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Asymptotic stability of scalar multi-D inviscid shock waves

Denis Serre
École Normale Supérieure de Lyon*

In memoriam Andrew J. Majda

Abstract

In several space dimensions, scalar shock waves between two constant states $u_\pm$ are not necessarily planar. We describe them in detail. Then we prove their asymptotic stability, assuming that they are uniformly non-characteristic. Our result is conditional for a general flux, while unconditional for the multi-D Burgers equation.

Key words: Scalar conservation laws, Burgers equation, shock waves, contraction semi-group, asymptotic stability.

MSC2010: 35L65, 35B40.

Notations. The norm in $L^p(\mathbb{R}^d)$ is $\|\cdot\|_p$ and that in $\mathbb{R}^d$ is $|\cdot|$. The positive/negative parts of a real number s are $s^\pm$. The ball of center $x \in \mathbb{R}^d$ and radius ε is denoted $B(x;\varepsilon)$. The cone $A(u_-,u_+)$ and its dual $A(u_-,u_+)^\circ$, defined in Section 2, are used repeatedly in the other parts.

*U.M.P.A., UMR CNRS–ENSL # 5669. 46 allée d'Italie, 69364 Lyon cedex 07. France. denis.serre@ens-lyon.fr

1 Introduction

We are concerned with inviscid scalar conservation laws, in arbitrary space dimension $d \geq 2$:

$$\partial_t u + \mathrm{div}_x f(u) = 0, \qquad t > 0, x \in \mathbb{R}^d. \tag{1}$$

The flux $f : \mathbb{R} \to \mathbb{R}^d$ is a smooth function. We shall assume the property of non-degeneracy, which we borrow from [6] :

(ND) For every $(\tau, \xi) \in \mathbb{R}^{1+d}$ such that $\tau^2 + |\xi|^2 = 1$ (or equivalently $\neq 0$),

$$\mathrm{meas}\{s \in \mathbb{R}\,;\, \tau + f'(s) \cdot \xi = 0\} = 0.$$

If $\xi \in \mathbb{R}^d$ is a non-zero vector, the directional flux $s \mapsto \xi \cdot f(s)$ cannot be affine on a non-trivial interval. Remark also that the graph of f may not be locally contained in a hyperplane. A stronger form of non-degeneracy occurs when the function $s \mapsto \det(f''(s), \ldots, f^{(d+1)}(s))$ does not vanish – a natural extension of Genuine Nonlinearity to several space dimensions. A paradigm of the latter situation is the flux

$$f^B(s) = (s^2, \ldots, s^{d+1}),$$

which characterizes the so-called *multi-D Burgers equation* :

$$\partial_t u + \partial_1(u^2) + \cdots + \partial_d(u^{d+1}) = 0. \tag{2}$$

We are interested in the Cauchy problem, where an initial datum

$$u(0,x) = a(x), \qquad x \in \mathbb{R}^d, \tag{3}$$

is prescribed. Since the seminal work by Kružkov [5], it has been known that the notion of solution must be understood in the sense of *entropy solutions.* Within this context, the forward Cauchy problem is well-posed for data in $L^\infty(\mathbb{R}^d)$. The existence and uniqueness allow us to define a semi-group $S_t : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$. Its most important properties, which we use below, are the comparison principle and the L^1-contraction :

Comp. If $a, b \in L^\infty(\mathbb{R}^d)$ are such that $a \leq b$ (inequalities are always understood almost everywhere), then for every $t > 0$, there holds $S_t a \leq S_t b$. In particular, the essential supremum of $S_t a$ is a non-increasing function of time.

Contr. If $a,b \in L^\infty(\mathbb{R}^d)$ are such that $b-a \in L^1(\mathbb{R}^d)$, then for every $t>0$, there holds $S_t b - S_t a \in L^1(\mathbb{R}^d)$, and the function $t \mapsto \|S_t b - S_t a\|_1$ is monotonous non-increasing.

The latter property allows us to extend by continuity the semi-group to data $a \in (L^\infty + L^1)(\mathbb{R}^d)$. Whether the corresponding flow $u(t) := S_t a$ is a solution in Kružkov's sense is still an open question ; this is discussed in [10] in the case of (2). The reader who is familiar with Kružkov's theory knows that the properties above are associated with the conservation of the mass (whence the terminology of conservation law),

$$\int_{\mathbb{R}^d} (S_t b - S_t a)\, dx \equiv \int_{\mathbb{R}^d} (b-a)\, dx.$$

This information will however be of little help in the multi-dimensional stability analysis. In one space dimension instead, it is used to determine the shift between the shocks at initial and final times, in terms of the mass of the initial disturbance.

The simplest solutions of (1), besides the constants, are planar waves $u(t,x) = U(x\cdot\nu - \sigma t)$, where $\nu \in S_{d-1}$ is the direction of propagation, and $\sigma \in \mathbb{R}$ is the normal velocity. The profile U obeys to $f(U)\cdot\nu - \sigma U = \text{cst}$, which implies, because of the non-degeneracy and the entropy condition, that U is piecewise constant. Generically, U takes only two values:

$$U(y) = \begin{cases} u_-, & \text{if} \quad y < y^*, \\ u_+, & \text{if} \quad y > y^*, \end{cases}$$

and we speak of a *planar shock wave*. Up to the flip $(u_-, u_+; \nu, \sigma) \leftrightarrow (u_+, u_-; -\nu, -\sigma)$, we may always assume that $u_+ < u_-$. Then the admissibility criterion given by the entropy condition writes

$$f(s)\cdot\nu - \sigma s \le f(u_\pm)\cdot\nu - \sigma u_\pm, \qquad \forall s \in (u_+, u_-). \tag{4}$$

Recall that by letting $s \to u_\pm$, (4) implies the Lax shock condition

$$f'(u_+)\cdot\nu \le \sigma \le f'(u_-)\cdot\nu. \tag{5}$$

A natural question is whether planar shock waves are stable under localized disturbances:

Given an initial data $a(x) = U(x\cdot\nu) + \phi(x)$, where $\phi \in \mathcal{D}(\mathbb{R}^d)$, how does the solution $u(t) = S_t a$ behave as $t \to +\infty$? We shall relax the restrictions on both U and ϕ in the statements below.

A naive, though deadly false, guess would be that $u(t) - U(x \cdot \nu - \sigma t) \to 0$ in some sense, perhaps up to a shift (orbital stability). We shall see that, whenever $d \geq 2$, the answer is not so simple. As described in Section 2, a planar shock is just one among all the shocks between u_- and u_+, without any special property. Their set can be split into equivalence classes in which two shock profiles differ only by an integrable disturbance, or by a compactly supported one. Our main result is that such classes are stable, a fact that can be interpreted as a generalized form of orbital stability. Mind that since $d \geq 2$, a shock profile and its shifts belong in general to distinct equivalence classes.

Since we shall exploit the L^1-contraction property of the Kružkov semigroup, we understand the stability in terms of an L^1-distance. Our most general result thus concerns disturbances ϕ that belong to $L^1(\mathbb{R}^d)$. Notice that because this perturbation can be arbitrarily wild, the shock structure is lost in the transient times, even though we recover it in the time asymptotics. Thus we shall not try to describe the evolution of the perturbed shock front.

We start our analysis by a description of general shock waves between constant states, for which we refer to Theorem 2.1 below. An important fact is that the shock fronts are Lipschitz graphs, without higher regularity in general. This lack of regularity makes it difficult, if not impossible, to prove their local-in-time stability in the spirit of A. Majda's memoirs [7, 8]. This flaw is closely related to the fact that, because $d \geq 2$, the Lopatinskiĭ condition for a scalar shock is satisfied only in a non-uniform way.

The other results address the asymptotic stability of such shocks, when they are uniformly non-characteristic. The first one deals with the case where the data take values in the interval $[u_+, u_-]$ defined by the end states ; it is valid for arbitrary fluxes. For the sake of simplicity, the shocks under consideration are stationary, in the sense that $f(u_+) = f(u_-)$. This does not limit the generality, since we may always assume this situation by choosing an appropriate moving frame, which amounts to adding a linear function to the flux.

Theorem 1.1 *We assume (***ND***). Let $U = U(x)$ be a steady shock between two constant states $u_+ < u_-$ (for a general description, see Theorem 2.1). We assume that the shock front is uniformly non-characteristic, in the sense of Definition 2.2. Let $\phi(x) \in L^1(\mathbb{R}^d)$ be an initial perturbation, such that the corresponding initial data $a = U + \phi$ still takes values in $[u_+, u_-]$ (in particular, ϕ is bounded).*

Then the solution $u(t) = S_t a$ tends in the L^1-distance towards another steady shock $\hat{U}$ in the

same class as U modulo $L^1(\mathbb{R}^d)$ *:*

$$\begin{array}{rcl} \hat{U} : \mathbb{R}^d & \mapsto & \{u_-, u_+\}, \\ \hat{U} - U & \in & L^1(\mathbb{R}^d), \\ \|u(t) - \hat{U}\|_1 & \stackrel{t\to+\infty}{\longrightarrow} & 0. \end{array}$$

The proof given in Section 3, reminiscent to dynamical systems theory, follows the strategy developed in [9]. We start with the subcase where the initial perturbation is compactly supported, and we prove that it remains so, uniformly in time. The kinetic theory of Lions, Perthame & Tadmor [6] ensures the compactness of the sequence $(u(\cdot+\tau,\cdot))_{\tau\to+\infty}$ and allows us to consider the ω-limit set Ω. The contraction property provides us with a large family of Lyapunov functions $u \mapsto \|W-u\|_1$, indexed by the shock waves W that belong to the same class as U. An element $\bar{u} \in \Omega$ is a solution of (1), still taking values in $[u_+, u_-]$, with the special property that each of these Lyapunov functions remains constant as the time evolves (Lasalle's Invariance Principle). We show that this implies that $\bar{u}$ is itself a steady shock in the same class.

Eventually, we remove the assumption that ϕ be compactly supported, by using a standard density / contraction argument.

When the initial data is not confined to the interval $[u_+, u_-]$, a reasonnable strategy is to prove that the "overhead" $\{x; u(t,x) > u_-\}$ disappears in finite time, as well as its counterpart $\{x; u(t,x) < u_+\}$. If so, we may apply Theorem 1.1 above to the data $u(T)$ for some $T > 0$ large enough. We achieve this goal in the case of the multi-D Burgers equation:

Theorem 1.2 *Consider the multi-D Burgers equation ($f = f^B$) and a (non-necessarily planar) shock $U(x-tv)$ between the states $u_+ < u_-$, whose shock front is uniformly non-characteristic. Let $\phi(x) \in L^\infty(\mathbb{R}^d)$ be a compactly supported initial perturbation. Let us form the initial data $a(x) = U(x) + \phi(x)$.*

Then there exists a finite time T such that, for every $t > T$, the solution $u(t) = S_t a$ takes values in $[u_+, u_-]$.

A uniform decay, at an algebraic rate, of the overhead is established as a consequence of a dispersion property, established by L. Silvestre and the author [10], given in the Appendix. Once its amplitude is small enough, the non-characteristicness forces it to move toward the shock, where it is absorbed.

Notice that if the shock was characteristic, for instance if $f'(u_-) \cdot \nu = 0$ in a direction ν normal to the shock, while $f''(u_-) \cdot \nu \neq 0$, then the statement would fail. A small amplitude

overhead would travel away from the shock and its mass could never be absorbed. This phenomenon occurs already in one-dimensional case, as described in Paragraph 3.4 of [9]. It is specific to the case of inviscid conservation law and does not happen when some amount of dissipation is added to the equation (see [2, 9] for $d = 1$ and [3, 4] for $d = 2$).

Combining the statements above, we obtain

Theorem 1.3 *Consider the multi-D Burgers equation ($f = f^B$) and a uniformly non-characteristic shock $U(x-tv)$ between the states $u_+ < u_-$. Let $\phi(x) \in L^1(\mathbb{R}^d)$ be an initial perturbation, with which we form the initial data $a(x) = U(x) + \phi(x)$.*

1. *Then the solution $u(t) = S_t a$ tends in the L^1-distance towards another shock in the same class as U* modulo $L^1(\mathbb{R}^d)$ *:*

$$\begin{array}{rcl} \hat{U} : \mathbb{R}^d & \mapsto & \{u_-, u_+\}, \\ \hat{U} - U & \in & L^1(\mathbb{R}^d), \\ \|u(t) - \hat{U}(\cdot - tv)\|_1 & \overset{t\to+\infty}{\longrightarrow} & 0. \end{array}$$

2. *If $\phi \in L^\infty(\mathbb{R}^d)$ is compactly supported, then so is $\hat{U} - U$.*

We emphasize the generality of our stability results. Neither the jump $u_+ - u_-$, nor the size of the amplitude or the mass of the perturbation ϕ need to be small. The width between a necessary condition – the non-characteristicness – and our sufficient one is very slim. The counterpart is that we do not know at which rate the convergence takes place. In this respect, our results differ significantly from those of Duchêne & Rodrigues [1], where the stability is induced by a lower-order term (balance law) and the rate is exponential. We cannot either characterize the limit $\hat{U}$ in terms of the initial data ; we can only say that

$$\int_{\mathbb{R}^d} (\hat{U}(x) - U(x))\, dx = \int_{\mathbb{R}^d} \phi(x)\, dx.$$

Remark that the non-characteristicness (in a uniform manner) is needed only away from a compact subset. As a matter of fact, we may always modify the profile U in a bounded set, where we can easily let it uniformly non-characteristic.

Plan of the paper. Shocks taking values in a doubleton $\{u_-, u_+\}$ are described in Section 2. Special attention is given to non-characteristic shocks. We prove Theorem 1.1 in Section 3, and Theorem 1.2 in Section 4. Two anterior results of the theory of scalar conservation laws are listed in Appendix A.

Remark. Almost everything in this paper is specific to the scalar case. Not only we use extensively the comparison principle and the L^1-contraction, the latter being a source of Lyapunov functions. But on another hand, a pair (u_-, u_+) of constant states yields a shock wave in every direction taken in some (usually wide) cone, denoted below $A(u_-, u_+)^\circ$. This contrasts with the system case, where not all pairs can be related by a single shock ; and even when they can be, it is usually in only one direction. For instance, two states of full gas dynamics with density $\rho_\pm$, pressure $p_\pm$ and specific internal energy $e_\pm$ must satisfy the compatibility condition

$$e_+ - e_- + \frac{p_+ + p_-}{2}\left(\frac{1}{\rho_+} - \frac{1}{\rho_-}\right) = 0$$

in order that a shock between them be possible ; and then the normal to the shock must be colinear to the jump $\vec{u}_+ - \vec{u}_-$ of the fluid velocity.

2 Non-planar shock waves

Recall that $f = (f_1, \dots, f_d) : \mathbb{R} \to \mathbb{R}^d$ is a smooth flux, which satisfies the non-degeneracy assumption (**ND**).

Given two real numbers $u_+ < u_-$, we define the velocity

$$v(u_-, u_+) := \frac{1}{u_+ - u_-}\,(f(u_+) - f(u_-)).$$

If $\xi \in S_{d-1}$, the Rankine-Hugoniot condition in the direction ξ provides us with a number

$$\sigma(u_-, u_+, \xi) := \xi \cdot v(u_-, u_+),$$

which is the normal velocity of a discontinuity $u_- \mapsto u_+$ along a hyperplane $\xi \cdot x = \text{cst}$. Then we extend the definition of σ to every $\xi \in \mathbb{R}^d$.

We say that ξ is an admissible direction if $\xi \cdot f$ satisfies also the Oleinik condition

$$\xi \cdot f(s) - \sigma(u_-, u_+; \xi)s \le \xi \cdot f(u_\pm) - \sigma(u_-, u_+; \xi)u_\pm, \qquad \forall s \in (u_+, u_-). \tag{6}$$

This is equivalent to saying that (1) admits the entropy solution $U(x\cdot\xi-\sigma t)$, where $U(y)\equiv u_\pm$, according to whether y is positive or negative. If $\xi\neq 0$, this is a planar shock wave in the direction ξ. Of course, $\xi=0$ is admissible.

The set of the admissible directions, denoted $A(u_-,u_+)$, is a closed convex cone. If it reduces to $\{0\}$, then a shock $u_-\mapsto u_+$ is not possible whatever the direction. Because of the non-degeneracy, $A(u_-,u_+)$ cannot contain an entire line.

Suppose now that $A(u_-,u_+)$ is non-trivial. We are interested in all the solutions u of (1) which take only the values $u_\pm$. Let us define the auxiliary flux $F(s):=f(s)-sv(u_-,u_+)$, for which we have $F(u_+)=F(u_-)$. This common value is denoted $\bar{F}$. For the sake of simplicity, we rewrite the conservation law in the moving frame $(t,x':=x-tv(u_-,u_+))$, where it becomes

$$\partial_t u+\operatorname{div}F(u)=0. \tag{7}$$

Our assumption that $u(t,x)\equiv u_\pm$ a.e. implies $F(u)\equiv\bar{F}$. The conservation law thus reduces to $\partial_t u=0$: the solution is steady in the moving frame. From now on, we write x instead of x' and $u(t,x)=u(x)$. Let us point out that if we replace f by F in the definition above, then $\sigma\equiv 0$.

The equation being understood in Kružkov's sense, we must also write the entropy inequalities, namely

$$\partial_t|u-k|+\operatorname{div}\left(\operatorname{sgn}(u-k)(F(u)-f(k))\right)\leq 0,$$

for every real parameter k. Since u is stationary and $F(u)$ is constant, this reduces to

$$(\bar{F}-F(k))\cdot\nabla\operatorname{sgn}(u-k)\leq 0.$$

When either $k>u_-$ or $k<u_+$, this inequality is trivial since $u-k$ is of constant sign. For $k\in(u_+,u_-)$ instead, it writes $(\bar{F}-F(k))\cdot\nabla\chi\geq 0$ where $\chi(x)=\pm 1$ according to whether $u(x)=u_\pm$. This amounts to saying that $\nabla\chi$ takes values in $A(u_-,u_+)$. The function χ is thus non-decreasing in the directions of the dual cone

$$A(u_-,u_+)^\circ=\left\{\vec{n}\in\mathbb{R}^d\,;\,\vec{n}\cdot\xi\geq 0,\forall\xi\in A(u_-,u_+)\right\}.$$

Remark that $A(u_-,u_+)^\circ$ is nothing but the convex cone spanned by the vectors $\bar{F}-F(s)$ as $s\in[u_+,u_-]$. We infer that

Lemma 2.1 *If u equals u_+ at some Lebesgue point x, then $u\equiv u_+$ in the cone $x+A(u_-,u_+)^\circ$. Likewise, if $u(x)=u_-$, then $u\equiv u_-$ in $x-A(u_-,u_+)^\circ$.*

Let us pick a vector W in the interior of $A(u_-,u_+)^\circ$, which exists because of (**ND**). Suppose that u is not a constant function. Thus there exists two points $x_\pm$ such that $u(x_\pm) = u_\pm$. Let L be any line of direction W. Then $L \cap (x_+ + A(u_-,u_+)^\circ) \neq \emptyset$ and therefore L contains a point at which $u = u_+$. Symmetrically, $L \cap (x_- - A(u_-,u_+)^\circ) \neq \emptyset$ and L must contain a point at which $u = u_-$. Thus there exists a point $X(L) \in L$ which divides L into two half-lines on which $u \equiv u_\pm$, respectively. We now pick an element λ of $A(u_-,u_+)$. The cone $A(u_-,u_+)^\circ$ is contained in a half-space delimited by the hyperplane H of equation $\lambda \cdot x = 0$. This allows us to define the coordinates $(y,r) \in H \times \mathbb{R}$ associated with the decomposition $\mathbb{R}^d = H \oplus \mathbb{R}W$. The set of lines L as above is parametrized by y, and the divider writes $X(L) = y + \psi(y)W$ for some function ψ. The shock front is therefore the *graph* of the function ψ above H. And because of Lemma 2.1, we have $|\psi(y') - \psi(y)| \leq \psi_0(y'-y)$, where ψ_0 is the function whose epigraph is $A(u_-,u_+)^\circ$. Since W is interior to $A(u_-,u_+)^\circ$, ψ_0 satisfies an inequality $\psi_0(y) \leq C|y|$ and thus ψ is globally Lipschitz.

We summarize our analysis in the following statement.

Theorem 2.1 *We assume (***ND***). Suppose that $A(u_-,u_+) \neq \{0\}$. Then every entropy solution u of (1) which takes values in $\{u_-,u_+\}$ is stationary and is of one of the following forms:*

Constants. *Either $u \equiv u_-$ in $\mathbb{R}^d$, or $u \equiv u_+$ in $\mathbb{R}^d$.*

Shocks. *A complete Lipschitz hypersurface Γ, whose normals belong to $A(u_-,u_+)$, separates $\mathbb{R}^d$ into two halves $D_\pm$, on which $u \equiv u_\pm$ respectively. In particular Γ is a graph in suitable linear coordinates.*

In the latter case, for every interior point W of $A(u_-,u_+)^\circ$, the family $(D_- + sW)_{s\in\mathbb{R}}$ is ordered by inclusion and we have

$$\bigcap_s (D_- + sW) = \emptyset, \qquad \bigcup_s (D_- + sW) = \mathbb{R}^d. \tag{8}$$

Notice that in the limit case where $A(u_-,u_+) = \mathbb{R}_+\lambda$ for some $\lambda \in S_{d-1}$, then the non-constant solutions described in the theorem above are rigid: they are planar shock waves $U(\lambda \cdot x - t\sigma(u_-,u_+;\lambda))$.

2.1 Characteristic shocks

Recall that Lipschitz hypersurfaces admit a normal direction at almost every point. The following definition deviates slightly from standards. We state it for globally defined shock profiles, though it can be used also for local ones.

Definition 2.1 *Consider a shock as described in Theorem 2.1. Let $x \mapsto \nu(x)$ be the measurable map defined over the shock front, where $\nu(x)$ is the unit normal, oriented towards the domain D_+. Recall that $\nu(x) \in A(u_-, u_+)$.*

We say that the shock is characteristic *at x if $\nu(x)$ belongs to the boundary of $A(u_-, u_+)$. It is* non-characteristic *otherwise, that is when $\nu(x)$ is an interior point of the cone.*

Two distinct phenomena may cause an admissible shock with normal direction $\nu \in S_{d-1}$ to be characteristic:

- One of the Lax inequalities (5) is non-strict: either $\nu \cdot F'(u_+) = 0$, or $\nu \cdot F'(u_-) = 0$.
- The (non-positive) function $s \mapsto \nu \cdot (F(s) - \bar{F})$ vanishes at some interior point $\bar{s} \in (u_+, u_-)$.

The former possibility is the one that most authors consider usually. It forbids the L^1-asymptotic stability of the corresponding shock, because it allows some charateristic lines to emerge tangentially from the shock. Some non-trivial mass in excess (above u_- or below u_+) can escape as the time increases. We think that the second possibility is equally important, because then a general initial disturbance lets the shock split into two subshocks $u_- \mapsto \bar{s}$ and $\bar{s} \mapsto u_+$, each one being characteristic in the ordinary sense.

We wish here to avoid both difficulties, whence the definition above. But in practice, because the shock front is not necessarily smooth, and also that it extends to infinity, we need some uniformity:

Definition 2.2 *With the same notations as in Definition 2.1, we say that the shock is* uniformly *non-characteristic if $A(u_-, u_+)$ has a non-empty interior, and if the Gauß map $x \mapsto \nu(x)$ takes its values in a compact subset of this interior.*

Obviously, a planar shock wave is uniformly non-characteristic if and only if it is non-characteristic.

The shock front $\{x; x = y + \psi(y)W\}$ is non-characteristic at a point $x_0 = y_0 + \psi(y_0)W$ when there exists a constant $0 < \rho < 1$ such that the function ψ satisfies the enhanced Lipschitz condition

$$|\psi(y) - \psi(y_0)| \leq \rho\, \psi_0(y - y_0)$$

in a neighbourhood of y_0. It is uniformly non-characteristic when one may choose the same constant $0 < \rho < 1$ almost everywhere. Since ψ_0 is subadditive (convex and homogeneous of degree 1), uniform non-characteristicness can be rewritten equivalently

$$\exists \rho < 1 \quad \text{s.t.} \quad |\psi(y') - \psi(y)| \leq \rho\, \psi_0(y' - y), \qquad \forall y, y' \in H. \tag{9}$$

The fact that the cone $A(u_-,u_+)$ depends continuously upon its arguments implies that the set of uniformly non-characteristic shock profiles with a given front is open:

Proposition 2.1 *Let the shock $U(x-tv(u_-,u_+))$, between the constant states $u_\pm$, be uniformly non-characteristic. Then there exists an $\eta>0$ such that, if $\hat{u}_\pm$ are chosen according to $|\hat{u}_\pm - u_\pm|<\eta$, then the function $\hat{U}$ defined by*

$$\hat{U}(x)=\begin{cases}\hat{u}_- & \text{if} \quad x\in D_-,\\ \hat{u}_+ & \text{if} \quad x\in D_+\end{cases}$$

is still the profile of a uniformly non-characteristic shock $\hat{U}(x-t\hat{v})$.

We point out that $\hat{U}$ and U have the same shock front though different end states. Of course, the modified velocity $\hat{v}$, given by the Rankine–Hugoniot condition, is

$$\hat{v}=v(\hat{u}_-,\hat{u}_+)=\frac{1}{\hat{u}_+-\hat{u}_-}\left(f(\hat{u}_+)-f(\hat{u}_-)\right)=v(u_-,u_+)+O(\eta). \tag{10}$$

Proposition 2.2 *We assume that the interior of $A(u_-,u_+)$ is not empty. Let u be a shock from u_- to u_+, as described in Theorem 2.1. Assume that it is uniformly non-characteristic. Denote $D_\pm$ the domains $\{x\in\mathbb{R}^d\,;\,u(x)=u_\pm\}$. Let $x\in\mathbb{R}^d$ be given.*

Then the domains

$$D_-\bigcap(x+A(u_-,u_+)^\circ),\qquad D_+\bigcap(x-A(u_-,u_+)^\circ)$$

are bounded.

Proof

It is enough to prove that the first intersection is bounded. In terms of the coordinates $(r,y)\in\mathbb{R}\times H$, D_- is given by $r<\psi(y)$. In particular it is contained in the domain defined by $r<\psi(0)+\rho\psi_0(y)$. On the other hand the cone $x+A(u_-,u_+)^\circ$ has equation $r-r_0\geq\psi_0(y-y_0)$, where $x=:r_0W+y_0$. A point in the intersection thus satisfies

$$\begin{aligned}\psi_0(y) &\leq \frac{1}{1-\rho}\left(\psi(0)+\psi_0(y_0)-r_0\right),\\ r_0-\psi_0(y_0)\leq\ \ r &\leq \frac{1}{1-\rho}\left(\psi(0)+\rho(\psi_0(y_0)-r_0)\right).\end{aligned}$$

Because the interior of $A(u_-,u_+)$ is not empty, ψ_0 is >0 away from the origin, and the first line above tells us that y belongs to a bounded set of H. The second line controls r.

■

3 Proof of Theorem 1.1

Let us assume in a first instance that ϕ is compactly supported. Because the interior of $A(u_-,u_+)^\circ$ is not empty, there exist two points $x^\pm$ such that

$$\operatorname{Supp}\phi \subset (x_- + A(u_-,u_+)^\circ)\bigcap(x_+ - A(u_-,u_+)^\circ).$$

Let us define a function

$$w(x) := \begin{cases} u_-, & \text{if} \quad x \in D_- \cup (x_+ - A(u_-,u_+)^\circ), \\ u_+, & \text{if} \quad \text{not.} \end{cases}$$

Likewise we define

$$z(x) := \begin{cases} u_+, & \text{if} \quad x \in D_+ \cup (x_- + A(u_-,u_+)^\circ), \\ u_-, & \text{if} \quad \text{not.} \end{cases}$$

Then $z \le a \le w$. Since both w and z are the profiles of steady solutions of (1), we infer (comparison principle) that our solution satisfies

$$w(x) \le u(t,x) \le z(x), \qquad \forall t > 0, x \in \mathbb{R}^d. \tag{11}$$

In other words, the perturbation $u(t) - U$ remains compactly supported, uniformly in time:

$$\operatorname{Supp}(u(t) - U) \subset (x_- + A(u_-,u_+)^\circ)\bigcap(x_+ - A(u_-,u_+)^\circ), \qquad \forall t > 0.$$

We now define the time-shifts of the solution, indexed by $\tau > 0$:

$$u^\tau(t,x) := u(t+\tau,x), \qquad (t,x) \in (-\tau,+\infty) \times \mathbb{R}^d.$$

Each u^τ is an entropy solution of (1). The sequence $(u^\tau)_{\tau>0}$ being uniformly bounded, is is relatively compact in $L^1_{\text{loc}}((A,+\infty)\times\mathbb{R}^d)$ for every $A \in \mathbb{R}$ according to Theorem 3[1] of [6], thanks to Assumption (**ND**). If we now consider the sequence $u^\tau - U$, which is compactly supported in space, the compactness holds instead in $L^1((A_1,A_2)\times\mathbb{R}^d)$ for every finite $A_1 < A_2$.

[1]The cited result makes in addition the spurious assumption that the sequence be bounded in $L^\infty_t(L^1(\mathbb{R}^d))$. Truncating $u^\tau(T)$ to some ball B and using the finite velocity of waves, we infer the compactness within a cone of basis $\{T\}\times B$ and fixed slopes. Such cones can be taken arbitrarily large.

Our next target is to characterize the ω-limit set Ω of this sequence as $\tau \to +\infty$. It is made of functions $\bar{u}$ that are limits of some subsequences u^{τ_k} (with $\tau_k \to +\infty$) in $L^1_{\rm loc}(\mathbb{R}^{1+d})$. Because the convergence is strong, one can pass to the limit in the equation and in the entropy inequalities. We thus find that such a limit is itself an entropy solution of (1), though in the entire space $\mathbb{R}^{1+d}$.

We now use the fact that (1) admits plenty of Lyapunov functions. Suppose that $R(x)$ is a shock from u_- to u_+, which coincides with U away from a bounded set. In other words, the shock front coincides with that of U away from a bounded set. Then $a - R \in L^1(\mathbb{R}^d)$, thus $u(t) - R \in L^1(\mathbb{R}^d)$ for all times, and $t \to \|u(t) - R\|_1$ is a non-increasing function.. Let ℓ_R be its limit as $t \to +\infty$. Then

$$\sup_{t\in[A_1,A_2]} \|u^{\tau_k}(t) - R\|_1 = \|u(A_1 + \tau_k) - R\|_1 \to \ell_R. \tag{12}$$

We infer that any $\bar{u} \in \Omega$ satisfies

$$\|\bar{u}(t) - R\|_1 \equiv \ell_R, \qquad t \in \mathbb{R}. \tag{13}$$

Recall that the decay of $\|\bar{u}(t) - R\|_1$ follows from integrating in space the inequality (that Kružkov proved as a consequence of entropy inequalities) between two solutions

$$\partial_t |\bar{u} - R| + \operatorname{div}\left[\operatorname{sgn}(\bar{u} - R)\,(f(\bar{u}) - f(R))\right] \le 0. \tag{14}$$

Thus the constancy (13) tells us that actually (14) is an equality:

$$\partial_t |\bar{u} - R| + \operatorname{div}\left[\operatorname{sgn}(\bar{u} - R)\,(f(\bar{u}) - f(R))\right] = 0. \tag{15}$$

We now exploit the flexibility given by the very large set of shocks R, to prove the following.

Lemma 3.1 *For every test function $\theta = \theta(x) \in C_K(\mathbb{R}^d)$, one has*

$$\partial_t \left[\theta\left(\bar{u} - \frac{u_- + u_+}{2}\right)\right] + \operatorname{div}\left[\theta\,(f(\bar{u}) - \bar{f})\right] = 0, \tag{16}$$

where $\bar{f}$ denotes the common value $f(u_\pm)$.

Proof

Let $\rho \in \mathcal{D}(\mathbb{R}^{1+d})$ be a test function. The PDE (15) tells us that

$$\int_{\mathbb{R}^{1+d}} \left(|\bar{u} - R|\partial_t \rho + \left[\operatorname{sgn}(\bar{u} - R)\,(f(\bar{u}) - f(R))\right] \cdot \nabla_x \rho\right) dx\,dt = 0.$$

Because R takes only the values $u_\pm$, this rewrites as

$$\int_{\mathbb{R}^{1+d}} \left(|\bar{u}-R|\partial_t\rho + [\mathrm{sgn}(\bar{u}-R)\,(f(\bar{u})-\bar{f})]\cdot\nabla_x\rho \right) dx\,dt = 0. \tag{17}$$

Let R' be another multi-D shock between the constants $u_\pm$ coinciding with U away from some bounded domain. Thus R and R' coincide away from a bounded domain D. Up to replacing R, R' by $\min(R,R')$ and $\max(R,R')$ respectively, we may assume $R \le R'$, which means that $R' \equiv u_-$ while $R \equiv u_+$ within D (recall that $u_+ \le u \le u_-$). Writing (17) for R and R' respectively, and taking the difference between both equalities, we obtain

$$\int_D \left((2\bar{u}-u_- - u_+)\partial_t\rho + 2(f(\bar{u})-\bar{f}))\cdot\nabla_x\rho \right) dx\,dt = 0. \tag{18}$$

The identity (18) is rather flexible: given two points $y_\pm \in \mathbb{R}^d$, one may take

$$D = (y_- + A(u_-,u_+)^\circ) \bigcap (y_+ - A(u_-,u_+)^\circ \tag{19}$$

by the following construction:

$$\begin{aligned} \{x\,;R(x)=u_-\} &= (D_- \bigcup (y_+ - A(u_-,u_+)^\circ) \setminus (y_- + A(u_-,u_+)^\circ), \\ \{x\,;R'(x)=u_-\} &= (D_- \setminus (y_- + A(u_-,u_+)^\circ) \bigcup (y_+ - A(u_-,u_+)^\circ). \end{aligned}$$

We have used above the set theoretic property

$$(B\cup S)\setminus T \subset (B\setminus T)\cup S, \qquad ((B\setminus T)\cup S)\setminus((B\cup S)\setminus T) = S\cap T,$$

together with the fact that both fronts are still Lipschitz hypersurfaces with normals in $A(u_-,u_+)$, so that both R and R' are admissible shocks. Thanks to Proposition 2.2, both R and R' are compact perturbations of U.

Remark. *In terms of the fronts, which are graphs of functions b,h,k respectively, we are forming the functions*

$$\hat{h}(y) = \min\{h(y), \max\{b(y),k(y)\}\}, \qquad \hat{k}(y) = \max\{k(y), \min\{b(y),h(y)\}\},$$

and we have $\hat{k}-\hat{h} = (k-h)^+$.

Chosing above y_+ in the interior of $A(u_-,u_+)^\circ$, and $y_- := -y_+$, we obtain a specific domain Δ which is a balanced convex neighbourhood of the origin. It is the unit ball for a suitable norm of $\mathbb{R}^d$. Then appropriate choices of $y_\pm$ yield domains $D = D(y_0,\mu) = y_0 + \mu\Delta$ (where $y_0 \in \mathbb{R}^d$ and $\mu > 0$) that are arbitrary balls for this norm. Writing (18) in the form

$$\int_D m(x)\,dx = 0, \qquad m(x) := \int_{\mathbb{R}} \big((2\bar{u} - u_- - u_+)\partial_t\rho + 2(f(\bar{u}) - \bar{f})\big)\cdot\nabla_x\rho\big)\,dt,$$

we obtain that m vanishes at every Lebesgue point, hence almost everywhere. Multiplying now by θ and integrating over $\mathbb{R}^d$, we obtain

$$\int_{\mathbb{R}^{1+d}} \theta(x)\,\big((2\bar{u} - u_- - u_+)\partial_t\rho + 2(f(\bar{u}) - \bar{f})]\cdot\nabla_x\rho\big)\,dx\,dt = 0,$$

which is the weak formulation of (16).

■

To conclude, we develop (16) into

$$\theta(\partial_t\bar{u} + \operatorname{div} f(\bar{u})) + (f(\bar{u}) - \bar{f})\cdot\nabla\theta = 0.$$

Because $\bar{u}$ solves the conservation law, there remains

$$(f(\bar{u}) - \bar{f})\cdot\nabla\theta = 0, \qquad \forall\theta \in C_K(\mathbb{R}^d).$$

This is equivalent to saying that $f(\bar{u}) \equiv \bar{f}$. Remembering that $\bar{u}(t,x) \in [u_+,u_-]$ and that $f(s) \neq \bar{f} = f(u_\pm)$ for $s \in (u_+,u_-)$, we infer $\bar{u}(t,x) \in \{u_-,u_+\}$ almost everywhere. According to Theorem 2.1, $\bar{u}$ is a stationary shock $\hat{U}(x)$.

So let $\tau_k \to +\infty$ be such that u^{τ_k} converges towards $\hat{U}$. Applying (12) to $\hat{U}$, we find $\ell_{\hat{U}} = 0$, that is

$$\lim_{t\to+\infty} \|u(t) - \hat{U}\|_1 = 0.$$

This proves the convergence of $u(t)$ towards the steady shock $\hat{U}$.

In the general case, where the perturbation is integrable but not compactly supported, we approach ϕ in $L^1(\mathbb{R}^d)$ by a sequence $(\phi_m)_{m\in\mathbb{N}}$ of compactly supported perturbations, such that $a_m := U + \phi_m$ still take values in $[u_+,u_-]$. Applying the Theorem, already proved under this restriction, to the data a_m, we have

$$\lim_{t\to+\infty} \|S_t a_m - Z_m\|_1 = 0,$$

for some steady shock Z_m such that $Z_m - U \in L^1(\mathbb{R}^d)$. By the contraction property, we have

$$\|Z_m - Z_p\|_1 = \lim_{t\to+\infty} \|S_t a_m - S_t a_p\|_1 \le \|a_m - a_p\|_1 = \|\phi_m - \phi_p\|_1 \overset{m,p\to+\infty}{\longrightarrow} 0.$$

The sequence $(Z_m)_{m\in\mathbb{N}}$ is thus Cauchy, hence convergent in $U + L^1(\mathbb{R}^d)$. Let $\hat{U}$ be its limit. From

$$\begin{aligned} \|u(t) - \hat{U}\|_1 &\le \|u(t) - S_t a_m\|_1 + \|S_t a_m - Z_m\|_1 + \|Z_m - \hat{U}\|_1 \\ &\le 2\|\phi - \phi_m\|_1 + \|S_t a_m - Z_m\|_1, \end{aligned}$$

we infer

$$\limsup_{t\to+\infty} \|u(t) - \hat{U}\|_1 \le 2\|\phi - \phi_m\|_1.$$

Passing to the limit as $m \to +\infty$, we obtain the desired result

$$\lim_{t\to+\infty} \|u(t) - \hat{U}\|_1 = 0.$$

4 Extinction of the overhead: proof of Theorem 1.2

This section deals with the multi-D Burgers equation (flux f^B). Remember that $u_+ < u_-$ and that the shock is uniformly non-characteristic. For technical reasons, we do not assume anymore a steady shock : the velocity $v(u_-, u_+)$ may be non-zero.

To begin with, we consider an auxiliary datum

$$a_+(x) = \max\{u_-, a(x)\}.$$

Thanks to the comparison principle, we know that

$$u(t) = S_t a \le S_t a_+.$$

By assumption, $a_+ - u_- = (a - u_-)^+$ is bounded, compactly supported.

It was remarked in [10] that there exists a uni-triangular matrix M and a constant vector Z, both depending upon u_-, such that $\tilde{v}(t,x) := (S_t a_+)(Mx - tZ) - u_-$ is again a solution of the multi-D Burgers equation. Applying Proposition A.2 to $\tilde{v}$, we infer

$$\|S_t a_+ - u_-\|_\infty \le c_d \|a_+ - u_-\|_1^\alpha t^{-\beta} = c_d \|(a - u_-)^+\|_1^\alpha t^{-\beta}$$

where β is a positive exponent. This shows that $\|S_t a_+ - u_-\|_\infty \to 0$ as $t \to +\infty$, and thus $\sup_x u(t) \to u_-$. The same trick shows that $\inf_x u(t) \to u_+$.

Velocity estimates. Because $u_- \leq S_t a_+ \leq \sup a_+$, Proposition A.1 tells us that the support of $S_t a_+ - u_-$ expands at most at finite velocity. More precisely, if $t, T > 0$, then

$$\mathrm{Supp}(S_{t+T} a_+ - u_-) \subset \mathrm{Supp}(S_T a_+ - u_-) + tC_T,$$

where C_T is the convex hull of

$$\left\{ \frac{1}{s_2 - s_1} \left(f(s_2) - f(s_1) \right); s_1, s_2 \in (u_-, \sup_x S_T a_+) \right\}.$$

From Taylor Formula, the diameter of C_T can be estimated as follow:

$$C_T \subset B(f'(u_-); c_f \|S_T a_+ - u_-\|_\infty), \tag{20}$$

where the finite constant c_f depends only upon the second differential of f over $(u_+, u_- + 1)$. Roughly speaking, the overhead travels approximately at the velocity $f'(u_-)$.

Likewise, the shock velocity between $\hat{u}_-$ and u_+ is a perturbation of that from u_- to u_+ :

$$|v(\hat{u}_-, u_+) - v(u_-, u_+)| \leq c_f |\hat{u}_- - u_-|. \tag{21}$$

Unless we encounter an ambiguity, we shall denote from now on v and $\hat{v}$ for both shock velocities, keeping in mind that v is given, but $\hat{v}$ depends upon our choice of $\hat{u}_-$.

Let $\eta \in (0,1)$ be a number, small enough that we can apply Proposition 2.1 to the profile $\hat{U}$ with the same front as U, and with end states $\hat{u}_- = u_- + \eta$ and $\hat{u}_+ = u_+$. Choose $T > 0$ large enough that $\sup_x S_T a_+ < \hat{u}_-$, which implies $u(T, \cdot) \leq \hat{u}_-$.

According to Theorem 2.1, there exists an $s > 0$ such that the (compact) support of $u(T) - U(\cdot - Tv)$ is contained in $D_- + Tv + sW$. Notice that the latter also contains $D_- + Tv$, the domain where $\hat{U}(\cdot - Tv) \equiv \hat{u}_-$. Thus either $u(T,x) = U(x - Tv) \leq \hat{U}(x - Tv) \leq \hat{U}(x - Tv - sW)$, or $x \in D_- + Tv + sW$, and then $u(T,x) \leq \hat{u}_- = \hat{U}(x - Tv - sW)$. We infer that $u(T) \leq \hat{U}(\cdot - Tv - sW)$. By comparison, there follows

$$u(t+T) \leq \hat{U}(\cdot - Tv - sW - t\hat{v}) =: \hat{U}(\cdot - z - t\hat{v}). \tag{22}$$

Coming back to velocities, we compare that of the disturbance $S_t a_+$ of u_- with that of the shock $\hat{u}_- \mapsto u_+$. We have

$$C_T - \hat{v} \subset B(f'(u_-) - v; 2c_f\eta) = B(F'(u_-); 2c_f\eta).$$

Let us use the coordinates $(r, y) \in \mathbb{R} \times H$, introduced in Section 2. We define a function $g : \mathbb{R}^d \to \mathbb{R}$ by $g(x) = r - \rho\psi_0(y)$, where $\rho < 1$ is the constant expressing the non-characteristicness of the shock front in (9). Because $F'(u_-) \in A(u_-, u_+)^\circ$, we have $r_{F'(u_-)} \geq \psi_0(y_{F'(u_-)})$. And since $F'(u_-) \neq 0$, because the shock is non-characteristic, this implies $g(F'(u_-)) > 0$. We may choose therefore $\eta > 0$ small enough that

$$\min\left\{g(x)\,;\, x \in B(F'(u_-); 2c_f\eta)\right\} =: \alpha > 0. \tag{23}$$

Lemma 4.1 *Let $\eta > 0$ be chosen so that Proposition 2.1 applies, and (23) holds true.*

Then for $t > 0$ large enough, we have

$$\mathrm{Supp}(S_T a_+ - u_-) + tC_T \subset D_+ + z + t\hat{v}.$$

Proof

It suffices to prove that if K is a compact set, then for t large enough,

$$K + t(C_T - \hat{v}) \subset D_+.$$

It is therefore enough to prove that

$$K + tB(F'(u_-); 2c_f\eta) \subset D_+.$$

Recall that D_+ is described by $r > \psi(y)$. Applying (9), we see that D_+ contains a conical domain D_ρ of equation $g(x) > \psi(0)$. Thus it will be sufficient to have, for t large enough,

$$K + tB(F'(u_-); 2c_f\eta) \subset D_\rho.$$

To this end, we evaluate the minimum value of g over the compact set $K + tB(F'(u_-); 2c_f\eta)$. Because ψ_0 is sub-additive (convex and homogeneous of degree one), we have for every x in this domain

$$g(x) \geq \min_K g + t\alpha.$$

There remains to choose

$$t > \frac{1}{\alpha}\,(\psi(0) - \min_K g).$$

■

Let $t > 0$ be as in Lemma 4.1. Then

$$\mathrm{Supp}(S_{t+T}a_+ - u_-) \subset D_+ + z + t\hat{v}.$$

If $u(t+T,x) > u_-$ then necessarily $S_{t+T}a_+(x) > u_-$ and thus $x \in D_+ + z + t\hat{v}$. This implies $\hat{U}(x - z - t\hat{v}) = u_+$. But since $S_{t+T}a_+ \leq \hat{U}(\cdot - z - t\hat{v})$, this means $S_{t+T}a_+(x) \leq u_+$, a contradiction. Therefore $u(t+T) \leq u_-$ everywhere.

The proof that $u(t) \geq u_+$ everywhere for t large enough is similar. Theorem 1.2 is proven.

A Auxiliary statements

The following result tells us that the support of bounded solutions propagates at finite velocity.

Proposition A.1 *Let $b_1, b_2 \in L^\infty(\mathbb{R}^d)$ take values in a bounded interval J and be such that* $\mathrm{Supp}(b_2 - b_1)$ *is compact. Denote C the convex hull of the compact set*

$$\left\{ \frac{1}{s_2 - s_1} \left(f(s_2) - f(s_1) \right) ; s_1, s_2 \in J \right\}$$

where the quotient is replaced by $f'(s_1)$ if $s_2 = s_1$. Let K be the convex hull of $\mathrm{Supp}(b_2 - b_1)$. *Then the support of $S_t b_2 - S_t b_1$ is contained in $K + tC$.*

Proof

For each direction $\xi \in S_{d-1}$, we consider a half-plane $H = \{x \in \mathbb{R}^d ; x \cdot \xi > \alpha_\xi\}$ separated from K. We also denote $c(\xi) = \max \xi \cdot C$. Denoting $u_j(t) = S_t b_j$, we integrate Kružkov's inequality

$$\partial_t |u_2 - u_1| + \mathrm{div}(\mathrm{sgn}(u_2 - u_1) \left(f(u_2) - f(u_1) \right)) \leq 0$$

over the domain

$$\left\{ (t,x) \in (0,T) \times \mathbb{R}^d ; x \cdot \xi > \alpha_\xi + c(\xi) t \right\}.$$

We obtain

$$\int_0^T \int_{x \cdot \xi = \alpha_\xi + c(\xi)t} \left(c(\xi) |u_2 - u_1| - \mathrm{sgn}(u_2 - u_1) \xi \cdot \left(f(u_2) - f(u_1) \right) \right) d\Sigma \, dt$$
$$+ \sqrt{1 + c(\xi)^2} \int_{x \cdot \xi > \alpha_\xi + c(\xi)T} |u_2 - u_1|(T,x) \, dx \quad \leq \quad 0,$$

where $d\Sigma$ is the $(d-1)$-dimensional Lebesgue measure. The first integral is non-negative because u_1, u_2 take values in J, and we conclude that

$$\int_{x\cdot\xi>\alpha_\xi+c(\xi)T} |u_2-u_1|(T,x)\,dx \le 0.$$

This implies that the support of $u_2(T)-u_1(T)$ is contained in the half-space

$$\{x\,;\,x\cdot\xi \le \alpha_\xi + c(\xi)T\}.$$

There remains to take the intersection as ξ runs over the unit sphere.

■

The following dispersion statement is taken from [10] (see Theorem 1.1).

Proposition A.2 *Consider the multi-D Burgers equation (flux f^B). Then there exists a universal constant $c_d < \infty$ and positive exponents*

$$\alpha(d) = \frac{2}{d^2+d+2}\,, \qquad \beta(d) = \frac{2d}{d^2+d+2}\,,$$

such that the solutions of (2) with integrable data obey to $\|S_t u_0\|_\infty \le c_d \|u_0\|_1^\alpha t^{-\beta}$.

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