# On constrictions of phase-lock areas in model of overdamped Josephson effect and transition matrix of double confluent Heun equation

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## Abstract

In 1973 B.Josephson received Nobel Prize for discovering a new fundamental effect concerning a *Josephson junction,* – a system of two superconductors separated by a very narrow dielectric: there could exist a supercurrent tunneling through this junction. We will discuss the model of the overdamped Josephson junction, which is given by a family of first order non-linear ordinary differential equations on two-torus depending on three parameters: a fixed parameter $\omega$ (the *frequency*); a pair of variable parameters $(B, A)$ that are called respectively the *abscissa,* and the *ordinate.* It is important to study the rotation number of the system as a function $\rho = \rho(B, A)$ and to describe the *phase-lock areas:* its level sets $L_r = \{\rho = r\}$ with non-empty interiors. They were studied by V.M.Buchstaber, O.V.Karpov, S.I.Tertychnyi, who observed in 2010 that the phase-lock areas exist only for integer values of the rotation number. It is known that each phase-lock area is a garland of infinitely many bounded domains going to infinity in the vertical direction; each two subsequent domains are separated by one point, which is called *constriction* (provided that it does not lie in the abscissa axis). Those points of intersection of the boundary $\partial L_r$ of the phase-lock area $L_r$ with the line $\Lambda_r = \{B = r\omega\}$ (which is called its *axis*) that are not constrictions are called *simple intersections.* It is known that our family of dynamical systems is related to appropriate family of double confluent Heun equations with the same parameters via Buchtaber–Tertychnyi construction. Simple intersections correspond

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1

to some of those parameter values for which the corresponding "conjugate" double confluent Heun equation has a polynomial solution (follows from results of a joint paper of V.M.Buchstaber and S.I.Tertychnyi and a joint paper of V.M.Buchstaber and the author). There is a conjecture stating that *all the constrictions of every phase-lock area $L_r$ lie in its axis $\Lambda_r$.* This conjecture was studied and partially proved in a joint paper of the author with V.A.Kleptsyn, D.A.Filimonov and I.V.Schurov. Another conjecture states that for any two subsequent constrictions in $L_r$ with positive ordinates the interval between them also lies in $L_r$. In this paper we present new results partially confirming both conjectures. The main result states that the phase-lock area $L_r$ contains the infinite interval of the axis $\Lambda_r$ issued upwards from the point of intersection $\partial L_r \cap \Lambda_r$ with the biggest possible ordinate that is not a constriction. The proof is done by studying the complexification of the system under question, which is the projectivization of a family of systems of second order linear equations with two irregular non-resonant singular points at zero and at infinity. We obtain new results on the transition matrix between appropriate canonical solution bases of the linear system; on its behavior as a function of parameters. A key result, which implies the main result of the paper, states that the off-diagonal terms of the transition matrix are both non-zero at each constriction. We reduce the above conjectures on constrictions to the conjecture on negativity of the ratio of the latter off-diagonal terms at each constriction.

# Contents

# 1   Introduction

## 1.1   Phase-lock areas in Josephson effect: history, main conjectures and main results

We study the family

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t, \ \omega > 0, \ B \geq 0. \tag{1.1}$$

of *nonlinear* equations, which arises in several models in physics, mechanics and geometry. Our main motivation is that it describes the overdamped model of the Josephson junction (RSJ - model) in superconductivity, see [24, 33, 28, 4, 29]. It arises in planimeters, see [18, 19]. Here $\omega$ is a fixed constant, and $(B, A)$ are the parameters. Set

$$\tau = \omega t, \ l = \frac{B}{\omega}, \ \mu = \frac{A}{2\omega}.$$

The variable change $t \mapsto \tau$ transforms (1.1) to a non-autonomous ordinary differential equation on the two-torus $\mathbb{T}^2 = S^1 \times S^1$ with coordinates $(\phi, \tau) \in \mathbb{R}^2/2\pi\mathbb{Z}^2$:

$$\dot{\phi} = \frac{d\phi}{d\tau} = -\frac{\sin\phi}{\omega} + l + 2\mu\cos\tau. \tag{1.2}$$

The graphs of its solutions are the orbits of the vector field

$$\begin{cases} \dot{\phi} = -\frac{\sin\phi}{\omega} + l + 2\mu\cos\tau \\ \dot{\tau} = 1 \end{cases} \tag{1.3}$$

3

on $\mathbb{T}^2$. The *rotation number* of its flow, see [1, p. 104], is a function $\rho(B, A)$ of parameters[1]:

$$\rho(B, A; \omega) = \lim_{k \to +\infty} \frac{\phi(2\pi k)}{2\pi k}.$$

Here $\phi(\tau)$ is a general $\mathbb{R}$-valued solution of the first equation in (1.3) whose parameter is the initial condition for $\tau = 0$. Recall that the rotation number is independent on the choice of the initial condition, see [1, p.104]. The parameter $B$ is called *abscissa,* and $A$ is called the *ordinate.* Recall the following well-known definition.

**Definition 1.1** (cf. [20, definition 1.1]) The *r-th phase-lock area* is the level set

$$L_r = \{\rho(B, A) = r\} \subset \mathbb{R}^2,$$

provided that it has a non-empty interior.

**Remark 1.2 : phase-lock areas and Arnold tongues.** H.Poincaré introduced the rotation number of a circle diffeomorphism. The rotation number of the flow of the field (1.3) on $\mathbb{T}^2$ equals (modulo $\mathbb{Z}$) the rotation number of the circle diffeomorphism given by its time $2\pi$ flow mapping restricted to the cross-section $S_\phi^1 \times \{0\}$. In Arnold family of circle diffeomorphisms $x \mapsto x + b + a \sin x$, $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ the behavior of its phase-lock areas for small $a$ demonstrates the tongues effect discovered by V.I. Arnold [1, p. 110]. That is why the phase-lock areas became "Arnold tongues", see [20, definition 1.1].

Recall that the rotation number has physical meaning of the mean voltage over a long time interval. The phase-lock areas of the family (1.2) were studied by V.M.Buchstaber, O.V.Karpov, S.I.Tertychnyi et al, see [6]–[16], [26], [20] and references therein. It is known that the following statements hold:

1) Phase-lock areas exist only for integer values of the rotation number (a "quantization effect" observed in [11] and later also proved in [22, 21]).

2) The boundary of the $r$-th phase-lock area consists of two analytic curves, which are the graphs of two functions $B = g_{r,\pm}(A)$ (see [12]; this fact was later explained by A.V.Klimenko via symmetry, see [26]).

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[1]There is a misprint, missing $2\pi$ in the denominator, in analogous formulas in previous papers of the author with co-authors: [20, formula (2.2)], [7, the formula after (1.16)].

4

3) The latter functions have Bessel asymptotics

$$\begin{cases} g_{r,-}(s) = r\omega - J_r(-\frac{s}{\omega}) + O(\frac{\ln|s|}{s}) \\ g_{r,+}(s) = r\omega + J_r(-\frac{s}{\omega}) + O(\frac{\ln|s|}{s}) \end{cases} \quad , \text{ as } s \to \infty \qquad (1.4)$$

(observed and proved on physical level in [30], see also [27, chapter 5], [4, section 11.1], [10]; proved mathematically in [26]).

4) Each phase-lock area is a garland of infinitely many bounded domains going to infinity in the vertical direction. In this chain each two subsequent domains are separated by one point. This follows from the above statement 3). Those of the latter separation points that lie in the horizontal $B$-axis are calculated explicitly, and we call them the *growth points*, see [12, corollary 3]. The other separation points, which lie outside the horizontal $B$-axis, are called the *constrictions*.

5) For every $r \in \mathbb{Z}$ the $r$-th phase-lock area is symmetric to the $-r$-th one with respect to the vertical $A$-axis.

6) Every phase-lock area is symmetric with respect to the horizontal $B$-axis. See Figures 1–5 below.

**Definition 1.3** For every $r \in \mathbb{Z}$ and $\omega > 0$ we consider the vertical line

$$\Lambda_r = \{B = \omega r\} \subset \mathbb{R}^2_{(B,A)}$$

and we will call it the *axis* of the phase-lock area $L_r$.

Numerical experiences made by V.M.Buchstaber, S.I.Tertychnyi, V.A.Kleptsyn, D.A.Filimonov, I.V.Schurov led to the following conjecture, which was stated and partially investigated in [20], see also [7, section 5].

**Conjecture 1.4** *([20, experimental fact A], [7, conjecture 5.17]). The upper part $L_r^+ = L_r \cap \{A \geq 0\}$ of each phase-lock area $L_r$ is a garland of infinitely many connected components separated by constrictions $\mathcal{A}_{r,1}, \mathcal{A}_{r,2} \ldots$ lying in its axis $\Lambda_r = \{B = r\omega\}$ and ordered by their ordinates $A$, see the figures below.*

**Remark 1.5** Conjecture 1.4 was proved in [20] for $\omega \geq 1$. It was proved in the same paper that for every $\omega > 0$ all the constrictions of every phase-lock area $L_r$ have abscissas $B = \omega l$, $l \in \mathbb{Z}$, $l \equiv r(mod2)$, $l \in [0, r]$. It is known that the zero phase-lock area $L_0$ contains the whole $A$-axis $\Lambda_0$, and all its constrictions lie in $\Lambda_0$; each point of the intersection $\partial L_0 \cap \Lambda_0$ is a constriction. This follows from symmetry of the phase-lock area $L_0$ with

5

Figure 1: Phase-lock areas and their constrictions for $\omega = 2$. The abscissa is $B$, the ordinate is $A$. Figure taken from [7, fig. 1a)]

respect to its axis $\Lambda_0$ and the fact that the interval $(-1, 1)$ of the $B$-axis is contained in $Int(L_r)$, see [7, proposition 5.22]. The two above statements together imply that for $r = \pm 1, \pm 2$ all the constrictions of the phase-lock area $L_r$ lie in its axis $\Lambda_r = \{B = r\omega\}$. But it is not known whether the latter statement holds for every $r \in \mathbb{Z} \setminus \{0\}$.

**Conjecture 1.6** *[7, conjecture 5.19]* $[\mathcal{A}_{r,j}, \mathcal{A}_{r,j+1}] \subset L_r$ *for every* $r \in \mathbb{Z}$ *and* $j \in \mathbb{N}$.

**Conjecture 1.7** *(see also [7, conjecture 5.26]) Each phase-lock area* $L_r$ *with* $r \in \mathbb{N}$ *lies on the right from the axis* $\Lambda_{r-1}$*: that is,* $B|_{L_r} > (l-1)\omega$.

The main results of the present paper are Theorems 1.8 and 1.12 stated below, which are partial results towards confirmation of Conjectures 1.4 and
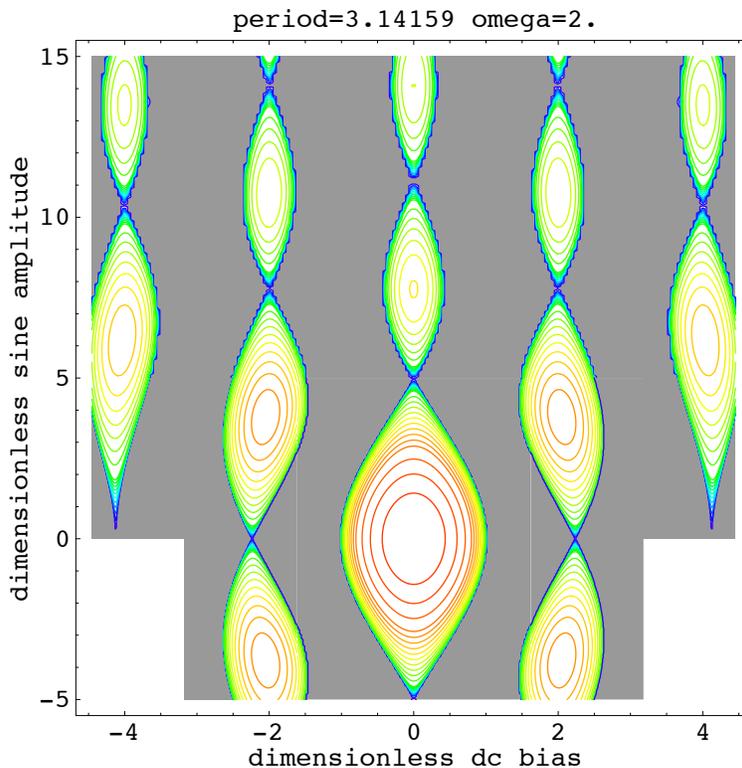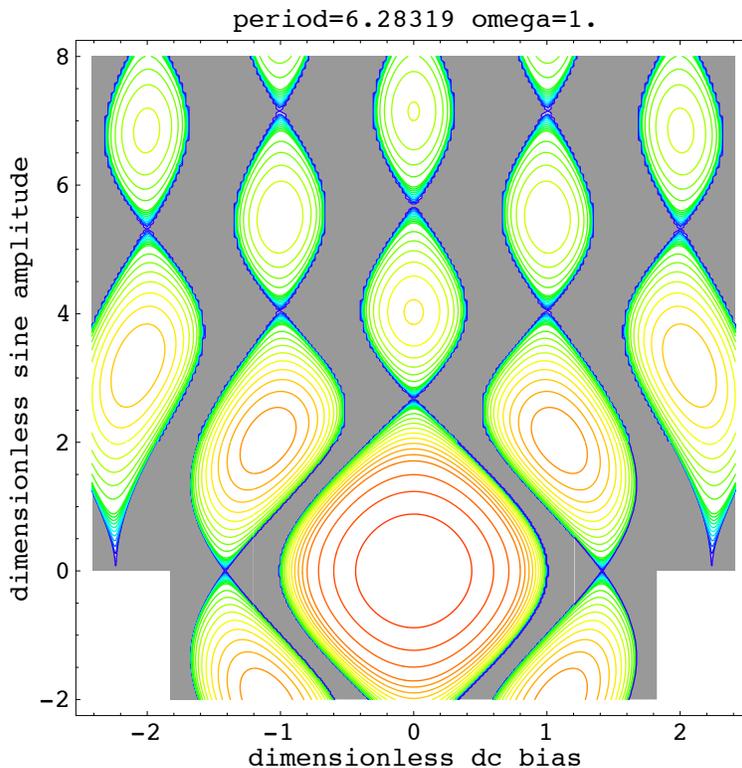
Figure 2: Phase-lock areas and their constrictions for $\omega = 1$. The abscissa is $B$, the ordinate is $A$. Figure taken from [7, fig. 1b)]

1.6.

**Theorem 1.8** *For every $\omega > 0$ and every constriction $(B_0, A_0)$ there exists a punctured neighborhood $U = U(A_0) \subset \mathbb{R}$ such that the punctured interval $B_0 \times (U \setminus \{A_0\}) \subset B_0 \times \mathbb{R}$ either lies entirely in the interior of a phase-lock area (then the constriction is called **positive**), or lies entirely outside the union of the phase-lock areas (then the constriction is called **negative**), see Fig.6.*

To state the second theorem, let us introduce the following definition.

**Definition 1.9** A *simple intersection* is a point of intersection of the boundary $\partial L_r$ with the axis $\Lambda_r$ that is not a constriction. The simple intersection with the maximal ordinate $A$ will be called the *higher simple intersection*

period=8.97598 mu=0.7
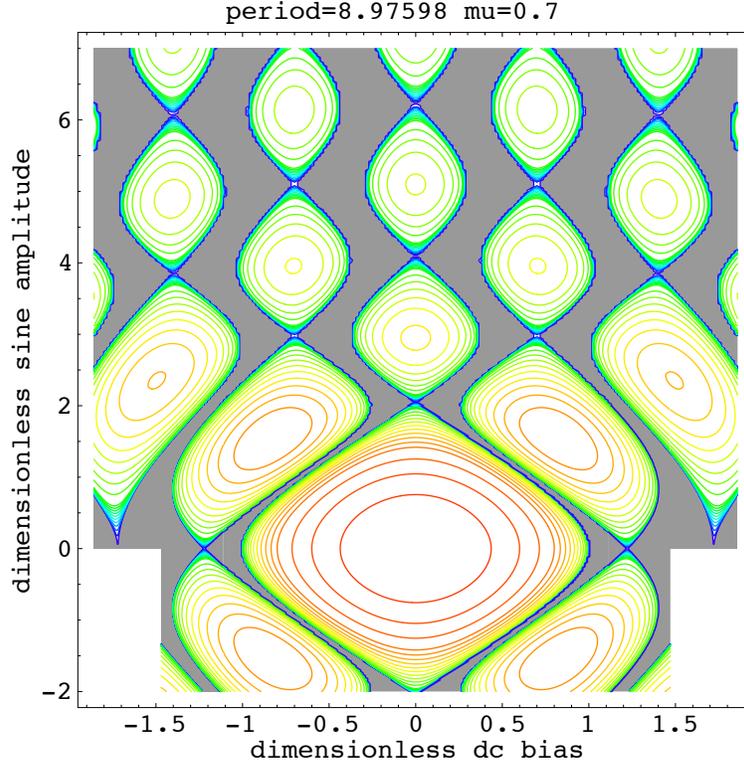
Figure 3: Phase-lock areas and their constrictions for $\omega = 0.7$. Figure taken from [14, p. 331], see also [7, fig. 1c)].

and denoted by $\mathcal{P}_r$. Set

$$Sr = \{\omega r\} \times [A(\mathcal{P}_r), +\infty) \subset \Lambda_r :$$

this is the vertical ray in $\Lambda_r$ issued from the point $\mathcal{P}_r$ in the direction of increasing of the coordinate $A$. See Fig.7.

**Conjecture 1.10** *For every $r \in \mathbb{Z}\setminus\{0\}$ the simple intersection with positive ordinate is unique.*

**Remark 1.11** There are no simple intersections for $r = 0$, since the intersection $\partial L_0 \cap \Lambda_0$ consists only of constrictions, see Remark 1.5. For every $r \in \mathbb{Z}\setminus\{0\}$ the ordinates of the simple intersections in $\Lambda_r$ lying in the upper

8
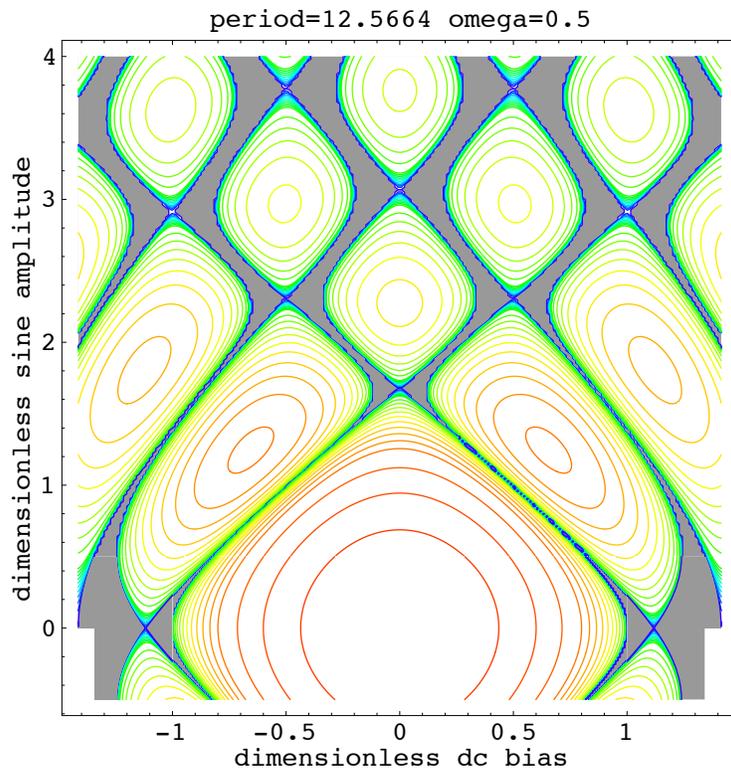
Figure 4: Phase-lock areas and their constrictions for $\omega = 0.5$. Figure taken from [7, fig. 1d)]

half-plane belong to the collection of roots of a known polynomial of degree $|r|$. This follows from results of [13, section 3] and [7, theorem 1.15]. This together with symmetry implies that for every given $r \in \mathbb{Z} \setminus \{0\}$ the number of the corresponding simple intersections is finite.

**Theorem 1.12** *For every $\omega > 0$ and every $r \in \mathbb{Z} \setminus \{0\}$ the corresponding simple intersections exist and do not lie in the $B$-axis; thus $\mathcal{P}_r$ and $Sr$ are well-defined. The ray $Sr$ is contained in the phase-lock area $L_r$.*

**Conjecture 1.13** *All the constrictions are positive.*

**Conjecture 1.14** *The intersection $L_r^+ \cap \Lambda_r$ coincides with $Sr$ for every $r \in \mathbb{Z} \setminus \{0\}$.*

9

period=20.944 omega=0.3
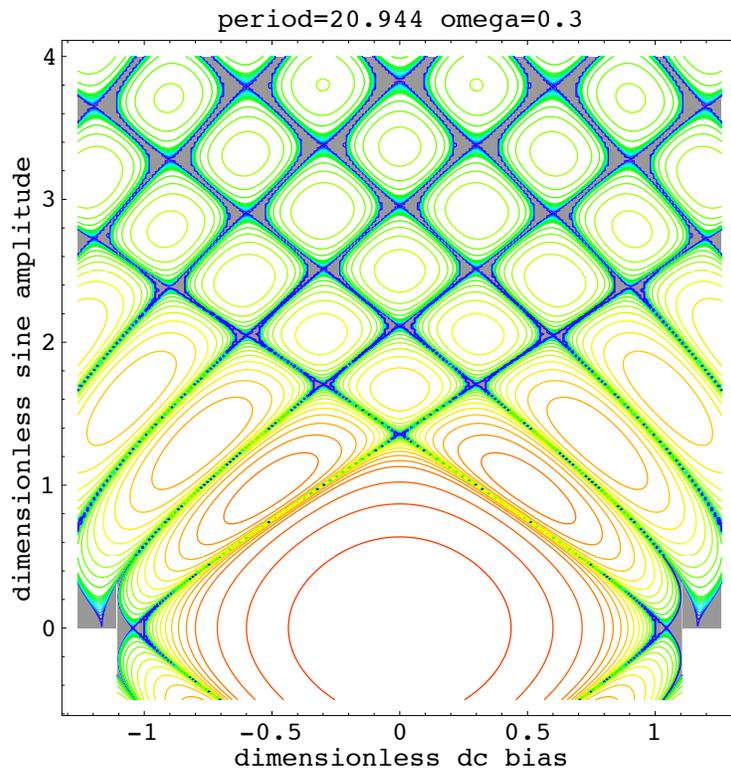
Figure 5: Phase-lock areas and their constrictions for $\omega = 0.3$. Figure taken from [7, fig. 1e)]

**Remark 1.15** Conjecture 1.14 obviously implies Conjecture 1.10. In Section 4 we show that any of Conjectures 1.14, 1.13 implies Conjectures 1.4 and 1.6, and we will discuss the relations between different conjectures in more details.

Theorem 1.12 will be deduced from Theorem 1.8 and the result of paper [26]. For the proof of Theorem 1.8 we complexify equation (1.1) and write it in the new complex variables

$$\Phi = e^{i\phi}, \ z = e^{i\tau} = e^{i\omega t},$$

set

$$l = \frac{B}{\omega}, \ \mu = \frac{A}{2\omega}, \ \lambda = \left(\frac{1}{2\omega}\right)^2 - \mu^2. \tag{1.5}$$

10

a) Positive constriction          b) Negative constriction
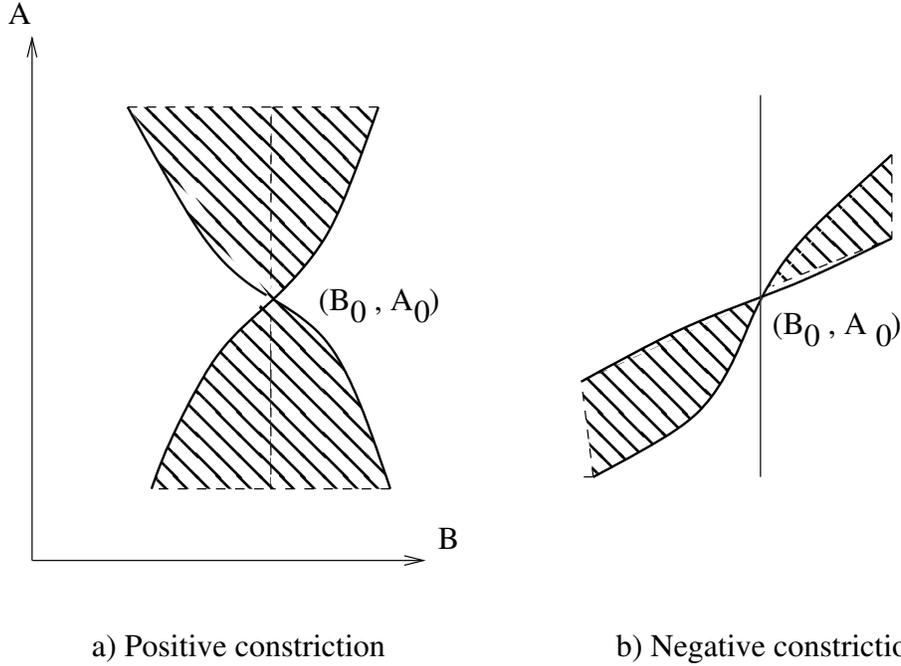
Figure 6: Positive and (conjecturally non-existing) negative constrictions. The shaded domains are phase-lock areas

The complexified equation (1.1) becomes the Riccati equation

$$\frac{d\Phi}{dz} = z^{-2}((lz + \mu(z^2 + 1))\Phi - \frac{z}{2i\omega}(\Phi^2 - 1)).  \qquad (1.6)$$

The latter is the projectivization of the following linear equation on a vector function $(u, v)$, $\Phi = \frac{v}{u}$:

$$\begin{cases} u' = z^{-2}(-(lz + \mu(1 + z^2))u + \frac{z}{2i\omega}v) \\ v' = \frac{1}{2i\omega z}u \end{cases} : \qquad (1.7)$$

each solution $\Phi(z)$ of equation (1.6) is a ratio $\frac{v}{u}$ of components of a solution of equation (1.7) and vice versa. See [6, sect. 3.2]. The above reduction to a linear system was obtained in slightly different terms in [9, 14, 18, 22].

After substitution $E(z) = e^{\mu z}v(z)$ system (1.7) becomes equivalent to the following *special double confluent Heun equation:*

$$z^2 E'' + ((l + 1)z + \mu(1 - z^2))E' + (\lambda - \mu(l + 1)z)E = 0. \qquad (1.8)$$

11
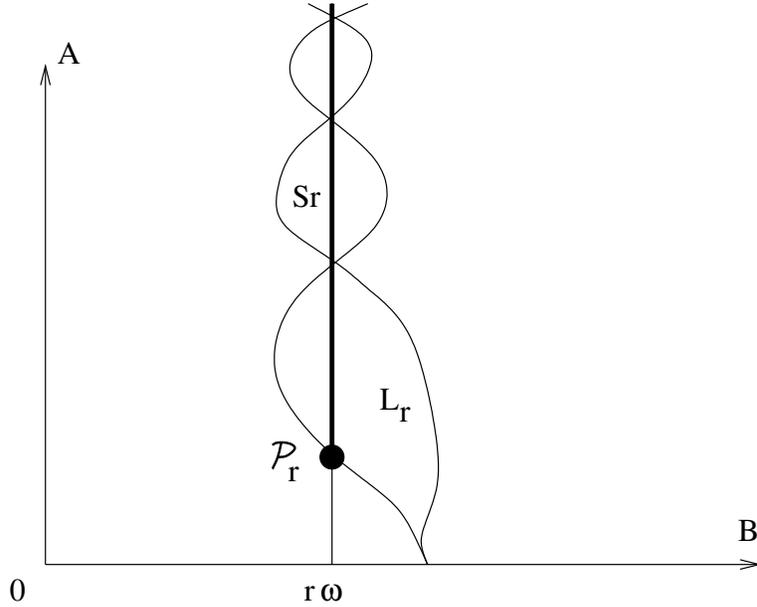
Figure 7: The higher simple intersection $\mathcal{P}_r$ and the corresponding ray $Sr$

Equation (1.8) belongs to the well-known class of double confluent Heun equations, see [32, formula (3.1.15)]. The reduction to equations (1.8) was obtained by V.M.Buchstaber and S.I.Tertychnyi [13, 14, 34, 35], who studied equations (1.8) and obtained many important results on their polynomial and entire solutions and symmetries in loc. cit. and in [15, 16, 17]). The complete description of equations (1.8) having entire solutions was started in [14] and finished in [6]. The description of their monodromy eigenvalues was obtained in the joint paper [7] of V.M.Buchstaber and the author as an explicit analytic transcendental equation relating one monodromy eigenvalue and the parameters. The following theorem was also proved in [7]. It concerns the "conjugate" double confluent Heun equations

$$z^2 E'' + ((-l+1)z + \mu(1-z^2))E' + (\lambda + \mu(l-1)z)E = 0 \qquad (1.9)$$

obtained from (1.8) by changing sign at the parameter $l$.

12

**Theorem 1.16** *[7, theorem 1.15]. Let $\omega > 0$, $(B, A) \in \mathbb{R}^2$, $B, A > 0$, $l = \frac{B}{\omega}$, $\mu = \frac{A}{2\omega}$, $\lambda = \frac{1}{4\omega^2} - \mu^2$, $\rho = \rho(B, A)$. The "conjugate" double confluent Heun equation (1.9) corresponding to the above $\lambda$, $\mu$ and $l$ has a polynomial solution, if and only if $l, \rho \in \mathbb{Z}$, $\rho \equiv l(mod2\mathbb{Z})$, $0 \leq \rho \leq l$, the point $(B, A)$ lies in the boundary of a phase-lock area and is not a constriction. In other terms, the points $(B, A) \in \mathbb{R}_+^2$ corresponding to equations (1.9) with polynomial solutions lie in boundaries of phase-lock areas and are exactly those their intersection points with the lines $\{B = m\omega\}$, $m \equiv \rho(mod2\mathbb{Z})$, $0 \leq \rho \leq m$ that are not constrictions. (For example, the statement of the theorem holds for simple intersections.)*

System (1.7) is a holomorphic linear differential equation on the Riemann sphere with two irregular non-resonant singular points of Poincaré rank 1 (pole of order 2) at zero and at infinity. The classical Stokes phenomena theory [2, 23, 3, 25, 31] yields canonical bases of its solutions at 0 and at $\infty$ in two appropriate sectors $S_\pm$ containing the punctured closed half-planes $\{\pm \text{Im} z \geq 0\} \setminus \{0\}$ and not containing the opposite imaginary semiaxes $i\mathbb{R}_\mp$. In each sector we have two canonical solution bases: one comes from zero, and the other one comes from infinity. The classical Stokes matrices at zero (infinity) compare appropriately normalized sectorial bases at zero (infinity) on components of the intersection $S_+ \cap S_-$. It is well-known that the Stokes matrices are triangular and unipotent; their triangular elements $c_0$ and $c_1$ are called the *Stokes multipliers*. In Subsection 2.2 we show that the Stoker multipliers "at zero" are real, whenever $l \in \mathbb{Z}$.

The statement of Theorem 1.8 deals with a constriction $(B_0, A_0)$, set $l = \frac{B_0}{\omega} \in \mathbb{Z}$, and says that its appropriate punctured neighborhood $U$ in the line $\{B = l\omega\}$ either entirely lies in a phase-lock area, or entirely lies outside the union of the phase-lock areas. Inclusion into a phase-lock area is equivalent to the statement that the monodromy operator of system (1.7), which is unimodular, has trace with modulus no less than 2. The trace of monodromy under question equals $2 + c_0c_1$, by the classical formula expressing the monodromy via the formal monodromy (which is trivial, since $l = \frac{B_0}{\omega} \in \mathbb{Z}$) and the Stokes matrices. To prove Theorem 1.8, we have to show that the product $c_0c_1$ has constant sign on appropriate punctured neighborhood $U$. The proof of this statement is based on studying of two appropriately normalized canonical solution bases and the transition matrix comparing them: the base on $S_+$ coming from zero, and the base on $S_-$ coming from infinity. The idea to study a transition matrix between canonical bases at zero and at infinity was suggested by V.M.Buchstaber.

We compare appropriate solution bases "at zero" on $S_+$ and "at in-

finity" on $S_-$ that are given by two fundamental solution matrices $W_+(z)$ and $\hat{W}_-(z)$ respectively. On the positive real semiaxis $\mathbb{R}_+ \subset S_+ \cap S_-$ the transition between these bases is defined by a constant matrix $Q$:

$$W_+(z) = \hat{W}_-(z)Q.$$

We show that one can normalize the bases $W_+$ and $\hat{W}_-$ under question so that $Q$ be an involution, and we prove a formula relating the coefficients of the matrix $Q$ and the Stokes multipliers. The key Lemma 2.11 says that the off-diagonal terms of the transition matrix are both non-zero at $(B_0, A_0)$. The latter formula and inequality together will imply that the ratio $\frac{c_1}{c_0}$ is a function analytic and nonvanishing on a neighborhood of the point $(B_0, A_0)$ in the line $\{B = B_0\}$. This will imply constance of sign of their product $c_0 c_1$ on the punctured neighborhood.

In Section 4 we state Conjecture 4.3 saying that the ratio of the off-diagonal elements of the transition matrix $Q$ is negative at each constriction $(B_0, A_0)$ with $B_0 \geq 0$, $A_0 > 0$. We show that Conjecture 4.3 would imply Conjectures 1.4 and 1.6 and discuss further relations between different conjectures.

In Section 5 we present additional technical results on the coefficients of the transition matrix $Q$, which will be used further on. The main result of Section 5 (Theorem 5.1) states that the upper triangular element of the matrix $Q$ is purely imaginary, whenever $l \in \mathbb{Z}_{\geq 0}$ and $A > 0$.

**Remark 1.17** Very recently Yulia Bibilo suggested a new approach to study the model of Josephson effect: to include system (1.7) into a general family of two-dimensional linear systems with two irregular singularities at zero and infinity and study isomonodromic conditions in the general family. Using this method, she have shown that an infinite collection of constrictions can be described as poles of Bessel solution of Painleve 3 equation with a special choice of parameters [5].

**Convention 1.18** In what follows, whenever the contrary is not specified, we consider that $l = \frac{B}{\omega} \geq 0$ and $\mu = \frac{A}{2\omega} > 0$: it suffices to treat the case of non-negative $l$ and positive $\mu$, by symmetry of the portrait of the phase-lock areas.

# 2 Linearization. Stokes and transition matrices and their relation to the rotation number

Here we study family (1.7) of linear systems equivalent to equations (1.1). In Subsection 2.1 we recall what are their formal normal forms, canonical sectorial solution bases and Stokes matrices and multipliers. In Subsection 2.2 we show that the Stokes multipliers are real. In Subsection 2.3 we recall results on symmetries of equation (1.8) and existence of its polynomial and entire solutions. In Subsection 2.4 we introduce the transition matrix between appropriate sectorial bases at zero and at infinity and prove a preliminary formula relating its coefficients to the Stokes multipliers.

## 2.1 Preliminaries: canonical solution bases, Stokes and transition matrices

All the results presented in this subsection are particular cases of classical results contained in [2, 23, 3, 25, 31].

**Definition 2.1** Two germs at 0 of linear systems of type

$$\dot{w} = \frac{A(z)}{z^{k+1}} w, \ w \in \mathbb{C}^n,$$

are *analytically (formally) equivalent,* if there exists a linear variable change $w = H(z)\widetilde{w}$ with $H(z)$ being a holomorphic invertible matrix function (respectively, a formal invertible matrix power series in $z$) that transforms one equation to the other.

The germs at zero and at infinity of linear system (1.7) are both formally equivalent to the germ of the diagonal system

$$\begin{cases} \widetilde{u}' = -z^{-2}(lz + \mu(1 + z^2))\widetilde{u} \\ \widetilde{v}' = 0, \end{cases} \tag{2.1}$$

which will be here called the *formal normal form.* The formal normal form has the canonical base in its solution space with the diagonal fundamental matrix

$$F(z) = \begin{pmatrix} z^{-l}e^{\mu(\frac{1}{z}-z)} & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

In general, the formal equivalence is not analytic: the normalizing power series diverges. On the other hand, there exist analytic normalizations on

appropriate sectors. Namely, let $S_+$, $S_-$ be sectors on the $z$-axis with vertex at 0 that contain the closed upper (respectively, lower) half-plane punctured at 0 so that the closure of the sector $S_\pm$ does not contain the opposite imaginary semiaxis $i\mathbb{R}_\mp$, see Fig. 8. In addition we consider that the sector $S_\pm$ is symmetric to $S_\mp$ with respect to the real axis. Set

$$w = (u, v), \ \widetilde{w} = (\widetilde{u}, \widetilde{v}).$$

There exist and unique invertible matrix functions $H_\pm(z)$ that are holomorphic on the sectors $S_\pm$ and $C^\infty$ on their closures $\overline{S}_\pm \subset \overline{\mathbb{C}}$ punctured at $\infty$, $H_\pm(0) = Id$, such that the variable change $w = H_\pm(z)\widetilde{w}$ transforms system (1.7) to its formal normal form (2.1) (the Sectorial Normalization Theorem).
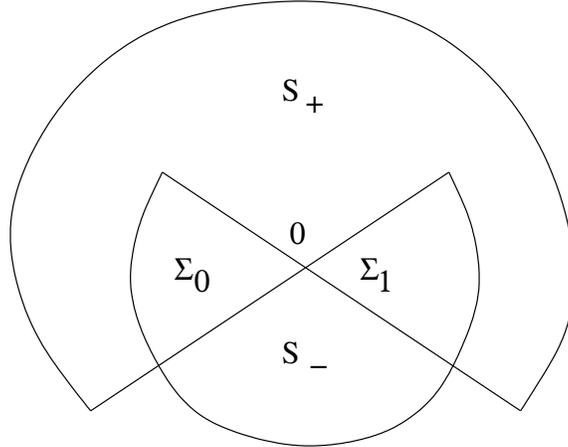


Figure 8: The sectors $S_\pm$ defining the Stokes matrices.

The canonical base "at zero" of system (1.7) in each sector $S_\pm$ is given by the fundamental matrix

$$W_\pm(z) = H_\pm(z)F(z), \ F(z) = \begin{pmatrix} z^{-l}e^{\mu(\frac{1}{z}-z)} & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.3}$$

In the definition of the above fundamental matrices we consider that the holomorphic branch of the fundamental matrix $F(z)$ of the formal normal form in $S_-$ is obtained from that in $S_+$ by counterclockwise analytic extension. We introduce yet another fundamental matrix $W_{+,1}(z) = H_+(z)F(z)$ of solutions of system (1.7) on $S_+$ where the holomorphic branch of the

16

matrix $F(z)$ is obtained from that in $S_-$ by counterclockwise analytic extension. Let $\Sigma_0$ ($\Sigma_1$) denote the left (respectively, right) component of the intersection $S_+ \cap S_-$. Over each component $\Sigma_j$ we have two fundamental matrices, $W_\pm(z)$ over $\Sigma_0$ and $W_-(z)$, $W_{+,1}(z)$ over $\Sigma_1$ that are related by a constant transition matrix, the *Stokes matrix* $C_j$:

$$W_-(z) = W_+(z)C_0 \text{ on } \Sigma_0; \quad W_{+,1}(z) = W_-(z)C_1 \text{ on } \Sigma_1. \qquad (2.4)$$

It is well-known that the Stokes matrices are triangular and unipotent:

$$C_0 = \begin{pmatrix} 1 & c_0 \\ 0 & 1 \end{pmatrix}, \ C_1 = \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}, \qquad (2.5)$$

whenever $\mu > 0$, which is our case (if $\mu < 0$, then their triangular types are opposite). Their upper (lower) triangular elements $c_0$, $c_1$ are called the *Stokes multipliers.*

Recall that the *monodromy operator* of a linear differential equation on the Riemann sphere acts on the space of germs of its solutions at a nonsingular point $z_0$. Namely, fix a closed path $\alpha$ starting at $z_0$ in the complement to the singular points of the equation. The monodromy operator along the path $\alpha$ sends each germ to the result of its analytic extension along the path $\alpha$. It is completely determined by the homotopy class of the path $\alpha$ in the complement to the singular points of the equation. Linear system (1.7) under consideration has exactly two singular points: zero and infinity. By the *monodromy operator of system* (1.7) we mean the monodromy operator along a counterclockwise circuit around zero. The monodromy matrix of the formal normal form (2.1) in the canonical solution base with a diagonal fundamental matrix is

$$M_N = \mathrm{diag}(e^{-2\pi i l}, 1). \qquad (2.6)$$

Note that it follows from definition that

$$W_{+,1}(z) = W_+(z)M_N. \qquad (2.7)$$

**Lemma 2.2** *[23, p. 35]. The monodromy matrix of system (1.7) in the sectorial canonical solution base $W_+$ equals*

$$M = M_N C_1^{-1} C_0^{-1}. \qquad (2.8)$$

## 2.2 Reality of the Stokes multipliers on the axes of phase-lock areas

**Theorem 2.3** *The Stokes multipliers $c_0$ and $c_1$ of system (1.7) are real, whenever $l = \frac{B}{\omega} \in \mathbb{Z}$.*

**Proof** System (1.7) admits the symmetry

$$J : (u, v)(z) \mapsto (\hat{u}, \hat{v})(z) = (-\bar{u}, \bar{v})(\bar{z}), \qquad (2.9)$$

which induces $\mathbb{C}$-antilinear automorphisms of the spaces of germs of its so-
lutions on $\mathbb{R}_{\pm}$ or equivalently, on $\Sigma_{0,1}$. Indeed, the right-hand side of system
(1.7) is the vector $(u, v)$ multiplied by $z^{-2}$ times an entire matrix function.
The diagonal terms of the latter matrix function are real polynomials, while
the off-diagonal terms are equal to $\frac{z}{2i\omega}$. This implies that $J$ is a symmetry
of system (1.7). For every vector function $f = (f_1, f_2)(z)$ by $J_* f$ we will
denote its image under the transformation $J$. Note that a vector function
$f$ holomorphic on a sector $S_{\pm}$ is transformed to the function $J_* f$ holomor-
phic on $S_{\mp}$, since the complex conjugation permutes the sectors $S_{\pm}$. Let
$f_{j\pm} = (f_{1j,\pm}, f_{2j,\pm})$, $j = 1, 2$, denote the canonical basic sectorial solutions
forming the fundamental matrices $W_{\pm} = (f_{ij,\pm}(z))$ in the sectors $S_{\pm}$.

**Proposition 2.4** *One has $J_* f_{1\pm} = -f_{1\mp}$, whenever $l \in \mathbb{Z}$, and $J_* f_{2\pm} = f_{2\mp}$ in the general case.*

**Proof** Note that $|f_{1\pm}(z)| = o(|f_{2\pm}(z)|)$, as $z \to 0$ along $\mathbb{R}_-$, by (2.3)
and since $z^{-l} e^{\frac{\mu}{z}} \to 0$; recall that $\mu > 0$, see Convention 1.18. Therefore,
$|J_* f_{1+}(z)| = |f_{1+}(\bar{z})| = o(|f_{2\pm}(\bar{z})|)$, by (2.9). Thus, both $J_* f_{1+}$ and $f_{1-}$ are
solutions of system (1.7) with the least asymptotics, as $z \to 0$ along $\mathbb{R}_-$. It
is classical that a solution of (1.7) with the least asymptotics is unique up to
constant factor (in particular, $f_{1+} = f_{1-}$ on $\mathbb{R}_-$). In our case this follows by
the above asymptotic formula, since the solution space is two-dimensional.
Therefore, $J_* f_{1+} = a_1 f_{1-}$, $a_1 = const \neq 0$. Similarly we get that $f_{2+}$ is the
solution with the least asymptotics on $\mathbb{R}_+$, and $J_* f_{2+} = a_2 f_{2-}$. Let us show
that $a_1 = -1$ and $a_2 = 1$.

    The vector function $f_{1\pm}$ is the product of the scalar function $z^{-l} e^{\mu(\frac{1}{z} - z)}$
and the first column of the matrix function $H_{\pm}(z)$, see (2.3). Recall that
$H_{\pm}(z)$ is $C^\infty$-smooth on $\overline{S}_{\pm} \setminus \{\infty\}$ and $H_{\pm}(0) = Id$; thus the upper ele-
ment of the latter column tends to 1 and dominates the lower one. Hence,
$f_{11,\pm}(z) \simeq z^{-l} e^{\mu(\frac{1}{z} - z)}$, $f_{21,\pm}(z) = o(f_{11,\pm}(z))$, as $z \to 0$ along $\mathbb{R}_-$. If $l \in \mathbb{Z}$,
then the first component of the vector function $J_* f_{1+}$ equals $-\overline{f_{11,+}(\bar{z})} \simeq$
$-z^{-l} e^{\mu(\frac{1}{z} - z)} \simeq -f_{11,-}(z)$. Therefore, $a_1 = -1$. The vector function $f_{2\pm}(z)$
equals the second column of the matrix function $H_{\pm}(z)$, by (2.3). The sec-
ond (lower) component of the latter column tends to 1, as $z \to 0$ along $\mathbb{R}_+$,
since $H_{\pm}(0) = Id$. The transformation $J$ acts on the second component of
a vector function by conjugation of the image and the preimage. The three
latter statements toghether imply that the second components of the vector

18

functions $f_{2-}(z)$ and $J_* f_{2+}(z)$ are asymptotic to each other, as $z \to 0$ along $\mathbb{R}_+$, and hence, $a_2 = 1$. This proves Proposition 2.4. $\qquad \square$

For every $z \in \mathbb{R}_-$ one has $f_{1+}(z) = f_{1-}(z)$ (unipotence and upper triangularity of the Stokes matrix $C_0$), hence,

$$-\overline{f_{11,\pm}(z)} = -f_{11,\mp}(z) = -f_{11,\pm}(z), \ \overline{f_{21,\pm}(z)} = -f_{21,\mp}(z) = -f_{21,\pm}(z),$$

by Proposition 2.4. Therefore,

$$f_{11,\pm}(z) = f_{11,\mp}(z) \in \mathbb{R}, \ f_{21,\pm}(z) = f_{21,\mp}(z) \in i\mathbb{R} \text{ for } z \in \mathbb{R}_-.$$

Similarly we get that

$$f_{12,\pm}(z) = f_{12,\mp}(z) \in i\mathbb{R}, \ f_{22,\pm}(z) = f_{22,\mp}(z) \in \mathbb{R} \text{ for } z \in \mathbb{R}_+, \qquad (2.10)$$

$$f_{12,-}(z) = -\overline{f_{12,+}(z)}, \ f_{21,+}(z) = -\overline{f_{21,-}(z)} \text{ for } z \in \mathbb{R}, \qquad (2.11)$$

by Proposition 2.4. One has

$$f_{12,-}(z) = f_{12,+}(z) + c_0 f_{11,+}(z) \text{ for } z \in \mathbb{R}_-,$$

$$f_{21,+}(z) = f_{21,-}(z) + c_1 f_{22,-}(z) \text{ for } z \in \mathbb{R}_+,$$

by definition, (2.4), (2.5), (2.7) and since in our case, when $l \in \mathbb{Z}$, one has $M_N = Id$ and $W_{+,1} = W_+$. Substituting (2.11) to the latter formulas one gets

$$c_0 f_{11,+}(z) = -2 \operatorname{Re} f_{12,+}(z) \in \mathbb{R} \text{ for } z \in \mathbb{R}_-,$$

$$c_1 f_{22,-}(z) = 2 \operatorname{Re} f_{21,+}(z) \in \mathbb{R} \text{ for } z \in \mathbb{R}_+.$$

This together with the reality of the values $f_{11,+}(z)$, $f_{22,-}(z)$ for $z \in \mathbb{R}_-$ and $z \in \mathbb{R}_+$ respectively and the fact that they do not vanish identically (being the dominant components of non-identically vanishing vector functions) implies that $c_0, c_1 \in \mathbb{R}$. Theorem 2.3 is proved. $\qquad \square$

## 2.3 Symmetries of system (1.7) and solutions of the special double confluent Heun equation

V.M.Buchstaber and S.I.Tertychnyi have constructed symmetries of double confluent Heun equation (1.8) [14, 16, 35], see also [17]. The symmetry $\# : E(z) \mapsto 2\omega z^{-l-1}(E'(z^{-1}) - \mu E(z^{-1}))$, which is an involution of its solution space, was constructed in [35, equations (32), (34)]. It can be obtained from the symmetry $(\phi, t) \mapsto (\pi - \phi, -t)$ of the nonlinear equation

(1.1); the latter symmetry was found in [26]. The symmetry $\#$ is equivalent to the symmetry

$$\mathbb{I} : (u, v)(z) \ \mapsto \ (\hat{u}, \hat{v})(z) = -iz^{-l}e^{\mu(\frac{1}{z}-z)}(-v(z^{-1}), u(z^{-1})) \qquad (2.12)$$

of system (1.7). The transformation

$$\diamond : E(z) \mapsto e^{\mu(z+z^{-1})}E(-z^{-1}) \qquad (2.13)$$

induces an isomorphism of the solution space of the special double confluent Heun equation (1.8) and that of the "conjugate" double confluent Heun equation (1.9). Recall that we consider that $l \geq 0$.

As it was shown in [14], equation (1.8) cannot have polynomial solutions, while (1.9) can have them only for $l \in \mathbb{N}$. In [16] Buchstaber and Tertychnyi have found new nontrivial symmetries of equation (1.8) in the case, when $l \in \mathbb{Z}_{\geq 0}$ and equation (1.9) does not have polynomial solutions.

The following theorem is a direct consequence of results of papers [14, 20].

**Theorem 2.5** *A point $(B, A) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_+$ is a constriction, if and only if one of the following equivalent statements holds.*

*1) $l = \frac{B}{\omega} \in \mathbb{Z}$ and system (1.7) is analytically equivalent at 0 to its formal normal form (2.1).*

*2) $l \in \mathbb{Z}$ and $c_0 = c_1 = 0$.*

*3) System (1.7) has trivial monodromy.*

*4) $l \in \mathbb{Z}$ and $c_0 = 0$.*

*5) $l \in \mathbb{Z}$ and equation (1.8) has an entire solution.*

**Proof** Each one of the statements 1), 2), 3) is equivalent to the statement that $(B, A)$ is a constriction, see [20, proposition 3.2, lemma 3.3]. Its equivalence with statement 5) was proved in [14, theorems 3.3, 3.5]. It is clear that statement 2) implies 4). Let us prove the converse: statement 4) implies statement 5), which is equivalent to 2). Condition $c_0 = 0$ (i.e., $C_0 = Id$) is equivalent to the statement that the second canonical basic solutions $H_{\pm}(z)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ of system (1.7) coming from different sectors paste together and form an entire vector solution $(u(z), v(z))$, $v(z) \not\equiv 0$, $u(z) = 2i\omega z v'(z)$. Hence, in this case equation (1.8) has entire solution $E(z) = e^{\mu z}v(z)$, thus statement 5) holds. Theorem 2.5 is proved. $\qquad \square$

**Theorem 2.6** *[6, theorem 3.10]. Let $l \in \mathbb{N}$, $B = \omega l$, $A > 0$. Let equation (1.9) have a polynomial solution. Then equation (1.8) has no entire solution; thus $(B, A)$ is not a constriction.*

20

**Theorem 2.7** *Equation (1.9) has a nontrivial polynomial solution, if and only if $l \in \mathbb{N}$, $c_1 = 0$ and $c_0 \neq 0$.*

**Proof** Let equation (1.9) have a polynomial solution $E(z)$. Then $l \in \mathbb{N}$, see [13, section 3], and the corresponding equation (1.8) has no entire solution, by Theorem 2.6. Hence, $c_0 \neq 0$, by Theorem 2.5. Let us show that $c_1 = 0$. Consider the involution

$$I : (\phi, t) \mapsto (-\phi, t + \pi),$$

which transforms equation (1.1) to the same equation with opposite sign at $B$. It induces the "right bemol" transformation from [16, the formula after (8)] (see also [14, formula (38)]) sending solutions of equation (1.8) to solutions of equation (1.9). It also induces the transformation[2]

$$G_l : (\hat{u}, \hat{v})(z) = z^l e^{\mu(\frac{1}{z} - z)}(v(-z), u(-z)), \tag{2.14}$$

which sends solutions of system (1.7) to those of the same system with opposite sign at $l$; the latter system (1.7), where $l$ is taken with the "-" sign, will be referred to, as $(1.7)_-$. And vice versa, the transformation $G_{-l}$ sends solutions of system $(1.7)_-$ to those of (1.7). It transforms canonical sectorial basic solutions of $(1.7)_-$ in $S_{\pm}$ (appropriately normalized by constant factors) to those of (1.7) in $S_{\mp}$ and changes the numeration of the canonical basic solutions. This follows from definition and (2.3). The polynomial solution $E(z) = e^{\mu z} v(z)$ of equation (1.9) is constructed from an entire vector solution $h(z) = (u(z), v(z))$ of system $(1.7)_-$, and the latter coincides with the second canonical solution of system $(1.7)_-$ in both sectors $S_{\pm}$, as in the proof of Theorem 2.5. Thus, its image $G_{-l}h(z)$, which is holomorphic on $\mathbb{C}^*$, coincides with the first canonical solution of system (1.7) in both sectors. Hence, the first basic solutions of system (1.7) coming from different sectors coincide, and thus, $C_1 = Id$, $c_1 = 0$, by definition and (2.4). Vice versa, let $l \in \mathbb{N}$, $c_1 = 0$ and $c_0 \neq 0$. Then the first canonical sectorial solution of system (1.7) in $S_+$ is invariant under the monodromy operator, by (2.6) and (2.8), and coincides with that coming from the sector $S_-$. Its image under the transformation $G_l$ is the second canonical solution of system $(1.7)_-$, the same in both sectors, and hence, it is holomorphic on $\mathbb{C}$. Let $\hat{E}(z) = e^{\mu z} v(z)$ be the corresponding solution of equation (1.9), which is an entire function. Let us show that it is polynomial. Indeed, in the contrary case the operator of taking $l$-th derivative would send it to a non-trivial entire solution

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[2]The transformation $-iG_l$ is equivalent to the right bemol transformation from [16, the formula after (8)]

of equation (1.8), by [14, lemma 3], which cannot exist since $c_0 \neq 0$ and by Theorem 2.5. Thus, equation (1.9) has a polynomial solution. Theorem 2.7 is proved. $\qquad\square$

### 2.4 Canonical sectorial solution bases at infinity and the transition matrix "zero-infinity"

The involution $z \mapsto z^{-1}$ permutes the sectors $S_\pm$ and $S_\mp$, as does the symmetry with respect to the real axis, by assumption, and it fixes each component $\Sigma_{0,1}$ of their intersection. The symmetry $\mathbb{I}$ of system (1.7), see (2.12), induces an isomorphism of the solutions spaces of system (1.7) on the sectors $S_\pm$ and $S_\mp$ and an automorphism of the two-dimensional space of solutions on the component $\Sigma_1$.

**Definition 2.8** The *canonical bases* of solutions of system (1.7) in $S_\pm$ "at infinity" are those obtained from appripriately normalized canonical solution bases at the opposite sectors $S_\mp$ by the automorphism $\mathbb{I}$; their fundamental matrices are denoted by $\hat{W}_\pm(z)$. In more detail, we set

$$\hat{W}_- = \mathbb{I}(W_+ M_N), \tag{2.15}$$

where the right-hand side is the result of application of the transformation $\mathbb{I}$ to the columns of the fundamental matrix under question.

**Proposition 2.9** The canonical bases $W_+ M_N$ and $\hat{W}_-$ in the space of solutions of system (1.7) in $\Sigma_1$ are related by the transition matrix $Q = Q(B, A)$ that is an involution:

$$W_+(z)M_N = \hat{W}_-(z)Q, \ Q^2 = Id. \tag{2.16}$$

**Proof** The automorphism $\mathbb{I}$ of the solution space of system (1.7) is an involution, as is #. It permutes the canonical bases with fundamental matrices $W_+(z)M_N$ and $\hat{W}_-(z)$. This implies that the transition matrix $Q$ between them, which is the matrix of the automorphism in the base $\hat{W}_-(z)$, is also an involution. $\qquad\square$

**Theorem 2.10** Let $l = \frac{B}{\omega} \geq 0$. The matrix $Q = Q(B, A)$ from (2.16) has the form

$$Q = \begin{pmatrix} -a & b \\ -c & a \end{pmatrix}. \tag{2.17}$$

*The coefficients of the matrix $Q$ satisfy the following system of equations:*

$$\begin{cases} a^2 = bc + 1 \\ a(1 - e^{2\pi i l} + c_0 c_1) = bc_1 e^{-2\pi i l} - cc_0 \end{cases}, \qquad (2.18)$$

*where $c_0 = c_0(B, A)$, $c_1 = c_1(B, A)$ are the Stokes multipliers. One has*

$$b, c \neq 0 \ \text{whenever } (B, A) \text{ is a constriction.} \qquad (2.19)$$

Theorem 2.10 is proved below. Its proof and the proof of the main results of the paper are based on the following key lemma.

**Lemma 2.11** *Let $Q = (q_{ij}(B, A))$ be the same, as in (2.16). If $(B, A)$ is a constriction, then $q_{12}(B, A), q_{21}(B, A) \neq 0$.*

**Proof** Let $f_1 = (f_{11}, f_{21})(z)$, $f_2 = (f_{12}, f_{22})(z)$ be the basis of solutions of system (1.7) with the fundamental matrix $W_+ M_N = (f_{ij})$, and let $g_1 = (g_{11}, g_{21})(z)$, $g_2 = (g_{12}, g_{22})(z)$ be the basis with the fundamental matrix $\hat{W}_- = (g_{ij})$. Let $E_j(z) = e^{\mu z} f_{2j}(z)$, $\hat{E}_j(z) = e^{\mu z} g_{2j}(z)$ be the solutions of equation (1.8) defined by $f_j$ and $g_j$ respectively. Recall that in our case, when $(B, A)$ is a constriction, one has $l = \frac{B}{\omega} \in \mathbb{Z}$, by Theorem 2.5. Note that all the functions $f_j, g_j, E_j, \hat{E}_j$ are holomorphic on $\mathbb{C}^*$, since the monodromy of the system (1.7) is trivial, by Theorem 2.5. Suppose the contrary: in the corresponding matrix $Q(B, A)$ from (2.16) one has $q_{12}q_{21} = 0$. We treat the two following cases separately.

Case 1): $q_{12} = 0$. This means that $f_2 = g_2$, $E_2 = \hat{E}_2$ up to constant factor. Note that $f_2(z)$ is an entire function, and hence, so are $E_2$ and $\hat{E}_2$ (see the proof of Theorem 2.5). On the other hand,

$$\hat{E}_2(z) = \#E_2(z) = 2\omega z^{-l-1}(E'_2(z^{-1}) - \mu E_2(z^{-1})),$$

which follows from the definition of the transformation $\mathbb{I}$, see the beginning of Subsection 2.3. It is clear that the right-hand side of the latter formula tends to zero, as $z \to \infty$, since $E_2$ is holomorphic at 0. Finally, $\hat{E}_2(z)$ is an entire function tending to 0, as $z \to \infty$. Therefore, $\hat{E}_2 = E_2 \equiv 0$, by Liouville Theorem, and the basic function $f_2$ is identically equal to zero. The contradiction thus obtained proves that $q_{12} \neq 0$.

Case 2): $q_{21} = 0$. This means that $f_1 = cg_1$, $c \equiv const \neq 0$, hence $E_1 = c\hat{E}_1$. Note that in our case the Stokes matrices are trivial, by Theorem 2.5, and hence, system (1.7) is analytically equivalent to its formal normal form. Therefore, the sectorial normalizations $H_\pm(z)$ coincide with one and

the same matrix function $H(z)$ holomorphic at 0, $H(0) = Id$. The canonical basic solution $f_1(z)$ is equal to $z^{-l}e^{\mu(\frac{1}{z}-z)}$ times the first column of the matrix $H(z)$; the upper element of the latter column equals 1 at 0 and the lower element vanishes at 0. Hence,

$$E_1(z) = e^{\mu z}f_{21}(z) = z^{1-l}e^{\frac{\mu}{z}}h(z), \text{ as } z \to 0, \tag{2.20}$$

where the function $h(z) = z^{-1}H_{21}(z)$ is holomorphic at 0. On the other hand,

$$\hat{E}_1(z) = (\#E_1)(z) = 2\omega z^{-l-1}(E_1'(z^{-1}) - \mu E_1(z^{-1})) = O(e^{\mu z}), \text{ as } z \to \infty,$$

by (2.20). This together with (2.20) and the equality $E_1 = c\hat{E}_1$ yields that

$$E_1(z) = O(z^{1-l}e^{\frac{\mu}{z}}), \text{ as } z \to 0; \ E_1(z) = O(e^{\mu z}), \text{ as } z \to \infty. \tag{2.21}$$

The transformation (2.13) sends $E_1$ to the solution $\mathcal{E}$ of the other Heun equation (1.9). We claim that $\mathcal{E}$ is a polynomial. Indeed,

$$\mathcal{E}(z) = e^{\mu(z+\frac{1}{z})}E_1(-z^{-1}),$$

$$\mathcal{E}(z) = O(z^{l-1}), \text{ as } z \to \infty, \ \mathcal{E}(z) = O(1), \text{ as } z \to 0,$$

by (2.21). Thus, $\mathcal{E}$ is an entire function with at most polynomial growth at infinity, and hence, is a polynomial solution of equation (1.9). Therefore, equation (1.8) has no entire solution, and the point $(B, A)$ under consideration is not a constriction, by Theorem 2.6. The contradiction thus obtained proves that $q_{12}, q_{21} \neq 0$. Lemma 2.11 is proved. $\qquad\square$

**Proof of Theorem 2.10.** One has

$$Q^2 = \begin{pmatrix} q_{11}^2 + q_{12}q_{21} & q_{12}(q_{11} + q_{22}) \\ q_{21}(q_{11} + q_{22}) & q_{22}^2 + q_{12}q_{21} \end{pmatrix} = Id,$$

since $Q$ is an involution. Let us prove that $Q$ has the same type, as in (2.17). By the above equation, there are two possible cases (we denote $a = q_{22}$):

 a) $a = q_{22} = -q_{11}$, $a^2 + q_{12}q_{21} = 1$;
 b) $q_{12} = q_{21} = 0$, $q_{11}, q_{22} = \pm1$.

Note that the equations of cases a) and b) are analytic in $(B, A) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_+$. Each one of the equations $q_{12}, q_{21} = 0$ does not hold whenever $(B, A)$ is a constriction (Lemma 2.11). Hence, these equations are both not satisfied on an open subset of points $(B, A) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_+$. Therefore, equations of case a) hold on the latter open subset, and thus, for all $(B, A) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_+$,

by analyticity. This implies that the matrix $Q$ has type (2.17) and satisfies the first relation in (2.18). Statement (2.19) of Theorem 2.10 follows from Lemma 2.11. Let us prove the second relation in (2.18). To this end, we compare the monodromy matrices of system (1.7) in bases $(f_1, f_2)$, $(g_1, g_2)$ represented by the fundamental matrices $W_+ M_N$ and $\hat{W}_-$ respectively and write the equation saying that they are conjugated by $Q$. The monodromy matrix in the base $W_+ M_N$ equals

$$M_+ = M_N^{-1} M_N C_1^{-1} C_0^{-1} M_N = C_1^{-1} C_0^{-1} M_N, \qquad (2.22)$$

see (2.8). We consider the monodromy operator under question as acting in the space of germs of solutions of system (1.7) at the point $z = 1$: a fixed point of the involution $z \mapsto z^{-1}$. To calculate the monodromy matrix in the base $(g_1, g_2)$, note that its fundamental matrix $\hat{W}_-$ is obtained from the matrix $W_+ M_N$ by the substitution $z \mapsto z^{-1}$, permutation of lines, change of sign of the first line and subsequent multiplication of the whole matrix by the scalar factor $-iz^{-l} e^{\mu(\frac{1}{z} - z)}$, see (2.12) and (2.15). Permutation of lines and change of sign of one line do not change the monodromy. The substitution $z \mapsto z^{-1}$ changes the monodromy to inverse. As $z$ makes one counterclockwise turn around zero, the latter scalar factor is multiplied by $e^{-2\pi i l}$. Finally, the monodromy matrix in the base $\hat{W}_-$ equals

$$\hat{M}_- = e^{-2\pi i l} M_+^{-1} = e^{-2\pi i l} M_N^{-1} C_0 C_1. \qquad (2.23)$$

On the other hand, $\hat{M}_- = Q M_+ Q^{-1}$, by (2.16). Substituting this formula and (2.22) to (2.23) and taking into account that $Q^{-1} = Q$ yields

$$M_+ Q = Q \hat{M}_- = C_1^{-1} C_0^{-1} M_N Q = e^{-2\pi i l} Q M_N^{-1} C_0 C_1.$$

For a matrix $Q$ of type (2.17), the latter equation is equivalent to the second relation in (2.18). Theorem 2.10 is proved. □

# 3 Rotation number and trace of monodromy. Proof of Theorems 1.8 and 1.12

## 3.1 Trace and phase-lock areas

In what follows we consider the monodromy matrix $M$ of system (1.7) written in the base $W_+$, see (2.8). Note that $\det M = e^{-2\pi i l}$. We normalize the matrix $M$ by the scalar factor $e^{\pi i l}$ to make it unimodular:

$$\widetilde{M} = e^{\pi i l} M = e^{\pi i l} M_N C_1^{-1} C_0^{-1} = \begin{pmatrix} e^{-\pi i l} & -e^{-\pi i l} c_0 \\ -e^{\pi i l} c_1 & e^{\pi i l}(1 + c_0 c_1) \end{pmatrix}. \qquad (3.1)$$

**Proposition 3.1** *The trace of the matrix $\widetilde{M}$ is real, that is, $e^{\pi i l}c_0 c_1 \in \mathbb{R}$ for all the real parameters $(B, A)$ with $A \neq 0$.*

**Proof** The projectivization of system (1.7), the Riccati equation (1.6) is the complex extension of a differential equation on the torus: the product of the unit circles in space and time variables. This implies that the monodromy of the Riccati equation along the unit circle in the time variable is an automorphism of the unit disk, and hence, so is the projectivized monodromy of system (1.7). Hence,

$$tr\widetilde{M} = 2\cos(\pi l) + e^{\pi i l}c_0 c_1 \in \mathbb{R}. \tag{3.2}$$

□

**Proposition 3.2** *A point $(B, A)$ belongs to a phase-lock area, if and only if*

$$|tr\widetilde{M}| = |2\cos(\pi l) + e^{\pi i l}c_0 c_1| \geq 2; \tag{3.3}$$

*$(B, A)$ belongs to its interior, if and only if the latter inequality is strict.*

**Proof** A point $(B, A)$ belongs to a phase-lock area, if and only if the time $2\pi$ flow map of the vector field (1.3), which acts as a diffeomorphism of the circle $S^1 = S^1_\phi \times 0$, has a fixed point. The latter diffeomorphism is the restriction to $S^1 = \partial D_1$ of a conformal automorphism of the disk $D_1$: the projectivized monodromy of system (1.7). A conformal automorphism of the disk $D_1$ represented by a Möbius transformation with unimodular matrix has a fixed point in $\partial D_1$, if and only if the trace of the latter matrix has module at least two. This together with formula (3.2) proves the first statement of the proposition. The module of the trace equals two, if and only if the Möbius transformation under question is either identity, or parabolic: has a unique fixed point in $\partial D_1$. The Möbius transformation has one of the two above types, if and only if the point $(B, A)$ lies in the boundary of a phase-lock area. This follows from strict monotonicity of the time $2\pi$ flow map of the vector field (1.3) as a function of the parameter $B$. Namely, if the time $2\pi$ flow map $S^1 \to S^1$ is either identity, or parabolic, then slightly deforming $B$ to the right or to the left, one can destroy all its fixed points in $S^1 = \partial D_1$ and thus, go out of the phase-lock area. Conversely, if the point $(B, A)$ lies in the boundary of a phase-lock area, then the corresponding above time $2\pi$ flow map is either identity, or parabolic, since it is Möbius and can have obviously neither attracting, nor repelling fixed points. Proposition 3.2 is proved. □

## 3.2 Behavior of phase-lock areas near constrictions. Proof of Theorems 1.8 and 1.12

Recall that without loss of generality we consider that $B \geq 0$, $A > 0$.

**Proof of Theorem 1.8.** The monodromy matrix, the matrix $\widetilde{M}$, see (3.1), and the Stokes multipliers $c_0$, $c_1$ are analytic functions of the parameters $(B, A)$ with $A \neq 0$. Fix a constriction $(B_0, A_0)$. Then $l = \frac{B_0}{\omega} \in \mathbb{Z}_{\geq 0}$, and $c_0(B_0, A_0) = c_1(B_0, A_0) = 0$, by Theorem 2.5. Consider the restrictions of the Stokes multiplier functions to the open ray $B_0 \times \mathbb{R}_+$:

$$s_j(t) := c_j(B_0, A_0 + t), \ j = 0, 1, \ t > -A_0; \ s_j(0) = 0.$$

**Claim 1.** *The difference $|tr\widetilde{M}(B_0, A_0 + t)| - 2$ is non-zero and has one and the same sign for all $t \neq 0$ small enough.*

**Proof** The trace under question equals

$$tr\widetilde{M}(B_0, A_0 + t) = (-1)^l(2 + s_0(t)s_1(t)).$$

The functions $s_j(t)$ are analytic on the interval $(-A_0, +\infty)$ and do not vanish identically: they may vanish only for those $t$, for which $(B_0, A_0 + t)$ is a point of intersection of the line $B_0 \times \mathbb{R}$ with boundaries of phase-lock areas (Theorems 2.5, 2.7 and 1.16). The set of the latter intersection points is discrete, since the boundary of each phase-lock area is the graph of a non-constant analytic function $A = g(B)$. Consider the coefficients $a$, $b$, $c$ in the matrix $Q = Q(B_0, A_0 + t)$, see (2.17), which are also analytic functions in $t$. In our case, when $l \in \mathbb{Z}$, the second relation in (2.18) with $c_j$ replaced by $s_j(t)$ takes the form

$$a(t)s_0(t)s_1(t) = b(t)s_1(t) - c(t)s_0(t), \ b(0), c(0) \neq 0,$$

by Theorem 2.10. This implies that

$$s_1(t) \simeq \frac{c(0)}{b(0)}s_0(t), \ \text{as } t \to 0, \tag{3.4}$$

$$s_0(t)s_1(t) \simeq \frac{c(0)}{b(0)}s_0^2(t) \simeq qt^{2n}, q \in \mathbb{R}, \ q \neq 0, \ \text{sign } q = \text{sign } \frac{c(0)}{b(0)},$$

$$(-1)^l tr\widetilde{M}(B_0, A_0 + t) = 2 + qt^{2n}(1 + o(1)). \tag{3.5}$$

The latter right-hand side is greater (less) than 2 for all $t \neq 0$ small enough, if $q > 0$ (respectively, $q < 0$). This proves the claim. $\square$

There exists a $\varepsilon > 0$ such that the punctured interval $B_0 \times ([A_0 - \varepsilon, A_0 + \varepsilon] \setminus \{A_0\})$ either lies entirely in the interior of a phase-lock area, or lies outside the union of the phase-lock areas. This follows from Proposition 3.2 and the above claim. Theorem 1.8 is proved. $\qquad\square$

Recall that for $r \in \mathbb{Z}$ by $L_r$ and $\Lambda_r = \{B = \omega r\}$ we denote respectively the $r$-th phase-lock area and its axis. In the proof of Theorem 1.12 we use the following immediate corollary of the main result of [26].

**Proposition 3.3** *For every $\omega > 0$ and every $r \in \mathbb{Z}_{\geq 0}$ there exists an $A \in \mathbb{R}_+$ that can be chosen arbitrarily large such that $(\omega r, A) \in Int(L_r)$.*

**Proof** The boundary $\partial L_r$ is a union of two graphs of functions $g_{r,\pm}(A)$ with Bessel asymptotics, by the main resulf of [26], see formula (1.4) at the beginning of the present paper. The functions $h_\pm(A) = g_{r,\pm}(A) - r\omega$ tend to zero roughly like sine function multiplied by $A^{-\frac{1}{2}}$, and $h_\pm(A) = -h_\mp(A) + o(A^{-\frac{1}{2}})$, as $A \to \infty$. In particular, there exists a sequence of local maxima $m_k$ of the function $h_+$, $m_k \to +\infty$, as $k \to \infty$, such that the values $h_+(m_k)$ are positive and asymptotically equivalent to $m_k^{-\frac{1}{2}}$ times a positive constant factor. Thus, for every $k$ large enough the values $h_\pm(m_k)$ are non-zero and have different signs. This implies that $(\omega r, m_k) \in Int(L_r)$. The proposition is proved. $\qquad\square$

**Proof of Theorem 1.12.** Let $r \in \mathbb{N}$. Let $A \in \mathbb{R}_+$ be such that $P = (r\omega, A) \in Int(L_r)$, and let $A$ be bigger than the ordinate of every simple intersection in $\Lambda_r = \{B = r\omega\}$ (if any): it exists by Proposition 3.3. Let $I \subset \Lambda_r$ denote the maximal interval containing $P$ that is contained in $L_r$.

**Claim 2.** *The interval $I$ is semi-infinite and bounded from below by a simple intersection. Simple intersections do not lie in the $B$-axis.*

**Proof** Suppose the contrary to the first statement of the claim. This means that either $I \neq \Lambda_r$ and $I$ has a boundary point $E \in \partial L_r$ that is not a simple intersection, or $I = \Lambda_r$. This alternative follows from the fact that the interval $I$ cannot be bounded by a simple intersection from above, by definition. The second case, when $I = \Lambda_r$, is impossible, since the intersection $L_r \cap \{A = 0\}$ is one point with abscissa $\sqrt{r^2\omega^2 + 1} > r\omega$ (see [22, section 3] and [12, corollary 3]), which does not lie in $\Lambda_r$. The latter statement also implies that if a simple intersection in $\Lambda_r$ exists, it does not lie in the $B$-axis. Therefore, the first case takes place. The boundary point $E$, which is not a simple intersection, is a constriction, by definition. It is a positive constriction, by Theorem 1.8 and since $E$ is adjacent to the interval $I \subset L_r$. Therefore, a small segment $J$ of the line $\{B = \omega r\}$ adjacent to

$E$ from the other side is also contained in $L_r$. Finally, $I \cup J \subset L_r$, – a contradiction to the maximality of the interval $I$. The claim is proved. □

Claim 2 implies the inclusion $Sr \subset I \subset L_r$ and hence, Theorem 1.12. □

# 4   Open problems and relations between conjectures

**Proposition 4.1** *Conjecture 1.14 implies Conjectures 1.13 and 1.7. Conjecture 1.7 implies Conjectures 1.4.*

**Proof**   Let us show that Conjecture 1.7 implies Conjecture 1.4. Conjecture 1.7 implies that for every $r \in \mathbb{N}$ all the constrictions of the phase-lock area $L_r$ have abscissas greater than $(r-1)\omega$. The latter abscissas are equal to $\omega l$, $l \in \mathbb{Z} \cap [0, r]$, see [20]. Hence, all of them are equal to $\omega r$ and thus, Conjecture 1.4 holds.

Now let us show that Conjecture 1.7 follows from Conjecture 1.14. The equality $L_r^+ \cap \Lambda_r = Sr$ given by Conjecture 1.14 implies that the set $L_r^+ \cap \{A < A(\mathcal{P}_r)\}$ lies on one side from the axis $\Lambda_r$: either on the right, or on the left. It should lie on the right, since the intersection of the phase-lock area $L_r$ with the $B$-axis is just one growth point with the abscissa $B_r = \sqrt{r^2\omega^2 + 1} > \omega r$ (see [22, section 3] and [12, corollary 3]), hence lying on the right. Finally, the right boundary of the upper phase-lock area $L_r^+$ is a connected curve issued from the point $B_r$ to infinity that lies on the right from the axis $\Lambda_r$, meets $\Lambda_r$ at constrictions and $\mathcal{P}_r$ and separates $\Lambda_r$ from the other upper phase-lock areas $L_k^+$, $k > r$. Applying this statement to $r$ replaced by $r - 1$ and taking into account symmetry and that $L_0$ contains the $A$-axis (by symmetry), we get that $L_r$ lies on the right from the axis $\Lambda_{r-1}$. This proves Conjecture 1.7.

Now let us prove that Conjecture 1.13 follows from Conjecture 1.14. Withour loss of generality we treat only the case of phase-lock areas $L_r$ with $r \in \mathbb{N}$ (by symmetry and positivity of all the constrictions in $L_0$, which also follows from symmetry). By Conjecture 1.14 and the implications proved above, Conjecture 1.4 holds: for every $r \in \mathbb{N}$ all the constrictions of the phase-lock area $L_r$ lie in $\Lambda_r$. Hence, those of them with $A > 0$ lie in $Sr = L_r^+ \cap \Lambda_r$, and thus, all of them are automatically positive, by connectivity of the ray $Sr$. This together with symmetry implies Conjecture 1.13. This finishes the proof of the proposition. □

29

**Proposition 4.2** *For every constriction* $(B, A) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_+$, *the ratio of the triangular terms in the corresponding transition matrix $Q$ from (2.17) is always real.*

The proposition follows from formula (3.4) and reality of the Stokes multipliers of system (1.7) for $l \in \mathbb{Z}$ (Theorem 2.3).

**Conjecture 4.3** *For every constriction* $(B, A) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_+$ *the ratio of the above triangular terms is negative, that is, $\frac{c}{b} > 0$.*

**Proposition 4.4** *Every constriction in* $\mathbb{R}_{\geq 0} \times \mathbb{R}_+$ *with* $\frac{c}{b} > 0$ *is positive.*

The proposition follows from formulas (3.4), (3.5) and Proposition 3.2.

**Definition 4.5** A negative constriction $(B_0, A_0)$ is called *queer,* if the germ at $(B_0, A_0)$ of the interior of the corresponding phase-lock area lies on one side from the vertical line $\{B = B_0\}$, and the origin $(0,0)$ lies on the different side.

**Conjecture 4.6** *(see a stronger conjecture: [7, conjecture 5.30]). There are no queer constrictions.*

**Proposition 4.7** *Conjecture 4.6 implies Conjecture 1.4.*

In the proof of Proposition 4.7 we use the following proposition.

**Proposition 4.8** *Incorrect constrictions cannot come "from infinity". That is, for every* $\omega_0 \in \mathbb{R}_+$ *and* $r \in \mathbb{N}$ *there exists no sequence* $\omega_n \to \omega_0$ *for which there exists a sequence of constrictions* $(B_n, A_n)$ *of the phase-lock areas* $L_r = L_r(\omega_n)$ *with* $B_n \neq \omega_n r$ *and* $A_n \to \infty$.

**Proof** Without loss of generality we consider that $r \geq 3$, since we already know that for $r = \pm 1, \pm 2$ Conjecture 1.4 holds: all the constrictions of the phase-lock area $L_r$ lie in $\Lambda_r$, see Remark 1.5. By symmetry, it suffices to prove the statement of the proposition with $A_n \to +\infty$. For every $r' \in \mathbb{N}$ the higher simple intersection $\mathcal{P}_{r'} \in L_{r'} \cap \Lambda_{r'}$ has ordinate $A(\mathcal{P}_{r'})$ bounded by the maximal root of an explicit family of monic polynomials depending on the parameter $\omega$, see [13, section 3]. Thus, for any given $r' \in \mathbb{N}$ and $\omega_0 \in \mathbb{R}_+$ the value $A(\mathcal{P}_{r'})$ is locally bounded from above as a function of $\omega$ on a neighborhood of the point $\omega_0$, say $A(\mathcal{P}_{r'}) < \alpha$ for $r' = r - 1$. On the other hand, the ray $R_\alpha = \omega r' \times [\alpha, +\infty) \subset Sr'$ lies in $L_{r'}$, by Theorem 1.12. Therefore, for $r' = r - 1$ the ray $R_\alpha$ is disjoint from the phase-lock

30

area $L_r$ and lies on the left from the intersection $L_r \cap \{A \geq \alpha\}$ All the possible "incorrect" constrictions of the upper phase-lock area $L_r^+$, that is, the constrictions with abscissas different from $\omega r$ have abscissas no greater than $\omega(r-2)$, see [20] and Remark 1.5. Hence, they should lie on the left from the line $\Lambda_{r'} = \{B = \omega r'\}$ containing the ray $R_\alpha$. Therefore, their ordinates should be less than $\alpha$. Indeed, if the ordinate of some "incorrect" constriction $X \in L_r^+$ were greater or equal to $\alpha$, then $X$ would be separated from the intersection $L_r \cap \{A \geq \alpha\}$ by the ray $R_\alpha$. Hence, $X$ would not lie in $L_r$, since $L_r \cap \{A \geq \alpha\}$ is connected (monotonicity of the rotation number function in $B$) – a contradiction. Thus, the "incorrect" constrictions cannot escape to $+\infty$, as $\omega \to \omega_0$ along some subsequence. This proves Proposition 4.8. □

**Proof of Proposition 4.7.** Here we repeat the arguments from [7] preceding conjecture 5.30. Recall that Conjecture 1.4 was proved in [20] for $\omega \geq 1$: for every $r \in \mathbb{Z}$ all the constrictions of the phase-lock area $L_r$ lie in its axis $\Lambda_r$. It was also shown in [20] that for arbitrary $\omega > 0$ for every $r \in \mathbb{Z}$ the abscissa of each constriction in $L_r$ equals $\omega l$ where $l \in \mathbb{Z} \cap [0, r]$ and $l \equiv r(mod 2)$. Let us now deform $\omega$ from 1 to 0: the phase-lock area portrait will change continuously in $\omega$. Fix an $r \in \mathbb{N}$. Suppose that there exists a certain "critical value" $\omega_0 \in (0, 1)$ such that for every $\omega > \omega_0$ all the constrictions in $L_r$ lie in $\Lambda_r$, and for $\omega = \omega_0$ a new constriction $(B_0, A_0)$ of the phase-lock area $L_r$ is born and it is "incorrect": it does not lie in $\Lambda_r$. This new constriction can be born only on an axis $\Lambda_l$ with $0 < l < r$, $l \equiv r(mod 2)$. Note that this is the only possible scenario of appearence of incorrect constrictions in $L_r$: they cannot come "from infinity", by Proposition 4.8. Then the boundary of the phase-lock area $L_r$ moves from the right to the left until it touches the axis $\Lambda_l$ for the first time. A point of intersection $\partial L_r \cap \Lambda_l$ cannot correspond to Heun equation (1.9) having a polynomial solution, by Theorem 1.16 and since $l < r$. Hence, it is a constriction, by the same theorem. Therefore, for $\omega = \omega_0$ the axis $\Lambda_l$ contains a constriction $(B_0, A_0)$ of the phase-lock area $L_r$ and separates $Int(L_r)$ from the origin. See Fig. 9. Thus, the newly born constriction $(B_0, A_0)$ is queer. Therefore, if Conjecture 4.6, which forbids queer constrictions, is true, then we would obtain a contradiction and this would prove Conjecture 1.4. □

**Proposition 4.9** *Conjecture 4.3 implies Conjecture 1.13. Conjecture 1.13 implies Conjecture 4.6. Conjecture 4.6 implies Conjecture 1.4.*

**Proof** The first implication follows from Proposition 4.4 and the symmetry of the phase-lock area portrait with respect to both coordinate axes.
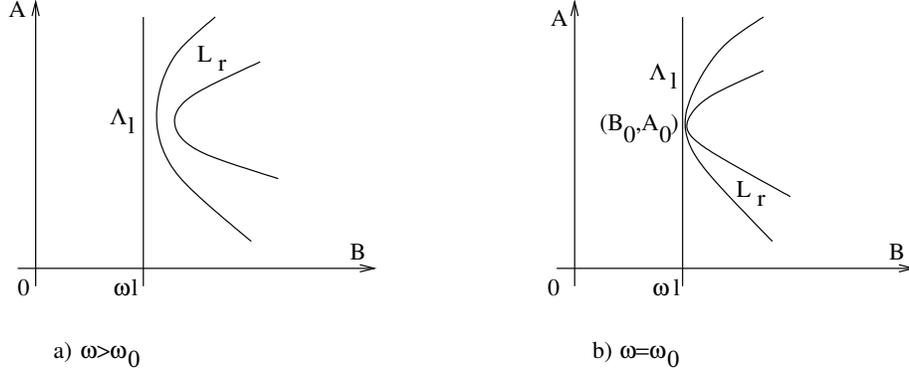
a) ω>ω₀       b) ω=ω₀

Figure 9: A conjecturally impossible scenario of appearense of an "incorrect" constriction of a phase-lock area $L_r$ in $\Lambda_l$, $0 < l < r$, as $\omega$ decreases to the critical value $\omega_0$.

Conjecture 1.13 obviously implies Conjecture 4.6, which in its turn implies Conjecture 1.4, by Proposition 4.7. □

# 5 Upper triangular element and further properties of the transition matrix $Q$

**Theorem 5.1** *Let $\omega > 0$, $l \geq 0$, $\mu > 0$, and let*

$$Q = \hat{W}_-^{-1}(z)W_+(z)M_N = \begin{pmatrix} -a & b \\ -c & a \end{pmatrix}$$

*be the corresponding transition matrix from (2.16). For every fixed $\omega > 0$ and $l \in \mathbb{Z}$ for every $\mu > 0$, set $(B, A) = (\omega l, 2\omega\mu)$, one has*

$$b = b(B, A) \in i\mathbb{R},$$

$$c = c(B, A) \in i\mathbb{R} \text{ whenever } (B, A) \text{ is a constriction.}$$

*For every $\omega > 0$ and $l \in \mathbb{N}$ the set of those parameter values $\mu > 0$, for which $c = c(B, A) \in i\mathbb{R}$, is discrete.*

In the proof of Theorem 5.1 we use the following formula for the determinant of the fundamental solution matrix $W_+(z)$.

32

**Lemma 5.2** *For every fundamental matrix $W(z)$ of solutions of system (1.7) one has*

$$\det W(z) = \mathcal{S} z^{-l} e^{\mu(\frac{1}{z}-z)}, \ \mathcal{S} \equiv const. \tag{5.1}$$

*The sectorial normalization matrices $H_\pm(z)$, see (2.3), are unimodular:*

$$\det H_\pm(z) \equiv 1, \tag{5.2}$$

$$\det W_\pm(z) \equiv z^{-l} e^{\mu(\frac{1}{z}-z)}. \tag{5.3}$$

*Here $W_\pm(z)$ are the canonical sectorial fundamental matrices from (2.3).*

**Proof** The proof of formulas (5.1) and (5.2) is based on the following Buchstaber–Tertychnyi formula for the Wronskian of Heun equation [14, proof of theorem 4]: *for every two solutions $E_1$ and $E_2$ of equation (1.8) one has*

$$E_1'(z)E_2(z) - E_2'(z)E_1(z) = \mathcal{W} z^{-l-1} e^{\mu(z+\frac{1}{z})}, \ \mathcal{W} \equiv const. \tag{5.4}$$

Let us first prove (5.1). Fix any pair of vector solutions $f_j(z) = (u_j(z), v_j(z))$ of system (1.7), $j = 1, 2$. Let $E_j(z) = e^{\mu z} v_j(z)$ denote the corresponding solutions of equation (1.8). One has $v_j(z) = e^{-\mu z} E_j(z)$,

$$u_j(z) = 2i\omega z v_j'(z) = 2i\omega z (e^{-\mu z} E_j(z))' = 2i\omega z e^{-\mu z}(E_j'(z) - \mu E_j(z)).$$

The determinant of the fundamental matrix $W(z)$ of the solutions $f_1(z)$ and $f_2(z)$ of system (1.7) equals

$$\det W(z) = u_1(z)v_2(z) - u_2(z)v_1(z) = 2i\omega z e^{-2\mu z}(E_1'(z)E_2(z) - E_2'(z)E_1(z))$$

$$= \mathcal{S} z^{-l} e^{\mu(\frac{1}{z}-z)}, \ \mathcal{S} = 2i\omega\mathcal{W},$$

by the latter formula and (5.4). This proves (5.1).

Let us now prove (5.2) and (5.3). Let $W_\pm(z) = H_\pm(z)F(z)$ denote the canonical sectorial fundamental matrix from (2.3). Its determinant equals $z^{-l} e^{\mu(\frac{1}{z}-z)}$ up to multiplicative constant, by (5.1), and the same statement holds for the matrix function $F(z)$:

$$F(z) = \mathrm{diag}(z^{-l} e^{\mu(\frac{1}{z}-z)}, 1), \ \det F(z) = z^{-l} e^{\mu(\frac{1}{z}-z)}.$$

Therefore, $\det H_\pm(z) \equiv const$. The latter constant equals one, since $H_\pm(0) = Id$. Formulas (5.2), (5.3) and Lemma 5.2 are proved. $\qquad\square$

**Proof of Theorem 5.1.** Let $W_+(z)$, $\hat{W}_-(z)$ be the fundamental matrices of solutions of system (1.7) from (2.16),

$$W_+ = \begin{pmatrix} f_{11,+} & f_{12,+} \\ f_{21,+} & f_{22,+} \end{pmatrix}, \tag{5.5}$$

see the notations from the proof of Theorem 2.3, $l \in \mathbb{Z}$. One has $M_N = Id$, since $l \in \mathbb{Z}$, hence $W_+ = W_+ M_N$,

$$\hat{W}_-(z) = \mathbb{I}(W_+)(z) = -iz^{-l}e^{\mu(\frac{1}{z}-z)} \begin{pmatrix} -f_{21,+} & -f_{22,+} \\ f_{11,+} & f_{12,+} \end{pmatrix}(z^{-1}), \tag{5.6}$$

by definition,

$$\det W_+(z) = f_{11,+}(z)f_{22,+}(z) - f_{12,+}(z)f_{21,+}(z) = z^{-l}e^{\mu(\frac{1}{z}-z)}, \tag{5.7}$$

by (5.3). Therefore,

$$D(z) = \det \begin{pmatrix} -f_{21,+} & -f_{22,+} \\ f_{11,+} & f_{12,+} \end{pmatrix}(z^{-1})$$

$$= (f_{11,+}f_{22,+} - f_{12,+}f_{21,+})(z^{-1}) = z^l e^{\mu(z-\frac{1}{z})},$$

by (5.7). Inverting the matrices in (5.6) yields

$$\hat{W}_-^{-1}(z) = iz^l e^{\mu(z-\frac{1}{z})} \begin{pmatrix} f_{12,+} & f_{22,+} \\ -f_{11,+} & -f_{21,+} \end{pmatrix}(z^{-1})D^{-1}(z)$$

$$= i \begin{pmatrix} f_{12,+} & f_{22,+} \\ -f_{11,+} & -f_{21,+} \end{pmatrix}(z^{-1}). \tag{5.8}$$

Calculating the product $Q = \hat{W}_-^{-1}(z)W_+(z)$ for $z = 1 \in \Sigma_1$ by formulas (5.5) and (5.8) yields

$$Q = \begin{pmatrix} -a & b \\ -c & a \end{pmatrix}, \quad b = i(f_{12,+}^2(1) + f_{22,+}^2(1)), \tag{5.9}$$

$$c = i(f_{11,+}^2(1) + f_{21,+}^2(1)), \quad a = -i(f_{11,+}(1)f_{12,+}(1) + f_{21,+}(1)f_{22,+}(1)).$$

One has $b \in i\mathbb{R}$, by (5.9) and since $f_{12,+}(z) \in i\mathbb{R}$, $f_{22,+}(z) \in \mathbb{R}$, whenever $z \in \mathbb{R}_+$, see (2.10). The first statement of Theorem 5.1 is proved. Its second statement, on the lower triangular element follows from the first statement and reality of the ratio $\frac{b}{c}$ at each constriction, see Proposition 4.2. Let us

34

prove the third statement of Theorem 5.1. In our case, when $l \in \mathbb{N}$, the second relation in (2.18) takes the form

$$bc_1 - cc_0 = ac_0c_1. \tag{5.10}$$

Recall that $c_0$ and $c_1$ are the Stokes multipliers, see (2.5), and they are real for $l \in \mathbb{Z}$, by Theorem 2.3. Writing relation (5.10) in the new parameters

$$\beta = -ib, \ \gamma = -ic, \ \sigma = \frac{c_1}{c_0}$$

yields

$$\beta\sigma - \gamma = -iac_1.$$

Substituting the latter formula to the first equation $a^2 = bc + 1$ in (2.18) yields

$$(\beta\sigma - \gamma)^2 = -c_1^2 a^2 = c_1^2(\beta\gamma - 1). \tag{5.11}$$

This yields the following quadratic equation in $\gamma$:

$$\gamma^2 - \beta(2\sigma + c_1^2)\gamma + \beta^2\sigma^2 + c_1^2 = 0, \tag{5.12}$$

which has a real solution if and only if its discriminant

$$\Delta = \beta^2(2\sigma + c_1^2)^2 - 4(\beta^2\sigma^2 + c_1^2) = c_1^2(\beta^2(4\sigma + c_1^2) - 4)$$

is non-negative, or equivalently,

$$\Delta_0 = \beta^2(4\sigma + c_1^2) - 4 \geq 0 \tag{5.13}$$

(whenever $c_1 \neq 0$). Note that the coefficients of the transition matrix $Q$ are analytic functions of the parameters on the complement to the hyperplane $\{A = 0\}$. Therefore, if, to the contrary, the coefficient $c$ were imaginary (or equivalently, $\gamma \in \mathbb{R}$) on some interval in $\Lambda_l^+ = \{B = l\omega\} \cap \{A > 0\}$, then $\gamma$ would be real on the whole interval $\Lambda_l^+$, and inequality (5.13) would hold on $\Lambda_l^+$, since $c_1 \not\equiv 0$ on $\Lambda_l^+$ (see Theorems 2.5 and 2.7). But this is impossible: $\Delta_0 < 0$ on a neighborhood of a simple intersection point $(B, A) \in \Lambda_l^+$, since $c_1 = \sigma = 0$, thus, $\Delta_0 = -4$ at this point. Indeed, a simple intersection corresponds to double confluent Heun equation (1.9) having a polynomial solution (Theorem 1.16), and hence, $c_1 = 0$ there, by Theorem 2.7. The contradiction thus obtained implies that $c$ is purely imaginary on a discrete subset in $\Lambda_l^+$. Theorem 5.1 is proved. $\qquad\square$

Theorem 5.1 and the above discussion imply the following corollary.

**Corollary 5.3** *For every $\omega > 0$, $l = \frac{B}{\omega} \in \mathbb{N}$ and arbitrary $\mu = \frac{A}{2\omega} > 0$ one has*

$$\Delta_0 = \beta^2(4\sigma + c_1^2) - 4 \leq 0, \ where \ \ \beta = -ib, \ \sigma = \frac{c_1}{c_0}. \qquad (5.14)$$

**Proof** If, to the contrary, $\Delta_0$ were positive on some interval in $\Lambda_l^+$, then the coefficient $c$ would be imaginary there, by the above discussion, – a contradiction to the last statement of Theorem 5.1. $\qquad\square$

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