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Laminations of a graph on a pair of pants

Sanjay Ramassamy

April 5, 2018

Abstract

A lamination of a graph embedded on a surface is a collection of pairwise disjoint non-contractible simple closed curves drawn on the graph. In the case when the surface is a sphere with three punctures (a.k.a. a pair of pants), we first identify the lamination space of a graph embedded on that surface as a lattice polytope, then we characterize the polytopes that arise as the lamination space of some graph on a pair of pants. This characterizes the image of a purely topological version of the spectral map for the vector bundle Laplacian for a flat connection on a pair of pants. The proof uses a graph exploration technique akin to the peeling of planar maps.

1 Introduction

The combinatorial study of the determinant of the vector bundle Laplacian on graphs was initiated by Forman [5] followed by Kenyon [7] as a generalization of the classical matrix-tree theorem [12]. While the (reduced) determinant of the usual Laplacian operator on a graph enumerates spanning trees of this graph, the determinant of the vector bundle Laplacian enumerates cycle-rooted spanning forests (CRSFs), which are spanning forests where each connected component is a unicycle (a connected graph with as many vertices as edges). The weight of a CRSF is the product over its cycles of a quantity related to the monodromy of the connection along each cycle.

Of particular interest is the case of a flat $SU(2, \mathbb{C})$ connection on a graph embedded on a surface [7], namely the case when the parallel transports are in $SU(2, \mathbb{C})$ and the monodromy of the connection along each cycle of the graph which is contractible on the surface has to be trivial. In that case, the only CRSFs which contribute to the determinant of the vector bundle Laplacian are those which have no contractible cycles. Such CRSFs are called incompressible CRSFs and the cycles of an incompressible CRSF form a *lamination* of the surface, i.e. a collection of pairwise disjoint non-contractible simple loops. The determinant of the vector bundle Laplacian in the flat connection case can be written as a polynomial in variables of the form $2 - Tr(w)$, where w is the monodromy along a non-contractible cycle on the surface [7]. Moreover these variables are free [3].

The most basic non-simply connected surfaces to consider are the annulus and the torus and was done in [7, 8, 6, 11, 9]. The next simplest case is probably the one of the pair of pants (aka three-holed sphere), briefly mentioned in [7]. It is one of the simplest surfaces for which the fundamental group is non abelian.

A non-contractible cycle on a pair of pants can be of three possible topological types, thus the determinant of the vector-bundle Laplacian associated with a flat $SU(2, \mathbb{C})$ connection on a graph embedded on that surface is a polynomial $P(X, Y, Z)$ of three independent variables.

The map which to a graph on a pair of pants associates the polynomial $P(X, Y, Z)$ is interesting to understand. We shall call it the *spectral map*, extending the terminology of the torus case [8] (this is a slight abuse of terminology, since the image of a graph under the spectral map should be the zero-locus of the polynomial together with a certain divisor on that algebraic variety [6]). Important questions include determining the image of the spectral map as well as the fiber of the spectral map above a given polynomial. This provides information on the probabilistic model associated with the uniform measure on incompressible CRSFs on the graph [7]. The polynomial $P(X, Y, Z)$ also plays an important role in relation with integrable systems, where it serves as the generating function of the integrals of motion [6]. The case of the annulus and the torus have been thoroughly investigated by Kenyon [7, 8]. For a different probabilistic model, the dimer model on bipartite graphs, the spectral map in the torus case is completely understood [10, 6, 4].

To any polynomial in n variables one can associate its Newton polytope, which is the convex hull in \mathbb{Z}^n of the n -tuples of integers (i_1, \dots, i_n) such that the monomial $X_1^{i_1} \dots X_n^{i_n}$ has a nonzero coefficient in the polynomial. We define the *topological spectral map*, which to a graph on a surface associates the Newton polytope of the polynomial produced by applying the spectral map. While the image under the spectral map depends on some weights that the edges of the graph may carry, the image under the topological spectral map only depend on the topological graph. The same questions can be asked about the topological spectral map : what is its image and what is the fiber above a given polytope ? These questions were answered in the case of the annulus and the torus [7, 8, 6]. In this article, we characterize the image of the topological spectral map for the pair of pants. The answer is much more involved than the annulus and torus cases. The next step would be to understand the fiber of this topological spectral map above a given polytope. Answering these questions for the spectral map itself in the pair of pants case seems to be much harder.

A monomial $X^i Y^j Z^k$ appears in $P(X, Y, Z)$ the determinant of the vector bundle Laplacian of a graph G on a pair of pants if and only if G has a lamination of type (i, j, k) , that is a lamination with i cycles around the first hole, j cycles around the second and k cycles around the third. Hence the image under the topological spectral map of G is the lamination space of G , i.e. the set of all (i, j, k) such that G admits a lamination of type (i, j, k) . The polytopes that arise in the image of the topological spectral map are exactly those that correspond to the lamination space of some graph on a pair of pants. The remainder of the paper will be formulated only in terms of laminations, no longer in terms of the determinant of the vector-bundle Laplacian, but the reader should keep in mind that the motivation behind this work comes from the spectral map associated with the vector-bundle Laplacian.

Organization of the paper

We introduce the relevant definitions and state our main results in Section 2. In Section 3 we describe an exploration process of a graph on a pair of pants and

use it to realize the lamination space of that graph as a polytope. In passing we define three collections of special loops. In Section 4 we investigate the structure of the intersection of two collections of special loops and deduce from this some necessary conditions for the polytopes arising as the lamination space of some graph. We show in Section 5 that these conditions are sufficient by constructing a class of graphs having as a lamination space a given polytope satisfying the aforementioned conditions.

2 Main results

We consider the three-holed sphere Σ obtained by removing from the sphere \mathbb{S}^2 three distinct points P_1, P_2 and P_3 . Every simple closed curve C on \mathbb{S}^2 which does not pass through the points P_i separates \mathbb{S}^2 into two hemispheres. For every $1 \leq i \leq 3$, we denote by $H_i(C)$ (resp. $H'_i(C)$) the connected component of $\mathbb{S}^2 \setminus C$ which contains P_i (resp. which does not contain P_i). A simple closed curve C is called *of type i* for $1 \leq i \leq 3$ if one of the hemispheres defined by C contains P_i and the other hemisphere contains the other two points, i.e. if

$$H_i(C) = H'_{i+1}(C) = H'_{i+2}(C).$$

In the previous equalities, as well as in the remainder of this article, the indices $1 \leq i \leq 3$ should be considered modulo 3. We will also denote by $\overline{H}_i(C)$ and $\overline{H}'_i(C)$ the closed hemispheres (containing C this time).

Let G be a connected nonempty graph embedded in \mathbb{S}^2 . The connected components of $\mathbb{S}^2 \setminus G$ are topological disks, they are called the faces of G and we denote by \mathcal{F} the set of faces of G . We say that G is a Σ -graph if there exist three distinct faces $(F_1, F_2, F_3) \in \mathcal{F}$ (called *marked faces*) such that P_i is in the interior of F_i for all $1 \leq i \leq 3$. A Σ -graph is more than just a graph embedded in Σ because we require that the graph actually separates the three punctures. A *lamination* of the Σ -graph G is a collection L of pairwise disjoint simple loops on G such that each loop in L is non-contractible on $\mathbb{S}^2 \setminus \{P_1, P_2, P_3\}$. By disjoint we mean having no vertex in common. For any non-negative integers m_1, m_2 and m_3 , a lamination is said to be *of type (m_1, m_2, m_3)* if for any $1 \leq i \leq 3$ it contains m_i loops of type i . The *lamination space* $\mathcal{L}(G)$ of a Σ -graph G is defined to be the set of all $(m_1, m_2, m_3) \in (\mathbb{Z}_+)^3$ such that G admit a lamination of type (m_1, m_2, m_3) . Below we will describe the lamination space of a given Σ -graph G as the integer points of a lattice polytope defined in terms of some geometric characteristics of G .

In order to simplify the inequalities defining the lamination space, we have allowed a lamination to be empty, in which case its topological type is $(0, 0, 0)$. Note however that the polynomials $P(X, Y, Z)$ arising in the image of the spectral map have no constant term, so the only difference between the image of a graph G under the topological spectral map and the lamination space of G will be the presence or absence of the point $(0, 0, 0)$.

We define a distance function d_G on \mathcal{F} such that any two faces sharing a vertex are at distance 1 for d_G . Let G^* be the dual graph of G (seen as a graph in \mathbb{S}^2). Construct \widetilde{G}^* by adding to G^* a dual edge between any two dual vertices such that the corresponding two primal faces share a primal vertex. The distance d_G is defined to be the usual graph distance on the vertex set of \widetilde{G}^* , which is canonically in bijection with \mathcal{F} . In the special case when all

the vertices of G have degree 3, then $\widetilde{G} = G^*$ and d_G is the classical distance between two faces corresponding to the graph distance on the dual graph. From now on, whenever we mention the distance between two faces of G , the distance function will implicitly be d_G .

Define $d_1(G) := d_G(F_2, F_3)$, $d_2(G) := d_G(F_1, F_3)$ and $d_3(G) := d_G(F_1, F_2)$. Also, for any $1 \leq i \leq 3$, define $M_i(G)$ to be the maximal number of pairwise disjoint simple loops of type i one can simultaneously draw on G . Given a Σ -graph G , we define the sextuple

$$s(G) := (M_1(G), M_2(G), M_3(G), d_1(G), d_2(G), d_3(G)) \in (\mathbb{Z}_+)^3 \times \mathbb{N}^3.$$

See Figure 1 for an example.

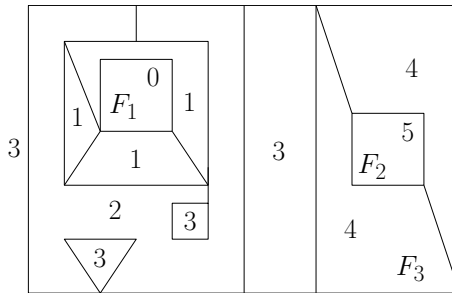


Figure 1: A Σ -graph G with each face labelled by its distance to the marked face F_1 . For this graph, $s(G) = (4, 1, 1, 1, 4, 5)$.

Given a sextuple of integers $\tau = (a, b, c, d, e, f) \in (\mathbb{Z}_+)^3 \times \mathbb{N}^3$, we define the convex lattice polytope \mathcal{P}_τ by

$$\mathcal{P}_\tau := \{(x, y, z) \in (\mathbb{Z}_+)^3 \mid x \leq a, y \leq b, z \leq c, y + z \leq d, x + z \leq e, x + y \leq f\}.$$

Proposition 2.1. *For any Σ -graph G , its lamination space $\mathcal{L}(G)$ is the polytope $\mathcal{P}_{s(G)}$.*

Proposition 2.1 is proved in Section 3.

Remark 2.2. The inequalities $m_i \leq M_i(G)$ are not redundant with the inequalities $m_i + m_{i+1} \leq d_{i+2}(G)$, as illustrated by Figure 2. On that picture, $d_1(G) = d_2(G) = d_3(G) = 2$ and $M_1(G) = M_2(G) = M_3(G) = 1$. The triple $(m_1, m_2, m_3) = (2, 0, 0)$ verifies the inequalities $m_i + m_{i+1} \leq d_{i+2}$, but that graph has no lamination of type $(2, 0, 0)$. This proposition corrects a statement made in [7], where the inequalities $m_i \leq M_i(G)$ were missing.

We can now characterize all the convex lattice polytopes that arise as the lamination space of some Σ -graph. By the previous proposition, it suffices to characterize the sextuples τ that arise as some $s(G)$.

Theorem 2.3. *Fix $\tau = (\mu_1, \mu_2, \mu_3, \delta_1, \delta_2, \delta_3) \in (\mathbb{Z}_+)^3 \times \mathbb{N}^3$. There exists a Σ -graph G such that $s(G) = \tau$ if and only if the following inequalities hold for all $1 \leq i \leq 3$:*

$$(T_1) \quad \max(\mu_{i+1}, \mu_{i+2}) \leq \delta_i \leq \mu_{i+1} + \mu_{i+2};$$

$$(T_2) \quad \min(\delta_{i+1} - \mu_{i+2}, \delta_{i+2} - \mu_{i+1}) + \left\lfloor \frac{\mu_{i+1} + \mu_{i+2} - \delta_i}{2} \right\rfloor \leq \mu_i.$$

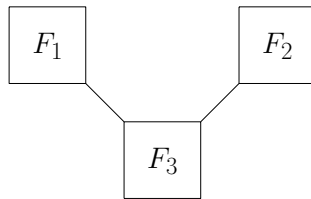


Figure 2: An example of a graph G illustrating the need to require the inequalities $m_i \leq M_i(G)$ in order to characterize the types of laminations that can arise.

The fact that conditions (T_1) and (T_2) are necessary is proved in Section 4, while the fact that they are sufficient is proved in Section 5 by explicitly constructing a Σ -graph G_τ such that $s(G_\tau) = \tau$ whenever τ satisfies the two conditions.

Remark 2.4. The second set of inequalities seems to suggest that the image of s in $(\mathbb{Z}_+)^3 \times \mathbb{N}^3$ may be a non-convex set. This is indeed the case, since $\tau_1 = (9, 6, 3, 9, 9, 12)$ and $\tau_2 = (9, 4, 9, 9, 9, 12)$ are both in the image of s , but $\frac{\tau_1 + \tau_2}{2}$ is not.

In order to prove Proposition 2.1 and the necessity of the conditions (T_1) and (T_2) in Theorem 2.3, we will explore any Σ -graph G starting from the face F_1 , then discover a first layer consisting of the faces at distance 1 from F_1 , then a second layer consisting of the faces at distance 2, etc. We will perform the same exploration starting from the faces F_2 and F_3 and understand how the boundaries of the layers arising in each of these three explorations interact with each other. In the case of simple triangulations, our construction is very similar to the layer decomposition of Krikun [13]. More generally, this construction resembles the peeling process for planar maps (see for example [2]). The difference is that here we use a distance which differs slightly from the graph distance on the dual graph. Instead of peeling an edge by discovering the face on the other side of the edge, we are peeling a vertex, by discovering all the unknown faces containing a vertex which is on the boundary of what we have already explored.

3 A collection of special loops around a puncture

In this section we first describe an exploration process of a Σ -graph G starting from a marked face, which will trace a collection of special loops on G centered around a marked face. Then we will use this exploration to prove Proposition 2.1.

3.1 Some special loops around a puncture

We start by an elementary observation, which we will be using several times. Let G be a connected planar graph and \tilde{G} be a subgraph of G . One defines the distance function $d_{\tilde{G}}$ on the set of the connected components of $\mathbb{S}^2 \setminus \tilde{G}$ in exactly the same way as the distance d_G was defined on the faces of G . Note that the

connected components of $\mathbb{S}^2 \setminus \tilde{G}$ do not have to be topological disks, they may be disks with multiple punctures or even the whole sphere if \tilde{G} is empty. Then we have the following result, the proof of which is easy and omitted.

Lemma 3.1. *Let G be a connected planar graph and \tilde{G} be a subgraph of G . Let F and F' be two faces of G and let \tilde{F} and \tilde{F}' be the two connected components of $\mathbb{S}^2 \setminus \tilde{G}$ containing respectively F and F' . Then $d_{\tilde{G}}(\tilde{F}, \tilde{F}') \leq d_G(F, F')$.*

Let G be a Σ -graph. For any $k \geq 0$ and $1 \leq i \leq 3$, define

$$A_i^k = \{F \in \mathcal{F} \mid d_G(F, F_i) = k\}. \quad (3.1)$$

For any $k \geq 1$ and $1 \leq i \leq 3$ such that A_i^k is nonempty, define B_i^k to be the boundary of the set $\bigcup_{j=0}^{k-1} A_i^j$ of faces that are at distance less than k to F_i . Each B_i^k is the union of simple loops that are not necessarily pairwise disjoint. The case when B_i^k consists in the union of several loops corresponds to a branching event in the peeling terminology, see e.g. [1]. The following lemma describes structural properties of the simple loops in B_i^k and will be used repeatedly in the remainder of the article.

Lemma 3.2. *Let G be a Σ -graph, let $1 \leq i \leq 3$ and let $k \geq 1$ be such that B_i^k is defined. Then, we have*

$$\bigcup_{j \geq k} A_i^j = \bigcup_{\substack{C \subset B_i^k \\ C \text{ simple loop}}} \overline{H'_i(C)}. \quad (3.2)$$

Furthermore, if C and C' are two distinct simple loops contained in B_i^k , then $\overline{H'_i(C)} \cap \overline{H'_i(C')} = \emptyset$. Finally, if $C \subset B_i^k$ is a simple loop, then the faces in $\overline{H'_i(C)}$ sharing an edge with C are in A_i^{k-1} .

Proof. Let $C \subset B_i^k$ be a simple loop and assume there is a face $F \subset \overline{H'_i(C)}$ such that $d_G(F_i, F) \leq k-1$. Then one can find a sequence (F^0, \dots, F^n) of faces such that $n = d_G(F_i, F)$, $F^0 = F_i$, $F^n = F$ and for every $1 \leq j \leq n$, the faces F^{j-1} and F^j share a vertex. By construction we have that for every $1 \leq j \leq n$, $d_G(F_i, F^j) = j$. Denoting by k' the largest j such that $F^j \subset \overline{H'_i(C)}$, we have that $d_G(F_i, F^{k'}) \leq k-2$ and by connectedness of the path of faces from the hemisphere $H_i(C)$ to the hemisphere $H'_i(C)$, $F^{k'}$ shares a vertex v with C . This yields a contradiction because all the faces containing the vertex v are at distance at most $k-1$ of F_i , hence B_i^k cannot pass through v so v cannot lie on C . We deduce from this that

$$\bigcup_{\substack{C \subset B_i^k \\ C \text{ simple loop}}} \overline{H'_i(C)} \subset \bigcup_{j \geq k} A_i^j.$$

Furthermore, by construction, every edge in B_i^k has on one side a face in A_i^{k-1} and on the other side a face in A_i^k . This implies that all the faces in $\overline{H'_i(C)}$ that contain an edge in C must be in A_i^{k-1} .

If C and C' are two distinct simple loops contained in B_i^k and $\overline{H'_i(C)} \cap \overline{H'_i(C')} \neq \emptyset$ then we can find a face F which satisfies one of the following two conditions :

1. $F \subset \overline{H_2}(C) \cap \overline{H'_2}(C')$ and F shares an edge with C ;
2. $F \subset \overline{H_2}(C') \cap \overline{H'_2}(C)$ and F shares an edge with C' .

This yields a contradiction because it implies on the one hand that $d_G(F, F_i) = k - 1$ and on the other hand that $d_G(F, F_i) \geq k$. Thus $H'_i(C)$ and $H'_i(C')$ must be disjoint.

Finally, let F be a face in A_i^j with $j \geq k$. We construct F^c the connected component of F in $\bigcup_{j \geq k} A_i^j$ as follows. We say that two faces in $\bigcup_{j \geq k} A_i^j$ are neighbors if they share an edge (not just a vertex) and F^c is the set of all faces in $\bigcup_{j \geq k} A_i^j$ that are reachable from F by walking across neighboring faces (these intermediate faces on the path must also lie in $\bigcup_{j \geq k} A_i^j$). Then F^c is a connected set, with boundary denoted by B^c . There exists a simple loop $C \subset B^c$ such that $F^c \subset \overline{H'_i}(C)$. By construction of B^c , for every edge e of C , the face adjacent to e in the hemisphere $\overline{H_i}(C)$ is in A_i^{k-1} and the face adjacent to e in the hemisphere $\overline{H'_i}(C)$ is in A_i^k . Hence $C \subset B_i^k$. We conclude that

$$\bigcup_{j \geq k} A_i^j \subset \bigcup_{\substack{C \subset B_i^k \\ C \text{ simple loop}}} \overline{H'_i}(C).$$

□

Fix $1 \leq i \leq 3$ and recall that indices $i + 1$ and $i + 2$ are considered modulo 3. Since $d_G(F_i, F_{i+1}) = d_{i+2}(G)$, Lemma 3.2 implies for any $1 \leq k \leq d_{i+2}(G)$ the existence of a unique simple loop $C_{i,i+1}^k \subset B_i^k$ such that $F_{i+1} \subset \overline{H'_i}(C_{i,i+1}^k)$. Similarly, for any $1 \leq k \leq d_{i+1}(G)$ there exists a unique simple loop $C_{i,i+2}^k \subset B_i^k$ such that $F_{i+2} \subset \overline{H'_i}(C_{i,i+2}^k)$. For $1 \leq k \leq \min(d_{i+1}(G), d_{i+2}(G)) - 1$, if $C_{i,i+1}^k \neq C_{i,i+2}^k$, then $C_{i,i+1}^{k+1} \neq C_{i,i+2}^{k+1}$ by Lemma 3.2. Thus there exists a unique integer $\widetilde{M}_i \geq 0$ such that for any $1 \leq k \leq \widetilde{M}_i$ we have $C_{i,i+1}^k = C_{i,i+2}^k$ and for any $\widetilde{M}_i + 1 \leq k \leq \min(d_{i+1}(G), d_{i+2}(G))$ we have $C_{i,i+1}^k \neq C_{i,i+2}^k$. If $1 \leq k \leq \widetilde{M}_i$ we denote simply by C_i^k the simple loop $C_{i,i+1}^k = C_{i,i+2}^k$. The following lemma gives the value of \widetilde{M}_i .

Lemma 3.3. *Let G be a Σ -graph. For any $1 \leq i \leq 3$, we have $\widetilde{M}_i = M_i(G)$. Furthermore, for any fixed $1 \leq i \leq 3$, the loops $(C_i^k)_{1 \leq k \leq M_i(G)}$ are pairwise disjoint.*

Proof. Fix $1 \leq i \leq 3$. Consider C_i^k and $C_i^{k'}$ for some $1 \leq k < k' \leq \widetilde{M}_i$. First observe that these loops are nested, i.e. $H_i(C_i^k) \subset H_i(C_i^{k'})$. If there exists a vertex v lying on both these loops, denoting by e an edge of C_i^k containing v and by F the face containing e and lying in $\overline{H_i}(C_i^k)$, it follows from Lemma 3.2 that $d_G(F, F_i) = k - 1$, which contradicts the fact that F shares the vertex v with some face F' lying in $\overline{H'_i}(C_i^{k'})$ because $d_G(F', F_i) \geq k' > k$ again by Lemma 3.2. So the loops $(C_i^k)_{1 \leq k \leq \widetilde{M}_i}$ are pairwise disjoint.

It remains to prove that $\widetilde{M}_i = M_i(G)$. Since the simple loops C_i^k of type i are pairwise disjoint for $1 \leq k \leq \widetilde{M}_i$, their union constitutes a lamination with \widetilde{M}_i loops of type i hence $\widetilde{M}_i \leq M_i(G)$.

Let L be a lamination consisting in $M_i(G)$ simple loops of type i denoted by C^k , $1 \leq k \leq M_i(G)$, which are nested in such a way that for any $1 \leq k \leq$

$M_i(G) - 1$, $H_i(C^k) \subset H_i(C^{k+1})$. If F is a face in $\overline{H}_i^j(C^k)$ for some $1 \leq k \leq M_i(G)$, then by Lemma 3.1, we have that $d_G(F_i, F) \geq k$ hence

$$\overline{H}_i^j(C^k) \subset \bigcup_{j \geq k} A_i^j. \quad (3.3)$$

This implies that

$$\bigcup_{j=0}^{k-1} A_i^j \subset \overline{H}_i(C^k).$$

Recalling that B_i^k is defined as the boundary of $\bigcup_{j=0}^{k-1} A_i^j$, we deduce that B_i^k is well-defined for all $1 \leq k \leq M_i(G)$ and that $B_i^k \subset \overline{H}_i(C^k)$. Since C^k is of type i , this implies that any simple loop contained in B_i^k is either contractible or of type i . So $k \leq \widetilde{M}_i$. This statement holds for every $1 \leq k \leq M_i(G)$ so $M_i(G) \leq \widetilde{M}_i$. \square

Remark 3.4. For any $1 \leq k \leq M_i(G)$, the simple loop C_i^k is of type i , for any $M_i(G) + 1 \leq k \leq d_{i+1}(G)$ the simple loop $C_{i,i+1}^k$ is of type $i + 1$ and for any $M_i(G) + 1 \leq k \leq d_{i+2}(G)$ the simple loop $C_{i,i+2}^k$ is of type $i + 2$.

The loops C_i^k are called *special loops of type i* . These special loops are optimal if one wants to pack as many disjoint simple loops of given type as possible. For example C_i^1 is the “tightest” simple loop of type i one can draw, C_i^2 is the “tightest” simple loop of type i one can draw which would be disjoint from C_i^1 , etc. See Figure 3 for an illustration.

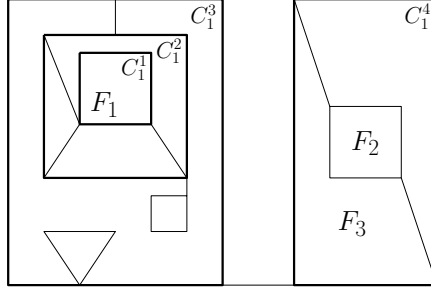


Figure 3: Representation in bold of the special loops C_1^k for the graph G of Figure 1.

3.2 Proof of Proposition 2.1

Proof. First, assume that G has a lamination L of type (m_1, m_2, m_3) . We will show that this triple of integers satisfies the six inequalities defining $\mathcal{P}_s(G)$. By symmetry, it suffices to prove the inequalities

$$m_1 \leq M_1(G) \quad (3.4)$$

and

$$m_2 + m_3 \leq d_1(G). \quad (3.5)$$

Inequality (3.4) directly follows from the definition of $M_1(G)$.

Let F'_2 (resp. F'_3) be the connected component of $\mathbb{S}^2 \setminus L$ containing P_2 (resp. P_3). Then, since L is a subgraph of G , by Lemma 3.1 we have that

$$d_L(F'_2, F'_3) \leq d_G(F_2, F_3). \quad (3.6)$$

Inequality (3.5) follows from the observation that $m_2 + m_3 = d_L(F'_2, F'_3)$.

Conversely, assume that we have a triple of integers (m_1, m_2, m_3) satisfying the six inequalities defining $\mathcal{P}_s(G)$. Set

$$L = \{C_1^1, \dots, C_1^{m_1}, C_2^1, \dots, C_2^{m_2}, C_3^1, \dots, C_3^{m_3}\}.$$

First, observe that for every $1 \leq i \leq 3$, C_i^k is well-defined because $k \leq m_i \leq M_i(G)$. It suffices to prove that these loops are pairwise disjoint to conclude that L is a lamination, its type will then be (m_1, m_2, m_3) by construction.

By Lemma 3.3, two loops of the same type are disjoint, so we only need to prove that loops of different types are also disjoint. We argue by contradiction and assume for example that we have $C_2^j \cap C_3^k \neq \emptyset$ for some $1 \leq j \leq m_2$ and $1 \leq k \leq m_3$. Then one can find two faces F and F' such that $F \subset A_2^{j-1}$, $F' \subset A_3^{k-1}$ and $d_G(F, F') = 1$. By the triangle inequality,

$$d_G(F_2, F_3) \leq d_G(F_2, F) + d_G(F, F') + d_G(F', F_3),$$

which can be rewritten as $d_1(G) \leq j + k - 1$. This contradicts the inequality $m_2 + m_3 \leq d_1(G)$. We conclude that L is a lamination of type (m_1, m_2, m_3) . \square

4 Intersection of two collections of special loops

In this section we will describe the structure of the intersection of two collections of special loops centered around two different marked faces. We will then use the knowledge of this structure to prove that conditions (T_1) and (T_2) in Theorem 2.3 are necessary conditions. In order to alleviate notation, we will drop the dependency of M_i and d_i on G in this section.

4.1 Structure of the intersection

We will describe how two collections of special loops of two different types intersect each other. Without loss of generality, we consider the collections of the C_1^k 's and the C_2^j 's. Actually, we will consider the broader collection of loops, $C_{1,2}^k$ and $C_{2,1}^j$ defined before Lemma 3.3.

Lemma 4.1. *Fix two integers $1 \leq j \leq d_3$ and $1 \leq k \leq d_3$. Then $C_{1,2}^k \cap C_{2,1}^j = \emptyset$ if and only if $j + k \leq d_3$. Furthermore, $H_1(C_{1,2}^k) \cap H_2(C_{2,1}^{d_3+1-k}) = \emptyset$.*

Proof. Assume that $C_{1,2}^k \cap C_{2,1}^j = \emptyset$. Let $L_{j,k}$ be the lamination consisting in the loops $(C_{1,2}^{k'})_{1 \leq k' \leq k}$ and $(C_{2,1}^{j'})_{1 \leq j' \leq j}$. These loops are indeed pairwise disjoint because $C_{1,2}^{k'} \subset \overline{H_1}(C_{1,2}^k)$ if $k' \leq k$, $C_{2,1}^{j'} \subset \overline{H_2}(C_{2,1}^j)$ if $j' \leq j$ and $\overline{H_2}(C_{2,1}^j) \subset \overline{H_1}(C_{1,2}^k)$. Applying Lemma 3.1 to the subgraph $L_{j,k}$ of G and to the connected components of $\mathbb{S}^2 \setminus L_{j,k}$ containing respectively P_1 and P_2 , we deduce that $j + k \leq d_3$.

Conversely, assume that $C_{1,2}^k \cap C_{2,1}^j \neq \emptyset$. Let v be a vertex in $C_{1,2}^k \cap C_{2,1}^j$, e be an edge in $C_{1,2}^k$ containing v , e' be an edge in $C_{2,1}^j$, F be the face in $\overline{H_1}(C_{1,2}^k)$ containing e and F' be the face in $\overline{H_2}(C_{2,1}^j)$ containing e' . Then by Lemma 3.2, $d_G(F_1, F) = k - 1$ and $d_G(F_2, F') = j - 1$. Since F and F' share the vertex v , we also have $d_G(F, F') = 1$. By the triangle inequality, we conclude that $d_G(F_1, F_2) \leq j + k - 1$. Thus $j + k > d_3$.

Finally, assume that $H_1(C_{1,2}^k) \cap H_2(C_{2,1}^{d_3+1-k}) \neq \emptyset$. Then we can find a face $F \subset \overline{H_1}(C_{1,2}^k) \cap \overline{H_2}(C_{2,1}^{d_3+1-k})$ which has at least one edge in common with $C_{2,1}^j$. By Lemma 3.2, we have $F \in A_2^{d_3-k}$ so $B_2^{d_3-k}$ intersects F (this intersection may be just a single vertex). On the other hand, it follows from Lemma 3.2 that $A_2^{d_3-k-1} \subset H_2(C_{2,1}^{d_3-k})$ so $B_2^{d_3-k} \subset \overline{H_2}(C_{2,1}^{d_3-k})$. Thus we obtain that $\overline{H_1}(C_{1,2}^k) \cap \overline{H_2}(C_{2,1}^{d_3-k}) \neq \emptyset$, and the previous paragraph would entail that $k + (d_3 - k) > d_3$. This is the desired contradiction, hence $H_1(C_{1,2}^k) \cap H_2(C_{2,1}^{d_3+1-k}) = \emptyset$. \square

In the remainder of this section, we prove one direction of Theorem 2.3. Let G be a Σ -graph. We will show that the six components of $s(G)$ satisfy the inequalities (T_1) and (T_2) of Theorem 2.3. By symmetry it suffices to consider the case $i = 1$.

4.2 Inequalities (T_1) are verified

The inequalities $M_2 \leq d_1$ and $M_3 \leq d_1$ follow from Lemma 3.3, thus

$$\max(M_2, M_3) \leq d_1.$$

To prove the other inequality, we distinguish several cases.

Case when $M_2 \geq 1$, $M_3 \geq 1$ and $C_2^{M_2} \cap C_3^{M_3} \neq \emptyset$.

Let v be a vertex in that intersection. Then one can find two faces F and F' containing v and such that $F \subset \overline{H_2}(C_2^{M_2})$, F shares an edge with $C_2^{M_2}$, $F' \subset \overline{H_3}(C_3^{M_3})$ and F' shares an edge with $C_3^{M_3}$. By the triangle inequality and Lemma 3.2 we have

$$d_G(F_2, F_3) \leq d_G(F_2, F) + d_G(F, F') + d_G(F', F_3) \leq (M_2 - 1) + 1 + (M_3 - 1),$$

thus, $d_1 \leq M_2 + M_3 - 1$ in that case.

Case when $M_2 \geq 1$, $M_3 \geq 1$ and $C_2^{M_2} \cap C_3^{M_3} = \emptyset$.

In that case $B_3^{M_3}$ is non-empty and there exists a simple loop $C \subset B_3^{M_3}$ such that $\overline{H_2}(C_2^{M_2}) \subset \overline{H_2}(C)$. Since C is disjoint from all the C_3^k with $1 \leq k \leq M_3$, it cannot be of type 3 (this would contradict the fact that M_3 is the maximal number of disjoint simple loops of type 3), thus $F_1 \subset \overline{H_3}(C)$. So C is of type 2, hence has to intersect $C_2^{M_2}$, otherwise this would contradict the fact that M_2 is the maximal number of disjoint simple loops of type 2. Considering the two non-disjoint simple loops $C \subset B_3^{M_3}$ and $C_2^{M_2} \subset B_2^{M_2}$, one concludes by selecting two appropriate faces F and F' as in the previous case and applying Lemma 3.2, which yields $d_1 \leq M_2 + M_3$.

Case when $M_2 = 0$ or $M_3 = 0$.

We first show that M_2 and M_3 cannot be both zero.

Lemma 4.2. *If $M_2 = 0$ then $M_1 \geq 1$ and $M_3 \geq 1$.*

Proof. Assume that $M_2 = 0$. The boundary B_2^1 of F_2 is nonempty even though it contains no simple loop of type 2. Since $d_G(F_3, F_2) \geq 1$ and $d_G(F_1, F_2) \geq 1$, by Lemma 3.2, there exist two simple loops C and C' contained in B_2^1 such that $F_3 \subset \overline{H_2'(C)}$ and $F_1 \subset \overline{H_2'(C')}$. Furthermore, $C \neq C'$ otherwise C would be of type 2. By Lemma 3.2, this implies that $H_2'(C) \cap H_2'(C') = \emptyset$, so C is a simple loop of type 3 and C' is a simple loop of type 1. Thus $M_3 \geq 1$ and $M_1 \geq 1$. \square

In the remainder of the proof we assume that $M_2 = 0$. As in the proof of Lemma 4.2, pick $C \subset B_2^1$ a simple loop of type 3. Then the special loop $C_3^{M_3}$ must intersect C in at least a vertex v , otherwise C would be an $(M_3 + 1)$ -st simple loop of type 3, being disjoint from all the special loops C_3^k with $1 \leq k \leq M_3$. Let e be an edge of $C_3^{M_3}$ containing the vertex v and let F be the face in $\overline{H_3(C_3^{M_3})}$ containing e . Then $d_G(F_3, F) = M_3 - 1$ and $d_G(F, F_2) = 1$ so $d_1 = d_G(F_2, F_3) \leq M_3$. Note that in this case we have equality, since $\max(0, M_3) \leq d_1 \leq 0 + M_3$.

4.3 Inequality (T_2) is verified

We begin by examining a special configuration of simple loops, which will then serve as a prototype for the general case.

Lemma 4.3. *Let D_2^1, \dots, D_2^n (resp. D_3^1, \dots, D_3^n) be n nested simple loops of type 2 (resp. of type 3), in the sense that for any $1 \leq k \leq n - 1$ we have $H_2(D_2^{k-1}) \subset H_2(D_2^k)$ (resp. $H_3(D_3^{k-1}) \subset H_3(D_3^k)$). Assume that for any $1 \leq k, k' \leq n$ we have $H_2(D_2^k) \cap H_3(D_3^{k'}) \neq \emptyset$. Then there exist n pairwise disjoint simple loops D_1^1, \dots, D_1^n of type 1 such that for any $1 \leq k \leq n$, $D_1^k \subset D_2^k \cup D_3^k$.*

See Figure 4 for an illustration.

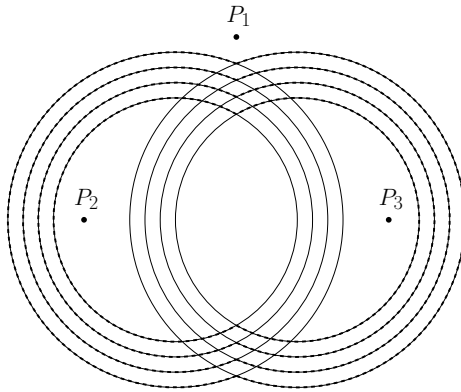


Figure 4: Illustration of the setting of Lemma 4.3. Here $n = 4$, the loops D_2^k (resp. D_3^k) are represented as concentric circles around P_2 (resp. P_3). The loops D_1^k are superimposed in bold dashed stroke.

Proof. Fix $1 \leq k \leq n$. The set $S_k := H_2(D_2^k) \cup H_3(D_3^k)$ is connected and contains P_2 and P_3 but not P_1 , thus the boundary of S_k contains a simple loop D_1^k of type 1. Since the boundary of S_k is contained in $D_2^k \cup D_3^k$, we have that $D_1^k \subset D_2^k \cup D_3^k$. Furthermore, the portion of D_1^k contained in D_2^k lies in $\overline{H_3}(D_3^k)$, hence is disjoint from every $D_3^{k'}$ with $1 \leq k' < k$. By pairwise disjointness of the D_2^j 's, the portion of D_1^k contained in D_2^k is also disjoint from every $D_2^{k'}$ with $1 \leq k' < k$. Similarly, the portion of D_1^k contained in D_3^k is disjoint from every $D_2^{k'}$ and $D_3^{k'}$ with $1 \leq k' < k$. Since $D_1^{k'} \subset D_2^{k'} \cup D_3^{k'}$, we conclude that D_1^k is disjoint from $D_1^{k'}$ for any $1 \leq k' < k$. \square

We apply Lemma 4.3 to the following simple loops. Set $n := \lfloor \frac{M_2 + M_3 - d_1}{2} \rfloor$. For any $1 \leq k \leq n$, write $D_2^k := C_2^{M_2+k-n}$ and $D_3^k := C_3^{M_3+k-n}$. First, note that $M_2 - n \geq 0$ and $M_3 - n \geq 0$ by inequalities (T_1) . Next, for any $1 \leq k, k' \leq n$, we have

$$(M_2 + k - n) + (M_3 + k' - n - 1) > d_1. \quad (4.1)$$

To prove that (4.1) holds true, it suffices to check it for $k = k' = 1$, in which case it is equivalent to

$$\left\lfloor \frac{M_2 + M_3 - d_1}{2} \right\rfloor < \frac{M_2 + M_3 - d_1}{2} - \frac{1}{2},$$

and the latter inequality is true since $M_2 + M_3 - d_1$ is an integer. Applying Lemma 4.1, we deduce from (4.1) that

$$C_2^{M_2+k-n} \cap C_3^{M_3+k'-n-1} \neq \emptyset.$$

Thus

$$H_2(C_2^{M_2+k-n}) \cap H_3(C_3^{M_3+k'-n-1}) \neq \emptyset,$$

which entails that the loops D_2^k and D_3^k satisfy the assumption of Lemma 4.3. Hence we obtain from this lemma n pairwise disjoint loops of type 1 D_1^1, \dots, D_1^n which are contained in the union of the loops C_2^k with $1 \leq k \leq M_2$ and C_3^k with $1 \leq k \leq M_3$.

Set $\nu := \min(d_2 - M_3, d_3 - M_2)$. Then $\mu + M_2 \leq d_3$, thus by Lemma 4.1, the simple loops C_1^j for $1 \leq j \leq \nu$ are disjoint from the loops C_2^k for $1 \leq k \leq M_2$. Similarly, the loops C_1^j for $1 \leq j \leq \nu$ are disjoint from the loops C_3^k for $1 \leq k \leq M_3$.

Putting everything together, the simple loops $D_1^1, \dots, D_1^n, C_1^1, \dots, C_1^\nu$ are pairwise disjoint and of type 1, thus $n + \nu \leq M_1$, which is the inequality (T_2) .

5 A class of graphs achieving any $s(G)$

In this section, given $\tau = (\mu_1, \mu_2, \mu_3, \delta_1, \delta_2, \delta_3) \in (\mathbb{Z}_+)^3 \times \mathbb{N}^3$ satisfying inequalities (T_1) and (T_2) , we construct a graph G_τ such that $s(G_\tau) = \tau$. The graphs G_τ will be constructed by gluing together several building blocks, most of which will be Young diagrams. Recall that the Young diagram $Y_{(\lambda_1, \dots, \lambda_n)}$ associated with the partition $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 1$ is (in French notation) the diagram consisting in n rows of left-aligned square boxes where the i -th row counted from the bottom contains λ_i boxes.

For any $1 \leq i \leq 3$, set $\nu_{i,i+1} := \mu_i + \mu_{i+1} - \delta_{i+2} - 1$. By the rightmost inequality in (T_1) , $\nu_{i,i+1} \geq -1$ for all $1 \leq i \leq 3$. For the sake of clarity, we distinguish three cases.

5.1 Case when $\mu_i \geq 1$ and $\nu_{i,i+1} \geq 0$ for every i

We first construct a graph Γ_τ which has the topology of the disk. We will then glue two identical copies of Γ_τ along their boundaries, like the construction of a pillowcase, which is topologically a sphere. However, we will only glue three disjoint arcs of the boundary of one graph with three disjoint arcs of the boundary of the other graph, hence the result will be a three-holed sphere.

We first define the following building blocks :

- the *connector* K , which is a triangle with edges called (in cyclic order) e'_1 , e'_2 and e'_3 . See Figure 5a.
- for every $1 \leq i \leq 3$, the Young diagram $Y_{(\mu_{i-1})}$ consisting in a single row, called a *leg*. We denote its vertical left edge by E_i , its vertical right edge by e_i , its bottom (resp. top) horizontal edges from right to left by $f_{i,i+1}^1, \dots, f_{i,i+1}^{\mu_i-1}$ (resp. $f_{i,i-1}^1, \dots, f_{i,i-1}^{\mu_i-1}$). See Figure 5b.
- for every $1 \leq i \leq 3$, the Young diagram $Y_{(\nu_{i,i+1}, \nu_{i,i+1}-1, \dots, 2, 1)}$ consisting in $\nu_{i,i+1}$ rows, called a *web*. We denote its horizontal edges on the bottom boundary from left to right by $f'_{i,i+1}^1, \dots, f'^{\nu_{i,i+1}}_{i,i+1}$ and its vertical edges on the left boundary from bottom to top by $f'_{i+1,i}^1, \dots, f'^{\nu_{i,i+1}}_{i+1,i}$. See Figure 5c.

Next, for every $1 \leq i \leq 3$, we glue the edges e_i with e'_i and for every $1 \leq k \leq \nu_{i,i+1}$ we glue $f_{i,i+1}^k$ with $f'^k_{i,i+1}$ and $f_{i+1,i}^k$ with $f'^k_{i+1,i}$. After gluing two edges together, the result is a single edge. See Figure 6 for an example. We call the resulting graph Γ_τ . It has the topology of the disk, with three distinguished edges E_1 , E_2 and E_3 on its boundary. Note that the fact that $\nu_{i,i+1} \geq \max(\mu_i, \mu_{i+1})$ is exactly the leftmost inequality of (T_1) .

Let Γ'_τ be an identical copy of Γ_τ . Each edge of the boundary of Γ'_τ is in canonical correspondence with an edge of the boundary of Γ_τ . In particular, Γ'_τ has three distinguished edges E'_1 , E'_2 and E'_3 on its boundary. We glue together each pair of corresponding edges, except the three pairs containing the distinguished edges. We call the resulting graph G_τ . For every $1 \leq i \leq 3$ we denote by F_i the digon with edges E_i and E'_i .

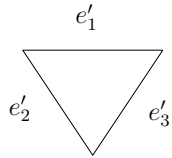
The components of $s(G_\tau)$ are computed in a straightforward fashion using the layer by layer exploration technique developed in Subsection 3.1 and one obtains that $s(G_\tau) = \tau$ in this case.

5.2 Case when $\mu_i \geq 1$ for every i and $\nu_{j,j+1} = -1$ for some j

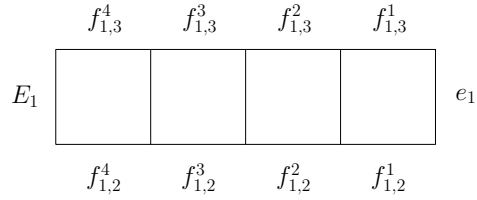
For each j such that $\nu_{j,j+1} = -1$, the connector K acquires an extra edge, placed in positive cyclic order between edge e'_j and edge e'_{j+1} , and the corresponding web is empty. For example, if $\nu_{1,2} = \nu_{2,3} = -1$ and $\nu_{3,1} \geq 0$, then the connector K is the pentagon depicted in Figure 7. From there, one performs exactly the same gluings as previously to construct Γ_τ (the extra edges of the connector are also glued in pairs) then G_τ and the conclusion follows similarly.

5.3 Case when $\mu_i = 0$ for some i

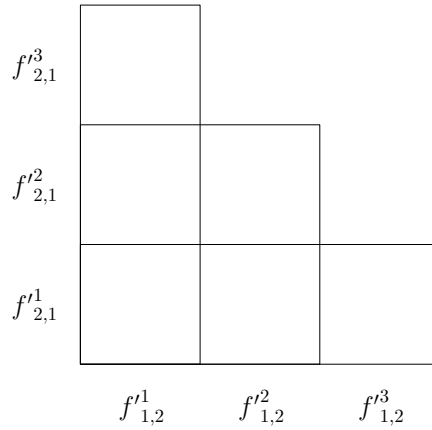
Without loss of generality we assume that $\mu_1 = 0$. Since $1 \leq \delta_2 \leq \mu_1 + \mu_3$ by inequality (T_1) , we have $\mu_3 \geq 1$. Similarly $\mu_2 \geq 1$. Furthermore, (T_1) also



(a) The connector K is a triangle when $\nu_{i,i+1} \geq 0$ for all $1 \leq i \leq 3$.



(b) The leg corresponding to $i = 1$ when $\mu_1 = 5$.



(c) The web between 1 and 2 when $\nu_{1,2} = 3$.

Figure 5: The different types of building blocks for the graph Γ_τ when $\mu_i \geq 1$ and $\nu_{i,i+1} \geq 0$ for every i .

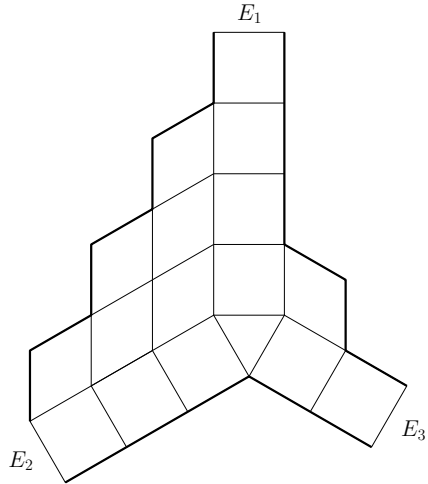


Figure 6: The graph Γ_τ for $\tau = (5, 4, 3, 6, 6, 5)$. In that case, $\nu_{1,2} = 3$, $\nu_{2,3} = 0$ and $\nu_{3,1} = 1$. The arcs of the boundary that will be glued to the corresponding arcs of an identical copy appear in bold stroke.

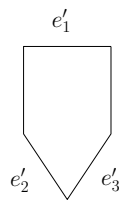


Figure 7: The connector K is a pentagon when $\nu_{1,2} = \nu_{3,1} = -1$ and $\nu_{2,3} \geq 0$.

implies that $\delta_2 = \mu_3$ and $\delta_3 = \mu_2$. Combining (T_1) and (T_2) both taken for $i = 1$, one deduces that $\delta_1 \in \{\mu_2 + \mu_3, \mu_2 + \mu_3 - 1\}$.

The case $\delta_1 = \mu_2 + \mu_3$ is achieved by a graph with two disjoint collections of simple loops, μ_2 around P_2 and μ_3 around P_3 and adding a loop separating the two collections and intersecting the outermost loop of each collection at exactly one vertex, as depicted in Figure 8.

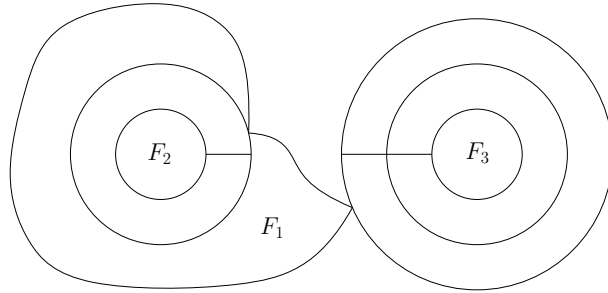


Figure 8: The graph G_τ for $\tau = (0, 2, 3, 5, 3, 2)$.

The case $\delta_1 = \mu_2 + \mu_3 - 1$ is achieved by a graph with two collections of simple loops, μ_2 around P_2 and μ_3 around P_3 such that the outermost loops of each collection intersect at a single vertex. Then one adds a loop surrounding both collections and intersecting only one of the outermost loops, with the intersection being a single vertex, as depicted in Figure 9.

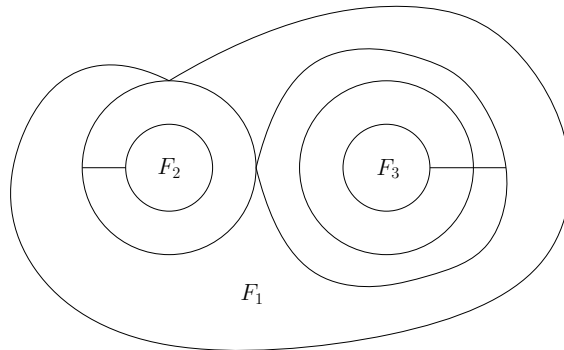


Figure 9: The graph G_τ for $\tau = (0, 2, 3, 4, 3, 2)$.

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