On the operator norm of non-commutative polynomials in deterministic matrices and iid Haar unitary matrices

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Abstract

Let $U^N = (U_1^N, \dots, U_p^N)$ be a d-tuple of $N \times N$ independent Haar unitary matrices and Z^{NM} be any family of deterministic matrices in $\mathbb{M}_N(\mathbb{C}) \otimes \mathbb{M}_M(\mathbb{C})$. Let P be a self-adjoint non-commutative polynomial. In [28], Voiculescu showed that the empirical measure of the eigenvalues of this polynomial evaluated in Haar unitary matrices and deterministic matrices converges towards a deterministic measure defined thanks to free probability theory. Let now f be a smooth function, the main technical result of this paper is a precise bound of the difference between the expectation of

$$\frac{1}{MN} \operatorname{Tr}_{\mathbb{M}_N(\mathbb{C})} \otimes \operatorname{Tr}_{\mathbb{M}_M(\mathbb{C})} \left(f(P(U^N \otimes I_M, Z^{NM})) \right),$$

and its limit when N goes to infinity. If f is seven times differentiable, we show that it is bounded by $M^2 \|f\|_{\mathcal{C}^7} N^{-2}$. As a corollary we obtain a new proof with quantitative bounds of a result of Collins and Male which gives sufficient conditions for the operator norm of a polynomial evaluated in Haar unitary matrices and deterministic matrices to converge almost surely towards its free limit. Our result also holds in much greater generality. For instance, it allows to prove that if U^N and Y^{M_N} are independent and $M_N = o(N^{1/3})$, then the norm of any polynomial in $(U^N \otimes I_{M_N}, I_N \otimes Y^{M_N})$ converges almost surely towards its free limit. Previous results required that M_N is bounded.

1 Introduction

Understanding the behaviour of random matrices in large dimension is the core of random matrix theory. In the early nineties Voiculescu showed that one could get very accurate results with the help of non-commutative probability theory. This theory is equipped with a notion of freeness, analogous to independence in classical probability theory, which often describes accurately the asymptotic behaviour of random matrices. In [27] he studied the asymptotic behaviour of independent matrices taken from the Gaussian Unitary Ensemble (GUE). In a later paper he proved a similar theorem for Haar unitary matrices, which are random matrices whose law is the Haar measure on the unitary group $\mathbb{U}_N$. In a nutshell, Voiculescu proved in [28] that given $U_1^N, \dots, U_p^N$ independent Haar unitary matrices, the renormalized trace of a polynomial P evaluated in these matrices converges towards a deterministic limit $\alpha(P)$. Specifically, the following holds true almost surely:

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}_N \left(P(U_1^N, \dots, U_p^N, U_1^{N*}, \dots, U_p^{N*}) \right) = \alpha(P). \tag{1}$$

Voiculescu computed the limit $\alpha(P)$ with the help of free probability. To give more detail, let B_N be a self-adjoint matrix of size N, then one can define the empirical measure of its (real) eigenvalues by

$$\mu_{B_N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i},$$

where δ_λ is the Dirac mass in λ and $\lambda_1, \dots, \lambda_N$ are the eingenvalue of B_N. In particular, if P is a self-adjoint polynomial, that is such that for any matrices $A_1, \dots, A_d$, $P(A_1, \dots, A_d, A_1^*, \dots, A_d^*)$ is a self-adjoint matrix, then one can define the random measure $\mu_{P(U_1^N,\dots,U_p^N,U_1^{N*},\dots,U_p^{N*})}$. In this case, Voiculescu's result (1) implies that there exists a measure μ_P with compact support such that almost surely $\mu_{P(U_1^N,\dots,U_p^N,U_1^{N*},\dots,U_p^{N*})}$ converges weakly towards μ_P: its moments are given by $\mu_P(x^k) = \alpha(P^k)$ for all integer numbers k.

However, the convergence of the empirical measure of the eigenvalues of a matrix does not say much about the local properties of its spectrum, in particular about the convergence of the norm of this matrix, or the local fluctuations of its spectrum. For a comprehensive survey of important milestones related to these questions, we refer to the introduction of our previous paper [11]. In a nutshell, when dealing with a single matrix, incredibly precise results are known. Typically, concerning the GUE, very precise results were obtained by Tracy and Widom in the early nineties in [26]. On the other hand, there are much less results available when one deals with a polynomial in several random matrices. One of the most notable result was found by Haagerup and Thorbjørnsen in 2005 in [16]: they proved the almost sure convergence of the norm of a polynomial evaluated in independent GUE matrices. Equivalently, for P a self-adjoint polynomial, they proved that almost surely, for N large enough,

$$\sigma\left(P(X_1^N, \dots, X_d^N)\right) \subset \operatorname{Supp} \mu_P + (-\varepsilon, \varepsilon), \tag{2}$$

where $\sigma(H)$ is the spectrum of H and $\operatorname{Supp} \mu_P$ the support of the measure μ_P. The result (2) was a major breakthrough in the context of free probability and was refined in multiple ways, see [25, 9, 1, 3, 23, 11]. Those results all have in common that the basic random matrix is always self-adjoint. Much less is known in the non self-adjoint case. However Collins and Male proved in [12] the same result as in [18] but with unitary Haar matrices instead of GUE matrices by using Male's former paper. With the exception of [12] and [11], all of these results are essentially based on the method introduced by Haagerup and Thorbjørnsen who relies on the so-called linearization trick. The main idea of this tool is that given a polynomial P, the spectrum of $P(X_1^N, \dots, X_d^N)$ is closely related to the spectrum of

$$L_N = a_0 \otimes I_N + \sum_{i=1}^{d} a_i \otimes X_i^N,$$

where $a_0, \dots, a_d$ are matrices of size k depending only on P. Thus we trade a polynomial of degree d with coefficient in $\mathbb{C}$ by a polynomial of degree 1 with coefficient in $\mathbb{M}_{k(d)}(\mathbb{C})$. In [12], the main idea was to view Haar unitary matrices as a random function of a GUE random matrix. Then the authors showed that almost surely this random function converges uniformly and they concluded by using the main result of [18]. An issue of this method is that it does not give any quantitative estimate. An important aim of this paper is to remedy to this problem. Our approach requires neither the linearization trick, nor the study of the Stieljes transform and attacks the problem directly without using previous results about the strong convergence of GUE random matrices. In this sense the proof is more direct and less algebraic. We will apply it to a generalization of Haar unitary matrices by tackling the case of Haar unitary matrices tensorized with deterministic matrices.

A usual strategy to study outliers, that are the eigenvalues going away from the spectrum, is to study the *non-renormalized* trace of smooth non-polynomial functions evaluated in independent Haar matrices i.e. if P is self-adjoint:

$$\operatorname{Tr}_N\left(f\left(P\left(U_1^N, \dots, U_p^N, U_1^{N*}, \dots, U_p^{N*}\right)\right)\right).$$

This strategy was also used by Haagerup, Thorbjørnsen and Male. Indeed it is easy to see that if f is a function which takes value 0 on $(-\infty, C-\varepsilon]$, 1 on $[C, \infty)$ and in $[0,1]$ elsewhere, then

$$\mathbb{P}\Big(\lambda_1(P(U_1^N, \dots, U_p^N, U_1^{N*}, \dots, U_p^{N*})) \geq C\Big) \leq \ \mathbb{P}\Big(\operatorname{Tr}_N\left(f(P(U_1^N, \dots, U_p^N, U_1^{N*}, \dots, U_p^{N*}))\right) \geq 1\Big).$$

Hence, if we can prove that $\operatorname{Tr}_N\left(f(P(U_1^N, \dots, U_p^N, U_1^{N*}, \dots, U_p^{N*}))\right)$ converges towards 0 in probability, this would already yield expected results. The above is just a well-known exemple, but one can get much more out of this strategy. Therefore, we need to study the non-renormalized trace. The case where f

is a polynomial function has already been studied a long time ago, starting with the pioneering works [8, 14], and later formalized by the concept of second order freeness [20, 19]. However here we have to deal with a function f which is at best C^∞. This makes things considerably more difficult and forces us to adopt a completely different approach. The main result is the following theorem (for the notations, we refer to Section 2 – for now, let us specify that $\frac{1}{N}\operatorname{Tr}_N$ denotes the usual renormalized trace on $N \times N$ matrices whereas τ denotes its free limit):

Theorem 1.1. *We define*

- $u = (u_1, \dots, u_p, u_1^*, \dots, u_p^*)$ *a family of p free Haar unitaries and their adjoints,*
- $U^N = (U_1^N, \dots, U_p^N, (U_1^N)^*, \dots, (U_p^N)^*)$ *random unitary i.i.d. matrices of size N whose law is invariant by multiplication by a matrix of $SU_N(\mathbb{R})$, and their adjoints.*
- $Z^{NM} = (Z_1^{NM}, \dots, Z_q^{NM})$ *deterministic matrices and their adjoint,*
- P *a self-adjoint polynomial,*
- $f : \mathbb{R} \mapsto \mathbb{R}$ *a smooth enough function.*

Then there exists a polynomial L_P which only depends on P such that for any N, M,

$$\left| \mathbb{E}\left[\frac{1}{MN} \operatorname{Tr}_{MN} \Big(f\left(P\left(U^N \otimes I_M, Z^{NM}\right)\right) \Big) \right] - \tau_N \otimes \tau_M \Big(f\left(P\left(u \otimes I_M, Z^{NM}\right)\right) \Big) \right|$$
$$\leq \frac{M^2}{N^2} L_P\left(\left\|Z^{NM}\right\|\right) \times \min\left\{ \ln^2(N) \|f\|_{\mathcal{C}^6}, \|f\|_{\mathcal{C}^7} \right\}.$$

where $\|f\|_{\mathcal{C}^6}$ is the sum of the supremum on $\mathbb{R}$ of the first six derivatives. Besides if $Z^{NM} = (I_N \otimes Y_1^M, \dots, I_N \otimes Y_q^M)$ and that these matrices commute, then we have the same inequality without the M^2.

This theorem is a consequence of the slightly sharper, but less explicit, Theorem 4.1. It is essentially the same statement, but instead of having the norm C^6 of f, we have the fourth moment of the Fourier transform of f. The above theorem calls for a few remarks.

- We assumed that the matrices Z^{NM} were deterministic, but thanks to Fubini's theorem we can assume that they are random matrices as long as they are independent from X^N. In this situation though, $L_P\left(\left\|Z^{NM}\right\|\right)$ in the right side of the inequality is a random variable (and thus we need some additional assumptions on the law of Z^{NM} if we want its expectation to be finite for instance).
- In Theorems 1.1 and 4.1 we have $U^N \otimes I_M$ and $u \otimes I_M$, however it is very easy to replace them by $U^N \otimes Y^M$ and $u \otimes Y^M$ for some matrices $Y_i^M \in M_M(\mathbb{C})$. Indeed we just need to apply Theorem 1.1 or 4.1 with $Z^{NM} = I_N \otimes Y^M$. Besides, in this situation, $L_P\left(\left\|Z^{NM}\right\|\right) = L_P\left(\left\|Y^M\right\|\right)$ does not depend on N. What this means is that if we have a matrix whose coefficients are polynomial in U^N, and that we replace U^N by u, we only change the spectra of this matrix by M^2N^{-2} in average.
- In the specific case where $Z^{NM} = (I_N \otimes Y_1^M, \dots, I_N \otimes Y_q^M)$ and the Y_i^M commute, as we stated in Theorem 1.1, we have the same inequality without the M^2. This shows that the M^2 term is really a non-commutative feature. Lowering the exponent in all generality would yield a direct improvement to Theorem 1.2. A lead to do so would be to prove a sharper version of Lemma 3.1. While this seems unrealistic for deterministic matrices, it might be possible to get some results when considering random matrices.

A detailed overview of the proof is given in Subsection 4.1. Similarly to [11], we interpolate Haar unitary matrices and free Haar unitaries with the help of a free Ornstein-Uhlenbeck process on the unitary group, i.e. the free unitary Brownian motion. For a reference, see Definition 2.7. However in [11] this idea was only to understand the intuition of the proof. In this paper the computations involved were quite different, indeed since we were considering the usual free Ornstein-Uhlenbeck process, we could use a computation trick to replace this process by a well-chosen interpolation between GUE matrices and free semicirculars. This means that we did not need to use free stochastic calculus. There is no such trick for the free unitary Brownian motion, hence the computations use much more advanced tools.

When using this process, the Schwinger-Dyson equations, which can be seen as an integration by part, appear in the computation. For more information about these equations we refer to [15] to find numerous applications. In the specific case of the unitary group it is worth checking the proof of Theorem 5.4.10 from [2]. Even though those equations only come into play in the proof of Lemma 4.3, they play a major role in the proof since we could get a theorem similar to Theorem 1.1 for any random matrices which satisfies those equations.

Theorem 1.1 is the crux of the paper and allows us to deduce many corollaries. Firstly we get the following result. The first statement is basically Theorem 1.4 from [12]. The second one is entirely new and let us tensorize by matrices whose size goes to infinity when until now we could only work with tensor of finite size. This theorem is about strong convergence of random matrices, that is the convergence of the norm of polynomials in these matrices, see Definition 2.1.

Theorem 1.2. *Let the following objects be given:*

- $U^N = (U_1^N, \dots, U_d^N)$ *independent unitary Haar matrices of size* N,
- $u = (u_1, \dots, u_d)$ *a system of free Haar unitaries,*
- $Y^M = (Y_1^M, \dots, Y_p^M)$ *random matrices of size* M*, which almost surely, as* M *goes to infinity, converges strongly in distribution towards a p-tuple* y *of non-commutative random variables in a* $\mathcal{C}^*$*- probability space* $\mathcal{B}$ *with a faithful trace* $\tau_{\mathcal{B}}$.
- $Z^N = (Z_1^N, \dots, Z_q^N)$ *random matrices of size* N*, which almost surely, as* N *goes to infinity, converges strongly in distribution towards a q-tuple* z *of non-commutative random variables in a* $\mathcal{C}^*$*-probability space with a faithful trace,*

then the following holds true.

- *If* U^N *and* Z^N *are independent, almost surely,* (U^N, Z^N) *converges strongly in distribution towards* $\mathcal{F} = (u, z)$*, where* $\mathcal{F}$ *belongs to a* $\mathcal{C}^*$*- probability space* $(\mathcal{A}, *, \tau_{\mathcal{A}}, \|.\|)$ *in which* u *and* z *are free.*
- *If* U^N *and* Y^{M_N} *are independent and* $M_N = o(N^{1/3})$*, almost surely,* $(U^N \otimes I_{M_N}, I_N \otimes Y^{M_N})$ *converges strongly in distribution towards* $\mathcal{F} = (u \otimes 1, 1 \otimes y)$*. The family* $\mathcal{F}$ *thus belongs to* $\mathcal{A} \otimes_{\min} \mathcal{B}$ *(see Definition 2.4). Besides if the matrices* Y^{M_N} *commute, then we can weaken the assumption on* M_N *by only assuming that* $M_N = o(N)$.

Understanding the Stieljes transform of a matrix gives a lot of information about its spectrum. This was actually a very important point in the proof of Haagerup and Thorbjørnsen's theorem. Our proof does not use this tool, however our final result, Theorem 4.1, allows us to deduce the following estimate. Being given a self-adjoint $NM \times NM$ matrix, we denote by G_A its Stieltjes transform:

$$G_A(z) = \frac{1}{NM} \operatorname{Tr}_{NM} \left(\frac{1}{z - A} \right).$$

This definition extends to the tensor product of free Haar unitaries with deterministic matrices by replacing Tr_{NM} by $\tau_N \otimes \tau_M$.

Corollary 1.1. *Given*

- $u = (u_1, \dots, u_p, u_1^*, \dots, u_p^*)$ *a family of* p *free Haar unitaries and their adjoints,*
- $U^N = (U_1^N, \dots, U_p^N, (U_1^N)^*, \dots, (U_p^N)^*)$ *random unitary i.i.d. matrices of size* N *whose law is invariant by multiplication by a matrix of* $SU_N(\mathbb{R})$*, and their adjoints.*
- $Y^M = (Y_1^M, \dots, Y_q^M, Y_1^{M*}, \dots, Y_q^{M*})$ *deterministic matrices of size* M *a fixed integer and their adjoints,*
- P *a self-adjoint polynomial,*

there exists a polynomial L_P such that for every Y^M, $z \in \mathbb{C}\backslash\mathbb{R}$, $N \in \mathbb{N}$,

$$\left|\mathbb{E}\left[G_{P(U^N\otimes I_M, I_N\otimes Y^M)}(z)\right] - G_{P(u\otimes I_M, 1\otimes Y^M)}(z)\right| \leq L_P\left(\|Y^M\|\right)\frac{M^2}{N^2}\left(\frac{1}{|\Im(z)|^5}+\frac{1}{|\Im(z)|^2}\right).$$

One of the limitation of Theorem 1.1 is that we need to pick f regular enough. Actually by approximating f, we can afford to take f less regular at the cost of a slower speed of convergence. In other words, we trade some degree of regularity on f for a smaller exponent in N. The best that we can achieve is to take f Lipschitz. Thus it makes sense to introduce the Lipschitz-bounded metric. This metric is compatible with the topology of the convergence in law of measure. Let $\mathcal{F}_{LU}$ be the set of Lipschitz function from $\mathbb{R}$ to $\mathbb{R}$, uniformly bounded by 1 and with Lipschitz constant at most 1, then

$$d_{LU}(\mu,\nu) = \sup_{f\in\mathcal{F}_{LU}}\left|\int_{\mathbb{R}} f d\mu - \int_{\mathbb{R}} f d\nu\right|.$$

For more information about this metric we refer to Annex C.2 of [2]. In this paper, we get the following result:

Corollary 1.2. *Under the same notations as in Corollary 1.1, there exists a polynomial L_P such that for every matrices Y^M and $M, N \in \mathbb{N}$,*

$$d_{LU}\left(\mathbb{E}[\mu_{P(U^N\otimes I_M, I_N\otimes Y_M)}], \mu_{P(u\otimes I_M, 1\otimes Y_M)}\right) \leq L_P\left(\|Y^M\|\right) M^2\left(\frac{\ln N}{N}\right)^{1/3}.$$

This paper is organized as follows. In section 2 we give many usual definitions and notations in free probability, commutative and non-commutative stochastic calculus. Section 3 contains the proof of many important properties which we will need later on. Section 4 contains the proof of Theorem 1.1. Finally in section 5 we prove all of the corollaries.

2 Framework and standard properties

2.1 Usual definitions in free probability

In order to be self-contained, we begin by reminding the following definitions of free probability.

Definition 2.1.
- *A $\mathcal{C}^*$-probability space $(\mathcal{A}, *, \tau, \|.\|)$ is a unital $\mathcal{C}^*$-algebra $(\mathcal{A}, *, \|.\|)$ endowed with a state τ, i.e. a linear map $\tau : \mathcal{A} \to \mathbb{C}$ satisfying $\tau(1_{\mathcal{A}}) = 1$ and $\tau(a^*a) \geq 0$ for all $a \in \mathcal{A}$. In this paper we always assume that τ is a trace, i.e. that it satisfies $\tau(ab) = \tau(ba)$ for any $a, b \in \mathcal{A}$. An element of $\mathcal{A}$ is called a (non commutative) random variable. We will always work with faithful trace, that is such that if $a \in \mathcal{A}$, $\tau(a^*a) = 0$ if and only if $a = 0$, in which case the norm is determined by τ thanks to the formula:*

$$\|a\| = \lim_{k\to\infty}\left(\tau\left((a^*a)^{2k}\right)\right)^{1/2k}.$$

- *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be $*$-subalgebras of $\mathcal{A}$, having the same unit as $\mathcal{A}$. They are said to be free if for all k, for all $a_i \in \mathcal{A}_{j_i}$ such that $j_1 \neq j_2$, $j_2 \neq j_3$, $\dots$, $j_{k-1} \neq j_k$:*

$$\tau\Big((a_1 - \tau(a_1))(a_2 - \tau(a_2))\dots(a_k - \tau(a_k))\Big) = 0.$$

Families of non-commutative random variable are said to be free if the $*$-subalgebras they generate are free.

- *Let $A = (a_1, \dots, a_k)$ be a k-tuple of non-commutative random variables. The joint distribution of the family A is the linear form $\mu_A : P \mapsto \tau\big[P(A, A^*)\big]$ on the set of polynomials in $2k$ non commutative indeterminates. By convergence in distribution, for a sequence of families of variables*

$(A_N)_{N\geq 1} = (a_1^N, \dots, a_k^N)_{N\geq 1}$ in $\mathcal{C}^*$-algebras $\big(\mathcal{A}_N, ^*, \tau_N, \|.\|\big)$, we mean the pointwise convergence of the map

$$\mu_{A_N} : P \mapsto \tau_N\big[P(A_N, A_N^*)\big],$$

and by strong convergence in distribution, we mean convergence in distribution, and pointwise convergence of the map

$$P \mapsto \big\|P(A_N, A_N^*)\big\|.$$

- *A non commutative random variable u is called a Haar unitary when it is unitary, that is $uu^* = u^*u = 1_{\mathcal{A}}$, and for all $n \in \mathbb{N}$, one has*

$$\tau(u^n) = \begin{cases} 1 & \textit{if } n = 0, \\ 0 & \textit{else.} \end{cases}$$

The strong convergence of non-commutative random variable is actually equivalent to the convergence of its spectrum for the Hausdorff distance. More precisely we have the following proposition whose proof can be found in [12] (see Proposition 2.1):

Proposition 2.1. *Let $\mathbf{x}_N = (x_1^N, \dots, x_p^N)$ and $\mathbf{x} = (x_1, \dots, x_p)$ be p-tuples of variables in $\mathcal{C}^*$-probability spaces, $(\mathcal{A}_N, .^*, \tau_N, \|\cdot\|)$ and $(\mathcal{A}, .^*, \tau, \|\cdot\|)$, with faithful states. Then, the following assertions are equivalent.*

- *$\mathbf{x}_N$ converges strongly in distribution to $\mathbf{x}$.*
- *For any self-adjoint variable $h_N = P(\mathbf{x}_N, \mathbf{x}_N^*)$, where P is a fixed polynomial, μ_{h_N} converges in weak-$*$ topology to μ_h where $h = P(\mathbf{x}, \mathbf{x}^*)$. Weak-$*$ topology means relatively to continuous functions on $\mathbb{C}$. Moreover, the spectrum of h_N converges in Hausdorff distance to the spectrum of h, that is, for any $\varepsilon > 0$, there exists N_0 such that for any $N \geq N_0$,*

$$\sigma(h_N) \ \subset\ \sigma(h) \ + (-\varepsilon, \varepsilon). \tag{3}$$

In particular, the strong convergence in distribution of a single self-adjoint variable is its convergence in distribution together with the Hausdorff convergence of its spectrum.

It is important to note that thanks to Theorem 7.9 from [22], that we recall below, one can consider free copy of any random variable.

Theorem 2.1. *Let $(\mathcal{A}_i, \phi_i)_{i\in I}$ be a family of $\mathcal{C}^*$-probability spaces such that the functionals $\phi_i : \mathcal{A}_i \to \mathbb{C}$, $i \in I$, are faithful traces. Then there exist a $\mathcal{C}^*$-probability space $(\mathcal{A}, \phi)$ with ϕ a faithful trace, and a family of norm-preserving unital $*$-homomorphism $W_i : \mathcal{A}_i \to \mathcal{A}$, $i \in I$, such that:*

- *$\phi \circ W_i = \phi_i$, $\forall i \in I$.*
- *The unital $\mathcal{C}^*$-subalgebras form a free family in $(\mathcal{A}, \phi)$.*

2.2 Non-commutative polynomials and derivatives

We set $\mathbb{C}\langle Y_1, \dots, Y_d\rangle$ the set of non-commutative polynomial in d indeterminates. We will also use $\mathcal{P}_d = \mathbb{C}\langle Y_1, \dots, Y_d, Y_1^*, \dots, Y_d^*\rangle$ the set of non-commutative polynomial in $2d$ indeterminates. We endow this vector space with the norm

$$\|P\|_A = \sum_{M \text{ monomial}} |c_M(P)| A^{\deg M}, \tag{4}$$

where $c_M(P)$ is the coefficient of P for the monomial M. One can define several maps which we use multiple times in the rest of the paper, but first let us set a few notations, for A, B, C non-commutative polynomials,

$$A \otimes B \# C = ACB,$$

$$A \otimes B \widetilde{\#} C = BCA,$$

$$m(A \otimes B) = BA.$$

Definition 2.2. *If $1 \leq i \leq d$, one set $\partial_i : \mathcal{P}_d \longrightarrow \mathcal{P}_d \otimes \mathcal{P}_d$ such that for $P, Q \in \mathcal{P}_d$,*

$$\partial_i(PQ) = \partial_i P \times 1 \otimes Q + P \otimes 1 \times \partial_i Q,$$

$$\partial_i Y_j = \mathbf{1}_{i=j}, \quad \partial_i Y_j^* = 0.$$

We also define $D_i : \mathcal{P}_d \longrightarrow \mathcal{P}_d$ by $D_i P = m \circ \partial_i P$. We similarly define ∂_i^* and D_i^* with the difference that for any j, $\partial_i^* Y_j^* = \mathbf{1}_{i=j}, \quad \partial_i Y_j = 0$.

Because they satisfy the Leibniz's rule, the maps ∂_i and ∂_i^* are called non-commutative derivatives. It is related to Schwinger-Dyson equations on semicircular variable, for more information see [2], Lemma 5.4.7. While we do not use those equations in this paper, we use those associated with Haar unitary matrices. To do so, we define the following non-commutative derivative.

Definition 2.3. *If $1 \leq i \leq d$, one set $\delta_i : \mathcal{P}_d \longrightarrow \mathcal{P}_d \otimes \mathcal{P}_d$ such that for $P, Q \in \mathcal{P}_d$,*

$$\delta_i(PQ) = \delta_i P \times 1 \otimes Q + P \otimes 1 \times \delta_i Q,$$

$$\delta_i Y_j = \mathbf{1}_{i=j} Y_i \otimes 1, \quad \delta_i Y_j^* = -\mathbf{1}_{i=j} 1 \otimes Y_i^*.$$

We also define $\mathcal{D}_i : \mathcal{P}_d \longrightarrow \mathcal{P}_d$ by $\mathcal{D}_i P = m \circ \delta_i P$.

We would like to apply the map δ_i to power series, more precisely the exponential of a polynomial, however since this is not well-defined in all generality we will need a few more definitions. Firstly, we need to define properly the operator norm of tensor of $\mathcal{C}^*$-algebras. Since we use it later in this paper, we work with the minimal tensor product also named the spatial tensor product. For more information we refer to chapter 6 of [21].

Definition 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{C}^*$-algebra with faithful representations $(H_{\mathcal{A}}, \phi_{\mathcal{A}})$ and $(H_{\mathcal{B}}, \phi_{\mathcal{B}})$, then if $\otimes_2$ is the tensor product of Hilbert spaces, $\mathcal{A} \otimes_{\min} \mathcal{B}$ is the completion of the image of $\phi_{\mathcal{A}} \otimes \phi_{\mathcal{B}}$ in $B(H_{\mathcal{A}} \otimes_2 H_{\mathcal{B}})$ for the operator norm in this space. This definition is independent of the representations that we fixed.*

Consequently if $P \in \mathcal{A}_d$, $z = (z_1, \dots, z_d)$ belongs to a $\mathcal{C}^*$-algebra $\mathcal{A}$, then $(\delta_i P^k)(z, z^*)$ belongs to $\mathcal{A} \otimes_{\min} \mathcal{A}$, and $\left\|(\delta_i P^k)(z, z^*)\right\| \leq C_P k \left\|P(z, z^*)\right\|^{k-1}$ for some constant C_P independent of k. Thus we can define

$$(\delta_i e^P)(z, z^*) = \sum_{k \in \mathbb{N}} \frac{1}{k!} (\delta_i P^k)(z, z^*). \tag{5}$$

While we will not always be in this situation during this paper, it is important to note that if $\mathcal{A} = \mathbb{M}_N(\mathbb{C})$, then up to isomorphism $\mathcal{A} \otimes_{\min} \mathcal{A}$ is simply $\mathbb{M}_{N^2}(\mathbb{C})$ with the usual operator norm. Now we prove the following property based on Duhamel formula.

Proposition 2.2. *Let $P \in \mathcal{P}_d$, $z = (z_1, \dots, z_d)$ elements of a $\mathcal{C}^*$-algebra $\mathcal{A}$, then*

$$\left(\delta_i e^P\right)(z, z^*) = \int_0^1 \left(e^{\alpha P} \ \delta_i P \ e^{(1-\alpha)P}\right)(z, z^*) \ d\alpha,$$

with convention

$$A \times (B \otimes C) \times D = (AB) \otimes (CD).$$

Proof. One has,

$$\begin{aligned}
\int_0^1 \left(e^{\alpha P} \ \delta_i P \ e^{(1-\alpha)P}\right)(z, z^*) \ d\alpha &= \sum_{n,m} \int_0^1 \frac{\alpha^n (1-\alpha)^m}{n! m!} d\alpha \ \left(P^n \ \delta_i P \ P^m\right)(z, z^*) \\
&= \sum_k \sum_{n+m=k} \int_0^1 \frac{\alpha^n (1-\alpha)^m}{n! m!} d\alpha \ \left(P^n \ \delta_i P \ P^m\right)(z, z^*).
\end{aligned}$$

But for any n, m,

$$\int_0^1 \frac{\alpha^n(1-\alpha)^m}{n!m!} d\alpha = \int_0^1 \frac{\alpha^{n+m}}{(n+m)!} d\alpha = \frac{1}{(m+n+1)!}.$$

Hence,

$$\int_0^1 \left(e^{\alpha P}\ \delta_i P\ e^{(1-\alpha)P} \right)(z,z^*)\ d\alpha = \sum_k \frac{1}{(k+1)!} \sum_{n+m=k} (P^n\ \delta_i P\ P^m)(z,z^*) = \left(\delta_i\ e^P\right)(z,z^*).$$

□

2.3 Free stochastic calculus

The main idea of this paper is to use an interpolation between Haar unitary matrices and their free limit. In order to do so, we will need some notion of free stochastic calculus. The main reference in this field is the paper [6] of Biane and Speicher to which we refer for most of the proofs in this subsection. That being said, we made the choice to be rigorous, but in the rest of the paper we will not use all of the notations and objects introduced here.

From now on, $(\mathcal{A}, \tau)$ is a W^*-non-commutative probability space, that is $\mathcal{A}$ is a von Neumann algebra, and τ is a faithful normal tracial state on $\mathcal{A}$. We take $\mathcal{A}$ filtered, that is there exists a family $(\mathcal{A}_t)_{t\in\mathbb{R}^+}$ of unital, weakly closed $*$-subalgebras of $\mathcal{A}$,such that $\mathcal{A}_s \subset A_t$ for all $s \leq t$. Besides we also assume that there exist p freely independent $(\mathcal{A}_t)_{t\in\mathbb{R}^+}$-free Brownian motions $(S_t)_{t\in\mathbb{R}^+}$. That is S_t^i is a self-adjoint element of $\mathcal{A}_t$ with semi-circular distribution of mean 0 and variance t, and for all $s \leq t$, $S_t^i - S_s^i$ is free with $\mathcal{A}_s$, and has semi-circular distribution of mean 0 and variance $t-s$. Besides since the state τ is tracial, for any unital, weakly closed $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$, there exists a unique conditional expectation onto $\mathcal{B}$. We shall denote it by $\tau(.|\mathcal{B})$. A map $t \in \mathbb{R}^+ \mapsto M_t \in \mathcal{A}$ will be called a martingale with respect to the filtration $(\mathcal{A}_t)_{t\in\mathbb{R}^+}$ if for every $s \leq t$ one has $\tau(M_t|\mathcal{A}_s) = M_s$.

We define the opposite algebra $\mathcal{A}^{\mathrm{op}}$ as the algebra $\mathcal{A}$ endowed of the same addition, norm and involution, but with the product $a \times b = b \cdot a$ where $\cdot$ is the product in $\mathcal{A}$. We can endow $\mathcal{A}^{\mathrm{op}}$ with a faithful normal tracial state τ^{op}, which if we view as a linear map on $\mathcal{A}$ is actually τ. Similarly to the minimal tensor product, we will denote $L^\infty(\tau \otimes \tau^{\mathrm{op}})$ the von Neuman algebra generated by $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ in $B(L^2(\mathcal{A}, \tau) \otimes_2 L^2(\mathcal{A}^{\mathrm{op}}, \tau^{\mathrm{op}}))$ where $\otimes_2$ is the usual tensor product for Hilbert spaces. Similarly to classical stochastic calculus, we now introduce piecewise constant maps.

Definition 2.5. *A simple biprocess is a piecewise constant map $t \mapsto U_t$ from $\mathbb{R}^+$ into the algebraic tensor product $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$, such that $U_t = 0$ for t large enough. Besides it is called adapted if for any $t \geq 0$, $U_t \in \mathcal{A}_t \otimes \mathcal{A}_t$.*

The space of simple biprocesses form a complex vector space that we can endow with the norm

$$\|U\|_{\mathcal{B}^\infty}^2 = \int_0^\infty \|U_s\|_{L^\infty(\tau\otimes\tau^{op})}^2\, ds. \tag{6}$$

We will denote by $\mathcal{B}_a^\infty$ the completion of the vector space of adapted simple biprocesses for this norm. Now that we have defined the notion of simple process, we can define its stochastic integral that we will later extend to the space $\mathcal{B}_a^\infty$.

Definition 2.6. *Let $(S_t)_{t\geq 0}$ be a free Brownian motion, U be a simple adapted biprocess, we can find a decomposition $U = \sum_{j=1}^n A^j \otimes B^j$ and $0 = t_0 \leq t_1 \leq \cdots \leq t_m$ such that for $t \in [t_i, t_{i+1})$, $A_t^j = A_{t_i}^j \in \mathcal{A}_{t_i}$ and $B_t^j = B_{t_i}^j \in \mathcal{A}_{t_i}^{\mathrm{op}}$. We define its stochastic integral by*

$$\int_0^\infty U_s \# dS_s = \sum_{i=0}^{m-1} U_{t_i} \#(S_{t_{i+1}} - S_{t_i}) = \sum_{j=1}^n \sum_{i=0}^{m-1} A_{t_i}^j (S_{t_{i+1}} - S_{t_i}) B_{t_i}^j.$$

This definition is independent of the decomposition chosen. Besides $t \mapsto \int_0^t U_s \# dX_s$ is a martingale.

Thanks to Burkholder-Gundy inequality, that is Theorem 3.2.1 of [6], if we see the stochastic integral as a linear map from the space of adapted simple biprocesses endowed with the norm $\|.\|_{\mathcal{B}^\infty}$ to $\mathcal{A}$, then this map is continuous. Hence we can extend it to $\mathcal{B}_a^\infty$ and the martingale property remains true. Before talking about Itô's formula, as in the classical case, we need to introduce the quadratic variation. We will not develop the idea, but by studying random matrices, in the case of simple tensors, we are prompted to define

$$\langle\langle a\otimes b, c\otimes d\rangle\rangle = a\ \tau(bc)\ d.$$

We call $\sharp$ the product law in $\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}$. If by contrast we want to use the usual product in $\mathcal{A}\otimes\mathcal{A}$, we will not put any sign. Let $\dagger$ be the linear application such that on simple tensors, $(a\otimes b)^\dagger = b\otimes a$. In all generality for any $Z,Y\in\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}$,

$$\langle\langle Z,Y\rangle\rangle = (\mathbf{1}_{\mathcal{A}}\otimes\tau^{\mathrm{op}})\left(Z\sharp(Y^\dagger)\right).$$

Since $\|\langle\langle Z,Y\rangle\rangle\| \leq \|Z\|_{L^\infty(\tau\otimes\tau^{\mathrm{op}})}\|Y\|_{L^\infty(\tau\otimes\tau^{\mathrm{op}})}$, we can extend this bilinear application to $Z,Y\in L^\infty(\tau\otimes\tau^{\mathrm{op}})$. Besides by Cauchy-Schwarz, for $U,V\in\mathcal{B}_a^\infty$, $\langle\langle U,V\rangle\rangle$ is integrable.

Now that we have defined all of the necessary object to do stochastic calculus, we can state Itô's formula. We will need to handle polynomials in several processes, however Biane and Speicher only stated Itô's formula for a product of two processes, that is if $X_0,Y_0\in\mathcal{A}$, $U^i,V^i\in\mathcal{B}_a^\infty$ and $K,L\in L^1(R^+,\mathcal{A})$, we set

$$Y_t = Y_0 + \int_0^t K_s ds + \sum_i \int_0^t U_s^i \# dS_s^i,$$

$$Z_t = Z_0 + \int_0^t L_s ds + \sum_i \int_0^t V_s^i \# dS_s^i,$$

then for any $t\geq 0$,

$$Y_tZ_t = Y_0Z_0 + \int_0^t \left(Y_sL_s + K_sZ_s + \sum_i \langle\langle U_s^i, V_s^i\rangle\rangle\right)\ ds + \sum_i\int_0^t \left((Y_s\otimes 1_{\mathcal{A}})V_s^i + U_s^i(1_{\mathcal{A}}\otimes Z_s)\right)\# dS_s^i. \tag{7}$$

To find a proof of this formula, see Theorem 4.1.2 in [6]. While this theorem only proves the case where $L=K=0$ and we only have a single Brownian motion, deducing equation (7) does not need much more work. Finally this formula let us prove the general Itô's formula. Even though this formula is used without a proof by Dabrowski in [13], we do not know of any satisfying reference. Hence we include a proof for self-containedness. Let us first fix a few notations.

- If $P\in\mathbb{C}\langle X_1,\dots,X_d\rangle$, $X\in(L^\infty(\mathbb{R}^+,\mathcal{A}))^d$ and $K\in(L^1(\mathbb{R}^+,\mathcal{A}))^d$, then $\partial P(X)\#K = \sum_i \partial_i P(X)\# K_i$.
- Similarly if $U\in(\mathcal{B}_a^\infty)^d$, then $\partial P(X)\sharp U = \sum_i \partial_i P(X)\sharp U_i$.
- Finally if $U,V\in\mathcal{B}_a^\infty$, $A,B,C\in L^\infty(\mathbb{R}^+,\mathcal{A})$, then $(A\otimes B\otimes C)\#\ (U,V) = ((A\otimes B)\sharp U, (1\otimes C)\sharp V)$.

Theorem 2.2. *Let $X_0\in\mathcal{A}^d$, P be a non-commutative polynomial in d indeterminates, for any $t\geq 0$, $K\in(L^1([0,t],\mathcal{A}))^d$ and $(\mathbf{1}_{s\leq t}U_s^i)_{s\in\mathbb{R}^+}\in(\mathcal{B}_a^\infty)^d$. With I the application identity on $\mathcal{P}_d$, we define*

$$X_t = X_0 + \int_0^t K_s ds + \sum_i \int_0^t U_s^i\# dS_s^i,$$

$$\Delta_U(P)(X) = \sum_i\sum_{j,k}\langle\langle\ (\partial_j\otimes I)\circ\partial_k P(X)\#(U^{i,j},U^{i,k})\ \ \rangle\rangle.$$

Then for any $t\geq 0$, $\partial P(X)\#K$ and $\Delta_U(P)(X)\in L^1([0,t],\mathcal{A}))$, and $(\mathbf{1}_{s\leq t}\partial P(X_s)\sharp U_s)_{s\in\mathbb{R}^+}\in\mathcal{B}_a^\infty$. Finally for any $t\geq 0$,

$$P(X_t) = P(X_0) + \int_0^t \partial P(X_s)\# K_s\ ds + \sum_i\int_0^t \partial P(X)\sharp U_s^i\ \# dS_s^i + \int_0^t \Delta_U(P)(X_s)\ ds.$$

Proof. Thanks to Burkholder-Gundy inequality, that is Theorem 3.2.1 of [6], we know that

$$\sup_{0\leq s\leq t}\left\|X_s^i\right\| \leq \left\|X_0^i\right\| + \left\|K_s^i\right\|_{L^1([0,t],\mathcal{A})} + \sum_j \left\|U^{i,j}\mathbf{1}_{[0,t]}\right\|_{\mathcal{B}_a^\infty}.$$

Thus for any $t\in\mathbb{R}^+$, $(X_s)_{s\in[0,t]}\in L^\infty([0,t],\mathcal{A})^d$, thus for any polynomial P, $\partial P(X)\#K\in L^\infty([0,t],\mathcal{A})$, and $(\mathbf{1}_{s\leq t}\partial P(X_s)\sharp U_s)_{s\in\mathbb{R}^+}\in\mathcal{B}_a^\infty$. With the help of inequality $\|\langle\langle Z,Y\rangle\rangle\|\leq\|Z\|_{L^\infty(\tau\otimes\tau^{\mathrm{op}})}\|Y\|_{L^\infty(\tau\otimes\tau^{\mathrm{op}})}$, we also have that $\Delta_U(P)(X)\in L^1([0,t],\mathcal{A}))$. Finally to prove the formula, we proceed recurrently. If P is of degree 1, there is nothing to prove. For larger degree, by linearity we only need to deal with the case where P is a monomial. Thus we can write $P=QR$ with Q and R monomials of smaller degree for which the formula is verified. Thus thanks to equation (7), we have that

$$\begin{aligned}
P(X_t) &= Q(X_0)R(X_0) + \int_0^t Q(X_s)\ \partial R(X_s)\#K_s + \partial Q(X_s)\#K_s\ R(X_s)ds \\
&= \int_0^t Q(X_s)\ \Delta_U(R)(X_s)\ + \Delta_U(Q)(X_s)\ R(X_s)\ + \sum_i \langle\langle\ \partial Q(X_s)\sharp U_s^i, \partial R(X_s)\sharp U_s^i\ \rangle\rangle\ ds \\
&= \sum_i \int_0^t (Q(X_s)\otimes\mathbf{1}_{\mathcal{A}})\ \partial R(X_s)\sharp U_s^i + \partial Q(X_s)\sharp U_s^i\ \mathbf{1}_{\mathcal{A}}\otimes R(X_s)\ \#dS_s^i.
\end{aligned}$$

It is clear that,

$$\begin{aligned}
\partial(QR)(X_s)\#K_s &= Q(X_s)\ \partial R(X_s)\#K_s + \partial Q(X_s)\#K_s\ R(X_s), \\
\partial(QR)(X_s)\sharp U_s^i &= (Q(X_s)\otimes I)\ \partial R(X_s)\sharp U_s^i + \partial Q(X_s)\sharp U_s^i\ \mathbf{1}_{\mathcal{A}}\otimes R(X_s).
\end{aligned}$$

And finally,

$$\begin{aligned}
\Delta_U(QR)(X) =& \sum_i\sum_{j,k}\langle\langle\ (\partial_j\otimes I)\circ\partial_k(QR)(X)\#(U^{i,j},U^{i,k})\ \ \rangle\rangle \\
=& \sum_i\sum_{j,k}\sum_{Q=AX_jBX_kC}\langle\langle\ A(X)\otimes B(X)\otimes C(X)\#(U^{i,j},U^{i,k})\ \ \rangle\rangle R(X) \\
&+ \sum_{R=AX_jBX_kC} Q(X)\langle\langle\ A(X)\otimes B(X)\otimes C(X)\#(U^{i,j},U^{i,k})\ \ \rangle\rangle \\
&+ \sum_{Q=AX_jB,R=CX_kD}\langle\langle\ A(X)\otimes(BC)(X)\otimes D(X)\#(U^{i,j},U^{i,k})\ \ \rangle\rangle \\
=& Q(X)\ \Delta_U(R)(X)\ + \Delta_U(Q)(X)\ R(X)\ + \sum_i\langle\langle\ \partial Q(X)\sharp U^i, \partial R(X)\sharp U^i\ \rangle\rangle.
\end{aligned}$$

□

Finally, one of the fundamental tool that we use in this paper is the free unitary Brownian motion, a good reference on the matter is [4]. In particular one can find a proof of its existence.

Definition 2.7. *Let $(S_t)_{t\geq 0}$ be a free Brownian motion, the free unitary Brownian motion $(u_t)_{t\geq 0}$ is the unique solution to the equation*

$$\forall t\geq 0,\quad u_t = 1_{\mathcal{A}} - \int_0^t \frac{u_s}{2}\ ds + \mathbf{i}\int_0^t u_s\otimes 1_{\mathcal{A}}\#dS_s.$$

In particular, for any $t\geq 0$, u_t is unitary, that is $u_tu_t^* = u_t^*u_t = 1_{\mathcal{A}}$.

2.4 Notations

Let us now fix a few notations concerning the spaces and traces that we use in this paper.

Definition 2.8. • $(\mathcal{A}_N, \tau_N)$ *is the free sum of* $\mathbb{M}_N(\mathbb{C})$ *with the von Neuman algebra* $\mathcal{A}$ *from the former subsection. To build* $\mathcal{A}_N$ *we use Theorem 2.1 and we get a* $\mathcal{C}^*$*-probability space* C *with a faithful trace* φ*. Since we want* $(\mathcal{A}_N, \tau_N)$ *to be a von Neuman algebra, we set* $L^2(C, \varphi)$ *as the completion of* C *for the norm* $a \mapsto \phi(a^*a)^{1/2}$*, we have an injective* $\mathcal{C}^*$*-algebra morphism from* C *to* $B(L^2(C,\varphi))$*. We then proceed to take* $\mathcal{A}_N$ *the closure of the image of* C *in this space for the weak topology. As for* τ_N*, since we can extend* $(x, y) \mapsto \varphi(x^*y)$ *to a scalar product* $\langle ., . \rangle_\varphi$ *on* $L^2(C, \varphi)$*, we set for* $a \in B(L^2(C, \varphi))$*,* $\tau_N(a) = \langle a(1), 1 \rangle_\varphi$*.*

- *Note that when restricted to* $\mathbb{M}_N(\mathbb{C})$*,* τ_N *is just the regular renormalized trace on matrices. As in the former subsection, the restriction of* τ_N *to the* $\mathcal{C}^*$*-algebra* $\mathcal{A}$ *is denoted as* τ*.*
- Tr_N *is the non-renormalized trace on* $\mathbb{M}_N(\mathbb{C})$*.*
- $E_{i,j}$ *is the matrix whose only non-zero coefficient is* (i, j) *and this coefficient has value* 1*, the size of the matrix* $E_{i,j}$ *will depend on the context.*
- *In general we identify* $\mathbb{M}_N(\mathbb{C}) \otimes \mathbb{M}_k(\mathbb{C})$ *with* $\mathbb{M}_{kN}(\mathbb{C})$ *through the isomorphism* $E_{i,j} \otimes E_{r,s} \mapsto E_{i+rN,j+sN}$*, similarly we identify* $\mathrm{Tr}_N \otimes \mathrm{Tr}_k$ *with* Tr_{kN}*.*
- *If* $A^N = (A_1^N, \dots, A_d^N)$ *and* $B^M = (B_1^M, \dots, B_d^M)$ *are two vectors of matrices, then we denote* $A^N \otimes B^M = (A_1^N \otimes B_1^M, \dots, A_d^N \otimes B_d^M)$ *and if* $M = N$*,* $A^N B^N = (A_1^N B_1^N, \dots, A_d^N B_d^N)$*. We typically use the notation* $X^N \otimes I_M$ *for the vector* $(X_1^N \otimes I_M, \dots, X_1^N \otimes I_M)$*.*
- *If* $P \in \mathcal{P}_d$*, in order to avoid cumbersome notations when evaluating* P *in* (X, X^*)*, instead of denoting* $P(X, X^*)$ *we will write* $\widetilde{P}(X)$*.*
- *We define* $(e_i)_{i\in[1,M]}$*,* $(g_i)_{i\in[1,N]}$ *and* $(f_i)_{i\in[1,k]}$ *the canonical basis of* $\mathbb{C}^M$*,* $\mathbb{C}^N$ *and* $\mathbb{C}^k$

We define an involution $*$ on $\mathcal{P}_d$ such that $(Y_i)^* = Y_i^*$, $(Y_i^*)^* = Y_i$ and we extend it to $\mathcal{P}_d$ with the formula $(\alpha PQ)^* = \overline{\alpha} Q^* P^*$. $P \in \mathcal{P}_d$ is said to be self-adjoint if $P^* = P$. Self-adjoint polynomials have the property that if $z_1, \dots, z_d$ are elements of a $\mathcal{C}^*$-algebra, then $P(z_1, \dots, z_d, z_1^*, \dots, z_d^*)$ is self-adjoint. Now that we have defined the notion of self-adjoint polynomial we give a property which justifies computations that we will do later on:

Proposition 2.3. *Let the following objects be given,*

- $u = (u_t^1, \dots, u_t^p)_{t\geq 0}$ *a family of* p *free unitary Brownian motions,*
- $U^N = (U_1^N, \dots, U_p^N)$ *matrices of size* N*,*
- $u_t^N = (U_1^N u_t^1, \dots, U_p^N u_t^p)$ *elements of* $\mathcal{A}_N$*,*
- Z^{NM} *matrices in* $\mathbb{M}_N(\mathbb{C}) \otimes \mathbb{M}_M(\mathbb{C})$*,*
- $f \in \mathcal{C}^0(\mathbb{R})$*,*
- P *a self-adjoint polynomial.*

Then this map is measurable:

$$(U^N, Z^{NM}) \mapsto \tau_N \otimes \tau_M \left(f \left(\widetilde{P} \left(u_t^N \otimes I_M, Z^{NM} \right) \right) \right).$$

For a full proof we refer to [11], Proposition 2.7. But in a few words, it is easy to see the measurability when f is a polynomial since then this map is also polynomial in the coefficient of U^N and Z^{NM}, and we conclude by density. Actually we could easily prove that this map is continuous, however we do not need it. The only reason we need this property is to justify that if U^N is a vector of random matrices, then the random variable $\tau_N \otimes \tau_M \left(f \left(\widetilde{P}(u_t^N \otimes I_M, Z^{NM}) \right) \right)$ is well-defined. To conclude this subsection we introduce different notations related to application defined on tensor spaces.

Definition 2.9. *Let* $n : A \otimes B \in \mathbb{M}_M(\mathbb{C}))^{\otimes 2} \mapsto AB \in \mathbb{M}_M(\mathbb{C}))$*, we define the linear application* $(\tau_N \otimes I_M) \bigotimes (\tau_N \otimes I_M) : (\mathcal{A}_N \otimes \mathbb{M}_M(\mathbb{C}))^{\otimes 2} \to M_M(\mathbb{C})$ *as*

$$(\tau_N \otimes I_M) \bigotimes (\tau_N \otimes I_M) = n \circ (\tau_N \otimes I_M)^{\otimes 2}.$$

We will also use the shorter notation $(\tau_N \otimes I_M)^{\bigotimes 2}$.

The following notation has similarities with the previous one, but its interest will be clear in section 4. For the reader familiar with it, this is close to Sweedler's convention.

Definition 2.10. *Let* $P \in \mathcal{P}_d$*,* $\mathcal{C}$ *be a* $\mathcal{C}^*$*-algebra. Then let* $\alpha : \mathcal{P}_d \to \mathcal{C}$ *and* $\beta : \mathcal{P}_d \to \mathcal{C}$ *be morphisms. We also set* $n : A \otimes B \in \mathcal{C} \otimes \mathcal{C} \mapsto AB \in \mathcal{C}$*. Then we use the following notation,*

$$\alpha(\delta_i^1 P) \boxtimes \beta(\delta_i^2 P) = n \circ (\alpha \otimes \beta(\delta_i P)).$$

This notation will be especially useful when our applications α and β are simply evaluation of P as it is the case in section 4. Indeed we will typically denote, $\delta_i^1 P(X) \boxtimes \delta_i^2 P(Y)$, rather than define $h_X : P \to P(X)$ and use the more cumbersome and abstract notation, $n \circ (h_X \otimes h_Y(\delta_i P))$.

2.5 Random matrix model

We conclude this section by giving the definition and a few properties on the models of random matrices that we will study.

Definition 2.11. *A Haar unitary matrix of size* N *is a random matrix distributed according to the Haar measure on the group of unitary matrices of size* N*.*

Definition 2.12. *A hermitian Brownian motion* $(X_t^N)_{t\in\mathbb{R}^+}$ *of size* N *is a self-adjoint matrix whose coefficients are random variables with the following laws:*

- *For* $1 \le i \le N$*, the random variables* $\sqrt{N}((X_t^N)_{i,i})_{t\in\mathbb{R}^+}$ *are independent Brownian motions.*
- *For* $1 \le i < j \le N$*, the random variables* $(\sqrt{2N}\ \Re(X_t^N)_{i,j})_{t\in\mathbb{R}^+}$ *and* $(\sqrt{2N}\ \Im(X_t^N)_{i,j})_{t\in\mathbb{R}^+}$ *are independent Brownian motions, independent of* $\sqrt{N}((X_t^N)_{i,i})_{t\in\mathbb{R}^+}$*.*

To study the free unitary Brownian motion, we will need to study its finite dimensional version. There are several ways to define it, but in this paper the handiest way to define it is through a stochastic differential equation.

Definition 2.13. *Let* X^N *be a hermitian Brownian motion, then* U^N *is said to be a unitary Brownian motion if it is the solution of the stochastic differential equation,*

$$dU_t^N = \mathbf{i} U_t^N dX_t^N - \frac{1}{2} U_t^N\ dt, \quad U_0^N = I_N, \tag{8}$$

where we use the convention $(U_t^N dX_t^N)_{i,j} = \sum_k (U_t^N)_{i,k} d(X_t^N)_{k,j}$*.*

The following property is typical of the kind of computation that we can do with unitary Brownian motion with classical stochastic calculus, see [10] for exemple.

Proposition 2.4. *Let* $U_1^N, \dots, U_p^N$ *be unitary Brownian motions of size* N*,* $A_{p+1}^N, \dots, A_d^N$ *be deterministic matrices,* $Q \in \mathcal{P}_d$ *be a monomial, we set* Q_s *the monomial evaluated in* $(U_{1,s}^N, \dots, U_{p,s}^N, A_{p+1}^N, \dots, A_d^N)$ *and their adjoints,* $|Q|_B$ *the degree of* Q *with respect to* $(U_1, \dots, U_p, U_1^*, \dots, U_p^*)$*. Then there exists a martingale* J *such that,*

$$
\begin{aligned}
d\operatorname{Tr}_N Q_s = dJ - \frac{|Q|_B}{2}\operatorname{Tr}_N Q_s &- \frac{1}{N}\sum_{i\le p,\ Q=AU_iBU_iC}\operatorname{Tr}_N(A_sU_{i,s}^NC_s)\operatorname{Tr}_N(B_sU_{i,s}^N)\\
&- \frac{1}{N}\sum_{i\le p,\ Q=AU_i^*BU_i^*C}\operatorname{Tr}_N\left(A_s{U_{i,s}^N}^*C_s\right)\operatorname{Tr}_N\left(B_s{U_{i,s}^N}^*\right)\\
&+ \frac{1}{N}\sum_{i\le p,\ Q=AU_iBU_i^*C}\operatorname{Tr}(A_sC_s)\operatorname{Tr}(B_s)\\
&+ \frac{1}{N}\sum_{i\le p,\ Q=AU_i^*BU_iC}\operatorname{Tr}(A_sC_s)\operatorname{Tr}(B_s).
\end{aligned}
$$

3 Preliminaries

3.1 A matricial inequality

We are indebted to Mikael de la Salle for supplying us with the proof of the following lemma.

Lemma 3.1 (de la Salle)**.** *Let $\mathcal{A}$ be a $\mathcal{C}^*$-algebra, $A_1, A_2 \in \mathcal{A}$, $B_1, B_2 \in \mathbb{M}_M(\mathbb{C})$, as in subsection 2.3 we define $(A_1 \otimes B_1)\sharp(A_2 \otimes B_2) = (A_1A_2) \otimes (B_2B_1)$. Then if $x, y \in \mathcal{A} \otimes \mathbb{M}_M(\mathbb{C})$, with the operator norm in their respective space,*

$$
\|x\sharp y\| \le M\,\|x\|\,\|y\|\,.
$$

Proof. We write $x = \sum_{1\le i,j\le M} x_{i,j} \otimes E_{i,j}$, $y = \sum_{1\le k,l\le M} y_{k,l} \otimes E_{k,l}$, then

$$
x\sharp y = \sum_{i,j,k} x_{k,j}y_{i,k} \otimes E_{i,j}.
$$

We define $A_k = \sum_{i,j} x_{k,j}y_{i,k} \otimes E_{i,j}$, $X_k = \sum_j x_{k,j} \otimes E_{k,j} \otimes I_M$, $Y_k = \sum_i y_{i,k} \otimes I_M \otimes E_{i,k}$. Then by using the fact that X_k and Y_k are band matrices, we have $\|X_k\| \le \|x\|$ and $\|Y_k\| \le \|y\|$. Besides $\|x\sharp y\| \le \sum_{1\le k\le M}\|A_k\|$. Finally we have for any k,

$$
\begin{aligned}
\|X_kY_k\|^2 &= \|X_kY_k(X_kY_k)^*\|\\
&= \left\|\left(\sum_{i,j} x_{k,j}y_{i,k}\otimes E_{k,j}\otimes E_{i,k}\right)\left(\sum_{i,j} x_{k,j}y_{i,k}\otimes E_{k,j}\otimes E_{i,k}\right)^*\right\|\\
&= \left\|\sum_{i,j,u,v} x_{k,j}y_{i,k}y_{u,k}^*x_{k,v}^*\otimes E_{k,j}E_{v,k}\otimes E_{i,k}E_{k,u}\right\|\\
&= \left\|\sum_{i,j,u} x_{k,j}y_{i,k}y_{u,k}^*x_{k,v}^*\otimes E_{k,k}\otimes E_{i,j}E_{v,u}\right\|\\
&= \|A_kA_k^*\otimes E_{k,k}\|\\
&= \|A_k\|^2\,.
\end{aligned}
$$

Thus $\|x\sharp y\| \le \sum_{1\le k\le M}\|A_k\| \le M\,\|x\|\,\|y\|$.

□

3.2 A Poincaré type equality

One of the main tool when dealing with GUE random matrices is the Poincaré inequality (see Definition 4.4.2 from [2]), which gives us a sharp majoration of the variance of a function in these matrices. Typically this inequality shows that the variance of the renormalized trace of a polynomial in GUE random matrices, which a priori is of order $\mathcal{O}(1)$, is of order $\mathcal{O}(N^{-2})$. In this paper we need a similar type of inequality but instead of working with independent GUE random matrices, we work with marginal of independent unitary Brownian motions at times t. We will follow the approach of [17], Proposition 6.1. We cannot however reuse directly their result since they only consider a single brownian motion and they prove an inequality instead of an equality.

Proposition 3.1. *Let $Q \in \mathcal{P}_d$, $(U_t^N)_{t\in\mathbb{R}^+}$, $(V_t^N)_{t\in\mathbb{R}^+}$, $(W_t^N)_{t\in\mathbb{R}^+}$ be independent vectors of p unitary Brownian motions of size N. Let A^N be a vector of deterministic matrices, with notations as in Definition 2.8, one has for any $T \geq 0$,*

$$\mathrm{Var}\Big(\,\mathrm{Tr}_N\left(\widetilde{Q}(U_T^N, A^N)\right)\Big) = \frac{1}{N}\sum_{k\leq p}\int_0^T \mathbb{E}\Big[\,\mathrm{Tr}_N\left(\widetilde{\mathcal{D}_k Q}(V_{T-t}^N U_t^N, A^N) \times \widetilde{\mathcal{D}_k Q}(W_{T-t}^N U_t^N, A^N)^*\right)\Big]dt.$$

Proof. To simplify notations, we will not write the index N in U_t^N, V_t^N, W_t^N and A^N. For $U \in \mathbb{M}_N(\mathbb{C})^p$, we set $f : (U, U^*) \mapsto \mathrm{Tr}_N(Q(U, A, U^*, A^*))$. We can view f as a polynomial in the coefficient of the matrices U and their conjuguate, since those are complex variables we use the notion of complex differential. That is if $g : (x, y) \in \mathbb{R}^2 \to g(x, y) \in \mathbb{C}$ is a differentiable function, we define $\partial_z g = \frac{1}{2}(\partial_x g - \mathbf{i}\partial_y g)$ and $\partial_{\overline{z}} g = \frac{1}{2}(\partial_x g + \mathbf{i}\partial_y g)$. If $u_k^{i,j}$ is the (i,j)-coefficient of the k-th matrix in U, we denote the differential of f with respect to $u_k^{i,j}$ by $\partial_{u_k^{i,j}} f$, and the differential of f with respect to the conjuguate of this coefficient by $\partial^*_{u_k^{j,i}} f$. In particular,

$$\begin{aligned}
&\partial_{u_k^{i,j}}((U_k)_{i,j}) = 1, \quad \partial^*_{u_k^{i,j}}((U_k^*)_{i,j}) = 1,\\
&\partial_{u_k^{i,j}}((U_k)_{a,b}) = 0, \quad \partial^*_{u_k^{i,j}}((U_k^*)_{a,b}) = 0, \text{ for all } (a,b) \neq (i,j),\\
&\partial_{u_k^{i,j}}((U_k^*)_{a,b}) = 0, \quad \partial^*_{u_k^{i,j}}((U_k)_{a,b}) = 0, \text{ for any } (a,b).
\end{aligned}$$

Next we introduce

$$M_t = P_{T-t} f(U_t, U_t^*),$$

where $P_{T-t} f(U, U^*) = \mathbb{E}_V[f(V_{T-t}U, (V_{T-t}U)^*)]$ with $(V_t)_{t\geq 0}$, p independent unitary Brownian motions of size N and E_V the expectation with respect to $(V_t)_{t\geq 0}$. We will follow the approach of [17], Proposition 6.1, and show that $(M_t)_{0\leq t\leq T}$ is a martingale. It will follow that,

$$\begin{aligned}
\mathrm{Var}\Big(\,\mathrm{Tr}_N\left(\widetilde{Q}(U_T^N, A^N)\right)\Big) &= \mathbb{E}[|f(U_T, U_T^*)|^2 - |\mathbb{E}[f(U_T, U_T^*)]|^2]\\
&= \mathbb{E}[M_T\overline{M_T} - M_0\overline{M_0}]\\
&= \mathbb{E}\left[\langle M_T, \overline{M_T}\rangle\right].
\end{aligned} \tag{9}$$

If we set $(X_t)_{t\geq 0}$, d independent hermitian Brownian motions of size N, and $f_t = P_{T-t} f$, then

$$\begin{aligned}
dM_t = (\partial_t f_t)(U_t, U_t^*)dt &+ \sum_{i,j,k}(\partial_{u_k^{i,j}} f_t)(U_t, U_t^*)\ d(U_{k,t})_{i,j} + (\partial^*_{u_k^{i,j}} f_t)(U_t, U_t^*)\ d(U_{k,t}^*)_{i,j}\\
&+ \frac{1}{2}\sum_{\substack{k,\ i,j,r,s\\ \varepsilon_1,\varepsilon_2\in\{1,*\}}} (\partial^{\varepsilon_1}_{u_k^{i,j}}\partial^{\varepsilon_2}_{u_k^{r,s}} f_t)(U_t, U_t^*)\ d\langle (U_{k,t}^{\varepsilon_1})_{i,j}, (U_{k,t}^{\varepsilon_2})_{r,s}\rangle_t.
\end{aligned}$$

By using equation (8), we can isolate the martingale term in the previous equation. We get that in order to show that $(M_t)_{0\leq t\leq T}$ is a martingale we have to show that if

$$
\begin{aligned}
\Lambda_t = \frac{1}{2} \sum_{i,j,k} & (\partial_{u_k^{i,j}} f_t)(U_t, U_t^*) \ (U_{k,t})_{i,j} + (\partial^*_{u_k^{i,j}} f_t)(U_t, U_t^*) \ (U^*_{k,t})_{i,j} \\
& + \frac{1}{2N} \sum_{k,\ i,j,r,s} (\partial_{u_k^{i,j}} \partial_{u_k^{r,s}} f_t)(U_t, U_t^*) \ (U_{k,t})_{i,s} (U_{k,t})_{r,j} \\
& + \frac{1}{2N} \sum_{k,\ i,j,r,s} (\partial^*_{u_k^{i,j}} \partial^*_{u_k^{r,s}} f_t)(U_t, U_t^*) \ (U^*_{k,t})_{i,s} (U^*_{k,t})_{r,j} \\
& - \frac{1}{N} \sum_{k,\ i,j} (\partial^*_{u_k^{i,j}} \partial_{u_k^{j,i}} f_t)(U_t, U_t^*),
\end{aligned}
$$

then $\Lambda_t = (\partial_t f_t)(U_t, U_t^*)$. We will use the fact that if $g_t : (V, V^*) \mapsto f(VU_t, U_t^* V^*)$,

$$
\begin{aligned}
\partial_{u_k^{i,j}} f_t(U_t, U_t^*) &= \mathbb{E}_V \left[\sum_q (V_{k,T-t})_{q,i} \ \partial_{u_k^{q,j}} f(V_{T-t} U_t, U_t^* V^*_{T-t}) \right], \\
\partial^*_{u_k^{i,j}} f_t(U_t, U_t^*) &= \mathbb{E}_V \left[\sum_q (V^*_{k,T-t})_{j,q} \ \partial^*_{u_k^{i,q}} f(V_{T-t} U_t, U_t^* V^*_{T-t}) \right], \\
\partial_{v_k^{a,b}} g_t(V_{T-t}, V^*_{T-t}) &= \sum_q (U_{k,t})_{b,q} \ \partial_{u_k^{a,q}} f(V_{T-t} U_t, U_t^* V^*_{T-t}), \\
\partial^*_{v_k^{a,b}} g_t(V_{T-t}, V^*_{T-t}) &= \sum_q (U^*_{k,t})_{q,a} \ \partial^*_{u_k^{q,b}} f(V_{T-t} U_t, U_t^* V^*_{T-t}).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\sum_{i,j} (\partial_{u_k^{i,j}} f_t)(U_t, U_t^*) \ (U_{k,t})_{i,j} &= \mathbb{E}_V \left[\sum_{i,j,q} (V_{k,T-t})_{q,i} \ \partial_{u_k^{q,j}} f(V_{T-t} U_t, U_t^* V^*_{T-t}) \ (U_{k,t})_{i,j} \right] \\
&= \mathbb{E}_V \left[\sum_{i,q} \left(\sum_j \partial_{u_k^{q,j}} f(V_{T-t} U_t, U_t^* V^*_{T-t}) \ (U_{k,t})_{i,j} \right) (V_{k,T-t})_{q,i} \right] \\
&= \mathbb{E}_V \left[\sum_{i,q} \partial_{v_k^{q,i}} g_t(V_{T-t}, V^*_{T-t}) \ (V_{k,T-t})_{q,i} \right].
\end{aligned}
$$

We also have,

$$
\begin{aligned}
\sum_{i,j} (\partial^*_{u_k^{i,j}} \partial_{u_k^{j,i}} f_t)(U_t, U_t^*) &= \sum_{i,j,s} (\partial^*_{u_k^{i,j}} \partial_{u_k^{j,s}} f_t)(U_t, U_t^*) \ \mathbf{1}_{i=s} \\
&= \sum_{i,j,s} (\partial^*_{u_k^{i,j}} \partial_{u_k^{j,s}} f_t)(U_t, U_t^*) \ (U_k^* U_k)_{i,s} \\
&= \sum_{i,j,s,q,a,b} \mathbb{E}_V \left[(\partial^*_{u_k^{i,a}} \partial_{u_k^{b,s}} f_t)(U_t, U_t^*) \ (V^*_{k,T-t})_{j,a} (V_{k,T-t})_{b,j} (U^*_{k,t})_{i,q} (U_{k,t})_{q,s} \right] \\
&= \sum_{j,q,a,b} \mathbb{E}_V \left[\left(\sum_{i,s} (\partial^*_{u_k^{i,a}} \partial_{u_k^{b,s}} f_t)(U_t, U_t^*) \ (U^*_{k,t})_{i,q} (U_{k,t})_{q,s} \right) (V^*_{k,T-t})_{j,a} (V_{k,T-t})_{b,j} \right] \\
&= \sum_{q,a,b} \mathbb{E}_V \left[(\partial^*_{v_k^{q,a}} \partial_{v_k^{b,q}} g_t)(V_{T-t}, V^*_{T-t}) \ \sum_j (V_{k,T-t})_{b,j} (V^*_{k,T-t})_{j,a} \right] \\
&= \mathbb{E}_V \left[\sum_{q,a} (\partial^*_{v_k^{q,a}} \partial_{v_k^{a,q}} g_t)(V_{T-t}, V^*_{T-t}) \right].
\end{aligned}
$$

With a few additional, but similar, computations we get that

$$\Lambda_t = -\frac{d}{ds}\Big(\mathbb{E}[g_t(V_s, V_s^*)]\Big)_{|s=T-t} = (\partial_t f_t)(U_t, U_t^*).$$

Hence

$$dM_t = \mathbf{i}\sum_{i,j,k} (\partial_{u_k^{i,j}} f_t)(U_t, U_t^*)\ (U_{k,t}\ dX_{k,t}^N)_{i,j} - (\partial^*_{u_k^{i,j}} f_t)(U_t, U_t^*)\ ((U_{k,t}\ dX_{k,t}^N)^*)_{i,j}, \tag{10}$$

and thus M_t is a martingale. Therefore, as we saw in equation (9), we are left with computing the bracket of M_t. To begin with we have,

$$\langle (U_{k,t}\ dX_{k,t}^N)_{i,j}, \overline{(U_{k,t}\ dX_{k,t}^N)_{r,s}}\rangle = \mathbf{1}_{i=r,j=s}\frac{dt}{N},$$

$$\langle ((U_{k,t}\ dX_{k,t}^N)^*)_{i,j}, \overline{((U_{k,t}\ dX_{k,t}^N)^*)_{r,s}}\rangle = \mathbf{1}_{i=r,j=s}\frac{dt}{N},$$

$$\langle (U_{k,t}\ dX_{k,t}^N)_{i,j}, \overline{((U_{k,t}\ dX_{k,t}^N)^*)_{r,s}}\rangle = (U_{k,t})_{i,r}(U_{k,t})_{s,j}\frac{dt}{N},$$

$$\langle ((U_{k,t}\ dX_{k,t}^N)^*)_{i,j}, \overline{(U_{k,t}\ dX_{k,t}^N)_{r,s}}\rangle = (U_{k,t}^*)_{s,j}(U_{k,t}^*)_{i,r}\frac{dt}{N}.$$

Let us remind that $f : (U, U^*) \mapsto \mathrm{Tr}_N(Q(U, A, U^*, A^*))$, thus one has

$$(\partial_{u_k^{i,j}} f_t)(U_t, U_t^*) = \mathbb{E}_V\left[\mathrm{Tr}_N(\widetilde{D_k Q}(V_{T-t}U_t, A)\ V_{k,T-t}E_{i,j})\right],$$

$$(\partial^*_{u_k^{i,j}} f_t)(U_t, U_t^*) = \mathbb{E}_V\left[\mathrm{Tr}_N(V_{k,T-t}^*\ \widetilde{D_k^* Q}(V_{T-t}U_t, A)\ E_{i,j})\right].$$

We will now compute four different brackets, and by summing them we will get the bracket of M_t (see equation (10)). First,

$$\begin{aligned}
&\left\langle \sum_{i,j,k} (\partial_{u_k^{i,j}} f_t)(U_t, U_t^*)\ (U_{k,t}\ dX_{k,t}^N)_{i,j}, \overline{\sum_{i,j,k} (\partial_{u_k^{i,j}} f_t)(U_t, U_t^*)\ (U_{k,t}\ dX_{k,t}^N)_{i,j}} \right\rangle \qquad (11)\\
&= \sum_k \sum_{i,j,r,s} (\partial_{u_k^{i,j}} f_t)(U_t, U_t^*)\ \overline{(\partial_{u_k^{r,s}} f_t)(U_t, U_t^*)} \left\langle (U_{k,t}\ dX_{k,t}^N)_{i,j}, \overline{(U_{k,t}\ dX_{k,t}^N)_{r,s}} \right\rangle \\
&= \frac{1}{N}\sum_k \sum_{i,j} (\partial_{u_k^{i,j}} f_t)(U_t, U_t^*)\ \overline{(\partial_{u_k^{i,j}} f_t)(U_t, U_t^*)}\ dt \\
&= \frac{1}{N}\sum_k \sum_{i,j} \mathbb{E}_V\left[\mathrm{Tr}_N(\widetilde{D_k Q}(V_{T-t}U_t, A)\ V_{k,T-t}E_{i,j})\right] \mathbb{E}_V\left[\mathrm{Tr}_N(E_{j,i}(V_{k,T-t})^* \widetilde{D_k Q}(V_{T-t}U_t, A)^*\ )\right]\ dt \\
&= \frac{1}{N}\sum_k \mathbb{E}_{V,W}\left[\mathrm{Tr}_N(\widetilde{D_k Q}(V_{T-t}U_t, A)\ V_{k,T-t}W_{k,T-t}^* \widetilde{D_k Q}(W_{T-t}U_t, A)^*)\right] dt.
\end{aligned}$$

Similarly one has,

$$\begin{aligned}
&\left\langle \sum_{i,j,k} (\partial^*_{u_k^{i,j}} f_t)(U_t, U_t^*)\ ((U_{k,t}\ dX_{k,t}^N)^*)_{i,j}, \overline{\sum_{i,j,k} (\partial^*_{u_k^{i,j}} f_t)(U_t, U_t^*)\ ((U_{k,t}\ dX_{k,t}^N)^*)_{i,j}} \right\rangle \qquad (12)\\
&= \frac{1}{N}\sum_k \mathbb{E}_{V,W}\left[\mathrm{Tr}_N\left(V_{k,T-t}^*\ \widetilde{D_k^* Q}(V_{T-t}U_t, A)\ \widetilde{D_k^* Q}(W_{T-t}U_t, A)^*\ W_{k,T-t}\right)\right] dt.
\end{aligned}$$

Next we have,

$$\left\langle \sum_{i,j,k} (\partial_{u_k^{i,j}} f_t)(U_t, U_t^*)\ (U_{k,t}\ dX_{k,t}^N)_{i,j}, \overline{\sum_{i,j,k} (\partial^*_{u_k^{i,j}} f_t)(U_t, U_t^*)\ ((U_{k,t}\ dX_{k,t}^N)^*)_{i,j}} \right\rangle \tag{13}$$

$$= \sum_k \sum_{i,j,r,s} (\partial_{u_k^{i,j}} f_t)(U_t, U_t^*)\ \overline{(\partial^*_{u_k^{r,s}} f_t)(U_t, U_t^*)} \left\langle (U_{k,t}\ dX_{k,t}^N)_{i,j}, \overline{((U_{k,t}\ dX_{k,t}^N)^*)_{r,s}} \right\rangle$$

$$= \frac{1}{N} \sum_k \sum_{i,j,r,s} (\partial_{u_k^{i,j}} f_t)(U_t, U_t^*)\ \overline{(\partial^*_{u_k^{r,s}} f_t)(U_t, U_t^*)}\ (U_{k,t})_{i,r} (U_{k,t})_{s,j}\ dt$$

$$= \frac{1}{N} \sum_k \sum_{i,j,r,s} \mathbb{E}_V \left[\mathrm{Tr}_N (\widetilde{D_k Q}(V_{T-t} U_t, A)\ V_{k,T-t} E_{i,j}) \right]$$

$$\times \mathbb{E}_W \left[\mathrm{Tr}_N (E_{s,r} \widetilde{D_k^* Q}(W_{T-t} U_t, A)^*\ W_{k,T-t}) \right]\ (U_{k,t})_{i,r} (U_{k,t})_{s,j}\ dt$$

$$= \frac{1}{N} \sum_k \mathbb{E}_{V,W} \left[\sum_{i,j,r,s} \left(\widetilde{D_k Q}(V_{T-t} U_t, A)\ V_{k,T-t} \right)_{j,i} (U_{k,t})_{i,r} \left(\widetilde{D_k^* Q}(W_{T-t} U_t, A)^*\ W_{k,T-t} \right)_{r,s} (U_{k,t})_{s,j} \right] dt$$

$$= \frac{1}{N} \sum_k \mathbb{E}_{V,W} \left[\mathrm{Tr}_N \left(\widetilde{D_k Q}(V_{T-t} U_t, A)\ V_{k,T-t} U_{k,t}\ \widetilde{D_k^* Q}(W_{T-t} U_t, A)^*\ W_{k,T-t} U_{k,t} \right) \right] dt.$$

And similarly,

$$\left\langle \sum_{i,j,k} (\partial^*_{u_k^{i,j}} f_t)(U_t, U_t^*)\ ((U_{k,t}\ dX_{k,t}^N)^*)_{i,j}, \overline{\sum_{i,j,k} (\partial_{u_k^{i,j}} f_t)(U_t, U_t^*)\ (U_{k,t}\ dX_{k,t}^N)_{i,j}} \right\rangle \tag{14}$$

$$= \frac{1}{N} \sum_k \mathbb{E}_{V,W} \left[\mathrm{Tr}_N \left((V_{k,T-t} U_t^k)^* \widetilde{D_k^* Q}(V_{T-t} U_t, A)\ (W_{k,T-t} U_t^k)^* \widetilde{D_k Q}(W_{T-t} U_t, A)^* \right) \right] dt.$$

We sum equations (11) to (14).

$$\begin{aligned}
&\mathrm{Var}\left(\mathrm{Tr}_N(\widetilde{Q}(U_T^N, A^N)) \right) \\
&= \frac{1}{N} \sum_k \int_0^T \mathbb{E}\Big[\mathrm{Tr}_N \Big(\widetilde{D_k Q}(V_{T-t} U_t, A)\ V_{k,T-t} U_{k,t}\ (W_{k,T-t} U_{k,t})^* \widetilde{D_k Q}(W_{T-t} U_t, A)^* \\
&\qquad + (V_{k,T-t} U_{k,t})^*\ \widetilde{D_k^* Q}(V_{T-t} U_t, A)\ \widetilde{D_k^* Q}(W_{T-t} U_t, A)^*\ W_{k,T-t} U_{k,t} \\
&\qquad - \widetilde{D_k Q}(V_{T-t} U_t, A)\ V_{k,T-t} U_{k,t}\ \widetilde{D_k^* Q}(W_{T-t} U_t, A)^*\ W_{k,T-t} U_{k,t} \\
&\qquad - (V_{k,T-t} U_t^k)^* \widetilde{D_k^* Q}(V_{T-t} U_t, A)\ (W_{k,T-t} U_t^k)^* \widetilde{D_k Q}(W_{T-t} U_t, A)^* \Big) \Big] dt \\
&= \frac{1}{N} \sum_k \int_0^T \mathbb{E}\Big[\mathrm{Tr}_N \Big(\left(\widetilde{D_k Q}(V_{T-t} U_t, A)\ V_{k,T-t} U_{k,t} - (V_{k,T-t} U_{k,t})^*\ \widetilde{D_k^* Q}(V_{T-t} U_t, A) \right) \\
&\qquad \times \left(\widetilde{D_k Q}(W_{T-t} U_t, A) W_{k,T-t} U_{k,t} - (W_{k,T-t} U_{k,t})^* \widetilde{D_k^* Q}(W_{T-t} U_t, A) \right)^* \Big) \Big] dt.
\end{aligned}$$

Hence the conclusion.

□

Corollary 3.1. *Let $P, Q \in \mathcal{P}_d$, $(U_t^N)_{t \in \mathbb{R}^+}$, $(V_t^N)_{t \in \mathbb{R}^+}$, $(W_t^N)_{t \in \mathbb{R}^+}$ be independent vectors of p unitary Brownian motions of size N. Let Z^{NM} be a vector of deterministic matrices. We also define the map*

$$h : x \otimes y \in (\mathbb{M}_N(\mathbb{C}) \otimes \mathbb{M}_M(\mathbb{C}))^{\otimes 2} \mapsto y \sharp x \in \mathbb{M}_N(\mathbb{C}) \otimes \mathbb{M}_M(\mathbb{C}).$$

With notations as in subsection 2.4, one has for any $T \geq 0$,

$$\mathbb{E}\Big[(\mathrm{Tr}_N \otimes I_M)^{\bigotimes 2}\left(\widetilde{P}(U_T^N \otimes I_M, Z^{NM}) \otimes \widetilde{Q}(U_T^N \otimes I_M, Z^{NM})\right)\Big]$$
$$- \mathbb{E}[\mathrm{Tr}_N \otimes I_M]^{\bigotimes 2}\left(\widetilde{P}(U_T^N \otimes I_M, Z^{NM}) \otimes \widetilde{Q}(U_T^N \otimes I_M, Z^{NM})\right)$$
$$= -\frac{1}{N}\sum_{k\leq p}\int_0^T \mathbb{E}\Big[\mathrm{Tr}_N \otimes I_M\Big(h\circ\delta_k\widetilde{P}(V_{T-t}^N U_t^N \otimes I_M, Z^{NM}) \times h\circ\delta_k\widetilde{Q}(W_{T-t}^N U_t^N \otimes I_M, Z^{NM})\Big)\Big]dt.$$

Proof. Let A^N be a vector of deterministic matrices, by polarization and the fact that $\mathcal{D}_k(Q^*)^* = -\mathcal{D}_k Q$, we have

$$\mathbb{E}\Big[\mathrm{Tr}_N\left(\widetilde{P}(U_T^N, A^N)\right)\mathrm{Tr}_N\left(\widetilde{Q}(U_T^N, A^N)\right)\Big] - \mathbb{E}\Big[\mathrm{Tr}_N\left(\widetilde{P}(U_T^N, A^N)\right)\Big]\mathbb{E}\Big[\mathrm{Tr}_N\left(\widetilde{Q}(U_T^N, A^N)\right)\Big]$$
$$= \mathbb{E}\left[\left(\mathrm{Tr}_N\left(\widetilde{P}(U_T^N, A^N)\right) - \mathbb{E}\Big[\mathrm{Tr}_N\left(\widetilde{P}(U_T^N, A^N)\right)\Big]\right)\overline{\left(\mathrm{Tr}_N\left(\widetilde{Q^*}(U_T^N, A^N)\right) - \mathbb{E}\Big[\mathrm{Tr}_N\left(\widetilde{Q^*}(U_T^N, A^N)\right)\Big]\right)}\right]$$
$$= \frac{1}{N}\sum_{k\leq p}\int_0^T \mathbb{E}\Big[\mathrm{Tr}_N\left(\widetilde{\mathcal{D}_k P}(V_{T-t}^N U_t^N, A^N) \times \widetilde{\mathcal{D}_k Q^*}(W_{T-t}^N U_t^N, A^N)^*\right)\Big]dt$$
$$= -\frac{1}{N}\sum_{k\leq p}\int_0^T \mathbb{E}\Big[\mathrm{Tr}_N\left(m\circ\delta_k\widetilde{P}(V_{T-t}^N U_t^N, A^N) \times m\circ\delta_k\widetilde{Q}(W_{T-t}^N U_t^N, A^N)\right)\Big]dt.$$

Now we want to study a polynomial in $(U_T^N \otimes I_M, Z^{NM})$ and their adjoints. By linearity we can assume that P is a monomial. One can also assume that $Z_i^{NM} = A_i \otimes B_i$ where $A_i \in \mathbb{M}_N(\mathbb{C})$ and $B_i \in \mathbb{M}_M(\mathbb{C})$. Then,

$$\widetilde{P}(U_T^N \otimes I_M, Z^{NM}) = \widetilde{P}(U_T^N, A) \otimes \widetilde{P}(I_M, B).$$

Thus assuming that P and Q are monomials, we have

$$\mathbb{E}\Big[(\mathrm{Tr}_N \otimes I_M)^{\bigotimes 2}\left(\widetilde{P}(U_T^N \otimes I_M, Z^{NM}) \otimes \widetilde{Q}(U_T^N \otimes I_M, Z^{NM})\right)\Big]$$
$$- \mathbb{E}[\mathrm{Tr}_N \otimes I_M]^{\bigotimes 2}\left(\widetilde{P}(U_T^N \otimes I_M, Z^{NM}) \otimes \widetilde{Q}(U_T^N \otimes I_M, Z^{NM})\right)$$
$$= \Big(\mathbb{E}[\mathrm{Tr}_N(\widetilde{P}(U_T^N, A))\,\mathrm{Tr}_N(\widetilde{Q}(U_T^N, A))] - \mathbb{E}[\mathrm{Tr}_N(\widetilde{P}(U_T^N, A))]\ \mathbb{E}[\mathrm{Tr}_N(\widetilde{Q}(U_T^N, A))]\Big) \otimes \widetilde{P}(I_M, B)\widetilde{Q}(I_M, B)$$
$$= -\frac{1}{N}\int_0^T \mathbb{E}\Big[\mathrm{Tr}_N\left(m\circ\delta_k\widetilde{P}(V_{T-t}^N U_t^N, A) \times m\circ\delta_k\widetilde{Q}(W_{T-t}^N U_t^N, A)\right)\Big] \otimes \widetilde{P}(I_M, B)\widetilde{Q}(I_M, B)\ dt$$
$$= -\frac{1}{N}\sum_{k\leq p}\int_0^T \mathbb{E}\Big[\mathrm{Tr}_N \otimes I_M\Big(h\circ\delta_k\widetilde{P}(V_{T-t}^N U_t^N \otimes I_M, Z^{NM}) \times h\circ\delta_k\widetilde{Q}(W_{T-t}^N U_t^N \otimes I_M, Z^{NM})\Big)\Big]dt.$$

Hence the conclusion by linearity.

□

3.3 Convergence of the submatrices of Haar unitary Brownian motions

The following lemma will be useful for a computation trick later in this paper. We prove the convergence in distribution of a family of random matrices by using a well-known trick. We first prove that the moments of the limit are the unique solution to some system of equations, then we prove that the moments of our random matrices satisfies the same system of equations up to a term which converges towards 0. We refer for exemple to the proof of Theorem 5.4.10 in [2] for a use of this method.

Proposition 3.2. *We define the following family of random matrix processes, let* $(X_{i,j}^k)_{1\leq i\leq j\leq N}$ *and* $(Y_{i,j}^k)_{1\leq i<j\leq N}$ *be independent hermitian Brownian motions of size* k. *We set* $H_{i,i}^k = X_{i,i}^k$, $H_{i,j}^k = 2^{-1/2}(X_{i,j}^k + \mathbf{i}Y_{i,j}^k)$ *for* $i<j$ *and* $H_{i,j}^k = 2^{-1/2}(X_{j,i}^k - \mathbf{i}Y_{j,i}^k)$ *for* $i>j$. *We then define*

$$\forall t \geq 0, \quad U^{k,t}_{i,j} = \mathbf{1}_{i=j} I_k - \int_0^t \frac{U^{k,s}_{i,j}}{2}\ ds + \mathbf{i}\frac{1}{\sqrt{N}} \sum_{l=1}^N \int_0^t U^{k,s}_{i,l} dH^{k,s}_{l,j}.$$

We similarly define $(H_{i,j})_{1\leq i,j\leq N}$ *where we replaced the independent hermitian Brownian motions by freely independent free Brownian motions. And similarly*

$$\forall t \geq 0, \quad u^{t}_{i,j} = \mathbf{1}_{i=j} 1_{\mathcal{A}} - \int_0^t \frac{u^{s}_{i,j}}{2}\ ds + \mathbf{i}\frac{1}{\sqrt{N}} \sum_{l=1}^N \int_0^t u^{s}_{i,l} \otimes 1_{\mathcal{A}} \# dH^{s}_{l,j}.$$

We consider $U^{k,t}$*, p independent copy of* $(U^{k,t}_{i,j})_{1\leq i,j\leq N}$*, and* $\mathbf{u}^t$*, p freely independent copy of* $(u^t_{i,j})_{1\leq i,j\leq N}$*. Then the family* $U^{k,t}$ *in the* $\mathcal{C}^*$*-algebra* $L^\infty(\Omega, \mathbb{M}_k(\mathbb{C}))$ *endowed with the trace* $X \mapsto \mathbb{E}[\tau_N(X)]$ *converges towards the family* $\mathbf{u}^t$ *in the sense of Definition 2.1.*

Proof. It is easy to see that $(U^{k,t}_{i,j})_{1\leq i,j\leq N}$ is a unitary Brownian motion of size kN, thus for any i,j and t, the operator norm of $U^{k,t}_{i,j}$ is smaller than 1. Next we proceed as usual, for a polynomial Q, we denote Q_s this monomial evaluated in $U^{k,s}$, we have for a monomial Q of degree n,

$$\begin{aligned}
\frac{d\mathbb{E}[\tau_k(Q_s)]}{dt} =& -\frac{n}{2}\ \mathbb{E}[\tau_k(Q_s)] \\
&- \frac{1}{N}\sum_a \sum_{i_1,j_1,i_2,j_2} \sum_{Q=AU_a^{i_1,j_1} BU_a^{i_2,j_2} C} \mathbb{E}[\tau_k((U^{k,s}_a)_{i_1,j_2} C_s A_s) \tau_k((U^{k,s}_a)_{i_2,j_1} B_s)] \\
&- \frac{1}{N}\sum_a \sum_{i_1,j_1,i_2,j_2} \sum_{Q=A(U_a^{i_1,j_1})^* B(U_a^{i_2,j_2})^* C} \mathbb{E}[\tau_k((U^{k,s}_a)^*_{i_1,j_2} C_s A_s)\ \tau_k((U^{k,s}_a)^*_{i_2,j_1} B_s)] \\
&+ \frac{1}{N}\sum_a \sum_{i,j} \sum_{Q=AU_a^{i,j} B(U_a^{j,i})^* C} \mathbb{E}[\tau_k(C_s A_s)\ \tau_k(B_s)] \\
&+ \frac{1}{N}\sum_a \sum_{i,j} \sum_{Q=A(U_a^{i,j})^* BU_a^{j,i} C} \mathbb{E}[\tau_k(C_s A_s)\ \tau_k(B_s)].
\end{aligned}$$

One can view $U^{k,t}_{i,j}$ as the matrix in the upper left corner of $I_k \otimes E_{1,i}\ (U^{k,t}_{i,j})_{1\leq i,j\leq N}\ I_k \otimes E_{j,1}$, thus for any polynomial P, there exists a polynomial S such that,

$$\tau_k(P(U^{k,t})) = \frac{1}{k}\operatorname{Tr}_{Nk}\left(\widetilde{S}\left(U^{k,t}, (I_k\otimes E_{1,i})_i\right)\right),$$

where we view $U^{k,t}$ as a vector of matrices of size kN. Since it has the law of a vector of unitary Brownian motions of size kN, thanks to Proposition 3.1, we get that

$$\begin{aligned}
&\mathbb{E}[\tau_k(Q_s)] = \mathbb{E}[\tau_k(Q_0)] \\
&+ \int_0^t e^{-\frac{n}{2}(t-s)} \Bigg(-\frac{1}{N}\sum_a \sum_{i_1,j_1,i_2,j_2} \sum_{Q=AU_a^{i_1,j_1} BU_a^{i_2,j_2} C} \mathbb{E}[\tau_k((U^{k,s}_a)_{i_1,j_2} C_s A_s)]\ \mathbb{E}[\tau_k((U^{k,s}_a)_{i_2,j_1} B_s)] \\
&\qquad - \frac{1}{N}\sum_a \sum_{i_1,j_1,i_2,j_2} \sum_{Q=A(U_a^{i_1,j_1})^* B(U_a^{i_2,j_2})^* C} \mathbb{E}[\tau_k((U^{k,s}_a)^*_{i_1,j_2} C_s A_s)]\ \mathbb{E}[\tau_k((U^{k,s}_a)^*_{i_2,j_1} B_s)] \\
&\qquad + \frac{1}{N}\sum_a \sum_{i,j} \sum_{Q=AU_a^{i,j} B(U_a^{j,i})^* C} \mathbb{E}[\tau_k(C_s A_s)]\ \mathbb{E}[\tau_k(B_s)] \\
&\qquad + \frac{1}{N}\sum_a \sum_{i,j} \sum_{Q=A(U_a^{i,j})^* BU_a^{j,i} C} \mathbb{E}[\tau_k(C_s A_s)]\ \mathbb{E}[\tau_k(B_s)] \Bigg)\ ds + \mathcal{O}(k^{-2}).
\end{aligned}$$

Similarly, thanks to Theorem 2.2, if Q_s is now the monomial Q evaluated in $\mathbf{u}^s$, we get that

$$\begin{aligned}\tau(Q_s) = \tau(Q_0) + \int_0^t e^{-\frac{n}{2}(t-s)} \Bigg(&- \frac{1}{N} \sum_a \sum_{i_1,j_1,i_2,j_2} \sum_{Q=AU_a^{i_1,j_1}BU_a^{i_2,j_2}C} \tau((\mathbf{u}_a^s)_{i_1,j_2} C_s A_s) \tau((\mathbf{u}_a^s)_{i_2,j_1} B_s) \\ &- \frac{1}{N} \sum_a \sum_{i_1,j_1,i_2,j_2} \sum_{Q=A(U_a^{i_1,j_1})^* B(U_a^{i_2,j_2})^* C} \tau((\mathbf{u}_a^s)^*_{i_1,j_2} C_s A_s)\ \tau((\mathbf{u}_a^s)^*_{i_2,j_1} B_s) \\ &+ \frac{1}{N} \sum_a \sum_{i,j} \sum_{Q=AU_a^{i,j}B(U_a^{j,i})^*C} \tau(C_s A_s)\ \tau(B_s) \\ &+ \frac{1}{N} \sum_a \sum_{i,j} \sum_{Q=A(U_a^{i,j})^*BU_a^{j,i}C} \tau(C_s A_s)\ \tau(B_s) \Bigg)\ ds.\end{aligned}$$

Thus, since for any k, $\mathbb{E}[\tau_k(Q_0)] = \tau(Q_0)$, we can show the convergence by induction.

□

3.4 Convergence of the free unitary Brownian motion

If u_t is a free unitary Brownian motion at time t, one can define μ_{u_t} as in Definition 2.1. Then thanks to Riesz theorem, there is a measure ν_t such that for any function polynomial P in two commuting indeterminates,

$$\tau(P(u_t, u_t^*)) = \int_{\mathbb{C}} f(z, z^*)\ d\nu_t(z).$$

The measure ν_t is well-known albeit not explicit, the proof of the following theorem can be found in [4].

Theorem 3.1. *For every $t > 0$, the measure ν_t is absolutely continuous with respect to the Haar measure on $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. For $t > 4$, the support of ν_t is equal to $\mathbb{T}$, and its density is positive on $\mathbb{T}$. We set $\kappa(t, \omega)$ the density of ν_t at the point $\omega \in \mathbb{T}$. Then for $t > 4$, $\kappa(t, \omega)$ is the real part of the only solution with positive real part of the equation,*

$$\frac{z-1}{z+1} e^{\frac{t}{2}z} = \omega. \tag{15}$$

This theorem let us prove that when t is large, we are exponentially close to a Haar unitary for the operator norm topology.

Proposition 3.3. *There exists a $\mathcal{C}^*$-algebra $\mathcal{C}$ which contains $u_t^1, \dots, u_t^p$ freely independent free unitary Brownian motions at time $t \geq 5$ and $\tilde{u}^1, \dots, \tilde{u}^p$ freely independent Haar unitaries such that for any i, $\left\| u_t^i - \tilde{u}_t^i \right\| \leq 4e^2\pi e^{-\frac{t}{2}}$.*

Proof. We view $B(L^2([0, 2\pi]))$ as the $\mathcal{C}^*$-algebra endowed with the state $\tau(u) = \langle u(\mathbf{1}_{[0,2\pi]}), \mathbf{1}_{[0,2\pi]} \rangle_{L^2([0,2\pi]}$. The endomorphism $x : f \mapsto (t \to tf(t))$ is self-adjoint and has distribution (as defined in 2.1) $\mu_x(f) = \int_{[0,2\pi]} f$. Consequently we set $g : s \to \kappa(t, e^{is})$ and $G : s \to \int_0^s g(u)\ du$. Since g is positive, we can define $u_t = e^{\mathbf{i}G^{-1}(x)}$ which has the distribution of a free unitary Brownian motion at time t, indeed for any polynomial P in two commuting indeterminates,

$$\tau(P(u_t, u_t^*)) = \int_0^{2\pi} P\left(e^{\mathbf{i}G^{-1}(s)}, e^{-\mathbf{i}G^{-1}(s)}\right) ds = \int_0^{2\pi} P(e^{\mathbf{i}s}, e^{-\mathbf{i}s}))g(s) ds = \int_{\mathbb{C}} f(z, z^*)\ d\nu_t(z).$$

And similarly, $u = e^{\mathbf{i}x}$ is a Haar unitary. Besides, since

$$u_t - u = \int_0^1 e^{\mathbf{i}\alpha G^{-1}(x)} (G^{-1}(x) - x) e^{\mathbf{i}(1-\alpha)x} d\alpha,$$

thanks to the fact that G is a dipheomorphism of $[0, 2\pi]$,

$$\|u_t - u\| \leq \left\| G^{-1}(x) - x \right\| = \sup_{s \in [0,2\pi]} |G^{-1}(s) - s| = \sup_{s \in [0,2\pi]} |s - G(s)| \leq 2\pi \sup_{s \in [0,2\pi]} |1 - g(s)|.$$

We set $y(s)$ the imaginary part of the only solution with positive real part of the equation (15). Then we have for any s,

$$\frac{(g(s)-1)^2+y(s)^2}{(g(s)+1)^2+y(s)^2} \leq e^{-tg(s)}.$$

However since $(g(s)-1)^2 \leq (g(s)+1)^2$, we have,

$$\frac{(g(s)-1)^2}{(g(s)+1)^2} \leq \frac{(g(s)-1)^2+y(s)^2}{(g(s)+1)^2+y(s)^2} \leq e^{-tg(s)}.$$

First in the case where $g(s) \geq 1$, then since we assumed $t \geq 4$, $|g(s)-1| \leq (|g(s)-1|+2)e^{-2|g(s)-1|}e^{-\frac{t}{2}}$, and since the function $u \to (u+2)e^{-2u}$ is decreasing, we have, $|g(s)-1| \leq 2e^{-\frac{t}{2}}$.
If $g(s) \leq 1$, then after studying the graph of the function $h : g \mapsto e^{-tg/2} - \frac{1-g}{1+g}$, we have that this function takes value 0 in in 0, is negative on $(0, c_t)$ for some $c_t \in (0,1)$, and finally is positive for $g > c_t$. Since we know that $g(s)$ is positive for $t > 4$ and $h(g(s)) \geq 0$, if we find g such that $h(g) \leq 0$, then $g(s) \geq g$. Besides for $t \geq 5$, we have that $h\left(\ln(t/2)\frac{2}{t}\right) \leq 0$. Thus necessarily $g(s) \geq \ln(t/2)\frac{2}{t}$, consequently since $g(s) \leq 1$, we know that $1 - g(s) \leq 2e^{-\frac{t}{2}g(s)}$. Hence,

$$1 - g(s) \leq 2e^{-\frac{t}{2}\times\ln(t/2)\frac{2}{t}} = \frac{4}{t}.$$

Thus by bootstrapping, for any s,

$$1 - g(s) \leq 2e^{-\frac{t}{2}\left(1-\frac{4}{t}\right)} = 2e^2e^{-\frac{t}{2}}.$$

Consequently $\|u_t - u\| \leq 4e^2\pi e^{-t/2}$, and thanks to Theorem 2.1, we take $\mathcal{C}$ the free sum of p copies of $B(L^2([0,2\pi]))$.

□

3.5 Free stochastic calculus and free unitary Brownian motion

As we defined in Proposition 2.3, we consider $u_t^N = (U_1^N u_t^1, \dots, U_p^N u_t^p) \otimes I_M$ and Z^{NM}. As we will see later, thanks to Proposition 3.3, this will let us interpolate between $U^N = (U_1^N, \dots, U_p^N)$ random unitary Haar matrices and $u = (u^1, \dots, u^p)$ free Haar unitaries. Concretely if $P \in \mathcal{P}_d$, we set

$$h(t) = \tau_N \otimes \tau_M \left(\tilde{P}(u_t^N, Z^{NM})\right).$$

Then,

$$h(0) = \frac{1}{NM}\operatorname{Tr}_{MN}\left(\tilde{P}(U^N \otimes I_M, Z^{NM})\right),$$
$$h(\infty) = \tau_N \otimes \tau_M \left(\tilde{P}(u \otimes I_M, Z^{NM})\right).$$

Consequently we would like to write,

$$\frac{1}{NM}\operatorname{Tr}_{MN}\left(\tilde{P}(U^N, Z^{NM})\right) - \tau_N \otimes \tau_M\left(\tilde{P}(u, Z^{NM})\right) = \int_0^\infty \frac{dh}{dt}(t)\, dt.$$

Hence we need to compute the differential of s with respect to t, which we do in the following proposition.

Proposition 3.4. *Let the following objects be given,*

- $u = (u_t^1, \dots, u_t^p)_{t\geq 0}$ *a family of p free unitary Brownian motions,*
- $U^N = (U_1^N, \dots, U_p^N)$ *matrices of size N,*
- $u_t^N = (U_1^N u_t^1, \dots, U_p^N u_t^p)$ *elements of $\mathcal{A}_N \otimes \mathbb{M}_M(\mathbb{C})$,*
- $Z^{NM} = (Z_{p+1}^{NM}, \dots, Z_d^{NM})$ *matrices in $\mathbb{M}_N(\mathbb{C}) \otimes \mathbb{M}_M(\mathbb{C})$,*
- $P \in \mathcal{P}_d$.

With notation as in subsection 2.4, the map $H : t \mapsto \tau_N \otimes \tau_M \left(\widetilde{P}\left(u_t^N \otimes I_M, Z^{NM}\right)\right)$ *is differentiable on* $\mathbb{R}^+$ *and,*

$$\frac{dH}{dt}(t) = -\frac{1}{2}\sum_{i\leq p} \tau_M \left((\tau_N \otimes I_M) \bigotimes (\tau_N \otimes I_M) \left(\delta_i \mathcal{D}_i \widetilde{P}\left(u_t^N \otimes I_M, Z^{NM}\right)\right)\right).$$

Proof. We want to use Theorem 2.2 to write H as an integral which we can then easily differentiate. We need to define $X_0 \in \mathcal{A}^d$, K such that for any $t \geq 0$, $K \in (L^1([0,t],\mathcal{A}))^d$, U such that for any $t \geq 0$, $(\mathbf{1}_{s\leq t} U_s^i)_{t\in\mathbb{R}^+} \in (\mathcal{B}_a^\infty)^d$, and then,

$$X_t = X_0 + \int_0^t K_s ds + \sum_i \int_0^t U_s^i \# dS_s^i.$$

By using the linearity of the trace and the non-commutative differential, we can assume that $Z_i^{NM} = A_i \otimes B_i$ where $A_i \in \mathbb{M}_N(\mathbb{C})$ and $B_i \in \mathbb{M}_M(\mathbb{C})$. We then set $X_t = (u_t^N, u_t^{N*}, A, A^*)$. Since (A, A^*) is free from $\mathcal{A}$, the processes K and U associated to (A, A^*) are zero. As for (u_t^N, u_t^{N*}), by definition of a free unitary Brownian motion, we have

$$\forall t \geq 0, \quad u_t^N = U^N - \int_0^t \frac{u_s^N}{2}\, ds + \mathbf{i} \int_0^t u_s^N \otimes 1_{\mathcal{A}} \# dS_s,$$

$$\forall t \geq 0, \quad (u_t^N)^* = U^N - \int_0^t \frac{(u_s^N)^*}{2}\, ds - \mathbf{i} \int_0^t 1_{\mathcal{A}} \otimes (u_s^N)^* \# dS_s.$$

To minimize cumbersome notations, we will forget about the N in u_t^N, and assimilate u_t^N with u_t. Consequently we set for any $s \geq 0$,

$$\begin{aligned}
&\forall i \in [1,p], \forall j \in [1,p], \quad K_s^j = -u_{j,s}/2, \quad U_s^{i,j} = \mathbf{i}\ \mathbf{1}_{i=j} u_{j,s} \otimes 1_{\mathcal{A}},\\
&\forall i \in [1,p], \forall j \in [p+1,2p], \quad K_s^j = -u_{j,s}^*/2, \quad U_s^{i,j} = -\mathbf{i}\ \mathbf{1}_{i=j} 1_{\mathcal{A}} \otimes u_{j,s}^*,\\
&\forall i \in [1,p], \forall j > 2p, \quad K_s^j = 0, \quad U_s^{i,j} = 0 \otimes 0.
\end{aligned}$$

Thus we have for any monomial Q,

$$\partial Q(X) \# K = -\frac{1}{2} \sum_{i\leq p} \partial_i Q(X) \# u_i + \partial_i^* Q(X) \# (u_i)^*,$$

$$\begin{aligned}
\Delta_U(Q)(X) = -\sum_{i\leq p} \quad &\langle\langle\ (\partial_i \otimes I) \circ \partial_i Q(X) \# (u_i \otimes \mathbf{1}_{\mathcal{A}}, u_i \otimes \mathbf{1}_{\mathcal{A}})\ \rangle\rangle\\
&- \langle\langle\ (\partial_i \otimes I) \circ \partial_i^* Q(X) \# (u_i \otimes \mathbf{1}_{\mathcal{A}}, \mathbf{1}_{\mathcal{A}} \otimes (u_i)^*\ \rangle\rangle\\
&- \langle\langle\ (\partial_i^* \otimes I) \circ \partial_i Q(X) \# (\mathbf{1}_{\mathcal{A}} \otimes (u_i)^*, u_i \otimes \mathbf{1}_{\mathcal{A}})\ \rangle\rangle\\
&+ \langle\langle\ (\partial_i^* \otimes I) \circ \partial_i^* Q(X) \# (\mathbf{1}_{\mathcal{A}} \otimes (u_i)^*, \mathbf{1}_{\mathcal{A}} \otimes (u_i)^*)\ \rangle\rangle.
\end{aligned}$$

And thanks to Theorem 2.2, we have for any $t \geq 0$,

$$Q(X_t) = Q(X_0) + \int_0^t \partial Q(X_s) \# K_s\ ds + \sum_i \int_0^t \partial Q(X) \sharp U_s^i\ \# dS_s^i + \int_0^t \Delta_U(Q)(X_s)\ ds.$$

Thus if we fix $t \in \mathbb{R}^+$, then for any $\varepsilon \geq -t$,

$$Q(X_{t+\varepsilon}) - Q(X_t) = \int_t^{t+\varepsilon} \partial Q(X_s) \# K_s\ ds + \sum_i \int_t^{t+\varepsilon} \partial Q(X) \sharp U_s^i\ \# dS_s^i + \int_t^{t+\varepsilon} \Delta_U(Q)(X_s)\ ds.$$

As we said in section 2.3, $\left(\sum_i \int_0^t \partial Q(X) \sharp U_s^i\ \# dS_s^i\right)_{t\geq 0}$ is a martingale, thus

$$\tau_N(Q(X_{t+\varepsilon})) - \tau_N(Q(X_t)) = \int_t^{t+\varepsilon} \tau_N(\partial Q(X_s) \# K_s)\ ds + \int_t^{t+\varepsilon} \tau_N(\Delta_U(Q)(X_s))\ ds.$$

Finally we have,

$$\frac{d\tau_N(Q(X_t))}{dt} = \tau_N(\partial Q(X_t)\# K_t) + \tau_N(\Delta_U(Q)(X_t)). \tag{16}$$

Besides,

$$\tau_N(\partial Q(X_t)\# K_t) = -\frac{1}{2}\sum_{i\leq p} \tau_N(D_i Q(X_t)\ u_{i,t}) + \tau_N(u_{i,t}^* D_i^* P(X_t)),$$

and,

$$\tau_N\left(\langle\langle\ (\partial_i \otimes I)\circ \partial_i Q(X)\#(u_i\otimes \mathbf{1}_{\mathcal{A}}, u_i\otimes \mathbf{1}_{\mathcal{A}})\ \ \rangle\rangle\right) = \sum_{Q=AY_iBY_iC} \tau_N(C(X)A(X)u_i)\tau_N(u_iB(X)),$$

$$\tau_N\left(\langle\langle\ (\partial_i \otimes I)\circ \partial_i^* Q(X)\#(u_i\otimes \mathbf{1}_{\mathcal{A}}, \mathbf{1}_{\mathcal{A}}\otimes u_i^*\ \ \rangle\rangle\right) = \sum_{Q=AY_iBY_i^*C} \tau_N(A(X)u_iu_i^*C(X))\tau_N(B(X)),$$

$$\tau_N\left(\langle\langle\ (\partial_i^* \otimes I)\circ \partial_i Q(X)\#(\mathbf{1}_{\mathcal{A}}\otimes u_i^*, u_i\otimes \mathbf{1}_{\mathcal{A}})\ \ \rangle\rangle\right) = \sum_{Q=AY_i^*BY_iC} \tau_N(A(X)C(X))\tau_N(u_iu_i^*B(X)),$$

$$\tau_N\left(\langle\langle\ (\partial_i^* \otimes I)\circ \partial_i^* Q(X)\#(\mathbf{1}_{\mathcal{A}}\otimes u_i^*, \mathbf{1}_{\mathcal{A}}\otimes u_i^*)\ \ \rangle\rangle\right) = \sum_{Q=AY_i^*BY_i^*C} \tau_N(C(X)A(X)u_i^*)\tau_N(u_i^*B(X)).$$

Besides we also have,

$$\begin{aligned}\tau_N\otimes\tau_N\left(\delta_i D_i Q(X)\times 1\otimes u_i\right) = 2 &\sum_{Q=AY_iBY_iC} \tau(B(X)u_i)\tau(C(X)A(X)u_i)\\ &- \sum_{Q=AY_i^*BY_iC} \tau(C(X)A(X))\tau(B(X)u_iu_i^*)\\ &- \sum_{Q=AY_iBY_i^*C} \tau(B(X))\tau(C(X)u_iu_i^*A(X)),\end{aligned}$$

$$\begin{aligned}\tau_N\otimes\tau_N\left(\delta_i D_i^* Q(X)\times u_i^*\otimes 1\right) = -\,2 &\sum_{Q=AY_i^*BY_i^*C} \tau(u_i^*B(X))\tau(C(X)A(X)u_i^*)\\ &+ \sum_{Q=AY_i^*BY_iC} \tau(B(X)u_iu_i^*)\tau(C(X)A(X))\\ &+ \sum_{Q=AY_iBY_i^*C} \tau(C(X)u_iu_i^*A(X))\tau(B(X)).\end{aligned}$$

Which means that

$$\tau_N\left(\Delta_U(Q)(X)\right) = -\frac{1}{2}\sum_{i\leq p}\tau_N\otimes\tau_N\left(\delta_i D_i Q(X)\times 1\otimes u_i\right) - \tau_N\otimes\tau_N\left(\delta_i D_i^* Q(X)\times u_i^*\otimes 1\right).$$

And with equation (16),

$$\begin{aligned}\frac{d\tau_N(Q(X_t))}{dt} &= -\frac{1}{2}\ \sum_{i\leq p} \tau_N(D_iQ(X_t)\ u_{i,t}) + \tau_N\otimes\tau_N\left(\delta_i D_i Q(X_t)\times 1\otimes u_{i,t}\right)\\ &\qquad + \tau_N(u_{i,t}^*D_i^*Q(X_t)) - \tau_N\otimes\tau_N\left(\delta_i D_i^* Q(X_t)\times u_{i,t}^*\otimes 1\right)\\ &= -\frac{1}{2}\ \sum_{i\leq p}\tau_N\otimes\tau_N\left(\delta_i\left(D_iQ(X_t)u_{i,t}\right)\right) - \tau_N\otimes\tau_N\left(\delta_i\left(u_{i,t}^*D_i^*Q(X_t)\right)\right)\\ &= -\frac{1}{2}\ \sum_{i\leq p}\tau_N\otimes\tau_N\left(\delta_i\mathcal{D}_iQ(X_t)\right).\end{aligned}$$

Now we want to study a polynomial in (u_t^N, Z^{NM}) and their adjoints. If P is a monomial, we have,

$$\widetilde{P}(u_t^N \otimes I_M, Z^{NM}) = \widetilde{P}(u_t^N, A) \otimes \widetilde{P}(I_M, B).$$

Therefore,

$$\frac{dH}{dt}(t) = -\frac{1}{2}\sum_{i\leq p} \tau_N \otimes \tau_N \left(\delta_i \mathcal{D}_i \widetilde{P}(u_t^N, A)\right) \times \tau_M\left(\widetilde{P}(I_M, B)\right).$$

And since for any $S, T \in \mathcal{P}_d$,

$$\tau_N \otimes \tau_N(\delta_i \widetilde{TS}(u_t^N, A)) \times \tau_M\left(\widetilde{ST}(I_M, B)\right) = \tau_M\left((\tau_N \otimes I_M)\bigotimes(\tau_N \otimes I_M)\left(\delta_i \widetilde{TS}\left(u_t^N \otimes I_M, A \otimes B\right)\right)\right).$$

Hence after summing,

$$\frac{d}{dt}\tau_N \otimes \tau_M\left(\widetilde{P}\left(u_t^N, Z^{NM}\right)\right) = -\frac{1}{2}\sum_{i\leq p}\tau_M\left((\tau_N \otimes I_M)\bigotimes(\tau_N \otimes I_M)\left(\delta_i \mathcal{D}_i \widetilde{P}\left(u_t^N, Z^{NM}\right)\right)\right),$$

and we conclude by linearity.

□

4 Proof of Theorem 1.1, the main result

4.1 Overview of the proof

If we take the point of view of free probability – for details we refer to the third point of Definition 2.1 – we have two families of non-commutative random variables, $(U^N \otimes I_M, Z^{NM})$ and $(u \otimes I_M, Z^{NM})$, and we want to study the difference between their distributions. As mentioned in the introduction the main idea of the proof is to interpolate those two families with the help of free unitary Brownian motions $u_t^N = \left(u_{i,t}^N\right)_i$ started in deterministic matrices $\left(U_i^N\right)_i$ of size N. A big difference with the case of the GUE which was treated in [11] is that we do not have an explicit expression of the law of u_t^N in function of U^N and u, which is why we had to introduce notions of free stochastic calculus.

Since our aim in this subsection is not to give a rigorous proof but to outline the strategy used in subsection 4.2, we assume that we have no matrix Z^{NM} and that $M = 1$. Now under the assumption that this is well-defined, if Q is a non-commutative polynomial,

$$\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}_N\left(Q\left(U^N\right)\right)\right] - \tau\Big(Q\left(u\right)\Big) = -\int_0^\infty \mathbb{E}\left[\frac{d}{dt}\Big(\tau_N\left(Q(u_t^N)\right)\Big)\right]\, dt.$$

In the classical case, if $(S_t)_{t\geq 0}$ is a Markov process with generator θ, then under the appropriate assumption we have

$$\frac{d}{dt}\mathbb{E}[f(S_t)] = \mathbb{E}[(\theta f)(S_t)].$$

And if the law of the process at time 0 is invariant for this Markov process we have that for any t, $\mathbb{E}[(\theta f)(S_t)] = 0$. Since $(u_t^N)_{t\geq 0}$ is a free Markov process, we expect to get similarly that

$$\frac{d}{dt}\Big(\tau_N\left(Q(u_t^N)\right)\Big) = \tau_N\left((\Theta Q)(u_t^N)\right),$$

for some generator Θ which we will compute with the help of Proposition 3.4. Besides the invariant law of a free Brownian motion is the law of free Haar unitaries. Thus if $(u_t)_{t\geq 0}$ is a free Brownian motion started in free Haar unitaries, we have that $\tau\left((\Theta Q)(u_t)\right) = 0$. Since unitary Haar matrices converges in distribution towards free Haar unitaries (see [2], Theorem 5.4.10), we have that $\tau_N\left((\Theta Q)(u_t^N)\right)$ converges towards $\tau\left((\Theta Q)(u_t)\right) = 0$. As we will see in this proof, the convergence happens at a speed of N^{-2}. To prove this, the main idea is to view free unitary Brownian motions started in U^N as the asymptotic limit when k goes to infinity of a unitary Brownian motion of size kN started in $U^N \otimes I_k$ (see Proposition 4.1).

Another issue is that to prove Theorem 1.1, we would like to set $Q = f(P)$ but since f is not polynomial this means that we need to extend the definition of operators such as δ_i. In order to do so we assume that there exists μ a measure on $\mathbb{R}$ such that,

$$\forall x \in \mathbb{R}, \quad f(x) = \int_{\mathbb{R}} e^{\mathbf{i}xy} \ d\mu(y).$$

While we have to assume that the support of μ is indeed on the real line, μ can be a complex measure. However we will usually work with measure such that $|\mu|(\mathbb{R})$ is finite. Indeed under this assumption we can use Fubini's theorem, and we get

$$\mathbb{E}\left[\frac{1}{M}\operatorname{Tr}_N\left(f\left(P(U^N)\right)\right)\right] - \tau\Big(f\left(P(u)\right)\Big) = \int_{\mathbb{R}} \left\{\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}_N\left(e^{\mathbf{i}yP(U^N)}\right)\right] - \tau\Big(e^{\mathbf{i}yP(u)}\Big)\right\} \ d\mu(y).$$

We can then set $Q = e^{\mathbf{i}yP}$. And even though this is not polynomial, since it is a power series, most of the properties associated to polynomials remain true with some assumption on the convergence. The main difficulty with this method is that we need to find a bound uniform in y, indeed we have terms of the form

$$\int_{\mathbb{R}} |y|^l \ d|\mu|(y)$$

which appear. Thanks to Fourier integration we can relate the exponent l to the regularity of the function f, thus we want to find a bound with l as small as possible. It turns out that with our proof $l = 4$.

4.2 Proof of Theorem 4.1

In this section we focus on proving the following theorem from which we deduce all of the important corollaries.

Theorem 4.1. *We define*

- $u = (u^1, \dots, u^p)$ *a family of p free Haar unitaries,*
- $U^N = (U_1^N, \dots, U_p^N)$ *random unitary i.i.d. matrices of size N whose law is invariant by multiplication by a matrix of $SU_N(\mathbb{R})$.*
- $Z^{NM} = (Z_{p+1}^{NM}, \dots, Z_d^{NM})$ *deterministic matrices,*
- $P \in \mathcal{P}_d$ *a self-adjoint polynomial,*
- $f : \mathbb{R} \mapsto \mathbb{R}$ *such that there exists a measure on the real line μ with $\int (1 + y^4) \ d|\mu|(y) < +\infty$ and for any $x \in \mathbb{R}$,*
$$f(x) = \int_{\mathbb{R}} e^{\mathbf{i}xy} \ d\mu(y).$$

Then there exists a polynomial L_P which only depends on P such that for any N, M,

$$\left|\mathbb{E}\left[\frac{1}{MN}\operatorname{Tr}_{MN}\left(f\left(\widetilde{P}\left(U^N \otimes I_M, Z^{NM}\right)\right)\right)\right] - \tau_N \otimes \tau_M\Big(f\left(\widetilde{P}\left(u \otimes I_M, Z^{NM}\right)\right)\Big)\right|$$
$$\leq \frac{M^2}{N^2} L_P\left(\left\|Z^{NM}\right\|\right) \times \min\left\{\ln^2(N)\int_{\mathbb{R}} (|y| + y^4) \ d|\mu|(y), \int_{\mathbb{R}} (|y| + |y|^5) \ d|\mu|(y)\right\}.$$

Even though we do not give an explicit expression for L_P, it is possible to compute it rather easily with the help of Lemma 4.4. In particular given a set of polynomials whose degree and coefficients are uniformly bounded, we can find a polynomial R such that for any P in this set and any matrices Z^{NM}, $L_P\left(\left\|Z^{NM}\right\|\right) \leq R\left(\left\|Z^{NM}\right\|\right)$. Besides, if we replace P by αP where $\alpha \in \mathbb{C}$, then up to a constant independent from α, we can bound $L_{\alpha P}$ by $(|\alpha| + |\alpha|^5)L_P$, or even $(|\alpha| + |\alpha|^4)L_P$ if one picks the first expression in the minimum. The first step to prove this theorem is the following lemma, who is a direct consequence of Proposition 3.4 and equation (5),

Lemma 4.1. *With the same notation as in Theorem 4.1, we define*

- $u = (u_t^1, \dots, u_t^p)_{t\geq 0}$ *a family of p free unitary Brownian motions,*
- $u_t^N = (U_1^N u_t^1, \dots, U_p^N u_t^p)$ *elements of* $\mathcal{A}_N$.

Then with notation as in subsection 2.4,

$$\begin{aligned}&\frac{1}{MN}\operatorname{Tr}_{MN}\left(f\left(\widetilde{P}\left(U^N\otimes I_M, Z^{NM}\right)\right)\right) - \tau_N\otimes\tau_M\left(f\left(\widetilde{P}\left(u_t^N\otimes I_M, Z^{NM}\right)\right)\right)\\ &= \frac{1}{2}\sum_{i\leq p}\int\int_0^t \tau_M\left((\tau_N\otimes I_M)\bigotimes(\tau_N\otimes I_M)\left(\delta_i\left(\mathcal{D}_i\ e^{\mathbf{i}y\widetilde{P}}\right)\left(u_t^N\otimes I_M, Z^{NM}\right)\right)\right) dt\ d\mu(y).\end{aligned}$$

Since $\mathcal{D}_i\ e^{\mathbf{i}yP} = \mathbf{i}y\ \delta_i P\ \widetilde{\#}\ e^{\mathbf{i}yP}$, this prompts us to define the following quantity.

Definition 4.1. *Let $A, B \in \mathcal{P}_d$, we set*

$$S_{t,y}^N(A,B) = \mathbb{E}\left[\tau_M\left((\tau_N\otimes I_M)\bigotimes(\tau_N\otimes I_M)\left(\delta_i\left(A\ e^{\mathbf{i}yP}\ B\right)\left(Z_t^N\right)\right)\right)\right],$$

where $Z_t^N = \left(u_t^N\otimes I_M, Z^{NM}, (u_t^N)^*\otimes I_M, (Z^{NM})^*\right)$.

The following proposition justifies why the family $(U^N\otimes I_M, u_t\otimes I_M, Z^{NM})$ has in the large k limit the distribution – in the sense of Definition 2.1 – of the family $(U^N\otimes I_{kM}, U_t^{kN}\otimes I_M, Z^{NM}\otimes I_k)$ where U_t^{kN} are independent unitary Brownian motions of size kN at time t.

Proposition 4.1. *If U_t^{kN} are unitary Brownian motions of size kN at time t, independent of U^N, we set*

$$Y_t^k = \left((U^N\otimes I_k\ U_t^{kN})\otimes I_M, Z^{NM}\otimes I_k, (U^N\otimes I_k\ U_t^{kN})^*\otimes I_M, (Z^{NM})^*\otimes I_k\right).$$

Then if $q = Ae^{\mathbf{i}yP}B$, we have that for any t,

$$(\tau_N\otimes I_M)\big(q(Z_t^N)\big) = \lim_{k\to\infty}\mathbb{E}_k\left[(\tau_{kN}\otimes I_M)\big(q(Y_t^k)\big)\right],$$

where $\mathbb{E}_k$ is the expectation with respect to $(U_t^{kN})_{t\geq 0}$.

Proof. It has been known for a long time that the unitary Brownian motion converges in distribution towards the free unitary Brownian motion, see [5]. However since we also have to consider deterministic matrices we will use Theorem 1.4 of [10]. This theorem states that if $(U_t^{kN})_{t\geq 0}$ are independent unitary Brownian motions and D^{kN} is a family of deterministic matrices which converges strongly in distribution towards a family of non-commutative random variables d, the family (U_t^{kN}, D^{kN}) in the non-commutative probability space $(\mathbb{M}_{kN}(\mathbb{C}), *, \mathbb{E}_k[\frac{1}{kN}\operatorname{Tr}])$ converges strongly in distribution towards the family (u_t, d) where u_t are freely independent free unitary Brownian motions at time t free from d. That being said, we do not use the convergence of the norm, we only need the convergence in distribution which is way easier to prove. We did not find a satisfying reference but for the reader interested, the proof of Proposition 3.2 is very similar. In our situation we can write for every i,

$$Z_i^{NM} = \sum_{1\leq r,s\leq N} E_{r,s}\otimes A_{r,s,i}^M.$$

Thus if $E^N = (E_{r,s})_{1\leq r,s\leq N}$, we fix $D^N = (U^N\otimes I_k, E^N\otimes I_k)$, and we can apply Theorem 1.4 of [10] to get that for any non-commutative polynomial P,

$$\lim_{k\to\infty}\mathbb{E}_k\left[\tau_{kN}(\widetilde{P}(U_t^{kN}, D^{kN}))\right] = \tau_N\left(\widetilde{P}(u_t, U^N, E^N)\right).$$

Consequently, for any non-commutative polynomial P, we also have

$$\lim_{k\to\infty}\mathbb{E}_k\left[\tau_{kN}\otimes I_M(\widetilde{P}(U_t^{kN}, D^{kN}, A^M))\right] = \tau_N\otimes I_M\left(\widetilde{P}(u_t, U^N, E^N, A^M)\right).$$

Hence for any $P \in \mathcal{P}_d$,

$$\lim_{k\to\infty} \mathbb{E}_k \left[\tau_{kN} \otimes I_M(P(Y_t^k))\right] = \tau_N \otimes I_M \left(P(Z_t^N)\right).$$

Now since U_t^{kN} are unitary matrices, we can find a polynomial L such that for any k, $\left\|P(Y_t^k)\right\| \leq C = L\left(\left\|U^N\right\|, \left\|Z^{NM}\right\|\right)$. Knowing this, let $f_\varepsilon \in \mathbb{C}[X]$ be a polynomial which is ε-close to $x \mapsto e^{\mathbf{i}yx}$ on the interval $[-C, C]$. Since one can always assume that $C > \left\|P(Z_t^N)\right\|$, we have a constant K such that

$$\begin{aligned} \left\|(\tau_N \otimes I_M)\big(q(Z_t^N)\big) - (\tau_N \otimes I_M)\big((Af_\varepsilon(P)B)(Z_t^N)\right\| &\leq K\varepsilon, \\ \left\|(\tau_N \otimes I_M)\big(q(Y_t^k)\big) - (\tau_N \otimes I_M)\big((Af_\varepsilon(P)B)(Y_t^k)\right\| &\leq K\varepsilon. \end{aligned}$$

Thus

$$\begin{aligned} &\left\|(\tau_N \otimes I_M)\big(q(Z_t^N)\big) - \mathbb{E}_k\left[(\tau_{kN} \otimes I_M)\big(q(Y_t^k)\big)\right]\right\| \\ &\leq \left\|(\tau_N \otimes I_M)\big((Af_\varepsilon(P)B)(Z_t^N)\big) - \mathbb{E}_k\left[(\tau_{kN} \otimes I_M)\big((Af_\varepsilon(P)B)(Y_t^k)\big)\right]\right\| + 2K\varepsilon. \end{aligned}$$

Consequently

$$\limsup_{k\to\infty} \left\|(\tau_N \otimes I_M)\big(q(Z_t^N)\big) - \mathbb{E}_R\left[(\tau_{kN} \otimes I_M)\big(q(Y_t^k)\big)\right]\right\| \leq 2K\varepsilon.$$

This completes the proof.

□

The next lemma shows that the non-diagonal coefficients can actually be neglected.

Lemma 4.2. *We define Y_t^k as in Proposition 4.1, $P_{1,2} = I_N \otimes E_{1,2} \otimes I_M$, $q = Ae^{\mathbf{i}yP}B$, then*

$$\lim_{k\to\infty} k^{1/2}\mathbb{E}_k\left[(\mathrm{Tr}_{kN} \otimes I_M)(q(Y_t^k)P_{1,2})\right] = 0.$$

Proof. Let us first define for $A, B \in \mathcal{P}_d$,

$$f_{A,B}^t(y) = \mathbb{E}_R\left[(\mathrm{Tr}_{kN} \otimes I_M)((A\ e^{\mathbf{i}yP}\ B)(Y_t^k)\ P_{1,2})\right],$$

$$d_n^t(y) = \sup_{\substack{A,B\in\mathcal{P}_d \text{ monomials,} \\ \deg(AB)\leq n \\ 0\leq s\leq t}} \left\|f_{A,B}^t(y)\right\|.$$

Consequently, there exists a constant D (which does depend on, N, $\left\|U^N\right\|$ and $\left\|Z^{NM}\right\|$) such that for any n, $d_n^t(y) \leq D^n$. Note that this constant D can be exponentially large in N, indeed it does not matter since in the end we will show that this quantity tends to 0 when k goes to infinity and the other parameters such as N, M or y are fixed. Next we define for a small enough,

$$g_{k,a}^t(y) = \sum_{n\geq 0} d_n^t(y)a^n.$$

But if we set $c_M(P)$ the coefficient associated to the monomial M in P, we have for any $s \leq t$,

$$\left|\frac{df_{A,B}^s(y)}{dy}\right| \leq \sum_{L \text{ monomials}} |c_L(P)|\ d_{\deg(AB)+\deg(L)}^t(y).$$

Thus if $\deg(AB) \leq n$, we have for any $y \geq 0$,

$$f_{A,B}^s(y) \leq f_{A,B}^s(0) + \sum_{L \text{ monomials}} |c_L(P)| \int_0^y d_{n+\deg(L)}^t(u)\ du.$$

Thus we have for $y \geq 0$ and any $n \geq 0$,

$$a^n d_n^t(y) \leq a^n d_n^t(0) + \sum_{L \text{ monomials}} |c_L(P)| a^{-\deg(L)} \int_0^y d_{n+\deg(L)}^t(u) a^{n+\deg(L)} \, du.$$

And with $\|.\|_{a^{-1}}$ defined as in (4), we have

$$g_{k,a}^t(y) \leq g_{k,a}^t(0) + \|P\|_{a^{-1}} \int_0^y g_{k,a}^t(u) du.$$

Thanks to Gronwall's inequality, we have for $y \geq 0$,

$$g_{k,a}^t(y) \leq g_{k,a}^t(0) e^{y\|P\|_{a^{-1}}}. \tag{17}$$

Now we need to find an estimate on $g_{k,a}(0)$, first for any j, one can write $Z_j^{NM} = \sum_{u,v} E_{u,v} \otimes I_k \otimes A_{u,v}^j$ for some matrices $A_{u,v}^j$, then we define

$$V_{N,k}^t = \Big(U_t^{kN}, (U_t^{kN})^*, U^N \otimes I_k, (U^N \otimes I_k)^*, (E_{u,v} \otimes I_k)_{u,v} \Big), \tag{18}$$

$$c_n^t = \sup_{\substack{\deg(L) \leq n, \ L \text{ monomial} \\ 0 \leq s \leq t}} \left| \mathbb{E}_k \left[\mathrm{Tr}_{kN}(L(V_{N,k}^s) \ P_{1,2}) \right] \right|.$$

Besides since Y_s^k is composed of

$$(U^N \otimes I_k) U_s^{kN} \otimes I_M, \quad Z_j^{NM} = \sum_{u,v} E_{u,v} \otimes I_k \otimes A_{u,v}^j,$$

and the adjoints of those matrices, if L is a monomial of degree n, then we can view $L(Y_s^k)$ as a sum of at most N^{2n} monomials of degree at most $2n$ in $U_s^{kN} \otimes I_M, U^N \otimes I_{kM}, E_{u,v} \otimes I_k \otimes A_{u,v}^j$ and their adjoints. We consider K the supremum over u, v, j of $\left\| A_{u,v}^j \right\|$, we also naturally assume that $K \geq 1$, then we have for $s \leq t$,

$$\left\| \mathbb{E}_k \left[\mathrm{Tr}_{Nk} \otimes I_M (L(U_s^k) P_{1,2}) \right] \right\| \leq N^{2n} K^n c_{2n}^t.$$

Thus if we set

$$f_k^t(a) = \sum_{n \geq 0} c_n^t a^n,$$

we have

$$g_{k,a}^t(0) \leq f_k^t(N\sqrt{K}a).$$

Now we need to study the behaviour of $f_k^t(a)$ when k goes to infinity for a small enough. Let Q be a monomial, we define Q_t as the monomial Q evaluated in $V_{N,k}^t$. Thanks to Lemma 2.4,

$$\begin{aligned}
\frac{d}{dt} \mathbb{E}_k \left[\mathrm{Tr}_{kN} \left(Q_t P_{1,2} \right) \right] = &- \frac{|Q|_B}{2} \ \mathbb{E}_k \left[\mathrm{Tr}_{kN} \left(Q_t P_{1,2} \right) \right] \\
&- \frac{1}{kN} \sum_{i \leq p, \ Q = A U_i B U_i C} \mathbb{E}_k \left[\mathrm{Tr}_{kN} \left(A_t U_{i,t}^{kN} C_t P_{1,2} \right) \mathrm{Tr}_{kN} \left(B_t U_{i,t}^{kN} \right) \right] \\
&- \frac{1}{kN} \sum_{i \leq p, \ Q = A U_i^* B U_i^* C} \mathbb{E}_k \left[\mathrm{Tr}_{kN} \left(A_t U_{i,t}^{kN^*} C_t P_{1,2} \right) \mathrm{Tr}_{kN} \left(B_t U_{i,t}^{kN^*} \right) \right] \\
&+ \frac{1}{kN} \sum_{i \leq p, \ Q = A U_i B U_i^* C} \mathbb{E}_k \left[\mathrm{Tr}_{kN} (A_t C_t P_{1,2}) \, \mathrm{Tr}_{kN} (B_t) \right] \\
&+ \frac{1}{kN} \sum_{i \leq p, \ Q = A U_i^* B U_i C} \mathbb{E}_k \left[\mathrm{Tr}_{kN} (A_t C_t P_{1,2}) \, \mathrm{Tr}_{kN} (B_t) \right].
\end{aligned}$$

Since $\mathbb{E}_k \left[\mathrm{Tr}_{kN} \left(Q_0 P_{1,2} \right) \right] = 0$, we have for any t,

$$\begin{aligned}\mathbb{E}_k\left[\mathrm{Tr}_{kN}\left(Q_tP_{1,2}\right)\right] = \int_0^t e^{-\frac{|Q|_B}{2}(t-s)} \Bigg(&-\frac{1}{kN}\sum_{i\leq p,\ Q=AU_iBU_iC} \mathbb{E}_k\left[\mathrm{Tr}_{kN}\left(A_sU_{i,s}^{kN}C_sP_{1,2}\right)\mathrm{Tr}_{kN}\left(B_sU_{i,s}^{kN}\right)\right] \\ &-\frac{1}{kN}\sum_{i\leq p,\ Q=AU_i^*BU_i^*C} \mathbb{E}_k\left[\mathrm{Tr}_{kN}\left(A_sU_{i,s}^{kN^*}C_tP_{1,2}\right)\mathrm{Tr}_{kN}\left(B_sU_{i,s}^{kN^*}\right)\right] \\ &+\frac{1}{kN}\sum_{i\leq p,\ Q=AU_iBU_i^*C} \mathbb{E}_k\left[\mathrm{Tr}_{kN}(A_sC_sP_{1,2})\,\mathrm{Tr}_{kN}(B_s)\right] \\ &+\frac{1}{kN}\sum_{i\leq p,\ Q=AU_i^*BU_iC} \mathbb{E}_k\left[\mathrm{Tr}_{kN}(A_sC_sP_{1,2})\,\mathrm{Tr}_{kN}(B_s)\right]\Bigg)ds.\end{aligned}$$

As in Proposition 3.1, we consider $(V_t^{kN})_{t\in\mathbb{R}^+}$ and $(W_t^{kN})_{t\in\mathbb{R}^+}$ independent vectors of p unitary Brownian motions of size kN, independent of $(U_t^{kN})_{t\in\mathbb{R}^+}$. We define $V_{N,k}^{r,1}$ and $V_{N,k}^{r,2}$ as $V_{N,k}^{r}$ (see (18)) but with $V_{s-r}^{kN}U_r^{kN}$ and $W_{s-r}^{kN}U_r^{kN}$ instead of U_r^{kN}. Thanks to Proposition 3.1, by polarization and the fact that $(\mathcal{D}_iQ^*)^* = -\mathcal{D}_iQ$, we have with $\mathrm{Cov}(X,Y) = \mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]$,

$$\mathrm{Cov}_k\left(\mathrm{Tr}_{kN}(A_sC_sP_{1,2}),\mathrm{Tr}_{kN}(B_s)\right) = -\frac{1}{kN}\sum_{i\leq p}\int_0^s \mathbb{E}_k\Big[\,\mathrm{Tr}_{kN}\left(\left(\delta_i(AC)\widetilde{\#}P_{1,2}\right)\left(V_{N,k}^{r,1}\right)\ (\mathcal{D}_iB)\left(V_{N,k}^{r,2}\right)\right)\Big]dr.$$

Since $P_{1,2}$ is a matrix of rank N and that we can assume $D\geq \max(1,\left\|U^N\right\|)$, we have

$$\left|\mathrm{Cov}_k\left(\mathrm{Tr}_{kN}(A_sC_sP_{1,2}),\mathrm{Tr}_{kN}(B_s)\right)\right| \leq \frac{s}{k}\deg(AC)\deg(B)\ D^{\deg(ABC)}.$$

We now assume that Q has degree at most n, then $\left|\mathrm{Cov}_k\left(\mathrm{Tr}_{kN}(A_sC_sP_{1,2}),\mathrm{Tr}_{kN}(B_s)\right)\right| \leq \frac{s}{k}n^2D^n$. Thus we have,

$$\begin{aligned}\left|\mathbb{E}_k\left[\mathrm{Tr}_{kN}\left(Q_tP_{1,2}\right)\right]\right| \leq{}& \frac{n^4t^2D^n}{k^2N} \\ &+\frac{1}{kN}\int_0^t e^{-\frac{|Q|_B}{2}(t-s)}\Bigg(\sum_{i\leq p,\ Q=AU_iBU_iC}\left|\mathbb{E}_k\left[\mathrm{Tr}_{kN}\left(A_sU_{i,s}^{kN}C_sP_{1,2}\right)\right]\mathbb{E}_k\left[\mathrm{Tr}_{kN}\left(B_sU_{i,s}^{kN}\right)\right]\right| \\ &\qquad\sum_{i\leq p,\ Q=AU_i^*BU_i^*C}\left|\mathbb{E}_k\left[\mathrm{Tr}_{kN}\left(A_sU_{i,s}^{kN^*}C_tP_{1,2}\right)\right]\mathbb{E}_k\left[\mathrm{Tr}_{kN}\left(B_sU_{i,s}^{kN^*}\right)\right]\right| \\ &\qquad\sum_{i\leq p,\ Q=AU_iBU_i^*C}\left|\mathbb{E}_k\left[\mathrm{Tr}_{kN}(A_sC_sP_{1,2})\right]\mathbb{E}_k\left[\mathrm{Tr}_{kN}(B_s)\right]\right| \\ &\qquad\sum_{i\leq p,\ Q=AU_i^*BU_iC}\left|\mathbb{E}_k\left[\mathrm{Tr}_{kN}(A_sC_sP_{1,2})\right]\mathbb{E}_k\left[\mathrm{Tr}_{kN}(B_s)\right]\right|\Bigg)ds.\end{aligned}$$

This means that,

$$\begin{aligned}
|\mathbb{E}_k\left[\mathrm{Tr}_{kN}\left(Q_tP_{1,2}\right)\right]| \leq \frac{n^4t^2D^n}{k^2N} + \int_0^t e^{-\frac{|Q|_B}{2}(t-s)} \Bigg(& \sum_{i\leq p,\ Q=AU_iBU_iC} c^t_{\deg(AC)+1} D^{\deg(B)+1} \\
& + \sum_{i\leq p,\ Q=AU_i^*BU_i^*C} c^t_{\deg(AC)+1} D^{\deg(B)+1} \\
& + \sum_{i\leq p,\ Q=AU_iBU_i^*C} c^t_{\deg(AC)} D^{\deg(B)} \\
& + \sum_{i\leq p,\ Q=AU_i^*BU_iC} c^t_{\deg(AC)} D^{\deg(B)} \Bigg) ds \\
\leq \frac{n^4t^2D^n}{k^2N} + \int_0^t |Q|_B e^{-\frac{|Q|_B}{2}s} ds & \sum_{0\leq d\leq n-1} D^d c^t_{n-1-d} \\
\leq \frac{n^4t^2D^n}{k^2N} + 2 \sum_{0\leq d\leq n-1} D^d c^t_{n-1-d}. &
\end{aligned}$$

Hence, for any $n \geq 1$,

$$c^t_n \leq \frac{n^4t^2D^n}{k^2N} + 2 \sum_{0\leq d\leq n-1} D^d c^t_{n-1-d}.$$

Since we are taking the trace of $L(V^s_{N,k})P_{1,2}$ with $P_{1,2} = I_N \otimes E_{1,2} \otimes I_M$, we have $c_0 = 0$. We fix $s : a \mapsto \sum_{n\geq 0} \frac{n^4t^2(aD)^n}{N}$, thus for a small enough,

$$\begin{aligned}
f^t_k(a) &\leq \frac{s(a)}{k^2} + 2\sum_{n\geq 1}\left(\sum_{0\leq d\leq n-1} D^d c^t_{n-1-d}\right) a^n \\
&\leq \frac{s(a)}{k^2} + \frac{2a}{1-aD} f^t_k(a)
\end{aligned}$$

Thus for a small enough, $f^t_k(a) \leq 2s(a)k^{-2}$. Which means that $f^t_k(a) = \mathcal{O}(k^{-2})$ and since as we have already said, $g^t_{k,a}(0) \leq f^t_k(N\sqrt{Ka})$, we have for any fixed t and y that $g^t_{k,a_k}(y) = \mathcal{O}(k^{-2})$. Since

$$\left| k^{3/2}\mathbb{E}_k\left[(\tau_{kN}\otimes I_M)(q(Y^k_t)P_{1,2})\right]\right| \leq k^{1/2}a^{-\deg(AB)} g^t_{k,a_k}(y),$$

we get the conclusion.

□

We can now prove the following intermediary lemma that will allow us to derive Lemma 4.4. This lemma is the only one where the law of U^N actually plays an important part. In other words the fact that U^N is a vector of random unitary i.i.d. matrices of size N whose law is invariant by multiplication by a matrix of $SU_N(\mathbb{R})$ will be the crux of the proof.

Lemma 4.3. *We define Y^k_t as in Proposition 4.1, we set*

- $P_{l,l'} = I_N \otimes E_{l,l'} \otimes I_M$,
- $q = A\ e^{\mathbf{i}yP}\ B$.

Then for every $M, N \in \mathbb{N}$, $t \in \mathbb{R}^+$ and $y \in \mathbb{R}$,

$$\begin{aligned}
S^N_{t,y}(A,B) = \lim_{k\to\infty} -\frac{1}{kN^2}\mathbb{E}\Bigg[\tau_M\Bigg(\sum_{1\leq l,l'\leq k} & \mathbb{E}_k\Big[(\mathrm{Tr}_{kN}\otimes I_M)^{\bigotimes 2}\left(\delta_i q(Y^k_t)\times P_{l',l}\otimes P_{l,l'}\right)\Big] \\
& - \mathbb{E}_k[\mathrm{Tr}_{kN}\otimes I_M]^{\bigotimes 2}\left(\delta_i q(Y^k_t)\times P_{l',l}\otimes P_{l,l'}\right)\Bigg)\Bigg]
\end{aligned}$$

Proof. Since all of our random variables are unitary matrices, thanks to Proposition 4.1 and the dominated convergence theorem,

$$S^N_{t,y}(A,B) = \lim_{k\to\infty} \mathbb{E}\left[\tau_M\left(\mathbb{E}_k[\tau_{kN}\otimes I_M] \bigotimes \mathbb{E}_k[\tau_{kN}\otimes I_M]\left(\delta_i\left(A\ e^{\mathbf{i}yP}\ B\right)\left(Y^k_t\right)\right)\right)\right], \tag{19}$$

where $\mathbb{E}_k[\tau_N\otimes I_M] \bigotimes \mathbb{E}_k[\tau_N\otimes I_M]\left(A\otimes B\left(Y^k_t\right)\right) = \mathbb{E}_k[\tau_N\otimes I_M(A(Y^k_t))]\mathbb{E}_k[\tau_N\otimes I_M(B(Y^k_t))]$. Since given $V\in SU_N(\mathbb{R})$, $(U^{kN}_{t,1}, U^N_1\otimes I_k, \dots, U^{kN}_{t,p}, U^N_p\otimes I_k)$ has the same law as $((V^*\otimes I_l)U^{kN}_{t,1}(V\otimes I_l), (U^N_1 V)\otimes I_l, U^{kN}_{t,2}, U^N_2\otimes I_k, \dots, U^{kN}_{t,p}, U^N_p\otimes I_k)$, we have

$$\begin{aligned}\mathbb{E}[q(Y^k_t)] = \mathbb{E}\Big[\widetilde{q}\Big(&(U^N_1\otimes I_k\ U^{kN}_{t,1})\otimes I_M\ (V\otimes I_{kM}),\\ &(U^N_2\otimes I_k\ U^{kN}_{t,2})\otimes I_M, \dots, (U^N_p\otimes I_k\ U^{kN}_{t,p})\otimes I_M, Z^{NM}\otimes I_k\Big)\Big].\end{aligned}$$

Hence let H be an skew-hermitian matrix, then for any $s\in\mathbb{R}$, $e^{sH}\in SU_N(\mathbb{R})$, thus by taking V this matrix and differentiating with respect to s we get that, $\mathbb{E}\Big[\delta_1 q(Y^k_t)\#(H\otimes I_{kM})\Big] = 0$. And similarly we get that for any i,

$$\mathbb{E}\Big[\delta_i q(Y^k_t)\#(H\otimes I_{kM})\Big] = 0.$$

Since every matrix is a linear combination of skew-Hermitian matrices, this is true for any matrix $H\in\mathbb{M}_N(\mathbb{C})$, and thus for any i,

$$\mathbb{E}\Big[(\mathrm{Tr}_N\otimes I_{kM})^{\bigotimes 2}\left(\delta_i q(Y^k_t)\right)\Big] = \sum_{1\le r,s\le N} g^*_r\otimes I_{kM}\ \mathbb{E}\Big[\delta_i q(Y^k_t)\#(E_{r,s}\otimes I_{kM})\Big] g_s\otimes I_{kM} = 0 \tag{20}$$

Let $S,T\in\mathbb{M}_{NkM}(\mathbb{C})$ be deterministic matrices, then

$$\begin{aligned}&\mathrm{Tr}_k\otimes I_M\left((\mathrm{Tr}_N\otimes I_{kM})^{\bigotimes 2}(S\otimes T)\right)\\ &= \sum_{1\le m,n\le N} \mathrm{Tr}_{Nk}\otimes I_M\left(S\ E_{m,n}\otimes I_{kM}\ T\ E_{n,m}\otimes I_{kM}\right)\\ &= \sum_{1\le l,l'\le k}\sum_{1\le m\le N} g^*_m\otimes f^*_l\otimes I_M\ S\ g_m\otimes f_{l'}\otimes I_M \sum_{1\le n\le N} g^*_n\otimes f^*_{l'}\otimes I_M\ T\ g_n\otimes f_l\otimes I_M\\ &= \sum_{1\le l,l'\le k} \mathrm{Tr}_N\otimes I_M(I_N\otimes f^*_l\otimes I_M\ S\ I_N\otimes f_{l'}\otimes I_M)\,\mathrm{Tr}_N\otimes I_M(I_N\otimes f^*_{l'}\otimes I_M\ T\ I_N\otimes f_l\otimes I_M)\\ &= \sum_{1\le l,l'\le k} \mathrm{Tr}_{kN}\otimes I_M\big(S\ I_N\otimes E_{l',l}\otimes I_M\big)\,\mathrm{Tr}_{kN}\otimes I_M\big(T\ I_N\otimes E_{l,l'}\otimes I_M\big).\end{aligned}$$

Thus by using equation (20), we have for any i,

$$\sum_{1\le l,l'\le k} \mathbb{E}\Big[(\mathrm{Tr}_{kN}\otimes I_M)^{\bigotimes 2}\left(\delta_i q(Y^k_t)\times P_{l',l}\otimes P_{l,l'}\right)\Big] = 0.$$

And consequently,

$$\begin{aligned}&\sum_{1\le l,l'\le k} \mathbb{E}\Big[(\mathrm{Tr}_{kN}\otimes I_M)^{\bigotimes 2}\left(\delta_i q(Y^k_t)\times P_{l',l}\otimes P_{l,l'}\right)\Big] - \mathbb{E}\left[\mathbb{E}_k[\mathrm{Tr}_{kN}\otimes I_M]^{\bigotimes 2}\left(\delta_i q(Y^k_t)\times P_{l',l}\otimes P_{l,l'}\right)\right]\\ &= -\sum_{1\le l,l'\le k} \mathbb{E}\left[\mathbb{E}_k[\mathrm{Tr}_{kN}\otimes I_M]^{\bigotimes 2}\left(\delta_i q(Y^k_t)\times P_{l',l}\otimes P_{l,l'}\right)\right].\end{aligned} \tag{21}$$

Let $V,W\in\mathbb{M}_k(\mathbb{C})$ be permutation matrices. Since $I_{NM}\otimes V$ commutes with $Z^{NM}\otimes I_k$ and $U^N\otimes I_{kM}$, and that the law of U^{kN}_t is invariant by conjugation by a unitary matrix, it follows that the law of every matrix of Y^k_t is invariant by conjugation by $I_{NM}\otimes V$ or $I_{NM}\otimes W$. Thus,

$$\mathbb{E}_k[\mathrm{Tr}_{kN}\otimes I_M]^{\bigotimes 2}\left(\delta_i q(Y^k_t)\times P_{l',l}\otimes P_{l,l'}\right) = \mathbb{E}_k[\mathrm{Tr}_{kN}\otimes I_M]^{\bigotimes 2}\left(\delta_i q(Y^k_t)\times VP_{l',l}V^*\otimes WP_{l,l'}W^*\right).$$

Thus by using well-chosen matrices, we get

- if $l = l'$, $\mathbb{E}_k[\mathrm{Tr}_{kN} \otimes I_M]^{\bigotimes 2}\left(\delta_i q(Y_t^k) \times P_{l',l} \otimes P_{l,l'}\right) = \mathbb{E}_k[\mathrm{Tr}_{kN} \otimes I_M]^{\bigotimes 2}\left(\delta_i q(Y_t^k) \times P_{1,1} \otimes P_{1,1}\right)$,
- if $l \neq l'$, $\mathbb{E}_k[\mathrm{Tr}_{kN} \otimes I_M]^{\bigotimes 2}\left(\delta_i q(Y_t^k) \times P_{l',l} \otimes P_{l,l'}\right) = \mathbb{E}_k[\mathrm{Tr}_{kN} \otimes I_M]^{\bigotimes 2}\left(\delta_i q(Y_t^k) \times P_{1,2} \otimes P_{1,2}\right)$.

Consequently, we have that

- equation (21) is equal to

$$\begin{aligned}
&\sum_{1\leq l,l'\leq k} \mathbb{E}\Big[(\mathrm{Tr}_{kN} \otimes I_M)^{\bigotimes 2}\left(\delta_i q(Y_t^k) \times P_{l',l} \otimes P_{l,l'}\right)\Big] - \mathbb{E}\left[\mathbb{E}_k[\mathrm{Tr}_{kN} \otimes I_M]^{\bigotimes 2}\left(\delta_i q(Y_t^k) \times P_{l',l} \otimes P_{l,l'}\right)\right] \\
&= -k\mathbb{E}\left[\mathbb{E}_k[\mathrm{Tr}_{kN} \otimes I_M]^{\bigotimes 2}\left(\delta_i q(Y_t^k) \times P_{1,1} \otimes P_{1,1}\right)\right] \\
&\quad - k(k-1)\mathbb{E}\left[\mathbb{E}_k[\mathrm{Tr}_{kN} \otimes I_M]^{\bigotimes 2}\left(\delta_i q(Y_t^k) \times P_{1,2} \otimes P_{1,2}\right)\right].
\end{aligned}$$

- Whereas the quantity inside the trace τ_M in equation (19) is equal to

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{E}_k[\tau_{kN} \otimes I_M] \bigotimes \mathbb{E}_k[\tau_{kN} \otimes I_M]\left(\delta_i q\left(Y_t^k\right)\right)\right] \\
&= \frac{1}{(kN)^2} \sum_{1\leq l,l'\leq k} \mathbb{E}\left[\mathbb{E}_k[\mathrm{Tr}_{kN} \otimes I_M]^{\bigotimes 2}\left(\delta_i q\left(Y_t^k\right) \times P_{l,l} \otimes P_{l',l'}\right)\right] \\
&= \frac{1}{N^2}\mathbb{E}\left[\mathbb{E}_k[\mathrm{Tr}_{kN} \otimes I_M]^{\bigotimes 2}\left(\delta_i q\left(Y_t^k\right) \times P_{1,1} \otimes P_{1,1}\right)\right].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
S_{t,y}^N(A,B) = \lim_{k\to\infty} -\frac{1}{kN^2}\tau_M\Bigg(&\sum_{1\leq l,l'\leq k} \mathbb{E}\Big[(\mathrm{Tr}_{kN} \otimes I_M)^{\bigotimes 2}\left(\delta_i q(Y_t^k) \times P_{l',l} \otimes P_{l,l'}\right)\Big] \\
&- \mathbb{E}\left[\mathbb{E}_k[\mathrm{Tr}_{kN} \otimes I_M]^{\bigotimes 2}\left(\delta_i q(Y_t^k) \times P_{l',l} \otimes P_{l,l'}\right)\right]\Bigg) \\
&-\frac{k-1}{N^2}\mathbb{E}\left[\tau_M\left(\mathbb{E}_k[\mathrm{Tr}_{kN} \otimes I_M]^{\bigotimes 2}\left(\delta_i q(Y_t^k) \times P_{1,2} \otimes P_{1,2}\right)\right)\right].
\end{aligned}$$

Thanks to Lemma 4.2 and Proposition 2.2, the second term converges towards 0, which gives the expected formula. □

Lemma 4.3 makes a covariance appears. Thus it is natural to want to use Corollary 3.1 to get an explicit expression for $S_{t,y}^N(A,B)$.

Lemma 4.4. *For any N, we have*

$$S_{t,y}^N(A,B) = \frac{t}{N^2}\sum_{j\leq p}\mathbb{E}\left[\tau_N \otimes \tau_M\Big((h\circ\delta_j)(\delta_i^1 q)(U^N u_t \otimes I_M, Z^{NM}) \boxtimes (h\circ\delta_j)(\delta_i^2 q)(U^N v_t \otimes I_M, Z^{NM})\Big)\right].$$

where u_t and v_t are vectors of free Brownian motions which are freely independent.

Proof. Thanks to Lemma 4.3 and 3.1 we have

$$\begin{aligned}
&S_{t,y}^N(A,B) \\
&= \lim_{k\to\infty} \frac{1}{k^2N^3}\sum_{j\leq p}\int_0^t \mathbb{E}\Bigg[\tau_M \otimes \mathrm{Tr}_{kN}\Bigg(\sum_{1\leq l,l'\leq k} P_{l',l}\ (h\circ\delta_j)(\delta_i^1 q)((U^N \otimes I_k)V_{t-s}^{kN}U_s^{kN} \otimes I_M, Z^{NM} \otimes I_k) \\
&\qquad \boxtimes P_{l,l'}\ (h\circ\delta_j)(\delta_i^2 q)((U^N \otimes I_k)W_{t-s}^{kN}U_s^{kN} \otimes I_M, Z^{NM} \otimes I_k)\Bigg)\Bigg]\ ds.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
&S^N_{t,y}(A,B)\\
&= \lim_{k\to\infty}\frac{1}{N^2}\sum_{j\le p}\int_0^t \mathbb{E}\Bigg[\tau_M\otimes\tau_N\Bigg(\tau_k\otimes I_{MN}\left((h\circ\delta_j)(\delta_i^1 q)((U^N\otimes I_k)V^{kN}_{t-s}U^{kN}_s\otimes I_M, Z^{NM}\otimes I_k)\right)\boxtimes\\
&\qquad\qquad \tau_k\otimes I_{MN}\left((h\circ\delta_j)(\delta_i^2 q)((U^N\otimes I_k)W^{kN}_{t-s}U^{kN}_s\otimes I_M, Z^{NM}\otimes I_k)\right)\Bigg)\Bigg]\ ds.
\end{aligned}
$$

Let $x,y\in\mathbb{M}_{kN}(\mathbb{C})\otimes\mathbb{M}_M(\mathbb{C})$, thanks to Lemma 3.1, $\|h(x\otimes y)\|\le M\,\|x\|\,\|y\|$. Moreover, let us remind that, with the convention $A\times(B\otimes C)\times D=(AB)\otimes(CD)$, we have for $q=Ae^{\mathbf{i}\beta yP}B$,

$$
\delta_i q=\delta_i A\ e^{\mathbf{i}\beta yP}\ B+\mathbf{i}\beta yA\int_0^1 e^{\mathbf{i}(1-u)\beta yP}\ \delta_i P\ e^{\mathbf{i}u\beta yP}\ Bdu+A\ e^{\mathbf{i}\beta yP}\ \delta_i B. \tag{22}
$$

Thus similarly to the proof of proposition 4.1, we can approximate the exponentials by polynomials and hence approximate $(h\circ\delta_j)(\delta_i^1 q)$ and $(h\circ\delta_j)(\delta_i^2 q)$ uniformly in k with polynomials in Y^k_t. If Q and T are such polynomials, then thanks to Corollary 3.1,

$$
\begin{aligned}
&\mathbb{E}_k\left[\tau_k\otimes I_{MN}(Q)\tau_k\otimes I_{MN}(T)\right]\\
&=\frac{1}{k^2}\sum_{1\le i,j,r,s\le N}\mathbb{E}_k\left[(\mathrm{Tr}_{kN}\otimes I_M(QI_k\otimes E_{j,i}\otimes I_M))\otimes E_{i,j}\ \ (\mathrm{Tr}_{kN}\otimes I_M(TI_k\otimes E_{s,r}\otimes I_M))\otimes E_{r,s}\right]\\
&=\frac{1}{k^2}\sum_{1\le i,j,r,s\le N}\mathbb{E}_k\left[(\mathrm{Tr}_{kN}\otimes I_M(QI_k\otimes E_{j,i}\otimes I_M))\ \ (\mathrm{Tr}_{kN}\otimes I_M(TI_k\otimes E_{s,r}\otimes I_M))\right]\otimes(E_{i,j}E_{r,s})\\
&=\mathcal{O}(k^{-2})+\frac{1}{k^2}\sum_{1\le i,j,r,s\le N}\Big(\mathbb{E}_k\left[(\mathrm{Tr}_{kN}\otimes I_M(QI_k\otimes E_{j,i}\otimes I_M))\right]\\
&\qquad\qquad \mathbb{E}_k\left[(\mathrm{Tr}_{kN}\otimes I_M(TI_k\otimes E_{s,r}\otimes I_M))\right]\Big)\otimes(E_{i,j}E_{r,s})\\
&=\mathcal{O}(k^{-2})+\mathbb{E}_k\left[\tau_k\otimes I_{MN}(Q)\right]\mathbb{E}_k\left[\tau_k\otimes I_{MN}(T)\right]
\end{aligned}
$$

Besides, since $(U_t)_{t\ge0}$ and $(V_t)_{t\ge0}$ are independent, $V^{kN}_{t-s}U^{kN}_s$ has the same law as U^{kN}_t, thus

$$
\begin{aligned}
&S^N_{t,y}(A,B)\\
&=\frac{t}{N^2}\sum_{j\le p}\mathbb{E}\Bigg[\tau_M\otimes\tau_N\Bigg(\lim_{k\to\infty}\mathbb{E}_k\Big[\tau_k\otimes I_{MN}\left((h\circ\delta_j)(\delta_i^1 q)((U^N\otimes I_k)U^{kN}_t\otimes I_M, Z^{NM}\otimes I_k)\right)\Big]\\
&\qquad\qquad\boxtimes\mathbb{E}_k\Big[\tau_k\otimes I_{MN}\left((h\circ\delta_j)(\delta_i^2 q)((U^N\otimes I_k)U^{kN}_t\otimes I_M, Z^{NM}\otimes I_k)\right)\Big]\Bigg)\Bigg].
\end{aligned}
$$

Thanks to Proposition 3.2, if τ is the trace on the $\mathcal{C}^*$-algebra where the variables $\mathbf{u}^t$ belongs, then viewing $\mathbf{u}^t$ as a vector of matrices of size N, we get

$$
\begin{aligned}
&\lim_{k\to\infty}\mathbb{E}_k\Big[\tau_k\otimes I_{MN}\left((h\circ\delta_j)(\delta_i^1 q)((U^N\otimes I_k)U^{kN}_t\otimes I_M, Z^{NM}\otimes I_k)\right)\Big]\\
&=\tau\otimes I_{MN}\left((h\circ\delta_j)(\delta_i^1 q)(U^N\mathbf{u}^t\otimes I_M, Z^{NM})\right)
\end{aligned}
$$

Thus if $\mathbf{v}^t$ is a freely independent copy of $\mathbf{u}^t$, we have

$$
\begin{aligned}
S^N_{t,y}(A,B)=\frac{t}{N^2}\sum_{j\le p}\mathbb{E}\Bigg[\tau_M\otimes\tau_N\otimes\tau\Bigg(&(h\circ\delta_j)(\delta_i^1 q)(U^N\mathbf{u}^t\otimes I_M, Z^{NM})\\
&\boxtimes(h\circ\delta_j)(\delta_i^2 q)(U^N\mathbf{v}^t\otimes I_M, Z^{NM})\Bigg)\Bigg].
\end{aligned}
$$

But then thanks to Proposition 3.2 again, we have

$$S^N_{t,y}(A,B) = \frac{t}{N^2}\sum_{j\leq p}\mathbb{E}\Bigg[\lim_{k\to\infty}\mathbb{E}_k\Big[\tau_M\otimes\tau_{kN}\Big((h\circ\delta_j)(\delta_i^1 q)((U^N\otimes I_k)U_t^{kN}\otimes I_M, Z^{NM}\otimes I_k)$$

$$\boxtimes (h\circ\delta_j)(\delta_i^2 q)((U^N\otimes I_k)V_t^{kN}\otimes I_M, Z^{NM}\otimes I_k)\Big)\Big]\Bigg].$$

And we conclude with Proposition 4.1.

□

The next step to prove Theorem 4.1 is to find an estimate for $S^N_{t,y}(A,B)$ in all of its parameters. In particular we have to be careful to pay attention to the estimation in t if we want to integrate from 0 to infinity.

Lemma 4.5. *There exists a polynomial L_P such that for any t, y, N, Z^{NM},*

$$|S^N_{t,y}(A,B)| \leq L_P\left(\|Z^{NM}\|\right)\frac{M^2}{N^2}(1+|y|^3)\ t \tag{23}$$

$$\left|S^N_{t,y}(A,B)\right| \leq L_P\left(\|Z^{NM}\|\right)\frac{M^2}{N^2}(1+y^4)\ te^{-\frac{t}{2}} \tag{24}$$

Proof. The first inequality is a straightforward consequence of Lemma 4.1 and 4.4. For the second one, we want to study the asymptotic of $S^N_{t,y}(A,B)$ when t go to infinity. Let u and v be vectors of freely independent Haar unitaries free between each other and free from $\mathbb{M}_N(\mathbb{C})$, then by invariance of Haar unitaries by multiplication by a free unitary,

$$\begin{aligned}&\tau_N\otimes\tau_M\Big((h\circ\delta_j)(\widetilde{\delta_i^1 q})(U^N u\otimes I_M, Z^{NM})\boxtimes(h\circ\delta_j)(\widetilde{\delta_i^2 q})(U^N v\otimes I_M, Z^{NM})\Big)\\ &=\tau_N\otimes\tau_M\Big((h\circ\delta_j)(\widetilde{\delta_i^1 q})(u\otimes I_M, Z^{NM})\boxtimes(h\circ\delta_j)(\widetilde{\delta_i^2 q})(v\otimes I_M, Z^{NM})\Big)\end{aligned} \tag{25}$$

Thanks to equation (5) and the remark preceding it, we can approximate equation (25) by replacing q by a polynomial in $\mathcal{P}_d$. We know that for any i, $Z_i^{NM} = \sum_{1\leq u,v\leq M} A_i^{u,v}\otimes E_{u,v}$, thus if we set $A = (A_i^{u,v})_{i,u,v}$, to show that (25) is zero, we only need to show that for any monomials Q,

$$\tau_N\Big(\mathcal{D}_j(\widetilde{\delta_i^1 Q})(u,A)\boxtimes\mathcal{D}_j(\widetilde{\delta_i^2 Q})(v,A)\Big) = 0. \tag{26}$$

In order to do so, if there exist S, T monomials and l such that $Q = SU_lU_l^*T$, then

$$\delta_i Q = \delta_i S\times 1\otimes U_lU_l^*T + SU_lU_l^*\otimes 1\times\delta_i T$$

In particular for any polynomial K, H,

$$\mathcal{D}_j(\widetilde{K}U_lU_l^*\widetilde{H})(u,A) = \delta_j\widetilde{K}(u,A)\ \widetilde{\#}\ \widetilde{H}(u,A) + \delta_j\widetilde{H}(u,A)\ \widetilde{\#}\ \widetilde{K}(u,A) = \mathcal{D}_j(\widetilde{KH})(u,A).$$

Consequently, we have

$$\begin{aligned}\tau_N\Big(\mathcal{D}_j(\widetilde{\delta_i^1 Q})(u,A)\boxtimes\mathcal{D}_j(\widetilde{\delta_i^2 Q})(v,A)\Big) &= \tau_N\Big(\mathcal{D}_j(\widetilde{\delta_i^1 S})(u,A)\boxtimes\mathcal{D}_j(\widetilde{\delta_i^2 S}\times U_lU_l^*\widetilde{T})(v,A)\Big)\\ &\quad+\tau_N\Big(\mathcal{D}_j(\widetilde{S}U_lU_l^*\times\widetilde{\delta_i^1 T})(u,A)\boxtimes\mathcal{D}_j(\widetilde{\delta_i^2 T})(v,A)\Big)\\ &=\tau_N\Big(\mathcal{D}_j(\widetilde{\delta_i^1 S})(u,A)\boxtimes\mathcal{D}_j(\widetilde{\delta_i^2 S}\times\widetilde{T})(v,A)\Big)\\ &\quad+\tau_N\Big(\mathcal{D}_j(\widetilde{S}\times\widetilde{\delta_i^1 T})(u,A)\boxtimes\mathcal{D}_j(\widetilde{\delta_i^2 T})(v,A)\Big)\\ &=\tau_N\Big(\mathcal{D}_j(\widetilde{\delta_i^1(ST)})(u,A)\boxtimes\mathcal{D}_j(\widetilde{\delta_i^2(ST)})(v,A)\Big)\end{aligned}$$

By induction one can show that Q is a linear combination of monomial of the form $A_0U_{i_1}^{n_1}A_1\dots A_{k-1}U_{i_k}^{n_k}A_k$ where for any s, $n_s\in\mathbb{Z}\setminus\{0\}$ and either $\tau_N(A_s)=0$, or $A_s=I_N$ and $i_s\neq i_{s+1}$. Thus δ_iQ is a linear combination of terms of one of the following form,

$$A_0U_{i_1}^{n_1}A_1\dots A_{k-1}U_{i_k}^{n_k}\otimes B_0U_{j_1}^{m_1}B_1\dots B_{l-1}U_{j_l}^{m_l}B_l,$$

$$A_0U_{i_1}^{n_1}A_1\dots A_{k-1}U_{i_k}^{n_k}A_k\otimes U_{j_1}^{m_1}B_1\dots B_{l-1}U_{j_l}^{m_l}B_l,$$

where for any s, $m_s,n_s\in\mathbb{Z}\setminus\{0\}$ and either $A_s=I_N$ and $i_s\neq i_{s+1}$, or $\tau_N(A_s)=0$, similarly for B_s. We are now going to prove (26). We restrict ourselves to the first case since the second one is similar. We need to prove that

$$\tau_N\Big(\mathcal{D}_j(A_0u_{i_1}^{n_1}A_1\dots A_{k-1}u_{i_k}^{n_k})\times\mathcal{D}_j(B_0v_{j_1}^{m_1}B_1\dots B_{l-1}v_{j_l}^{m_l}B_l)\Big)=0.$$

This is a linear combination of terms of the form,

$$\tau_N\Big(B_0v_{j_1}^{m_1}B_1\dots B_{t-1}v_{j_t}^{\alpha_2}u_{i_r}^{\beta_1}A_r\dots u_{i_k}^{n_k}A_0u_{i_1}^{n_1}A_1\dots A_{r-1}u_{i_r}^{\alpha_1}v_{j_t}^{\beta_2}B_t\dots B_{l-1}v_{j_l}^{m_l}B_l\Big),$$

where $\alpha_1+\beta_1=n_r$ and $\alpha_2+\beta_2=m_t$. If $(\alpha_2,\beta_1)\neq(0,0)$ and $(\alpha_1,\beta_2)\neq(0,0)$, then this is equal to zero by freeness of u,v and $\mathbb{M}_N(\mathbb{C})$. Otherwise if $(\alpha_2,\beta_1)=(0,0)$ for example, then

$$\begin{aligned}
&\tau_N\Big(B_0v_{j_1}^{m_1}B_1\dots B_{t-1}v_{j_t}^{\alpha_2}u_{i_r}^{\beta_1}A_r\dots u_{i_k}^{n_k}A_0u_{i_1}^{n_1}A_1\dots A_{r-1}u_{i_r}^{\alpha_1}v_{j_t}^{\beta_2}B_t\dots B_{l-1}v_{j_l}^{m_l}B_l\Big)\\
&=\tau_N\Big(B_0v_{j_1}^{m_1}B_1\dots v_{j_{t-1}}^{m_{t-1}}(B_{t-1}A_r-\tau_N(B_{t-1}A_r))u_{i_{r+1}}^{n_{r+1}}\dots u_{i_k}^{n_k}A_0u_{i_1}^{n_1}A_1\dots A_{r-1}u_{i_r}^{\alpha_1}v_{j_t}^{\beta_2}B_t\dots B_{l-1}v_{j_l}^{m_l}B_l\Big)\\
&\quad+\tau_N(B_{t-1}A_r))\tau_N\Big(B_0v_{j_1}^{m_1}B_1\dots v_{j_{t-1}}^{m_{t-1}}u_{i_{r+1}}^{n_{r+1}}\dots u_{i_k}^{n_k}A_0u_{i_1}^{n_1}A_1\dots A_{r-1}u_{i_r}^{\alpha_1}v_{j_t}^{\beta_2}B_t\dots B_{l-1}v_{j_l}^{m_l}B_l\Big)\\
&=0.
\end{aligned}$$

Thus we get that,

$$\tau_N\otimes\tau_M\Big((h\circ\delta_j)(\delta_i^1q)(U^Nu\otimes I_M,Z^{NM})\boxtimes(h\circ\delta_j)(\delta_i^2q)(U^Nv\otimes I_M,Z^{NM})\Big)=0.$$

Consequently, with Lemma 4.4 and 3.3, we have that

$$\begin{aligned}
S_{t,y}^N(A,B)&=\frac{t}{N^2}\sum_{j\leq p}\mathbb{E}\Big[\tau_N\otimes\tau_M\Big((h\circ\delta_j)(\delta_i^1q)(U^Nu_t\otimes I_M,Z^{NM})\boxtimes(h\circ\delta_j)(\delta_i^2q)(U^Nv_t\otimes I_M,Z^{NM})\Big)\\
&\qquad-\tau_N\otimes\tau_M\Big((h\circ\delta_j)(\delta_i^1q)(U^N\tilde{u}_t\otimes I_M,Z^{NM})\boxtimes(h\circ\delta_j)(\delta_i^2q)(U^N\tilde{v}_t\otimes I_M,Z^{NM})\Big)\Big]\\
&=\frac{t}{N^2}\sum_{j\leq p}\mathbb{E}\Big[\tau_N\otimes\tau_M\Big(h\Big(\delta_j(\delta_i^1q)(U^Nu_t\otimes I_M,Z^{NM})-\delta_j(\delta_i^1q)(U^N\tilde{u}_t\otimes I_M,Z^{NM})\Big)\\
&\qquad\boxtimes(h\circ\delta_j)(\delta_i^2q)(U^Nv_t\otimes I_M,Z^{NM})\\
&\qquad+(h\circ\delta_j)(\delta_i^2q)(U^N\tilde{u}_t\otimes I_M,Z^{NM})\\
&\qquad\boxtimes h\Big(\delta_j(\delta_i^1q)(U^Nv_t\otimes I_M,Z^{NM})-\delta_j(\delta_i^1q)(U^N\tilde{v}_t\otimes I_M,Z^{NM})\Big)\Big]\Big].
\end{aligned}$$

Hence by using Duhamel formula with Lemmas 3.3 and 3.1, we have that there is a polynomial L_P which only depends on P such that

$$\left|S_{t,y}^N(A,B)\right|\leq L_P\left(\left\|Z^{NM}\right\|\right)\frac{M^2}{N^2}(1+y^4)\ te^{-\frac{t}{2}}.$$

□

We now have the tools to prove Theorem 4.1.

Proof of Theorem 4.1. Thanks to Lemma 4.1, and since $\mathcal{D}_i\ e^{\mathbf{i}yP} = \mathbf{i}y\ \delta_i P\ \widetilde{\#}\ e^{\mathbf{i}yP}$, there exist a family of monomials $(A_k, B_k)_k$ and a constant C which only depends on P such that,

$$\left|\mathbb{E}\left[\frac{1}{MN}\operatorname{Tr}_{MN}\left(f\left(\widetilde{P}\left(U^N\otimes I_M, Z^{NM}\right)\right)\right) - \tau_N\otimes\tau_M\Big(f\left(\widetilde{P}\left(u_t^N, Z^{NM}\right)\right)\Big)\right]\right|$$
$$\leq C\sum_k \int |y| \int_0^t \left|S_{t,y}^N(A_k,B_k)\right| ds\ d|\mu|(y).$$

Thanks to equation (24), by letting t got to infinity, we get that for some polynomial L_P,

$$\left|\mathbb{E}\left[\frac{1}{MN}\operatorname{Tr}_{MN}\left(f\left(\widetilde{P}\left(U^N\otimes I_M, Z^{NM}\right)\right)\right)\right] - \tau_N\otimes\tau_M\Big(f\left(\widetilde{P}\left(u, Z^{NM}\right)\right)\Big)\right|$$
$$\leq \frac{M^2}{N^2} L_P\left(\left\|Z^{NM}\right\|\right)\times \int_{\mathbb{R}} (|y|+|y|^5)\ \ d|\mu|(y)\ .$$

Whereas with equation (23), we get that

$$\left|\mathbb{E}\left[\frac{1}{MN}\operatorname{Tr}_{MN}\left(f\left(\widetilde{P}\left(U^N\otimes I_M, Z^{NM}\right)\right)\right) - \tau_N\otimes\tau_M\Big(f\left(\widetilde{P}\left(u_t^N, Z^{NM}\right)\right)\Big)\right]\right|$$
$$\leq t^2\ \frac{M^2}{N^2} L_P\left(\left\|Z^{NM}\right\|\right)\times \int_{\mathbb{R}} (|y|+y^4)\ \ d|\mu|(y)\ .$$

Besides, thanks to Proposition 3.3, thanks to Duhamel formula we can find a polynomial L_P' such that

$$\left|\tau_N\otimes\tau_M\Big(e^{\mathbf{i}y\widetilde{P}\left(u,Z^{NM}\right)}\Big) - \tau_N\otimes\tau_M\Big(e^{\mathbf{i}y\widetilde{P}\left(u_t^N,Z^{NM}\right)}\Big)\right|$$
$$= \left|\tau_N\otimes\tau_M\Big(e^{\mathbf{i}y\widetilde{P}\left(U^N u,Z^{NM}\right)}\Big) - \tau_N\otimes\tau_M\Big(e^{\mathbf{i}y\widetilde{P}\left(U^N u_t,Z^{NM}\right)}\Big)\right|$$
$$\leq e^{-t/2} L_P'\left(\left\|Z^{NM}\right\|\right)\times |y|.$$

Hence the conclusion by fixing $t = 4\ln(N)$.

□

We can finally prove Theorem 1.1.

Proof of Theorem 1.1. We want to use Theorem 4.1. To do so we would like to take the Fourier transform of f and use Fourier inversion formula. However we did not assume that f is integrable. Thus the first step of the proof is to show that we can assume that f has compact support. Since U^N and u are unitaries, there exists a polynomial H which only depends on P such that $\left\|\widetilde{P}\left(U^N\otimes I_M, Z^{NM}\right)\right\| \leq H\left(\left\|Z^{NM}\right\|\right)$. Consequently since we also have that $\left\|\widetilde{P}(u\otimes I_M, Z^{NM})\right\| \leq H\left(\left\|Z^{NM}\right\|\right)$, we can replace f by fg where g is a $\mathcal{C}^\infty$ function which takes value in $[0,1]$, takes value 1 in $\left[-H\left(\left\|Z^{NM}\right\|\right), H\left(\left\|Z^{NM}\right\|\right)\right]$ and 0 outside of $\left[-H\left(\left\|Z^{NM}\right\|\right)-1, H\left(\left\|Z^{NM}\right\|\right)+1\right]$. Since f can be differentiated six times, we can take its Fourier transform and then invert it so that with the convention $\hat{f}(y) = \frac{1}{2\pi}\int_{\mathbb{R}} f(x)e^{-\mathbf{i}xy}dx$, we have

$$\forall x\in\mathbb{R},\quad f(x) = \int_{\mathbb{R}} e^{\mathbf{i}xy}\widehat{f}(y)\ dy.$$

Besides since if f has compact support bounded by $H\left(\left\|Z^{NM}\right\|\right)+1$, then $\left\|\hat{f}\right\|_\infty \leq 2\left(H\left(\left\|Z^{NM}\right\|\right)+1\right)\|f\|_\infty$, we have

$$\int_{\mathbb{R}} (|y|+y^4)\ \left|\widehat{f}(y)\right|\ dy \leq \int_{\mathbb{R}} \frac{|y|+|y|^3+y^4+y^6}{1+y^2}\ \left|\widehat{f}(y)\right|\ dy$$

$$\leq \int_{\mathbb{R}} \frac{\left|\widehat{(f)^{(1)}}(y)\right| + \left|\widehat{(f)^{(3)}}(y)\right| + \left|\widehat{(f)^{(4)}}(y)\right| + \left|\widehat{(f)^{(6)}}(y)\right|}{1+y^2}\ dy$$

$$\leq 2\left(H\left(\left\|Z^{NM}\right\|\right)+1\right)\|f\|_{\mathcal{C}^6} \int_{\mathbb{R}} \frac{1}{1+y^2}\ dy$$

$$\leq C\left(H\left(\left\|Z^{NM}\right\|\right)+1\right)\|f\|_{\mathcal{C}^6},$$

for some absolute constant C. Hence it satisfies the hypothesis of Theorem 4.1 with $\mu(dy) = \widehat{f}(y)dy$, thus we have

$$\left|\mathbb{E}\left[\frac{1}{MN}\operatorname{Tr}\left(f\left(\widetilde{P}\left(U^N\otimes I_M, Z^{NM}\right)\right)\right)\right] - \tau\left(f\left(\widetilde{P}\left(u\otimes I_M, Z^{NM}\right)\right)\right)\right|$$

$$\leq \frac{M^2\ln^2(N)}{N^2} L_P\left(\left\|Z^{NM}\right\|\right)\int_{\mathbb{R}}(|y|+y^4)\ \left|\widehat{f}(y)\right|\ dy$$

$$\leq \frac{M^2\ln^2(N)}{N^2}\ C L_P\left(\left\|Z^{NM}\right\|\right)\left(H\left(\left\|Z^{NM}\right\|\right)+1\right)\|f\|_{\mathcal{C}^6}.$$

We similarly get the case where f is a $\mathcal{C}^7$ function.

□

Finally to conclude this subsection, it is worth noting the following corollary from Lemma 4.1 and 4.4.

Corollary 4.1. *We define,*

- $u = (u^1, \dots, u^p)$ *a family of* p *free Haar unitaries,*
- $U^N = (U_1^N, \dots, U_p^N)$ *random unitary i.i.d. matrices of size* N *whose law is invariant by multiplication by a matrix of* $SU_N(\mathbb{R})$*,*
- $u_t = (u_t^1, \dots, u_t^p)$ *a family of* p *free unitary Brownian motions at times* t*,*
- Y^{M_N} *be a family of commuting matrices which converges in distribution towards a family* y*,*
- $P \in \mathcal{P}_d$ *a self-adjoint polynomial.*

Then,

$$\lim_{N\to\infty} N^2\left(\mathbb{E}\left[\frac{1}{MN}\operatorname{Tr}_{MN}\left(f\left(\widetilde{P}\left(U^N\otimes I_{M_N}, I_N\otimes Y^{M_N}\right)\right)\right)\right] - \tau_N\otimes\tau_M\left(f\left(\widetilde{P}\left(u\otimes I_{M_N}, I_N\otimes Y^{M_N}\right)\right)\right)\right)$$

$$= \frac{1}{2}\sum_{1\leq i,j\leq p}\int\int_0^\infty t\ \tau^{\otimes 2}\left(\mathcal{D}_j\delta_i^1\left(\mathcal{D}_i\ e^{\mathbf{i}y\widetilde{P}}\right)(uu_t\otimes 1, 1\otimes y)\ \boxtimes\ \mathcal{D}_j\delta_i^2\left(\mathcal{D}_i\ e^{\mathbf{i}y\widetilde{P}}\right)(uv_t\otimes 1, 1\otimes y)\right)dt\ d\mu(y).$$

5 Proof of Corollaries

5.1 Proof of Corollary 1.1

We could directly apply Theorem 1.1 to $f_z : x \to (z-x)^{-1}$, however for z such that $\Im z$ is small, we have $\|f\|_{\mathcal{C}^6} = O\left((\Im z)^{-7}\right)$ when we want $O\left((\Im z)^{-5}\right)$ instead. Since P is self-adjoint, $\overline{G_P(z)} = G_P(\overline{z})$, thus we can assume that $\Im z < 0$, but then

$$f_z(x) = \int_0^\infty e^{\mathbf{i}xy} \left(\mathbf{i}e^{-\mathbf{i}yz}\right) \, dy.$$

Consequently with $\mu_z(dy) = \mathbf{i}e^{-\mathbf{i}yz} \, dy$, we have

$$\int_0^\infty (y+y^4) \, d|\mu_z|(y) = \frac{1}{|\Im z|^2} + \frac{24}{|\Im_z|^5}.$$

Thus by applying Theorem 4.1 with $Z^{NM} = \left(I_N \otimes Y_1^M, \dots, I_N \otimes Y_p^M\right)$, P and f_z, we have

$$\left|\mathbb{E}\left[G_{P(U^N \otimes I_M, I_N \otimes Y^M)}(z)\right] - G_{P(u \otimes I_M, 1 \otimes Y^M)}(z)\right| \leq \frac{M^2 \ln^2(N)}{N^2} L_P\left(\left\|Z^{NM}\right\|\right) \int_{\mathbb{R}} (1+y^4) \ \ d|\mu_z|(y).$$

Now since $\left\|Z^{NM}\right\| = \left(\left\|Y_1^M\right\|, \dots, \left\|Y_p^M\right\|\right)$ which does not depend on N, we finally have

$$\left|\mathbb{E}\left[G_{P(U^N \otimes I_M, I_N \otimes Y^M)}(z)\right] - G_{P(u \otimes I_M, 1 \otimes Y^M)}(z)\right| \leq \frac{M^2 \ln^2(N)}{N^2} L_P\left(\left\|Y_1^M\right\|, \dots, \left\|Y_p^M\right\|\right) \left(\frac{1}{|\Im z|^2} + \frac{24}{|\Im_z|^5}\right).$$

5.2 Proof of Corollary 1.2

Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function uniformly bounded by 1 and with Lipschitz constant at most 1, we want to find an upper bound on

$$\left|\mathbb{E}\left[\frac{1}{MN} \mathrm{Tr}_{NM}\Big(f\left(P\left(U^N \otimes I_M, I_N \otimes Y_M\right)\right)\Big)\right] - \tau \otimes \tau_M\Big(f\left(P\left(u \otimes I_M, 1 \otimes Y_M\right)\right)\Big)\right|. \tag{27}$$

Firstly, since U^N are unitary matrices, we can assume that the support of f is bounded by a constant $S = H(\left\|Y^M\right\|)$ for some polynomial H independent of everything. However we cannot apply directly Theorem 1.1 since f is not regular enough. In order to deal with this issue we use the convolution with gaussian random variable, thus let G be a centered gaussian random variable, we set

$$f_\varepsilon : x \to \mathbb{E}[f(x+\varepsilon G)].$$

Since f has Lipschitz constant 1, we have for any $x \in \mathbb{R}$,

$$|\mathbb{E}[f(x+\varepsilon G)] - f(x)| \leq \varepsilon.$$

Since f_ε is regular enough we could now apply Theorem 1.1, however we a get better result by using Theorem 4.1. Indeed we have

$$\begin{aligned} f_\varepsilon(x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x+\varepsilon y) e^{-y^2/2} \ \ dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) \frac{e^{-\frac{(x-y)^2}{2\varepsilon^2}}}{\varepsilon} \ \ dy \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} f(y) \int_{\mathbb{R}} e^{\mathbf{i}(x-y)u} e^{-(u\varepsilon)^2/2} \, du \, dy. \end{aligned}$$

Since the support of f is bounded, we can apply Fubini's theorem:

$$f_\varepsilon(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\mathbf{i}ux} \int_{\mathbb{R}} f(y) e^{-\mathbf{i}yu} \ dy \ e^{-(u\varepsilon)^2/2} \ du.$$

And so with the convention $\hat{h}(u) = \frac{1}{2\pi} \int_{\mathbb{R}} h(y) e^{-\mathbf{i}uy} dy$, we have

$$f_\varepsilon(x) = \int_{\mathbb{R}} e^{\mathbf{i}ux} \hat{f}(u) e^{-(u\varepsilon)^2/2} \ du.$$

Thus if we set $\mu_\varepsilon(dy) = \hat{f}(y) e^{-(y\varepsilon)^2/2} \ dy$, then, since $\|f\|_\infty \leq 1$,

$$\int_{\mathbb{R}} (1+y^4) d|\mu_\varepsilon|(y) \leq 2S \int_{\mathbb{R}} (1+y^4) e^{-y^2/2} \ dy \ \varepsilon^{-5}.$$

Consequently we can apply Theorem 4.1 with f_ε and since $\|f - f_\varepsilon\|_\infty \leq \varepsilon$, there exists a polynomial R_P such that (27) can be bounded by

$$2\varepsilon + R_P\left(\|Y^M\|\right) \frac{M^2 \ln^2(N)}{N^2 \varepsilon^5}.$$

Thus we can now fix $\varepsilon = (N^{-1}\ln(N))^{1/3}$ and we get that for any f Lipschitz function uniformly bounded by 1 and with Lipschitz constant at most 1, (27) can be bounded by

$$2R_P\left(\|Y^M\|\right) M^2 \left(\frac{\ln N}{N}\right)^{1/3}.$$

5.3 Proof of Theorem 1.2

Firstly, we need to set the operator norm of tensor of $\mathcal{C}^*$-algebras we will work with. When writing the proof it appears that it is the minimal tensor product as defined in 2.4. The following two lemmas were used in [11], see Lemma 4.1.8 from [7] for a proof of the first one and Lemma 4.3 from [11] for the second one. In order to learn more about the second lemma, especially how to weaken the hypothesis, we refer to [23].

Lemma 5.1. *Let $(\mathcal{A}, \tau_{\mathcal{A}})$ and $(\mathcal{B}, \tau_{\mathcal{B}})$ be $\mathcal{C}^*$-algebra with faithful traces, then $\tau_{\mathcal{A}} \otimes \tau_{\mathcal{B}}$ extends uniquely to a faithful trace $\tau_{\mathcal{A}} \otimes_{\min} \tau_{\mathcal{B}}$ on $\mathcal{A} \otimes_{\min} \mathcal{B}$.*

Lemma 5.2. *Let $\mathcal{C}$ be an exact $\mathcal{C}^*$-algebra endowed with a faithful state $\tau_{\mathcal{C}}$, let $Y^N \in \mathcal{A}_N$ be a sequence of family of noncommutative random variable in a $\mathcal{C}^*$-algebra $\mathcal{A}_N$ which converges strongly towards a family Y in a $\mathcal{C}^*$-algebra $\mathcal{A}$ endowed with a faithful state $\tau_{\mathcal{A}}$. Let $S \in \mathcal{C}$ be a family of noncommutative random variable, then the family $(S \otimes 1, 1 \otimes Y^N)$ converges strongly in distribution towards the family $(S \otimes 1, 1 \otimes Y)$.*

In order to prove Theorem 1.2 we use well-known concentration properties of unitary Haar matrices coupled with an estimation of the expectation, let us begin by stating the concentration properties that we will use.

Proposition 5.1. *Let f be a continuous function on $\mathbb{U}_N^p$, such that for any $X, Y \in \mathbb{U}_N^p$,*

$$|f(X) - f(Y)| \leq C \sum_i \mathrm{Tr}_N\left((X_i - Y_i)(X_i - Y_i)^*\right)^{1/2}.$$

Then if W is a vector of p independent random matrices distributed according to the Haar measure on SU_N, and U a vector of independent unitary Haar matrices of size N, we have,

$$\mathbb{P}\left(|f(U) - \mathbb{E}_W\left[f(WU)\right]| \geq \delta\right) \leq 4p \ e^{-\left(\frac{\delta}{2pC}\right)^2 N}.$$

Proof. We want to use Corollary 4.4.28 from [2], in order to do so let us first assume that f takes real values. We then set,

$$f^i_{U_{i+1},\dots,U_p} : U_i \to \mathbb{E}_{W_1,\dots,W_{i-1}}\left[f(W_1U_1,\dots,W_{i-1}U_{i-1},U_i,U_{i+1},\dots,U_p)\right].$$

Thus,

$$f(U) - \mathbb{E}_W\left[f(WU)\right] = \sum_{1\le i\le p} f^i_{U_{i+1},\dots,U_p}(U_i) - \mathbb{E}_{W_i}\left[f^i_{U_{i+1},\dots,U_p}(W_iU_i)\right].$$

Besides for any U_i, V_i, we have that $|f^i_{U_{i+1},\dots,U_p}(U_i) - f^i_{U_{i+1},\dots,U_p}(V_i)| \le C\,\mathrm{Tr}_N\left((U_i-V_i)(U_i-V_i)^*\right)^{1/2}$. Thus thanks to Corollary 4.4.28 from [2] we have that,

$$\begin{aligned}\mathbb{P}\left(|f(U) - \mathbb{E}_W\left[f(WU)\right]| \ge \delta\right) &\le \sum_i \mathbb{P}\left(\left|f^i_{U_{i+1},\dots,U_p}(U_i) - \mathbb{E}_{Y_i}\left[f^i_{U_{i+1},\dots,U_p}(W_iU_i)\right]\right| \ge \frac{\delta}{p}\right)\\ &\le 2p\; e^{-\left(\frac{\delta}{pC}\right)^2 N}.\end{aligned}$$

Finally we conclude by taking the real and imaginary part of f.

□

We can now prove the concentration inequality that we will use in the rest of this paper. To simplify notations we will write M instead of M_N. We also set $Z^{NM} = (Z^N\otimes I_M, I_N\otimes Y^M)$ and $Z = (z\otimes 1, 1\otimes y)$.

Proposition 5.2. *Let $P\in\mathcal{P}_d$, there are polynomials H_P and K_P which only depends on P such that for any N, M,*

$$\mathbb{P}\left(\ \left|\ \left\|\widetilde{P}(U^N\otimes I_M, Z^{NM})\right\| - \mathbb{E}\left[\left\|\widetilde{P}(U^N\otimes I_M, Z^{NM})\right\|\right]\ \right| \ge \delta + \frac{K_P(\|Z^{NM}\|)}{N}\right) \le e^{-\frac{\delta^2 N}{H_P(\|Z^{NM}\|)}}.$$

Proof. We set $G_N : X\mapsto \left\|\widetilde{P}(X\otimes I_M, Z^{NM})\right\|$. One can find a polynomial L_P such that for any N and Z^{NM},

$$|G_N(X) - G_N(Y)| \le L_P\left(\|Z^{NM}\|\right)\sum_i \|X_i - Y_i\|,$$

where $\|.\|$ is the operator norm. Besides

$$\sum_i \|X_i - Y_i\| \le \sum_i \mathrm{Tr}_N\left((X_i-Y_i)^*(X_i-Y_i)\right)^{1/2}.$$

Hence with Proposition 5.1, there is a polynomial H_P which only depends on P such that for any N, M,

$$\mathbb{P}\Big(\ \Big|\ \left\|\widetilde{P}(U^N\otimes I_M, Z^{NM})\right\| - \mathbb{E}_W\left[\left\|\widetilde{P}(WU^N\otimes I_M, Z^{NM})\right\|\right]\ \Big| \ge \delta\Big) \le e^{-\frac{\delta^2 N}{H_P(\|Z^{NM}\|)}}.$$

Besides for any matrix $U\in\mathbb{U}_N$, there exist $S\in SU_N$ and $\theta\in[0,2\pi]$ such that $U = e^{\mathbf{i}\frac{\theta}{N}}S$. Indeed we just have to pick θ such that $e^{\mathbf{i}\theta} = \det(U)$. Thus there is a polynomial K_P such that

$$\left|\mathbb{E}_W\left[\left\|\widetilde{P}(WU^N\otimes I_M, Z^{NM})\right\|\right] - \mathbb{E}\left[\left\|\widetilde{P}(U^N\otimes I_M, Z^{NM})\right\|\right]\right| \le \frac{K_P(\|Z^{NM}\|)}{N}.$$

This concludes the proof.

□

We can now prove Theorem 1.2. Firstly, we can assume that Z^N and Y^M are deterministic matrices by Fubini's theorem. The convergence in distribution is a well-known theorem, we refer to [2], Theorem 5.4.10. We set g a function of class $\mathcal{C}^\infty$ which takes value 0 on $(-\infty, 1/2]$ and value 1 on $[1,\infty)$, and belongs to $[0,1]$ otherwise. Let us define $f_\varepsilon : t\mapsto g\left(\varepsilon^{-1}\left(t - \left\|\widetilde{P}\widetilde{P}^*(u\otimes 1, Z)\right\|\right)\right)$. By Theorem 1.1, there exists a constant C which only depends on P, $\sup_M \left\|Y^M\right\|$ and $\sup_N \left\|Z^N\right\|$ (which is finite thanks to the strong convergence assumption on Y^M and Z^N) such that,

$$\left|\mathbb{E}\left[\mathrm{Tr}_{MN}\left(f_\varepsilon\left(\widetilde{P}\widetilde{P}^*\left(U^N\otimes I_M, Z^{NM}\right)\right)\right)\right] - MN\tau_N\otimes\tau_M\Big(f_\varepsilon\left(\widetilde{P}\widetilde{P}^*\left(u\otimes I_M, Z^{NM}\right)\right)\Big)\right|$$
$$\leq C\varepsilon^{-7}\frac{M^3}{N}.$$

According to Theorem A.1 from [18], $(u, Z^N)_{N\geq 1}$ converges strongly in distribution towards (u, z) since, given a system of free semi-circular variable, we can write $u_i = f(x_i)$ for a specific function f built with the help of Lemma 3.1 of [12]. Besides thanks to Lemma 5.2 and Corollary 17.10 from [24], we have that $(u\otimes I_M, 1\otimes Y^M)_{M\geq 1}$ converges strongly in distribution towards $(u\otimes 1, 1\otimes y)$. In Theorem 1.2, we are interested in the situation where $Z^{NM} = Z^N\otimes I_M$ or $Z^{NM} = I_N\otimes Y^M$. So, without loss of generality, we restrict ourselves to this kind of Z^{NM}. We know that $(u\otimes I_M, Z^{NM})$ converges strongly towards $(u\otimes 1, Z)$, but since the support of f_ε is disjoint from the spectrum of $\widetilde{P}\widetilde{P}^*(u\otimes 1, Z)$, thanks to Proposition 2.1, for N large enough, $\tau_N\otimes\tau_M\Big(f_\varepsilon\left(\widetilde{P}\widetilde{P}^*\left(u\otimes I_M, Z^{NM}\right)\right)\Big) = 0$ and therefore,

$$\mathbb{E}\left[\mathrm{Tr}_{MN}\left(f_\varepsilon\left(\widetilde{P}\widetilde{P}^*\left(U^N\otimes I_M, Z^{NM}\right)\right)\right)\right] \leq C\varepsilon^{-7}\frac{M^3}{N}. \tag{28}$$

Hence, we deduce for N large enough,

$$\begin{aligned}
&\mathbb{E}\left[\left\|\widetilde{P}\widetilde{P}^*\left(U^N\otimes I_M, Z^{NM}\right)\right\|\right] - \left\|\widetilde{P}\widetilde{P}^*(u\otimes 1, Z)\right\| \\
&\leq \varepsilon + \int_\varepsilon^\infty \mathbb{P}\left(\left\|\widetilde{P}\widetilde{P}^*\left(U^N\otimes I_M, Z^{NM}\right)\right\| \geq \left\|\widetilde{P}\widetilde{P}^*(u\otimes 1, Z)\right\| + \alpha\right)\ d\alpha \\
&\leq \varepsilon + \int_\varepsilon^K \mathbb{P}\left(\mathrm{Tr}_{NM}\left(f_\alpha\left(\widetilde{P}\widetilde{P}^*\left(U^N\otimes I_M, Z^{NM}\right)\right)\right) \geq 1\right)\ d\alpha \\
&\leq \varepsilon + C'\varepsilon^{-6}\frac{M^3}{N}.
\end{aligned}$$

Finally we get that almost surely,

$$\limsup_{N\to\infty}\ \mathbb{E}\left[\left\|\widetilde{P}\widetilde{P}^*\left(U^N\otimes I_M, Z^{NM}\right)\right\|\right] \leq \left\|\widetilde{P}\widetilde{P}^*(u\otimes 1, Z)\right\|.$$

Thanks to Proposition 5.2, by taking $\delta_N = N^{-1/4}$ and using Borel-Cantelli lemma, we get that almost surely,

$$\lim_{N\to\infty}\left\|\widetilde{P}\widetilde{P}^*\left(U^N\otimes I_M, Z^{NM}\right)\right\| - \mathbb{E}\left[\left\|\widetilde{P}\widetilde{P}^*\left(U^N\otimes I_M, Z^{NM}\right)\right\|\right] = 0$$

And consequently almost surely,

$$\limsup_{N\to\infty}\left\|\widetilde{P}\widetilde{P}^*\left(U^N\otimes I_M, Z^{NM}\right)\right\| \leq \left\|\widetilde{P}\widetilde{P}^*(u\otimes 1, Z)\right\|.$$

Besides, we know thanks to Theorem 5.4.10 of [2] that if h is a continuous function taking positive values on $\left(\left\|\widetilde{P}\widetilde{P}^*(u\otimes 1, Z)\right\| - \varepsilon, \infty\right)$ and taking value 0 elsewhere. Then $\frac{1}{MN}\mathrm{Tr}_{MN}(h(\widetilde{P}\widetilde{P}^*(U^N\otimes I_M, Z^{NM})))$ converges almost surely towards $\tau_{\mathcal{A}}\otimes_{\min}\tau_{\mathcal{B}}(h(\widetilde{P}\widetilde{P}^*(u\otimes 1, Z)))$. If this quantity is positive, then almost surely for N large enough so is $\frac{1}{MN}\mathrm{Tr}_{MN}(h(\widetilde{P}\widetilde{P}^*(U^N\otimes I_M, Z^{NM})))$, thus

$$\left\|\widetilde{P}\widetilde{P}^*(U^N\otimes I_M, Z^{NM})\right\| \geq \left\|\widetilde{P}\widetilde{P}^*(u\otimes 1, Z)\right\| - \varepsilon.$$

Since h is non-negative and the intersection of the support of h with the spectrum of $\widetilde{P}\widetilde{P}^*(u\otimes 1, Z)$ is non-empty, we have that $h(\widetilde{P}\widetilde{P}^*(u\otimes 1, Z)) \geq 0$ and is not 0. Besides, we know that the trace on the space where z is defined is faithful, and so is the trace on the $\mathcal{C}^*$-algebra generated by a single semicircular variable, hence by Theorem 2.1, so is $\tau_{\mathcal{A}}$. Thus, since both $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ are faithful, by Lemma 5.1, so is $\tau_{\mathcal{A}}\otimes_{\min}\tau_{\mathcal{B}}$ and $\tau_{\mathcal{A}}\otimes_{\min}\tau_{\mathcal{B}}(h(\widetilde{P}\widetilde{P}^*(u\otimes 1, Z))) > 0$. As a consequence, almost surely,

$$\liminf_{N\to\infty}\left\|\widetilde{P}\widetilde{P}^*\left(U^N\otimes I_M, Z^{NM}\right)\right\| \geq \left\|\widetilde{P}\widetilde{P}^*(u\otimes 1, Z)\right\|.$$

We finally conclude thanks to the fact that for any y in a $\mathcal{C}^*$-algebra, $\|yy^*\| = \|y\|^2$.

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