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On spectral curves and complexified boundaries of the phase-lock areas in a model of Josephson junction

A. A. Glutsyuk ^{*†‡} I. V. Netay ^{§¶}

September 17, 2019

Abstract

The paper deals with a three-parameter family of special double confluent Heun equations that was introduced and studied by V. M. Buchstaber and S. I. Tertychnyi as an equivalent presentation of a model of overdamped Josephson junction in superconductivity. The parameters are $l, \lambda, \mu \in \mathbb{R}$. Buchstaber and Tertychnyi described those parameter values, for which the corresponding equation has a polynomial solution. They have shown that for $\mu \neq 0$ this happens exactly when $l \in \mathbb{N}$ and the parameters (λ, μ) lie on an algebraic curve $\Gamma_l \subset \mathbb{C}_{(\lambda, \mu)}^2$ called the *l-spectral curve* and defined as zero locus of determinant of a remarkable three-diagonal $l \times l$ -matrix. They studied the real part of the spectral curve and obtained important results with applications to model of Josephson junction, which is a family of dynamical systems on 2-torus depending on real parameters $(B, A; \omega)$; the parameter ω , called the frequency, is fixed. One of main problems on the above-mentioned model is to study the geometry of boundaries of its phase-lock areas in $\mathbb{R}_{(B, A)}^2$ and their evolution, as ω decreases to 0. An approach to this problem suggested in the present paper is to study the complexified boundaries. We prove irreducibility of the complex spectral curve Γ_l for every $l \in \mathbb{N}$. We also calculate its genus for $l \leq 20$

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||The results of Sections 1.2, 1.3, 3 are obtained by I. V. Netay

and present a conjecture on general genus formula. We apply the irreducibility result to the complexified boundaries of the phase-lock areas of model of Josephson junction. The family of real boundaries taken for all $\omega > 0$ yields a *countable* union of two-dimensional analytic surfaces in $\mathbb{R}^3_{(B,A,\omega^{-1})}$. We show that, unexpectedly, its complexification is a complex analytic subset consisting of just *four* two-dimensional irreducible components, and we describe them. This is done by using the representation of some special points of the boundaries (the so-called generalized simple intersections) as points of the real spectral curves and the above irreducibility result. We also prove that the spectral curve has no real ovals. We present a Monotonicity Conjecture on the evolution of the phase-lock area portraits, as ω decreases, and a partial positive result towards its confirmation.

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1 Introduction and main results

The paper deals with the family of special double confluent Heun equations

$$z^2 E'' + ((-l + 1)z + \mu(1 - z^2))E' + (\lambda + \mu(l - 1)z)E = 0; \quad l, \lambda, \mu \in \mathbb{C}. \quad (1.1)$$

The above family, which belong to the well-known class of Heun equations, see [35], was studied by V.M. Buchstaber and S.I. Tertychnyi in [37, 11, 12, 13]. They have shown in [11, 12] that its restriction to real parameters satisfying the inequality $\lambda + \mu^2 > 0$ is equivalent to a model of overdamped Josephson junction in superconductivity. In [11] they have described those complex parameters (l, λ, μ) with $\mu \neq 0$ for which equation (1.1) has a polynomial solution: this holds exactly, when $l \in \mathbb{N}$ and the point (λ, μ) lies on an algebraic curve $\Gamma_l \subset \mathbb{C}_{(\lambda, \mu)}^2$ called the *spectral curve*. Namely, Γ_l is the zero locus of the determinant of a remarkable 3-diagonal $l \times l$ -matrix, see [11, formula (21)] and Theorem 1.1 below.

In the present paper we show that for every $l \in \mathbb{N}$ the spectral curve Γ_l is irreducible (Theorems 1.2, 1.3 stated in Subsection 1.1 and proved in Subsection 2.1).

In Subsection 1.2 we state a conjecture on formula for genera of curves Γ_l and confirm it for $l \leq 20$ via a computer-assisted proof. In Section 3 we discuss this genus formula conjecture in more detail, with some more pictures for the real parts of the spectral curves. We prove that the conjectured genus formula provides an upper estimate for the genus. We show that the genus formula conjecture is equivalent to regularity of appropriate curve in $\mathbb{P}^1 \times \mathbb{P}^1$ birationally equivalent to Γ_l .

In Subsection 1.3 we study the real parts of the spectral curves. We prove that the upper half-plane $\{\mu > 0\}$ intersects Γ_l by l non-intersecting

smooth curves without vertical tangent lines. Then we deduce absence of real ovals in Γ_l .

In Subsections 1.4 and 1.5 we present background material on model of Josephson junction and its relation to special double confluent Heun equations. This model is given by a family of dynamical systems on two-torus depending on three real parameters: one of them, the frequency $\omega > 0$ is fixed; two other parameters (B, A) are variable. The rotation number of dynamical system is a function $\rho = \rho(B, A; \omega)$. The phase-lock areas are those level sets $\{\rho = r\}$ that have non-empty interiors. They exist only for $r \in \mathbb{Z}$, see [9]. It is interesting to study dependence of the rotation number on the parameters, the phase-lock area portrait in $\mathbb{R}_{(B,A)}^2$ and its evolution, as ω changes.

Conjecturally for every $r \in \mathbb{N}$ the upper half $L_r^+ = L_r \cap \{A \geq 0\}$ of each phase-lock area L_r satisfies the following statements (confirmed numerically, see Figures 3 and 4 taken from [5]):

C1) The upper area L_r^+ intersects the line $\Lambda_r = \{B = \omega r\}$ (which is called its axis) by a ray Sr going upwards and bounded from below by a point \mathcal{P}_r ; $Sr = \{\omega r\} \times [A(\mathcal{P}_r), +\infty)$ (see a partial result and a survey in [20, theorem 1.12 and section 4]).

C2) The complement

$$Ir := \Lambda_r^+ \setminus Sr = \{\omega r\} \times [0, A(\mathcal{P}_r)) \quad (1.2)$$

lies on the left from the area L_r^+ .

C3) As $\omega > 0$ decreases, in the renormalized coordinates ($l = \frac{B}{\omega}, \mu = \frac{A}{2\omega}$) the point \mathcal{P}_r moves up and the lower part $L_r \cap \{0 < A < A(\mathcal{P}_r)\}$ of the upper area L_r^+ "moves from the left to the right".

In Subsection 1.7 we discuss a Monotonicity Conjecture on evolution of the family of phase-lock area portraits, as $\omega > 0$ decreases. It deals with the intersection points of their boundaries with a segment Ir , see (1.2), $r \in \mathbb{N}$. Roughly speaking, the conjecture states that topologically the intersection points appear (disappear) in the same way, as if the above statement C3) were true. More precisely, it states that as $\omega > 0$ decreases, no new intersection point may be born from a part of a boundary curve moving from the right to the left in the renormalized coordinates. We present a partial result towards its confirmation: a proof for some special points of intersection $\partial L_s \cap Ir$ with $s \equiv r \pmod{2}$, $s \neq r$; the so called *generalized simple intersections* introduced in Subsection 1.5 (except for those with $s = l$). Namely, for every given $l \in \mathbb{N}$ and $\omega > 0$ consider the intersection points of the real part of the l -th spectral curve Γ_l with the curve $\{\lambda + \mu^2 = \frac{1}{4\omega^2}\}$. The generalized

simple intersections are exactly those points in the line Λ_l , whose $\mu = \frac{A}{2\omega}$ -coordinates coincide with the μ -coordinates of the above intersection points in the l -th spectral curve. The Monotonicity Conjecture will be proved in Section 4 for all the generalized simple intersections in Λ_l , except for those lying in $\Lambda_l \cap \partial L_l$.

In Subsection 1.5 we state new results on generalized simple intersections (Theorem 1.18 and Corollary 1.19). Corollary 1.19 states that for every $l \in \mathbb{N}$ and every $\omega > 0$ small enough (dependently on l) there are exactly l distinct generalized simple intersections in Λ_l , and they depend analytically on $\omega > 0$; they obviously tend to the A -axis, as $\omega \rightarrow 0$.

The boundaries of the phase-lock areas form a countable union of analytic curves in $\mathbb{R}_{(B,A)}^2$. Their families depending on the frequency parameter ω form a countable union of two-dimensional analytic surfaces in $\mathbb{R}_{(B,A,\omega^{-1})}^3$. In Subsection 1.6 we present the following **unexpected result** (Theorem 1.22): *the complexification* of the above *countable* union of surfaces (families of boundaries) is a two-dimensional analytic subset in \mathbb{C}^3 consisting of just *four irreducible components!* The proof of Theorem 1.22 given in Subsection 2.3 is based on the irreducibility of spectral curves Γ_l and a result on the mentioned above simple intersections: on irreducible components of their complexified families (Theorem 1.20 stated in Subsection 1.5 and proved in Subsection 2.2).

At the end of the paper we discuss some open questions on the complexified (unions of) boundaries of the phase-lock areas. And also about the complex family of double confluent Heun equations (1.1): on the locus of those parameter values in \mathbb{C}^3 , for which the corresponding monodromy operator has a multiple eigenvalue. We provide a partial result on irreducible components of the latter locus, which is an immediate corollary of main results.

1.1 Irreducibility of loci of special double confluent Heun equations with polynomial solutions (A. A. Glutsyuk)

Let us recall the description of the parameters corresponding to equations (1.1) with polynomial solutions. To do this, consider the three-diagonal $l \times l$ -matrix

$$H_l = \begin{pmatrix} 0 & \mu & 0 & 0 & 0 & 0 \dots & 0 \\ \mu(l-1) & 1-l & 2\mu & 0 & 0 & \dots & 0 \\ 0 & \mu(l-2) & -2(l-2) & 3\mu & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 2\mu & -2(l-2) & (l-1)\mu \\ 0 & \dots & 0 & 0 & 0 & \mu & 1-l \end{pmatrix} : (1.3)$$

$$\begin{aligned} H_{l;ij} &= 0 \text{ if } |i-j| \geq 2; & H_{l;jj} &= (1-j)(l-j+1); \\ H_{l;j,j+1} &= \mu j; & H_{l;j,j-1} &= \mu(l-j+1). \end{aligned}$$

The matrix H_l belongs to the class of the so called Jacobi matrices that arise in different questions of mathematics and mathematical physics [24].

Theorem 1.1 [11, section 3] *A special double confluent Heun equation (1.1) with $\mu \neq 0$ has a polynomial solution, if and only if $l \in \mathbb{N}$ and the three-diagonal matrix $H_l + \lambda \text{Id}$ has zero determinant. For every $l \in \mathbb{N}$ the determinant $\det(H_l + \lambda \text{Id})$ is a polynomial in $(u, v) = (\lambda, \mu^2)$ of degree l :*

$$\det(H_l + \lambda \text{Id}) = P_l(\lambda, \mu^2). \quad (1.4)$$

See also [5, remark 4.13] for non-existence of polynomial solutions for $l \notin \mathbb{N}$ and $\mu \neq 0$.

Theorem 1.2 *The polynomial $P_l(u, v)$ from (1.4) is irreducible.*

Theorem 1.3 *For every l the l -th spectral curve*

$$\Gamma_l := \{P_l(\lambda, \mu^2) = 0\} \subset \mathbb{C}_{(\lambda, \mu)}^2$$

is irreducible.

Theorems 1.2 and 1.3 will be proved in Subsection 2.1. One can see examples of real parts of some curves Γ_l on Fig. 1.

Remark 1.4 Equation (1.4) defining the curve Γ_l belongs to a remarkable class of determinantal representations of plane curves: equations

$$\det(x_1 L_1 + x_2 L_2 + x_3 L_3) = 0,$$

where $(x_1 : x_2 : x_3) \in \mathbb{CP}^2$ and L_1, L_2, L_3 are $l \times l$ -matrices. Determinantal representations of curves arise in algebraic geometry and integrable systems, see [38, 39] and references therein. Complete description of determinantal representations of smooth complex irreducible projective curves was obtained in [38]. Self-adjoint determinantal representations of real smooth plane curves were described in [39].

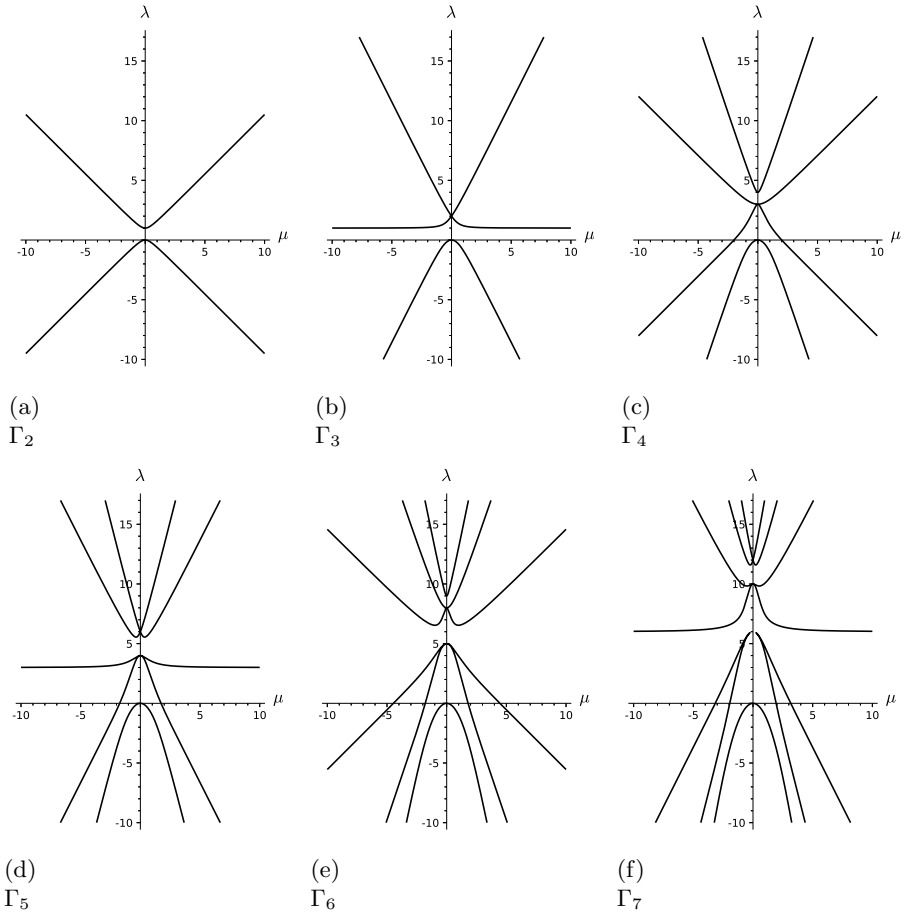


Figure 1: Some examples of curves Γ_l (I.V.Netay).

1.2 Genera of spectral curves of low degrees and a genus formula conjecture (I. V. Netay)

Recall that the *geometric genus* of an irreducible algebraic curve is the genus of the Riemann surface parametrizing it bijectively (except for possible singularities), i. e., the genus of its normalization.

Conjecture 1.5 *The geometric genus of the curve Γ_l equals*

$$\begin{cases} \left(\frac{l-2}{2}\right)^2, & l \text{ even;} \\ \left(\frac{l-1}{2}\right)\left(\frac{l-3}{2}\right), & l \text{ odd.} \end{cases}$$

We will prove in Section 3 that the estimate from above holds, i. e. the genera of curves Γ_l do not exceed these values (see Proposition 3.4).

For small l one can get the above formulas directly (see Section 3). For some next values (for $l \leq 20$; $g(\Gamma_{20}) = 81$) the Conjecture holds and can be verified by direct computation (the author of the present subsection computed this in the **SageMath** open source computer algebra system).

The function `geometric_genus` refers to the computation in the package **Singular** (free available computer algebra system for polynomial computations with special emphasis on the needs of commutative algebra, algebraic geometry, and singularity theory.) It calculates the *Hilbert polynomial* (see more details and examples in [15]) of the normalization of the curve.

Let $C \subset \mathbb{C}\mathbb{P}^n$ be a projective curve. Then its *Hilbert polynomial* is

$$h_C(t) = d \cdot t - p_a(C) + 1,$$

where d is the degree of the curve C , $p_a(C)$ is its arithmetic genus¹. The geometric genus is the arithmetic genus of the normalization C_n of C . If we are able to compute the normalization, we can compute the geometric genus. The most time consuming operation here is normalization. In the case of Γ_{20} it takes about a few hours. See the code for computing genera at Fig. 2.

1.3 Real spectral curve $\{P_l(\lambda, \mu^2) = 0\}$: topology and absence of ovals (I. V. Netay)

Here we prove some results on topology of the real spectral curves Γ_l .

¹See <https://www.win.tue.nl/~aeb/2WF02/hilbert.pdf>, p.5

```

K = PolynomialRing(QQ, 3, names='theta,mu,r')
theta, mu, r = K.gens()
P.<theta,mu,r> = ProjectiveSpace(QQ, 2)

d = lambda i, j: 1 if i == j else 0

def h(n,i,j):
    if i == j:
        return -i*(n - i + 1)*r
    elif i + 1 == j:
        return (i + 1)*mu
    elif i == j + 1:
        return (n - i + 1)*mu
    else:
        return 0
H = lambda n: matrix(
    [[h(n,i,j) + theta*d(i,j)
      for j in range(n+1)]
     for i in range(n+1)]
)

for i in range(1, 20)
    C = P.curve([K(det(H(i)))]])
    print(C.geometric_genus())

```

Figure 2: Listing of the code computing genera of some curves Γ_l .

Lemma 1.6 *The intersection of Γ_l with open half-plane $\{\mu > 0\}$ consists of l non-intersecting smooth curves without horizontal tangents $\{\mu = \text{const}\}$.*

Definition 1.7 An *oval* of a real planar projective algebraic curve Γ is its subset analytically parametrized by circle; the parametrization should be bijective except for possible singularities of the curve. (An oval may contain singularities, and the parametrization may have zero derivative at some singularities.)

Theorem 1.8 *The real curve $\Gamma_l = \{P_l(\lambda, \mu^2) = 0\}$ has no ovals in the affine plane $\mathbb{R}_{(\lambda, \mu)}^2$ and no ovals in closure of the half-plane $\{\mu > 0\}$ in the ambient projective plane $\mathbb{RP}^2 \supset \mathbb{R}_{(\lambda, \mu)}^2$.*

The above lemma and theorem are basically implied by the following key result of V. M. Buchstaber and S. I. Tertychnyi and Proposition 1.10 stated below.

Theorem 1.9 [11, p.974, theorem 1]. For $\mu \neq 0$ all the eigenvalues of the matrix H_l are real and simple; that is, the polynomial $P_l(\lambda, \mu^2)$ with fixed μ considered as a polynomial in one variable λ has l real and simple roots.

Proof of Lemma 1.6. For any fixed $\mu' > 0$ the intersection $\Gamma_l \cap \{\mu = \mu'\}$ consists of l distinct points, by Theorem 1.9. At the same time by Bézout Theorem the intersection of a complex curve of degree l with any complex line consists of l points if any point is counted with its multiplicity.

For real μ' we already have l distinct real intersection points. Also Theorem 1.9 implies that all these intersections are simple. This implies that they are regular points of the curve Γ_l and at each of these points the line $\{\mu = \mu'\}$ is not tangent to Γ_l . This proves Lemma 1.6. \square

Proof of Theorem 1.8. Lemma 1.6 immediately implies absence of ovals in the affine plane $\mathbb{R}_{(\lambda, \mu)}^2$ and in the upper half-plane $\{\mu > 0\} \subset \mathbb{R}^2$. Moreover, Lemma 1.6 together with Roll Theorem imply that ovals lying in its closure in \mathbb{RP}^2 (if any) should intersect both its boundary lines: the axis $\{\mu = 0\}$ and the infinity line. To prove absence of the latter ovals, we use the following proposition.

Proposition 1.10 *The asymptotic directions of branches of the complex curve Γ_l at infinity correspond to ratios λ/μ equal $l-1, l-3, \dots, -(l-1)$. In particular, Γ_l does not contain the point of intersection of the λ -axis with the infinity line in $\mathbb{CP}^2 \supset \mathbb{C}_{(\lambda, \mu)}^2$.*

Proof Asymptotic directions correspond to the intersection points of the curve Γ_l with the infinity line. That is, to zeros of the higher homogeneous part of the polynomial $\det(H_l + \lambda \text{Id})$, which is equal to the determinant of the $l \times l$ -matrix

$$\begin{pmatrix} \lambda & \mu & & & & \\ (l-1)\mu & \lambda & 2\mu & & & \\ & (l-2)\mu & \lambda & \ddots & & \\ & & \ddots & \ddots & (l-1)\mu & \\ & & & \mu & \lambda & \end{pmatrix}. \quad (1.5)$$

The matrix (1.5) coincides with the matrix of the operator

$$L_{\lambda, \mu} := \lambda + \mu S, \quad S := y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \quad (1.6)$$

acting on the space of homogeneous polynomials of degree $l-1$ in x and y . The vector fields $y \frac{\partial}{\partial x}$ and $x \frac{\partial}{\partial y}$ generate the Lie algebra $\mathfrak{sl}(2)$ of divergence

free linear vector fields. The space of homogeneous polynomials of degree $l-1$ as a representation of $\mathfrak{sl}(2)$ is the $(l-1)$ -th symmetric power of its representation acting on linear forms. In the space of linear forms the eigenvalues of the operator S in (1.6) are $\{-1, 1\}$. Therefore the eigenvalues of its action on the $(l-1)$ -th symmetric power are $l-1, l-3, \dots, -(l-1)$. Hence, the determinant of the matrix (1.5) of the operator $L_{\lambda, \mu}$ is equal to $(\lambda - (l-1)\mu)(\lambda - (l-3)\mu) \dots (\lambda + (l-1)\mu)$. This proves Proposition 1.10. \square

The ratios $\lambda/\mu = l-1, l-3, \dots, -(l-1)$ from Proposition 1.10 corresponding to the intersection of the curve Γ_l with the infinity line form l distinct numbers. Therefore, the curve Γ_l intersects the infinity line transversely at its l distinct regular points (being a curve of degree l). The latter intersection points are distinct from the point of intersection of the infinity line with the λ -axis (Proposition 1.10). Hence, each local branch of the real curve Γ_l at any of the latter points crosses the infinity line from the half-plane $\{\mu > 0\}$ to the opposite half-plane $\{\mu < 0\}$. This implies that Γ_l cannot have ovals intersecting the infinity line and lying in the closure of the upper half-plane. This finishes the proof of Theorem 1.8. \square

Proposition 1.11 *The curve Γ_l intersects the line $\{\mu = 0\} \subset \mathbb{RP}^2$ at points with λ -abscissas $0, 1 \cdot (l-1), 2(l-2), \dots, (l-1) \cdot 1$. All these intersection points are singular points of the curve Γ_l , except for the points with abscissas $\frac{l^2}{4}$ (if l is even) and 0 , which are regular points of orthogonal intersection.*

Proof The first statement of the proposition follows from definition and Proposition 1.10 (which implies that the point of the intersection of the line $\{\mu = 0\}$ with the infinity line does not lie in Γ_l). Note that all the above abscissas are roots of the polynomial $P_l(\lambda, 0)$, and the multiplicity of each of them equals two (except for possible roots 0 and $\frac{l^2}{4}$, where the multiplicity is equal to 1). Recall that the curve Γ_l is invariant under the symmetry $(\lambda, \mu) \mapsto (\lambda, -\mu)$, and hence, so are its germs at all the above intersection points. This implies that the germ of the curve Γ_l at any of the points $(\frac{l^2}{4}, 0)$ (if l is even) and $(0, 0)$ consists of just one local branch orthogonal to the line $\{\mu = 0\}$. The germ of the curve Γ_l at any other intersection point contains at least two local branches, and hence, is singular. Indeed, otherwise if it consisted of just one regular branch, then this branch would be tangent to the line $\{\mu = 0\}$ (the intersection point being of multiplicity two). The latter branch obviously cannot coincide with the line and thus, cannot be invariant under the above symmetry. The proposition is proved. \square

1.4 Model of the overdamped Josephson effect, phase-lock areas and associated family of Heun equations

Our results are motivated by applications to the family

$$\frac{d\varphi}{dt} = -\sin \varphi + B + A \cos \omega t; \quad \omega > 0, \quad B, A \in \mathbb{R}, \quad (1.7)$$

of nonlinear equations, which arises in several models in physics, mechanics and geometry: in a model of the Josephson junction in superconductivity (our main motivation), see [25, 36, 14, 3, 31, 32], [4]-[12]; in planimeters, see [16, 17]. Here ω is a fixed constant, and (B, A) are the parameters; B is called *abscissa* and A *ordinate*. Set

$$\tau = \omega t, \quad l = \frac{B}{\omega}, \quad \mu = \frac{A}{2\omega}.$$

The variable change $t \mapsto \tau$ transforms (1.7) to a non-autonomous ordinary differential equation on the two-torus $\mathbb{T}^2 = S^1 \times S^1 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ with coordinates (φ, τ) :

$$\dot{\varphi} = \frac{d\varphi}{d\tau} = -\frac{\sin \varphi}{\omega} + l + 2\mu \cos \tau. \quad (1.8)$$

The graphs of its solutions are the orbits of the vector field

$$\begin{cases} \dot{\varphi} = -\frac{\sin \varphi}{\omega} + l + 2\mu \cos \tau \\ \dot{\tau} = 1 \end{cases} \quad (1.9)$$

on \mathbb{T}^2 . The *rotation number* of its flow, see [1, p. 104], is a function $\rho(B, A) = \rho(B, A; \omega)$ of the parameters of the vector field. It is given by the formula

$$\rho(B, A; \omega) = \lim_{k \rightarrow +\infty} \frac{\varphi(2\pi k)}{2\pi k},$$

where $\varphi(\tau)$ is an arbitrary solution of equation (1.8). (Normalization convention: the rotation number of a usual circle rotation equals the rotation angle divided by 2π .)

The *phase-lock areas* are those level subsets of the rotation number function in the (B, A) -plane that have non-empty interiors. They have been studied by V.M. Buchstaber, O.V. Karpov, S.I. Tertychnyi et al, see [4]-[12], [19, 22, 23, 26, 27] and references therein. It is known that the *phase-lock areas exist only for integer values of the rotation number function* [9], contrarily to the Arnold tongues picture [1, p.110].

Remark 1.12 Fix an $\omega > 0$. For a given $(B, A) \in \mathbb{R}^2$ consider the time 2π flow map

$$h_{(B,A)} : S_\varphi^1 \rightarrow S_\varphi^1$$

of the corresponding dynamical system (1.9) on the torus acting on the transversal circle $S_\varphi^1 = \{\tau = 0\} = \mathbb{R}/2\pi\mathbb{Z}$. It coincides with the corresponding Poincaré map of the transversal circle. It is known, see, e. g., [27], that (B, A) lies in the boundary of a phase-lock area, if and only if $h_{(B,A)}$ either has a parabolic fixed point (and then it has a unique fixed point $\pm \frac{\pi}{2} \pmod{2\pi\mathbb{Z}}$), or is identity.

In what follows the phase-lock area corresponding to a rotation number s will be denoted by L_s .

It is known that the phase-lock areas satisfy the following geometric statements:

- (i) The boundary ∂L_s of the s -th phase-lock area is a union of two graphs $\partial L_{s,\pm}$ of two analytic functions $B = g_{s,\pm}(A)$, see [10]. This fact was later explained by A. V. Klimenko via symmetry, see [27], where it was shown that each graph $\partial L_{s,\pm}$ consists exactly of those points in L_s for which the corresponding Poincaré map $h_{(B,A)}$ fixes the point $\pm \frac{\pi}{2} \pmod{2\pi\mathbb{Z}}$.
- (ii) The latter functions have Bessel asymptotics, as $A \rightarrow \infty$ (observed and proved on physical level in [34], see also [28, chapter 5], [3, section 11.1], [8]; proved mathematically in [27]).
- (iii) Each phase-lock area is an infinite chain of bounded domains going to infinity in the vertical direction, in this chain each two subsequent domains are separated by one point. This follows from Bessel asymptotics, see the references mentioned in (ii). Those separation points that lie in the horizontal B -axis are calculated explicitly, and we call them *exceptional*, or *growth points*, see [10, corollary 3]. The other separation points lie outside the horizontal B -axis and are called the *constrictions*, see Fig. 3, 4. *The Poincaré map $h_{(B,A)}$ is identity if and only if (B, A) is either a constriction, or a growth point* [19, proposition 2.2].
- (iv) Each phase-lock area L_r is symmetric itself with respect to the B -axis and is symmetric to L_{-r} with respect to the A -axis. This follows from the fact that the transformations $(\phi, \tau) \mapsto (\phi, \tau + \pi)$ and $(\phi, \tau) \mapsto (-\phi, \tau + \pi)$ inverse the sign at μ (respectively, at l) in the vector field (1.9), see [5, p.76].

One of the main open conjectures on the geometry of phase-lock area portrait is the following.

Conjecture 1.13 (experimental fact, see [19]). *In every phase-lock area L_ρ all the constrictions lie in the line $\{B = \rho\omega\}$.*

It was shown in [19] that at all the constrictions in L_ρ one has $l = \frac{B}{\omega} \in \mathbb{Z}$, $l \equiv \rho \pmod{2}$ and $l \in [0, \rho]$. See a survey on Conjecture 1.13 and related open problems in [19, 5, 20].

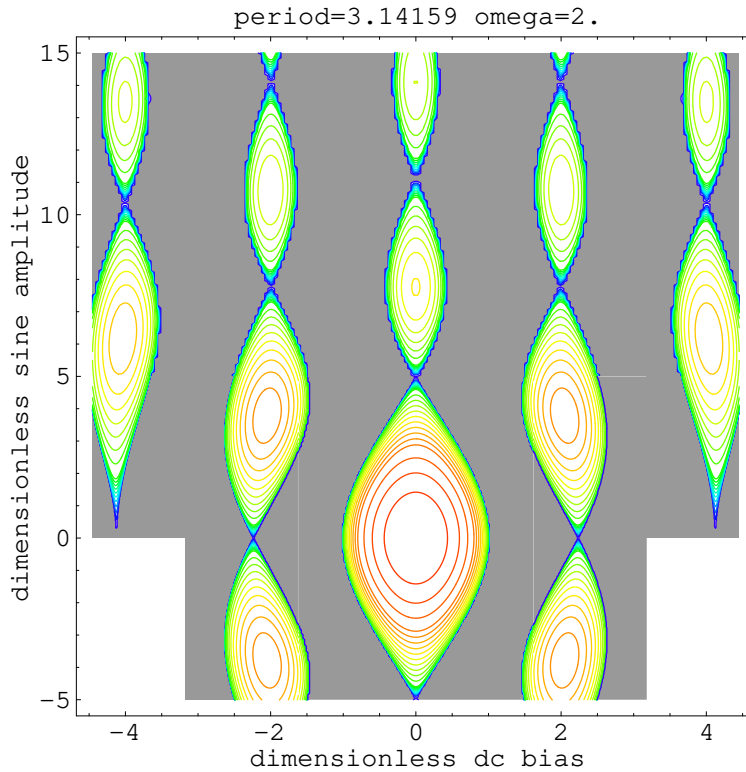


Figure 3: Phase-lock areas and their constrictions for $\omega = 2$. The abscissa is B , the ordinate is A . Figure taken from [5, fig. 1a)]

The family of non-linear equations (1.7) was reduced in [7, 16, 12, 23] to a family of linear equations, which were written in form (1.1) (with opposite sign at l) in [11, 12, 37]. Let us recall an equivalent reduction given in [4, subsection 3.2]. Set

$$\Phi = e^{i\varphi}, \quad z = e^{i\tau} = e^{i\omega t}. \quad (1.10)$$

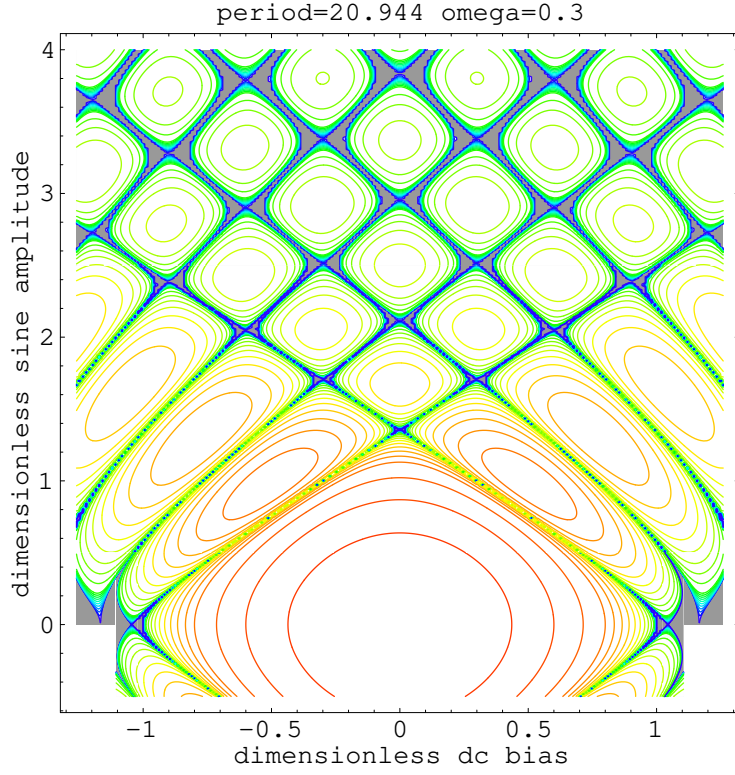


Figure 4: Phase-lock areas and their constrictions for $\omega = 0.3$. Figure taken from [5, fig. 1e)]

Considering equation (1.7) with complex time t we get that transformation (1.10) sends it to the Riccati equation

$$\frac{d\Phi}{dz} = z^{-2}((lz + \mu(z^2 + 1))\Phi - \frac{z}{2i\omega}(\Phi^2 - 1)). \quad (1.11)$$

This equation is the projectivization of the following linear system in vector function (u, v) with $\Phi = \frac{v}{u}$, see [4, subsection 3.2]:

$$\begin{cases} v' = \frac{1}{2i\omega z}u \\ u' = z^{-2}(-(lz + \mu(1 + z^2))u + \frac{z}{2i\omega}v) \end{cases} \quad (1.12)$$

Set

$$\lambda = \left(\frac{1}{2\omega}\right)^2 - \mu^2. \quad (1.13)$$

It is easy to check that a function $v(z)$ is the component of a solution of system (1.12), if and only if the function $E(z) = e^{\mu z}v(z)$ satisfies the second order linear equation obtained from (1.1) by changing sign at l :

$$\mathcal{L}E = z^2 E'' + ((l+1)z + \mu(1-z^2))E' + (\lambda - \mu(l+1)z)E = 0. \quad (1.14)$$

Remark 1.14 Heun equations (1.1) and (1.14) corresponding to the family (1.9) of dynamical systems on torus are those corresponding to real parameters l, ω, μ , and thus, real λ . In the present paper we are dealing with general equation (1.1), with arbitrary complex parameters λ, μ .

1.5 Generalized simple intersections and polynomial solutions (A. A. Glutsyuk)

Let us recall the following definition.

Definition 1.15 [20, definition 1.3] The *axis* of a phase-lock area $L_s = L_s(\omega)$ is the vertical line

$$\Lambda_s = \Lambda_s(\omega) = \{B = s\omega\} \subset \mathbb{R}_{(B,A)}^2; \quad s \in \mathbb{Z}.$$

Definition 1.16 (compare with [20, definition 1.9]) A *generalized simple intersection* is a point of intersection of an axis Λ_l with the boundary of a phase-lock area L_s with $s \equiv l \pmod{2}$ that is not a constriction, see Fig. 5.

Theorem 1.17 [11, 5] *The generalized simple intersections with $B \geq 0$ and $A \neq 0$ correspond exactly to those real parameter values $(B, A; \omega)$ with $\omega, B > 0, A \neq 0$ for which Heun equation (1.1) with $l = \frac{B}{\omega}, \mu = \frac{A}{2\omega}, \lambda = \frac{1}{4\omega^2} - \mu^2$ has a polynomial solution. Generalized simple intersections in Λ_l may exist only for $l \in \mathbb{Z} \setminus \{0\}$ and those $s \equiv l \pmod{2}$ that lie in the segment $[0, l]$.*

We prove the following new result on generalized simple intersections lying in *upper semiaxis*

$$\Lambda_l^+ = \Lambda_l^+(\omega) := \Lambda_l(\omega) \cap \{A \geq 0\}.$$

Theorem 1.18 *For every $l \in \mathbb{N}$ and every $\omega > 0$ the semiaxis $\Lambda_l^+ = \Lambda_l^+(\omega)$ contains at least one generalized simple intersection lying in $\partial L_l = \partial L_l(\omega)$. For every $l, s \in \mathbb{N}, l \geq 3, s \equiv l \pmod{2}, 0 < s < l$, and every $\omega > 0$ small enough (dependently on l) the semiaxis Λ_l^+ intersects each component $\partial L_{s,\pm}$ of the boundary of the phase-lock area L_s . For every $l \geq 2$ and every $\omega > 0$ small enough the boundary component $\partial L_{0,+}$ intersects Λ_l^+ . See Fig. 5.*

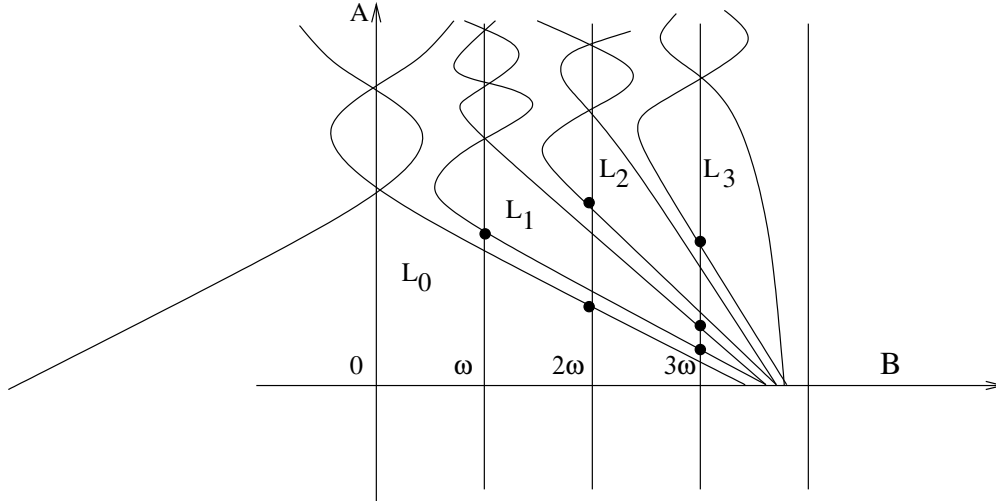


Figure 5: Approximate phase-lock areas for $\omega \simeq 0.27$; the marked points are generalized simple intersections. This hand-made figure, which is taken from [5, fig. 2], illustrates open Conjecture 1.13: for every $r \in \mathbb{Z}$ all the constrictions of the phase-lock area L_r lie in its axis Λ_r .

Corollary 1.19 *For every $l \in \mathbb{N}$ and every $\omega > 0$ small enough (dependently on l) the semiaxis $\Lambda_l^+(\omega)$ contains exactly l distinct generalized simple intersections; exactly one of them lies in ∂L_l . They depend analytically on ω and tend to the A -axis, as $\omega \rightarrow 0$: their common abscissa ωl tends to zero.*

Set

$$r = \frac{1}{2\omega}; \quad \mu = \frac{A}{2\omega} = Ar.$$

For given $r > 0$ (or equivalently, $\omega > 0$) and $l \in \mathbb{N}$, the set of the μ -coordinates of the generalized simple intersections in the axis $\Lambda_l(\omega)$ will be denoted by $\text{SI}_l(r)$. We set

$$\text{SI}_l := \cup_{r>0} (\text{SI}_l(r) \times \{r\}) \subset \mathbb{R}_{(\mu,r)}^2.$$

Recall that for every $(\mu, r) \in \text{SI}_l$ the corresponding point $(B = \frac{l}{2r}, A = \frac{\mu}{r})$ is not a constriction, but a priori it can be a growth point. The complement of the set SI_l to growth points is an open and dense subset $\text{SI}_l^0 \subset \text{SI}_l$ consisting of those parameters for which the corresponding Poincaré map $h_{(B,A)}$ has a unique fixed point $\pm \frac{\pi}{2} \pmod{2\pi\mathbb{Z}}$, see Remark 1.12 and statement (iii) in Subsection 1.4. Thus, SI_l^0 is split into two disjoint parts $\text{SI}_{l,\pm}^0$ of those

parameter values for which the above fixed point is $\pm\frac{\pi}{2} \pmod{2\pi\mathbb{Z}}$. Set

$$\begin{aligned}\widehat{\text{SI}}_l &:= \text{the complex Zariski closure of the subset } \text{SI}_l \text{ in } \mathbb{C}_{(\mu,r)}^2, \\ \widehat{\text{SI}}_{l,\pm} &:= \text{the complex Zariski closure of the subset } \text{SI}_{l,\pm}^0.\end{aligned}\tag{1.15}$$

Theorem 1.20 *For every $l \in \mathbb{N}$ the image of the subset $\widehat{\text{SI}}_l \subset \mathbb{C}_{(\mu,r)}^2$ under the mapping $(\mu, r) \mapsto (\mu, r^2)$ is an irreducible algebraic curve. The algebraic subset $\widehat{\text{SI}}_l$ consists of two irreducible components: the algebraic curves $\widehat{\text{SI}}_{l,\pm}$.*

Theorems 1.18, 1.20 and Corollary 1.19 will be proved in Subsection 2.2, together with the following new result used in the proof of Theorem 1.20.

Theorem 1.21 *For every $s \in \mathbb{Z}$ the symmetry $(B, A) \mapsto (B, -A)$ with respect to the B -axis preserves the boundary curves $\partial L_{s,\pm}$ for even s and interchanges them for odd s . In particular, for every even s the curves $\partial L_{s,\pm}$ are orthogonal to the B -axis at the growth point of the phase-lock area L_s . The symmetry $(B, A) \mapsto (-B, A)$ with respect to the A -axis permutes the curves $L_{s,\pm}$ and $L_{-s,\mp}$ for even s and the curves $L_{s,\pm}, L_{-s,\pm}$ for odd s .*

1.6 Boundaries of phase-lock areas and the complexified union of their families (A. A. Glutsyuk)

For every $\omega > 0$ let $\mathcal{L}_{\pm}^{\text{odd}}(\omega), \mathcal{L}_{\pm}^{\text{even}}(\omega) \subset \mathbb{R}_{(B,A)}^2$ denote the unions of those points in the boundaries of the phase-lock areas with odd (respectively, even) rotation numbers, for which the corresponding Poincaré map $h_{(B,A)}$ fixes the point $\pm\frac{\pi}{2} \pmod{2\pi\mathbb{Z}}$. Recall that we denote $r = \frac{1}{2\omega}$. Set

$$\mathcal{L}_{\pm}^{\text{odd}(\text{even})} := \bigcup_{\omega>0} (\mathcal{L}_{\pm}^{\text{odd}(\text{even})}(\omega) \times \{r\}) \subset \mathbb{R}_{(B,A,r)}^3,$$

$\widehat{\mathcal{L}}_{\pm}^{\text{odd}(\text{even})} :=$ the minimal analytic subset in $\mathbb{C}_{(B,A,r)}^3$ containing $\mathcal{L}_{\pm}^{\text{odd}(\text{even})}$.

Theorem 1.22 *The analytic subsets $\widehat{\mathcal{L}}_{\pm}^{\text{odd}}, \widehat{\mathcal{L}}_{\pm}^{\text{even}}$ are distinct two-dimensional irreducible analytic subsets. The union of these four irreducible analytic subsets is the minimal complex analytic subset in $\mathbb{C}_{(B,A,r)}^3$ containing $\bigcup_{\omega>0} \bigcup_{s \in \mathbb{Z}} (\partial L_s(\omega) \times \{\frac{1}{2\omega}\})$.*

Theorem 1.22 will be proved in Subsection 2.3.

1.7 On family of phase-lock area portraits, the Monotonicity Conjecture and constrictions (A.A.Glutsyuk)

In this subsection we consider the model of Josephson effect and study the evolution of the phase-lock area portrait, as $\omega > 0$ decreases. We present the Monotonicity Conjecture, its relation with Conjecture 1.13 on abscissas of constrictions and a partial result towards confirmation of the Monotonicity Conjecture.

Definition 1.23 Let $\omega_0 > 0$, $l \in \mathbb{N}$, $\mu > 0$, $B_0 = l\omega_0$, $A_0 = 2\mu\omega_0$. Let $s \in \mathbb{Z}_{\geq 0}$. Let $(B_0, A_0) \in \partial L_{s,\pm}(\omega_0)$ for appropriately chosen sign \pm . Let $\partial L_{s,\pm}(\omega_0)$ be tangent to the axis $\Lambda_l(\omega_0) = \{B = l\omega_0\}$ at (B_0, A_0) . Let the germ of the curve $\partial L_{s,\pm}$ at (B_0, A_0) lie on the right from the line $\Lambda_l(\omega_0)$. We say that the point $(B_0, A_0) \in \mathbb{R}_+^2$ is a *left-moving tangency* (for $\omega = \omega_0$), if the point (B_0, A_0) of intersection $\partial L_{s,\pm}(\omega) \cap \Lambda_l(\omega)$ with $\omega = \omega_0$ disappears for $\omega > \omega_0$: there exists a neighborhood U of the point (B_0, A_0) such that for every $\omega > \omega_0$ close enough to ω_0 (dependently on U) the intersection $\partial L_{s,\pm}(\omega) \cap \Lambda_l(\omega)$ contains no points in U .

Conjecture 1.24 (Monotonicity Conjecture). *There are no left-moving tangencies in model of Josephson effect.*

Remark 1.25 The Monotonicity Conjecture means that as $\omega > 0$ decreases, the intersections of the axes $\Lambda_k(\omega)$, $k \in \mathbb{N}$, with boundaries of phase-lock areas cannot be born from those pieces of boundaries that move "from the right to the left" in the renormalized coordinates ($l = \frac{B}{\omega}, \mu = \frac{A}{2\omega}$). That is why we call it the Monotonicity Conjecture.

Remark 1.26 Let $s \equiv l \pmod{2}$, $s \geq 0$, $\omega > 0$. Every point of intersection $\Lambda_l(\omega) \cap \partial L_{s,\pm}(\omega)$ (if any; in particular, every left-moving tangency lying there) is either a generalized simple intersection, or a constriction, see Figures 6 and 7. This follows from definition. If $s \equiv l + 1 \pmod{2}$, then the above intersection points are neither generalized simple intersections (by definition), nor constrictions (by [19, theorem 3.17]).

Proposition 1.27 *Let $l \in \mathbb{N}$, $s \equiv l \pmod{2}$, $\omega_0 > 0$, and let some left-moving tangency (B_0, A_0) for $\omega = \omega_0$ be a constriction. Then for every $\omega < \omega_0$ close enough to ω_0 the points of intersection $\Lambda_l(\omega) \cap \partial L_{s,\pm}(\omega)$ newly born from (B_0, A_0) (if any) are constrictions. See Fig. 7.*

Proof If for every $\omega < \omega_0$ close enough to ω_0 some of the above newly born intersection points were not a constriction, then it would be a generalized simple intersection, see Remark 1.26. Hence, the corresponding double confluent Heun equation (1.1) would have a polynomial solution. Passing to limit, as $\omega \rightarrow \omega_0$, we get that equation (1.1) corresponding to a constriction (B_0, A_0) also has a polynomial solution. On the other hand, the corresponding equation (1.14) has an entire solution, since (B_0, A_0) is a constriction, see [12, p. 332, §2], [4, theorem 3.3 and subsection 3.2]. But the above properties of equations (1.1) and (1.14) to have polynomial (entire) solution are incompatible for real parameter values, by [4, theorem 3.10]. The contradiction thus obtained proves the proposition. \square

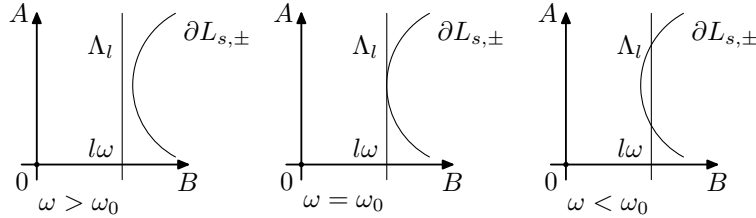


Figure 6: Left-moving tangencies (conjecturally non-existing) that are simple intersections

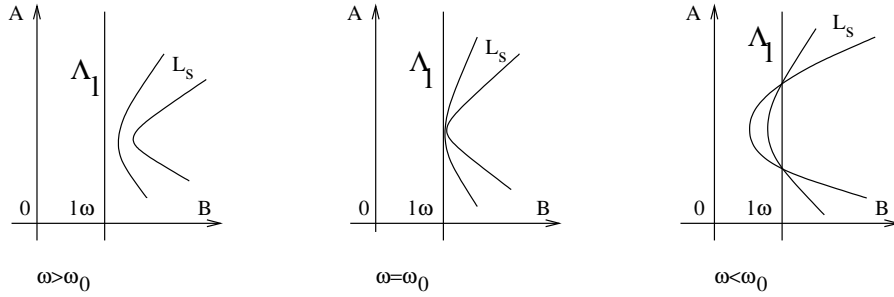


Figure 7: Left-moving tangencies (conjecturally non-existing) that are constrictions

Proposition 1.28 [20, proposition 4.7] *Let no left-moving tangency (if any) be a constriction. Then Conjecture 1.13 holds.*

Corollary 1.29 *Monotonicity Conjecture 1.24 implies Conjecture 1.13.*

The main result of the present subsection is the following theorem giving a partial positive result towards the Monotonicity Conjecture.

Theorem 1.30 *No left-moving tangency of a boundary curve $\partial L_{s,\pm}$ with Λ_l , $s \neq l$, can be a generalized simple intersection.*

Theorem 1.30 is proved in Section 4.

1.8 Historical remarks

Model of overdamped Josephson junction and its phase-lock area portraits were studied in [4] – [12], [19, 20, 22, 23, 26, 27, 37]. In the case, when $\omega > 0$, $B \geq 0$ and $A \neq 0$, Buchstaber and Tertychnyi have shown that the constrictions in model of Josephson junction correspond exactly to those parameter values (B, A) with $A \neq 0$, for which $l = \frac{B}{\omega}$ is integer and equation (1.14) has a non-trivial holomorphic solution at 0 (which is automatically an entire solution: holomorphic on \mathbb{C}); see the statement in [12, p. 332, §2] and the proof in [4, theorem 3.3 and subsection 3.2]. They have explicitly constructed a family of holomorphic solutions for parameters satisfying an explicit functional equation $\xi_l(\lambda, \mu) = 0$, see [12, theorem 2]. They have conjectured that the latter functional equation describes the constrictions completely. They have reduced this conjecture to another one saying that if equation (1.1) has a polynomial solution, then equation (1.14) does not have an entire solution. Both conjectures were proved in [4].

V. M. Buchstaber and S. I. Tertychnyi have constructed symmetries of double confluent Heun equation (1.1) [12, 13]. The symmetry $\# : E(z) \mapsto 2\omega z^{-l-1}(E'(z^{-1}) - \mu E(z^{-1}))$, which is an involution of its solution space, was constructed in [37, equations (32), (34)]. It corresponds to the symmetry $(\varphi, t) \mapsto (\pi - \varphi, -t)$ of the nonlinear equation (1.7); the latter symmetry was found in [27]. In [13] they have found new nontrivial symmetries in the case, when $l \in \mathbb{N}$ and equation (1.1) has no polynomial solutions.

For a survey of Conjecture 1.13 on abscissas of constrictions and related results see [5, 19, 20] and references therein.

Numerical experiences made by V.M.Buchstaber, S.I.Tertychnyi, I.V.Schurov, V.A.Kleptsyn, D.A.Filimonov allowed to observe that as $\omega \rightarrow 0$, the "upper part" of the phase-lock area portrait converges to a parquet-like structure in the renormalized coordinates (l, μ) . More precisely, the complement to the union of the phase-lock areas is a very thin subset whose upper part tends to the boundary of parquet pieces. See Fig. 4 and also [26]. This

statement is an open problem. In [26] V.A.Kleptsyn, O.L.Romaskevich and I.V.Schurov proved results on smallness of gaps between the phase-lock areas for small ω and the rate of their convergence to zero, as $\omega \rightarrow 0$, using methods of slow-fast systems.

2 Complex spectral curves and boundaries. Proofs of Theorems 1.2, 1.3, 1.18, 1.20, 1.22 (A. A. Glutsyuk)

2.1 Irreducibility of determinant. Proof of Theorems 1.2 and 1.3

First let us prove Theorem 1.2. For $l = 1$ the irreducibility is obvious, since the determinant under question coincides with the monomial λ . Thus, everywhere below we consider that $l \geq 2$. We prove irreducibility of the determinant, by finding its Newton diagram at 0 in appropriate new affine coordinates (λ, R) .

Conjugation by diagonal matrix $\text{diag}(1, \mu^{-1}, \dots, \mu^{1-l})$ transforms the matrix H_l to the same matrix without multipliers μ below the diagonal and with the multipliers μ above the diagonal being replaced by μ^2 . The above conjugation together with subsequent substitution

$$\mu^2 = R - \lambda, \quad R := r^2 = \left(\frac{1}{2\omega}\right)^2$$

transform the matrix $H_l + \lambda Id$ to the matrix

$$M(\lambda, R) = M_1(\lambda) + M_2(R), \tag{2.1}$$

$$M_1(\lambda) := \begin{pmatrix} \lambda & -\lambda & 0 & 0 & \dots & 0 \\ l-1 & \lambda - (l-1) & -2\lambda & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 2 & \lambda - 2(l-2) & -(l-1)\lambda & \dots \\ 0 & \dots & 0 & 1 & \lambda - (l-1) & \dots \end{pmatrix},$$

$$M_2(R) = \begin{pmatrix} 0 & R & 0 & 0 & \dots & 0 \\ 0 & 0 & 2R & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & (l-1)R \\ 0 & \dots & \dots & \dots & \dots & 0 \end{pmatrix}.$$

Proposition 2.1 *The determinant $\det M(\lambda, R)$ is a polynomial containing the monomials λ^l and R with non-zero coefficients and containing no monomials λ^k with $k \neq l$.*

Proof The coefficients at monomials λ^k in the determinant $\det M(\lambda, R)$ are equal to the corresponding coefficients in the determinant $\det M_1(\lambda)$. Let us calculate $\det M_1(\lambda)$. Let us replace the second column of the matrix $M_1(\lambda)$ by its sum with the first column. This cancels the first element $-\lambda$ and the free term $-(l-1)$ in the second element: the rest of the second column remains unchanged. In the new matrix thus obtained let us replace the third column by its sum with the second column multiplied by two. This cancels its second element -2λ and the free term $-2(l-2)$ in the third element. Repeating similar operations, we finally get a lower-triangular matrix with diagonal elements λ . We finally get that

$$\det M_1(\lambda) = \lambda^l. \quad (2.2)$$

This implies that the only monomial λ^k entering the polynomial $\det M(\lambda, R)$ with non-zero coefficient is λ^l .

Now it remains to show that the monomial R enters the polynomial $\det M(\lambda, R)$ with non-zero coefficient. To do this, consider the auxiliary matrix

$$N(R) = \begin{pmatrix} 0 & R & 0 & 0 & \dots & 0 \\ l-1 & 1-l & 2R & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 2 & -2(l-2) & (l-1)R \\ 0 & \dots & 0 & 0 & 1 & 1-l \end{pmatrix},$$

whose above-diagonal elements (elements next to the diagonal) are equal to jR , $j = 1, \dots, l-1$, and whose other elements are equal to the corresponding free terms in the entries of the matrix $M_1(\lambda)$. The coefficient at the monomial R is the same in both polynomials $\det M(\lambda, R)$ and $\det N(R)$. Thus, it suffices to show that the corresponding coefficient in $\det N(R)$ is non-zero. The only non-zero element in the first line of the matrix $N(R)$ is its second element R . Therefore, its determinant is equal to the product of the number $-(l-1)R$ and its principal minor formed by its columns number 3, 4, \dots , l . The non-zero free terms in the latter minor exist exactly in the diagonal and immediately below it. This implies that the coefficient at the monomial R in the polynomial $\det N(R)$ is equal to the product of the number $1-l$ and the diagonal terms in the above minor. The latter product is non-zero.

Therefore, the coefficient at R in the determinant $\det N(R)$ (and hence, in $\det M(\lambda, R)$) is non-zero. The proposition is proved. \square

Corollary 2.2 *The Newton diagram of the polynomial $Q(\lambda, R) = \det M(\lambda, R)$ consists of just one edge E with vertices $(l, 0)$ and $(0, 1)$.*

The corollary follows immediately from the proposition and the definition of Newton diagram.

Proof of Theorem 1.2. Suppose the contrary: the polynomial $P_l(u, v)$ is not irreducible. Then the above polynomial $Q(\lambda, R)$, which is obtained from the polynomial $P(u, v)$ by affine variable change $(u, v) = (\lambda, R - \lambda)$, is also not irreducible: it is a product of two polynomial factors P_1 and P_2 . At least one of them, say P_1 , vanishes at 0. The Newton diagram of each polynomial factor vanishing at 0 should consist of an edge parallel to the above edge E and have vertices with integer non-negative coordinates. Moreover, the latter edge should either coincide with E , or lie below E . This follows from the well-known fact that the upper component of the Newton diagram of product of two germs of analytic functions is the Minkovski sum of analogous components of the factors. But the only possible edge parallel to E with integer vertices and lying no higher than E in the positive quadrant is the edge E itself, since one of its vertices is $(0, 1)$.

Therefore, only P_1 vanishes at 0, and its Newton diagram coincides with the edge E . Thus, P_1 contains the monomials λ^l and R . The polynomial P_2 cannot be non-constant: otherwise the product $Q = P_1 P_2$ would have degree greater than l in (λ, R) , while $\det M(\lambda, R) = Q(\lambda, R)$ is clearly a polynomial of degree l . This proves irreducibility of the polynomial $Q(\lambda, R)$, and hence, $P(u, v)$. Theorem 1.2 is proved. \square

Proof of Theorem 1.3. The case, when $l = 1$, being obvious ($P_1(\lambda, v) = \lambda$), we consider that $l \geq 2$. The curve Γ_l is the preimage of the irreducible zero locus $W_l = \{P_l(\lambda, v) = 0\}$ under the degree two mapping $F: (\lambda, \mu) \mapsto (\lambda, \mu^2)$. Therefore, Γ_l is an algebraic curve consisting of either one, or two irreducible components. To show that it is just one component, it suffices to show that the two preimages of some point in W_l close to the origin are connected by a path lying in the regular part of the curve W_l . The germ at 0 of the curve W_l is regular and tangent to the line $R = \lambda + v = \text{const}$: the linear part at the origin of the polynomial P_l is equal to R times a non-zero constant, by Corollary 2.2 and since $l \geq 2$. Thus, the germ under question is transversal to the line $\{v = 0\}$: the critical value line of the mapping F . Therefore, a small circuit around the origin in the above germ lifts via F to a path in

the regular part of the curve W_l that connects two different preimages. This together with the above discussion proves Theorem 1.3. \square

2.2 Generalized simple intersections. Proof of Theorems 1.18, 1.20, 1.21 and Corollary 1.19

Proof of Theorem 1.18. The first statement of the theorem follows from [11, appendix C] and [20, theorem 1.12], and the latter theorem also states that a semi-infinite interval $Sl \subset \Lambda_l^+$ lying in the upper half-plane and bounded by the highest generalized simple intersection in Λ_l^+ lies entirely in L_l . Let $s \in \mathbb{N}$, $s < l$. The phase-lock area L_s lies on the left from the phase-lock area L_l . Hence, its upper part consisting of the points with ordinates A large enough lies on the left from the above interval Sl . On the other hand, its growth point, the intersection $L_s \cap \{A = 0\}$, has abscissa $B = \sqrt{s^2\omega^2 + 1}$, see [10, corollary 3]. Hence, it lies on the right from the axis $\Lambda_l = \{B = l\omega\}$, whenever ω is small enough. Finally, the upper part of each boundary component, $\partial L_{s,\pm} \cap \{A \geq 0\}$, intersects Λ_l^+ , whenever ω is small enough, since it contains a point on the left from the semiaxis Λ_l^+ and the above growth point lying on its right. For $s = 0$ the intersection point of the boundary ∂L_0 with the positive B -semiaxis is the point $(1, 0)$. This follows, e. g., from arguments in [5, example 5.23]. The corresponding differential equation (1.8) has the form $\dot{\varphi} = \frac{1 - \sin \varphi}{\omega}$; $l = \frac{1}{\omega}$, $\mu = 0$. Hence, the Poincaré map of the corresponding vector field (1.9) is parabolic with fixed point $\frac{\pi}{2} \pmod{2\pi\mathbb{Z}}$, and thus, $(1, 0) \in \partial L_{0,+}$. For every $l \in \mathbb{N}$ and every $\omega > 0$ small enough (dependently on l) the latter point $(1, 0)$ lies on the right from the axis $\Lambda_l = \{B = l\omega\}$. Hence, $\partial L_{0,+}$ intersects Λ_l^+ , as in the above discussion. This proves Theorem 1.18. \square

Proof of Corollary 1.19. The number of generalized simple intersections lying in a given semiaxis Λ_l^+ is no greater than l , since they are defined by a polynomial in (λ, μ^2) of degree l , $\mu = \frac{A}{2\omega}$. For small ω the number of generalized simple intersections in Λ_l^+ mentioned in Theorem 1.18 is at least l : at least one lies in ∂L_l ; at least two lie in ∂L_s for $0 < s < l$, $s \equiv l \pmod{2}$; at least one lies in ∂L_0 , if l is even. Therefore, their number is exactly equal to l , exactly one of them lies in ∂L_l , and they depend analytically on ω , since the squares of their μ -coordinates are distinct roots of an analytic family of polynomials of degree l . Their abscissas are equal to the same number ωl , which tends to 0, as $l \rightarrow 0$. This proves Corollary 1.19. \square

Recall that we denote $r = \frac{1}{2\omega}$. The mapping $\pi: \mathbb{C}^2 \rightarrow \mathbb{C}^2$, $\pi(\mu, r) = (\lambda = r^2 - \mu^2, \mu)$, sends the (μ, r) -coordinates of the generalized simple intersections

in $\Lambda_l(\omega)$ to the zero locus $\Gamma_l = \{P_l(\lambda, \mu^2) = 0\}$ and vice versa: for every real point $(\lambda, \mu) \in \Gamma_l$ with $\lambda + \mu^2 > 0$ its preimage with $r > 0$ is a generalized simple intersection (Theorems 1.17 and 1.1). Set $\widehat{\Gamma}_l = \pi^{-1}(\Gamma_l)$. One has $\widehat{\Sigma}_l \subset \widehat{\Gamma}_l$ and $\pi(\widehat{\Sigma}_l) = \Gamma_l$, by the above discussion, and since the image of the algebraic set $\widehat{\Sigma}_l$ under degree two rational branched cover π is algebraic (Remmert Proper Mapping Theorem and Chow Theorem [21]). We already know that the complex algebraic curve Γ_l is irreducible (Theorem 1.3). This proves the first statement of Theorem 1.20. The mapping π has degree two. Therefore, the set $\widehat{\Gamma}_l$, which contains $\widehat{\Sigma}_l$, is an algebraic curve that either is irreducible, or has two irreducible components. We already know that $\widehat{\Sigma}_l = \widehat{\Sigma}_{l,+} \cup \widehat{\Sigma}_{l,-}$. Let us show that the sets $\widehat{\Sigma}_{l,\pm}$ are non-empty analytic curves. This will imply their irreducibility, the equality $\widehat{\Gamma}_l = \widehat{\Sigma}_l$ and Theorem 1.20.

Lemma 2.3 *For every $l \in \mathbb{N}$, every $\omega > 0$ small enough and every sign \pm there exists an $s \in \mathbb{Z}_{\geq 0}$, $s \equiv l \pmod{2}$, $s \leq l$, such that the boundary component $\partial L_{s,\pm}$ intersects Λ_l .*

For $l \geq 3$ the statement of the lemma holds for every $s \equiv l \pmod{2}$, $0 < s < l$, by Theorem 1.18. Below we prove it for $l = 1, 2$. In the proofs of Lemma 2.3 and Theorems 1.20, 1.21 we use the following proposition.

Proposition 2.4 [26, p. 40, the discussion around formulas (10)-(11)] *For every $s \in \mathbb{Z}$ and every point $(B, A) \in \partial L_{s,\pm}$ the orbit of the corresponding vector field (1.9) starting at $(\varphi, \tau) = (\pm \frac{\pi}{2}, 0)$ is 2π -periodic and invariant under the symmetry $I: (\varphi, \tau) \mapsto (\pi - \varphi, -\tau)$ (found by A. V. Klimenko in [27]) of the field (1.9). In particular, the value $\pmod{2\pi}$ of the corresponding solution of differential equation on a function $\varphi(\tau)$ at the half-period $\tau = \pi$ is the initial value $\pm \frac{\pi}{2}$, if s is even, and $\mp \frac{\pi}{2}$, if s is odd.*

Proof The orbit under question is 2π -periodic, since $(B, A) \in \partial L_{s,\pm}$, and hence, $\pm \frac{\pi}{2}$ is a fixed point of the Poincaré map of the field (1.9). On the other hand, the points $\pm \frac{\pi}{2}$ are the only fixed points of the circle involution $\varphi \mapsto \pi - \varphi$, and $0, \pi$ are the only fixed points of the circle involution $\tau \mapsto -\tau$. These statements imply the proposition. \square

Proof of Theorem 1.21. The transformation $J: (\varphi, \tau) \mapsto (\varphi, \tau + \pi)$ sends a vector field (1.9) corresponding to a point (B, A) to the similar vector field denoted (1.9)* with changed sign at μ , i. e., corresponding to the point $(B, -A)$, see [5, p.76]. The rotation number obviously remains the same; let us denote it by s . The symmetry J transforms an orbit O_1 of the field (1.9) through the point $(\pm \frac{\pi}{2}, 0)$ to an orbit O_2 of the field (1.9)* through the

point $(\pm\frac{\pi}{2}, \pi)$. The orbit O_2 is 2π -periodic, if and only if so is O_1 . Fix a sign \pm , and let O_1 be periodic; then $s \in \mathbb{Z}$. In the case, when s is even, the orbit O_1 passes through both points $(\pm\frac{\pi}{2}, 0)$ and $(\pm\frac{\pi}{2}, \pi)$, by the last statement of Proposition 2.4. Hence, both of them are fixed by the time 2π flow map of the field. This together with the above discussion implies that the orbit O_2 is periodic and also passes through the same two points. Finally, both vector fields (1.9) corresponding to the points (B, A) and $(B, -A)$ have the same periodic point $(\pm\frac{\pi}{2}, 0)$ and the same rotation number. This implies that the symmetry $(B, A) \mapsto (B, -A)$ leaves each curve $\partial L_{s,\pm}$ invariant. Similarly in the case, when s is odd, the orbit O_1 starting at $(\pm\frac{\pi}{2}, 0)$ passes through the point $(\mp\frac{\pi}{2}, \pi)$, by Proposition 2.4. Hence, the orbit $O_2 = J(O_1)$ passes through the point $(\mp\frac{\pi}{2}, 0)$ and is periodic. This implies that the symmetry $(B, A) \mapsto (B, -A)$ interchanges the curves $\partial L_{s,\pm}$.

Now let us prove the last statement of Theorem 1.21, on the symmetry $(B, A) \mapsto (-B, A)$. The transformation $(\varphi, \tau) \mapsto (-\varphi, \tau + \pi)$ sends a vector field (1.9) corresponding to a point (B, A) to the same field with opposite sign at l , i.e., corresponding to $(-B, A)$; the latter field will be denoted by $(1.9)^*$. It changes the sign of the rotation number. It transforms an orbit O_1 of the field (1.9) through the point $(\pm\frac{\pi}{2}, 0)$ to an orbit O_2 of the field $(1.9)^*$ through the point $(\mp\frac{\pi}{2}, \pi)$. Fix a sign \pm , and let O_1 be periodic. Let s denote its rotation number; then $s \in \mathbb{Z}$. In the case, when s is even, the orbit O_1 passes through both points $(\pm\frac{\pi}{2}, 0)$ and $(\pm\frac{\pi}{2}, \pi)$; the orbit O_2 is periodic and passes through the two points $(\mp\frac{\pi}{2}, 0)$ and $(\mp\frac{\pi}{2}, \pi)$, as in the above discussion. Finally, the vector fields (1.9) corresponding to the points (B, A) and $(-B, A)$ have periodic points $(\pm\frac{\pi}{2}, 0)$ and $(\mp\frac{\pi}{2}, 0)$ and opposite rotation numbers. Thus, the symmetry $(B, A) \mapsto (-B, A)$ interchanges the curves $\partial L_{s,\pm}$ and $\partial L_{-s,\mp}$. In the case, when s is odd, the orbit O_1 passes through the points $(\pm\frac{\pi}{2}, 0)$ and $(\mp\frac{\pi}{2}, \pi)$; the orbit O_2 is periodic and passes through the two points $(\mp\frac{\pi}{2}, \pi)$ and $(\pm\frac{\pi}{2}, 0)$. Hence, the vector fields (1.9) corresponding to the points (B, A) and $(-B, A)$ have the same periodic point $(\pm\frac{\pi}{2}, 0)$ and opposite rotation numbers. Thus, the symmetry interchanges the curves $\partial L_{s,\pm}$ and $\partial L_{-s,\pm}$. This proves Theorem 1.21. \square

Proof of Lemma 2.3. Recall that it suffices to prove Lemma 2.3 for $l = 1, 2$ (Theorem 1.18). Let $l = 1$. At least one boundary component $\partial L_{1,\pm}$ intersects Λ_1 (Theorem 1.18). The other component is symmetric to it with respect to the B -axis (Theorem 1.21), and hence, also intersects Λ_1 . This proves the statement of Lemma 2.3 for $l = 1$. The case, when $l = 2$, was already treated in [11, appendix A, subsection A.1]. In more detail, let $(1.8)^*$ and $(1.9)^*$ denote respectively the differential equation (1.8) and

the vector field (1.9) where the parameter l is taken with opposite sign. It was shown in loc. cit. that for every positive $\omega < \frac{1}{2}$ there are two positive values $\mu_{\pm} = (2\omega)^{-1}\sqrt{1 \mp 2\omega}$ of the parameter μ for which the corresponding equation (1.1) has a polynomial solution; these polynomial solutions induce solutions $\varphi(\tau)$ of equation (1.8)* with $\varphi(0) = \mp \frac{\pi}{2}$, $\varphi(2\pi) = \varphi(0) - 2\pi \pm 2\pi$. Hence, the corresponding trajectories of dynamical systems (1.9)* on torus are periodic with initial conditions $\mp \frac{\pi}{2} \pmod{2\pi}$ and rotation numbers 0 (for μ_+) and -2 (for μ_-).

Set $B = 2\omega$, $A_{\pm} = 2\mu_{\pm}\omega$. By construction, $(B, A_{\pm}) \in \Lambda_2$ are generalized simple intersections, and the above system (1.9)* is the system (1.9) corresponding to the parameters $(-B, A_{\pm})$. The above discussion implies that $(-B, A_+) \in \partial L_{0,-}$ and $(-B, A_-) \in \partial L_{-2,+}$. Therefore, $(B, A_+) \in \partial L_{0,+} \cap \Lambda_2$, $(B, A_-) \in \partial L_{2,-} \cap \Lambda_2$, by Theorem 1.21. This proves Lemma 2.3. \square

Lemma 2.3 together with the discussion before it imply that the algebraic set $\widehat{\mathbb{S}}_l$, which consists of at most two irreducible components, is the union of two non-trivial algebraic curves $\widehat{\mathbb{S}}_{l,\pm}$. Therefore, the latter curves are irreducible and the number of components under question is exactly two. This proves Theorem 1.20.

2.3 Boundaries of phase-lock areas. Proof of Theorem 1.22.

Recall that the complexification of equation (1.8) is the corresponding Riccati equation (1.11). For every collection of complex parameters l, μ, ω in (1.11) with $\omega \neq 0$ the *monodromy transformation* Mon of the Riccati equation (1.11) acts on the space of initial conditions at $z = 1$ by analytic continuation of solutions along the positively oriented unit circle in the z -line. It is a Möbius transformation of the Riemann sphere, since the equation under question is the projectivization of a linear equation. For real parameter values the corresponding Poincaré map $h_{(B,A)}$ acts on the circle $\mathbb{R}/2\pi\mathbb{Z}$ with coordinate φ , and the variable change $\varphi \mapsto \Phi = e^{i\varphi}$ sends it to the unit circle in the Riemann sphere. The monodromy Mon is the complexification of the Poincaré map considered as a unit circle diffeomorphism. Thus, the Poincaré map is a *Möbius circle diffeomorphism*: a unit circle diffeomorphism that extends as a Möbius transformation of the Riemann sphere.

Recall that the condition saying that $h_{(B,A)}$ is parabolic implies that it fixes some of the points $\pm \frac{\pi}{2} \pmod{2\pi\mathbb{Z}}$, see Remark 1.12. This implies the following corollary.

Corollary 2.5 *For every $\omega > 0$, $s \in \mathbb{N}$ and sign \pm if a point $(B, A) \in \mathbb{R}^2$ lies in $\partial L_{s,\pm}$, then the monodromy Mon of the corresponding Riccati equation (1.11) fixes the point $\pm i$. Thus, the subsets $\mathcal{L}_{\pm}^{\text{odd}(\text{even})}$, and hence, their complexifications $\widehat{\mathcal{L}}_{\pm}^{\text{odd}(\text{even})}$ lie in the hypersurfaces*

$$\Sigma_{\pm} \subset \mathbb{C}_{(B,A,r)}^3, \quad r = \frac{1}{2\omega},$$

of those parameter values, for which the monodromy Mon fixes the point $\pm i$.

For every $l \in \mathbb{Z}$ set

$$\mathcal{L}_{l,\pm} = \bigcup_{\omega>0} \left(\partial L_{l,\pm}(\omega) \times \left\{ \frac{1}{2\omega} \right\} \right) \subset \mathbb{R}_{(B,A,r)}^3,$$

$\widehat{\mathcal{L}}_{l,\pm} :=$ the minimal analytic subset in \mathbb{C}^3 containing $\mathcal{L}_{l,\pm}$.

Recall that for every $l \in \mathbb{N}$ by $\text{SI}_{l,+}$ we denote the union of families of tuples $(\mu, r = \frac{1}{2\omega}) \in \mathbb{R}^2$ that correspond to the generalized simple intersections with $\omega = \frac{1}{2r}$, $B = l\omega$, $A = 2\mu\omega$. Consider the transformation

$$g_l : \mathbb{C}_{(\mu,r)}^2 \mapsto \mathbb{C}_{(B,A,r)}^3 : g_l(\mu, r) = \left(\frac{l}{2r}, \frac{\mu}{r}, r \right).$$

Set

$\widetilde{\Gamma}_{l,\pm} := g_l(\widehat{\text{SI}}_{l,\pm})$, $\widehat{\Gamma}_{l,\pm} :=$ the closure of the curve $\widetilde{\Gamma}_{l,\pm}$ in the usual topology.

It is clear that $\widehat{\Gamma}_{l,\pm}$ is an irreducible algebraic curve, since so is $\widehat{\text{SI}}_{l,\pm}$ (Theorem 1.20) and the mapping g_l is rational and injective on the set $r \neq 0$.

Proposition 2.6 *Fix an $\omega > 0$. Consider the family of the Poincaré maps $h_{(B,A)}$ as a family of elements of Möbius circle transformation group. Let $(B_0, A_0) \in \mathbb{R}^2$ lie in the boundary curve $\partial L_{s,\pm}$ of a phase-lock area L_s . The derivative in B of the Poincaré map family $h_{(B,A)}$ at (B_0, A_0) is a vector transversal to the hypersurface of Möbius circle transformations fixing $\pm \frac{\pi}{2} \pmod{2\pi\mathbb{Z}}$. In particular $\mathcal{L}_{s,\pm}$ is a smooth two-dimensional surface.*

The proposition follows from monotonicity of the Poincaré map as a function of the parameter B (classical).

Proposition 2.7 *Each complex analytic set $\widehat{\mathcal{L}}_{l,\pm}$ is irreducible and purely two-dimensional. For every $s \in \mathbb{N}$, $l > s$, $l \equiv s \pmod{2\mathbb{Z}}$ and sign \pm the surface $\widehat{\mathcal{L}}_{s,\pm}$ contains the curve $\widehat{\Gamma}_{l,\pm}$. The same statement also holds for $s = 0$, the sign $+$ and even $l \in \mathbb{N}$.*

Proof The minimal complex analytic subset $\widehat{\mathcal{L}}_{s,\pm}$ containing $\mathcal{L}_{s,\pm}$ is irreducible and at least two-dimensional, since $\mathcal{L}_{s,\pm}$ is smooth and two-dimensional, by the last statement of Proposition 2.6. It cannot have bigger dimension, since it is already contained in a two-dimensional analytic subset Σ_{\pm} . This implies that it is purely two-dimensional. Let us prove the inclusion (the second and third statements) of Proposition 2.7. The surface $\mathcal{L}_{s,\pm}$ contains a family of simple intersections in $\Lambda_l(\omega)$ with small ω (Theorem 1.18). The minimal complex analytic subset in \mathbb{C}^3 containing the above family of simple intersections is the curve $\widehat{\Gamma}_{l,\pm}$, by definition and irreducibility (Theorem 1.20). This implies that the irreducible complex surface $\widehat{\mathcal{L}}_{s,\pm}$ contains the irreducible curve $\widehat{\Gamma}_{l,\pm}$ and proves Proposition 2.7. \square

Proposition 2.8 *The analytic surface Σ_{\pm} is regular at each its real point corresponding to a simple intersection.*

Proposition 2.8 follows from Proposition 2.6.

Proposition 2.9 *For every sign \pm and every $s_1, s_2 \in \mathbb{N}$, $s_1 - s_2$ being even, one has $\widehat{\mathcal{L}}_{s_1,\pm} = \widehat{\mathcal{L}}_{s_2,\pm}$. The same statement holds for the sign "+" in the case, when $s_{1,2} \in \mathbb{Z}_{\geq 0}$ and $s_i = 0$ for some i . For every sign \pm and every $s_1, s_2 \in \mathbb{Z}_{< 0}$, $s_1 - s_2$ being even, one has $\widehat{\mathcal{L}}_{s_1,\pm} = \widehat{\mathcal{L}}_{s_2,\pm}$. The same statement holds for the sign "-" in the case, when $s_{1,2} \in \mathbb{Z}_{\leq 0}$ and $s_i = 0$ for some i .*

Proof Let us prove only the statements of the proposition for $s_1, s_2 \geq 0$; its statements for $s_1, s_2 \leq 0$ then follow by symmetry and Theorem 1.21.

The surfaces $\widehat{\mathcal{L}}_{s_i,\pm}$ contain the irreducible analytic curve $\widehat{\Gamma}_{l,\pm}$, whenever l is greater than each s_i and has the same parity (Proposition 2.7). Their germs at each point $p \in \widehat{\Gamma}_{l,\pm}$ representing a generalized simple intersection coincide with the germ of the ambient surface $\Sigma_{\pm} \supset \widehat{\mathcal{L}}_{s_i,\pm}$, since the latter germ is regular and two-dimensional (Proposition 2.8). Finally, the surfaces $\widehat{\mathcal{L}}_{s_i,\pm}$, $i = 1, 2$, are irreducible and have the same germ at p . Therefore, they coincide. This proves Proposition 2.9. \square

Proposition 2.10 *For every $s \in \mathbb{N}$ one has $\widehat{\mathcal{L}}_{s,\pm} = \widehat{\mathcal{L}}_{-s,\pm}$.*

Proof Recall that the growth point of the phase-lock area L_s , $s \in \mathbb{N}$, i. e., its intersection with the horizontal B -axis has abscissa $B_s(\omega) \operatorname{sign}(s)$,

$$B_s(\omega) := \sqrt{s^2 \omega^2 + 1},$$

see [10, corollary 3]. Therefore, the corresponding complex surface $\widehat{\mathcal{L}}_{s,\pm}$ contains the algebraic curve

$$G_s := \{A = 0, B^2 = s^2\omega^2 + 1\} \subset \mathbb{C}_{(B,A,\frac{1}{2\omega})}^3.$$

The curve G_s is obviously irreducible and contained in both surfaces $\widehat{\mathcal{L}}_{s,\pm}$ and $\widehat{\mathcal{L}}_{-s,\pm}$. For every given $\omega > 0$ the curve G_s contains two real symmetric points q_+ and q_- with coordinates $B = B_s(\omega)$ and $B = -B_s(\omega)$. The germ of the ambient surface Σ_{\pm} at a real point in G_s (i. e., at a point corresponding to a growth point) is regular, as in Proposition 2.8. This implies that the germ of the surface $\widehat{\mathcal{L}}_{s,\pm}$ at q_+ and its germ obtained by analytic extension to q_- along a path in G_s coincide with the germs of the ambient surface Σ_{\pm} , as in the proof of Proposition 2.9. Similar statement should hold for the surface $\widehat{\mathcal{L}}_{-s,\pm}$. Finally, the irreducible surfaces $\widehat{\mathcal{L}}_{s,\pm}$ and $\widehat{\mathcal{L}}_{-s,\pm}$ have coinciding germs at the point q_- . Hence, they coincide. This proves the proposition. \square

Proposition 2.11 *The surfaces $\widehat{\mathcal{L}}_{\pm}^{\text{even(odd)}}$ are irreducible.*

Proof For every $s_1, s_2 \in \mathbb{Z}$ having the same parity the minimal analytic sets $\widehat{\mathcal{L}}_{s_j,\pm}$ containing the real surfaces $\mathcal{L}_{s_j,\pm}$, $j = 1, 2$, coincide, by Propositions 2.9 and 2.10. This implies the statement of Proposition 2.11. \square

Proposition 2.12 *The four irreducible surfaces $\widehat{\mathcal{L}}_{\pm}^{\text{even(odd)}}$ are distinct.*

Proof It is clear that the above surfaces with different signs indices should be distinct, as are the ambient surfaces Σ_{\pm} . Let us prove that $\widehat{\mathcal{L}}_{\pm}^{\text{even}} \neq \widehat{\mathcal{L}}_{\pm}^{\text{odd}}$.

For every point $q \in \Sigma_{\pm}$ let $f_q(z)$ denote the corresponding solution of the Riccati equation (1.11) with $f_q(1) = \pm i$. It is meromorphic on \mathbb{C}^* , since the monodromy fixes the point $\pm i$ and hence, fixes its initial branch at $z = 1$. In the case, when the function f_q has neither zeros, nor poles in the unit circle $S^1 = \{|z| = 1\}$, let ρ_q denote the index of the restriction $f_q : S^1 \rightarrow \mathbb{C}^*$: the argument increment divided by 2π .

Proposition 2.13 *Each surface $\widehat{\mathcal{L}}_{\pm}^{\text{even(odd)}}$ contains a nowhere dense real-analytic subset \mathcal{S} consisting of those points q for which the function $f_q(z)$ has either zeros, or poles in S^1 . For $q \in \widehat{\mathcal{L}}_{\pm}^{\text{even(odd)}} \setminus \mathcal{S}$ the indices ρ_q are even (respectively, odd).*

Proof In the case, when q is a real point, the function f_q takes values in the unit circle on S^1 (thus, has neither zeros, nor poles there), and the number

ρ_q is clearly equal to the rotation number of the corresponding dynamical system (1.9) on torus. This together with irreducibility implies the first statement of the proposition. For every $q \in \mathcal{S}$ if the function $f_q(z)$ has a zero at a point z of the unit circle, then it has a pole at its conjugate $z^{-1} = \bar{z}$, and the multiplicities of zero and pole are equal. This follows from the symmetry of the Riccati equation (1.11) under the involution $(\Phi, z) \mapsto (-\Phi^{-1}, z^{-1})$, which is the complexification of the symmetry $(\varphi, \tau) \mapsto (\pi - \varphi, -\tau)$ of equation (1.8). The above statement together with symmetry imply that when q crosses the real codimension one hypersurface \mathcal{S} , the parity of the index ρ_q does not change. This proves the proposition. \square

Proposition 2.13 implies Proposition 2.12. \square

Proposition 2.12 implies Theorem 1.22.

3 Spectral curves: genera and real parts (I. V. Netay)

Let us consider some other curves birationally equivalent to Γ_l but more convenient. Let us introduce an $l \times l$ -matrix

$$\mathcal{G}_l = \begin{pmatrix} 0 & \dots & 0 & \mu \\ \vdots & & \ddots & -(l-1) \\ 0 & \ddots & \ddots & \\ \mu & -1 & & 0 \end{pmatrix}.$$

The following relation (found in [11, §3, eq. (30)]) holds:

$$H_l + (r^2 - \mu^2) \text{Id} = -(\mathcal{G}_l + r \text{Id})(\mathcal{G}_l - r \text{Id})$$

for a variable r . Let us put $\lambda = r^2 - \mu^2$. Then we obtain

$$\det(H_l + \lambda \text{Id}) = (-1)^l \det(\mathcal{G}_l + r \text{Id}) \det(\mathcal{G}_l - r \text{Id}).$$

Then the mapping

$$\pi: (\mu, r) \mapsto (\lambda = r^2 - \mu^2, \mu)$$

maps points of the curve $\{\det(\mathcal{G}_l + r \text{Id}) \det(\mathcal{G}_l - r \text{Id}) = 0\}$ of affine plane $\mathbb{A}_{\mu, r}$ to points of the curve $\Gamma_l = \{\det(H_l + \lambda \text{Id}) = 0\} \subset \mathbb{A}_{\lambda, \mu}$.

Denote the curves $\{\det(\mathcal{G}_l \pm r \text{Id}) = 0\}$ by Ξ_l^\pm . Then $\pi^{-1}(\Gamma_l)$ is the union of the curves Ξ_l^\pm . On the other hand, the latter preimage is the union of two irreducible curves $\widehat{\text{S}}\widehat{\Gamma}_{l, \pm}$ (Theorem 1.20). Hence, the curves Ξ_l^\pm are irreducible

and coincide with the curves $\widehat{S}\widehat{I}_{l,\pm}$ (up to transposition, i. e., sign change), and π maps each of Ξ_l^- and Ξ_l^+ birationally to Γ_l .

Let us look at the projective closures $\overline{\Xi}_l^\pm$ of some of the curves Ξ_l^\pm :

$l = 1$: the curve is a line;

$l = 2$: the curve is a smooth conic and is rational;

$l = 3$: the curve is a singular cubic and therefore is rational;

$l = 4$: it is a quartic with two simple self-intersections, and hence, an elliptic curve.

Consider the following diagram of morphisms:

$$\begin{array}{ccccc}
 \widehat{\Xi}_l^\pm \hookrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 & \xleftarrow{\text{Bl}_{\{\theta=0\}}} & X_7 & \\
 \downarrow & \uparrow & \swarrow & \downarrow \text{Bl}_{p_\pm} & \\
 \Xi_l^\pm \hookrightarrow & \mathbb{A}_{\mu,r}^2 \hookrightarrow & \mathbb{P}_{\mu:r:\theta}^2 & & \\
 \downarrow 2:1 & \downarrow \pi: r \mapsto \lambda = r^2 - \mu^2 & \downarrow \tilde{\pi} & & \\
 \Gamma_l \hookrightarrow & \mathbb{A}_{\lambda,\mu}^2 \hookrightarrow & \mathbb{P}_{\lambda:\mu:\zeta}^2 & &
 \end{array}$$

Let us describe the diagram. The natural compactification of the curve Γ_l lies in $\mathbb{P}_{\lambda:\mu:\zeta}^2$. At the same time the natural compactification of the curve Ξ_l^\pm lies in $\mathbb{P}_{\mu:r:\theta}^2$. The rational map $\tilde{\pi} : \mathbb{P}_{\mu:r:\theta}^2 \dashrightarrow \mathbb{P}_{\lambda:\mu:\zeta}^2$ is defined as the map π on an open subset, the affine chart $\mathbb{A}_{\mu,r}^2 = \mathbb{C}_{(\mu,r)}^2$, and has degree 2. It is easy to define the map in homogeneous coordinates:

$$\tilde{\pi} : \begin{cases} \lambda & \mapsto r^2 - \mu^2, \\ \mu & \mapsto \mu\theta, \\ \theta & \mapsto \theta^2. \end{cases}$$

The rational mapping $\tilde{\pi}$ is not well-defined exactly at two points, namely, $(\mu : r : \theta) = (1 : \pm 1 : 0)$. Let us denote these points by p_\pm correspondingly.

If we blow them up, we get del Pezzo surface X_7 of degree 7. It has three (-1) -curves: preimages of points p_\pm (which will be denoted by C_\pm) and the strict transform C_∞ of the line $\{\theta = 0\}$ passing through them. If we blow C_∞ down, then we get del Pezzo surface of degree 8, namely, $\mathbb{P}^1 \times \mathbb{P}^1$. Consider the composition $\mathbb{A}_{\mu,r}^2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ of the inclusion $\mathbb{A}_{\mu,r}^2 \rightarrow \mathbb{P}_{\mu:r:\theta}^2$ with

the above blows up and down. It is well-defined, because the points blown up lie at infinity. Preimages of the two line families $\{\text{pt}\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{\text{pt}\}$ are lines $r \pm \mu = \text{const}$.

Proposition 3.1 *The only real singular points of projective closures $\overline{\Xi_l^\pm}$ are $(1 : \pm 1 : 0)$ (except smooth curves for $l = 1, 2$ and the smooth point p_+ for Ξ_l^+ and p_- for Ξ_l^- for $l = 3$). In particular, the affine curves $\Xi_l^\pm(\mathbb{R})$ are smooth.*

Proof Absence of real singularities except for those lying in the infinity line and the line $\{\mu = 0\}$ is guaranteed by smoothness of Γ_l outside the line $\{\mu = 0\}$ (because the image of a singular point is singular). Also smoothness of $\Xi_l^- \cup \Xi_l^+$ outside the line $\{\mu = 0\}$ is proved in [11, §3, Cor.4]. The only points of Ξ^\pm at infinity are $(1 : \pm 1 : 0)$. The multiplicities of these two points are calculated in the following proposition.

It remains only to consider the line $\{\mu = 0\}$. For $\mu = 0$ one has

$$\det(\mathcal{G}_n + r \text{Id}) = \begin{cases} r \prod_{k=1}^{\frac{l-1}{2}} (r^2 - k(l-k)) & \text{if } l \text{ is odd,} \\ r(r - \frac{l}{2}) \prod_{k=1}^{\frac{l-2}{2}} (r^2 - k(l-k)) & \text{if } l \text{ is even,} \end{cases} \quad (3.1)$$

since appropriate permuting of lines and columns of the matrix in the above left-hand side makes it block-diagonal with obvious blocks of dimension two or one. The above polynomial has no multiple roots. Therefore all the intersections of a given curve Ξ^\pm with the line $\{\mu = 0\}$ are simple. In particular, this implies that there are no singular points among the intersections. \square

Definition 3.2 Let us denote the *multiplicity* of a point P on a curve C by $\mu_P(C)$: this is its intersection index at P with a generic line through P . Recall that in the case when P is an intersection of k pairwise transversal smooth local branches, its multiplicity is equal to k ; in this case we will call P a *pairwise transversal self-intersection*. A regular point, which corresponds to one branch, will be also treated as a pairwise transversal self-intersection.

Proposition 3.3 *Each of the points p_+, p_- is a pairwise transversal self-intersection for every curve $\overline{\Xi_l^\pm}$, and one has $\mu_{p_\pm}(\overline{\Xi_l^\pm}) = \lfloor \frac{l}{2} \rfloor$; $\mu_{p_\pm}(\overline{\Xi_l^\mp}) = l - \lfloor \frac{l}{2} \rfloor$. Moreover, all the local branches of the curve $\overline{\Xi_l^+} \cup \overline{\Xi_l^-}$ at p_\pm are transversal to the infinity line $\{\theta = 0\}$.*

Proof Consider a germ of analytic curve C at O in an affine chart. Recall that to show that O is a pairwise transversal self-intersection of multiplicity

k , it suffices to prove that the function f defining C (i.e., generating the ideal of germs of functions vanishing at C) has lower Taylor homogeneous part of degree k that is a product of k pairwise non-proportional linear forms. In the case, when $O = p_{\pm}$, to show that each local branch is transversal to the infinity line, one has to show that the above lower homogeneous part is not divisible by θ .

The symmetry $r \leftrightarrow -r$ swaps pairs (p_+, p_-) and $(\overline{\Xi}_l^+, \overline{\Xi}_l^-)$ for each l . So let us consider only the curve $\overline{\Xi}_l^+$.

Let us consider the affine chart $\mu = 1$. At first, consider the point $p_+ = (1 : 1 : 0)$. So, we substitute $r = 1 + a$ with a being small.

Suppose l is even. In the new coordinates the curve $\overline{\Xi}_l^+$ is defined by

$$\begin{vmatrix} 1+a & 0 & \cdots & \cdots & 0 & 1 \\ 0 & \ddots & & & \ddots & -(l-1)\theta \\ \vdots & & 1+a & 1 & \ddots & 0 \\ \vdots & & 1 & 1+a-\frac{l}{2}\theta & \vdots & \vdots \\ 0 & \ddots & \ddots & & \ddots & 0 \\ 1 & -\theta & 0 & \cdots & 0 & 1+a \end{vmatrix} =$$

Let us subtract columns k from $l-k$ for $k = 1, \dots, l/2$:

$$= \begin{vmatrix} 1+a & 0 & \cdots & \cdots & 0 & -a \\ 0 & \ddots & & & \ddots & -(l-1)\theta \\ \vdots & & 1+a & -a & \ddots & 0 \\ \vdots & & 1 & a-\frac{l}{2}\theta & \vdots & \vdots \\ 0 & \ddots & \ddots & (\frac{l}{2}-1)\theta & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ 1 & -\theta & 0 & \cdots & \theta & a \end{vmatrix} =$$

Note that the constants now are only in the left half of matrix. Now let us do the same operation with the matrix rows:

$$= \det \left(\begin{array}{cc|ccc} 1+a & 0 & & & \\ & \ddots & & & * \\ 0 & & 1+a & & \\ \hline & & & 2a-\frac{l}{2}\theta & (\frac{l}{2}+1)\theta & 0 \\ & * & & (\frac{l}{2}-1)\theta & 2a & \ddots \\ & & & & \ddots & \ddots & (l-1)\theta \\ & & & 0 & & \theta & 2a \end{array} \right).$$

Here the left-lower and the right-upper parts of the matrix marked by $*$ do not contain constant terms. Therefore, the lowest homogeneous part of the function $f = \det(\mathcal{G}_l + r \text{Id})$ defining Ξ_l^+ is equal to the determinant of the lower right matrix part being a three-diagonal matrix.

The determinant of the latter matrix is obviously not divisible by θ , which follows from (3.1). Let us show that it is a product of pairwise non-proportional linear forms.

Fix a $\theta > 0$. We look for those a for which the determinant vanishes; they are equal to minus half-eigenvalues of the same matrix without the $2a$ on the diagonal. Since pairs of elements symmetric w.r. t. the diagonal have positive coefficients at θ , this matrix can be conjugated by a constant diagonal matrix in such a way that the result would be a symmetric three-diagonal matrix with positive entries below and above the diagonal.

From [24, §1.3.10] the eigenvalues of such matrix are real and do not coincide. This implies that for every given $\theta > 0$ the above determinant vanishes if and only if a takes one of $\frac{l}{2}$ distinct real values. Therefore, the linear forms in its product decomposition are pairwise non-proportional.

So we conclude that for even l the equality $\mu_{p_+}(\overline{\Xi}_l^+) = l/2$ holds, the point has $l/2$ tangent directions and $\theta = 0$ is not one of them.

Now consider the case of l odd. We process the same transformations with matrix (except the central row and column) and obtain

$$\det \left(\begin{array}{ccc|ccc} 1+a & & 0 & & & \\ & \ddots & & & & \\ & & 1+a & & & * \\ 0 & & 2+a & & & \\ \hline & & & 2a & \frac{l+3}{2}\theta & 0 \\ & & * & \frac{l-3}{2}\theta & 2a & \ddots \\ & & & & \ddots & \ddots & (l-1)\theta \\ & & & 0 & \theta & 2a & \end{array} \right).$$

In this case the lowest homogeneous component has degree $\frac{l-1}{2} = \lceil \frac{l}{2} \rceil = \mu_{p_+}(\overline{\Xi}_l^+)$. Analogously, all the eigenvalues are real and do not coincide.

Now let us consider the point p_- . Here we substitute $r = a - 1$. We make the same transformations, but we add columns and rows instead of subtracting.

In the case of l even we obtain the same three-diagonal matrix with θ replaced by $-\theta$. All the rest of reasoning is the same.

The case of l odd for p_- is slightly different, because the constants at the central element of matrix vanish:

$$\det \left(\begin{array}{cc|ccc} 1+a & 0 & & & \\ & \ddots & & & \\ 0 & & 1+a & & \\ \hline & & & a & \frac{l+1}{2}\theta & 0 \\ & * & & \frac{l-1}{2}\theta & 2a & \ddots \\ & & & & \ddots & \ddots & (l-1)\theta \\ & & & 0 & \theta & 2a & \end{array} \right).$$

In this case the lowest component has degree $l - \frac{l-1}{2} = l - \lfloor \frac{l}{2} \rfloor = \mu_{p_-}(\overline{\Xi}_l^+)$. \square

Proposition 3.4 *The genus of $\widehat{\Xi}_l^\pm$ does not exceed*

$$\begin{cases} \left(\frac{l-2}{2} \right)^2, & \text{for } l \text{ even,} \\ \left(\frac{l-1}{2} \right) \left(\frac{l-3}{2} \right), & \text{for } l \text{ odd.} \end{cases}$$

The equality takes place it and only if the curves $\widehat{\Xi}_l^\pm$ are non-singular, which is equivalent to the statement that the complex affine curve $\Gamma_l \subset \mathbb{C}^2$ has no singularities outside the line $\{\mu = 0\}$.

Proof Recall that the curves $\widehat{\Xi}_l^\pm$ lie in $\mathbb{P}^1 \times \mathbb{P}^1$. We use the following formula for the geometric genus of an irreducible curve of bidegree (d_1, d_2) in $\mathbb{P}^1 \times \mathbb{P}^1$, see [33, §2, Prop. 3, Ex 2.2]:

$$g(C) = (d_1 - 1)(d_2 - 1) - \delta, \quad (3.2)$$

where $\delta = \sum_P \delta(P)$ is the sum of so-called δ -invariants of singular points of C . The analytic invariant $\delta(P)$ is positive if and only if the point P is singular, see [33, §1, Prop. 1]. The δ -invariant admits several definitions. Let us recall the geometric definition. Let (α, P) be a germ of analytic curve, $P \in \mathbb{C}_{x,y}^2$, and let $f(x, y) = 0$ be its equation. Let V be a small ball centered at P . Then $\alpha_\varepsilon := \{f(x, y) = \varepsilon\} \cap V$, for $0 < |\varepsilon| \ll 1$, is a smooth real two-dimensional closed surface with r holes (*Milnor fiber*). The δ -invariant $\delta(P)$ topologically can be defined as the genus of the closed surface obtained by attaching a sphere with r holes to the surface α_ε . See [30].

Let us now apply formula (3.2) to the curves $\widehat{\Xi}_l^\pm \subset \mathbb{P}^1 \times \mathbb{P}^1$, say, to the curve with the sign $+$.

Claim 1. *The bidegree of the curve $\widehat{\Xi}_l^+$ is equal to the vector of multiplicities $(\mu_{p_+}(\overline{\Xi}_l^+), \mu_{p_-}(\overline{\Xi}_l^+))$ up to permutation.*

Proof Blowing up each singularity p_{\pm} separates the branches through it and erases the singularity of the blown up curve (i.e., its strict transform). It yields a (-1) -curve C_{\pm} intersecting the above strict transform transversely in $\mu_{p_{\pm}}$ distinct points. The latter intersection points do not lie in the strict transform C_{∞} of the infinity line, since the branches at p_{\pm} (before the blow up) are transversal to the infinity line (the last statement of Proposition 3.3). Therefore, blowing down C_{∞} does not create additional singularities. Moreover, the strict transforms of the above (-1) -curves C_{\pm} are vertical and horizontal fibers in the product structure $\mathbb{P}^1 \times \mathbb{P}^1$ of the blown down surface. They intersect the curve $\widehat{\Xi}_l^+$ transversely at $\mu_{p_{\pm}}$ distinct smooth points, by the above argument. This implies the statement of the claim. \square

Substituting the formulas for the multiplicities from Proposition 3.3 to the bidegree vector in the above claim and substituting everything to genus formula (3.2) yields the statement of Proposition 3.4. Its last statement follows from construction and the above blow-up argument. \square

Corollary 3.5 *The strict transforms of curves $\overline{\Xi}_l^{\pm}$ for any l on X_7 have no real singularities. Also, their projections $\widehat{\Xi}_l^{\pm} \subset \mathbb{P}^1 \times \mathbb{P}^1$ are smooth over \mathbb{R} .*

The corollary follows from the blow up argument in the proof of Claim 1.

One can see examples of curves Ξ_x^+ on Fig. 8

Corollary 3.6 *Conjecture 1.5 on genus formula is equivalent to smoothness of some (any) of the curves $\widehat{\Xi}_l^{\pm}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ over \mathbb{C} and also equivalent to the smoothness of the complex curve $\Gamma_l \setminus \{\mu = 0\} \subset \mathbb{C}^2$.*

4 The Monotonicity Conjecture: proof for generalized simple intersections with smaller rotation numbers (A.A.Glutsyuk)

Here we prove Theorem 1.30. For its proof let us first recall that for every $\mu > 0$ the polynomial $P_l(\lambda, \mu^2)$ with fixed μ and variable λ is a polynomial of degree l that has l distinct real roots

$$\lambda_1 < \lambda_2 < \dots < \lambda_l; \quad \lambda_j = \lambda_j(\mu), \quad (4.1)$$

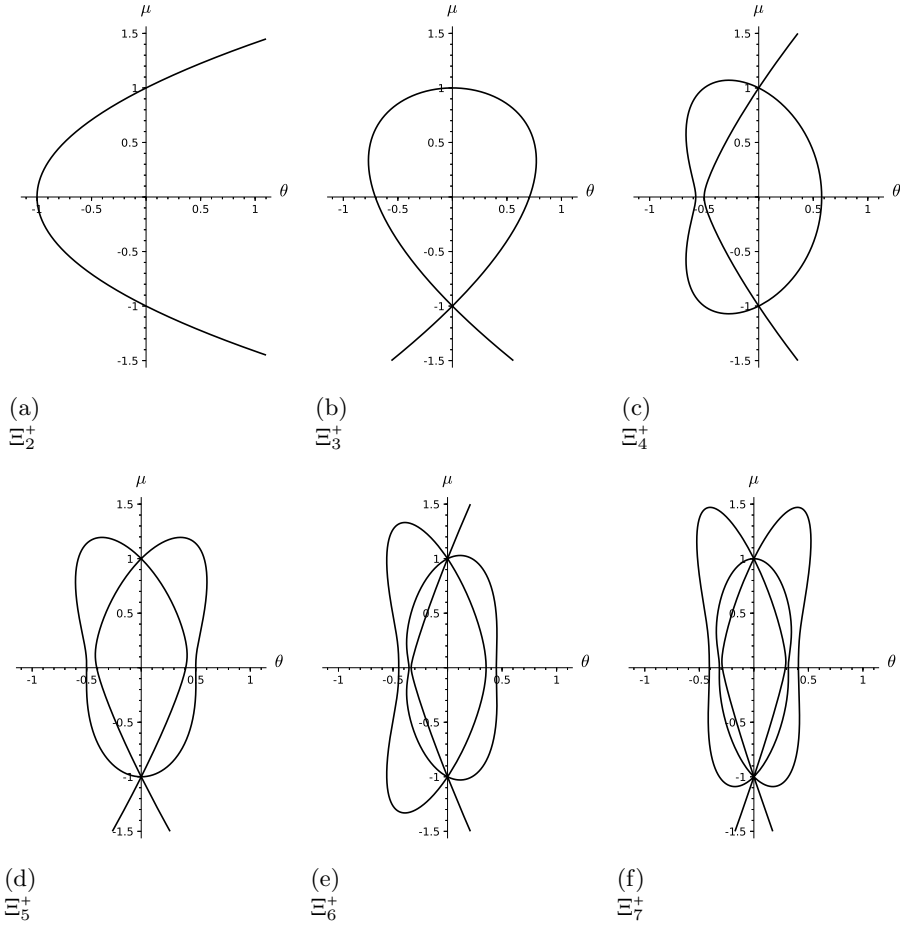


Figure 8: Some examples of curves Ξ_l^+ in the affine chart $r \neq 0$.

by Buchstaber–Tertychnyi Theorem 1.9 (see [11, p.974, theorem 1]). Set

$$R_j(\mu) := \lambda_j(\mu) + \mu^2.$$

In the case, when $R_j(\mu) > 0$, set

$$\begin{aligned} \omega_j(\mu) &:= \sqrt{\frac{1}{4R_j(\mu)}} > 0, \quad B_j(\mu) := l\omega_j(\mu), \quad A_j(\mu) := 2\mu\omega_j(\mu), \\ \Pi_j(\mu) &:= (B_j(\mu), A_j(\mu)). \end{aligned} \tag{4.2}$$

Then the point $\Pi_j(\mu)$ is a generalized simple intersection for $\omega = \omega_j(\mu)$.

Our first goal is to show that $R_j(\mu) > 0$ for every $\mu > 0$ and to describe the rotation numbers $s = s(j)$ of the generalized simple intersections defined by the roots $\lambda_j(\mu)$. This is done in the three following propositions.

Proposition 4.1 *For every $\mu > 0$ one has $R_j(\mu) \neq 0$; thus, each function $R_j(\mu)$ in $\mu > 0$ has constant sign.*

Proof If, to the contrary, $R_j(\mu) = 0$ for some $\mu > 0$, then

$$0 = P_l(\lambda_j(\mu), \mu^2) = P_l(\lambda_j, -\lambda_j)(\mu) = \det M_1(\lambda_j(\mu)) = \lambda_j^l(\mu),$$

see the definition of the matrix function $M_1(\lambda)$ and formula (2.2) for its determinant in Subsection 2.1. Finally,

$$\lambda_j(\mu) = 0, \quad \mu^2 = R_j(\mu) - \lambda_j(\mu) = 0,$$

– a contradiction. The proposition is proved. \square

Proposition 4.2 *Let $l \in \mathbb{N}$. For every j such that $R_j > 0$ there exists a unique pair (s, \pm) of a number $s = s(j) \equiv l \pmod{2}$, $0 \leq s \leq l$, and a sign \pm (for $s(j) = 0$ the corresponding sign is always "+") such that for every $\mu > 0$ the point $\Pi_j(\mu)$, see (4.2), is a generalized simple intersection of the axis $\Lambda_l(\omega_j(\mu))$ with the boundary curve $\partial L_{s,\pm}(\omega_j(\mu))$.*

Proof It follows by construction that each point $\Pi_j(\mu)$ is a generalized simple intersection depending continuously on $\mu > 0$. The corresponding rotation number $s = s(j)(\mu)$ is integer-valued, by construction, and also continuous in μ . Therefore, it is constant in $\mu > 0$ and hence, depends only on j . Continuity and thus, constance of the sign $\pm = \pm(\mu)$ follows by the same argument and by well-definedness of the sign at a generalized simple intersection. Indeed, a generalized simple intersection with $\mu > 0$ cannot lie in two distinct boundary components of a phase-lock area, since it is neither a growth point, nor a constriction. The proposition is proved. \square

Proposition 4.3 *Let $l \in \mathbb{N}$.*

- 1) *For every $j = 1, \dots, l$ one has $R_j > 0$.*
- 2) *The corresponding rotation number $s(j)$ from Proposition 4.2 is given by the formula*

$$s(j) = \begin{cases} l - j + 1 & \text{for odd } j, \\ l - j & \text{for even } j. \end{cases} \quad (4.3)$$

3) For every s , $0 < s < l$, choose arbitrary sign \pm . For $s = 0$ choose the sign $+$. There exists a (unique) choice of sign $\pm = (\pm)_l$ for $s = l$ such that for every pair (s, \pm) as above with $s \equiv l \pmod{2}$ and every $\mu > 0$ there exists a unique $\omega > 0$ for which $(l\omega, 2\mu\omega)$ is a generalized simple intersection of the axis $\Lambda_l(\omega)$ and the boundary curve $\partial L_{s,\pm}(\omega)$. The latter ω is equal to some $\omega_j(\mu)$. The above j and s are related by formula (4.3); the number j is uniquely determined by the pair (s, \pm) .

Proof For every s , $0 < s < l$, every choice of sign \pm and every $\omega > 0$ small enough the boundary curve $\partial L_{s,\pm}(\omega)$ intersects $\Lambda_l(\omega)$ at some point $\Pi^{s,\pm}(\omega) = (l\omega, A^{s,\pm}(\omega))$. A version of this statement holds for $s = 0$ and the sign “+”, and for $s = l$ and exactly one choice of sign. Both statements follow from Theorem 1.18 and Corollary 1.19. The number of pairs (s, \pm) with $s \equiv l \pmod{2}$ for which the above statement holds (including $s = 0$ and $s = l$) is equal to l , by Corollary 1.19. Set

$$\mu^{s,\pm} := \frac{A^{s,\pm}(\omega)}{2\omega}, \quad s \equiv l \pmod{2}.$$

Then $\Pi^{s,\pm}(\omega) = \Pi_j(\mu^{s,\pm})$ for some $j \in \{1, \dots, l\}$, and one has $R_j(\mu^{s,\pm}) > 0$. This follows from construction. The numbers $j = j(s, \pm)$ are uniquely defined by (s, \pm) , and they are distinct for distinct pairs (s, \pm) . This follows from Proposition 4.2. The number of indices j corresponding to pairs (s, \pm) is equal to l , as is the number of pairs (s, \pm) , and $R_j > 0$ for these j . The total number of indices j is equal to l . Therefore, the correspondence $j \mapsto (s, \pm)$ is one-to-one, and for $s = l$ the corresponding sign is uniquely determined, as in Theorem 1.18. This also implies that $R_j > 0$ for all $j = 1, \dots, l$. For every given $l \in \mathbb{N}$, $\mu > 0$ and $\omega > 0$ the point $(B = l\omega, A = 2\mu\omega)$ is a generalized simple intersection, if and only if it coincides with some of $\Pi_j(\mu)$, and then $\omega = \omega_j(\mu)$. This follows from construction.

The above discussion proves Proposition 4.3, except for formula (4.3) and the last part of statement 3), which follows from this formula. Let us now prove (4.3).

Claim. *The numbers $s(j)$ form a non-increasing sequence.*

Proof Note that for every integer $s_1 < s_2$ the phase-lock area L_{s_1} lies on the left from the phase-lock area L_{s_2} . For every $s \in \mathbb{N}$ consider the growth point $(\sqrt{s^2\omega^2 + 1}, 0)$ of the phase-lock area L_s , see [10, corollary 3]. In the renormalized coordinates $(l = \frac{B}{\omega}, \mu = \frac{A}{2\omega})$ its abscissa $\sqrt{s^2 + \frac{1}{\omega^2}}$ moves from the left to the right, as ω decreases. Therefore, for every $l \in \mathbb{N}$ and $s_1 < s_2 \leq l$, as $\omega > 0$ decreases, an intersection of the axis $\Lambda_l(\omega)$ with the boundary ∂L_{s_2} will appear earlier than its intersection with ∂L_{s_1} . The

above intersections are generalized simple intersections, if $s_1 \equiv s_2 \equiv l \pmod{2}$ and $s_2 < l$. Indeed, they are either constrictions, or generalized simple intersections (by definition), and they cannot be constrictions by inequality $s_1 < s_2 < l$ and [19, theorem 1.2].

We already know that for given $l \in \mathbb{N}$, $\mu > 0$ and $\omega > 0$ a point $(B = l\omega, A = 2\mu\omega)$ is a generalized simple intersection, if and only if $\omega = \omega_j(\mu)$ for some $j \in \{1, \dots, l\}$. One has $\omega_1(\mu) > \dots > \omega_l(\mu)$, by (4.1). This together with the above discussion implies that the sequence $s(j)$ of the corresponding rotation numbers does not increase and proves the claim. \square

The claim implies that the pairs $(s, \pm)(1), \dots, (s, \pm)(l)$ are ordered so that the pair $(l, (\pm)_l)$ goes first, then two pairs $((l-1), \pm)$ with both signs \pm , etc. This implies formula (4.3) and proves Proposition 4.3. \square

Proof of Theorem 1.30. Suppose the contrary: for certain $k \in \mathbb{N}$, $\mu_0 > 0$ and $j \in \{1, \dots, k\}$ a generalized simple intersection $z = \Pi_j(\mu_0) \in \Lambda_k(\omega) \cap \partial L_{s, \pm}(\omega)$, is a left-moving tangency; here $s = s(j)$, $\omega = \omega_j(\mu_0)$. We use the rescaled coordinates $(l = \frac{B}{\omega}, \mu = \frac{A}{2\omega})$ on \mathbb{R}^2 . Recall that we consider that $s < k$, $s \equiv k \pmod{2}$, by the conditions of Theorem 1.30 and since generalized simple intersections of a boundary curve $\partial L_{s, \pm}$ and an axis Λ_k may exist only for $s \in [0, k]$, $s \equiv k \pmod{2}$, see Theorem 1.17. Then for $\omega > \omega_j(\mu_0)$ the boundary curve $\partial L_{s, \pm}(\omega)$ intersects the horizontal line $\mu = \mu_0$ on the right from the point $z = \Pi_j(\mu_0)$, by the definition of left-moving tangency (see Fig. 6). On the other hand, for every $\omega > 0$ large enough the phase-lock area $L_s(\omega)$ should lie on the left from the line $\{l = k\}$, by [19, proposition 3.4] (which in its turn follows from Chaplygin's comparison theorem [29, 10]). Hence, there exists a $\omega^* > \omega_j(\mu_0)$ such that the boundary curve $\partial L_{s, \pm}(\omega^*)$ contains z , by continuity. Then the point z is a generalized simple intersection of the curve $\partial L_{s, \pm}(\omega^*)$ and the line $\Lambda_k(\omega^*)$, and hence, $\omega^* = \omega_m(\mu_0)$ for a certain $m \in \{1, \dots, k\}$. One has $m < j$, since $\omega_m(\mu_0) = \omega^* > \omega_j(\mu_0)$, by construction and the sequence $\omega_n(\mu)$ decreases. Finally, two distinct indices m and j correspond to the same pair (s, \pm) , - a contradiction to the last statement of Proposition 4.3. This proves Theorem 1.30. \square

5 Some open problems and a corollary (A.A.Glutsyuk)

First we discuss some open problems on model of Josephson effect: on boundaries of phase-lock areas, constrictions and their complexifications.

Afterwards we reformulate Theorem 1.22 in terms of family (1.14) of special double confluent Heun equations and state an open question about them.

5.1 Open problems related to geometry of phase-lock area portrait

To our opinion the main results of the paper and solutions of the next open problems might be useful in study of the geometry of the family of phase-lock area portrait in the model of Josephson effect and dependence of the constrictions on the parameter ω .

Let us formulate a complex version of Conjecture 1.13 and related questions. To do this, for a given $l \in \mathbb{Z}$ consider the one-dimensional complex analytic subset

$$C_l = \{(\mu, \omega) \in \mathbb{C}^2 \setminus \{\mu = 0\} \mid \text{Riccati equation (1.11) has trivial monodromy}\}.$$

This is a complex analytic subset in

$$\mathbb{C}_{\mu \neq 0}^2 := \mathbb{C}_{\mu, \omega}^2 \setminus \{\mu = 0\}.$$

Remark 5.1 Let $\omega > 0$, $\mu \in \mathbb{R}$, $l \in \mathbb{Z}$, $B = l\omega$, $A = 2\mu\omega$. Let (B, A) be a constriction for the model of Josephson effect. Then $(\mu, \omega) \in C_l$. Thus, the set C_l can be viewed as the set of *complex constrictions* in the complex axis $\Lambda_l = \{B = \omega l\} \simeq \mathbb{C}$. The numerical pictures (see, e. g., Figs. 1, 2) show that the real constrictions viewed as points in C_l depend continuously on the parameter ω .

Proposition 5.2 *For every $l \in \mathbb{Z}$ the analytic subset C_l has pure dimension one and is given by zeros of a global holomorphic function $\mathbb{C}_{\mu \neq 0}^2 \rightarrow \mathbb{C}$. For $l \geq 0$ ($l < 0$) the latter function is the Stokes multiplier c_0 (respectively, c_1) of the irregular singular point 0 of linear system (1.12), see [20, formula (2.2)] for the definition of the Stokes multipliers $c_{0,1}$.*

Proof It is known that for $l \in \mathbb{Z}$ triviality of the monodromy transformation of Riccati equation (1.11) is equivalent to vanishing of both Stokes multipliers, $c_0 = c_1 = 0$, see [19, proof of lemma 3.3]. In the case, when $l \in \mathbb{Z}_{\geq 0}$, this is equivalent to the existence of an entire solution of the double confluent Heun equation (1.14), see [5, theorem 4.10]. On the other hand, vanishing of just one Stokes multiplier c_0 implies the existence of the latter solution. See the proof of the real analogue of the latter statement in [20,

proof of theorem 2.5], which remains valid for complex parameters as well. The case, when $l \in \mathbb{Z}_{<0}$, is reduced to the case, when $l \in \mathbb{Z}_{\geq 0}$, via linear isomorphism [20, (2.14)] between solution spaces of system (1.12) and the same system with opposite sign at l . The latter isomorphism conjugates the corresponding monodromies and inverses the numeration of the canonical basic solutions and thus, of the Stokes multipliers. Therefore, vanishing of the Stokes multiplier c_0 for system (1.12) with non-negative l is equivalent to vanishing of the Stokes multiplier c_1 for the same system with opposite sign at l . This proves the proposition. \square

Question 1. How many irreducible components does the curve C_l have?

Question 2. Let \widehat{C}_l denote the normalization of the curve C_l : the Riemann surface bijectively parametrizing C_l (except for self-intersections). Is it true that the projection

$$\widehat{C}_l \rightarrow \mathbb{C}, (\mu, \omega) \mapsto \omega$$

is a (ramified) covering over \mathbb{C} ? Describe its ramification points (if any).

Conjecture 5.3 The above projection has no real critical points (μ, ω) .

Remark 5.4 A positive solution of Conjecture 5.3 would imply the solution of Conjecture 1.13. This implication follows from [5, proposition 5.31] and the discussion before it.

Recall that Σ_{\pm} is the locus of those parameter values for which the monodromy of Riccati equation (1.11) fixes the point $\pm i$. The curve C_l is included into the intersection $\Sigma_+ \cap \Sigma_-$ via the rational mapping

$$(\mu, \omega) \mapsto (l\omega, 2\mu\omega, \frac{1}{2\omega}).$$

Proposition 5.5 *The surface Σ_{\pm} consists exactly of those points, for which the monodromy of the Riccati equation (1.11) is a parabolic (or identical) Möbius transformation fixing $\pm i$. Each point in $\mathbb{C}_{(B,A,r)}^3$ corresponding to a parabolic (or identical) monodromy lies in the union $\Sigma = \Sigma_+ \cup \Sigma_-$.*

Proof The symmetry $(\Phi, z) \mapsto (-\Phi^{-1}, z^{-1})$ of Riccati equation (1.11) (see the proof of Proposition 2.13) restricted to the invariant fiber $\{z = 1\}$ conjugates the monodromy transformation of the fiber with its inverse. Its fixed points are exactly $\pm i$. For every point in Σ_{\pm} the monodromy and its inverse

both fix the corresponding point $\pm i$. Therefore, their germs at the fixed point are analytically conjugated, and hence, are either parabolic, or identical. Vice versa, if the monodromy is parabolic, then its unique fixed point should be also fixed by the above symmetry and hence, should coincide with some of $\pm i$. The proposition is proved. \square

It could be useful to study analogues of Conjecture 5.3 for the projections of the surfaces Σ_{\pm} .

Question 3. Is it true that the projections of the normalizations $\widehat{\Sigma}_{\pm}$ of the surfaces $\Sigma_{\pm} \subset \mathbb{C}_{(B,A,r)}^3$ to $\mathbb{C}_{(B,r)}^2$, $(B, A, r) \mapsto (B, r)$, are (ramified) coverings? Find their ramification loci (if any).

The surfaces $\widehat{\mathcal{L}}_{\pm}$ were constructed as the minimal analytic subsets containing the union of families of \pm -boundary components of the phase-lock areas for the model of Josephson effect in $\mathbb{R}_{(B,A,\frac{1}{2\omega})}^3$. It follows from definition that $\widehat{\mathcal{L}}_{\pm} = \widehat{\mathcal{L}}_{\pm}^{even} \cup \widehat{\mathcal{L}}_{\pm}^{odd} \subset \Sigma_{\pm}$, and all the surfaces under question are purely two-dimensional.

Remark 5.6 Theorem 1.22 implies that $\widehat{\mathcal{L}}_{\pm}^{even(odd)}$ are some irreducible components of the surfaces Σ_{\pm} . Thus, *the surface $\Sigma = \Sigma_+ \cup \Sigma_-$ consists of at least four irreducible components. Each Σ_{\pm} splits into two parts $\Sigma_{\pm}^{even(odd)}$, which are unions of some irreducible components, by the index parity argument from the proof of Proposition 2.13.*

Question 4. Is it true that $\widehat{\mathcal{L}}_{\pm} = \Sigma_{\pm}$? In other terms, is it true that each surface Σ_{\pm} is a union of two irreducible components $\Sigma_{\pm}^{even(odd)} = \widehat{\mathcal{L}}_{\pm}^{even(odd)}$?

5.2 A corollary and an open problem on special double confluent Heun equation

Recall that the *monodromy operator* of linear equation (1.14) (or linear system (1.12)) is a linear operator acting on the space of its solutions by analytic extension along a counterclockwise circuit around the origin in the z -line. The monodromy operators of equation (1.14) and the corresponding linear system (1.12) are conjugated via the transformation

$$E(z) \mapsto (u(z), v(z)) = (2i\omega z e^{-\mu z} (E'(z) - \mu E(z)), e^{-\mu z} E(z)), \quad \omega^2 = \frac{1}{4(\lambda + \mu^2)}$$

sending solutions of (1.14) to those of system (1.12). The ratio

$$\Phi(z) = \frac{v(z)}{u(z)} = \frac{E(z)}{2i\omega z (E'(z) - \mu E(z))} \quad (5.1)$$

is a solution of the Riccati equation (1.11).

The transformation

$$\Psi : (B, A, r = \frac{1}{2\omega}) \mapsto (l = 2Br, \lambda = r^2(1 - A^2), \mu = Ar).$$

sends the parameters of model of Josephson effect (or equivalently, the parameters of Riccati equation (1.11)) to the parameters (l, λ, μ) of the corresponding special double confluent Heun equation (1.14).

Remark 5.7 A point $(B, A, r) \in \mathbb{C}^3$ lies in $\Sigma = \Sigma_+ \cup \Sigma_-$, if and only if the corresponding Heun equation (1.14) has monodromy operator with multiple eigenvalue. This follows from Proposition 5.5. If in this case the monodromy operator is a Jordan cell, then its eigenfunction $E(z)$ (unique up to constant factor) defines a meromorphic solution $\Phi(z)$, see (5.1), of Riccati equation (1.11), for which $\Phi(1) = \pm i$. If we change ω by $-\omega$ (which does not change the parameters (l, λ, μ) of Heun equation), then the corresponding value $\Phi(1) = \pm i$ will change the sign.

Theorem 1.22, Proposition 5.5 and Question 4 can be equivalently reformulated as follows in terms of double confluent Heun equations (1.14).

Theorem 5.8 Let $\tilde{\Sigma} \subset \mathbb{C}_{(l, \lambda, \mu)}^3$ denote the subset of those parameters $(l, \lambda, \mu) \in \mathbb{C}^3$ for which the corresponding special double confluent Heun equation (1.14) has a monodromy operator with multiple eigenvalue. The set of real points of the surface $\tilde{\Sigma}$ satisfying the inequality $\lambda + \mu^2 > 0$ (which thus come from boundaries of the phase-lock areas) is contained in the union of two irreducible components $\tilde{\mathcal{L}}^{\text{even(odd)}}$ of the surface $\tilde{\Sigma}$.

Addendum to Theorem 5.8. In each of the above components $\tilde{\mathcal{L}}^{\text{even(odd)}}$ there exists an open and dense subset of those points for which

- the monodromy operator of equation (1.14) is a Jordan cell;
- the monodromy eigenfunction $E(z)$ defines a solution (5.1) of Riccati equation (1.11) meromorphic on \mathbb{C}^* that has neither zeros, nor poles on the unit circle $S^1 = \{|z| = 1\}$ and has even (respectively, odd) index along S^1 .

Proof The minimal analytic subset containing the real points under question coincides with $\Psi(\cup_{\pm} \widehat{\mathcal{L}}_{\pm})$, which follows from definition. The mapping Ψ is polynomial of degree two, and its ramification locus is the surface

$\lambda + \mu^2 = 0$. The surface Σ is the preimage of the surface $\tilde{\Sigma}$, by definition, Proposition 5.5 and Remark 5.7. The sign change $\omega \mapsto -\omega$ permutes the sets Σ_{\pm} and the sets $\widehat{\mathcal{L}}_{\pm}$. This implies that Ψ sends the four irreducible components $\widehat{\mathcal{L}}_{\pm}^{even(odd)}$ to two components $\tilde{\mathcal{L}}^{even(odd)}$. The statements of the addendum follow from construction. \square

Question 5. Is it true that the whole surface $\tilde{\Sigma} \subset \mathbb{C}_{(l,\lambda,\mu)}^3$ is the union of the two above irreducible components $\tilde{\mathcal{L}}^{even(odd)}$?

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