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EXTENSIONS OF PARTIAL CYCLIC ORDERS AND CONSECUTIVE COORDINATE POLYTOPES

ARVIND AYYER, MATTHIEU JOSUAT-VERGÈS,
AND SANJAY RAMASSAMY

ABSTRACT. We introduce several classes of polytopes contained in $[0, 1]^n$ and cut out by inequalities involving sums of consecutive coordinates, extending a construction by Stanley. We show that the normalized volumes of these polytopes enumerate the extensions to total cyclic orders of certain classes of partial cyclic orders. We also provide a combinatorial interpretation of the Ehrhart h^* -polynomials of some of these polytopes in terms of descents of total cyclic orders. The Euler numbers, the Eulerian numbers and the Narayana numbers appear as special cases.

1. INTRODUCTION

Lattice polytopes, i.e. polytopes with vertices in \mathbb{Z}^n , have a volume which is an integer multiple of $\frac{1}{n!}$, which is the volume of the smallest simplex with vertices in \mathbb{Z}^n . An important question is to find a combinatorial interpretation of the integers arising as the normalized volume (the volume multiplied by factorial of the dimension) of some natural classes of lattice polytopes. The most celebrated instance is probably the Chan-Robbins-Yuen polytope [CRY00], the normalized volume of which was conjectured by [CRY00] and shown by Zeilberger [Zei99] to be equal to a product of Catalan numbers. This was later generalized to flow polytopes, see for example [CKM17] and the references therein. Another class of polytopes is that of the poset polytopes [Sta86]: to any poset one can associate two polytopes, the order polytope and the chain polytope of the poset, whose normalized volumes are equal to the number of linear extensions of the poset. Refined enumeration results involve the notion of Ehrhart h^* -polynomial of the polytope, which has the property that its coefficients are non-negative integers which sum to the normalized volume of the polytope [Sta80]. See [BR15] for some background about Ehrhart theory.

This article is close in spirit to the poset polytopes construction. We define several classes of polytopes, obtained as subsets of $[0, 1]^n$ and cut

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out by inequalities comparing the sum of some consecutive coordinates to the value 1. Some of these polytopes were introduced by Stanley in [Sta12] exercise 4.56(c), asking for a computation of the generating series of their normalized volumes. We show that the normalized volumes of these polytopes enumerate extensions of some partial cyclic orders to total cyclic orders (see below for some background on cyclic orders). We also find a combinatorial interpretation of the Ehrhart h^* -polynomials of some of these polytopes in terms of descents in the total cyclic orders. Remarkably enough, the Euler up/down numbers and the Eulerian numbers both arise, the former as the volumes of some polytopes and the latter as the coefficients of the h^* -polynomial of other polytopes. The Catalan and Narayana numbers also arise, as some stationary values for the volumes and coefficients of the h^* -polynomials of some polytopes.

A cyclic order Z on a set X is a subset of triples in X^3 satisfying the following three conditions, respectively called cyclicity, asymmetry and transitivity:

- (1) $\forall x, y, z \in X, (x, y, z) \in Z \Rightarrow (y, z, x) \in Z$;
- (2) $\forall x, y, z \in X, (x, y, z) \in Z \Rightarrow (z, y, x) \notin Z$;
- (3) $\forall x, y, z, u \in X, (x, y, z) \in Z$ and $(x, z, u) \in Z \Rightarrow (x, y, u) \in Z$.

A cyclic order is called total if for every triple of distinct elements $(x, y, z) \in X^3$, either $(x, y, z) \in Z$ or $(z, y, x) \in Z$. Intuitively a total cyclic order Z on X is a way of placing all the elements of X on a circle such that a triple (x, y, z) lies in Z whenever y lies on the cyclic interval from x to z when turning around the circle in the positive direction. See Figure 1 for an example. In this article, we consider certain classes of total cyclic orders on $\{0, \dots, n\}$ where we prescribe the relative position on the circle of certain consecutive integers. This corresponds to considering the set of all total cyclic orders extending a given partial cyclic order. Although the set of total cyclic orders on $\{0, \dots, n\}$ is naturally in bijection with the set of permutations on $\{1, \dots, n\}$, the conditions defining the subsets under consideration are expressed more naturally in terms of extensions of partial cyclic orders. Not every partial cyclic order admits an extension to total cyclic order, as was shown by Megiddo [Meg76]. The classes of partial cyclic orders considered in this article build upon those introduced in [Ram18], which are the first classes for which positive enumerative results were obtained concerning extensions to total cyclic orders.

Organization of the paper. In Section 2 we introduce several classes of polytopes and partial cyclic orders then state the main results relating the volumes and the Ehrhart h^* -polynomials of the former to

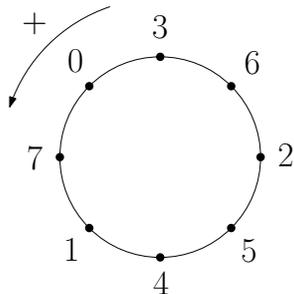


FIGURE 1. An example of a graphical representation of a total cyclic order on $\{0, \dots, 7\}$. The arrow indicates the positive direction of rotation on the circle. This total cyclic order contains for example the triples $(0, 4, 2)$ and $(6, 1, 2)$ but not the triple $(7, 5, 4)$.

the enumeration and refined enumeration of the latter. In Section 3 we prove that our polytopes are lattice polytopes. In Section 4 we introduce a transfer map which maps the original polytopes to other polytopes, from which we deduce the statement about their volumes. Section 5 is devoted to the interpretation of the coefficients of the Ehrhart h^* -polynomials of the polytopes. Some of these polynomials are shown in Section 6 to be palindromic and in Section 7 to stabilize to the Narayana polynomials. Finally in Section 8 we explain how to use the multidimensional boustrophedon construction to compute the volumes of the polytopes.

2. MAIN RESULTS

2.1. Volumes of polytopes. For any $n \geq 1$, denote by $[n]$ the set $\{0, 1, \dots, n\}$ and by \mathcal{Z}_n the set of total cyclic orders on $[n]$. If $m \geq 3$ and $Z \in \mathcal{Z}_n$, the m -tuple $(x_1, \dots, x_m) \in [n]^m$ is called a Z -chain if for every $2 \leq i \leq m - 1$, we have $(x_1, x_i, x_{i+1}) \in Z$. In words, this means that if we place all the numbers on a circle in the cyclic order prescribed by Z and turn around the circle in the positive direction starting at x_1 , we will first see x_2 , then x_3 , etc, before coming back to x_1 . We extend this definition to the case $m = 2$ by declaring that any pair $(x_1, x_2) \in [n]^2$ forms a Z -chain. For example, for the total cyclic order Z depicted on Figure 1, $(0, 1, 2, 3)$ and $(1, 5, 6, 3, 7)$ are Z -chains but $(1, 2, 3, 4)$ is not a Z -chain.

For any $n \geq 1$ and $2 \leq k \leq n + 1$, define $A_{k,n}$ to be the set of total cyclic orders $Z \in \mathcal{Z}_n$ such that for any $0 \leq i \leq n + 1 - k$, the k -tuple $(i, i + 1, \dots, i + k - 1)$ forms a Z -chain.

For any $n \geq 1$ and $2 \leq k \leq n + 1$, define the convex polytope $B_{k,n}$ as follows. It is the set of all $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that:

- for any $1 \leq i \leq n$, we have $x_i \geq 0$;
- for any $1 \leq i \leq n + 2 - k$, we have $x_i + x_{i+1} + \dots + x_{i+k-2} \leq 1$.

These polytopes were introduced by Stanley in [Sta12] exercise 4.56(c). Stanley [Sta12] gives in the solution to the exercise some discrete difference equations for polynomials which can be used to compute the volumes of $B_{k,n}$. The polytopes $B_{3,n}$ were also extensively studied by Diaconis and Wood [DW13], arising as spaces of random doubly stochastic tridiagonal matrices. Our first result relates the normalized volumes of $B_{k,n}$ to the enumeration of $A_{k,n}$:

Theorem 1. *For any $n \geq 1$ and $2 \leq k \leq n + 1$, the polytope $B_{k,n}$ is a lattice polytope and we have*

$$(1) \quad n! \operatorname{vol}(B_{k,n}) = \#A_{k,n}$$

Remark 1. *The cases $k = 2$ and $k = 3$ are already known. When $k = 2$, $A_{2,n} = \mathcal{Z}_n$, which has cardinality $n!$ and $B_{2,n} = [0, 1]^n$, which has volume 1. When $k = 3$, it was shown in [Ram18] that $\#A_{3,n}$ is equal to the n -th Euler up/down number E_n . On the other hand, it follows from [Sta12] exercise 4.56(c) that $n! \operatorname{vol}(B_{3,n}) = E_n$ (see also [SMN79]). In that case, the polytope $B_{3,n}$ arises as the chain polytope of the zigzag poset [Sta86].*

Theorem 1 admits a generalization where the lengths of the chains defining the partial cyclic order (resp. the number of coordinates appearing in each inequality defining the polytope) do not have to be all equal. For any $n \geq 1$, let P_n be the set of all pairs $(i, j) \in [n]^2$ such that $i < j$. To any subset $I \subset P_n$, we associate the set A_I of all the total cyclic orders $Z \in \mathcal{Z}_n$ such that for every $(i, j) \in I$, $(i, i + 1, \dots, j)$ forms a chain in Z . Furthermore, to any subset $I \subset P_n$, we associate the polytope B_I defined as the set of all $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that:

- for any $1 \leq i \leq n$, we have $0 \leq x_i \leq 1$;
- for any $(i, j) \in I$, we have $x_{i+1} + \dots + x_{j-1} + x_j \leq 1$.

Then we have:

Theorem 2. *For any $n \geq 1$ and $I \subset P_n$, the polytope B_I is a lattice polytope and we have*

$$(2) \quad n! \operatorname{vol}(B_I) = \#A_I.$$

If $I = \{(i, i + k - 1)\}_{0 \leq i \leq n - k + 1}$, we recover $A_I = A_{k,n}$ and $B_I = B_{k,n}$, hence Theorem 1 follows as a corollary of Theorem 2.

Remark 2. *If some pair $(i, j) \in I$ is nested inside another pair $(i', j') \in I$, then the condition on Z -chains imposed by (i, j) (resp. the inequality imposed by (i, j)) is redundant in the definition of A_I (resp. B_I). Without loss of generality, we can thus restrict ourselves to considering sets I with no nested pairs, which provides a minimal way of describing A_I and B_I .*

The case $k = 3$ of Theorem 1 can be generalized in the following way. To any word $s = (s_1, \dots, s_n) \in \{+, -\}^n$ with $n \geq 0$, following [Ram18], one can associate the *cyclic descent class* A_s , defined as the set of all $Z \in \mathcal{Z}_{n+1}$ such that for every $1 \leq i \leq n$, we have $(i-1, i, i+1) \in Z$ (resp. $(i+1, i, i-1) \in Z$) if $s_i = +$ (resp. if $s_i = -$). For example, if $s_i = +$ for every $1 \leq i \leq n$, then $A_s = A_{3, n+1}$. On the other hand, one can associate to any word $s = (s_1, \dots, s_n) \in \{+, -\}^n$ the polytope B_s defined as the set of all $(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ such that:

- for any $1 \leq i \leq n+1$, we have $x_i \geq 0$;
- for any $1 \leq i \leq n$ such that $s_i = +$, we have $x_i + x_{i+1} \leq 1$;
- for any $1 \leq i \leq n$ such that $s_i = -$, we have $x_i + x_{i+1} \geq 1$.

For example, if $s_i = +$ for every $1 \leq i \leq n$, then $B_s = B_{3, n+1}$. We have the following result:

Theorem 3. *For any $n \geq 0$ and $s = (s_1, \dots, s_n) \in \{+, -\}^n$, the polytope B_s is a lattice polytope and we have*

$$(3) \quad (n+1)! \operatorname{vol}(B_s) = \#A_s.$$

While these cyclic descent classes are in bijection with descent classes by [Ram18], the polytopes B_s are usually not the chain polytopes associated with the corresponding descent class.

2.2. Ehrhart h^* -polynomials. One can actually refine Theorem 2 by considering the Ehrhart h^* -polynomials of the polytopes B_I , whose evaluation at 1 give the volume of the polytope.

Definition 1. *If $P \subset \mathbb{R}^n$ is a lattice polytope, its Ehrhart function is defined for any integer $t \geq 0$ by:*

$$E(P, t) := \#(t \cdot P) \cap \mathbb{Z}^n$$

where $t \cdot P$ is the dilation of P by a factor t , i.e. $t \cdot P = \{t \cdot v \mid v \in P\}$.

The function $E(P, t)$ is actually a polynomial function of t , hence we call it the *Ehrhart polynomial* of P . This function may in fact also be defined if $P \subset \mathbb{R}^n$ is not a lattice polytope, this point of view will be useful later.

Definition 2. If $P \subset \mathbb{R}^n$ is a lattice polytope, we set

$$E^*(P, z) := (1 - z)^{n+1} \sum_{t=0}^{\infty} E(P, t) z^t.$$

The function $E^*(P, z)$ is a polynomial function of z , which is called the Ehrhart h^* -polynomial of P .

By a result of Stanley [Sta80], the coefficients of this polynomial are nonnegative integers. We provide a combinatorial interpretation of the coefficients of the h^* -polynomial of B_I in terms of descents in the elements of A_I .

To every total cyclic order $Z \in \mathcal{Z}_n$ we associate the word Z^w of length $n + 1$ obtained by placing the elements of Z on a circle in the cyclic order imposed by Z and reading them in the positive direction, starting from 0. For example, for the total cyclic order Z depicted on Figure 1, we have $Z^w = (0, 7, 1, 4, 5, 2, 6, 3)$. We denote by \mathcal{W}_n the set of words of length $n + 1$ with letters in $[n]$ that are all distinct and starting with 0. Then $Z \in \mathcal{Z}_n \mapsto Z^w \in \mathcal{W}_n$ is a bijection.

Given a word $w = (w_0, \dots, w_n) \in \mathcal{W}_n$ and an integer $0 \leq i \leq n - 1$, we say that w has a descent at position i if $w_{i+1} < w_i$. We denote by $\delta(w)$ the number of positions at which w has a descent. For example, the word $w = (0, 3, 4, 1, 5, 2)$ has two descents, at positions 2 and 4, thus $\delta(w) = 2$. We have the following generalization of Theorem 2:

Theorem 4. For any $n \geq 1$ and $I \subset P_n$, we have

$$(4) \quad E^*(B_I, z) = \sum_{Z \in A_I} z^{\delta(Z^w)}.$$

Remark 3. In the case $k = 2$, $B_{2,n}$ is the unit hypercube $[0, 1]^n$ and its h^* -polynomial is well-known to be the n -th Eulerian polynomial, whose coefficients enumerate the permutations of $\{1, \dots, n\}$ refined by their number of descents (see e.g. [HJV16]). This is consistent with the fact that $A_{2,n}$ is in bijection with the set of all permutations of $\{1, \dots, n\}$, arising upon removing the initial 0 from each word Z^w for $Z \in A_{2,n}$.

Remark 4. Recall that a polynomial $R(z) = \sum_{k=0}^d a_k z^k$ of degree d is called palindromic if its sequence of coefficients is symmetric, i.e. for every $0 \leq k \leq d$, we have $a_k = a_{d-k}$. It follows from [LJ15] that for every $n \geq 1$ and $2 \leq k \leq n + 1$ the polynomial $E^*(B_{k,n}, z)$ is palindromic. See Section 6 for more details. However, in general, the polynomials $E^*(B_I, z)$ are not palindromic.

For every $1 \leq k \leq n$, define the *Narayana numbers* (sequence A001263 in [Slo18])

$$(5) \quad N(n, k) := \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$$

and the *Narayana polynomials*

$$(6) \quad Q_n(z) := \sum_{k=1}^n N(n, k) z^{k-1}.$$

The Narayana numbers are a well-known refinement of Catalan numbers, counting for example the number of Dyck paths with prescribed length and number of peaks (see e.g. [FS09, Example III.13]). We have the following stabilization result of the Ehrhart h^* -polynomials of $B_{k,n}$ to the Narayana polynomials:

Theorem 5. *For any $k \geq 2$ and $2 \leq n \leq 2k - 2$, we have*

$$E^*(B_{k,n}, z) = Q_{n-k+2}(z).$$

3. LATTICE POLYTOPES

In this section we show that the polytopes B_I and B_s are lattice polytopes. A rectangular matrix M is said to be *totally unimodular* (TU) if every square nonsingular submatrix of M is unimodular, i.e. has determinant ± 1 . By [Sch86, Theorem 19.1], if M is TU then for every integral vector c , the polyhedron defined by

$$(7) \quad \{x \mid x \geq 0, Mx \leq c\}$$

is integral, i.e. it is equal to the convex hull of its integer points. In the case of polytopes, which are bounded polyhedra, the integrality property is equivalent to being a lattice polytope. Thus it suffices to realize the polytopes B_I and B_s in the form of (7) involving a TU matrix to conclude that these polytopes are lattice polytopes.

Lemma 6. *For any $n \geq 1$ and $I \subset P_n$, there exists a TU matrix M_I and a vector c_I such that*

$$B_I = \{x \mid x \geq 0, M_I x \leq c_I\}.$$

Proof. Fix $n \geq 1$ and $I \subset P_n$. Write

$$I = \{(i_1, j_1), \dots, (i_m, j_m)\},$$

where $m \geq 1$ is the cardinality of I . Define

$$\tilde{M}_I := \left(\mathbf{1}_{i_p < q \leq j_p} \right)_{\substack{1 \leq p \leq m \\ 1 \leq q \leq n}}.$$

In words, \tilde{M}_I is the $m \times n$ matrix such that for any $1 \leq p \leq m$, the p -th row of \tilde{M}_I contains a 1 in positions comprised between $i_p + 1$ and j_p and 0 elsewhere. Set M_I to be the $(m+n) \times n$ matrix whose first n rows consist of the identity and whose last m rows consist of \tilde{M}_I . Let c_I be the vector in \mathbb{R}^{m+n} with all coordinates equal to 1. Then

$$B_I = \{x \mid x \geq 0, M_I x \leq c_I\}.$$

The matrix M_I has the property that it is a matrix with entries in $\{0, 1\}$ where the 1's in each line are arranged consecutively. Such matrices are called *interval matrices* and form a well-known class of TU matrices [Sch86, Example 7 of Chapter 19]. \square

Lemma 7. *For any $n \geq 1$ and $s \in \{+, -\}^n$, there exists a TU matrix M_s and a vector c_s such that*

$$B_s = \{x \mid x \geq 0, M_s x \leq c_s\}.$$

Proof. Fix $n \geq 1$ and $s \in \{+, -\}^n$. Define \tilde{M}_s to be the matrix of size $n \times (n+1)$ such that for every $1 \leq i \leq n$, the entries in positions (i, i) and $(i, i+1)$ of \tilde{M}_s are equal to 1 (resp. -1) if $s_i = +$ (resp. $s_i = -$), and all the other entries of \tilde{M}_s are zero. Set M_s to be the $(2n+1) \times (n+1)$ matrix whose $n+1$ first rows consist of the identity matrix and whose last n rows consist of \tilde{M}_s . Set c_s to be the vector of \mathbb{R}^{2n+1} with the i -th coordinate equal to 1 (resp. -1) if $s_i = +$ (resp. $s_i = -$) for any $1 \leq i \leq n$. Then

$$B_s = \{x \mid x \geq 0, M_s x \leq c_s\}.$$

The matrix M_s can be realized as an interval matrix (with entries in $\{0, 1\}$) where some rows have been multiplied by -1 . Since the interval matrix is TU, the matrix M_s is also TU. \square

4. THE TRANSFER MAP

In this section we prove Theorem 2 and Theorem 3 by introducing a transfer map F_n from $[0, 1]^n$ to itself, which is piecewise linear, measure-preserving and bijective outside of a set of measure 0. In order to avoid confusion, we will denote by (x_1, \dots, x_n) (resp. (y_1, \dots, y_n)) the coordinates on the source (resp. target) space of F_n . We will show that the image under F_n of the polytopes of type B_I for $I \subset P_n$ and B_s for $s \in \{+, -\}^n$ are some convex polytopes whose normalized volumes are easily seen to enumerate the wanted sets.

For any $n \geq 1$, we define the map

$$F_n : (x_1, \dots, x_n) \in [0, 1]^n \rightarrow \left(\sum_{j=1}^i x_j \pmod{1} \right)_{1 \leq i \leq n} \in [0, 1]^n$$

and the sets of measure zero

$$X_n = \{(x_1, \dots, x_n) \in [0, 1]^n \mid \exists j \in \{1, \dots, n\}, \sum_{i=1}^j x_i \in \mathbb{N}\}$$

and

$$Y_n = \{(y_1, \dots, y_n) \in [0, 1]^n \mid \exists i \in \{1, \dots, n\}, y_i \in \{0, y_{i-1}, 1\}\}.$$

Lemma 8. *The map F_n is a piecewise linear measure-preserving map from $[0, 1]^n$ to itself. Furthermore, F_n is a bijection from $[0, 1]^n \setminus X_n$ to $[0, 1]^n \setminus Y_n$.*

Proof. For any $n \geq 1$, define the map

$$G_n : (y_1, \dots, y_n) \in [0, 1]^n \rightarrow (x_1, \dots, x_n) \in [0, 1]^n,$$

where $x_1 := y_1$ and for any $2 \leq i \leq n$,

$$x_i := \begin{cases} y_i - y_{i-1} & \text{if } y_i \geq y_{i-1} \\ 1 + y_i - y_{i-1} & \text{if } y_i < y_{i-1}. \end{cases}$$

It is straightforward to check that G_n is a left- and right-inverse to F_n outside the sets X_n and Y_n . In each connected component of $[0, 1]^n \setminus X_n$, the map F_n coincides with a translate of the map

$$\tilde{F}_n : (x_1, \dots, x_n) \in [0, 1]^n \rightarrow \left(\sum_{j=1}^i x_j \right)_{1 \leq i \leq n}.$$

Since the matrix of \tilde{F}_n in the canonical basis is upper triangular with 1 on the diagonal, \tilde{F}_n is a measure-preserving linear map and F_n is also measure-preserving. \square

Remark 5. *A map with a definition very similar to G_n was introduced by Stanley in [Sta77] in order to show that the volumes of hypersimplices are given by Eulerian numbers.*

Definition 3. *The cyclic standardization of an n -tuple $y = (y_1, \dots, y_n) \in [0, 1]^n \setminus Y_n$ (i.e. y is an n -tuple of distinct numbers in $(0, 1)^n$) is defined to be the unique word $cs(y) = w_0 \dots w_n \in \mathcal{W}_n$ such that for any $1 \leq i, j \leq n$, $w_i < w_j$ if and only if $y_i < y_j$.*

For example, if $y = (0.2, 0.7, 0.4, 0.1)$, then $cs(y) = 02431$. For any $Z \in \mathcal{Z}_n$, we define S_Z to be the closure of the set of $y \in [0, 1]^n$ whose cyclic standardization is Z^w :

$$S_Z := \overline{\{y \in [0, 1]^n \mid cs(y) = Z^w\}}.$$

It is not hard to see that for every $Z \in \mathcal{Z}_n$, S_Z is a simplex defined by $n + 1$ inequalities. Furthermore, for any $Z \neq Z' \in \mathcal{Z}_n$, the simplices S_Z and $S_{Z'}$ have disjoint interiors. By symmetry, all the S_Z have the same volume. Since $[0, 1]^n = \cup_{Z \in \mathcal{Z}_n} S_Z$, we deduce that $\text{vol}(S_Z) = \frac{1}{n!}$ for every $Z \in \mathcal{Z}_n$.

Proposition 9. *For every $n \geq 1$ and every $I \subset P_n$, we have*

$$(8) \quad F_n(B_I) = \cup_{Z \in A_I} S_Z.$$

For every $n \geq 0$ and every $s \in \{+, -\}^n$, we have

$$(9) \quad F_{n+1}(B_s) = \cup_{Z \in A_s} S_Z.$$

Proof. Observe that each variable x_i measures the gap between y_{i-1} and y_i (where by convention $y_0=0$). It follows that for any $1 \leq i < j \leq n$, outside of the sets X_n and Y_n , we have $x_{i+1} + \dots + x_j < 1$ if and only if $(y_i, y_{i+1}, \dots, y_j)$ forms a chain in the circle obtained by quotienting out the interval $[0, 1]$ by the relation $0 \sim 1$ and equipping it with the standard cyclic order on the circle (see Figure 2). It follows immediately that for any $I \subset P_n$ and for every $(x_1, \dots, x_n) \in [0, 1]^n \setminus X_n$, (x_1, \dots, x_n) lies in B_I if and only if the cyclic standardization of $F_n(x_1, \dots, x_n)$ is in A_I . This proves equality (8).

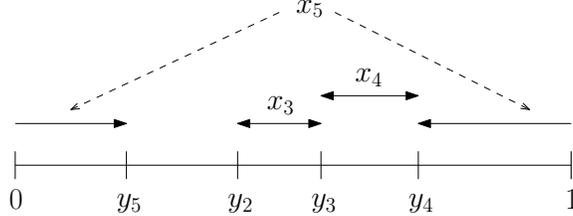


FIGURE 2. The inequality $x_3 + x_4 + x_5 < 1$ is equivalent to the fact that (y_2, y_3, y_4, y_5) forms a chain in $[0, 1]/\sim$ equipped with its standard cyclic order.

Equality (9) follows similarly, by observing that the knowledge of the sign of $x_i + x_{i+1} - 1$ is equivalent to the knowledge of the relative positions of y_{i-1} , y_i and y_{i+1} on the circle $[0, 1]/\sim$. \square

Theorems 2 and 3 follow immediately from Lemma 8 and Proposition 9, since the polytopes involved in the unions on the r.h.s. of (8) and (9) have disjoint interiors.

Remark 6. *The sets A_s and the polytopes B_s constitute a generalization of the sets $A_{3,n}$ and the polytopes $B_{3,n}$. There does not seem to be a straightforward extension of these generalized sets and polytopes when $k \geq 4$ for the following reason. If we have an inequality of the type $x_i + x_{i+1} > 1$, it determines completely the relative cyclic position of y_i , y_{i+1} and y_{i+2} . However, if we have the inequality $x_i + x_{i+1} + x_{i+2} > 1$, it does not determine completely the relative cyclic position of y_i , y_{i+1} , y_{i+2} and y_{i+3} .*

5. THE EHRHART h^* -POLYNOMIAL

In this section, we prove Theorem 4 about the combinatorial interpretation for the h^* -polynomial of B_I in terms of descents of elements in A_I . To alleviate notations, we only consider the special case of the polytopes $B_{k,n}$, but the proof for general B_I works along the same lines. We introduce the notion of *standardization*, defined as follows: if $v \in \mathbb{R}^n$, then $\text{std}(v)$ is the unique permutation $\sigma \in \mathfrak{S}_n$ such that if $i < j$, $v_i > v_j$ if and only if $\sigma_i > \sigma_j$. Note that this standardization is different from the cyclic standardization introduced in Section 4.

The first step consists in relating our polytope $B_{k,n}$ with its “half-open” version $B'_{k,n}$:

Definition 4. *We define $B'_{k,n}$ as the set of $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that*

- *for any $1 \leq i \leq n$, we have $x_i \geq 0$;*
- *for any $1 \leq i \leq n + 2 - k$, we have $x_i + x_{i+1} + \dots + x_{i+k-2} < 1$.*

Though $B'_{k,n}$ is not a polytope (it is obtained from the polytope $B_{k,n}$ by removing some faces), we can define its Ehrhart polynomial and h^* -polynomial by:

$$E(B'_{k,n}, t) = \#(t \cdot B'_{k,n} \cap \mathbb{Z}^n) \text{ if } t \in \mathbb{Z}, t \geq 1,$$

$$E^*(B'_{k,n}, z) = (1 - z)^{n+1} \sum_{t=1}^{\infty} E(B'_{k,n}, t) z^t.$$

Note that there is no general result to guarantee that $E^*(B'_{k,n}, z)$ is a polynomial with nonnegative coefficients. But we have the following:

Lemma 10. *If $t \in \mathbb{Z}$, $t \geq 1$, then $E(B'_{k,n}, t) = E(B_{k,n}, t - 1)$. Moreover:*

$$(10) \quad E^*(B'_{k,n}, z) = z \cdot E^*(B_{k,n}, z).$$

Proof. The first equality follows from:

$$(t \cdot B'_{k,n}) \cap \mathbb{Z}^n = ((t - 1) \cdot B_{k,n}) \cap \mathbb{Z}^n.$$

To see why this is true, it suffices to notice that the condition $x_i + x_{i+1} + \cdots + x_{i+k-2} < t$ is equivalent to $x_i + x_{i+1} + \cdots + x_{i+k-2} \leq t - 1$ when $(x_i)_{1 \leq i \leq n} \in \mathbb{Z}^n$. The second equality follows from the first one. \square

It remains only to show that $E^*(B'_{k,n}, z)$ is the descent generating function of $A_{k,n}$ (up to this factor z). The first tool is the transfer map.

Proposition 11. *The transfer map F_n defines a bijection from $B'_{k,n}$ to the set of $(y_i)_{1 \leq i \leq n} \in \mathbb{R}^n$ satisfying:*

- $0 \leq y_i < 1$ for all i ,
- $y_1 \leq \cdots \leq y_{k-1}$,
- if $1 \leq i \leq n - k + 2$, there exists $i \leq j \leq i + k - 2$ such that

$$y_i \leq y_{i+1} \leq \cdots \leq y_j, \quad \text{and } y_{j+1} \leq y_{j+2} \leq \cdots \leq y_{i+k-2} < y_i.$$

Proof. Let $(x_i)_{1 \leq i \leq n} \in B'_{k,n}$ and let $(y_i)_{1 \leq i \leq n}$ denote its image by F_n . We first check that $(y_i)_{1 \leq i \leq n}$ satisfies the conditions above.

First, we have $0 \leq y_i < 1$ by definition of the representative modulo 1. Because $\sum_{i=1}^{k-1} x_i < 1$, we get $0 \leq \sum_{i=1}^j x_i < 1$ if $1 \leq j \leq k - 1$, so that $y_j = \sum_{i=1}^j x_i$. So $y_{i+1} - y_i = x_i \geq 0$ if $1 \leq i < k - 2$ and the second point follows. Then, let $1 \leq i \leq n - k + 2$. We have $x_i + \cdots + x_{i+k-2} < 1$. Using that we see that the partial sums $x_i + \cdots + x_j$ modulo 1 wrap around the circle and this gives the third point.

The same arguments show that if the sequence $(y_i)_{1 \leq i \leq n}$ satisfies the condition, its preimage $(x_i)_{1 \leq i \leq n}$ under the transfer map is in $B'_{k,n}$. \square

Note also that the transfer map preserves integrality. More precisely, for $v \in [0, 1]^n$ and an integer $t \geq 1$, we have $t \cdot v \in \mathbb{Z}^n$ if and only if $t \cdot F_n(v) \in \mathbb{Z}^n$. It follows that the h^* -polynomial of $B'_{k,n}$ can be obtained from that of its image by F_n (which again is not a lattice polytope, but its h^* -polynomial is defined in the same way as that of $B'_{k,n}$).

Proposition 12. *Let $y = (y_i)_{1 \leq i \leq n} \in \mathbb{R}^n$ with $0 \leq y_i < 1$. Then $y \in F_n(B'_{k,n})$ if and only if the permutation $\sigma = \text{std}(y)$ satisfies:*

- $\sigma_1 < \cdots < \sigma_{k-1}$,
- if $1 \leq i \leq n - k + 2$, there exists j such that

$$\sigma_{j+1} < \sigma_{j+1} < \cdots < \sigma_{i+k-2} < \sigma_i < \sigma_{i+1} < \cdots < \sigma_j.$$

Equivalently, σ^{-1} is in $A_{k,n}$ (upon seeing a permutation $\tau \in \mathfrak{S}_n$ as the total cyclic order Z such that $Z^w = (0, \tau_1, \tau_2, \dots, \tau_n)$).

Proof. This follows from Proposition 11 and the properties of standardization. \square

It follows that $F_n(B'_{k,n})$ can be seen as a disjoint union over the sets \mathcal{S}_σ where σ runs through permutations satisfying the conditions in the previous proposition, and

$$\mathcal{S}_\sigma = \{y \in [0, 1]^n \mid \text{std}(y) = \sigma\}.$$

Next, we need to know the h^* -polynomial of \mathcal{S}_σ . We have the following, which is exactly [HJV16, Lemma 4]:

Lemma 13.

$$(11) \quad E^*(\mathcal{S}_\sigma, z) = z^{\text{des}(\sigma^{-1})+1}.$$

Taking the sum of (11) over permutations σ such that σ^{-1} is in $A_{k,n}$, we obtain the h^* -polynomial of $F_n(B'_{k,n})$, hence that of $B'_{k,n}$. Using (10) we get $E^*(B_{k,n}, z)$.

6. PALINDROMICITY OF THE h^* -POLYNOMIALS OF $B_{k,n}$

The first few values of the h^* -polynomials of $B_{k,n}$ are displayed in Figure 3. One observation is that all the polynomials have a palindromic sequence of coefficients. We explain in this section how this can be deduced from the geometry of the polytopes.

Definition 5. *If a lattice polytope P contains the origin in its interior, the dual polytope is:*

$$P^* = \{v \in \mathbb{R}^n \mid \langle v, w \rangle \geq -1 \text{ for all } w \in P\}.$$

Moreover, we say that P is reflexive if P^* is also a lattice polytope.

Hibi's palindromic theorem [Hib92] states that if a lattice polytope P contains an integral interior point v , then $E^*(P, z)$ has palindromic coefficients if and only if $P - v$ is reflexive. But in the present case, our polytopes $B_{k,n}$ do not contain integral interior point, as their vertices are vectors only containing 0's and 1's.

In [LJ15], Lee and Ju define *regular positive reflexive polytopes* and show that their h^* -polynomials are palindromic. It is immediate to check that $B_{k,n}$ satisfies the conditions for being regular positive reflexive polytopes and as a direct application of [LJ15, Theorem 4] we obtain the palindromicity of the h^* -polynomials of $B_{k,n}$.

Rather than repeating here the results of Lee and Ju, it is worth giving more context. In general, palindromicity of h^* -polynomial characterizes the class of *Gorenstein polytopes*. We refer to [BN08] about this subject. A lattice polytope P is Gorenstein if and only if there exists an integer $t \geq 1$ and an integer vector v , such that $t \cdot P - v$ is

reflexive. In the present case, the elements $v \in k \cdot B_{k,n} - (1, \dots, 1)$ are characterized by the inequalities:

$$\begin{aligned} v_i &\geq -1 && \text{for } 1 \leq i \leq n, \\ v_i + \dots + v_{i+k-2} &\leq 1 && \text{for } 1 \leq i \leq n + 2 - k. \end{aligned}$$

Since there are 1 and -1 on the right-hand sides of the inequalities, the coefficients in the left hand sides are the coordinates of the vertices of P^* . So we can deduce from these inequalities that $(k \cdot B_{k,n} - (1, \dots, 1))^*$ is a lattice polytope. So $k \cdot B_{k,n} - (1, \dots, 1)$ is reflexive and $B_{k,n}$ is Gorenstein.

7. STABILIZATION TO NARAYANA POLYNOMIALS

In this section, we first prove Theorem 5 about the stabilization of the h^* -polynomials of $B_{k,n}$ to Narayana polynomials. This is done combinatorially, using the connection with $A_{k,n}$. This is also illustrated in Figure 3. Then we provide some geometric insight as to why the Ehrhart h^* -polynomials of $B_{k,n}$ stabilize.

Proof of Theorem 5. We will first prove the result for $A_{n+1,2n}$ via a bijective correspondence with nondecreasing parking functions. An $(n+1)$ -tuple of nonnegative integers (p_0, \dots, p_n) is called a *nondecreasing parking function* if the following two conditions hold:

- (1) for any $0 \leq i \leq n-1$, we have $p_i \leq p_{i+1}$;
- (2) for any $0 \leq i \leq n$, we have $0 \leq p_i \leq i$.

We denote by \mathcal{P}_n the set of all $(n+1)$ -tuples that are nondecreasing parking functions. It is well-known that the cardinality of \mathcal{P}_n is the $(n+1)$ -st Catalan number; see [Sta99, Exercise 6.19(s)], for example. A nondecreasing parking function (p_0, \dots, p_n) is said to have an *ascent* at position $0 \leq i \leq n-1$ if $p_i < p_{i+1}$. It follows from [Sch09, Corollary A.3] that the number of nondecreasing parking functions in \mathcal{P}_n with k ascents is the Narayana number $N(n+1, k+1)$. To complete the proof in the case of $A_{n+1,2n}$, we will define a bijection H_n between $A_{n+1,2n}$ and \mathcal{P}_n such that the number of descents of $Z \in A_{n+1,2n}$ equals the number of ascents of $H_n(Z)$.

We call the numbers in $\{0, \dots, n\}$ “small numbers”. Given $Z \in A_{n+1,2n}$, we set $H_n(Z)$ to be the $(n+1)$ -tuple (p_0, \dots, p_n) defined as follows. For any $0 \leq i \leq n$, p_i is defined to be the first small number to the right of $n+i$ in the word Z^w if there exists a small number to the right of $n+i$. Otherwise p_i is defined to be 0. In other words, visualizing Z as the placement of the numbers from 0 to $2n$ on a circle, p_i is the next small number after $n+i$, turning in the positive direction. For example, if $n=3$ and $Z^w = (0, 4, 5, 1, 2, 6, 3)$, then $H_n(Z) = (0, 1, 1, 3)$.

$k \backslash n-k$	-1	0	1	2	3	4
2	1	$z+1$	z^2+4z+1	$z^3+11z^2+11z+1$	$z^4+26z^3+66z^2+26z+1$	$z^5+57z^4+302z^3+302z^2+57z+1$
3	1	$z+1$	z^2+3z+1	z^3+7z^2+7z+1	$z^4+14z^3+31z^2+14z+1$	$z^5+26z^4+109z^3+109z^2+26z+1$
4	1	$z+1$	z^2+3z+1	z^3+6z^2+6z+1	$z^4+14z^3+31z^2+14z+1$	$z^5+19z^4+71z^3+71z^2+19z+1$
5	1	$z+1$	z^2+3z+1	z^3+6z^2+6z+1	$z^4+10z^3+20z^2+10z+1$	$z^5+16z^4+56z^3+56z^2+16z+1$
6	1	$z+1$	z^2+3z+1	z^3+6z^2+6z+1	$z^4+10z^3+20z^2+10z+1$	$z^5+15z^4+50z^3+50z^2+15z+1$
7	1	$z+1$	z^2+3z+1	z^3+6z^2+6z+1	$z^4+10z^3+20z^2+10z+1$	$z^5+15z^4+50z^3+50z^2+15z+1$
8	1	$z+1$	z^2+3z+1	z^3+6z^2+6z+1	$z^4+10z^3+20z^2+10z+1$	$z^5+15z^4+50z^3+50z^2+15z+1$

$k \backslash n-k$	5	6
2	$z^6+120z^5+1191z^4+2416z^3+1191z^2+120z+1$	$z^7+247z^6+4293z^5+15619z^4+15619z^3+4293z^2+247z+1$
3	$z^6+46z^5+334z^4+623z^3+334z^2+46z+1$	$z^7+79z^6+937z^5+2951z^4+2951z^3+937z^2+79z+1$
4	$z^6+31z^5+191z^4+340z^3+191z^2+31z+1$	$z^7+49z^6+472z^5+1365z^4+1365z^3+472z^2+49z+1$
5	$z^6+25z^5+140z^4+242z^3+140z^2+25z+1$	$z^7+38z^6+322z^5+881z^4+881z^3+322z^2+38z+1$
6	$z^6+22z^5+115z^4+195z^3+115z^2+22z+1$	$z^7+32z^6+249z^5+656z^4+656z^3+249z^2+32z+1$
7	$z^6+21z^5+105z^4+175z^3+105z^2+21z+1$	$z^7+29z^6+211z^5+540z^4+540z^3+211z^2+29z+1$
8	$z^6+21z^5+105z^4+175z^3+105z^2+21z+1$	$z^7+28z^6+196z^5+490z^4+490z^3+196z^2+28z+1$
9	$z^6+21z^5+105z^4+175z^3+105z^2+21z+1$	$z^7+28z^6+196z^5+490z^4+490z^3+196z^2+28z+1$
10	$z^6+21z^5+105z^4+175z^3+105z^2+21z+1$	$z^7+28z^6+196z^5+490z^4+490z^3+196z^2+28z+1$

FIGURE 3. Some values of the h^* -polynomial of $B_{k,n}$. This illustrates in particular the stabilization property: the sequence of polynomials in a particular column is stationary.

Fix $Z \in A_{n+1,2n}$, we first show that $H_n(Z)$ is in \mathcal{P}_n . To begin with, note that the entries $0, 1, \dots, n$ appear in that order in Z^w , so n is the rightmost number in $\{1, \dots, n\}$ to appear. If $0 \leq i \leq n$, since $(i, i+1, \dots, n+i)$ forms a chain in Z , then $n+i$ must lie either to the right of n or to the left of i in Z^w , hence $p_i \leq i$. Let $0 \leq i \leq n-1$. If $n+i$ lies to the right of n in Z^w , then $p_i = 0$ and $p_i \leq p_{i+1}$. Otherwise, $n+i$ lies to the left of i and from the fact that $(i+1, \dots, n+i, n+i+1)$ forms a chain in Z , we deduce that $n+i+1$ lies between $n+i$ and $i+1$. Thus $p_i \leq p_{i+1}$ again. This concludes the proof that $H_n(Z)$ is in \mathcal{P}_n .

Next, given $p = (p_0, \dots, p_n) \in \mathcal{P}_n$, we define a total cyclic order $\tilde{H}_n(p) \in \mathcal{Z}_{2n}$ as follows. First we place all the small numbers on the circle in such a way that $(0, 1, \dots, n)$ form a chain. Then we place each number $n+i$ for $1 \leq i \leq n$ in the cyclic interval from $p_i - 1$ to p_i if $p_i \geq 1$ and in the cyclic interval from n to 0 if $p_i = 0$. This determines for $1 \leq i \leq n$ the position of each number $n+i$ with respect to the small numbers. Since the sequence (p_0, \dots, p_n) is weakly increasing, it is possible to arrange the numbers $n+i$ for $1 \leq i \leq n$ in such a way that $(n, n+1, \dots, 2n)$ forms a chain. This determines uniquely the position of all the numbers on the circle and yields (by definition) $\tilde{H}_n(p)$.

Now we need to check that $Z := \tilde{H}_n(p) \in A_{n+1,2n}$. By construction, we already have that $(i, i+1, \dots, n+i)$ are Z -chains when $i = 0$ and $i = n$. Fix $1 \leq i \leq n-1$. Then $(i, i+1, \dots, n)$ (resp. $(n, n+1, \dots, n+i)$) forms a Z -chain, as a subchain of $(0, 1, \dots, n)$ (resp. $(n, n+1, \dots, 2n)$). In the case when $p_i = 0$ then for every $1 \leq j \leq i$ we also have $p_j = 0$, so all the numbers $n+1, n+2, \dots, n+i$ are to the right of n in Z^w and in this order. Hence $(i, i+1, \dots, n+i)$ is a Z -chain in this case. In the case when $p_i > 0$, then

$$p_0 \leq p_1 \leq \dots \leq p_i \leq i$$

thus all the numbers $n+1, n+2, \dots, n+i$ are to the left of i in Z^w and in this order. Hence $(i, i+1, \dots, n+i)$ is a Z -chain in this case too. This concludes the proof that $\tilde{H}_n(p)$ belongs to $A_{n+1,2n}$. Clearly, H_n and \tilde{H}_n are inverses to each other, hence H_n is a bijection from $A_{n+1,2n}$ to \mathcal{P}_n .

Finally, it is immediate to check that two numbers $n+i$ and $n+i+1$ with $0 \leq i \leq n-1$ are consecutive in Z^w with $Z \in A_{n+1,2n}$ if and only if for $p = H_n(Z)$, we have $p_i = p_{i+1}$. The fact that descents in Z^w are in one-to-one correspondence with ascents in $H_n(Z)$ follows from the

observation that a descent in Z^w is always from a number larger than n to a small number. This concludes the proof of Theorem 5 for $A_{n+1,2n}$.

As for $A_{k,n}$ where k is bigger than $n/2 + 1$, the cycle conditions ensure that $n - k + 1, \dots, k - 1$ are consecutive elements in the list (the statement is non-void if $n - k + 1 < k - 1$, i.e. $n \leq 2k - 2$). For any $Z \in A_{k,n}$, by replacing these consecutive entries with $n - k + 1$ and “standardizing” the other elements in the natural way, we obtain an element in $Z' \in A_{n-k+2,2n-2k+2}$. It is not difficult to see that this is a bijection which preserves the number of descents. But now we note that $Z' \in A_{m+1,2m}$, where $m = n - k + 1$, and we appeal to the first part of the proof. \square

It is worth observing that one can also see the stabilization property of the h^* -polynomials as coming directly from the geometric property that $B_{k+1,n+1}$ as a cone over $B_{k,n}$.

Lemma 14. *If $k > \frac{n+1}{2}$, we have $E^*(B_{k,n}, z) = E^*(B_{k+1,n+1}, z)$.*

Proof. First note that $k > \frac{n+1}{2}$ is equivalent to $k \geq n + 2 - k$, so there exists ℓ such that $k \geq \ell \geq n + 2 - k$.

Consider the map $\alpha : \mathbb{Z}^{n+1} \rightarrow \mathbb{Z} \times \mathbb{Z}^n$ defined by

$$\alpha(v_1, \dots, v_{n+1}) = (v_\ell, (v_1, \dots, \hat{v}_\ell, \dots, v_{n+1}))$$

where \hat{v}_ℓ means v_ℓ is omitted in the sequence. It is clearly bijective. We claim that

$$(12) \quad \alpha((t \cdot B_{k+1,n+1}) \cap \mathbb{Z}^{n+1}) = \bigsqcup_{u=0}^t \{t - u\} \times ((u \cdot B_{k,n}) \cap \mathbb{Z}^n).$$

Let $v = (v_i)_{1 \leq i \leq n} \in \mathbb{Z}^{n+1}$. By definition of $B_{k+1,n+1}$, we have $v \in t \cdot B_{k+1,n+1}$ if and only if $v_i \geq 0$ and

$$(13) \quad v_i + \dots + v_{i+k-1} \leq t, \quad \text{for } 1 \leq i \leq n + 2 - k.$$

Then, note that $k \geq \ell \geq n + 2 - k$ ensures that v_ℓ appears in all the sums in (13). So the equations in (13) are equivalent to

$$(14) \quad v_i + \dots + \hat{v}_\ell + \dots + v_{i+k-1} \leq t - v_\ell, \quad \text{for } 1 \leq i \leq n + 2 - k.$$

But these equations precisely say that $(v_1, \dots, \hat{v}_\ell, \dots, v_{n+1}) \in (t - v_\ell) \cdot B_{k,n}$, (knowing that $v_i \geq 0$). Thus we get (12).

Then, from (12), we have:

$$E(B_{k+1,n+1}, t) = \sum_{u=0}^t E(B_{k,n}, u).$$

By summing, we get:

$$\sum_{t \geq 0} E(B_{k+1, n+1}, t) z^t = \sum_{0 \leq u \leq t} E(B_{k, n}, u) z^t = \sum_{u \geq 0} E(B_{k, n}, u) \frac{z^u}{1 - z}.$$

After multiplying by $(1 - z)^{n+1}$, we get $E^*(B_{k, n}, z) = E^*(B_{k+1, n+1}, z)$. \square

Note that in Equation (12), u ranges among integers between 0 and t . But the argument in the proof also shows that

$$\alpha(B_{k+1, n+1}) = \bigsqcup_{0 \leq x \leq 1} \{1 - x\} \times (x \cdot B_{k, n}),$$

where x runs through reals numbers between 0 and 1. This precisely says that $B_{k+1, n+1}$ is a cone over $B_{k, n}$.

8. ENUMERATING $A_{k, n}$

In this section we show how to use the multidimensional boustrophedon construction introduced in [Ram18] to compute the cardinalities of $A_{k, n}$, which by Theorem 1 are equal to the normalized volumes of $B_{k, n}$.

For any total order $Z \in \mathcal{Z}_n$, $i \neq j$ two elements of $[n]$, define the length of the arc from i to j in Z to be

$$(15) \quad L_Z(i, j) := 1 + \#\{h \in [n] \mid (i, h, j) \in Z\}.$$

This notion is related to the notion of content $c_Z(i, j)$ of the arc from i to j introduced in [Ram18] by the relation $L_Z(x, y) = 1 + c_Z(x, y)$.

Example 1. Take $n = 6$ and take the cyclic order Z associated with the cyclic permutation 0351624. Then

$$\begin{aligned} L_Z(3, 5) &= 1 \\ L_Z(3, 2) &= 4 \\ L_Z(2, 3) &= 3. \end{aligned}$$

For any $d \geq 1$ and $N \geq d + 1$, define the simplex of dimension d and order N to be

$$(16) \quad T_N^d := \{(i_1, \dots, i_{d+1}) \in \mathbb{N}^{d+1} \mid i_1 + \dots + i_{d+1} = N\}.$$

When $d = 1$, T_N^1 is a row of $N - 1$ elements. When $d = 2$ (resp. $d = 3$), T_N^d is a triangle (resp. tetrahedron) of side length $N - 2$ (resp. $N - 3$). In general, T_N^d is a d -dimensional simplex of side length $N - d$.

For any $3 \leq k \leq n + 1$ and $\underline{i} = (i_1, \dots, i_{k-1}) \in T_{n+1}^{k-2}$, we define $A_{\underline{i}}$ to be the set of all total cyclic orders $Z \in A_{k, n}$ such that the following conditions hold:

- For any $1 \leq j \leq k-2$, we have $L_Z(n+1+j-k, n+2+j-k) = i_j$
- $L_Z(n, n+2-k) = i_{k-1}$.

It is not hard to see that

$$(17) \quad A_{k,n} = \bigsqcup_{\underline{i} \in T_{n+1}^{k-2}} A_{\underline{i}}.$$

Define $a_{\underline{i}} := \#A_{\underline{i}}$. We will provide linear recurrence relations for the $(a_{\underline{i}})_{\underline{i} \in T_{n+1}^{k-2}}$, which are arrays of numbers indexed by some T_N^d . We first need to define linear operators Ψ and Ω which transform one array of numbers indexed by some T_N^d into another array of numbers, indexed by T_{N+1}^d in the case of Ψ and by T_N^d in the case of Ω .

We define the map τ which to an element $\underline{i} = (i_1, \dots, i_{d+1}) \in T_{N+1}^d$ associates the subset of all the $\underline{i}' = (i'_1, \dots, i'_{d+1}) \in T_N^d$ such that:

- $1 \leq i'_1 \leq i_1 - 1$;
- $i'_{d+1} = i_{d+1} + i_1 - i'_1 - 1$;
- $i'_j = i_j$ for any $2 \leq j \leq d$.

Example 2. *Case $d = 2$, $N = 6$. Then*

$$(18) \quad \tau(1, 2, 4) = \emptyset$$

$$(19) \quad \tau(4, 2, 1) = \{(1, 2, 3), (2, 2, 2), (3, 2, 1)\}$$

Define the map Ψ which sends an array of numbers $(b_{\underline{i}})_{\underline{i} \in T_N^d}$ to the array of numbers $(c_{\underline{i}})_{\underline{i} \in T_{N+1}^d}$, where for any $\underline{i} \in T_{N+1}^d$, we have

$$(20) \quad c_{\underline{i}} := \sum_{\underline{i}' \in \tau(\underline{i})} b_{\underline{i}'}$$

Remark 7. *In the language of [Ram18], the map Ψ would be called $\Phi_{1,1,d+1}$. Note that the Φ operators of [Ram18] were acting on generating functions, while the Ψ operator here acts on arrays of numbers, but that is essentially the same, since there is a natural correspondence between arrays of numbers and their generating functions.*

We also introduce the operator Ω , which sends an array of numbers $(b_{\underline{i}})_{\underline{i} \in T_N^d}$ to the array of numbers $(c_{\underline{i}})_{\underline{i} \in T_N^d}$, where for any $\underline{i} \in T_N^d$, we have

$$(21) \quad c_{(i_1, \dots, i_{d+1})} := b_{(i_{d+1}, i_1, i_2, \dots, i_d)}.$$

The operator Ω acts by cyclically permuting the indices.

We can now state the following recurrence relation for the arrays of numbers $(a_{\underline{i}})_{\underline{i} \in T_{n+1}^{k-2}}$:

Theorem 15. *For any $n \geq 3$ and $3 \leq k \leq n$, we have*

$$(22) \quad (a_{\underline{i}})_{\underline{i} \in T_{n+1}^{k-2}} = \Omega \circ \Psi(a_{\underline{i}})_{\underline{i} \in T_n^{k-2}}.$$

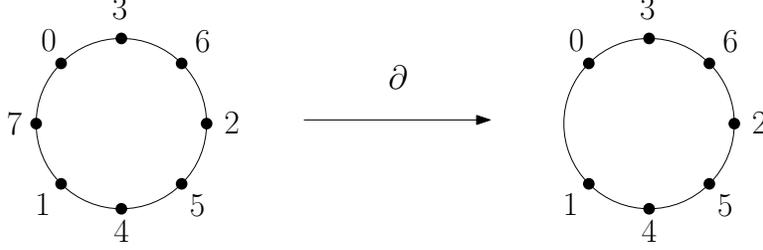


FIGURE 4. Illustration of the action of ∂ on a total order on $[7]$, yielding a total order on $[6]$.

Proof. Consider the map ∂ which to an element $Z \in A_{\underline{i}}$ associates the element $Z' \in A_{k,n-1}$ obtained by deleting the number n from the circle. For any $Z \in A_{\underline{i}}$ with $\underline{i} = (i_1, \dots, i_{k-1})$, the element Z belongs to some $A_{\underline{j}}$, where $\underline{j} = (j_1, \dots, j_{k-1})$ satisfies the following conditions:

$$(23) \quad 1 \leq j_1 \leq i_{k-1} - 1;$$

$$(24) \quad j_{k-1} = i_{k-2} + i_{k-1} - j_1 - 1;$$

$$(25) \quad j_m = i_{m-1} \text{ for any } 2 \leq m \leq k-2.$$

Furthermore, the map ∂ is a bijection between $A_{\underline{i}}$ and $\sqcup_{\underline{j}} A_{\underline{j}}$, where the union is taken over all the multi-indices \underline{j} satisfying conditions (23)-(25), because starting from any element Z' in some $A_{\underline{j}}$ with \underline{j} satisfying conditions (23)-(25), there is a unique way to add back the number n on the circle to obtain Z such that $L_Z(n-1, n) = i_{k-2}$. This concludes the proof. \square

We can use this to compute the cardinality of any $A_{k,n}$ inductively on n . We start at $n = k - 2$. In this case, the simplex T_{k-1}^{k-2} has a single element, and we start with the array consisting of a single entry equal to 1. Then we apply formula (22) to reach the desired value of n , and we take the sum of all the entries in the corresponding array of numbers.

Remark 8. *In the case $k = 3$, we recover the classical boustrophedon used to compute Entringer numbers (which are the numbers $a_{\underline{i}}$). The appearance of the operator Ω explains why each line is read alternatively from left to right or from right to left. For $k > 3$, the numbers $a_{\underline{i}}$ may*

thus be seen as higher-dimensional versions of the Entringer numbers and the numbers

$$(26) \quad a_{k,n} := \sum_{i \in T_{n+1}^{k-2}} a_i$$

may be seen as higher-dimensional Euler numbers (where the number k is the dimension parameter).

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DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE, BANGALORE
- 560012, INDIA

E-mail address: `arvind@iisc.ac.in`

LABORATOIRE D'INFORMATIQUE GASPARD MONGE, CNRS AND UNIVERSITÉ PARIS-
EST MARNE-LA-VALLÉE, FRANCE

E-mail address: `matthieu.josuut-verges@u-pem.fr`

UNITÉ DE MATHÉMATIQUES PURES ET APPLIQUÉES, ÉCOLE NORMALE SUPÉRIEURE
DE LYON, 46 ALLÉE D'ITALIE, 69364 LYON CEDEX 07, FRANCE

E-mail address: `sanjay.ramassamy@ens-lyon.fr`