infty \right\}$. For $t \geq 1$, we set $\zeta_t = 2^{t-1}\beta$. If $\bar{k}_t$ is the kernel of $\bar{P}^{2^{t-1}}$, we have,

$$\bar{k}_2(x,y) \leq I_{\zeta_1}(x,y).$$

Now, if β is an in (36) with $\ell = 1$, then $2\zeta_1 = 2\beta > 1$ and from (37), $\bar{P}^2$ is Hilbert-Schmidt since $\bar{k}_2(x,y) \leq Cx^{-\beta/2}y^{-\beta/2}$ and $\beta < 1$. If $\ell \geq 2$, we find from (37),

$$\bar{k}_2(x,y) \leq C|x-y|^{\zeta_2-1}(x\vee y)^{-\zeta_2/2},$$

and

$$\bar{k}_3(x,y) \leq CI_{\zeta_2}(x,y).$$

We deduce easily by recursion that for some new constant $C > 0$,

$$\bar{k}_\ell(x,y) \leq C|x-y|^{\zeta_\ell-1}(x\wedge y)^{-\zeta_\ell/2}.$$

and

$$\bar{k}_{\ell+1}(x,y) \leq CI_{\zeta_\ell}(x,y) \leq C'x^{-\zeta_\ell/2}y^{-\zeta_\ell/2}. \tag{38}$$

Now, since $\zeta_\ell/2 = 2^{\ell-2}\beta$ and $2^{\ell-1}\beta < 1$, the operator $\bar{P}^{2^\ell}$ is Hilbert-Schmidt in $L^2_\kappa(S_1^+)$. It proves the first statement of the proposition.

For the second statement, we proceed similarly but starting with the kernel P^3 instead of P^2. First, if $2\beta < 1$ but $3\beta > 1$, from (37) the kernel of P^3, which is bounded by $I_{\beta,2\beta}$, is in $L^2_\kappa(S_1^+)$. For $\beta \in (3^{-1}2^{-\ell}, 3^{-1}2^{-\ell+1}), \ell \geq 1$, we can proceed by recursion as above. □

We recall that if an operator is Hilbert-Schmidt, its square is trace class. Hence, Proposition 4.7 implies that

Corollary 4.8. *Let $\kappa \in (0,1)$. Let $V = \{\alpha \in \mathbb{C} : \Re(\alpha) \in (0,2)\backslash\{1/2,1\}\}$, there exists $m \in \mathbb{N}$ finite such that H_α^m is trace class in the Hilbert space $L^2_\kappa(X)$. More precisely, let V_ℓ and W_ℓ be as in Proposition 4.7. If $\alpha \in V_\ell$, $\ell \geq 0$, $H_\alpha^{2^{\ell+1}}$ is trace class and if $\alpha \in W_\ell$, $\ell \geq 1$, $H_\alpha^{3\cdot 2^{\ell+1}}$ is trace class.*

Step 3 : Analyticity of the Fredholm determinant. From Corollary 4.8, the Fredholm determinant of H_α^m is then properly defined for $\alpha \in V$ and m large enough. We now want to justify that $\alpha \mapsto \det(I - H_\alpha^m)$ is analytic on V. Consider a trace class operator Q in $L^2_\kappa(X)$, written for some measurable kernel $q : X^2 \to \mathbb{C}$,

$$Qf(x) = \int_X q(x,y)f(y)d\rho(y),$$

where ρ is the underlying measure of our L^2 space, that is $L^2_\kappa(X) = L^2(X,\rho)$ with

$$\int f d\rho = \int_{[0,\pi/2]} f_0(e^{i\psi})d\psi + \sum_{\varepsilon\in\{1,i\}} \int_{[0,\pi/2]} f_\varepsilon(e^{i\psi}) \frac{d\psi}{|i.e^{i\psi}|^\kappa}\,.$$

There is a specific choice of q which is especially relevant. If $x = (e^{i\omega},\varepsilon)$ and $y = (e^{i\psi},\delta)$, $0 < \omega, \psi < \pi/2$, we consider the average value around (x,y), for $r > 0$,

$$q(x,y,r) = \frac{1}{r^2}\int_{C_r} q((e^{i\omega},\varepsilon),(e^{i\psi},\delta))d\psi d\omega,$$

with $C_r = ([\omega - r/2, \omega + r/2] \times [\psi - r/2, \psi + r/2]) \cap [0,\pi/2]^2$. We define the Lebesgue value of q as $\tilde{q}(x,y) = \lim_{r\downarrow 0} q(x,y,r)$. From Lebesgue's Theorem, $\rho\otimes\rho$-a.e., $\tilde{q}(x,y) = q(x,y)$. From Brislawn's Theorem [18], $\tilde{q}(x,x)$ exists ρ-a.e. and we have $\mathrm{Tr}(Q) = \int_X \tilde{q}(x,x)d\rho(x)$. As a consequence, Simon [30, Theorem 3.7] implies

$$\det(I+Q) = \sum_{n=0}^{\infty} \frac{1}{n!}\int_{X^n} \det\left(\tilde{q}(x_i,x_j)\right)_{1\leq i,j\leq n} \prod_{i=1}^n d\rho(x_i). \tag{39}$$

This representation of the Fredholm determinant will be used in the proof of the following lemma.

Lemma 4.9. *Let $f \in L^2(X,\rho)$, Ω be an open connected set of $\mathbb{C}$ and for $z \in V$, let Q_z be a trace class operator in $L^2(X,\rho)$ with a measurable kernel $q_z(x,y)$. Assume that $\rho\otimes\rho$-a.e. (i) for all $z \in \Omega$, $|q_z(x,y)| \leq f(x)f(y)$ and (ii) $z \mapsto q_z(x,y)$ is analytic on Ω. Then $z \mapsto \det(I+Q_z)$ is analytic on Ω.*

We will use repeatedly the following elementary consequence of Cauchy's formula and Lebesgue dominated convergence.

Lemma 4.10. *Let Ω be an open set of $\mathbb{C}$ and for each $z \in \Omega$, let $x \mapsto f(x,z)$ be a measurable function in a measure space (X,ρ). Assume that there exists $g \in L^1(X,\rho)$ such that ρ-a.e. (i) for all $z \in \Omega$, $|f(x,z)| \leq g(x)$ and (ii) $z \mapsto f(x,z)$ is analytic on Ω. Then $z \mapsto \int_X f(x,z)d\rho(x)$ is analytic on Ω.*

Proof of Lemma 4.9. Since $f \in L^2(X,\rho)$, it also belongs to $L^1(X,\rho)$. From (i)-(ii) and Lemma 4.10 for any $r>0$, $\rho\otimes\rho$-a.e., $z\mapsto q_z(x,y,r)$ is analytic on Ω. From Vitali's convergence theorem, it follows that, $\rho\otimes\rho$-a.e., $z\mapsto \tilde{q}_z(x,y)$ is analytic on Ω.

Now, we let $p_z(x,y)=\tilde{q}_z(x,y)/(f(x)f(y))$. By assumption (i), $\rho\otimes\rho$-a.e., for all $z\in\mathbb{C}$ $|p_z(x,y)|\le 1$. From the multi-linearity of the determinant,

$$D_z(x_1,\dots,x_n)=\det\left(\tilde{q}_z(x_i,x_j)\right)_{1\le i,j\le n}=\det\left(p_z(x_i,x_j)\right)_{1\le i,j\le n}\prod_{i=1}^n f(x_i)^2.$$

However, from Hadamard's inequality, $\rho^{\otimes n}$-a.e.,

$$\left|\det\left(p_z(x_i,x_j)\right)_{1\le i,j\le n}\right|\le n^{n/2}.$$

Hence, $\rho^{\otimes n}$-a.e., for all $z\in\Omega$, $|D_z(x_1,\dots,x_n)|\le n^{n/2}\prod_{i=1}^n f(x_i)^2$. From (ii), $\rho^{\otimes n}$-a.e., $z\mapsto D_z(x_1,\dots,x_n)$ is analytic. We deduce by a new application of Lemma 4.10 that

$$z\mapsto \frac{1}{n!}\int_{X^n} D_z(x_1,\dots,x_n)\prod_{i=1}^n d\rho(x_i)$$

is analytic on Ω and bounded by $\|f\|_{L^2(X)}^{2n}n^{n/2}/n!\le n^{-n/2}C^n$, with $C=e\|f\|_{L^2(X)}^2$. The series $\sum_n n^{-n/2}C^n$ being convergent, a new application of Vitali's convergence theorem proves the analycity on Ω of

$$z\mapsto\sum_{n=0}^\infty\frac{1}{n!}\int_{X^n} D_z(x_1,\dots,x_n)\prod_{i=1}^n d\rho(x_i).$$

It remains to use (39). □

Proposition 4.11. *For integer $\ell\ge 0$, let V_ℓ and W_ℓ be as in Proposition 4.7. The function $\alpha\mapsto\det(I-H_\alpha^{2^{\ell+1}})$ is analytic on V_ℓ. For $\ell\ge 1$, $\alpha\mapsto\det(I-H_\alpha^{3\cdot 2^{\ell+1}})$ is analytic on W_ℓ.*

Proof. We only treat the first case, the second being similar. Let $\Omega\subset V_\ell$ be an open set with $\bar{\Omega}\subset V_\ell$. In particular for all $\alpha\in\Omega$,

$$2^{-\ell}<\beta_0<\Re(\alpha)<\beta_1<2^{-\ell+1}.$$

Consider the operator P defined in (26) on $L^2_\kappa(S_1^+)$. We denote the kernel of $P^{2^{s-1}}$ by k_s (it depends implicitly on $\alpha\in V_\ell$). We will prove that for all $\omega,\psi\in(0,\pi/2)^2$, $\alpha\mapsto k_{\ell+1}(\omega,\psi)$ is analytic on Ω and that $|k_{\ell+1}(\omega,\psi)|\le Cf(\sin(2\psi))f(\sin(2\omega))$ with

$$f(x)=\begin{cases} x^{\beta_0/2-1} & \text{if } \ell=0\\ x^{-2^{\ell-2}\beta_1} & \text{if } \ell\ge 1.\end{cases}$$

We will then argue that we have the same properties for the kernels of $H_\alpha^{2^\ell}$ and $H_\alpha^{2^{\ell+1}}$. It will remain to apply Corollary 4.8 and Lemma 4.9 to conclude the proof.

Iterated kernels of P, case $\ell = 0$. From the explicit form of the kernel $k = k_1$ in (27), we see by Lemma 4.10 that, for all ω, ψ, $\alpha \mapsto k_1(\omega, \psi)$ is analytic on Ω. From (32), $|k_1(\omega, \psi)| \leq C(\sin(2\psi) \wedge \sin(2\omega))^{\beta_0/2-1} \leq Cf(\sin(2\psi))f(\sin(2\omega))$.
Iterated kernels of P, $\ell \geq 1$. First, from the explicit form of the kernel $k = k_1$ in (27) and Lemma 4.10, for all $\omega \neq \psi$, $\alpha \mapsto k_1(\omega, \psi)$ is analytic on Ω and from (32), $|k_1(\omega, \psi)| \leq C|\psi - \omega|^{\beta_0 - 1}(\sin(2\psi) \wedge \sin(2\omega))^{-\beta_1/2}$.

We set $\delta_\ell = 2^{\ell-1}\beta_1$ and $\zeta_\ell = 2^{\ell-1}\beta_0$. We use the following inequality (which generalizes (37)) for $0 < a, b < 1$, $0 < \delta, \zeta < 1$,

$$J_{\zeta,\delta}(a,b) = \int_0^1 |x-a|^{\zeta-1}|x-b|^{\zeta-1}(x \wedge a)^{-\delta/2}(x \wedge b)^{-\delta/2}dx \leq \begin{cases} Ca^{-\delta}(b-a)^{2\zeta-1} & \text{if } 0 < 2\zeta < 1, \\ Ca^{-\delta/2}b^{-\delta/2} & \text{if } 1 < 2\zeta. \end{cases}$$

We may argue as in the proof of Proposition 4.7 and use Lemma 4.10 by recursion on $1 \leq s \leq \ell$. We obtain that $|k_s(\omega, \psi)| \leq C_s|\psi - \omega|^{\zeta_s - 1}(\sin(2\psi) \wedge \sin(2\omega))^{-\delta_s/2}$ and $\alpha \mapsto k_s(\omega, \psi)$ analytic on Ω for all $\omega \neq \psi$ in $(0, \pi/2)$. Since $2\zeta_\ell = \zeta_{\ell+1} = 2^\ell\beta_0 > 1$, we find that $k_{\ell+1}(\omega, \psi) \leq Cf(\sin(2\psi))f(\sin(2\omega))$ and is analytic in $\alpha \in \Omega$ for all $(\omega, \psi) \in (0, \pi/2)^2$.
Iterated kernels of H_α. For $x = (e^{i\omega}, \varepsilon), y = (e^{i\psi}, \delta) \in X$, let $h_s(x, y)$ be the kernel of $H_\alpha^{2^{s-1}}$. We now study the analyticity of the kernels of $\alpha \mapsto h_s(x, y)$ for $s \geq 1$. First, in (30), the function $\alpha \mapsto c'_\alpha$ is analytic on U. It thus suffices to study $\hat{S} = SJ$. If $\hat{P} = PJ$, we have from (29) in matrix form

$$\hat{S} = \begin{pmatrix} M_{00} & M_{01} & M_{0i} \\ M_{10} & 1 & 0 \\ M_{i0} & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{P} & 0 & 0 \\ 0 & \hat{P} & 0 \\ 0 & 0 & \hat{P} \end{pmatrix} \begin{pmatrix} N_1 & 0 & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{pmatrix} J$$

In particular, the kernel of $\hat{S}$ at $x = (e^{i\omega}, \varepsilon), y = (e^{i\psi}, \delta) \in X$ is equal to $\sigma(x, y) = g^\alpha_{\varepsilon,\delta}(\psi)k_1(\psi, \omega)f^\alpha_{\varepsilon,\delta}(\omega)$, where $g^\alpha_{\varepsilon,\delta}(\psi), f^\alpha_{\varepsilon,\delta}(\omega)$ are bounded and analytic in $\alpha \in \Omega$. This factors $g^\alpha_{\varepsilon,\delta}(\psi), f^\alpha_{\varepsilon,\delta}(\psi)$ is harmless and the above argument carries over easily to this more general situation. We find that the kernel $\sigma_\ell(x, y)$ of $\hat{S}^{2^{\ell-1}}$ satisfies for all $\omega, \psi \in (0, \pi/2)^2$, $\alpha \mapsto \sigma_{\ell+1}(x, y)$ is analytic on Ω and $|\sigma_{\ell+1}(x, y)| \leq Cf(\sin(2\psi))f(\sin(2\omega))$. By a new application Lemma 4.10, the same holds for $\sigma_{\ell+1}$ replaced by $\sigma_{\ell+2}$. Finally, since $H_\alpha = c'_\alpha\hat{S}$, the same holds for $h_{\ell+2}$. By Corollary 4.8, $H_\alpha^{2^{\ell+1}}$ is trace class. Hence, the conditions of Lemma 4.9 are fulfilled for $-H_\alpha^{2^{\ell+1}}$, we obtain that $\alpha \mapsto \det(I - H_\alpha^{2^{\ell+1}})$ is analytic on Ω. □

Step 4 : Proof of Theorem 4.5. We need a final lemma before proving Theorem 4.5

Lemma 4.12. *Let $\ell \geq 0$ be an integer and $2^{-\ell} < \beta < 2^{-\ell+1}$. If $m \geq 2^\ell$ then,*

$$\lim_{t \to \infty} \|H^m_{\beta+it}\| = 0,$$

where $\|\cdot\|$ denotes the operator norm in $L^2_\kappa(X)$. Similarly, the same conclusion holds if $2^{-\ell} < 3\beta < 2^{-\ell+1}$ with $\ell \geq 1$ and $m \geq 3.2^\ell$.

Proof. We observe that $H_{\beta+it} = f(t)\hat{S}$, where $\hat{S} = SJ$ and $f(t) = \frac{2c_{\beta+it}}{\alpha a_0^2}$ goes to zero when t goes to infinity by definition (14). Moreover, the operator norm of $\hat{S}^m = \hat{S}_\alpha^m$ is bounded by the Hilbert-Schmidt norm of $\hat{S}^m_{\Re(\alpha)}$ which is finite by Proposition 4.7 if m is large enough. □

We are now ready to prove Theorem 4.5. Let $\ell \geq 0$ be an integer, V_ℓ, W_ℓ be as in Proposition 4.7 and F be a closed subset of V_ℓ. From (31) and Lemma 4.6, if $I + K_\alpha$ is not an isomorphism of $\mathcal{H}_{\alpha/2}$ then -1 is an eigenvalue of H_α in $C_\kappa(X)$ and 1 is an eigenvalue of H_α^m in $C_\kappa(X)$ for any m even. If $\alpha \in V_\ell$, we take $m = 2^{\ell+1}$, then by Corollary 4.8, H_α^m is a trace class operator in $L^2_\kappa(X)$. Since $C_\kappa(X) \subset L^2_\kappa(X)$, from Theorem [30, Theorem 3.7], we have $\det(I - H_\alpha^m) = 0$. In summary,

$$\mathcal{A} \cap F \subset \{\alpha \in F : \det(I - H_\alpha^m) = 0\}.$$

However, by Proposition 4.11, $\varphi : \alpha \mapsto \det(I - H_\alpha^m)$ is analytic on V_ℓ and by Lemma 4.12, $\det(I - H_\alpha^m)$ is non-zero for $\Im(\alpha)$ large enough. It follows that the level set 0 of φ cannot have accumulation points in F. Similarly, if $F \subset W_\ell$, F cannot have accumulation points. This proves Theorem 4.5. □

5 Local law : proofs of Proposition 3.2 and Theorem 1.2

In this section, we conclude the proof of Theorem 1.2. We will first establish Proposition 3.2, the main technical steps are an adaptation of [16]. Then, using the result of the previous section, we will derive a quantitative estimate on the resolvent of A.

5.1 Concentration and deviation inequalities

We start by recalling a lemma on the concentration for the diagonal of the resolvent of random matrices. The Lipschitz norm of $f : \mathbb{C} \to \mathbb{C}$ is

$$\|f\|_L = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Lemma 5.1 ([16, Lemma C.3])**.** *Let $\beta \in (0,1)$ and $z = E + i\eta \in \mathbb{C}_+$. Let $B \in M_n(\mathbb{C})$ be a random Hermitian matrix and $G = (B - z)^{-1}$. Let us assume that the vectors $(B_i)_{1 \leq i \leq n}$, where $B_i := (B_{ij})_{1 \leq j \leq i} \in \mathbb{C}^i$, are independent. Then for any $f : \mathbb{C} \to \mathbb{R}$ and every $t \geq 0$,*

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{k=1}^n f(G_{kk}) - \mathbb{E}\frac{1}{n}\sum_{k=1}^n f(G_{kk})\right| \geq t\right) \leq 2\exp\left(-\frac{n\eta^2 t^2}{8\|f\|_L^2}\right).$$

We will need concentration inequalities for the norm $\|.\|_\kappa$. Our first claim checks that the norm $\|\cdot\|_\kappa$ has some Hölder regularity.

Lemma 5.2. *Let $\beta \in (0,1)$ and for $k \in \{1,2\}$, $x_k \in \mathcal{K}_1$, $|x_k| \leq \eta^{-1}$, $\eta \leq 1$.*

(i) We define $\gamma_k \in \bar{\mathcal{H}}^0_\beta$ by

$$\gamma_k(u) = (x_k.u)^\beta.$$

There exists a constant $c = c(\beta)$ such that for any $0 < \delta < \beta$,

$$\|\gamma_k\|_{1-\beta+\delta} \leq c|x_k|^\beta \quad \text{and} \quad \|\gamma_1 - \gamma_2\|_{1-\beta+\delta} \leq c\eta^{-\beta}\Big(|x_1 - x_2|^\beta + \eta^\delta |x_1 - x_2|^\delta\Big).$$

(ii) Assume additionally that $\Re(x_k) \geq s$. We set $\gamma_k(u) = (x_k^{-1}.u)^\beta$. Then, for some $c = c(\beta)$,

$$\|\gamma_1 - \gamma_2\|_{1-\beta+\delta} \leq cs^{\beta-2}\eta^{2\beta-1}|x_1 - x_2|. \tag{40}$$

Proof. We prove the second statement of (i). The first statement is easy. We shall use two simple bounds. First, for any $\beta \in (0, 1]$, for all $x, y \in \mathcal{K}_1$

$$|x^\beta - y^\beta| \leq |x - y|^\beta. \tag{41}$$

Secondly, for any $u \in \mathcal{K}_1^+$, $h \in \mathcal{K}_1$,

$$|h||i.u| \leq |h.u| \leq \sqrt{2}|h||u|, \tag{42}$$

(see [16, Equation (55)]). From (41)-(42), we get

$$|\gamma_1(u) - \gamma_2(u)| \leq |x_1.u - x_2.u|^\beta \leq 2^{\beta/2}|x_1 - x_2|^\beta. \tag{43}$$

Similarly, we get

$$|\gamma_1(u) - \gamma_1(u')| \leq 2^{\beta/2}\eta^{-\beta}|u - u'|^\beta. \tag{44}$$

Hence,

$$\sup_{u \in S_1^+} |\gamma_1(u) - \gamma_2(u)| \leq c|x_1 - x_2|^\beta.$$

We also need to control the derivative of $\gamma_1 - \gamma_2$. For $\varepsilon \in \{1, i\}$,

$$\partial_\varepsilon \gamma_k(u) = \beta(x_k.u)^{\beta-1}(x_k.\varepsilon).$$

We consider the function $g \in \mathcal{H}_\beta$ defined for $x \in \mathcal{K}_1$ by $g(x) = (x.u)^{\beta-1}(x.\varepsilon)$. For $s > 0$ to be fixed later, let $\phi : \mathbb{C} \to \mathbb{C}$ be equal to one for $|z| \geq 2s$, equal to zero for $|z| \leq s$ and growing linearly with the modulus in between. Thus ϕ is Lipschitz with constant $1/s$. We write

$$g(x) = (1 - \phi(x.u))g(x) + \phi(x.u)g(x) = g_1(x) + g_2(x).$$

From (42),

$$|x.\varepsilon| \leq \sqrt{2}|x| \leq \sqrt{2}\frac{|x.u|}{|i.u|}. \tag{45}$$

If follows that the function g_1 is bounded by $c_0 s^\beta/|i.u|$. Moreover, the derivative with respect to x of $g(x)$ is equal for $\varepsilon' \in \{1, i\}$,

$$\partial_{\varepsilon'} g(x) = (\beta-1)(x.u)^{\beta-2}(\varepsilon'.u)(x.\varepsilon) + (x.u)^{\beta-1}(\varepsilon'.\varepsilon),$$

where ∂_1 is the derivative with respect to the real part of x and ∂_i is the derivative with respect to the imaginary part of x. Using (45), we deduce that on the support of the function $x \mapsto \phi(x.u)$, g is Lipschitz with constant $cs^{\beta-1}/|i.u|$. Thus g_2 is Lipschitz with constant $cs^{\beta-1}/|i.u|$ (for some new constant $c > 0$). It follows that

$$|\partial_\varepsilon \gamma_1(u) - \partial_\varepsilon \gamma_2(u)| \le |g_1(x_1)| + |g_1(x_2)| + |g_2(x_1) - g_2(x_2)| \le c_1 s^\beta |i.u|^{-1} + c_1 s^{\beta-1}|x_1 - x_2||i.u|^{-1}.$$

We choose $s = |x_1 - x_2|$, we find for some new constant $c > 0$.

$$|\partial_\varepsilon \gamma_1(u) - \partial_\varepsilon \gamma_2(u)| \le c|x_1 - x_2|^\beta |i.u|^{-1}. \tag{46}$$

The above bound is not quite enough due to the factor $|i.u|^{-1}$. Let $\beta' = \beta - \delta$ and $a = (1-\beta)/(1-\beta') < 1$ From (41) for any $u, v \in S_1^+$, $x \in \mathcal{K}_1$,

$$\left| \left(\frac{(x.u)^a}{i.u} \right)^{\beta'-1} - \left(\frac{(x.v)^a}{i.v} \right)^{\beta'-1} \right| \le \left| \frac{i.u}{(x.u)^a} - \frac{i.v}{(x.v)^a} \right|^{1-\beta'}.$$

If $|i.u| \ge |i.v|$, we find from (42)

$$\left| \frac{i.u}{(x.u)^a} - \frac{i.v}{(x.v)^a} \right| \le \left| \frac{(i.u - i.v)}{(x.u)^a} \right| + \left| \frac{(i.v)((x.v)^a - (x.u)^a)}{(x.u)^a (x.v)^a} \right| \le \frac{|u-v|}{|x|^a |i.u|^a} + \frac{\sqrt{2}^a |u-v|^a}{|x|^a |i.u|^{2a-1}}$$

where at the last step, we have used (41) (for $\beta = a < 1$). Using that $|x_k| \le \eta^{-1}$, we arrive at

$$|(i.u)^{1-\beta'} \partial_\varepsilon \gamma_k(u) - (i.v)^{1-\beta'} \partial_\varepsilon \gamma_k(v)| \le \eta^{-\beta} \left(\frac{|u-v|}{(|i.u| \vee |i.v|)^a} \right)^{1-\beta'} + \eta^{-\beta} \left(\frac{|u-v|}{(|i.u| \vee |i.v|)^{2-1/a}} \right)^{1-\beta} \tag{47}$$

Now, we fix s to be chosen later. For any $v \in S_1^+$, there exists $u \in S_1^+$, such that $|u - v| \le s$ and $|i.u| \ge s/2$, see (35). Let $t = \eta^{-\beta}(s/(s/2)^a)^{1-\beta'} + \eta^{-\beta}(s/(s/2)^{2-1/a})^{1-\beta}$, we have from (47) that

$$\left| (i.v)^{1-\beta'} (\partial_\varepsilon \gamma_1(v) - \partial_\varepsilon \gamma_2(v)) \right| \le 2t + \left| (i.u)^{1-\beta'} (\partial_\varepsilon \gamma_1(u) - \partial_\varepsilon \gamma_2(u)) \right|.$$

Observe that $s^{\beta-\beta'} = s^\delta = ct\eta^\beta$. Hence, from (46), we get

$$\left| (i.v)^{1-\beta'} (\partial_\varepsilon \gamma_1(v) - \partial_\varepsilon \gamma_2(v)) \right| \le c\eta^{-\beta} s^\delta + c|x_1 - x_2|^\beta s^{-\beta'}.$$

We choose $s = \eta|x_1 - x_2|$. The second statement of (i) follows.

To prove the claim (ii), observe that the derivative of $x \mapsto (x^{-1}.u)^\beta$ in the direction $\varepsilon \in \{1, i\}$ is

$$-\beta x^{-2}(\varepsilon.u)(x^{-1}.u)^{\beta-1}.$$

We observe that $\Re(x.u) \geq \Re(x) \geq s$ and $\Re(x^{-1}) = \Re(x)/|x|^2$. This derivative is bounded by $c(\Re x)^{\beta-1}|x|^{-2\beta}$. Thus, we find from the intermediate value theorem that $|\gamma_1(u) - \gamma_2(u)| \leq cs^{\beta-1}\eta^{-2\beta}|x_1 - x_2|$.

Similarly, the derivative of $x \mapsto (x^{-1}.u)^{\beta-1}(x^{-1}.\varepsilon')$ in the direction $\varepsilon \in \{1, i\}$ is

$$-(\beta-1)x^{-2}(\varepsilon.u)(x^{-1}.u)^{\beta-2}(x^{-1}.\varepsilon') - x^{-2}(x^{-1}.u)^{\beta-1}(\varepsilon.\varepsilon').$$

The derivative is bounded by $c|x|^{1-2\beta}\Re(x)^{\beta-2} + c|x|^{-2\beta}\Re(x)^{\beta-1} \leq 2c|x|^{1-2\beta}\Re(x)^{\beta-2}$. □

Lemma 5.3. *Let $\beta \in (0,1)$ and $z = E + i\eta \in \mathbb{C}_+$. Let $B \in M_n(\mathbb{C})$ be a random Hermitian matrix, $G = (B-z)^{-1}$, and we define $\gamma \in \bar{\mathcal{H}}_\beta^0$ by*

$$\gamma(u) = \frac{1}{n}\sum_{k=1}^n (-iG_{kk}.u)^\beta.$$

Let us assume that the vectors $(B_i)_{1\leq i\leq n}$ are independent, where $B_i = (B_{ij})_{j\leq i} \in \mathbb{C}^i$. Then, there exists a constant $c = c(\beta)$ such that for any $0 < \delta < \beta$ and $t \geq 0$,

$$\mathbb{P}(\|\gamma - \mathbb{E}\gamma\|_{1-\beta+\delta} \geq t) \leq c(\eta^\beta t)^{-\frac{1}{\delta}} \exp\Big(-cn(\eta^\beta t)^{\frac{2}{\delta}}\Big).$$

Proof. For $1 \leq k \leq n$, we set $h_k = -iG_{kk}$. We also set $\underline{\gamma}(u) = \gamma(u) - \mathbb{E}\gamma(u)$. The first step is to find a concentration inequality for the function $\gamma(u)$ and its derivatives for any fixed $u \in S_1^+$. It will follow from Lemma 5.1. We consider the function $f \in \mathcal{H}_\beta$ defined for $x \in \mathcal{K}_1$ by $f(x) = (x.u)^\beta$. For $s > 0$ to be fixed later, let $\phi : \mathbb{C} \to \mathbb{C}$ be equal to one for $|z| \geq 2s$, equal to zero for $|z| \leq s$ and growing linearly with the modulus in between. Thus ϕ is Lipschitz with constant $1/s$. We write,

$$f(x) = (1 - \phi(x.u))f(x) + \phi(x.u)f(x) = f_1(x) + f_2(x).$$

The function f_1 is bounded by $(2s)^\beta$. Let $t > 0$, we set s such that $(2s)^\beta = t/4$. We get

$$\begin{aligned}
\mathbb{P}\big(|\underline{\gamma}(u)| \geq t\big) &= \mathbb{P}\left(\left|\frac{1}{n}\sum_{k=1}^n f(h_k) - \mathbb{E}\frac{1}{n}\sum_{k=1}^n f(h_k)\right| \geq t\right) \\
&\leq \mathbb{P}\left(\left|\frac{1}{n}\sum_{k=1}^n f_2(h_k) - \mathbb{E}\frac{1}{n}\sum_{k=1}^n f_2(h_k)\right| \geq t/2\right) \\
&\leq 2\exp\Big(-cn\eta^2 t^{2/\beta}\Big), \qquad (48)
\end{aligned}$$

where the last line follows from Lemma 5.1 and that f_2 is Lipschitz with constant $cs^{\beta-1} = c't^{1-1/\beta}$.

Similarly, for $\varepsilon \in \{1, i\}$, we have

$$\partial_\varepsilon\gamma(u) = \frac{1}{n}\sum_{k=1}^n \beta(h_k.u)^{\beta-1}(h_k.\varepsilon).$$

We consider the function $g \in \mathcal{H}_\beta$ defined for $x \in \mathcal{K}_1$ by $g(x) = (x.u)^{\beta-1}(x.\varepsilon)$. As in the proof of Lemma 5.2, we decompose $g(x) = g_1(x) + g_2(x)$ with g_1 bounded by $c_0 s^\beta/|i.u|$ and g_2 Lipschitz with constant $cs^{\beta-1}/|i.u|$. Now, for fixed $t > 0$, we choose $s > 0$ such that $c_0 s^\beta/|i.u| = t/4$. Using Lemma 5.1, we find

$$\begin{aligned}\mathbb{P}\big(|\partial_\varepsilon \underline{\gamma}(u)| \geq t\big) &\leq \mathbb{P}\left(\left|\frac{1}{n}\sum_{k=1}^n g_2(h_k) - \mathbb{E}\frac{1}{n}\sum_{k=1}^n g_2(h_k)\right| \geq t/2\right)\\ &\leq 2\exp\Big(-cn\eta^2 t^{2/\beta}|i.u|^{2/\beta}\Big).\end{aligned}$$

Hence, for any $t \geq 0$,

$$\mathbb{P}\Big(\Big|(i.u)^{1-\beta+\delta}\partial_\varepsilon \underline{\gamma}(u)\Big| \geq t\Big) \leq 2\exp\Big(-cn\eta^2 t^{2/\beta}|i.u|^{2-2\delta/\beta}\Big). \tag{49}$$

In the second and final step of the proof, we use a net argument to obtain the concentration for the norm. If F is a s-net of S_+^1 with $2^{\beta/2}\eta^{-\beta}s^\beta = t/4$, from (44), we deduce that for any $t \geq 0$,

$$\mathbb{P}\left(\sup_{u\in S_1^+}|\underline{\gamma}(u)| \geq t\right) \leq \mathbb{P}\left(\sup_{u\in F}|\underline{\gamma}(u)| \geq t/2\right).$$

There exists a net F of cardinal bounded by $1/s$. From the union bound and (48), we deduce that

$$\mathbb{P}\left(\sup_{u\in S_1^+}|\underline{\gamma}(u)| \geq t\right) \leq \frac{c}{\eta t^{1/\beta}}\exp\Big(-cn\eta^2 t^{2/\beta}\Big).$$

Similarly, consider F is a s-net of S_+^1 of cardinal at most $1/s$ such that $|i.u| \geq s/2$ for all $u \in F$. Set $\beta' = \beta - \delta$. From (47), if $\eta^{-\beta}(s/(s/2)^a)^{1-\beta'} + \eta^{-\beta}(s/(s/2)^{2-1/a})^{1-\beta} = t/4$, we have that for any $t \geq 0$,

$$\mathbb{P}\left(\sup_{u\in S_1^+}\Big|(i.u)^{1-\beta'}\partial_\varepsilon \underline{\gamma}(u)\Big| \geq t\right) \leq \mathbb{P}\left(\sup_{u\in F}\Big|(i.u)^{1-\beta'}\partial_\varepsilon \underline{\gamma}(u)\Big| \geq t/2\right).$$

Observe that $s^{\beta-\beta'} = s^\delta = ct\eta^\beta$. From the union bound and (49), we deduce that

$$\mathbb{P}\left(\sup_{u\in S_1^+}\Big|(i.u)^{1-\beta'}\partial_\varepsilon \underline{\gamma}(u)\Big| \geq t\right) \leq c(\eta t^{1/\beta})^{-\beta/\delta}\exp\Big(-cn\eta^{2\beta/\delta}t^{2/\delta}\Big).$$

It concludes the proof. □

The following lemma is a variant of the preceding statement.

Lemma 5.4. *Let $0 < \alpha < 2$, $\beta \in (0,1)$, $(h_k)_{1\leq k\leq n} \in \mathcal{K}_1^n$ and $(g_k)_{1\leq k\leq n}$ be iid standard normal variables. We define $\gamma \in \bar{\mathcal{H}}_\beta^0$ by*

$$\gamma(u) = \frac{1}{n}\sum_{k=1}^n (h_k.u)^\beta |g_k|^\alpha.$$

Assume that for all $k \geq 1$, $|h_k| \leq \eta^{-1}$. Then, there exists a constant $c = c(\alpha, \beta)$ such that for any $0 < \delta < \beta$ and $0 \leq t \leq \eta^{-\beta}$,

$$\mathbb{P}(\|\gamma - \mathbb{E}\gamma\|_{1-\beta+\delta} \geq t) \leq c(\eta^\beta t)^{-\frac{1}{\delta}} \exp\Big(-cn(\eta^\beta t)^{\frac{2}{\delta}}\Big).$$

Proof. We start by a preliminary concentration inequality. For $0 < \alpha < 2$, the variable $|g_k|^\alpha$ is sub-exponential. From Bernstein's inequality, for any $t \geq 0$, we have

$$\mathbb{P}\left(\left|\sum_k x_k|g_k|^\alpha - \mathbb{E}\sum_k x_k|g_k|^\alpha\right| \geq t\right) \leq 2\exp\left(-c\frac{t^2}{\|x\|_2^2} \wedge \frac{t}{\|x\|_\infty}\right).$$

Since $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$, if $\varphi(t) = t \wedge t^2$,

$$\mathbb{P}\left(\left|\sum_k x_k|g_k|^\alpha - \mathbb{E}\sum_k x_k|g_k|^\alpha\right| \geq nt\right) \leq 2\exp\left(-cn\varphi\left(\frac{t}{\|x\|_\infty}\right)\right). \tag{50}$$

In particular, for any $u \in S_1^+$,

$$\mathbb{P}(|\gamma(u) - \mathbb{E}\gamma(u)| \geq t) \leq 2\exp\Big(-cn\varphi(t\eta^\beta)\Big),$$

and for $\varepsilon \in \{1, i\}$, from (45),

$$\mathbb{P}(|\partial_\varepsilon\gamma(u) - \mathbb{E}\partial_\varepsilon\gamma(u)| \geq t) \leq 2\exp\Big(-cn\varphi(t\eta^\beta|i.u|)\Big).$$

We then repeat the net argument used in Lemma 5.1. Setting, $L = \frac{1}{n}\sum_k |g_k|^\alpha$, the inequalities (43)- (47) are replaced respectively by

$$|\gamma(u) - \gamma(v)| \leq L2^{\beta/2}\eta^{-\beta}|u - v|^\beta.$$

and

$$|(i.u)^{1-\beta'}\partial_\varepsilon\gamma(u) - (i.v)^{1-\beta'}\partial_\varepsilon\gamma(v)| \leq L\eta^{-\beta}\left(\frac{|u-v|}{(|i.u| \vee |i.v|)^a}\right)^{1-\beta'} + L\eta^{-\beta}\left(\frac{|u-v|}{(|i.u| \vee |i.v|)^{2-1/a}}\right)^{1-\beta}$$

Arguing as in Lemma 5.1, we find that for any $t \leq \eta^{-\beta}$,

$$\mathbb{P}(\|\gamma - \mathbb{E}\gamma\|_{1-\beta+\delta} \geq t\ell) \leq c(\eta^\beta t)^{-\frac{1}{\delta}} \exp\Big(-cn(\eta^\beta t)^{\frac{2}{\delta}}\Big) + \mathbb{P}(L \geq \ell).$$

From Bernstein's inequality, for $\ell = 2\mathbb{E}|g_1|^\alpha$, $\mathbb{P}(L \geq \ell) \leq \exp(-cn)$. □

We conclude this subsection with a perturbation inequality for the resolvent.

Lemma 5.5. *Let $\beta \in (0, 1]$, $B, B' \in M_n(\mathbb{C})$ be hermitian matrices, $z = E + i\eta$ and $R = (B - z)^{-1}$ and $R' = (B' - z)^{-1}$. Then,*

$$\sum_{k=1}^n |R_{kk} - R'_{kk}|^\beta \leq 2n^{1-\beta}\eta^{-\beta}\mathrm{rank}(B - B').$$

Proof. It is a variant of the proof of [16, Equation (91)]. The resolvent identity asserts that $M = R - R' = R(B' - B)R'$. It follows that $r = \mathrm{rank}(M) \leq \mathrm{rank}(B - B')$. We notice also that $\|M\| \leq 2\eta^{-1}$. Hence, in the singular value decomposition of $M = UDV$, at most r entries of $D = \mathrm{diag}(s_1, \cdots, s_n)$ are non zero and they are bounded by $\|M\|$. We denote by $u_1, \cdots, u_r$ and $v_1, \cdots, v_r$ the associated orthonormal vectors so that

$$M = \sum_{i=1}^{r} s_i u_i v_i^*,$$

and

$$\left|R_{kk} - R'_{kk}\right| = |M_{kk}| = \left|\sum_{i=1}^{r} s_i \langle u_i, e_k\rangle\langle v_i, e_k\rangle\right| \leq \|M\| \sum_{i=1}^{r} |\langle u_i, e_k\rangle||\langle v_i, e_k\rangle|.$$

From the subadditivity of $x \mapsto |x|^\beta$, we get

$$\left|R_{kk} - R'_{kk}\right|^\beta \leq \|M\|^\beta \sum_{i=1}^{r} |\langle u_i, e_k\rangle|^\beta |\langle v_i, e_k\rangle|^\beta.$$

Finally, from Hölder inequality,

$$\sum_{k=1}^{n} \left|R_{kk} - R'_{kk}\right|^\beta \leq \|M\|^\beta \sum_{i=1}^{r} n^{1-\beta} \left(\sum_{k=1}^{n} |\langle u_i, e_k\rangle|^2\right)^{\beta/2} \left(\sum_{k=1}^{n} |\langle v_i, e_k\rangle|^2\right)^{\beta/2} = r\|M\|^\beta n^{1-\beta}.$$

It completes the proof. □

5.2 Properties of α-stable variables

In this subsection, we let $(X_k)_{1\leq k\leq n}$ be iid symmetric α-stable random variables with distribution $\mathrm{Stab}_\alpha(0, \sigma)$ for some $\sigma > 0$ and $0 \leq \alpha < 2$. More precisely, for all $t \in \mathbb{R}$,

$$\mathbb{E}\exp(itX) = \exp\left[-\sigma^\alpha |t|^\alpha\right] \tag{51}$$

Our first lemma is a deviation inequality for quadratic forms with heavy tails (in a slightly modified form).

Lemma 5.6 ([16, Lemma 3.3])**.** *Let $B \in M_n(\mathbb{C})$. For any $0 < \alpha < 2$, there exists a constant $c = c(\alpha) > 0$ such that for $n \geq 2$ and $t \geq 0$,*

$$\mathbb{P}\left(\left|n^{-\frac{2}{\alpha}} \sum_{1\leq k\neq \ell\leq n} X_k X_\ell B_{k\ell}\right| \geq \sigma^2 \sqrt{\frac{\mathrm{Tr}(BB^*)}{n^2}} t\right) \leq ct^{-\alpha} \log\left(n(t \vee 2)\right) \log(t \vee 2).$$

We also use the following identity (which is special case of a more general distributional identity).

Lemma 5.7 ([16, Corollary B.2]). *Let $(h_k)_{1\le k\le n}\in\mathcal{K}_1^n$. Then*

$$\mathbb{E}\exp\left(-\sum_{k=1}^n h_k X_k^2\right)=\mathbb{E}\exp\left(-2^{\frac{\alpha}{2}}\sigma^\alpha\sum_{k=1}^n h_k^{\frac{\alpha}{2}}|g_k|^\alpha\right),$$

where $(g_1,\cdots,g_n)$ is a standard gaussian vector $N(0,I)$.

In the proof of Proposition 3.2, we shall use the following key corollary.

Corollary 5.8. *Let $(h_k)_{1\le k\le n}\in\mathcal{K}_1^n$ and $z\in\mathbb{C}_+$. Define the function $f\in\bar{\mathcal{H}}^0_{\alpha/2}$,*

$$f(u) \;=\; \Gamma\left(1-\frac{\alpha}{2}\right)\mathbb{E}\left(\left(-iz+n^{-\frac{2}{\alpha}}\sum_{k=1}^n X_k^2h_k\right)^{-1}.u\right)^{\frac{\alpha}{2}}.$$

and for $p\ge 0$, the scalars

$$\zeta=\mathbb{E}\left|-iz+n^{-\frac{2}{\alpha}}\sum_{k=1}^n X_k^2h_k\right|^{-p}\quad\text{and}\quad\xi=\mathbb{E}\left(-iz+n^{-\frac{2}{\alpha}}\sum_{k=1}^n X_k^2h_k\right)^{-p}.$$

We have $f=\mathbb{E}G_z(Z)$, $\zeta=\mathbb{E}r_{p,z}(Z)$ and $\xi=\mathbb{E}s_{p,z}(Z')$ with

$$Z(u)=2^{\frac{\alpha}{2}}\sigma^\alpha\times\frac{1}{n}\sum_{k=1}^n(h_k.u)^{\frac{\alpha}{2}}|g_k|^\alpha\quad\text{and}\quad Z'=2^{\frac{\alpha}{2}}\sigma^\alpha\times\frac{1}{n}\sum_{k=1}^n h_k^{\frac{\alpha}{2}}|g_k|^\alpha,$$

where $(g_1,\cdots,g_n)$ is a standard Gaussian vector $N(0,I)$.

Proof. The proof is again essentially contained in [16], we reproduce it. We set $h=-iz$, $\Re(h)>0$. We have

$$f(u)=\Gamma\left(1-\frac{\alpha}{2}\right)\mathbb{E}\left(\frac{h.\check{u}+n^{-\frac{2}{\alpha}}\sum_{k=2}^n X_{1k}^2h_k.\check{u}}{\left|h+n^{-\frac{2}{\alpha}}\sum_{k=2}^n X_{1k}^2h_k\right|^2}\right)^{\frac{\alpha}{2}}.$$

We use the formulas, for all $w\in\mathcal{K}_1$, $\beta>0$,

$$\begin{aligned}|w|^{-2\beta}=(w)^{-\beta}(\bar{w})^{-\beta} &= \Gamma(\beta)^{-2}\int_{[0,\infty)^2}dxdy\,x^{\beta-1}y^{\beta-1}e^{-xw-y\bar{w}}\\ &= \Gamma(\beta)^{-2}2^{1-\beta}\int_0^{\frac{\pi}{2}}d\theta\sin(2\theta)^{\beta-1}\int_0^\infty dr\,r^{2\beta-1}e^{-rw.e^{i\theta}}.\end{aligned}$$

and for $0<\beta<1$,

$$w^\beta \;=\; \beta\Gamma(1-\beta)^{-1}\int_0^\infty dx\,x^{-\beta-1}(1-e^{-xw}).$$

With $h=-iz$, we find that $f(u)$ is equal to

$$\begin{aligned}&c_\alpha\int_0^{\frac{\pi}{2}}d\theta\sin(2\theta)^{\frac{\alpha}{2}-1}\int_0^\infty dx\,x^{-\frac{\alpha}{2}-1}\\ &\quad\times\int_0^\infty dr\,r^{\alpha-1}\mathbb{E}\left(e^{-rh.e^{i\theta}-n^{-\frac{2}{\alpha}}\sum X_k^2rh_k.e^{i\theta}}-e^{-h.(re^{i\theta}+x\check{u})-n^{-\frac{2}{\alpha}}\sum X_k^2h_k.(re^{i\theta}+x\check{u})}\right).\end{aligned}$$

The above integrals are absolutely integrable since $\Re(h) > 0$. It remains to perform the change of variable $x = ry$ and use Corollary 5.7.

Similarly, with $h = -iz$,the above formula for $|w|^{-2\beta}$ with $\beta = p/2$ asserts that

$$\zeta = \Gamma(p/2)^{-2} 2^{1-p/2} \int_0^{\frac{\pi}{2}} d\theta \sin(2\theta)^{p/2-1} \int_0^{\infty} dr r^{p-1} \mathbb{E} e^{-rh.e^{i\theta} - n^{-\frac{2}{\alpha}} \sum X_k^2 r h_k . e^{i\theta}}.$$

We then use Lemma 5.7 and that $\Gamma(1/2) = \sqrt{\pi}$. The proof of the last statement is identical. □

The inverse of a non-negative $\alpha/2$-stable random variable has a light tail.

Lemma 5.9 ([16, Lemma B.3])**.** *Let S be a non-negative $\alpha/2$-stable variable with Laplace transform, $t \geq 0$, $\mathbb{E}\exp(-tS) = \exp(-t^{\alpha/2})$. Then, there exists $c > 0$ such that $\mathbb{E}\exp(cS^{-\alpha/(2-\alpha)}) < \infty$. In particular, for some $C > 0$, for any $t \geq 1$, $\mathbb{E}S^{-t} < (Ct^{\alpha/(2-\alpha)})^t$.*

5.3 Proof of Proposition 3.2

We first introduce the variable which depend on n and $z \in \mathbb{C}_+$,

$$M_z = \frac{\operatorname{Tr} R^{(1)}(z) R^{(1)}(z)^*}{n^2} = \frac{\operatorname{Tr} \Im(R^{(1)}(z))}{n^2 \Im z}.$$

We recall that $\gamma_z \in \bar{\mathcal{H}}^0_{\alpha/2}$ was defined for $u \in S_1^+$ by

$$\gamma_z(u) = \Gamma\Big(1 - \frac{\alpha}{2}\Big) \times \frac{1}{n} \sum_{k=1}^{n} (-iR_{kk}(z).u)^{\frac{\alpha}{2}}.$$

For short notation, as in the statement of Proposition 3.2, their expectations are denoted by

$$\bar{M}_z = \mathbb{E} M_z \quad \text{and} \quad \bar{\gamma}_z = \mathbb{E}\gamma_z.$$

Finally, we set $\eta = \Im(z)$ and

$$\widetilde{\gamma}_z = \bar{\gamma}_z(1) = \Gamma(1 - \frac{\alpha}{2}) \mathbb{E}(-iR_{kk}(z))^{\frac{\alpha}{2}}.$$

We will use that the resolvent R and $R^{(1)}$ are close. For example, we observe that

$$\left| M_z - \frac{\operatorname{Tr} \Im(R(z))}{n^2 \eta} \right| \leq \frac{3}{n^2 \eta^2} \quad \text{and} \quad \left| \bar{M}_z - \frac{\mathbb{E}\Im R_{11}(z)}{n\eta} \right| \leq \frac{3}{n^2 \eta^2}, \tag{52}$$

(we apply Lemma 5.5 to the matrix B' given by $B'_{ij} = B_{ij}\mathbf{1}(i, j \geq 2)$. Its resolvent R'_{ij} coincides with $R^{(1)}_{ij}$ for all $i, j \geq 2$ and $R'_{11} = -1/z$). We will also need a technical lemma.

Lemma 5.10. *If $h \in \mathcal{K}_1$ and $0 < \beta < 1$, $\Re(h^\beta) \geq (\Re h)^\beta$.*

Proof. We may assume without loss of generality that $|h| = 1$ and $\Im(h) \geq 0$. Then the lemma is equivalent to the inequality, for any $x \in [0, \pi/2]$,

$$\cos(\beta x) \geq (\cos x)^\beta.$$

which can easily be checked by showing that $f(x) = x^{-1} \log\cos(x)$ is decreasing (for instance by showing that $f'(0) < 0$ and $f''(x) \leq 0$). □

As explained in subsection 3.2, the approximate fixed point for γ will come from Schur's complement formula (9). We recall that

$$R_{11}(z) \;=\; -(ih + iQ_z + T_z)^{-1}, \tag{53}$$

where we have set $h = -iz \in \mathcal{K}_1$, $H_k = -iR^{(1)}_{kk}(z)$ and

$$\begin{aligned} Q_z &= n^{-\frac{2}{\alpha}} \sum_{2\leq k\leq n} X_{1k}^2 H_k \\ T_z &= n^{-\frac{1}{\alpha}} X_{11} + n^{-\frac{2}{\alpha}} \sum_{2\leq k\neq \ell\leq n} X_{1k} X_{1\ell} R^{(1)}_{k\ell}(z). \end{aligned}$$

We introduce the function in $\mathcal{H}_{\alpha/2}$,

$$I_z(u) \;=\; \Gamma\left(1 - \frac{\alpha}{2}\right) \mathbb{E}\left((h + Q_z)^{-1}.u\right)^{\frac{\alpha}{2}}$$

We will often drop the explicit dependence in z. Finally, $\mathcal{F}$ is the σ-algebra generated by the variables (X_{ij}), $i, j \geq 2$. Note that Q and T are $\mathcal{F}$-measurable and (X_{1k}), $k \geq 1$, is independent of $\mathcal{F}$.

The proof of Proposition 3.2 is divided in four steps.

Step 1 : from I_z to $G_z(\bar\gamma)$. From Corollary 5.8, we have

$$I_z = \mathbb{E}G_z(Z). \tag{54}$$

with Z given by

$$Z(u) = \Gamma\Big(1 - \frac{\alpha}{2}\Big) \times \frac{1}{n} \sum_{k=2}^{n} (H_k.u)^{\frac{\alpha}{2}} \frac{|g_k|^\alpha}{\mathbb{E}|g_k|^\alpha},$$

where $(g_1, \cdots, g_n)$ is a standard Gaussian vector $N(0, I)$ independent of $\mathcal{F}$. Indeed, with our choice $\sigma^\alpha = \pi/(2\sin(\pi\alpha/2)\Gamma(\alpha))$ in (51), we have

$$2^{\alpha/2}\sigma^\alpha = 2^{\alpha/2-1}\Gamma(\alpha/2)\Gamma(1-\alpha/2)/\Gamma(\alpha) = \Gamma(1-\alpha/2)/\mathbb{E}|g_1|^\alpha,$$

where we used the classical identities $\mathbb{E}|g_1|^\alpha = 2^{\alpha/2}\Gamma(1/2+\alpha/2)/\sqrt{\pi}$ and, for $0 < \beta < 1$, $\Gamma(\beta)\Gamma(1-\beta) = \pi/\sin(\pi\beta)$, $\Gamma(\beta)\Gamma(1/2+\beta) = \sqrt{\pi}2^{1-2\beta}\Gamma(2\beta)$. We set

$$\bar Z(u) = \mathbb{E}_{\mathcal{F}} Z(u) = \Gamma\Big(1 - \frac{\alpha}{2}\Big) \times \frac{1}{n} \sum_{k=2}^{n} (H_k.u)^{\frac{\alpha}{2}}.$$

From Lemma 5.4, with overwhelming probability, we get

$$\|Z - \bar{Z}\|_{1-\alpha/2+\delta} \leq (\log n)^c \eta^{-\alpha/2} n^{-\delta/2}.$$

Similarly, from Lemma 5.3, with overwhelming probability, we have

$$\|\gamma - \bar{\gamma}\|_{1-\alpha/2+\delta} \leq (\log n)^c \eta^{-\alpha/2} n^{-\delta/2}.$$

We consider the matrix B' given by $B'_{ij} = B_{ij}\mathbf{1}(i,j \geq 2)$. Its resolvent R'_{ij} coincides with $R^{(1)}_{ij}$ for all $i,j \geq 2$ and $R'_{11} = -1/z$. Using Lemma 5.2, as $\bar{Z}$ equals γ up to replacing R by $R^{(1)}$,

$$\|\bar{Z} - \gamma\|_{1-\alpha/2+\delta} \leq c\eta^{-\alpha/2} n^{-1} \left(1 + \sum_{k=1}^{n} |R_{kk}(z) - R'_{kk}(z)|^{\alpha/2} + \eta^{\delta} \sum_{k=1}^{n} |R_{kk} - R'_{kk}|^{\delta} \right).$$

We apply Lemma 5.5 to the matrix B' and use the triangle inequality, it gives

$$\|\bar{Z} - \gamma\|_{1-\alpha/2+\delta} \leq c\eta^{-\alpha/2} n^{-1} + c\eta^{-\alpha} n^{-\alpha/2} + c\eta^{-\alpha/2} n^{-\delta} \leq c\eta^{-\alpha/2} n^{-\delta}.$$

So finally, with overwhelming probability, we deduce

$$\|Z - \bar{\gamma}\|_{1-\alpha/2+\delta} \leq (\log n)^c \eta^{-\alpha/2} n^{-\delta/2}. \tag{55}$$

Observe that the right hand side is $o(1)$ for our range of η (we can take δ as close from $\alpha/2$ as wished). The assumption $\mathbb{E}|R_{11}(z)| \leq \varepsilon^{-1}$ and Lemma 5.2 imply that for some $c > 0$, $\|\bar{\gamma}\|_{1-\alpha/2+\delta} \leq c$. On the other end, Lemma 5.2 also implies the rough bound $\|Z\|_{1-\alpha/2+\delta} \leq L\eta^{-\alpha/2}$ with $L = (c/n)\sum_{k=2}^{n} |g_k|^{\alpha}$. For any $p \geq 0$, $\mathbb{E}L^p$ being bounded, we deduce from (55) that

$$\left(\mathbb{E}\|Z - \bar{\gamma}\|^p_{1-\alpha/2+\delta} \right)^{1/p} \leq (\log n)^c \eta^{-\alpha/2} n^{-\delta/2} \quad \text{and} \quad \mathbb{E}\|Z\|^p_{1-\alpha/2+\delta} \leq c. \tag{56}$$

We now claim for that for some $\varepsilon' > 0$, with overwhelming probability, for all $u \in S_1^+$

$$\Re(Z.u) > \varepsilon' \quad \text{and} \quad \Re(\bar{\gamma}.u) > \varepsilon' \tag{57}$$

(that is $Z, \bar{\gamma} \in \mathcal{H}^{\varepsilon'}_{\alpha/2,\delta}$). First, since $\|Z - \bar{\gamma}\|_{\infty} \leq \|Z - \bar{\gamma}\|_{1-\alpha/2+\delta} = o(1)$, it suffices to prove that $\bar{\gamma} \in \mathcal{H}^{\varepsilon'}_{\alpha/2,\delta}$ up to modifying the value of ε'. We should thus check that (for some new $\varepsilon' > 0$), for all $u \in S_1^+$

$$\Re\mathbb{E}(-iR_{11}.u)^{\alpha/2} > \varepsilon'.$$

The assumption $\mathbb{E}(\Re(-iR_{11}))^{\alpha/2} \geq \varepsilon$ implies that for any $u \in S_1^+$, $\mathbb{E}(\Re(-iR_{11}.u))^{\alpha/2} \geq \varepsilon$ (since $\Re(h.u) \geq \Re(h)$). Using Lemma 5.10, we deduce that $\Re\mathbb{E}(-iR_{11}.u)^{\alpha/2} \geq \varepsilon$. It proves (57).

We now want to apply (54). Let V be the event of overwhelming probability such that (57) holds. On V, we apply Lemma 4.3, while on V^c, we use Lemma 4.1(16). It gives

$$\begin{aligned}
\|I_z - G_z(\bar{\gamma})\|_{1-\alpha/2+\delta} &\leq \mathbb{E}\|G_z(Z) - G_z(\bar{\gamma})\|_{1-\alpha/2+\delta} \\
&\leq c\mathbb{E}(1 + \|\bar{\gamma}\|_{1-\alpha/2+\delta} + \|Z\|_{1-\alpha/2+\delta})\|Z - \bar{\gamma}\|_{1-\alpha/2+\delta} \\
&\qquad + c\eta^{-\alpha}\mathbb{E}\mathbf{1}_{V^c}\big(\|Z\|_{1-\alpha/2+\delta} + \|\bar{\gamma}\|_{1-\alpha/2+\delta}\big) \\
&\leq (\log n)^c \eta^{-\alpha/2} n^{-\delta/2}\,,
\end{aligned} \tag{58}$$

where the last inequality follows from (56) and Cauchy-Schwartz inequality.

Step 2 : bounds on Q and T. Recall that $Z(e^{i\pi/4}) = (c/n)\sum_{k=2}^n \Re(H_k)^{\frac{\alpha}{2}}|g_k|^\alpha$. From Lemma 5.7, given $\mathcal{F}$, with $Q = Q_{e^{i\pi/4}}$, we have

$$\Re(Q) \stackrel{d}{=} cS[Z(e^{i\pi/4})]^{2/\alpha}, \tag{59}$$

where S is a non-negative $\alpha/2$-stable variable independent of Z. By Lemma 5.9, for $a > 0$ and $b > 0$ small enough,

$$\mathbb{P}(S \le (\log n)^{-a}) = \mathbb{P}\left(e^{bS^{-\frac{\alpha}{2-\alpha}}} \ge e^{b(\log n)^{\frac{a\alpha}{2-\alpha}}}\right) \le ce^{-b(\log n)^{\frac{a\alpha}{2-\alpha}}}.$$

Hence, if $a\alpha/(2-\alpha) > 1$, the event $\{S \le (\log n)^{-a}\}$ holds with overwhelming probability. Putting together (55) and (59), we deduce that for some $c_0 > 0$, with overwhelming probability,

$$\Re(Q) \ge 2(\log n)^{-c_0}. \tag{60}$$

Similarly, from Lemma 5.7, given $\mathcal{F}$,

$$|Q| \le cn^{-\frac{2}{\alpha}} \sum_{k=2}^n |H_k| X_{1k}^2 \stackrel{d}{=} cSY^{2/\alpha}, \tag{61}$$

where, given $\mathcal{F}$, S is as above and independent of Y given by

$$Y = \frac{1}{n}\sum_{k=2}^n |H_k|^{\frac{\alpha}{2}} \frac{|g_k|^\alpha}{\mathbb{E}|g_k|^\alpha},$$

with $(g_1, \cdots, g_n)$ iid standard normal variables. We set

$$\bar{Y} = \mathbb{E}_{\mathcal{F}} Y = \frac{1}{n}\sum_{k=2}^n |H_k|^{\frac{\alpha}{2}}.$$

From (50), with overwhelming probability,

$$|Y - \bar{Y}| \le (\log n)^c n^{-1/2}\eta^{-\alpha/4} = o(1).$$

By Lemma 5.5,

$$|\bar{Y} - \rho| \le cn^{-\alpha/2}\eta^{-\alpha/2} = o(1).$$

where

$$\rho = \frac{1}{n}\sum_{k=1}^n |R_{kk}|^{\alpha/2}.$$

Let $\bar{\rho} = \mathbb{E}\rho = \mathbb{E}|R_{11}|^{\alpha/2}$. The proof of (48) gives for any $t \ge 0$,

$$\mathbb{P}(|\rho - \bar{\rho}| \ge t) \le 2\exp(-cn\eta^2 t^{4/\alpha}).$$

Hence, with overwhelming probability,

$$|\rho - \bar{\rho}| \leq (\log n)^c n^{-\alpha/4} \eta^{-\alpha/2} = o(1).$$

By assumption $\bar{\rho} \leq \varepsilon^{-1}$. It follows that with overwhelming probability,

$$|Y| \leq 2\varepsilon^{-1}.$$

We denote by F the event

$$\{|Y| \leq c_0 \text{ and } \Re(Q) \geq 2(\log n)^{-c_0}\}.$$

From what precedes, if c_0 is large enough, the event F holds with overwhelming probability. We also consider the event :

$$E = \left\{|T| \leq (\log n)^{-c_0}\right\}$$

Since $M_z \leq 1/(n\eta^2) \leq 1$, by Lemma 5.6, if $0 \leq t \leq 1$

$$\mathbb{P}_{\mathcal{F}}(|T| \geq t) \leq cM^{\alpha/2} t^{-\alpha} (\log n), \tag{62}$$

and, consequently, for $\beta > \alpha$ and $0 \leq t \leq 1$, for some constant $c = c(\alpha)$,

$$\mathbb{E}_{\mathcal{F}} |T|^\beta \mathbf{1}(|T| \leq t) \leq \frac{c}{\beta - \alpha} M^{\alpha/2} t^{\beta - \alpha} (\log n). \tag{63}$$

Thus by Chebychev's inequality for some $c > 0$,

$$\mathbb{P}_{\mathcal{F}}(E^c) \leq M^{\alpha/2} (\log n)^c. \tag{64}$$

Step 3 : from Schur's formula (53) **to I_z.** In (61), up to enlarge our probability space, we may assume that the variables (Y, S, Q, T) are defined on the same probabilisty space. From (53), Jensen's inequality and Lemma 5.2, for any $s > 0$,

$$\begin{aligned} \|\bar{\gamma} - I_z\|_{1-\alpha/2+\delta} &\leq c\mathbb{E}\|((h+Q-iT)^{-1}.u)^{\alpha/2} - ((h+Q)^{-1}.u)^{\alpha/2}\|_{1-\alpha/2+\delta} \\ &\leq (\log n)^c \mathbb{E} S^{\alpha-1} \mathbf{1}_{S \leq s} |T| \mathbf{1}_E + c\eta^{-\alpha/2} (\mathbb{P}(S \geq s) + \mathbb{P}(E^c \cup F^c)), \end{aligned} \tag{65}$$

where we have used that on $E \cap F$, $|h + Q - iT|$ and $|h + Q|$ are $O(S)$ and $\Re(h + Q - iT) \geq \Re(Q) - |T| \geq (\log n)^{-c_0}$.

We first consider the case $0 < \alpha \leq 1$ in (65) where we take $s = \infty$. Let $p = \log n$ and $1/q + 1/p = 1$. We have $q = 1 + 1/\log n + o(1/\log n)$. From Hölder inequality and Lemma 5.9

$$\mathbb{E} S^{\alpha-1} |T| \mathbf{1}_E \leq \left(\mathbb{E} S^{p(\alpha-1)}\right)^{1/p} (\mathbb{E}|T|^q \mathbf{1}_E)^{1/q} \leq (\log n)^c (\mathbb{E}|T|^q \mathbf{1}_E)^{1/q}.$$

Recall that $E = \{|T| \leq (\log n)^{-c_0}\}$. Also Jensen's inequality implies that $\mathbb{E} M^{\alpha/2} \leq \bar{M}^{\alpha/2}$. We deduce from (63) that

$$(\mathbb{E}|T|^q \mathbf{1}_E)^{1/q} \leq \left((\log n)^c \bar{M}^{\alpha/2}\right)^{1/q} \leq (\log n)^c \bar{M}^{\alpha/2},$$

where the last inequality comes from $\bar{M} \geq cn^{-1}$ which follows from (52) and the assumptions of Proposition 3.2. Hence, from (64)-(65),

$$\|\bar{\gamma} - I_z\|_{1-\alpha/2+\delta} \leq (\log n)^c \bar{M}^{\alpha/2} + (\log n)^c \eta^{-\alpha/2} \bar{M}^{\alpha/2} \leq (\log n)^{c'} \eta^{-\alpha/2} \bar{M}^{\alpha/2}.$$

Similarly, for $1 < \alpha < 2$, we choose $s = M^{-1}$ in (65). Let $p = \alpha/(2(\alpha-1)) > 1$ and $q = \alpha/(2-\alpha) > \alpha$. Since $1/q + 1/p = 1$, from Hölder inequality, we find for n large enough,

$$\mathbb{E} S^{\alpha-1} \mathbf{1}_{S \leq s} |T| \mathbf{1}_E \leq \left(\mathbb{E} S^{\alpha/2} \mathbf{1}_{S \leq s} \right)^{1/p} (\mathbb{E}|T|^q \mathbf{1}_E)^{1/q} \leq (\log n)(\mathbb{E}|T|^q \mathbf{1}_E)^{1/q},$$

where we have used that $\mathbb{E} S^{\alpha/2} \mathbf{1}(S \leq s) \leq c \log(s \vee 2)$ and $\bar{M} \geq cn^{-1}$. Since $\alpha/(2q) = 1 - \alpha/2$, we get from (63)-(64),

$$\|\bar{\gamma} - I_z\|_{1-\alpha/2+\delta} \leq (\log n)^c \bar{M}^{1-\alpha/2} + (\log n)^c \eta^{-\alpha/2} \bar{M}^{\alpha/2}.$$

We deduce from (58) that the first statement of Proposition 3.2 holds.

Step 4 : from Schur's formula (53) **to** $\mathbb{E}|R_{11}|^p$ **and** $\mathbb{E}R_{11}^p$**.** We only treat the case $\mathbb{E}|R_{11}|^p$. The case of $\mathbb{E}R_{11}^p$ is identical. Let $J_z = \mathbb{E}|h + Q|^{-p}$. From Corollary 5.8

$$J_z = \mathbb{E} r_{p,z}(Z),$$

where Z is as above defined in (54). We drop the parameters (p, z). From Jensen's inequality,

$$|J - r(\bar{\gamma})| \leq \mathbb{E}|r(Z) - r(\bar{\gamma})|.$$

We then argue as in (58) : when the event V holds, we apply Lemma 4.3 and when V fails to hold, we use instead that $|r(g)| \leq c\eta^{-p}$ for $g \in \bar{\mathcal{H}}_\alpha^0$ (Lemma 4.1). It gives in conjunction with (56)

$$|J - r(\bar{\gamma})| \leq c\mathbb{E}\|Z - r(\bar{\gamma})\|_\infty + c\eta^{-p}\mathbb{P}(E^c) \leq (\log n)^c \eta^{-\alpha/2} n^{-\delta/2}. \tag{66}$$

We may now repeat the third step. Recall the variable S defined in (61) and the events E, F. We observe that for any $x, y \in \mathbb{C}$,

$$\left| \frac{1}{|x|^p} - \frac{1}{|y|^p} \right| \leq \frac{|x-y|}{|x|^p |y|^p} \sum_{k=0}^{p-1} |x|^k |y|^{p-k-1} \leq p \frac{|x-y|}{s^{p+1}},$$

where $s = \Re(x) \wedge \Re(y)$. Hence, from (53), Jensen's inequality,

$$\begin{aligned} |\mathbb{E}|R_{11}|^p - J| &\leq \mathbb{E}\big| |h + Q - iT|^{-p} - |h+Q|^{-p} \big| \\ &\leq (\log n)^c \mathbb{E}|T| \mathbf{1}_E + 2\eta^{-p}\mathbb{P}(E^c \cup F^c), \end{aligned}$$

where we have used that on $E \cap F$, $\Re(h + Q - iT) \geq \Re(Q) - |T| \geq (\log n)^{-c_0}$. Using (63)-(64), we arrive at

$$|\mathbb{E}|R_{11}|^p - J| \leq (\log n)^c \eta^{-p} \bar{M}^{\alpha/2}.$$

Together with (66), it implies the second statement of the proposition. □

5.4 Local law in neighborhood of the origin

We denote by

$$\widetilde{\gamma}_z^\star = \gamma_z^\star(1) = \Gamma(1-\alpha/2)\mathbb{E}(-iR_\star)^{\alpha/2},$$

where $R_\star$ was defined by (7). Lemma 4.4 asserts that $\widetilde{\gamma}_z^\star = \Gamma(1-\alpha/2)s_{\alpha/2,z}(\widetilde{\gamma}_z^\star)$. We prove the following theorem :

Theorem 5.11. *For $\alpha \in (0,2)\backslash\mathcal{A}$, there exist $\tau > 0$, $c > 0$ and $0 < \delta < \alpha/2$ such that if $|z| \le \tau$ and $\Im(z) \ge (\log n)^c n^{-\alpha/(2+\alpha)}$, then*

$$\|\bar{\gamma}_z - \gamma_z^\star\|_{1-\alpha/2+\delta} = o(1) \quad \text{and} \quad |\mathbb{E}|R_{11}| - r_{1,z}(\gamma_z^\star)| = o(1).$$

Finally, if $p \ge \alpha/2$ is such that $\Im(z) \ge (\log n)^c n^{-\alpha/(2p+\alpha)}$,

$$\begin{aligned} |\mathbb{E}|R_{11}(z)|^p - r_{p,z}(\gamma_z^\star)| &= o(1) \\ |\mathbb{E}(-iR_{11}(z))^p - s_{p,z}(\widetilde{\gamma}_z^\star)| &= o(1). \end{aligned}$$

We first check that the assumptions of Proposition 3.2 are met for any fixed $z \in \mathbb{C}_+$.

Lemma 5.12. *If $z \in \mathbb{C}_+$, there exists $c = c(z) > 0$ such that $|R_{11}(z)| \le c^{-1}$ and $\mathbb{E}\Im(R_{11}(z))^{\alpha/2} \ge c$.*

Proof. Recall that $|R_{11}| \le 1/\Im(z)$. For the lower bound, let $X_1 = (X_{1k})_{2\le k\le n}$. Since $\Im(R^{(1)}) \ge 0$, we have from (53)

$$\Im(R_{11}) = \frac{\Im(z) + \langle X_1, \Im R^{(1)} X_1\rangle}{|z + iQ + T|^2} \ge \frac{1}{3}\frac{\Im(z)}{|z|^2 + |Q|^2 + |T|^2}.$$

By (62), since $M \le n^{-1}\Im(z)^{-2} \le c/n$, we have $\mathbb{P}(|T| \ge 1) \le cn^{-\alpha/2}(\log n) \le 1/4$ for n large enough. Finally, from (61), $\mathbb{E}|Q|^\delta < c$ if $\delta < \alpha/2$. We deduce that $\mathbb{P}(|Q| \ge t) \le 1/4$ for t large enough. Then, with probability at least $1/2$, $\Im(R_{11}) \ge \Im(z)/(3|z|^2 + 3t + 3)$. □

Proof of Theorem 5.11. Set $r_z = r_{1,z}$ and $\eta_{\min} = (\log n)^{c_0} n^{-\alpha/(2+\alpha)}$ for some $c_0 > 0$ to be chosen later on. Since $\alpha/(2+\alpha) < 1/2$, we may choose $0 < \delta < \alpha/2$ such that for all $\eta \ge \eta_{\min}$ and fixed $c > 0$,

$$(\log n)^c \eta^{-\alpha/2} n^{-\delta/2} = o(1). \tag{67}$$

We fix such parameter δ. We use a continuity argument. Assume that for some $z_1 = E + i\eta_1 \in \mathbb{C}_+$, with $|z_1| \le \tau$ and $\eta_1 \ge \eta_{\min}$, we have that

$$\|\bar{\gamma}_{z_1} - \gamma_{z_1}^\star\|_{1-\alpha/2+\delta} \le \varepsilon \quad \text{and} \quad |\mathbb{E}|R_{11}(z_1)| - r_{z_1}(\gamma_{z_1}^\star)| \le \varepsilon \tag{68}$$

where $\varepsilon \le \tau/2$ is an arbitrarily small constant and τ is as in Proposition 3.4. Let $z = E + i\eta$ with $\eta_1 - n^{-3/\delta} \le \eta \le \eta_1$. Using $\eta \ge 1/\sqrt{n}$, we have $|R_{kk}(z) - R_{kk}(z_1)| \le |\eta - \eta_1|/\eta^2 \le 2n^{-3/\delta+1} \le 2n^{-2/\delta}$. Hence, by Lemma 5.2, for $\beta \in (0,1)$, as $|R_{kk}(z)| \le \eta^{-1} \le \sqrt{n}$,

$$\|\bar{\gamma}_{z_1} - \bar{\gamma}_z\|_{1-\beta+\delta} \le cn^{\beta/2} n^{-2\beta/\delta} + cn^{\beta/2-\delta/2} n^{-2} = o(1).$$

The same holds for $\|\gamma^\star_{z_1} - \gamma^\star_z\|_{1-\beta+\delta}$. Taking $\beta = \alpha/2$, we get from the triangle inequality

$$\|\bar\gamma_z - \gamma^\star_z\|_{1-\alpha/2+\delta} \le 2\varepsilon \le \tau. \tag{69}$$

Similarly, using the continuity of $z \mapsto r_z(f)$ for $f \in \mathcal{H}^0_{\alpha/2}$, we get

$$|\mathbb{E}|R_{11}(z)| - r_z(\gamma^\star_z)| \le 2\varepsilon.$$

By Proposition 3.3, there exists a constant $c_1 > 1$ such that for all $z \in \mathbb{C}$ with $|z| \le \tau$,

$$|r_z(\gamma^\star_z)| \le c_1 \quad \text{and} \quad \gamma^\star_z(e^{i\pi/4}) \ge c_1^{-1}.$$

Hence, if ε is small enough, we deduce that

$$\mathbb{E}|R_{11}(z)| \le 2c_1 \quad \text{and} \quad \mathbb{E}(\Im R_{11}(z))^{\alpha/2} \ge \frac{1}{2c_1\Gamma(1-\alpha/2)}.$$

We may thus apply Proposition 3.2. The first inequality of Proposition 3.2, in conjunction with Corollary 3.4 and (69), implies that

$$\|\bar\gamma_z - \gamma^\star_z\|_{1-\alpha/2+\delta} \le (\log n)^c\Big(\eta^{-\alpha/2}\bar M^{\alpha/2} + \eta^{-\alpha/2}n^{-\delta/2} + \bar M^{1-\alpha/2}\mathbf{1}_{\alpha>1}\Big). \tag{70}$$

The second inequality of Proposition 3.2 gives thanks to Lemma 4.3 and the above inequality gives

$$|\mathbb{E}|R_{11}| - r_z(\gamma^\star_z)| \le (\log n)^c\Big(\eta^{-1}\bar M^{\alpha/2} + \eta^{-\alpha/2}n^{-\delta/2}\Big). \tag{71}$$

From (52), we have

$$\bar M = \frac{\mathbb{E}\Im(R^{(1)}_{22})}{n\eta} \le \frac{\mathbb{E}|R_{11}|}{n\eta} + \frac{3}{n^2\eta^2} \le \frac{2c_1+3}{n\eta}.$$

We deduce from (70)-(71) and (67) that

$$\|\bar\gamma_z - \gamma^\star_z\|_{1-\alpha/2+\delta} \le (\log n)^c\eta^{-\alpha}n^{-\alpha/2} + o(1) \quad \text{and} \quad |\mathbb{E}|R_{11}| - r_z(\gamma^\star_z)| \le (\log n)^c\eta^{-1-\alpha/2}n^{-\alpha/2} + o(1).$$

It is easy to check that if c_0 in choice of $\eta_{\min}$ is small enough, the above terms are $o(1)$. It follows that z satisfies (68) with the same ε.

We may thus use of continuity argument, for any fixed z with $|z| \le \tau$, (68) holds by Lemma 5.12, Proposition 3.2 and Lemma 3.4. By iteration, it proves that (68) holds for all $z \in \mathbb{C}_+$ with $|z| \le \tau$ and $\Im(z) \ge \eta_{\min}$. It proves the first statement of Theorem 5.11. The second statement is then a direct consequence of a new application of Proposition 3.2. □

Theorem 5.11 implies a control of the number of eigenvalues in an interval. Recall that $|\Lambda_I|$ is the number of eigenvalues of A in the interval I. We denote by $\mu_\star$ the probability measure on $\mathbb{R}$ whose Cauchy-Stieltjes transform is $is_{1,z}(\widetilde\gamma^\star_z) = \mathbb{E}R_\star$ (defined by (7)) : for any $z \in \mathbb{C}_+$,

$$\int \frac{d\mu_\star(\lambda)}{\lambda - z} = is_{1,z}(\widetilde\gamma^\star_z).$$

It is notably shown in [12, 11, 15] that $\mu_\star$ has a bounded positive continuous density which is explicit at 0.

Corollary 5.13. *For $\alpha \in (0,2)\backslash\mathcal{A}$, there exist $\tau > 0$ and $c > 0$ such that if $I \subset [-\tau,\tau]$ is an interval of length at least $(\log n)^c n^{-\alpha/(2+\alpha)}$, then with overwhelming probability,*

$$\left|\frac{|\Lambda_I|}{n} - \mu_\star(I)\right| = o(1)|I|.$$

Proof. Define the probability measure $\mu(I) = |\Lambda_I|/n$. By construction, the Cauchy-Stieltjes transform of μ is $\frac{1}{n}\mathrm{Tr}\ R$. By [16, Lemma 3.7], up to modifying the value of c and τ, it is sufficient to prove that with overwhelming probability,

$$\frac{1}{n}\mathrm{Tr}\ (-iR_z) = s_{1,z} + o(1)$$

for all z such that $|z| \le \tau$ and $\Im(z) \ge (\log n)^c n^{-\alpha/(2+\alpha)}$. By Lemma 5.1, with overwhelming probability

$$\left|\frac{1}{n}\mathrm{Tr}\ (-iR_z) - \mathbb{E}\frac{1}{n}\mathrm{Tr}\ (-iR_z)\right| \le (\log n)n^{-1/2}\eta^{-1} = o(1).$$

Since $\mathbb{E}\frac{1}{n}\mathrm{Tr}\ (-iR) = \mathbb{E}(-iR_{11})$, the conclusion follows from Theorem 5.11. □

Corollary 5.14. *For $\alpha \in (0,2)\backslash\mathcal{A}$, there exist $\tau > 0$ and $c > 0$ such that if $|z| \le \tau$, $\Im(z) \ge (\log n)^c(n^{-\alpha/(4+\alpha)} \vee n^{-1/4})$, then with overwhelming probability,*

$$\frac{1}{n}\sum_{k=1}^{n} |R_{kk}(z)|^2 \le c.$$

Proof. Set $z = E + i\eta$. We apply Lemma 5.1 with $f(x) = |x|^2 \wedge \eta^{-2}$. It is Lispchitz with constant $2\eta^{-1}$. We deduce that with overwhelming probability,

$$\left|\frac{1}{n}\sum_{k=1}^{n} |R_{kk}(z)|^2 - \mathbb{E}|R_{11}(z)|^2\right| \le (\log n)n^{-1/2}\eta^{-2} = o(1).$$

It remains to apply Theorem 5.11. □

5.5 Proof of Theorem 1.2

Since $\mu_\star$ has a positive continuous density, there exists $c > 0$ such that

$$\mu_\star(I) \ge c|I|$$

for any interval $I \subset [-\tau,\tau]$. Hence, Corollary 5.13 implies that $|\Lambda_I| \ge cn|I|/2$ with overwhelming probability. Thus Lemma 3.1 implies that Q_I is bounded by $\frac{4}{c^2 n}\sum_{k=1}^{n} |R_{kk}(z)|^2$ Thus, Theorem 1.2 is an immediate consequence of Corollary 5.14. □

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