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ABSTRACT. We construct elements in the group K_4 of modular curves using the polylogarithmic complexes of weight 3 defined by Goncharov and De Jeu. The construction is uniform in the level and uses new modular units obtained as cross-ratios of division values of the Weierstraß $\wp$ function. These units provide explicit triangulations of the 3-term relations in K_2 of modular curves, which in turn give rise to elements in K_4. Based on numerical computations and on recent results of W. Wang, we conjecture that these elements are proportional to the Beilinson elements defined using the Eisenstein symbol.

1. INTRODUCTION

The motivic cohomology of algebraic varieties is a fundamental invariant which appears, for example, in the statement of Beilinson's general conjectures on special values of L-functions. However this invariant is very difficult to handle in general: no universal recipe is known to produce non-trivial elements in motivic cohomology groups. At the same time, for varieties defined over number fields, finite generation results for motivic cohomology seem to be completely out of reach in general.

Let us describe the situation for fields in more detail. Let F be an arbitrary field. The motivic cohomology group $H^i_{\mathcal{M}}(F, \mathbf{Q}(n))$ is isomorphic to the Adams eigenspace $K^{(n)}_{2n-i}(F)$ of Quillen's K-group $K_{2n-i}(F) \otimes \mathbf{Q}$. The groups $K_0(F)$ and $K_1(F)$ are isomorphic to $\mathbf{Z}$ and $F^\times$ respectively. The group $K_2(F)$ is described by Matsumoto's theorem, which gives generators and relations for this group:

$$K_2(F) \cong \frac{F^\times \otimes_{\mathbf{Z}} F^\times}{\langle x \otimes (1-x) : x \in F \backslash \{0,1\} \rangle}.$$

The class of $x \otimes y$ in $K_2(F)$ is denoted by $\{x, y\}$ and is called a Milnor symbol. The relations $\{x, 1-x\} = 0$ are called the Steinberg relations.

The group $K_3(F)$ has a Milnor part $K^M_3(F)$, generated by symbols $\{x, y, z\}$ subject to Steinberg relations. The motivic-to-K-theory spectral sequence shows that $K_3(F)/K^M_3(F)$ is isomorphic to $H^1_{\mathcal{M}}(F, \mathbf{Z}(2))$. If F is infinite, Suslin has shown that $K_3(F)/K^M_3(F)$ is isomorphic, up to torsion, to the Bloch group of F (see Definition 5.2).

The higher K-groups of F are even more difficult to deal with. For any weight $n \geq 1$, Goncharov has defined in [17] a polylogarithmic motivic complex $\Gamma(F, n)$ whose cohomology in degree $1 \leq i \leq n$ is expected to compute $H^i_{\mathcal{M}}(F, \mathbf{Q}(n))$. In the case F is a number field,

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$i = 1$ and $n \geq 2$, a map from $H^1(\Gamma(F, n))$ to $H^1_{\mathcal{M}}(F, \mathbf{Q}(n))$ has been constructed [2, 23, 37] and is expected to be an isomorphism. This is related to Zagier's conjecture for the Dedekind zeta value $\zeta_F(n)$ [42, 2], which has been proved for $n \leq 4$ [15].

We are mainly interested here with the motivic cohomology of modular curves. The groups of interest are $H^2_{\mathcal{M}}(Y(N), \mathbf{Q}(n)) \cong K^{(n)}_{2n-2}(Y(N))$ for $n \geq 2$, where $Y(N)$ is the modular curve of level N. For $n = 2$, taking the cup-product of two modular units provides the so-called Beilinson-Kato elements. Via the regulator map, they are related to the L-values $L(f, 2)$ for modular forms f of weight 2 and level N. These elements have important applications: for example, they are used in Kato's construction of a Euler system for modular forms [26]. For $n \geq 3$, Beilinson has constructed classes in $K^{(n)}_{2n-2}(Y(N))$ using his theory of the Eisenstein symbol [1]. Their images under the regulator map are related to the values $L(f, n)$.

In this article, we concentrate on the case $n = 3$ and construct new elements in $K^{(3)}_4(Y(N))$ using the polylogarithmic complex of weight 3 attached to the function field of $Y(N)$. One key ingredient is certain modular units obtained as cross-ratios of N-division values of the Weierstraß $\wp$ function. These units are solutions to the S-unit equation for $X(N)$, where S is the set of cusps. Another input is the 3-term relations for the Beilinson-Kato elements in $K_2(Y(N))$, proved in [7] and [20] with $\mathbf{Q}$-coefficients (see also the new approach of Sharifi and Venkatesh [32] to these relations). We give an effective proof of these relations with $\mathbf{Z}[1/6N]$-coefficients, and use them to build degree 2 cocycles in the Goncharov complex. One important feature of this construction is that it is uniform in the level N.

At this point, we apply De Jeu's results [24] to map these cocycles to $K^{(3)}_4(Y(N))$, making use of his wedge complexes. These complexes are defined similarly but involve K-theory in a more direct way, which is crucial in our construction. We also devise a method to compute numerically with PARI/GP [31] the image of these K_4 elements under Beilinson's regulator map. Our strategy is to integrate the regulator 1-form along modular symbols joining cusps. This is similar to the approach taken in [13] for hyperelliptic curves, but differs from [24, 25], where the regulator 1-form gets integrated against a holomorphic 1-form. This enables us to check numerically Beilinson's conjecture on $L(E, 3)$ for every elliptic curve E over $\mathbf{Q}$ of conductor $N \leq 50$. This extends a result of De Jeu [24, Section 6], who considered a specific elliptic curve of conductor 20.

As mentioned above, special elements in K_4 of modular curves have already been defined by Beilinson through a different method, and their regulators are known to be related to L-values of modular forms [1]. On the other hand, it is a difficult open problem to relate regulators on the Goncharov complex in the non-Milnor case, to special values of L-functions. In this direction, we conjecture, based on numerical evidence, that our elements coincide (up to a simple rational factor) with the Beilinson elements. Proving this would have interesting consequences, for example on the Mahler measure of certain 3-variable polynomials; see Conjecture 9.5 and [9, Chapter 8].

The outline of this paper is as follows. In Section 2, we prove, using a theorem of Mason, that the S-unit equation for curves has only finitely many solutions. In Section 3, we define the modular units which are used in our construction. Section 4 gives an "effective" proof of the 3-term relations in K_2 of modular curves. These relations are then used in Sections 5 and 6 to construct the K_4 elements. In Sections 7 and 8, we explain how to compute numerically their regulators. Finally, we formulate in Section 9 the conjecture relating our elements and the Beilinson elements.

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2. The S-unit equation for curves

Let X be a smooth proper connected curve over $\mathbf{C}$. Let S be a finite set of closed points of X. The S-unit equation for X is the equation $f+g=1$, where f, g are non-constant rational functions on X whose zeros and poles are contained in S. Geometrically, this amounts to find the non-constant morphisms $f\colon X\backslash S \to \mathbf{P}^1\backslash\{0,1,\infty\}$.
Solving the S-unit equation for curves has two potential applications:

(1) Prove relations in K_2 of curves;
(2) Construct elements in K_4 of curves.

Namely, each solution (f,g) to the S-unit equation provides the Steinberg relation $\{f,g\}=0$ in the group $K_2(X\backslash S)$. Moreover, as we shall see in the particular case of modular curves, relations in K_2 can be used to construct elements in K_4; see Section 6.
We first recall the following bound on the degrees of the solutions to the S-unit equation, due to Mason [29, p. 222].

Theorem 2.1 (Mason)**.** *If (f,g) is a solution to the S-unit equation for X, then* $\deg(f) \leq 2g_X - 2 + |S|$*, where g_X is the genus of X.*

Corollary 2.2. *The set of solutions to the S-unit equation for X is finite.*

Proof of Corollary 2.2. By Theorem 2.1, there are only finitely many possibilities for the divisors of f and g. Moreover, if (f,g) is a solution, then g must vanish at some point $p \in S$, which implies $f(p)=1$. This shows that for a given divisor D, there are only finitely many solutions (f,g) such that $\operatorname{div}(f)=D$. □

As I learnt from A. Javanpeykar [22], the finiteness of solutions to the S-unit equation for curves can also be proved using the de Franchis-Severi theorem for hyperbolic curves.
The proof of Corollary 2.2 above actually provides an algorithm to find all the solutions to the S-unit equation. I implemented this algorithm in Magma [6]. In the case of elliptic curves, one may view this algorithm as an extension of Mellit's technique of parallel lines [30]. Namely, the rational functions appearing in [30] have degree at most 3, while here the degree is arbitrary. Of course, looping over the possible divisors becomes impracticable when the cardinality of S or the Mason bound is too large; in practice we are only able to deal with rather small degrees.
Regarding the S-unit equation, here are some interesting situations:

- $X=\mathbf{P}^1$ and $S=\{0,\infty\}\cup\mu_N$, where μ_N denotes the Nth roots of unity (see the recent work of Zhao [43]);
- $X=E$ is an elliptic curve, and S is a finite subgroup of E;

- X is the Fermat curve with projective equation $x^N + y^N = z^N$, and S is the set of points with one coordinate equal to 0;
- X is a modular curve, and S is the set of cusps of X.

In this article, we will concentrate on the case of modular curves.

3. Modular S-units

We denote by $\mathcal{H}$ the Poincaré upper half-plane. Let $N \geq 1$ be an integer. For any $a = (a_1, a_2) \in (\mathbf{Z}/N\mathbf{Z})^2$, $a \neq (0,0)$, we define

$$\wp_a(\tau) = \wp\Big(\tau, \frac{a_1\tau + a_2}{N}\Big) \qquad (\tau \in \mathcal{H}),$$

where $\wp$ is the Weierstraß function. We have the transformation formula $\wp_a|_2\gamma = \wp_{a\gamma}$ for any $\gamma \in \mathrm{SL}_2(\mathbf{Z})$, where $|_2$ is the slash action in weight 2. Moreover $\wp_a$ is holomorphic at the cusps, so that $\wp_a$ is a modular form of weight 2 on the principal congruence subgroup $\Gamma(N) = \ker(\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}))$.
Since the Weierstraß $\wp$-function has a double pole at the origin, we also set $\wp_0 = \infty$. Note that since $\wp$ is even, we have $\wp_{-a} = \wp_a$ for every $a \in (\mathbf{Z}/N\mathbf{Z})^2$.

Definition 3.1. Let a, b, c, d be distinct elements of $(\mathbf{Z}/N\mathbf{Z})^2/\pm 1$. We define $u(a,b,c,d)$ as the cross-ratio of the modular forms $\wp_a, \wp_b, \wp_c, \wp_d$:

$$u(a,b,c,d) = [\wp_a, \wp_b, \wp_c, \wp_d] = \frac{\wp_c - \wp_a}{\wp_c - \wp_b} \Big/ \frac{\wp_d - \wp_a}{\wp_d - \wp_b}.$$

Since the functions $\wp_x$ are modular forms of weight 2, the function $u(a,b,c,d)$ is invariant under $\Gamma(N)$, and is meromorphic at the cusps. That is, $u(a,b,c,d)$ is a modular function for $\Gamma(N)$.

Lemma 3.2. *The function $u(a,b,c,d)$ has no zeros or poles on $\mathcal{H}$. In other words, it is a modular unit for $\Gamma(N)$.*

Proof. We know from the theory of elliptic functions that $\wp(\tau, z) = \wp(\tau, z')$ if and only if $z' = \pm z \bmod \mathbf{Z} + \tau\mathbf{Z}$. Since a, b, c, d are distinct in $(\mathbf{Z}/N\mathbf{Z})^2/\pm 1$, the function $u(a,b,c,d)$ has no zeros or poles on $\mathcal{H}$. □

Modular units of the form $(\wp_a - \wp_b)/(\wp_c - \wp_d)$ are called Weierstraß units in [27, Chapter 2, Section 6]. Here $u(a,b,c,d)$ is a quotient of two Weierstraß units, but is not a priori a Weierstraß unit. Recently, Bolbachan considered the cross-ratio of Weierstraß functions in relation with the elliptic dilogarithm [4]. The cross-ratio is viewed there as an elliptic function, not as a modular one.
If Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$, we denote by $Y(\Gamma) = \Gamma\backslash\mathcal{H}$ the modular curve of level Γ, and by $X(\Gamma) = Y(\Gamma) \cup \{\text{cusps}\}$ the compactification of $Y(\Gamma)$. We also write $Y(N) = Y(\Gamma(N))$ and $X(N) = X(\Gamma(N))$.
The definition of $u(a,b,c,d)$ as a cross-ratio makes it clear that

$$u(a,b,c,d) + u(a,c,b,d) = 1.$$

It follows that $u(a,b,c,d)$ is a solution to the S-unit equation for the modular curve $X(N)$, where S is the set of cusps of $X(N)$. We call $u(a,b,c,d)$ a modular S-unit. Since a, b, c, d are arbitrary, this provides us with plenty of solutions to this equation, of the order of $N^8/64$ (taking into account that permuting a, b, c, d gives rise to only 6 distinct units).

By specialising a, b, c, d, we also get S-units for the modular curve $X_1(N) = X(\Gamma_1(N))$, with

$$\Gamma_1(N) = \big\{ g \in \mathrm{SL}_2(\mathbf{Z}) : g \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \bmod N \big\}.$$

Definition 3.3. For any distinct elements a, b, c, d in $(\mathbf{Z}/N\mathbf{Z})/\pm 1$, we define

$$u_1(a, b, c, d) = u((0, a), (0, b), (0, c), (0, d)).$$

Writing $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, we have $\wp_{(0,a)}|_2 T = \wp_{(0,a)T} = \wp_{(0,a)}$. This implies that $u_1(a, b, c, d)$ is invariant under T, hence is a modular unit for the larger group $\Gamma_1(N)$.
It turns out that the modular units $u_1(a, b, c, d)$ on $X_1(N)$ have remarkably low degree. The following facts illustrate this; (a), (b) and (e) have been obtained using PARI/GP.

(a) A Hauptmodul for $\Gamma_1(N)$ is given by $u_1(0, 1, 2, 3)$ for $N \in \{6, 7, 8\}$, and by $u_1(1, 2, 3, 5)$ for $N \in \{9, 10, 12\}$ (these are the integers $N \geq 6$ such that $X_1(N)$ has genus 0).
(b) For N prime, $7 < N < 300$, $N \neq 31$, the lowest degree among the units $u_1(a, b, c, d)$ is attained for the quadruplet $(a, b, c, d) = (1, 2, 3, 5)$.
(c) Up to composing by an homography, the unit $u_1(1, 2, 3, 5)$ is equal to the unit F_7/F_8 studied by Van Hoeij and Smith in [21]; they prove that $\deg(F_7/F_8) = [11N^2/840]$ for $N > 7$ prime, where $[\cdot]$ denotes the nearest integer.
(d) In fact, for $N > 7$ prime, the unit $u_1(1, 2, 3, 5)$ yields the lowest known degree for a non-constant map $X_1(N) \to \mathbf{P}^1$ defined over $\mathbf{Q}$, except for $N \in \{31, 67, 101\}$, where the lowest known degree is one less [12, Table 1].
(e) For N prime, $7 < N < 300$, the highest degree among the $u_1(a, b, c, d)$ is attained for the quadruplet $(0, 1, 3, 4)$, the degree appearing to be $[N^2/35]$.

The degrees above can be compared to the Mason bound for $X_1(N)$, which is equal, for $N \geq 5$ prime, to $2g_{X_1(N)} - 2 + |\mathrm{cusps}| = (N^2 - 1)/12$.
Our next goal is to express the modular units $u(a, b, c, d)$ in terms of the classical Siegel units. Let us first recall the definition of these units. Let $(x, y) \in \mathbf{Z}^2$ with $(x, y) \not\equiv (0, 0) \bmod N$. Consider the following infinite product

$$\theta_{x,y}(\tau) = \prod_{n \geq 0} \Big(1 - e\Big(n\tau + \frac{x\tau + y}{N}\Big)\Big) \prod_{n \geq 1} \Big(1 - e\Big(n\tau - \frac{x\tau + y}{N}\Big)\Big) \qquad (\tau \in \mathcal{H}), \tag{1}$$

where $e(z) = e^{2\pi i z}$. Following Yang [41, Theorem 1], we let

$$E_{x,y}(\tau) = q^{B_2(x/N)/2} \theta_{x,y}(\tau) \qquad (\tau \in \mathcal{H}), \tag{2}$$

where $q^\alpha = e(\alpha\tau)$ and $B_2(t) = t^2 - t + 1/6$ is the second Bernoulli polynomial. Up to multiplication by a root of unity, this is the Siegel function considered in [27, p. 29]. The function $E_{x,y}^{12N}$ is a modular unit for $\Gamma(N)$ [27, Chapter 2, Theorem 1.2].
For any $(a, b) \in (\mathbf{Z}/N\mathbf{Z})^2$, $(a, b) \neq (0, 0)$, the Siegel unit $g_{a,b}$ is defined by $g_{a,b} = E_{\tilde{a},\tilde{b}}$, where $\tilde{a}, \tilde{b}$ are representatives of a, b with $0 \leq \tilde{a} \leq N - 1$. By [41, (4)], we then have, for any $(x, y) \in \mathbf{Z}^2$, $(x, y) \not\equiv (0, 0) \bmod N$,

$$E_{x,y} = (-\zeta_N^{-y})^{\lfloor x/N \rfloor} \cdot g_{\overline{x},\overline{y}}, \tag{3}$$

where $\zeta_N = e^{2\pi i/N}$, and $\lfloor \cdot \rfloor$ denotes the floor function.
The following lemma expresses $u(a, b, c, d)$ in terms of the Siegel functions.

Lemma 3.4. *Let a, b, c, d be distinct elements of $(\mathbf{Z}/N\mathbf{Z})^2/\pm 1$, with representatives x, y, z, t in $\mathbf{Z}^2$. Then*

$$u(a,b,c,d) = \frac{E_{z+x}E_{z-x}E_{t+y}E_{t-y}}{E_{z+y}E_{z-y}E_{t+x}E_{t-x}}.$$

Proof. Write $x = (x_1, x_2)$, and similarly for y, z, t. The function $\wp_a - \wp_b$ can be expressed in terms of the Weierstraß σ-function [33, Corollary I.5.6(a)] and thus as the following infinite product by [33, Theorem I.6.4]:

$$\wp_a(\tau) - \wp_b(\tau) = -(2\pi i)^2 q^{y_1/N}\zeta_N^{y_2}\prod_{n\geq 1}(1-q^n)^4 \cdot \frac{\theta_{x+y}(\tau)\theta_{x-y}(\tau)}{\theta_x(\tau)^2\theta_y(\tau)^2}.$$

It follows that the Weierstraß units can be expressed as

$$\frac{\wp_a - \wp_b}{\wp_c - \wp_d} = \zeta_N^{y_2-t_2}\frac{E_{x+y}E_{x-y}}{E_x^2E_y^2}\frac{E_z^2E_t^2}{E_{z+t}E_{z-t}}. \tag{4}$$

The definition of $u(a,b,c,d)$ as a quotient of Weierstraß units gives the desired result. □

Remark 3.5. We would like to point out an error in [7]: the first equation of p. 288 is off by a root of unity. This root of unity can be determined from (3) and (4).

From now on, we consider the modular curve $Y(N)$ as an algebraic curve over $\mathbf{Q}$, defined as in [26, Section 1]. The field of constants of $Y(N)$ is $\mathbf{Q}(\zeta_N)$, and the group $\mathcal{O}(Y(N))^\times$ can be identified with the group of modular units for $\Gamma(N)$ whose Fourier expansion at infinity has coefficients in $\mathbf{Q}(\zeta_N)$. The group $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ acts from the left on $Y(N)$, inducing a right action on $\mathcal{O}(Y(N))^\times$. Under the above identification, a matrix $\gamma \in \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$ acts on $u \in \mathcal{O}(Y(N))^\times$ by $u|\gamma = u \circ \tilde{\gamma}$, where $\tilde{\gamma} \in \mathrm{SL}_2(\mathbf{Z})$ is any representative of γ. Moreover, for any $\lambda \in (\mathbf{Z}/N\mathbf{Z})^\times$, we have $u|\left(\begin{smallmatrix}1 & 0\\ 0 & \lambda\end{smallmatrix}\right) = \sigma_\lambda(u)$, where $\sigma_\lambda(u)$ is obtained by applying the automorphism $\zeta_N \mapsto \zeta_N^\lambda$ to the Fourier coefficients of u.

Proposition 3.6. *The unit $u(a,b,c,d)$ defines an element of $\mathcal{O}(Y(N))^\times$. Moreover, we have the following transformation formula:*

$$u(a,b,c,d)|\gamma = u(a\gamma, b\gamma, c\gamma, d\gamma) \qquad (\gamma \in \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})).$$

Proof. Lemmas 3.2 and 3.4 show that $u(a,b,c,d)$ is a modular unit for $\Gamma(N)$ whose Fourier coefficients belong to $\mathbf{Q}(\zeta_N)$. Hence $u(a,b,c,d)$ defines an element of $\mathcal{O}(Y(N))^\times$. The transformation formula holds for $\gamma \in \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$ because $\wp_a|_2\gamma = \wp_{a\gamma}$ for every $a \in (\mathbf{Z}/N\mathbf{Z})^2$. It remains to consider the case $\gamma = \left(\begin{smallmatrix}1 & 0\\ 0 & \lambda\end{smallmatrix}\right)$ with $\lambda \in (\mathbf{Z}/N\mathbf{Z})^\times$. Going back to the definition (2) of $E_{x,y}$, we have $\sigma_\lambda(E_{x,y}) = E_{x,y\tilde{\lambda}} = E_{(x,y)\left(\begin{smallmatrix}1 & 0\\ 0 & \tilde{\lambda}\end{smallmatrix}\right)}$, where $\tilde{\lambda} \in \mathbf{Z}$ is a representative of λ. The formula then follows from Lemma 3.4. □

We finally express the modular S-units in terms of Siegel units. Since the functions $g_{a,b}^{12N}$ are modular units for $\Gamma(N)$ with Fourier coefficients in $\mathbf{Q}(\zeta_N)$, we may consider the Siegel units $g_{a,b}$ as elements of $\mathcal{O}(Y(N))^\times \otimes \mathbf{Z}[\frac{1}{6N}]$.

Proposition 3.7. *Let a, b, c, d be elements of $(\mathbf{Z}/N\mathbf{Z})^2$ whose images in $(\mathbf{Z}/N\mathbf{Z})^2/\pm 1$ are pairwise distinct. Then*

$$u(a,b,c,d) = \frac{g_{c+a}g_{c-a}g_{d+b}g_{d-b}}{g_{c+b}g_{c-b}g_{d+a}g_{d-a}} \qquad \text{in } \mathcal{O}(Y(N))^\times \otimes \mathbf{Z}\Big[\frac{1}{6N}\Big].$$

Proof. This follows from Lemma 3.4, noting that the root of unity in (3) has order dividing $2N$. □

We have similar results for the modular units $u_1(a,b,c,d)$. The modular curve $Y_1(N)$ is the quotient of $Y(N)$ by the subgroup of matrices $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$. By Proposition 3.6, these matrices fix $u_1(a,b,c,d)$, so that $u_1(a,b,c,d)$ belongs to $\mathcal{O}(Y_1(N))^\times$. Note that the group $\mathcal{O}(Y_1(N))^\times$ can be identified with the group of modular units for $\Gamma_1(N)$ whose Fourier expansion at the cusp 0 has rational coefficients.

Proposition 3.8. *Let a,b,c,d be elements of $\mathbf{Z}/N\mathbf{Z}$ whose images in $(\mathbf{Z}/N\mathbf{Z})/\pm 1$ are pairwise distinct. We have the following identity between functions on $\mathcal{H}$:*

$$u_1(a,b,c,d) = \frac{g_{0,c+a}g_{0,c-a}g_{0,d+b}g_{0,d-b}}{g_{0,c+b}g_{0,c-b}g_{0,d+a}g_{0,d-a}}. \tag{5}$$

In particular, (5) holds in $\mathcal{O}(Y_1(N))^\times \otimes \mathbf{Z}[\frac{1}{6N}]$.

Proof. This follows from Lemma 3.4, since $E_{0,y} = g_{0,\overline{y}}$ for any $y \not\equiv 0 \bmod N$ by (3). □

Let us also mention that the transform of $u_1(a,b,c,d)$ under the Atkin-Lehner involution $W_N\colon \tau \mapsto -1/N\tau$ has the following simple expression:

$$u_1(a,b,c,d)|W_N = \frac{\tilde{g}_{c+a}\tilde{g}_{c-a}\tilde{g}_{d+b}\tilde{g}_{d-b}}{\tilde{g}_{c+b}\tilde{g}_{c-b}\tilde{g}_{d+a}\tilde{g}_{d-a}} \tag{6}$$

where for any $a \in \mathbf{Z}/N\mathbf{Z}$, $a \neq 0$, we put

$$\tilde{g}_a(\tau) = q^{NB_2(\tilde{a}/N)/2} \prod_{\substack{n\geq 1 \\ n\equiv a \bmod N}} (1-q^n) \prod_{\substack{n\geq 1 \\ n\equiv -a \bmod N}} (1-q^n),$$

and $\tilde{a}$ is the representative of a in $\mathbf{Z}$ such that $0 \leq \tilde{a} \leq N-1$. The identity (6) can be proved by determining the action of W_N on the Siegel units $g_{0,x}$, using [41, Theorem 1] with the transformation matrix $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Note that the unit $u_1(a,b,c,d)|W_N$ has rational Fourier coefficients by (6), which is is convenient for computations, but it is not defined over $\mathbf{Q}$ for this model of $Y_1(N)$.

Remark 3.9. Unlike the Siegel units g_a, the functions $u(a,b,c,d)$ and $u_1(a,b,c,d)$ are true modular units, not just roots of modular units. This is important for the K_4 construction in Section 6.

4. The 3-term relations in K_2 of modular curves

In order to give some context, we first recall the classical Manin relations in the homology of modular curves. For any two cusps $\alpha \neq \beta$ in $\mathbf{P}^1(\mathbf{Q})$, the modular symbol $\{\alpha,\beta\}$ is the hyperbolic geodesic from α to β in the upper half-plane $\mathcal{H}$. For any congruence subgroup Γ of $\mathrm{SL}_2(\mathbf{Z})$, the symbol $\{\alpha,\beta\}$ defines an element of the first homology group of $\overline{\Gamma\backslash\mathcal{H}}$ relative to the cusps. It is known that this group is generated by the Manin symbols $[g] = \{g0, g\infty\}$ with $g \in \Gamma\backslash\mathrm{SL}_2(\mathbf{Z})$. They satisfy the following relations:

$$[g] + [g\sigma] = 0, \qquad [g] + [g\tau] + [g\tau^2] = 0,$$

where the matrices $\sigma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\tau = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ have order 4 and 3, respectively. In the upper half-plane, the cycle $[g] + [g\tau] + [g\tau^2]$ is the boundary of an (ideal) hyperbolic triangle with vertices $\{g0, g\infty, g1\}$.

We now turn to K_2 and state the main result of this section. We work with the modular curve $Y(N)$ and the Siegel units g_a with $a \in (\mathbf{Z}/N\mathbf{Z})^2$, $a \neq (0,0)$. By convention, we put $g_{0,0} = 1$.

Theorem 4.1. *For any $a, b, c \in (\mathbf{Z}/N\mathbf{Z})^2$ such that $a+b+c=0$, we have*

$$\{g_a, g_b\} + \{g_b, g_c\} + \{g_c, g_a\} = 0 \qquad \text{in } K_2(Y(N)) \otimes \mathbf{Z}\Big[\frac{1}{6N}\Big]. \tag{7}$$

Theorem 4.1 was previously known with $\mathbf{Q}$-coefficients; see [7] when $(N,3)=1$, and [20] in general. The analogy with the Manin 3-term relations goes as follows. For any two row vectors x, y in $(\mathbf{Z}/N\mathbf{Z})^2$, consider the matrix $M = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ with rows x and y, and let $\rho(M) = \{g_x, g_y\}$. Then the relation (7) is equivalent to $\rho(M) + \rho(\tau M) + \rho(\tau^2 M) = 0$ with $M = \left(\begin{smallmatrix} a \\ -b \end{smallmatrix}\right)$ (here we use the relation $g_{-x} = g_x$).
Theorem 4.1 says that a particular element in K_2 is trivial. We will actually find explicit Steinberg relations explaining the identity (7), see Theorem 4.3 below. These Steinberg relations will be a key ingredient in the construction of elements of $K_4^{(3)}(Y(N))$ in Section 6.

Definition 4.2. For any distinct elements a, b, c, d in $(\mathbf{Z}/N\mathbf{Z})^2/\pm 1$, define

$$\tilde{\delta}(a,b,c,d) = u(a,b,c,d) \wedge u(a,c,b,d) \in \Lambda^2 \mathcal{O}(Y(N))^\times.$$

If a, b, c, d are not distinct, we put $\tilde{\delta}(a,b,c,d) = 0$. Moreover, we denote by $\delta(a,b,c,d)$ the image of $\tilde{\delta}(a,b,c,d)$ in $\Lambda^2 \mathcal{O}(Y(N))^\times \otimes \mathbf{Z}\big[\frac{1}{6N}\big]$.

Note that $u(a,c,b,d) = 1 - u(a,b,c,d)$, so that the image of $\tilde{\delta}(a,b,c,d)$ in $K_2(Y(N))$ is trivial. Theorem 4.1 is a consequence of the following theorem.

Theorem 4.3. *Let G be a subgroup of $(\mathbf{Z}/N\mathbf{Z})^2$, and let $a, b, c \in G$ with $a+b+c=0$. We have the following equality in $\Lambda^2 \mathcal{O}(Y(N))^\times \otimes \mathbf{Z}\big[\frac{1}{6N}\big]$:*

$$\begin{aligned} g_a \wedge g_b + g_b \wedge g_c + g_c \wedge g_a = {} & \frac{1}{|G|} \sum_{x \in G} \delta(0, x, a-x, b+x) \\ & - \frac{1}{4|G|^2} \sum_{x,y \in G} \delta(0, a, c+2x, y) + \delta(0, c, b+2x, y) + \delta(0, b, a+2x, y). \end{aligned} \tag{8}$$

In the case $|G|$ is odd, this simplifies to

$$g_a \wedge g_b + g_b \wedge g_c + g_c \wedge g_a = \frac{1}{|G|} \sum_{x \in G} \delta(0, x, a-x, b+x). \tag{9}$$

In analogy with the classical Manin relations, we refer to (8) and (9) as *triangulations* of the 3-term relation in K_2. Note that the group G is arbitrary. For example, we may take $G = \{0\} \times \mathbf{Z}/N\mathbf{Z}$ when working with the modular curve $Y_1(N)$.

Proof. Let a, b, c, d be distinct elements of $(\mathbf{Z}/N\mathbf{Z})^2/\pm 1$. Using Proposition 3.7, we get

$$\begin{aligned} \delta(a,b,c,d) = {} & (g_{c+a} g_{c-a} \cdot g_{d+b} g_{d-b}) \wedge (g_{b+a} g_{b-a} \cdot g_{d+c} g_{d-c}) \\ & + (g_{b+a} g_{b-a} \cdot g_{d+c} g_{d-c}) \wedge (g_{c+b} g_{c-b} \cdot g_{d+a} g_{d-a}) \\ & + (g_{c+b} g_{c-b} \cdot g_{d+a} g_{d-a}) \wedge (g_{c+a} g_{c-a} \cdot g_{d+b} g_{d-b}). \end{aligned} \tag{10}$$

This expression is antisymmetric in (a,b,c,d) and is zero when $b=\pm a$ (recall that $g_{0,0}=1$ and $g_{-x}=g_x$ for every x). Therefore (10) holds for any $a,b,c,d\in(\mathbf{Z}/N\mathbf{Z})^2$. Expanding the right-hand side of (10) with respect to the dots, we get

$$\delta(a,b,c,d)=\varphi(a,b,c)+\varphi(c,d,a)+\varphi(b,a,d)+\varphi(d,c,b), \tag{11}$$

where we have set

$$\varphi(x,y,z)=g_{z+x}g_{z-x}\wedge g_{y+x}g_{y-x}+g_{y+x}g_{y-x}\wedge g_{z+y}g_{z-y}+g_{z+y}g_{z-y}\wedge g_{z+x}g_{z-x}.$$

The functions δ and φ are antisymmetric with respect to their arguments.

Lemma 4.4. *For any $y,z\in G$, we have $\sum_{x\in G}\varphi(x,y,z)=0$.*

Proof. A simple computation shows that the sum simplifies to

$$\sum_{x\in G}\varphi(x,y,z)=\sum_{x\in G}g_{z+x}g_{z-x}\wedge g_{y+x}g_{y-x}=2\sum_{x\in G}g_{z+x}\wedge g_{y+x}+g_{z+x}\wedge g_{y-x}.$$

Let us first consider $S=\sum_{x\in G}g_{z+x}\wedge g_{y+x}$. Changing variables $x=-y-z-x'$, we get

$$S=\sum_{x'\in G}g_{-y-x'}\wedge g_{-z-x'}=\sum_{x'\in G}g_{y+x'}\wedge g_{z+x'}=-S,$$

so that $2S=0$. A similar argument using the change of variables $x=y-z-x'$ shows that the second part of the sum vanishes. □

Lemma 4.5. *For any $a,b,c\in G$, we have*

$$\varphi(a,b,c)=\frac{1}{|G|}\sum_{d\in G}\delta(a,b,c,d).$$

Proof. It follows from summing (11) over $d\in G$ and using Lemma 4.4. □

Let $\psi(a,b)=g_a\wedge g_b+g_b\wedge g_c+g_c\wedge g_a$, where c is chosen so that $a+b+c=0$. Our next task is to show that $\psi(a,b)$ is a linear combination of values of φ. The definition of φ gives us

$$\varphi(x,y,z)=\psi(z+x,-y-x)+\psi(z+x,y-x)+\psi(z-x,-y+x)+\psi(z-x,y+x). \tag{12}$$

Changing variables and putting $a=z+x$ and $b=-y-x$, this becomes

$$\begin{aligned}\varphi(x,-b-x,a-x)&=\psi(a,b)+\psi(a,-b-2x)+\psi(a-2x,b+2x)+\psi(a-2x,-b)\\&=\psi(a,b)+\psi(-a+b+2x,a)+\psi(b+2x,c)+\psi(a-2x,-b).\end{aligned}$$

Here we used $\psi(u,v)=\psi(v,-u-v)=\psi(-u-v,u)$. Summing over $x\in G$, we get

$$\sum_{x\in G}\varphi(x,-b-x,a-x)=|G|\cdot\psi(a,b)+R_{-a+b}(a)+R_b(c)+R_a(-b),$$

where $R_u(v)=\sum_{x\in G}\psi(u+2x,v)$. One checks the relations $R_{u+2w}(v)=R_u(v)$ for any $w\in G$, and $R_u(-v)=R_{-u}(v)=R_u(v)$. Therefore

$$\sum_{x\in G}\varphi(x,-b-x,a-x)=|G|\cdot\psi(a,b)+R_c(a)+R_b(c)+R_a(b). \tag{13}$$

Lemma 4.6. *For any $u,v\in G$, we have $R_u(v)=\frac{1}{4}\sum_{x\in G}\varphi(0,v,u+2x)$.*

Proof. Taking $x = 0$ in (12), we obtain

$$\varphi(0, y, z) = 2(\psi(z, y) + \psi(z, -y)) = 2(\psi(z, y) + \psi(-z, y)).$$

Specialising to $y = v$, $z = u + 2x$, and summing over $x \in G$ gives

$$\sum_{x \in G} \varphi(0, v, u + 2x) = 2 \sum_{x \in G} \psi(u + 2x, v) + \psi(-u - 2x, v) = 4R_u(v). \qquad \square$$

Using Lemmas 4.5 and 4.6, the equation (13) becomes

$$\begin{aligned} \psi(a, b) = {} & \frac{1}{|G|^2} \sum_{x,y \in G} \delta(x, -b - x, a - x, y) \\ & - \frac{1}{4|G|^2} \sum_{x,y \in G} \delta(0, a, c + 2x, y) + \delta(0, c, b + 2x, y) + \delta(0, b, a + 2x, y). \end{aligned} \tag{14}$$

We now wish to simplify the first sum. For this, we will use the fact that δ satisfies the 5-term relations. More precisely, we have:

Lemma 4.7. *Let $(a_j)_{j \in \mathbf{Z}/5\mathbf{Z}}$ be a family of elements of $(\mathbf{Z}/N\mathbf{Z})^2/\pm 1$. Then*

$$\sum_{j \in \mathbf{Z}/5\mathbf{Z}} \tilde{\delta}(a_j, a_{j+1}, a_{j+2}, a_{j+3}) = 0 \quad \text{in } \Lambda^2 \mathcal{O}(Y(N))^\times \otimes \mathbf{Z}\Big[\frac{1}{2}\Big]. \tag{15}$$

Proof. Recall that $\tilde{\delta}(a, b, c, d) = u(a, b, c, d) \wedge (1 - u(a, b, c, d))$ where $u(a, b, c, d)$ is the cross-ratio $[\wp_a, \wp_b, \wp_c, \wp_d]$. The classical 5-term relation [17, Section 1.8] implies that the left-hand side of (15) is 2-torsion in $\Lambda^2 F^\times$, where F is the field generated over $\mathbf{Q}$ by the modular forms $\wp_a$ with $a \in (\mathbf{Z}/N\mathbf{Z})^2$, $a \neq 0$. A tedious computation shows that it is actually 2-torsion in $\Lambda^2 W$, where W is the multiplicative group generated by the modular forms $\wp_a - \wp_b$ with $a, b \in (\mathbf{Z}/N\mathbf{Z})^2$, $a, b \neq 0$ and $b \neq \pm a$. These modular forms have weight 2, so the weight provides a homomorphism $w \colon W \to 2\mathbf{Z}$. The group $W_0 = \ker(w)$ is a direct factor of W, so that the left-hand side of (15) is zero in $\Lambda^2 W_0 \otimes \mathbf{Z}[\frac{1}{2}]$. But W_0 is generated by the Weierstraß units $(\wp_a - \wp_b)/(\wp_c - \wp_d)$, which belong to $\mathcal{O}(Y(N))^\times$ because their q-expansions have coefficients in $\mathbf{Q}(\zeta_N)$ by (4). $\square$

Lemma 4.7 gives in particular:

$$\begin{aligned} & \delta(x, -b - x, a - x, y) + \delta(-b - x, a - x, y, 0) + \delta(a - x, y, 0, x) \\ & \qquad + \delta(y, 0, x, -b - x) + \delta(0, x, -b - x, a - x) = 0. \end{aligned} \tag{16}$$

Lemma 4.8. *For any $\alpha, \beta, z, t \in G$, we have $\sum_{x \in G} \delta(\alpha + x, \beta + x, z, t) = 0$.*

Proof. Denote this sum by S. The change of variables $x = -\alpha - \beta - x'$ gives

$$S = \sum_{x' \in G} \delta(-\beta - x', -\alpha - x', z, t) = \sum_{x' \in G} \delta(\beta + x', \alpha + x', z, t) = -S. \qquad \square$$

Note that $\delta(\pm a, \pm b, \pm c, \pm d) = \delta(a, b, c, d)$. From (16) and Lemma 4.8, we obtain

$$\sum_{x \in G} \delta(x, -b - x, a - x, y) = -\sum_{x \in G} \delta(0, x, b + x, a - x) = \sum_{x \in G} \delta(0, x, a - x, b + x).$$

Together with (14), this proves (8). Finally, let us suppose that $|G|$ is odd. For any $\alpha \in G$, the map $x \mapsto \alpha + 2x$ is a bijection of G. Therefore, for any $z, t \in G$, we have

$$\sum_{x,y \in G} \delta(z,t,\alpha+2x,y) = \sum_{x,y\in G} \delta(z,t,x,y) = 0$$

by antisymmetry with respect to (x,y). Therefore the second line of (8) vanishes. This finishes the proof of Theorem 4.3. □

Remarks 4.9. (1) Thanks to the 5-term relation, every $\delta(a,b,c,d)$ is a linear combination of elements $\delta(0,x,y,z)$, and we have $u(0,x,y,z) = (\wp_z - \wp_x)/(\wp_y - \wp_x)$. These modular units appear in certain diophantine problems [27, Chapter 8].

(2) The proof of Theorem 4.1 can be made to work in the group $\Lambda^2\mathcal{O}(Y(N))^\times$ (without inverting $6N$), provided both sides of (8) are multiplied by $(24|G|N)^2$. In order for the statement to make sense, the terms $(12N)^2 g_x \wedge g_y$ must be interpreted as $g_x^{12N} \wedge g_y^{12N}$, since only g_x^{12N} defines an element of $\mathcal{O}(Y(N))^\times$ in general.

5. The polylogarithmic complex

In this section, we mainly present the various tools and results in the literature needed to construct elements in K_4 of curves, and compute their regulators.

5.1. **Goncharov's complexes.** We begin by recalling Goncharov's theory of polylogarithmic complexes [17]. For an abelian group A, we set $A_{\mathbf{Q}} = A \otimes_{\mathbf{Z}} \mathbf{Q}$. Let F be a field, and let $n \geq 1$ be an integer. Goncharov constructs a *weight n polylogarithmic motivic complex* $\Gamma(F,n)$ of the following shape:

$$B_n(F) \to B_{n-1}(F) \otimes F_{\mathbf{Q}}^\times \to B_{n-2}(F) \otimes \Lambda^2 F_{\mathbf{Q}}^\times \to \cdots \to B_2(F) \otimes \Lambda^{n-2} F_{\mathbf{Q}}^\times \to \Lambda^n F_{\mathbf{Q}}^\times,$$

where $B_n(F)$ is defined as the quotient of the $\mathbf{Q}$-vector space $\mathbf{Q}[\mathbf{P}^1(F)]$ with basis $\mathbf{P}^1(F)$, by a certain subspace related to the functional equations of the n-logarithm.
The complex $\Gamma(F,n)$ sits in cohomological degrees 1 to n and is expected to compute the weight n motivic cohomology of $\operatorname{Spec} F$. More precisely, combining [17, Conjecture A and Conjecture 1.17, p. 222–223], we have:

Conjecture 5.1 (Goncharov)**.** *The group $H^i(\Gamma(F,n))$ is isomorphic to $H^i_{\mathcal{M}}(F,\mathbf{Q}(n))$ for every $1 \leq i \leq n$.*

In this article, we only use the polylogarithmic complexes of weight 2 and 3. Actually, we will take the version of these complexes where $B_n(F)$ is defined using explicit relations [17, Section 1.8], rather than defined inductively [17, Section 1.9]. In particular, the group $B_2(F)$ will be the quotient of $\mathbf{Q}[F\backslash\{0,1\}]$ by the subspace generated by the so-called 5-term relations [17, p. 218]. The group $B_3(F)$ has a similar explicit definition. For any $x \in F\backslash\{0,1\}$, we denote by $\{x\}$ the associated basis element of $\mathbf{Q}[F\backslash\{0,1\}]$, and by $\{x\}_n$ its image in $B_n(F)$.
We will still denote by $\Gamma(F,2)$ and $\Gamma(F,3)$ the resulting complexes (they are denoted by $B_F(2)$ and $B_F(3)$ in [17, Section 1.8]).

The complex $\Gamma(F,2)$ is none other than the Bloch-Suslin complex, in degrees 1 and 2:

$$\begin{aligned}\Gamma(F,2): \qquad B_2(F) &\longrightarrow \Lambda^2 F_{\mathbf{Q}}^{\times}\\ \{x\}_2 &\longmapsto (1-x)\wedge x.\end{aligned}$$

By Matsumoto's theorem, we have $H^2(\Gamma(F,2)) \cong K_2(F)_{\mathbf{Q}}$.

Definition 5.2. The *Bloch group* of F (tensored with $\mathbf{Q}$) is the group $H^1(\Gamma(F,2))$.

By Suslin's theorem, for $|F| \geq 4$, the Bloch group of F is isomorphic to the quotient of $K_3(F)_{\mathbf{Q}}$ by its Milnor part $K_3^M(F)_{\mathbf{Q}}$, see [17, Theorem 1.13, p. 219] and [40, VI, Theorem 5.2]. Thanks to the motivic-to-K-theory spectral sequence [40, VI, 4.3.1], this implies that $H^1(\Gamma(F,2))$ is isomorphic to $H^1_{\mathcal{M}}(F,\mathbf{Q}(2))$.
In weight 3, the polylogarithmic complex, placed in degrees 1 to 3, is as follows:

$$\begin{aligned}\Gamma(F,3): \qquad B_3(F) &\longrightarrow B_2(F)\otimes F_{\mathbf{Q}}^{\times} \longrightarrow \Lambda^3 F_{\mathbf{Q}}^{\times}\\ \{x\}_3 &\longmapsto \{x\}_2\otimes x\\ \{x\}_2\otimes y &\longmapsto (1-x)\wedge x\wedge y\end{aligned}$$

In degree 2, Goncharov's conjecture says that $H^2(\Gamma(F,3))$ is isomorphic to $H^2_{\mathcal{M}}(F,\mathbf{Q}(3))$, in other words to $K_4^{(3)}(F)$. In support of this, Goncharov constructs a canonical map

$$K_4(F)_{\mathbf{Q}} \to H^2(\Gamma(F,3)),$$

see [17, Section 6]. This map should induce an isomorphism $K_4^{(3)}(F) \xrightarrow{\cong} H^2(\Gamma(F,3))$ [17, Conjectures 1.15 and 1.17].

5.2. **De Jeu's map for fields.** We now recall the work of De Jeu [23, 24, 25] in which a map in the other direction is constructed. This map is essential for us, in order to actually produce elements in K_4.

Theorem 5.3 (De Jeu)**.** *For any field F of characteristic zero, there is a map*

$$H^2(\Gamma(F,3)) \to K_4^{(3)}(F), \tag{17}$$

which is canonical up to sign. Moreover, it is possible to choose the sign consistently for all fields so that the map (17) becomes functorial in F.

Let us explain, without going into the technical details, the way De Jeu's map is constructed. De Jeu builds in [23] a complex $\mathcal{M}^{\bullet}_{(3)}(F)$ in degrees 1 to 3, given as follows:

$$\begin{aligned}\mathcal{M}^{\bullet}_{(3)}(F): \qquad M_{(3)}(F) &\longrightarrow M_{(2)}(F)\otimes F_{\mathbf{Q}}^{\times} \longrightarrow F_{\mathbf{Q}}^{\times}\otimes \Lambda^2 F_{\mathbf{Q}}^{\times}\\ [x]_3 &\longmapsto [x]_2\otimes x\\ [x]_2\otimes y &\longmapsto (1-x)\otimes(x\wedge y).\end{aligned}$$

(We conform with the notation of [25, Section 2].) Here $M_{(n)}(F)$ is a certain $\mathbf{Q}$-vector space with a generating family $([x]_n)$ indexed by $x \in F\backslash\{0,1\}$. He also constructs [25, Section 2] a quotient complex $\widetilde{\mathcal{M}}^{\bullet}_{(3)}(F)$:

$$\begin{aligned} \widetilde{\mathcal{M}}^{\bullet}_{(3)}(F): \qquad \widetilde{M}_{(3)}(F) &\longrightarrow \widetilde{M}_{(2)}(F)\otimes F^{\times}_{\mathbf{Q}} \longrightarrow \Lambda^3 F^{\times}_{\mathbf{Q}} \\ [x]_2\otimes y &\longmapsto (1-x)\wedge x\wedge y, \end{aligned}$$

where $\widetilde{M}_{(n)}(F)$ is the quotient of $M_{(n)}(F)$ by the relations $[x]_n + (-1)^n[1/x]_n = 0$ for all $x \in F\backslash\{0,1\}$. The class of $[x]_n$ in $\widetilde{M}_{(n)}(F)$ is still denoted by $[x]_n$. The canonical map $H^n(\mathcal{M}^{\bullet}_{(3)}(F)) \to H^n(\widetilde{\mathcal{M}}^{\bullet}_{(3)}(F))$ is an isomorphism for $n \in \{2,3\}$ (see [24, p. 529]). De Jeu constructs a map

$$H^2(\widetilde{\mathcal{M}}^{\bullet}_{(3)}(F)) \xleftarrow{\cong} H^2(\mathcal{M}^{\bullet}_{(3)}(F)) \to K_4^{(3)}(F). \tag{18}$$

One hopes that the second map in (18) is an isomorphism, similarly as in Goncharov's Conjecture 5.1.

The relation with the Goncharov complex $\Gamma(F,3)$ is the following. By [25, Lemma 5.2], there is a commutative diagram

$$\begin{array}{lccc} \Gamma(F,3): & B_3(F) & \longrightarrow & B_2(F)\otimes F^{\times}_{\mathbf{Q}} \longrightarrow \Lambda^3 F^{\times}_{\mathbf{Q}} \\ & & & \downarrow \qquad\quad \| \\ \widetilde{\mathcal{M}}^{\bullet}_{(3)}(F): & \widetilde{M}_{(3)}(F) & \longrightarrow & \widetilde{M}_{(2)}(F)\otimes F^{\times}_{\mathbf{Q}} \longrightarrow \Lambda^3 F^{\times}_{\mathbf{Q}} \end{array}$$

where the vertical map in degree 2 sends $\{x\}_2\otimes y$ to $[x]_2\otimes y$. In degree 1, the map $\{x\}\mapsto[x]_3$ should factor through $B_3(F)$, but this is not known. Since the differentials of degree 1 in $\Gamma(F,3)$ and $\widetilde{\mathcal{M}}^{\bullet}_{(3)}(F)$ have the same expression, the above diagram suffices to induce a map $H^2(\Gamma(F,3)) \to H^2(\widetilde{\mathcal{M}}^{\bullet}_{(3)}(F))$. Composing with (18), this gives the map of Theorem 5.3.

5.3. **De Jeu's map for curves.** Let X be a smooth proper geometrically connected curve defined over a number field k, and let $F = k(X)$ be its function field. Let S be a finite set of closed points of X, and let $Y = X\backslash S$. To construct elements of $K_4^{(3)}(Y)$, we use the localisation exact sequence

$$0 \to K_4^{(3)}(Y) \to K_4^{(3)}(F) \xrightarrow{\text{Res}} \bigoplus_{x\in Y} K_3^{(2)}(k(x)), \tag{19}$$

which follows from Quillen's localisation theorem (see [40, V, 6.12]). The injectivity on the left comes from the fact that K_4 of a number field is torsion, a consequence of Borel's theorem. The map $K_4^{(3)}(F) \to K_3^{(2)}(k(x))$ is called the residue map at x. There are also residue maps at the level of Goncharov's complexes [17, Sections 1.14–15] and De Jeu's complexes [24, Proposition 5.1]. More precisely, for any closed point $x \in X$, there is a morphism of complexes

$$\text{Res}_x\colon\ \Gamma(F,3) \to \Gamma(k(x),2)[-1]$$

which, in degree 2, sends $\{f\}_2\otimes g$ to $\text{ord}_x(g)\{f(x)\}_2$, with the convention $\{0\}_2 = \{1\}_2 = \{\infty\}_2 = 0$ in $B_2(k(x))$. Goncharov then defines the complex $\Gamma(Y,3)$ as the simple of the

morphism of complexes

$$\bigoplus_{x\in Y} \mathrm{Res}_x\colon\ \Gamma(F,3) \to \bigoplus_{x\in Y} \Gamma(k(x),2)[-1].$$

We thus have an exact sequence

$$0 \to H^2(\Gamma(Y,3)) \to H^2(\Gamma(F,3)) \to \bigoplus_{x\in Y} H^1(\Gamma(k(x),2)),$$

which should be isomorphic to (19). De Jeu has proved that the map $H^2(\Gamma(F,3)) \to K_4^{(3)}(F)$ commutes with the residue maps up to a small indeterminacy coming from $K_3^{(2)}(k)$, see [24, Corollary 5.4] for the precise statement. In the case k is totally real, this indeterminacy vanishes by Borel's theorem, giving rise to a map

$$H^2(\Gamma(Y,3)) \to K_4^{(3)}(Y).$$

For a general number field k, the following result will suffice for our needs.

Theorem 5.4 (De Jeu)**.** *Let Y be a smooth (not necessarily proper) geometrically connected curve over a number field k, with function field $F = k(Y)$. Let $\xi = \sum_i n_i\{f_i\}_2 \otimes g_i$ be a degree 2 cocycle in the complex $\Gamma(F,3)$, with $f_i, g_i \in F^\times$ and $n_i \in \mathbf{Q}$. Assume that all the functions $f_i, 1-f_i$ and g_i are invertible on Y. Then the image of ξ under De Jeu's map (17) belongs to $K_4^{(3)}(Y)$.*

This theorem essentially follows from [24, Theorem 5.2]. Let us indicate the details.

Proof. Let $\xi' = \sum_i n_i[f_i]_2 \otimes g_i$ considered in $\mathcal{M}^2_{(3)}(F)$. Its boundary is $\delta_2(\xi') = \sum_i n_i \cdot (1-f_i) \otimes (f_i \wedge g_i)$ in $F^\times_{\mathbf{Q}} \otimes \Lambda^2 F^\times_{\mathbf{Q}}$. Let W be the subspace of $F^\times_{\mathbf{Q}}$ generated by the functions $f_i, 1-f_i, g_i$. Since ξ is a cocycle, $\sum_i n_i \cdot (1-f_i) \otimes f_i \otimes g_i$ maps to 0 in $\Lambda^3 W$. This implies that $\delta_2(\xi')$ is a linear combination of symbols $u \otimes (u \wedge v)$ with $u, v \in W$. But such a symbol is the boundary of $([u]_2 + [u^{-1}]_2) \otimes v$, which maps to 0 in $\widetilde{\mathcal{M}}^2_{(3)}(F)$ by definition of the quotient complex. We may thus modify ξ' by such elements $([u]_2 + [u^{-1}]_2) \otimes v$ to get a cocycle ξ'' in $\mathcal{M}^2_{(3)}(F)$. Since $f_i, 1-f_i, g_i$ are invertible on Y, we conclude by applying [24, Theorem 5.2] to ξ'' with $U = Y$. □

5.4. **Regulator on the polylogarithmic complex.** Goncharov has defined completely explicit regulator maps for complex algebraic varieties at the level of his polylogarithmic complexes [19]. We will use these regulator formulas in the case of curves.
Let us keep the same setting as in Section 5.3. Goncharov [19, Theorem 2.2] has defined an explicit regulator

$$r_3(2)\colon H^2(\Gamma(Y,3)) \to H^1(Y(\mathbf{C}), \mathbf{R}(2))^+,$$

where $Y(\mathbf{C})$ denotes the complex points of $Y \times_{\mathbf{Q}} \mathbf{C}$, and the superscript $+$ denotes the invariants with respect to complex conjugation acting on the second factor of $Y \times_{\mathbf{Q}} \mathbf{C}$. The map $r_3(2)$ is defined at the level of cocycles by means of explicit differential forms, see (29) for the precise formula. We will use these forms in Section 7 to compute numerically the regulator in the case of modular curves.

It is expected that De Jeu's map (17) is compatible with taking regulators. More precisely, there should be a commutative diagram

$$\begin{array}{ccccc} H^2(\Gamma(Y,3)) & \longrightarrow & H^2(\widetilde{\mathcal{M}}^\bullet_{(3)}(Y)) & \longrightarrow & K_4^{(3)}(Y) \\ & \searrow^{r_3(2)} & \downarrow & \swarrow_{\frac{1}{2}r_B} & \\ & & H^1(Y(\mathbf{C}),\mathbf{R}(2))^+ & & \end{array} \tag{20}$$

which commutes up to sign. Here r_B is Beilinson's regulator, and the complex $\widetilde{\mathcal{M}}^\bullet_{(3)}(Y)$ is defined as in [25, Section 4, p. 164], by considering the residue maps at all the closed points of Y. The existence of the left horizontal map follows from [25, Lemma 5.2].
De Jeu has proved a slightly weaker version of (20), where the regulator maps are composed with the map[1]

$$H^1(Y(\mathbf{C}),\mathbf{R}(2))^+ \to \mathrm{Hom}_{\mathbf{C}}(\Omega^1(X(\mathbf{C})),\mathbf{C}), \qquad \eta \mapsto \left(\omega \mapsto \int_{X(\mathbf{C})} \eta\wedge\omega\right) \tag{21}$$

and, moreover, $K_4^{(3)}(Y)$ is replaced by $K_4^{(3)}(Y)+(K_3^{(2)}(k)\cup F^\times)$, the last group being viewed inside $K_4^{(3)}(F)$. In this weaker setting, the right horizontal map is obtained from [25, Theorem 2], and the commutativity of the diagram follows from [25, Theorem 3.5, Remark 3.7], given that [25, (3.1)] agrees with Goncharov's formula [18, Theorem 3.3] up to a rational factor.[2]
Since the map (21) is injective when $Y = X$ is proper, this proves the existence and commutativity of (20) when Y is proper and k is totally real [25, Theorem 5.4].

5.5. **Finding cocycles in Goncharov's complex.** We now describe a method to construct elements in $K_4^{(3)}$ of curves, using De Jeu's Theorem 5.4. One obstacle is that the $\mathbf{Q}$-vector spaces appearing in the Goncharov complexes are infinite-dimensional, making explicit computations a priori difficult. The main idea, explained below, is to reduce the problem to finite-dimensional linear algebra.
In view of Theorem 5.4, we search for degree 2 cocycles in the Goncharov complex $\Gamma(F,3)$ in the following way. We consider linear combinations of symbols $\{f\}_2\otimes g$ with the additional condition that f, $1-f$ and g are invertible on Y. This means that f should be a solution to the S-unit equation for X. Note that the S-unit equation has only finitely many solutions by Corollary 2.2. Moreover g lives in the $\mathbf{Q}$-vector space $\mathcal{O}(Y)^\times_{\mathbf{Q}}$, which is finite-dimensional after quotienting by $k^\times_{\mathbf{Q}}$. Finally, the cocycle condition takes place in $\Lambda^3\mathcal{O}(Y)^\times_{\mathbf{Q}}$, which is also finite-dimensional after modding out by the constants. This essentially reduces our search to a linear algebra problem.
This strategy can be in principle implemented on a computer for any given curve, although the S-unit equation may not have any solution if S is too small. In this case, one may try to enlarge S, but it is not clear a priori which points should be added to S in order to find non-trivial cocycles.
On the other hand, the method turns out to work well for the modular curves $Y_1(N)$ and $Y(N)$, taking S to be the set of cusps. One reason is that the modular units $u(a,b,c,d)$

[1] The map (21) is well-defined since $H^1(Y(\mathbf{C}),\mathbf{C})$ can be computed using forms with logarithmic singularities along infinity [38, Proposition 8.18].

[2] The factor c_3 in [18, Theorem 3.3] is $\frac{4}{3}$ and the rational factor in [25, (3.1)] is $\pm\frac{8}{3}$, hence the factor $\frac{1}{2}$ in the diagram (20).

defined in Section 3 provide plenty of solutions to the S-unit equation. In this way, we were able to construct cocycles on these curves for small values of N. We could also show that the associated cohomology classes were non-trivial, by computing numerically their images under the regulator map. However, this construction only worked case by case and, from a computational point of view, became impractical for larger N. Moreover, no pattern emerged in these cocycles. A general and uniform construction, working for every N, will be given in the next section.

6. Constructing the elements in K_4

In this section, we construct elements of $K_4^{(3)}(Y(N))$ and $K_4^{(3)}(Y_1(N))$, using the results from Sections 4 and 5. We will switch to the cohomological notation and work with motivic cohomology, writing $H^2_{\mathcal{M}}(Y, \mathbf{Q}(3))$ instead of $K_4^{(3)}(Y)$.

6.1. **Definition of the elements.** Let G be a subgroup of $(\mathbf{Z}/N\mathbf{Z})^2$. The group $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ acts by right multiplication on the set of row vectors $(\mathbf{Z}/N\mathbf{Z})^2$. Let

$$\Gamma_G = \{\gamma \in \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}) : \forall g \in G,\ g\gamma = g\}.$$

We may then consider the modular curve $Y(\Gamma_G) := \Gamma_G \backslash Y(N)$. We view $Y(\Gamma_G)$ as a curve defined over its field of constants k_G, so that $Y(\Gamma_G)$ is geometrically connected. Let F_G be the function field of $Y(\Gamma_G)$.
For example, the group $G = \{0\} \times (\mathbf{Z}/N\mathbf{Z})$ gives rise to the modular curve $Y_1(N)$, since $\Gamma_G = \{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)\}$ in this case. Let $X(\Gamma_G)$ be the compactification of $Y(\Gamma_G)$, and let S be the set of cusps, seen as a closed subscheme of $X(\Gamma_G)$.

Construction 6.1. Let $a, b, c \in G$ with $a + b + c = 0$. Write the triangulation (8) from Theorem 4.3 as follows:

$$g_a \wedge g_b + g_b \wedge g_c + g_c \wedge g_a = \sum_i m_i \cdot u_i \wedge (1 - u_i) \qquad \text{in } \Lambda^2 F_G^\times \otimes \mathbf{Q},$$

with coefficients $m_i \in \mathbf{Q}$ and modular units $u_i \in \mathcal{O}(Y(\Gamma_G))^\times$. Then the element

$$\widetilde{\xi}_G(a, b) := \sum_i m_i \{u_i\}_2 \otimes \left(\frac{g_b}{g_a}\right)$$

is a degree 2 cocycle in the Goncharov complex $\Gamma(F_G, 3)$.

Indeed, the boundary of $\widetilde{\xi}_G(a, b)$ is given by

$$\begin{aligned} \delta\widetilde{\xi}_G(a, b) &= \sum_i m_i \cdot (1 - u_i) \wedge u_i \wedge \left(\frac{g_b}{g_a}\right) = -\big(g_a \wedge g_b + g_b \wedge g_c + g_c \wedge g_a\big) \wedge \left(\frac{g_b}{g_a}\right) \\ &= g_b \wedge g_c \wedge g_a - g_c \wedge g_a \wedge g_b = 0. \end{aligned}$$

Definition 6.2. For any $a, b \in G$, we denote by $\xi_G(a, b) \in H^2_{\mathcal{M}}(Y(\Gamma_G), \mathbf{Q}(3))$ the image of $\widetilde{\xi}_G(a, b)$ under De Jeu's map (17).

Note that $\xi_G(a, b)$ indeed belongs to $H^2_{\mathcal{M}}(Y(\Gamma_G), \mathbf{Q}(3))$ thanks to Theorem 5.4.
In the special cases $G = (\mathbf{Z}/N\mathbf{Z})^2$ and $G = \{0\} \times (\mathbf{Z}/N\mathbf{Z})$, we write

$$\begin{aligned} \widetilde{\xi}(a, b) &:= \widetilde{\xi}_{(\mathbf{Z}/N\mathbf{Z})^2}(a, b) && \big(a, b \in (\mathbf{Z}/N\mathbf{Z})^2\big), \\ \widetilde{\xi}_1(a, b) &:= \widetilde{\xi}_{\{0\}\times(\mathbf{Z}/N\mathbf{Z})}((0, a), (0, b)) && (a, b \in \mathbf{Z}/N\mathbf{Z}). \end{aligned}$$

The associated motivic cohomology classes are denoted by $\xi(a,b) \in H^2_{\mathcal{M}}(Y(N), \mathbf{Q}(3))$ and $\xi_1(a,b) \in H^2_{\mathcal{M}}(Y_1(N), \mathbf{Q}(3))$, respectively.
We will see in Sections 8 and 9 that the classes $\xi_1(a,b)$ can be non-trivial, by computing numerically their images under Beilinson's regulator.
If $|G|$ is odd, then the triangulation (9) leads to the same cocycle $\widetilde{\xi}_G(a,b)$, and thus to the same class $\xi_G(a,b)$. Indeed, for any $f \in F_G^\times \backslash \{1\}$, we have $\{1/f\}_2 = -\{f\}_2$ in $B_2(F_G)$ by [40, VI, Lemma 5.4(b)]. Then for fixed $\alpha, \beta \in G$, we have in $B_2(F_G)$:

$$\sum_{x,y \in G} \{u(\alpha,\beta,x,y)\}_2 = \sum_{x,y \in G} \{u(\alpha,\beta,y,x)^{-1}\}_2 = -\sum_{x,y \in G} \{u(\alpha,\beta,y,x)\}_2$$

so that $\sum_{x,y \in G} \{u(\alpha,\beta,x,y)\}_2 = 0$ (in these sums we only keep the terms where α, β, x, y are distinct in $G/\pm 1$). For the curve $Y_1(N)$ with N odd, the cocycle simplifies to

$$\widetilde{\xi}_1(a,b) = \frac{1}{N} \sum_{x \in \mathbf{Z}/N\mathbf{Z}} \{u(0,x,a-x,b+x)\}_2 \otimes \left(\frac{g_b}{g_a}\right),$$

with the abuse of notation $x = (0,x)$ for $x \in \mathbf{Z}/N\mathbf{Z}$, and the convention $\{u(x,y,z,t)\}_2 = 0$ if x, y, z, t are not distinct in $(\mathbf{Z}/N\mathbf{Z})/\pm 1$.
Note that we could have chosen another triangulation in the above construction. As we shall see later in Section 6.3, the class $\xi_G(a,b)$ depends in general on the triangulation. In the rest of this article, in particular for the regulator computations in Sections 8 and 9, we will use the triangulation provided by Theorem 4.3.
It is possible to produce K_4 classes for modular curves associated to arbitrary congruence subgroups, but the cocycles are not explicit anymore. More precisely, if Γ is a subgroup of $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ and $Y(\Gamma) := \Gamma\backslash Y(N)$ is the associated modular curve, we may consider the images of the elements $\xi(a,b)$ under the trace map $H^2_{\mathcal{M}}(Y(N), \mathbf{Q}(3)) \to H^2_{\mathcal{M}}(Y(\Gamma), \mathbf{Q}(3))$. However, consider the modular curve $X_0(p)$ with p prime. This curve has only two cusps, hence, essentially, only one modular unit. Thus it is not possible to write down the cocycles using only modular units.

6.2. **Extension to the compactification.** The classes $\xi_G(a,b)$, $a, b \in G$, live on the open modular curve $Y(\Gamma_G)$. We now explain how to construct classes on the compactification $X(\Gamma_G)$. Recall that S denotes the finite subscheme of cusps of $X(\Gamma_G)$. There is a localisation exact sequence [11, Theorem 1.3(5)]
(22)

$$0 \to H^2_{\mathcal{M}}(X(\Gamma_G), \mathbf{Q}(3)) \to H^2_{\mathcal{M}}(Y(\Gamma_G), \mathbf{Q}(3)) \xrightarrow{\mathrm{Res}} H^1_{\mathcal{M}}(S, \mathbf{Q}(2)) \to H^3_{\mathcal{M}}(X(\Gamma_G), \mathbf{Q}(3)),$$

where the first map in (22) is injective because the group $H^0_{\mathcal{M}}(S, \mathbf{Q}(2))$ is zero (by Borel's theorem). We say that $\xi_G(a,b)$ *extends to* $X(\Gamma_G)$ if it is the image of a class in $H^2_{\mathcal{M}}(X(\Gamma_G), \mathbf{Q}(3))$, in other words, if its residues at the cusps are trivial.
Here is a sufficient condition for $\xi_1(a,b)$ to extend to $X_1(N)$.

Lemma 6.3. *If $N = p$ or $N = 2p$ with p prime, then the class $\xi_1(a,b)$ extends to $X_1(N)$ for every $a, b \in \mathbf{Z}/N\mathbf{Z}$.*

Proof. For N arbitrary, the Galois action on the cusps of $X_1(N)$ is described in [34, Section 1.3]. A set of representatives of the Galois orbits is given by the cusps $1/v$ with $0 \leq v \leq N/2$. Among them, the real cusps are given by $v = 0$, $v = N/2$ (for even N), and the integers

$0 < v < N/2$ such that $\gcd(v, N) \in \{1, 2\}$. It follows that in the cases $N = p$ and $N = 2p$ with p prime, all the cusps are totally real. But for a totally real number field k, the group $H^1_{\mathcal{M}}(k, \mathbf{Q}(2))$ is zero by Borel's theorem, hence the residues are automatically trivial. □

In general, the classes $\xi_1(a, b)$ do not extend to $X_1(N)$. This happens for example in the case $N = 15$ (see the computations of Section 8).
Nevertheless, we can modify the classes $\xi_G(a, b)$ in order for them to extend, as follows.

Proposition 6.4. *The restriction map $H^2_{\mathcal{M}}(X(\Gamma_G), \mathbf{Q}(3)) \to H^2_{\mathcal{M}}(Y(\Gamma_G), \mathbf{Q}(3))$ admits a natural retraction.*

This proposition is an analogue of Bloch's trick to construct an element of $H^2_{\mathcal{M}}(X, \mathbf{Q}(2))$ from an element of $H^2_{\mathcal{M}}(Y, \mathbf{Q}(2))$, where X is a smooth proper curve and Y is the complement of a finite set of points of finite order in the Jacobian of X (see [3, (8.2)] for the case of elliptic curves). Proposition 6.4 relies crucially on the Manin–Drinfel'd theorem, asserting that the cusps of $X(\Gamma_G)$ are torsion in the Jacobian of $X(\Gamma_G)$.

Proof. Write $Y = Y(\Gamma_G)$ and $X = X(\Gamma_G)$. Let k be the field of constants of X, and let k'/k be a splitting field of S. Since motivic cohomology with $\mathbf{Q}$-coefficients satisfies Galois descent [11, (1.3)], it suffices to prove that the map $H^2_{\mathcal{M}}(X_{k'}, \mathbf{Q}(3)) \to H^2_{\mathcal{M}}(Y_{k'}, \mathbf{Q}(3))$ has a natural $\mathrm{Gal}(k'/k)$-equivariant retraction. Write $i\colon S_{k'} \hookrightarrow X_{k'}$ and $\pi\colon X_{k'} \to \operatorname{Spec} k'$. We have the following diagram

$$\begin{array}{ccccccc} 0 \longrightarrow H^2_{\mathcal{M}}(X_{k'}, \mathbf{Q}(3)) & \longrightarrow & H^2_{\mathcal{M}}(Y_{k'}, \mathbf{Q}(3)) & \xrightarrow{\mathrm{Res}} & \bigoplus_{x \in S(k')} H^1_{\mathcal{M}}(k', \mathbf{Q}(2)) & \xrightarrow{i_*} & H^3_{\mathcal{M}}(X_{k'}, \mathbf{Q}(3)) \\ & & & & & \searrow^{\Sigma} & \downarrow{\scriptstyle \pi_*} \\ & & & & & & H^1_{\mathcal{M}}(k', \mathbf{Q}(2)) \end{array}$$

where the first row is exact, and the diagonal arrow Σ is the sum map because the morphism $\pi \circ i\colon S_{k'} \to \operatorname{Spec} k'$ consists of copies of the identity map of $\operatorname{Spec} k'$.
Consider symbols of the form $\lambda \cup u$ where $\lambda \in H^1_{\mathcal{M}}(k', \mathbf{Q}(2))$ and $u \in \mathcal{O}(Y_{k'})^\times$ a modular unit. Let T be the subspace of $H^2_{\mathcal{M}}(Y_{k'}, \mathbf{Q}(3))$ generated by these symbols. Since λ comes from the base, the residue of $\lambda \cup u$ is given by $\lambda \otimes \operatorname{div}(u)$ (see [10, 1.3.2.(2)] applied to the closed immersion i and to $Y = \operatorname{Spec} k'$). Since the cusps are torsion in the Jacobian of X [14], this implies that $\mathrm{Res}(T) = \ker(\Sigma)$. We claim that

$$H^2_{\mathcal{M}}(Y_{k'}, \mathbf{Q}(3)) = H^2_{\mathcal{M}}(X_{k'}, \mathbf{Q}(3)) \oplus T. \tag{23}$$

The fact that $H^2_{\mathcal{M}}(Y_{k'}, \mathbf{Q}(3))$ is generated by $H^2_{\mathcal{M}}(X_{k'}, \mathbf{Q}(3))$ and T follows from the localisation sequence above. Now consider the composite map

$$H^1_{\mathcal{M}}(k', \mathbf{Q}(2)) \otimes \mathcal{O}(Y_{k'})^\times \xrightarrow{\cup} T \xrightarrow{\mathrm{Res}} \bigoplus_{x \in S(k')} H^1_{\mathcal{M}}(k', \mathbf{Q}(2)).$$

The kernel of this map is $H^1_{\mathcal{M}}(k', \mathbf{Q}(2)) \otimes k'^\times$. Therefore the intersection of $H^2_{\mathcal{M}}(X_{k'}, \mathbf{Q}(3))$ and T is contained in $H^2_{\mathcal{M}}(k', \mathbf{Q}(3))$, which is zero by Borel's theorem. The decomposition (23) provides the desired retraction over k'. Finally, we note that T is stable under $\mathrm{Gal}(k'/k)$, hence the retraction descends. □

Definition 6.5. For $a, b \in G$, we denote by $\xi'_G(a, b)$ the image of $\xi_G(a, b)$ in $H^2_{\mathcal{M}}(X(\Gamma_G), \mathbf{Q}(3))$ under the retraction of Proposition 6.4.

For the modular curves $X(N)$ and $X_1(N)$, we write $\xi'(a,b)$, $a,b \in (\mathbf{Z}/N\mathbf{Z})^2$ and $\xi'_1(a,b)$, $a,b \in \mathbf{Z}/N\mathbf{Z}$, for the classes associated to $\xi(a,b)$ and $\xi_1(a,b)$, respectively.

6.3. **Dependence on the triangulation.** We now study how the classes $\xi_G(a,b)$ and $\xi'_G(a,b)$ depend on the triangulation chosen. Say we have two triangulations

$$g_a \wedge g_b + g_b \wedge g_c + g_c \wedge g_a = \sum_i m_i \cdot u_i \wedge (1-u_i) = \sum_j n_j \cdot v_j \wedge (1-v_j) \quad \text{in } \Lambda^2 F_G^\times \otimes \mathbf{Q},$$

where F_G denotes the function field of $Y(\Gamma_G)$. Then

$$\sum_j n_j \{v_j\}_2 - \sum_i m_i \{u_i\}_2 \in H^1(\Gamma(F_G, 2))$$

defines an element in the Bloch group of F_G. Suslin's rigidity conjecture [36, Conjecture 5.4] asserts that the Bloch group of F_G is isomorphic to the Bloch group of k_G, the field of constants of F_G. If k_G is totally real, we have $H^1(\Gamma(k_G,2)) = 0$ by Borel's theorem. Therefore we obtain the following (conditional) independence result.

Proposition 6.6. *Let $N \geq 1$ be an integer. Assuming Suslin's rigidity conjecture, the classes $\xi_1(a,b)$, $a,b \in \mathbf{Z}/N\mathbf{Z}$, do not depend on the choice of triangulation.*

In general k_G is not totally real,[3] and the triangulation of $g_a \wedge g_b + g_b \wedge g_c + g_c \wedge g_a$ can be modified by an arbitrary element λ of $H^1(\Gamma(k_G,2))$. It should be the case that the class $\xi_G(a,b)$ in $H^2_{\mathcal{M}}(Y(\Gamma_G), \mathbf{Q}(3))$ gets modified by $\lambda \cup (g_b/g_a)$, where we used the isomorphism $H^1(\Gamma(k_G,2)) \cong H^1_{\mathcal{M}}(k_G, \mathbf{Q}(2))$. This would follow from the compatibility of De Jeu's map (17) with the cup-product $K_3^{(2)} \times K_1^{(1)} \to K_4^{(3)}$. As in the proof of Proposition 6.4, the residue of $\lambda \cup (g_b/g_a)$ at a cusp x is equal to $\mathrm{ord}_x(g_b/g_a)\lambda$, where ord_x denotes the order of vanishing at x. This shows that $\xi_G(a,b)$ can depend on the triangulation.
On the other hand, the element $\lambda \cup (g_b/g_a)$ is killed by the retraction of Proposition 6.4. Therefore, the class $\xi'_G(a,b)$ in $H^2_{\mathcal{M}}(X(\Gamma_G), \mathbf{Q}(3))$ should not depend on the triangulation.
Finally, we investigate how the classes $\xi_G(a,b)$ and $\xi'_G(a,b)$ depend on G. It will be convenient to work with the tower of modular curves $Y(N)$. Observe that for $(\alpha,\beta) \in \mathbf{Z}^2$ and $N \geq 1$, the Siegel unit $g_{(\alpha,\beta) \bmod N}$ from Section 3 depends only on the classes of $\frac{\alpha}{N}$ and $\frac{\beta}{N}$ in $\mathbf{Q}/\mathbf{Z}$. In this way, we may define the Siegel unit g_x for any $x \in (\mathbf{Q}/\mathbf{Z})^2$, living in the direct limit

$$\mathcal{O}(Y(\infty))^\times_{\mathbf{Q}} := \varinjlim_{N \geq 1} \mathcal{O}(Y(N))^\times_{\mathbf{Q}},$$

where the transition maps are the pull-backs associated to the canonical projection maps $Y(N') \to Y(N)$ for N dividing N'.
Now, let a,b be two elements of $(\mathbf{Q}/\mathbf{Z})^2$. Choose a finite subgroup G of $(\mathbf{Q}/\mathbf{Z})^2$ containing a and b, and choose an integer $N \geq 1$ such that G is killed by N. Then G identifies with a subgroup of $(\mathbf{Z}/N\mathbf{Z})^2$, and Construction 6.1 gives us classes $\xi_G(a,b)$ and $\xi'_G(a,b)$ on $Y(\Gamma_G)$ and $X(\Gamma_G)$, respectively. Because of the expected independence on triangulations discussed above, the class

$$\xi'_G(a,b) \in H^2_{\mathcal{M}}(X(\infty), \mathbf{Q}(3)) := \varinjlim_{N \geq 1} H^2_{\mathcal{M}}(X(N), \mathbf{Q}(3)) \tag{24}$$

should not depend on the choice of G and N.

[3] For example, the field of constants of $Y(N)$ is $\mathbf{Q}(\zeta_N)$.

In the special case of the modular curves $X_1(N)$, we may consider similarly, for $a, b \in \mathbf{Q}/\mathbf{Z}$, the images of $\xi_1(a,b)$ and $\xi_1'(a,b)$ in

$$\begin{aligned} H^2_{\mathcal{M}}(Y_1(\infty), \mathbf{Q}(3)) &:= \varinjlim_{N\geq 1} H^2_{\mathcal{M}}(Y_1(N), \mathbf{Q}(3)) \\ \text{and } H^2_{\mathcal{M}}(X_1(\infty), \mathbf{Q}(3)) &:= \varinjlim_{N\geq 1} H^2_{\mathcal{M}}(X_1(N), \mathbf{Q}(3)), \end{aligned} \tag{25}$$

respectively. By Proposition 6.6 we have the following result.

Proposition 6.7. *Let $a, b \in \mathbf{Q}/\mathbf{Z}$. Assuming Suslin's rigidity conjecture, the images of $\xi_1(a,b)$ and $\xi_1'(a,b)$ in the cohomology groups (25) do not depend on the choice of N such that $Na = Nb = 0$.*

We will discuss briefly this idea of taking cohomology at infinite level when trying to compare these motivic classes with the ones constructed by Beilinson (see Remark 9.4).

6.4. **Analogy with modular symbols.** To conclude this section, we show that the classes $\xi_G(a,b)$ satisfy relations analogous to modular symbols. Note that the elements $\xi(a,b)$ satisfy the transformation formula

$$\xi(a,b)|\gamma = \xi(a\gamma, b\gamma) \qquad (a,b \in (\mathbf{Z}/N\mathbf{Z})^2,\ \gamma \in \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})).$$

This is actually true at the level of the cocycles $\widetilde{\xi}(a,b)$, thanks to Proposition 3.6 and the transformation formula $g_a|\gamma = g_{a\gamma}$ for Siegel units [26, Lemma 1.7(1)].
The classes $\xi_G(a,b)$ also satisfy 3-term relations.

Proposition 6.8. *Let G be a subgroup of $(\mathbf{Z}/N\mathbf{Z})^2$. We have*

$$\xi_G(a,b) + \xi_G(b,c) + \xi_G(c,a) = 0 \qquad (a,b,c \in G,\ a+b+c=0), \tag{26}$$

and similarly for $\xi_G'(a,b)$. If $|G|$ is odd, we also have $\xi_G(b,a) = \xi_G(a,b)$ for any $a, b \in G$.

Proof. We show that (26) actually holds for the cocycles $\widetilde{\xi}_G(a,b)$. Write

$$\begin{aligned} T(a,b) :=& \frac{1}{|G|}\sum_{x\in G}\{u(0,x,a-x,b+x)\}_2 \\ &- \frac{1}{4|G|^2}\sum_{x,y\in G}\{u(0,a,c+2x,y)\}_2 + \{u(0,c,b+2x,y)\}_2 + \{u(0,b,a+2x,y)\}_2 \end{aligned} \tag{27}$$

for the triangulation of the Manin relation given in Theorem 4.3. The second line of (27) is invariant under cyclic permutation of (a,b,c). Regarding the first line, we have

$$\begin{aligned} \sum_{x\in G}\{u(0,x,a-x,b+x)\}_2 &= \sum_{x\in G}\{u(0,-x,-a+x,b+x)\}_2 \\ &\overset{x=y-b}{=} \sum_{y\in G}\{u(0,b-y,c+y,y)\}_2 \\ &= \sum_{y\in G}\{u(0,y,b-y,c+y)\}_2 \end{aligned}$$

since $\{1-1/f\}_2 = \{f\}_2$ by [40, VI, Lemma 5.4]. Thus $T(a,b) = T(b,c) = T(c,a)$ and

$$\widetilde{\xi}_G(a,b) + \widetilde{\xi}_G(b,c) + \widetilde{\xi}_G(c,a) = T(a,b) \otimes \left(\frac{g_b}{g_a}\frac{g_c}{g_b}\frac{g_a}{g_c}\right) = 0.$$

The proof of $\widetilde{\xi}_G(b,a) = \widetilde{\xi}_G(a,b)$ for $|G|$ odd is similar. □

Numerical experiments suggest that the classes $\xi'_G(a,b)$ also satisfy the 2-term relations $\xi'_G(a,b) + \xi'_G(b,-a) = 0$, as well as $\xi'_G(-a,b) = \xi'_G(a,-b) = -\xi'_G(a,b)$. We could not find a proof – maybe another triangulation is needed. We also do not know whether these relations also hold for the classes $\xi_G(a,b)$.
Proposition 6.8 gives some hope to find an inductive procedure to construct motivic classes in $H^2_{\mathcal{M}}(X(N), \mathbf{Q}(n))$ for $n \geq 4$.

7. Numerical computation of the regulator

The aim of this section is to explain how to compute numerically Goncharov's regulator integrals [19] in our case of interest, namely modular curves, using generalised Mellin transforms. This works in theory for any weight $n \geq 2$, but we implemented this computation in PARI/GP [31] only in the case $n = 3$.

7.1. **Convergence of Goncharov's integrals.** We first show that Goncharov's regulator integrals are absolutely convergent in the case of curves (Proposition 7.2 and Corollary 7.3). Let X be a smooth proper connected curve over $\mathbf{C}$, and let F be the function field of X. Let $n \geq 3$. According to Goncharov's theory, the motivic cohomology group $H^2_{\mathcal{M}}(F, \mathbf{Q}(n))$ should be isomorphic to a certain subquotient of $B_{n-1}(F) \otimes F^\times_{\mathbf{Q}}$. Goncharov constructs in [19, Theorem 2.2] a regulator map

$$r_n(2)\colon B_{n-1}(F) \otimes F^\times_{\mathbf{Q}} \to \mathcal{A}^1(\eta_X)(n-1)$$

where $\mathcal{A}^1(\eta_X)(n-1)$ is the space of $(2\pi i)^{n-1}\mathbf{R}$-valued differential 1-forms on X which are regular outside a finite subset of X. Concretely, for $f \in F\backslash\{0,1\}$ and $g \in F^\times$, we have

$$\begin{aligned} r_n(2)(\{f\}_{n-1} \otimes g) = i\widehat{\mathcal{L}}_{n-1}(f)\,\mathrm{d}\arg g - \frac{2^{n-1}B_{n-1}}{(n-1)!}\alpha(1-f,f)\cdot\log^{n-3}|f|\log|g| \\ - \sum_{k=2}^{n-2}\frac{2^k B_k}{k!}\widehat{\mathcal{L}}_{n-k}(f)\log^{k-2}|f|\,\mathrm{dlog}\,|f|\cdot\log|g|, \end{aligned} \tag{28}$$

where $\widehat{\mathcal{L}}_m\colon \mathbf{P}^1(\mathbf{C}) \to (2\pi i)^{m-1}\mathbf{R}$ is the single-valued polylogarithm defined in [19, Section 2.1][4], B_k is the k^{th} Bernoulli number, and

$$\alpha(f,g) = -\log|f|\,\mathrm{dlog}\,|g| + \log|g|\,\mathrm{dlog}\,|f|.$$

In particular $\widehat{\mathcal{L}}_2 = iD$, where $D\colon \mathbf{P}^1(\mathbf{C}) \to \mathbf{R}$ is the Bloch-Wigner dilogarithm [42, Section 2]. For $m \geq 2$, the function $\widehat{\mathcal{L}}_m$ is real-analytic outside $\{0,1,\infty\}$ and is continuous on $\mathbf{P}^1(\mathbf{C})$ with $\widehat{\mathcal{L}}_m(0) = \widehat{\mathcal{L}}_m(\infty) = 0$. It follows that the 1-form $r_n(2)(\{f\}_{n-1} \otimes g)$ is defined and real-analytic outside the set of zeros and poles of f, $1-f$ and g.

[4]There is a misprint in [19, Section 2.1]: $\log^{n-k}|z|$ should be replaced by $\log^k|z|$ in the definition of $\widehat{\mathcal{L}}_n(z)$.

For $n = 3$, we have explicitly

$$r_3(2)(\{f\}_2 \otimes g) = -D(f)\operatorname{darg} g - \frac{1}{3}\alpha(1-f, f)\cdot \log|g|. \tag{29}$$

This defines the map $r_3(2)$ in the diagram (20).

Lemma 7.1. *Let f, g be non-zero rational functions on X. Let $z = re^{i\theta}$ be a local coordinate at $p \in X$. In a neighbourhood of the point p, we have*

$$\alpha(f,g) = \Big(-\frac{\log|\partial_p(f,g)|}{r} + O(\log r)\Big)\cdot dr + O(r\log r)d\theta$$

where $\partial_p(f,g) = (-1)^{\operatorname{ord}_p(f)\operatorname{ord}_p(g)}\big(f^{\operatorname{ord}_p(g)}/g^{\operatorname{ord}_p(f)}\big)(p)$ is the tame symbol of (f,g) at p.

Proof. Write $f(z) \sim az^m$, $g(z) \sim bz^n$ with $a, b \in \mathbf{C}^\times$ and $m, n \in \mathbf{Z}$. A direct computation gives

$$\operatorname{dlog} f = (m + O(z))\frac{dz}{z} = (m+O(z))\frac{dr}{r} + (im + O(z))d\theta.$$

Taking the real and imaginary parts, we get

$$\begin{aligned} \operatorname{dlog}|f| &= (\frac{m}{r} + O(1))dr + O(r)d\theta, \\ \operatorname{darg} f &= O(1)dr + (m + O(r))d\theta. \end{aligned} \tag{30}$$

On the other hand $\log|f| = \log|a| + m\log r + O(r)$. Putting things together, we arrive at

$$\alpha(f,g) = \Big(-\frac{1}{r}\log\Big|\frac{a^n}{b^m}\Big| + O(\log r)\Big)dr + O(r\log r)d\theta. \qquad \square$$

Proposition 7.2. *Let $f \in F\backslash\{0,1\}$ and $g \in F^\times$. Let S be the set of zeros and poles of the functions f, $1-f$ and g. Let $\gamma\colon [0,1] \to X$ be a C^∞ path such that*

(a) *γ avoids S except possibly at the endpoints;*
(b) *If $p \in S$ is an endpoint of γ, then the argument of $\gamma(t)$ with respect to a local coordinate at p is of bounded variation when $\gamma(t)$ approaches p.*

Then for every $n \geq 3$, the integral $\int_\gamma r_n(2)(\{f\}_{n-1}\otimes g)$ converges absolutely.

Proof. As noted above, the integrand is C^∞ outside S. We are going to show the convergence of the integral at the endpoint $t = 0$ (the case $t = 1$ is identical). Let $z = re^{i\theta}$ be a local coordinate at $p = \gamma(0)$. Assumption (b) means that the form $d\theta$ is (absolutely) integrable along γ near $t = 0$. Moreover $dr = e^{-i\theta}dz - ird\theta$, so that dr is also integrable along γ. Using (30), we deduce that $\operatorname{darg} g$ and the first term of (28) are integrable. Regarding the second term, Lemma 7.1 and the fact that $\partial_p(1-f, f) = 1$ give

$$\alpha(1-f,f) = O(\log r)dr + O(r\log r)d\theta.$$

It follows that the second term in (28) has at worst logarithmic singularities, hence is integrable. Finally, the integrability of the third term in (28) can be proved similarly, noting that $\widehat{\mathcal{L}}_m(z) = O(|z|\log^{m-1}|z|)$ when $z \to 0$ for any $m \geq 2$, and using the functional equation $\widehat{\mathcal{L}}_m(1/z) = (-1)^{m-1}\widehat{\mathcal{L}}_m(z)$ to get the asymptotic behaviour when $z \to \infty$. $\square$

Corollary 7.3. *Let X be the modular curve associated to a congruence subgroup Γ of $\mathrm{SL}_2(\mathbf{Z})$, and let u, v be modular units on X such that $1-u$ is also a modular unit. Then for any $n \geq 3$ and any two cusps $\alpha \neq \beta$ in $\mathbf{P}^1(\mathbf{Q})$, the integral of $r_n(2)(\{u\}_{n-1}\otimes v)$ along the modular symbol $\{\alpha, \beta\}$ converges absolutely.*

Proposition 7.2 also holds for $n = 2$; in fact in this case we don't need to include the function $1 - f$ in the definition of S. This follows from a similar computation, the regulator being defined by $(f, g) \mapsto \log|f| \operatorname{darg} g - \log|g| \operatorname{darg} f$. As a consequence, Corollary 7.3 also holds in the case $n = 2$, without assuming that $1 - u$ is a modular unit.

Remark 7.4. We emphasise that the integral considered in Proposition 7.2 depends on the direction from which γ approaches the endpoints. This is because the differential 1-form $r_n(2)(\{f\}_{n-1} \otimes g)$ may have non-trivial residues at the points of S. As a consequence, in the setting of Corollary 7.3, the formula $\int_\alpha^\gamma = \int_\alpha^\beta + \int_\beta^\gamma$ does not always hold. A convenient framework to deal with this issue is Stevens's theory of extended modular symbols [35].

7.2. **Generalised Mellin transforms.** Let u, v be modular units for $\Gamma(N)$ such that $1 - u$ is also a modular unit. Let $n \geq 3$. For any two cusps $\alpha \neq \beta$ in $\mathbf{P}^1(\mathbf{Q})$, we would like to compute the integral

$$\int_\alpha^\beta r_n(2)(\{u\}_{n-1} \otimes v). \tag{31}$$

The modular symbol $\{\alpha, \beta\}$ may be written as a linear combination $\sum_i \{g_i 0, g_i \infty\}$ for some elements $g_i \in \mathrm{SL}_2(\mathbf{Z})$. Therefore the computation of (31) reduces to the case $\alpha = g0$ and $\beta = g\infty$ with $g \in \mathrm{SL}_2(\mathbf{Z})$, together with the computation of some residues at the cusps. Moreover, we have

$$\int_{g0}^{g\infty} r_n(2)(\{u\}_{n-1} \otimes v) = \int_0^\infty r_n(2)(\{u|g\}_{n-1} \otimes v|g)$$

and the functions $u|g$, $v|g$ are also modular units for $\Gamma(N)$. We are thus reduced to the case $\alpha = 0$ and $\beta = \infty$. In this case, let us write

$$\int_0^\infty r_n(2)(\{u\}_{n-1} \otimes v) = \int_0^\infty \phi(y) dy,$$

where $\phi\colon]0, +\infty[\to \mathbf{C}$ is a C^∞ function. We have seen in Corollary 7.3 that ϕ is absolutely integrable. We are going to show that ϕ belongs to a specific class of functions, for which the (generalised) Mellin transform can be computed rapidly.

Definition 7.5. Let $\mathcal{P}$ be the class of functions $\phi\colon]0, +\infty[\to \mathbf{C}$ such that

$$\phi(y) = \sum_{j=0}^{j_\infty} y^j \sum_{n=0}^{\infty} a_n^{(j)} e^{-2\pi n y/N}, \quad \text{and} \quad \phi\Big(\frac{1}{y}\Big) = \sum_{j=0}^{j_0} y^j \sum_{n=0}^{\infty} b_n^{(j)} e^{-2\pi n y/N}, \tag{32}$$

for some integers $j_\infty, j_0 \geq 0$, and where the sequences $(a_n^{(j)})_{n\geq 0}$ and $(b_n^{(j)})_{n\geq 0}$ have polynomial growth when $n \to \infty$.

By considering the asymptotic expansion, it is easy to see that the coefficients $a_n^{(j)}$ and $b_n^{(j)}$ are uniquely determined by ϕ. Moreover, the function ϕ is absolutely integrable on $]0, +\infty[$ if and only if $a_0^{(j)} = 0$ for $j \geq 0$ and $b_0^{(j)} = 0$ for $j \geq 1$.

Recall that the generalised Mellin transform of a function $\phi \in \mathcal{P}$ is defined as follows

$$\mathcal{M}(\phi, s) = \int_0^\infty \phi(y) y^s \frac{dy}{y} := \text{a.c.}\Big(\int_1^\infty \phi(y) y^s \frac{dy}{y}\Big) + \text{a.c.}\Big(\int_0^1 \phi(y) y^s \frac{dy}{y}\Big)$$
$$= \text{a.c.}\Big(\int_1^\infty \phi(y) y^s \frac{dy}{y}\Big) + \text{a.c.}\Big(\int_1^\infty \phi\Big(\frac{1}{y}\Big) y^{-s} \frac{dy}{y}\Big),$$

where $s \in \mathbf{C}$ and "a.c." means analytic continuation with respect to s. Note that the first integral converges for $\mathrm{Re}\,(s) \ll 0$ while the second integral converges for $\mathrm{Re}\,(s) \gg 0$; both have a meromorphic continuation to $s \in \mathbf{C}$ thanks to (32).
In the case ϕ is absolutely integrable, the integral $\int_0^\infty \phi = \mathcal{M}(\phi, 1)$ is given by the following series with exponential decay:
(33)

$$\int_0^\infty \phi = \sum_{n=1}^\infty \sum_{j=0}^{j_\infty} a_n^{(j)} \Big(\frac{N}{2\pi n}\Big)^{j+1} \Gamma\Big(j+1, \frac{2\pi n}{N}\Big) + \sum_{n=1}^\infty \sum_{j=0}^{j_0} b_n^{(j)} \Big(\frac{N}{2\pi n}\Big)^{j-1} \Gamma\Big(j-1, \frac{2\pi n}{N}\Big) + b_0^{(0)},$$

where $\Gamma(s, x)$ is the incomplete gamma function. So the integral of ϕ over $]0, +\infty[$ can be computed efficiently, provided sufficiently many coefficients $a_n^{(j)}$ and $b_n^{(j)}$ are known.
The class $\mathcal{P}$ is a $\mathbf{C}$-algebra stable under the differentiation $\frac{d}{dy}$. However it is not stable under taking primitive (e.g. consider a constant function). In fact, a function $\phi \in \mathcal{P}$ has a primitive in $\mathcal{P}$ if and only if $b_n^{(0)} = b_n^{(1)} = 0$ for all $n \geq 0$. This criterion shows that the image $\mathcal{P}'$ of the operator $\frac{d}{dy} : \mathcal{P} \to \mathcal{P}$ is an ideal of $\mathcal{P}$.

7.3. **The modular case.** We are now going to show, in the case of modular curves, that the regulators defined by Goncharov belong to $\mathcal{P}$.

Lemma 7.6. *For any modular unit u for $\Gamma(N)$, we have $\log u \in \mathcal{P}$, where $\log u$ is any determination of the logarithm of u on $\mathcal{H}$. In particular $\log|u| \in \mathcal{P}$, and the forms $\mathrm{dlog}\,|u|$ and $\arg u$ belong to $d\mathcal{P} = \mathcal{P}'dy$.*

Proof. Since the group of modular units is generated by the Siegel units g_a modulo the constants, it suffices to prove the result for them. For the asymptotic expansion of $\log g_a(iy)$ when $y \to +\infty$, this follows from taking the logarithm of (1) and (2), and expanding as a power series in $e^{-2\pi y/N}$. The expansion when $y \to 0$ also has the correct shape since $g_a(-1/\tau)$ is (a root of) a modular unit for $\Gamma(N)$. □

Proposition 7.7. *Let u be a modular unit for $\Gamma(N)$ such that $1-u$ is also a modular unit. For every $n \geq 2$, we have $\widehat{\mathcal{L}}_n(u) \in \mathcal{P}$.*

Proof. We will prove this by complete induction on n. For $n = 2$, we have $\widehat{\mathcal{L}}_2 = iD$, where D is the Bloch-Wigner dilogarithm [42, Section 2]. We have

$$dD(u) = \log|u| \arg(1-u) - \log|1-u| \arg(u).$$

From Lemma 7.6, it follows that $dD(u)(iy) \in d\mathcal{P}$, hence $D(u) \in \mathcal{P}$.

Now let $n \geq 3$. By the commutative diagram in [19, Theorem 2.2], we have

$$\begin{aligned} d\widehat{\mathcal{L}}_n(u) &= r_n(2)(\{u\}_{n-1} \otimes u) \\ &= i\widehat{\mathcal{L}}_{n-1}(u) \operatorname{darg} u - \frac{2^{n-1}B_{n-1}}{(n-1)!}\alpha(1-u,u) \cdot \log^{n-2}|u| \\ &\quad - \sum_{k=2}^{n-2} \frac{2^k B_k}{k!} \widehat{\mathcal{L}}_{n-k}(u) \log^{k-1}|u| \operatorname{dlog}|u| \end{aligned}$$

By the induction hypothesis $\widehat{\mathcal{L}}_m(u)$ belongs to $\mathcal{P}$ for $m < n$. The result now follows from Lemma 7.6 and the fact that $\mathcal{P}'$ is an ideal of $\mathcal{P}$. □

Note that the proof of Proposition 7.7 provides a way to compute the Fourier coefficients of $\widehat{\mathcal{L}}_n(u)$ (inductively on n): we first compute the Fourier expansions of $d\widehat{\mathcal{L}}_n(u)$ at 0 and ∞, and then integrate term by term. The constant of integration is determined by computing the value of $\widehat{\mathcal{L}}_n(u)$ at ∞ (note that this value is always finite). It should be equal to the coefficient $a_0^{(0)}$ of the expansion.

Theorem 7.8. *Let $n \geq 3$. Let u, v be two modular units for $\Gamma(N)$ such that $1-u$ is also a modular unit. Write*

$$r_n(2)(\{u\}_{n-1} \otimes v)|_{\{0,\infty\}} = \phi(y)dy.$$

Then ϕ belongs to $\mathcal{P}$.

Proof. This follows from (28), Lemma 7.6 and Proposition 7.7. □

In the case $u = u(a,b,c,d)$ and $v = g_e$, Proposition 3.7, the equation (28) and the proof of Proposition 7.7 actually provide an algorithm to compute the asymptotic expansion of $r_n(2)(\{u\}_{n-1} \otimes v)$ at 0 and ∞, and thus the associated regulator integral by (33).

8. Numerical verification of Beilinson's conjecture

In this section, we check numerically Beilinson's conjecture on $L(E,3)$ for elliptic curves E over $\mathbf{Q}$, using the K_4 elements constructed in Section 6. This conjecture relates $L(E,3)$ to a certain regulator map defined on the group $H^2_{\mathcal{M}}(E, \mathbf{Q}(3)) \cong K_4^{(3)}(E)$; we refer to [24, Section 4] for the precise statement.

Beilinson [1] has proved a weak form of this conjecture using his theory of the Eisenstein symbol (and assuming the curve E was modular). Namely, he shows the existence of a non-zero element of $K_4^{(3)}(E)$ whose regulator is related to $L(E,3)$.

Let us outline our strategy. Let $\phi\colon X_1(N) \to E$ be a modular parametrisation of E. We consider the elements $\xi'_1(a,b)$ of $K_4^{(3)}(X_1(N))$ from Section 6 and, as in Beilinson's approach, we apply to them the trace map $\phi_*\colon K_4^{(3)}(X_1(N)) \to K_4^{(3)}(E)$. The regulators of the resulting elements can be expressed[5] using concrete integrals of the form

$$\int_{\gamma_E} r_3(2)(\widetilde{\xi}_1(a,b)) \tag{34}$$

where $\widetilde{\xi}_1(a,b)$ is the cocycle of Construction 6.1, and γ_E is a 1-cycle on $X_1(N)(\mathbf{C})$ in the Hecke eigenspace corresponding to E. The cycle γ_E will be given as a $\mathbf{Z}$-linear combination

[5] To be precise, the expression may also involve residues, but we omit them for simplicity.

of Manin symbols $\{g0, g\infty\}$ with $g \in \mathrm{SL}_2(\mathbf{Z})$, so that the integral (34) can be computed using the techniques of Section 7.

Let us mention that De Jeu [24, Section 6] also did explicit computations in K_4 of elliptic curves. In [24, Example 6.2], he constructed two elements of $K_4^{(3)}(E)$ for an elliptic curve E of conductor 20, and computed their regulators, matching $L(E,3)$ numerically. His construction also uses explicit cocycles in a complex related to $\widetilde{\mathcal{M}}^{\bullet}_{(3)}(E)$. The difference with our approach is that he computed integrals of the form $\int_{E(\mathbf{C})} r_3(2)(\cdot) \wedge \omega_E$, where ω_E is the invariant differential form on $E(\mathbf{C})$, while we consider line integrals of the form (34). The latter task seems to be computationally easier.

One word of caution is in order here. The modular symbols $\{g0, g\infty\}$ live in the homology of the modular curve $X_1(N)$ relative to the cusps, while the regulator of $\widetilde{\xi}_1(a,b)$ is a closed 1-form on $Y_1(N)(\mathbf{C})$ which may have non-trivial residues at the cusps. Since there is no natural pairing between the relative homology of $X_1(N)$ and the cohomology of $Y_1(N)$, general integrals of the form (34) are a priori not cohomological, and will depend on choices of representatives. The same issue will arise in Section 9.

Before describing how to proceed practically, we need the following lemma.

Lemma 8.1. *For any $a, b \in \mathbf{Z}/N\mathbf{Z}$, the differential 1-form $r_3(2)(\widetilde{\xi}_1(a,b))$ is invariant under the complex conjugation acting on $Y_1(N)(\mathbf{C})$.*

Proof. It suffices to show that for two modular units u, v on $Y_1(N)$ defined over $\mathbf{Q}$, the 1-form $r_3(2)(\{u\}_2 \otimes v)$ defined in (29) is invariant under the complex conjugation c. A computation gives $D(u \circ c) = D(\overline{u}) = -D(u)$ and $\mathrm{darg}(v \circ c) = \mathrm{darg}\,\overline{v} = -\mathrm{darg}\,v$. The other term involving $\alpha(1-u,u)$ is dealt with similarly. □

Given a 1-cycle γ on $X_1(N)(\mathbf{C})$, write $[\gamma]$ for its class in the group $H_1(X_1(N)(\mathbf{C}), \mathbf{Q})$, and $[\gamma] = [\gamma]^+ + [\gamma]^-$ for the decomposition in the $+1$ and -1 eigenspaces of complex conjugation. Lemma 8.1 indicates that the integral of the regulator along an anti-invariant cycle is zero. Therefore in (34), we must choose a cycle γ_E satisfying $[\gamma_E]^+ \neq 0$.

Using the implementation of modular symbols in Magma, we obtain an explicit 1-cycle γ_E on $X_1(N)(\mathbf{C})$, given as a linear combination of paths $\{g0, g\infty\}$ with $g \in \mathrm{SL}_2(\mathbf{Z})$, such that $[\gamma_E]$ is in the Hecke eigenspace attached to E, and $[\gamma_E]^+ \neq 0$.[6]

We compute the integral (34) as follows. Write

$$\gamma_E = \sum_i n_i \{g_i 0, g_i \infty\}$$
$$\widetilde{\xi}_1(a,b) = \sum_j m_j \{u_j\}_2 \otimes v_j$$

for some $g_i \in \mathrm{SL}_2(\mathbf{Z})$ and some modular units u_j, v_j on $Y_1(N)$, with $n_i, m_j \in \mathbf{Q}$. Then

$$\int_{\gamma_E} r_3(2)(\widetilde{\xi}_1(a,b)) = \sum_{i,j} n_i m_j \int_0^{\infty} r_3(2)\big(\{u_j|g_i\}_2 \otimes (v_j|g_i)\big) \tag{35}$$

where the integrand is defined as in (29). This last integral (which is absolutely convergent by Corollary 7.3) can be computed numerically using Sections 7.2 and 7.3.

[6] The Magma code and the resulting cycles γ_E for all elliptic curves E of conductor up to 100 are available online [8], see the files `HomologyBasis.m` and `dataH1ell`.

As explained above, the integral (35) is a priori not cohomological, because $\xi_1(a,b)$ lives only on $Y_1(N)$, and may have residues at the cusps. The residue of $\widetilde{\xi}_1(a,b)$ at a cusp $x \in X_1(N)$ is determined using the formula from Section 5.3:

$$\mathrm{Res}_x(\{f\}_2 \otimes g) = \mathrm{ord}_x(g)\{f(x)\}_2 \in B_2(\mathbf{Q}(x)).$$

Let $\sigma_1, \dots, \sigma_{r_2}$ be the non-real complex embeddings of $\mathbf{Q}(x)$ (taking only one in each complex conjugate pair). The Bloch-Wigner dilogarithm $D\colon \mathbf{P}^1(\mathbf{C}) \to \mathbf{R}$ induces by linearity a map

$$\widetilde{D}\colon B_2(\mathbf{Q}(x)) \to \mathbf{R}^{r_2}, \quad \{a\}_2 \mapsto (D(\sigma_i(a)))_{1\le i\le r_2}$$

We will say that the residue of $\widetilde{\xi}_1(a,b)$ at x is numerically zero if its image under $\widetilde{D}$ is numerically zero. It is known that the restriction of $\widetilde{D}$ to $H^1(\Gamma(\mathbf{Q}(x),2))$ is injective, because the composition of $\widetilde{D}$ with Suslin's isomorphism $K_3(\mathbf{Q}(x))_{\mathbf{Q}} \cong H^1(\Gamma(\mathbf{Q}(x),2))$ identifies with a multiple of Borel's regulator map on $K_3(\mathbf{Q}(x))_{\mathbf{Q}}$ (see [17, Section 2]). Moreover, the residue of $\widetilde{\xi}_1(a,b)$ at x is a cocycle in the Bloch-Suslin complex of $\mathbf{Q}(x)$. We thus have a credible way to detect triviality (or non-triviality) of residues.
For each elliptic curve E of conductor $N \le 50$, we searched for a pair $(a,b) \in (\mathbf{Z}/N\mathbf{Z})^2$ satisfying the following two conditions:

- The residues of $\xi_1(a,b)$ at the cusps of $X_1(N)$ are numerically zero;
- The integral (35) is non-zero.

For the first pair (a,b) found, the integral (35) always turned out to be, numerically, a simple rational multiple of $\frac{\pi^2}{N} \cdot L'(E,-1)$.

Theorem 8.2. *For every elliptic curve E of conductor $N \le 50$, there exist $a,b \in \mathbf{Z}/N\mathbf{Z}$ such that the residues of $\widetilde{\xi}_1(a,b)$ at the cusps of $X_1(N)$ are numerically zero, and*

$$\int_{\gamma_E} r_3(2)(\widetilde{\xi}_1(a,b)) \stackrel{?}{=} \frac{r\pi^2}{N} \cdot L'(E,-1) \tag{36}$$

where $r \in \mathbf{Q}^\times$ is a non-zero rational number of small height, and $\stackrel{?}{=}$ means equality to at least 40 decimal places.

Curiously, for our choice of γ_E and (a,b), the rational factor r in (36) was always ± 3, except for the curves $38a1$ ($r=9$) and the curves $42a1, 43a1$ ($r=\pm 6$). We don't have an explanation for this. Also, for the elliptic curve $E = 36a1$ and the pair $(a,b)=(1,4)$, some residues are non-trivial, yet (35) is non-zero and proportional to $L'(E,-1)$. So having trivial residues at the cusps is not always necessary to get (36).
Theorem 8.2 was obtained using PARI/GP scripts available online [8], see the file `K4.gp`. The results are stored in the file `checkBeilinson50.txt`. The computation took approximately 7 hours on a desktop computer.
Let E be an elliptic curve of conductor $N \le 50$, and let $(a,b) \in (\mathbf{Z}/N\mathbf{Z})^2$ be the pair found in Theorem 8.2. Assume that the residues of $\widetilde{\xi}_1(a,b)$ are trivial.[7] Then $\widetilde{\xi}_1(a,b)$ is a cocycle in $\Gamma(X_1(N),3)$ and by [25, Theorem 2], the class $\xi_1(a,b)$ belongs to $K_4^{(3)}(X_1(N))$. By the discussion in Section 5.4, and since $X_1(N)/\mathbf{Q}$ is proper and geometrically connected, the Beilinson regulator $r_B(\xi_1(a,b))$ is equal to $\pm 2r_3(2)(\widetilde{\xi}_1(a,b))$. Now, let $j\colon Y_1(N) \to X_1(N)$

[7] This is automatically true if N is prime or twice a prime, by Lemma 6.3.

denote the canonical open immersion. Since the cycle γ_E passes through cusps, we modify it slightly to avoid them, as shown in Figure 1.

FIGURE 1. Modifying the path γ_E at a cusp

The modified cycle γ_ε (depending on a radius ε for fixed choices of local coordinates at the cusps) belongs to $H_1(Y_1(N)(\mathbf{C}), \mathbf{Z})$, and by construction we have $[\gamma_E] = j_*[\gamma_\varepsilon]$. Thus

$$\begin{aligned}\langle [\gamma_E]^+, r_B(\xi_1(a,b))\rangle &= \pm 2\langle [\gamma_E]^+, r_3(2)(\widetilde{\xi}_1(a,b))\rangle \\ &= \pm 2\langle [\gamma_\varepsilon]^+, j^* r_3(2)(\widetilde{\xi}_1(a,b))\rangle.\end{aligned}$$

The latter pairing converges to the integral (35) when $\varepsilon \to 0$, because the form $r_3(2)(\widetilde{\xi}_1(a,b))$ is defined on $Y_1(N)(\mathbf{C})$ and has trivial residues at the cusps. Therefore

$$\langle [\gamma_E]^+, r_B(\xi_1(a,b))\rangle = \pm 2 \int_{\gamma_E} r_3(2)(\widetilde{\xi}_1(a,b)).$$

In this case (35) has a cohomological interpretation.
Moreover, let ξ_E be the image of $\xi_1(a,b)$ under the trace map $\phi_*\colon K_4^{(3)}(X_1(N)) \to K_4^{(3)}(E)$. Write $[\gamma_E]^+ = \phi^* \alpha_E^+$, where α_E^+ is a generator of $H_1(E(\mathbf{C}), \mathbf{Q})^+$. Then

$$\langle \alpha_E^+, r_B(\xi_E)\rangle = \langle [\gamma_E]^+, r_B(\xi_1(a,b))\rangle = \pm 2 \int_{\gamma_E} r_3(2)(\widetilde{\xi}_1(a,b)).$$

So assuming $\widetilde{\xi}_1(a,b)$ has trivial residues, Theorem 8.2 implies that the regulator of ξ_E is numerically proportional to $\pi^2 \cdot L'(E,-1)$, as predicted by Beilinson's conjecture.

9. Comparison with Beilinson's elements and applications

In this section, we compare the elements $\xi_1(a,b)$ with the Beilinson elements defined using the Eisenstein symbol. To this end, we compute numerically their images under Beilinson's regulator map, using a recent result of W. Wang [39, Theorem 0.1.3] for the regulator of the Beilinson elements. Our computations suggest that the elements $\xi_1(a,b)$ and the Beilinson elements are proportional (Conjecture 9.3).
Let us first recall the definition of the Beilinson elements. Let $p\colon E_1(N) \to Y_1(N)$ denote the universal elliptic curve. In [1, Section 3], Beilinson constructs Eisenstein symbols

$$\mathrm{Eis}^1(0,a) \in H^2_{\mathcal{M}}(E_1(N), \mathbf{Q}(2)) \qquad (a \in \mathbf{Z}/N\mathbf{Z}).$$

Taking cup-product and pushing forward along p, one gets for any $a, b \in \mathbf{Z}/N\mathbf{Z}$:

$$\mathrm{Eis}^{0,0,1}(a,b) := p_*\big(\mathrm{Eis}^1(0,a) \cup \mathrm{Eis}^1(0,b)\big) \in H^2_{\mathcal{M}}(Y_1(N), \mathbf{Q}(3)).$$

The regulator computation goes as follows. We first use Magma to find a basis of the homology group $H_1(X_1(N)(\mathbf{C}),\mathbf{Q})^+$.[8] The integral of the regulator of $\xi_1(a,b)$ along a given cycle in the basis is then computed as in Section 8.
Regarding the Beilinson elements, Wang computed the integral of the regulator of $\mathrm{Eis}^{0,0,1}(a,b)$ along a modular symbol $\{g0, g\infty\}$ with $g \in \mathrm{SL}_2(\mathbf{Z})$. Note that this integral is a priori not cohomological, so a certain representative $\mathrm{Eis}_{\mathcal{D}}^{0,0,1}(a,b)$ of the Beilinson regulator of $\mathrm{Eis}^{0,0,1}(a,b)$ has to be chosen [39, Proposition 2.4.2]. Also, for simplicity, we state Wang's result only in the case $g = 1$. The case of an arbitrary $g \in \mathrm{SL}_2(\mathbf{Z})$ can be treated using the relation $\mathrm{Eis}^{0,0,1}(a,b)|g = \mathrm{Eis}^{0,0,1}((0,a)g,(0,b)g)$; here the classes live on the modular curve $Y(N)$. We refer to [39, Theorem 0.1.3] for the general statement.

Theorem 9.1 (W. Wang). *For any integer $N \geq 3$ and any $a, b \in (\mathbf{Z}/N\mathbf{Z})\backslash\{0\}$, we have*

$$\int_0^\infty \mathrm{Eis}_{\mathcal{D}}^{0,0,1}(a,b) = -\frac{36\pi^2}{N^3} L'(\tilde{s}_a \tilde{s}_b, -1),$$

where the $\tilde{s}_x$ are Eisenstein series of weight 1 and level $\Gamma_1(N)$ defined by

$$\tilde{s}_x(\tau) = \frac{1}{2} - \left\{\frac{x}{N}\right\} + \sum_{\substack{m,n\geq 1 \\ n\equiv x \bmod N}} q^{mn} - \sum_{\substack{m,n\geq 1 \\ n\equiv -x \bmod N}} q^{mn} \qquad (q = e^{2\pi i\tau}),$$

and $\{\cdot\}$ denotes the fractional part.

The proof of Theorem 9.1 uses the Rogers–Zudilin method, for which we refer to [9, Chapter 9]. In fact, Wang proves a much more general statement concerning the regulators of motivic classes $\mathrm{Eis}^{k_1,k_2,j}(u_1,u_2)$ on Kuga-Sato varieties. These classes were defined by Beilinson, Deninger–Scholl [11, 5.7] and Gealy [16]. The formula involves the completed L-function of a modular form of weight $k_1 + k_2 + 2$ evaluated at $s = -j$ [39, Chapter 6].
The Eisenstein series $\tilde{s}_x$ appear in the work of Borisov and Gunnells [5, 3.18]. Moreover, the image of $\tilde{s}_x$ under the Fricke involution W_N is a multiple of the Eisenstein series denoted by s_x in [5, 3.5]. From this, we can use the standard W_N-trick to compute numerically the L-value $L'(\tilde{s}_a\tilde{s}_b, -1)$, and thus the regulator of $\mathrm{Eis}^{0,0,1}(a,b)$ in the case $g = 1$. The computation for an arbitrary modular symbol $\{g0, g\infty\}$ proceeds similarly, using Eisenstein series of level $\Gamma(N)$ instead of $\Gamma_1(N)$.

Theorem 9.2. *For every integer $N \leq 28$ and every $a, b \in \mathbf{Z}/N\mathbf{Z}$, we have*

$$\int_{\gamma_i} r_3(2)\big(\widetilde{\xi_1}(a,b)\big) \stackrel{?}{=} \frac{N^2}{6} \int_{\gamma_i} \mathrm{Eis}_{\mathcal{D}}^{0,0,1}(a,b) \qquad (1 \leq i \leq g_N)$$

where $\gamma_1, \ldots, \gamma_{g_N}$ are the cycles computed by Magma representing a basis of $H_1(X_1(N)(\mathbf{C}),\mathbf{Q})^+$, and $\stackrel{?}{=}$ means equality to at least 40 decimal places.

Theorem 9.2 was obtained using PARI/GP scripts available online [8], see the file `K4.gp`. The computation took approximately 4 days on a desktop computer.
Recall that $\mathrm{Eis}_{\mathcal{D}}^{0,0,1}(a,b)$ is a representative of $r_B(\mathrm{Eis}^{0,0,1}(a,b))$. Based on Theorem 9.2 and on the diagram (20), we formulate the following conjecture.

[8] The Magma code and the resulting data for $N \leq 50$ are available online [8], see the files `HomologyBasis.m` and `dataH1`, respectively.

Conjecture 9.3. *For every integer $N \geq 1$ and every $a, b \in \mathbf{Z}/N\mathbf{Z}$, we have*

$$\xi_1(a,b) = \pm\frac{N^2}{3}\mathrm{Eis}^{0,0,1}(a,b).$$

Conjecture 9.3 relates two motivic classes $\mathrm{Eis}^{0,0,1}(a,b)$ and $\xi_1(a,b)$ whose constructions are quite different. However, both classes are of modular nature, and we expect that there is a modular proof of the relation between them. As a first step, one could investigate the residues of $\xi_1(a,b)$ and $\mathrm{Eis}^{0,0,1}(a,b)$ at the cusps.

Remark 9.4. Here is one way to approach Conjecture 9.3, using the cohomology at infinite level introduced in Section 6. Let us identify $\mathbf{Z}/N\mathbf{Z}$ with a subgroup of $(\mathbf{Q}/\mathbf{Z})^2$ by mapping x to $(0, \tilde{x}/N)$, where $\tilde{x}$ is a representative of x in $\mathbf{Z}$. For any $a, b \in \mathbf{Z}/N\mathbf{Z}$ and any level N' divisible by N, the group $G = (\frac{1}{N'}\mathbf{Z}/\mathbf{Z})^2$ contains a and b. Moreover, we saw in Section 6 that the class

$$\xi'_G(a,b) \in H^2_{\mathcal{M}}(X(\infty), \mathbf{Q}(3))$$

conjecturally does not depend on N' (see the discussion before (24)). Assuming N' odd for simplicity, the element $\xi_G(a,b)$ is the image under De Jeu's map of the cocycle

$$\widetilde{\xi}_G(a,b) = \sum_{x \in G} \{u(0,x,a-x,b+x)\}_2 \otimes \left(\frac{g_b}{g_a}\right). \tag{37}$$

One may view G as the full N'-torsion subgroup of the universal elliptic curve $E_1(N)$ over $Y_1(N)$. Applying the regulator map and taking the limit when $N' \to \infty$, the sum (37) becomes an integral along the fibres of $E_1(N) \to Y_1(N)$. This is reminiscent of the definition of $\mathrm{Eis}^{0,0,1}(a,b)$, which is (after applying the regulator) also obtained by integrating along the fibres of the universal elliptic curve. It would be also interesting to interpolate the regulator of $\xi_G(a,b)$ as a continuous function of $a, b \in \mathbf{R}/\mathbf{Z}$.

We finally come to a potential application of the motivic elements $\xi_1(a,b)$ regarding the Mahler measure of certain 3-variable polynomials. Recall that the (logarithmic) Mahler measure of a polynomial $P \in \mathbf{C}[x_1, \ldots, x_n]$ is defined by

$$m(P) = \frac{1}{(2\pi i)^n}\int_{T^n} \log|P(x_1,\ldots,x_n)|\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n},$$

where T^n is the n-torus $|x_1| = \cdots = |x_n| = 1$.
Boyd and Rodriguez Villegas have formulated conjectures relating the Mahler measure of specific 3-variable polynomials and L-functions of elliptic curves evaluated at $s = 3$ (see [9, Section 8.4] and [28]). Here is one example of these identities.

Conjecture 9.5 (Boyd–Rodriguez Villegas)**.** *We have $m\big((1+x)(1+y)+z\big) = -2L'(E,-1)$, where $E = 15a8$ is the elliptic curve of conductor 15 defined by the affine equation*

$$E : (1+x)\Big(1+\frac{1}{x}\Big)(1+y)\Big(1+\frac{1}{y}\Big) = 1.$$

Lalín has shown in [28] that

$$m\big((1+x)(1+y)+z\big) = \frac{1}{4\pi^2}\int_{\gamma_E^+} r_3(2)\big(\{-x\}_2 \otimes y - \{-y\}_2 \otimes x\big),$$

where γ_E^+ is a generator of $H_1(E(\mathbf{C}), \mathbf{Z})^+$. The symbol $\xi := \{-x\}_2 \otimes y - \{-y\}_2 \otimes x$ defines, via De Jeu's map, an element of $K_4^{(3)}(E)$. Moreover, the curve E is isomorphic to $X_1(15)$ and one can show that

$$-x = u_1(1, 2, 3, 7), \qquad -y = u_1(2, 4, 6, 1).$$

It would be interesting to express ξ in terms of the symbols $\xi_1(a, b)$ with $a, b \in \mathbf{Z}/15\mathbf{Z}$. Conjecture 9.5 would then be a consequence of Theorem 9.1 and Conjecture 9.3.

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