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Abstract

A rectangular matrix is called *totally positive,* (according to F.R.Gantmacher and M.G.Krein) if all its minors are positive. A point of a real Grassmanian manifold $G_{l,m}$ of l-dimensional subspaces in $\mathbb{R}^m$ is called *strictly totally positive* (according to A.E.Postnikov) if one can normalize its Plücker coordinates to make all of them positive. Clearly if a $k \times m$-matrix, $k < m$, is totally positive, then each collection of its $l \leq k$ raws generates an l-subspace represented by a strictly totally positive point of the Grassmanian manifold $G_{l,m}$. The totally positive matrices and the strictly totally positive Grassmanians, that is, the subsets of strictly totally positive points in Grassmanian manifolds arise in many domains of mathematics, mechanics and physics. F.R.Gantmacher and M.G.Krein considered totally positive matrices in the context of classical mechanics. S.Karlin considered them in a wide context of analysis, differential equations and probability theory. As it was shown in a joint paper by M.Boiti, F.Pemperini and A.Pogrebkov, each matrix of appropriate dimension with positive minors of higher dimension generates a multisoliton solution of the Kadomtsev-Petviashvili (KP) partial differential equation. There exist several approaches of construction of totally positive matrices due to F.R. Gantmacher, M.G.Krein, S.Karlin, A.E.Postnikov and ourselves. In our previous paper we have proved that certain determinants formed by modified Bessel functions of the first kind are positive on the positive semi-axis. This yields a one-dimensional family of totally positive points in all the Grassmanian manifolds.

In the present paper we provide a construction of multidimensional families of totally positive points in all the Grassmanian manifolds, again using modified Bessel functions of the first kind but different from the above-mentioned construction. These families represent images of explicit injective mappings of the convex open subset $\{x = (x_1, \ldots, x_l) \in \mathbb{R}^l_+ \mid x_1 < \cdots < x_l\} \subset \mathbb{R}^l$ to the Grassmanian manifolds $G_{l,m}$, $l < m$.

S.Karlin presented in his book examples of totally non-negative matrices given in terms of just one modified Bessel function I_α. We present a new result that gives totally positive matrices whose columns are defined by modified Bessel functions with different non-negative integer indices.

1 Introduction

1.1 Brief survey on totally positive matrices. Main result

The following notion was introduced in the classical books [14, 15] in the context of the classical mechanics.

Definition 1.1 [1, 14, 23], [15, p.289 of the russian edition] A rectangular $l \times m$-matrix is called *totally positive* (nonnegative), if all its minors of all the dimensions are positive (nonnegative).

Example 1.2 It is known that every generalized Vandermonde matrix

$$(f(x_i, y_j))_{i=1,\ldots,m; \ j=1,\ldots,n}, \ f(x,y) = x^y,$$

$$0 < x_1 < \cdots < x_m, \ 0 \le y_1 < y_2 < \cdots < y_n$$

is totally positive, see [11, chapter XIII, section 8].

The study of $n \times n$ matrices with positive elements goes back to Perron [22] who had shown that for such a matrix the eigenvalue that is largest in the module is simple, real and positive, and the corresponding eigenvector can be normalized to have all the components positive (1907). Later in 1908 his result was generalized by G.Frobenius [11, chapter 13, section 2] to irreducible matrices with non-negative coefficients. For these matrices he had proved that its complex eigenvalues of maximal module are roots of a polynomial $P(\lambda) = \lambda^h - r^h$, all of them are simple and at least one of them is real and positive.

In 1935–1937 F.R.Gantmacher and M.G.Krein [12, 13] observed that if the matrix under question satisfies a stronger condition, that is, total positivity, then *all* its eigenvalues are simple, real and positive. Earlier in 1930 I.Schoenberg [25] studied totally positive matrices in connection with a problem of Pólya.

Many results on characterization and properties of strictly totally positive matrices and their relations to other domains of mathematics (e.g., combinatorics, dynamical systems, geometry and topology, probability theory and Fourier analysis), mechanics and physics are given in [1, 11, 12, 13, 14, 15, 23, 8, 9, 16, 18, 19, 20, 21] and in [24, 2, 3, 10, 17] (see also references in all these papers and books). F.R.Gantmacher and M.G.Krein [14, 15] considered

totally positive matrices in the context of applications to mechanical problems. S.Karlin [18] considered them in a wide context of analysis, differential equations and probability theory. In 2008 G.Lusztig suggested an analogue of the theory of total positivity for the Lie group context [19].

As it was shown by M.Boiti, F.Pempinelli, A.Pogrebkov [4, section II], each matrix of appropriate dimension with positive minors of higher dimension generates a multisoliton solution of the Kadomtsev-Petviashvili (KP) differential equation. There exist several approaches of construction of totally positive matrices, see [14, 24, 5], [15, p.290 of Russian edition]. In the previous paper [5] we have constructed a class of explicit one-dimensional families of totally positive matrices given by a finite collection of double-sided infinite vector functions, whose components are modified Bessel functions of the first kind. Matrices of such kind arised in a paper of V.M.Buchstaber and S.I.Tertychnyi in the construction of appropriate solutions on the non-linear differential equations in a model of overdamped Josephson junction in superconductivity, see [6] and references therein. It was shown in [5] that the nature of the modified Bessel functions as coefficients of appropriate generating function implies that the infinite vector formed by appropriate minors of the above-mentioned matrices satisfy the *differential-difference heat equation* with positive constant potential.

In the present paper we provide a new construction of explicit multidimensional family of totally positive matrices formed by a finite collection of one-sided infinite vector functions. This family is parametrized by a domain in $\mathbb{R}^l$. Each raw of the matrix corresponds to a coordinate x_i in $\mathbb{R}^l$, and its elements are modified Bessel functions of this coordinate.

Recall that the modified Bessel functions $I_j(y)$ of the first kind are Laurent series coefficients for the family of analytic functions

$$g_y(z) = e^{\frac{y}{2}(z+\frac{1}{z})} = \sum_{j=-\infty}^{+\infty} I_j(y) z^j.$$

Equivalently, they are defined by the integral formulas

$$I_j(y) = \frac{1}{\pi} \int_0^\pi e^{y\cos\phi} \cos(j\phi) d\phi, \ j \in \mathbb{Z}.$$

Example 1.3 The infinite matrix $(A_{km})_{k,m\in\mathbb{Z}}$ with $A_{km} = I_{m-k}(x)$ is totally positive for every $x > 0$, see [5, theorem 1.3].

Set
$$X_l = \{x = (x_1, \ldots, x_l) \in \mathbb{R}_+^l \mid x_1 < x_2 < \cdots < x_l\};$$
$$K_m = \{k = (k_1, \ldots, k_m) \in \mathbb{Z}_{\geq 0}^m \mid k_1 < k_2 < \cdots < k_m\}.$$
For every $x \in X_l$ and $k \in K_m$ set

$$A_{k,x} = (a_{ij})_{i=1,\ldots,l; \ j=1,\ldots m}, \ a_{ij} = I_{k_j}(x_i). \tag{1.1}$$

In the special case, when $l = m$, set

$$f_k(x) = \det A_{k,x}. \tag{1.2}$$

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Theorem 1.4 *For every $m \in \mathbb{N}$, $k \in K_m$ and $x \in X_m$ one has $f_k(x) > 0$.*

Theorem 1.4 will be proved in Section 2.

Corollary 1.5 *For every $x = (x_1, \ldots, x_l) \in X_l$ the one-sided infinite matrix formed by the values $a_{ij} = I_j(x_i)$, $i = 1, \ldots, l$, $j = 0, 1, 2, \ldots$ is totally positive.*

This corollary follows immediately from the theorem.

Remark 1.6 The above square matrices $A_{k,x}$ belong to a more general class of matrices given by a function $K(x, y)$ defined on a product of two subsets in $\mathbb{R}$: the matrices $A_K = (K(x_i, y_j))_{i,j=1,\ldots,m}$, where $x_1 < \cdots < x_m$, $y_1 < \cdots < y_m$. A function K is called *totally positive (strictly totally positive) kernel,* see [18, chapter 2, definition 1.1, p.46], if all the corresponding above matrices have non-negative (respectively, positive) determinants. Various necessary and sufficient conditions on a kernel K to be (strictly) totally positive were stated and proved in S.Karlin's book [18, chapter 2]. In the case, when $K(x, y)$ is defined on a product of two intervals and is smooth enough, a sufficient condition for its strict total positivity says that appropriate matrix formed by appropriate (higher order) partial derivatives of the function K is positive everywhere [18, chapter 2, theorem 2.6, p. 55]. (The same condition written in the form of non-strict inequality is necessary for total positivity.) In [18, chapter 3, p. 109] S.Karlin presented an example of totally positive kernel coming from one modified Bessel function of the first kind. Namely, set

$$\kappa_\alpha(x; \lambda) = \begin{cases} e^{-(x+\lambda)} \left(\frac{x}{\lambda}\right)^{\frac{\alpha}{2}} I_\alpha(2\sqrt{x\lambda}) \text{ for } x \geq 0 \\ 0 \text{ for } x < 0 \end{cases},$$

$$K_\alpha(x, y) = \kappa_\alpha(x - y; \lambda).$$

It was shown in loc. cit. (just after corollary 2.1) that for every $\alpha > 1$ every $m \times m$- matrix A_{K_α} with $m < \alpha + 2$ has non-negative determinant.

We prove Theorem 1.4 by induction in m. For the proof of the induction step we consider the sequence of all the determinants $f_k(x)$ for all $k \in K_m$ as an infinite-dimensional vector function in new variables $y = (x_1, w)$, $w = (w_2, \ldots, w_m)$, $w_j = x_j - x_1$. We fix w and consider the latter vector function as a function of one variable $x_1 \geq 0$. Analogously to the arguments from [5, section 2], we show that it satisfies an ordinary differential equation given by a linear bounded vector field on the Hilbert space l_2 with coordinates f_k, $k \in K_m$ such that the positive quadrant $\{f_k \geq 0 \mid k \in K_m\}$ is invariant for its flow. We show that the initial value of the vector function for $x_1 = 0$ lies in this quadrant and is non-zero. This will imply positivity of all the functions $f_k(x_1, w)$ for all $x_1 > 0$, as in loc. cit.

It is known that the modified Bessel functions $I_\nu(x)$ of the first kind are given by the series

$$I_\nu(x) = \left(\frac{1}{2}x\right)^\nu \sum_{k=0}^\infty \frac{\left(\frac{1}{4}z^2\right)^k}{k! \Gamma(\nu + k + 1)},$$

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and the latter series extends them to all the real values of the index ν.

Thus, the modified Bessel functions of the first kind yield examples of totally positive (non-negative) matrices of two following different kinds. Karlin's example yields a totally positive kernel $K_\alpha(x,y) = \kappa_\alpha(x-y;\lambda)$ constructed from just one modified Bessel function I_α with arbitrary given real index $\alpha > 1$. Our main result gives other matrices defined by the function $K(y,s) = I_s(y)$ in $y \in \mathbb{R}_+$ and $s \in \mathbb{Z}_{\geq 0}$, which appears to be a strictly totally positive kernel.

Open Question. *Is it true that the determinants $f_k(x)$ in (1.2) with $x \in X_m$ are all positive for every $m \in \mathbb{N}$ and every $k = (k_1, \ldots, k_m)$ with* (**may be non-integer**) *$k_j \in \mathbb{R}_+$, $k_1 < \cdots < k_m$? In other terms, is it true that the kernel $K(y,s) = I_s(y)$ is strictly totally positive (in the sense of the above remark) as a function in $(y,s) \in \mathbb{R}_+^2$?*

1.2 A brief survey on total positivity in Grassmanian manifolds and Lie groups

A point L of Grassmanian manifold $G_{l,m}$ of l-subspaces in $\mathbb{R}^m$, $m > l$ is represented by an $l \times m$-matrix, whose lines form a basis of the subspace represented by the point L. Recall that the *Plücker coordinates* of the point L are the l-minors of the latter matrix. The Plücker coordinates of the point are well-defined up to multiplication by a common factor, and they are considered as homogeneous coordinates representing a point of a projective space $\mathbb{RP}^N$, $N = \binom{n}{k} - 1$. The Plücker coordinates induce the Plücker embedding of the Grassmanian manifold to $\mathbb{RP}^N$.

Definition 1.7 A point $L \in G_{l,m}$ is called *strictly totally positive,* if one can normalize its Plücker coordinates to make all of them positive. Or equivalently, if it can be represented by a matrix with all of higher minors positive.

A.E.Postnikov's paper [24] deals with the matrices $l \times m$, $m \geq l$ or rank l satisfying the condition of nonnegativity of just higher rank minors. One of its main results provides an explicit combinatorial cell decomposition of the corresponding subset in the Grassmanian $G_{l,m}$, called the *totally nonnegative Grassmanian.* The cells are coded by combinatorial types of appropriate planar networks. K.Talaska [26] obtained further development and generalization of Postnikov's result. In particular, for a given point of the totally nonnegative Grassmanian the results of [26] allow to decide what is its ambient cell and what are its affine coordinates in the cell. S.Fomin and A.Zelevinsky [10] studied a more general notion of total positivity (nonnegativity) for elements of a semisimple complex Lie group with a given double Bruhat cell decomposition. They have proved that the totally positive parts of the double Bruhat cells are bijectively parametrized by the product of the positive quadrant $\mathbb{R}_+^m$ and the positive subgroup of the maximal torus. For other results on totally positive (nonnegative) Grassmanians see [17].

Theorem 1.4 of the present paper implies the following corollary.

Corollary 1.8 *For every $l, m \in \mathbb{N}$, $l < m$, and every $k \in K_m$ the mapping $H_k : X_l \to G_{l,m}$ sending x to the l-subspace in $\mathbb{R}^m$ generated by the vectors*

$$v_k(x_i) = (I_{k_1}(x_i) \dots I_{k_m}(x_i)), \ i = 1, \dots, l$$

is well-defined and injective. Its image is contained in the open subset of l-subspaces with positive Plücker coordinates.

Proof The well-definedness and positivity of Plücker coordinates are obvious, since the l-minors of the matrix $A_{k,x}$ are positive, by Theorem 1.4. Let us prove injectivity. Fix some two distinct $x, y \in X_l$. Let us show that $H_k(x) \neq H_k(y)$. Fix a component y_i that is different from every component x_j of the vector x. Then the vectors $v_k(x_1), \dots, v_k(x_l), v_k(y_i)$ are linearly independent: every $(l+1)$-minor of the matrix formed by them is non-zero, by Theorem 1.4. Hence, $v_k(y_i)$ is not contained in the l-subspace $H_k(x)$, which is generated by the vectors $v_k(x_1), \dots, v_k(x_l)$. Thus, $H_k(y) \neq H_k(x)$. The corollary is proved. $\square$

Example 1.9 Consider the infinite matrix with elements

$$A_{ms} = I_{s-m}(x), \quad m, s \in \mathbb{Z}.$$

It is shown in [5, theorem 1.3] that this matrix is totally positive for every $x > 0$, see Example 1.3 and the reference therein: all its minors are positive. Therefore, the subspace generated by any of its l raws is l-dimensional, and it represents a strictly totally positive point of the infinite-dimensional Grassmanian manifold of l-subspaces in the infinite-dimensional vector space. Every submatrix in A_{ms} given by its l raws and a finite number $r > l$ of columns represents a strictly totally positive point of the finite-dimensional Grassmanian manifold G_{lr}.

2 Positivity. Proof of Theorem 1.4

In the proof of Theorem 1.4 we use the following classical properties of the modified Bessel functions I_j of the first kind, see [27, section 3.7].

$$I_j = I_{-j}; \tag{2.1}$$

$$I_j|_{y>0} > 0; \ I_j(0) = 0 \text{ for } j \neq 0; \ I_0(0) > 0; \tag{2.2}$$

$$I_0' = I_1; \ I_j' = \frac{1}{2}(I_{j-1} + I_{j+1}); \tag{2.3}$$

We prove Theorem 1.4 by induction in m.

Induction base. For $m = 1$ the statement of the theorem is obvious and follows from inequality 2.2.

Induction step. Let the statement of the theorem be proved for $m = m_0$. Let us prove it for $m = m_0 + 1$. To do this, consider the sequence of all the

determinants $f_k(x)$ for all $k \in K_m$ as an infinite-dimensional vector function in the new variables

$$(x_1, w), \quad w = (w_2, \ldots, w_m), \quad w_j = x_j - x_1; \quad w \in X_{m-1}.$$

The next two propositions and corollary together imply that for every fixed $k \in K_m$ the vector function $(f_k(x))_{k \in K_m}$ with fixed w and variable x_1 is a solution of a bounded linear ordinary differential equation in the Hilbert space l_2 of infinite sequences $(f_k)_{k \in K_m}$: a phase curve of a bounded linear vector field. We show that the positive quadrant $\{f_k \geq 0 \mid k \in K_m\} \subset l_2$ is invariant under the positive flow of the latter field, and the initial value $(f_k(0, w))_{k \in K_m}$ lies there. This implies that $f_k(x_1, w) \geq 0$ for all $x_1 \geq 0$, and then we easily deduce that the latter inequality is strict for $x_1 > 0$.

Let us recall how the discrete Laplacian Δ_{discr} acts on the space of functions $f = f(k)$ in $k \in \mathbb{Z}^m$. For every $j = 1, \ldots, m$ let T_j denote the corresponding shift operator:

$$(T_j f)(k) = f(k_1, \ldots, k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_l).$$

Then

$$\Delta_{discr} = \sum_{j=1}^{m} (T_j + T_j^{-1} - 2). \tag{2.4}$$

Thus, one has

$$(\Delta_{discr} f)(p) = \sum_{s=1}^{m} (f(p_1, \ldots, p_{s-1}, p_s - 1, p_{s+1}, \ldots, p_m)$$

$$+ f(p_1, \ldots, p_{s-1}, p_s + 1, p_{s+1}, \ldots, p_m)) - 2mf(p). \tag{2.5}$$

Remark 2.1 We will deal with the class of sequences $f(k)$ with the following properties:

(i) $f(k) = 0$, whenever $k_i = k_j$ for some $i \neq j$;

(ii) $f(k)$ is an even function in each component k_i.

This class includes $f(k) = f_k(x_1, w)$: statement (i) is obvious; statement (ii) follows from equality (2.1). In this case the discrete Laplacian is well-defined by the above formulas (2.4), (2.5) on the restrictions of the latter sequences $f(k)$ to $k \in K_m$, as in [5, remark 2.1]. In more detail, in formula (2.5) written for $p \in K_m$ with $p_1 = 0$ the first term in the right-hand side equals $f_{(p_1 - 1, p_2, \ldots, p_m)} = f_{(-1, p_2, \ldots, p_m)} = f_{(1, p_2, \ldots, p_m)}$, by (ii).

Proposition 2.2 (analogous to [5, proposition 2.2]). For every $m \geq 1$ and $w \in \mathbb{R}^{m-1}$ the vector function $(f(x_1, k) = f_k(x_1, w))_{k \in K_m}$ satisfies the following linear differential-difference equation:

$$\frac{\partial f}{\partial x_1} = \Delta_{discr} f + 2mf. \tag{2.6}$$

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Equation (2.6) follows immediately from definition, equation (2.3) and Remark 2.1, as in [5, section 2].

Remark 2.3 (analogous to [5, remark 23]). For every $k \in K_m$ the k-th component of the right-hand side in (2.6) is a linear combination with strictly positive coefficients of the components $f(x_1, k')$ with $k' \in K_m$ obtained from $k = (k_1, \ldots, k_m)$ by adding ± 1 to some k_i. This follows from (2.5), (2.6).

Proposition 2.4 [5, proposition 2.4]. *For every constant $R > 1$ and every $j \geq R^2$ one has*

$$|I_j(z)| < \frac{R^j}{j!} \text{ for every } 0 \leq z \leq R. \tag{2.7}$$

Corollary 2.5 (analogous to [5, corollary 2.6]). *For every $k \in K_m$ and $x \in \mathbb{R}^m$ one has $(f_k(x_1, w))_{k \in K_m} \in l_2$. Moreover, there exists a function $C(R) > 0$ in $R > 1$ such that*

$$\sum_{k \in K_m} |f_k(x_1, w)|^2 < C(R) \text{ for every } 0 \leq x \leq R. \tag{2.8}$$

Proof The proof of Corollary 2.5 repeats the proof of [5, corollary 2.6] with minor changes. Fix an $R > 1$. Set

$$M = \max_{j \in \mathbb{Z}, \ 0 \leq z \leq R} |I_j(z)|.$$

The number M is finite, by (2.7) and [5, remark 2.5]. Recall that $0 \leq k_1 < \cdots < k_m$ for every $k = (k_1, \ldots, k_m) \in K_m$. For every $k \in K_m$ one has

$$|f_k(x_1, w)| < m! \frac{R^{k_m}}{k_m!} M^{m-1} \text{ whenever } |x_1| + |w| \leq R; \text{ here } |w| = |w_1| + \cdots + |w_m|. \tag{2.9}$$

Indeed, the last column of the matrix $A_{k,x}$ consists of the values $I_{k_m}(x_i) = I_{k_m}(x_1 + w_i)$, which are no greater than $\frac{R^{k_m}}{k_m!}$, whenever $|x_1| + |w| \leq R$, by inequality (2.7). The other matrix elements are no greater that M on the same set. Therefore, the module $|f_{k,n}(x)|$ of its determinant defined as sum of $m!$ products of functions I_j satisfies inequality (2.9). This implies that the sum in (2.8) through $k \in K_m$ is no greater than

$$C(R) = m! M^{m-1} \sum_{k \in K_m} \frac{R^{k_m}}{k_m!} < +\infty.$$

The corollary is proved. □

Definition 2.6 [5, definition 2.7]. Let Ω be the closure of an open convex subset in a Banach space. For every $x \in \partial\Omega$ consider the union of all the rays issued from x that intersect Ω in at least two distinct points (including x). The closure of the latter union of rays is a convex cone, which will be here referred to, as the *generating cone* $K(x)$.

Proposition 2.7 *[5, proposition 2.8]. Let H be a Banach space, $\Omega \subset H$ be as above. Let v be a C^1 vector field on a neighborhood of the set Ω in H such that $v(x) \in K(x)$ for every $x \in \partial\Omega$. Then the set Ω is invariant under the flow of the field v: each positive semitrajectory starting at Ω is contained in Ω.*

Now the proof of the induction step in Theorem 1.4 is analogous to the argument from [5, end of section 2]. The right-hand side of differential equation (2.6) is a bounded linear vector field on the Hilbert space l_2 of sequences $(f_k)_{k \in K_m}$. We will denote the latter vector field by v. Let $\Omega \subset l_2$ denote the "positive quadrant" defined by the inequalities $f_k \geq 0$. For every point $y \in \partial\Omega$ the vector $v(y)$ lies in its generating cone $K(y)$: the components of the field v are non-negative on Ω, by Remark 2.3. The vector function $(f_k(x_1) = f_k(x_1, w))_{k \in K_m}$ in $x_1 \geq 0$ is an l_2-valued solution of the corresponding differential equation, by Corollary 2.5. One has $(f_k(0))_{k \in K_m} \in \Omega$:

$$f_k(0) = 0 \text{ whenever } k_1 > 0; \; f_{(0,k_2,\ldots,k_m)}(0) = I_0(0) f_{(k_2,\ldots,k_m)}(w_2, \ldots w_m) > 0. \tag{2.10}$$

The latter equality and inequality follow from definition, (2.2) and the induction hypothesis. This together with Proposition 2.7 implies that

$$f_k(x_1, w) \geq 0 \text{ for every } k \in K_m \text{ and } x_1 \geq 0. \tag{2.11}$$

Now let us prove that the inequality is strict for all $k \in K_m$ and $x_1 > 0$. Indeed, let $f_p(x_0) = 0$ for some $p = (p_1, \ldots, p_m) \in K_m$ and $x_0 > 0$. All the derivatives of the function f_p are non-negative, by (2.6), Remark 2.3 and (2.11). Therefore, $f_p \equiv 0$ on the segment $[0, x_0]$. This together with (2.6), Remark 2.3 and (2.11) implies that $f_{p'} \equiv 0$ on $[0, x_0]$ for every $p' \in K_m$ obtained from p by adding ± 1 to some component. We then get by induction that $f_{(0,k_2,\ldots,k_m)}(0) = 0$, – a contradiction to (2.10). The proof of Theorem 1.4 is complete.

3 Acknowledgments

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