Mahler measures, special values of L-functions and complex multiplication

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PhD thesis

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To Giulio Regeni, whose PhD studies
were brutally interrupted by the cruelty
which continues to subjugate too many people
and to oppress too many lands.
Giulio lives in our hearts.
Giulio fa cose!

Abstract

This thesis studies the relations between special values of L-functions of arithmetic objects and heights, as well as the arithmetic of torsion points on elliptic curves with complex multiplication. The first of the main results of this thesis, exposed in its last chapter, shows that the special value $L^*(E,0)$ of the L-function associated to an elliptic curve E defined over $\mathbb{Q}$ which has complex multiplication can be expressed as an explicit rational linear combination of a logarithm of an algebraic number and the Mahler measure of a polynomial. The other main result of this thesis, exposed in its penultimate chapter and obtained in collaboration with Francesco Campagna, shows that the family of p^∞-division fields associated to an elliptic curve E defined over a number field F containing the CM field K becomes linearly disjoint after removing a finite and explicit subfamily of fields, which we expect to be never linearly disjoint over F as soon as it contains more than one element, and E satisfies a technical condition (see Definition 7.1.30). We prove this expectation if $F = K$ and E is the base-change of an elliptic curve defined over $\mathbb{Q}$.

The content of this thesis is articulated in the following chapters:

- the *first chapter* contains background material on the notion of height, and on Diophantine properties of heights;
- the *second chapter* contains background material on motives, motivic cohomology and regulators;
- the *third chapter* contains background material on L-functions, together with some results concerning the finiteness of the family of L-functions having bounded special values, which is based on joint work in progress with Fabien Pazuki;
- the *fourth chapter* contains background material on the Mahler measure, as well as some computations concerning explicit families of polynomials;
- the *fifth chapter* contains the outline of an ongoing project joint with François Brunault, whose aim is to give a geometric interpretation of results by Lalín, inspired by an insight from Maillot, concerning the Mahler measures associated to polynomials satisfying a suitable exactness condition;
- the *sixth chapter*, which is based on joint work in progress with Francesco Campagna, introduces the notion of ray class fields associated to orders in algebraic number fields. This is probably well known to the experts but not so well documented in the literature;
- the *seventh chapter* contains background material on elliptic curves and abelian varieties with complex multiplication, together with the proof of an optimal upper bound for the index of the image of the Galois representation attached to the torsion points of an elliptic curve with complex multiplication, which is based on joint work in progress with Francesco Campagna;
- the *eight* and *ninth chapter* contain the expositions of the main results of this thesis, which were described in the previous paragraph.
- the *appendix* contains the tables mentioned in the main body of the thesis

Key words: L-functions, heights, special values, Mahler measure, complex multiplication, elliptic curves.

2020 Mathematics Subject Classification: 11G05, 14K22, 11G15, 11S15, 11F80, 11R06, 11S40, 14K22, 19F27.

Resumé

Denne afhandling studerer relationer mellem specielle værdier for L-funktioner af aritmetiske objekter og højder, og aritmetikken af torsionspunkter på elliptiske kurver med kompleks multiplikation. Det første af hovedresultaterne i denne afhandling, fremsat i dens sidste kapitel, viser, at den specielle værdi $L^*(E, 0)$ for L-funktionen associeret til en elliptisk kurve, E, defineret over $\mathbb{Q}$, som har kompleks multiplikation, kan udtrykkes som en eksplicit, rationel, lineær kombination af en logaritme af et algebraisk tal og Mahlermålet af et polynomium. Det andet hovedresultat i denne afhandling, fremsat i næstsidste kapitel og opnået i samarbejde med Francesco Campagna, viser, at familien af p^∞-divisionslegemer associeret til en elliptisk kurve, E, defineret over et tallegeme, F, som indeholder CM-legemet K, bliver lineært disjunkt efter at en endelig og eksplicit underfamilie af legemer udelukkes. Vi forventer at denne underfamilie aldrig er lineært disjunkt over F, så snart den indeholder mere end ét element og E opfylder en teknisk betingelse (se Definition 7.1.30). Vi beviser denne forventning hvis $F = K$ og E er basisskiftet af en elliptisk kurve defineret over $\mathbb{Q}$.

Indeholdet i denne afhandling er struktureret i følgende kapitler:

- det *første* kapitel indeholder baggrundsmateriale om begrebet højde og om diofantine egenskaber af højder;
- det *andet* kapitel indeholder baggrundsmateriale om motiver, motivisk cohomologi og regulatorer;
- det *tredje* kapitel indeholder baggrundsmateriale om L-funktioner sammen med nogle resultater vedrørende endeligheden af familien af L-funktioner med begrænsede specielle værdier, som er baseret på et igangværende samarbejde med Fabien Pazuki;
- det *fjerde* kapitel indeholder baggrundsmateriale om Mahlermål sammen med nogle beregninger forbundet med eksplicitte familier af polynomier;
- det *femte* kapitel indeholder en skitse af et igangværende projekt i samarbejde med François Brunault, hvis formål er at give en geometrisk fortolkning af resultater fra Lalín, inspirerede af en indsigt af Maillot, vedrørende Mahlermålet associeret til polynomier, der opfylder en passende eksakthedsbetingelse;
- det *sjette* kapitel, som er baseret på et igangværende samarbejde med Francesco Campagna, introducerer idéen om stråleklasselegemer associeret til ordner i algebraiske tallegemer. Dette er sandsynligvis velkendt for eksperter, men ikke så veldokumenteret i litteraturen;
- det *syvende* kapitel indeholder baggrundsmateriale om elliptiske kurver og abelske varieteter med kompleks multiplikation sammen med et bevis for en optimal øvre grænse for indekset af billedet af Galois repræsentationen forbundet med torsionspunkterne på en elliptisk kurve med kompleks multiplikation. Dette er også baseret på et samarbejde med Francesco Campagna;
- det *ottende* og *niende* kapitel indeholder en fremstilling af hovedresultaterne i denne afhandling, som var beskrevet i de foregående paragraffer;
- *bilaget* indeholder tabellerne nævnt i afhandlingens hovedtekst.

Nøgleord: L-funktioner, højder, specielle værdier, Mahlermål, kompleks multiplikation, elliptiske kurver.

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> I am not myself free or human until or unless I recognise the freedom and humanity of all my fellowmen.
>
> Mikhail Bakunin, *Man, society and freedom*

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Preface: an outline of this PhD thesis

> Freedom of the spirit is beyond price, but this world wants to impose a price on everything.
>
> Wu Ming, *Q*

It was probably very cold in Saint Petersburg on the 5th of December 1735, when a paper entitled *De summis serierum reciprocarum* was read to the Saint Petersburg Academy of sciences. This paper, presented by the 28 year old Swiss mathematician Leonhard Euler, computes the values of the infinite series

$$\zeta(s) := \sum_{n=1}^{+\infty} \frac{1}{n^s} \tag{0.1}$$

evaluated at even natural numbers $s = 2n$, with $n \in \mathbb{N}$. More precisely, Euler proves that

$$\frac{\zeta(2n)}{\pi^{2n}} = \frac{(-1)^{n+1} 2^{2n-1} B_{2n}}{(2n)!} \in \mathbb{Q}^\times \tag{0.2}$$

where $B_{2n} \in \mathbb{Q}^\times$ is the $2n$-th Bernoulli number, defined by the generating series

$$\sum_{m=0}^{+\infty} \frac{B_m}{m!} t^m = \frac{t}{e^t - 1}$$

which was introduced in Jakob Bernoulli's book *Ars Conjectandi*, published in Basel in 1713, eight years after the death of Jakob Bernoulli and twenty-two years before Euler's remarkable discovery. Euler's proof solved in particular the so-called *Basel problem*, which asked to find the explicit value of $\zeta(2)$. This was posed in 1650, thus more than eighty years earlier than Euler's solution, by the Italian mathematician Pietro Mengoli. We refer the interested reader to Raymond Ayoub's survey [Ayo74] for a thorough exposition of Euler's insights on the series (0.1).

Euler's theorem was a profound breakthrough, and contributed to bring him the fame that lasts until this very day. Moreover, Euler himself extended the study of the series (0.1) to the real values $s \in \mathbb{R}$ such that $s > 1$, but it wasn't until Bernhard Riemann's 1859 work *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* that this function was shown to have a meromorphic continuation to the whole complex plane, with a pole only at $s = 1$, corresponding to the fact that (0.1) degenerates to the notoriously divergent harmonic series in this case. As it can be

evinced from the German title of Riemann's paper, this work also focused on the profound relations between the function $\zeta(s)$, which is today known as Riemann's ζ-function, and the distribution of prime numbers. These relations occur in the form of Riemann's explicit formula, which links the prime counting function $\pi(x)$ to the complex numbers $s \in \mathbb{C}$ such that $\zeta(s) = 0$ and $0 < \Re(s) < 1$. Riemann then conjectured that all such complex numbers must lie on the vertical line $\Re(s) = 1/2$. This problem, which is known as the Riemann hypothesis, remains unsolved to this day, despite numerous attempts and partial results towards it, among which lies the fundamental *prime number theorem*, which asserts that $\zeta(s) \neq 0$ on the vertical line $\Re(s) = 1$. We refer the interested reader to Alain Connes's survey [Con16] for more historical background on the Riemann hypothesis.

Going back to the equality (0.2), one might wonder if similar formulas exist for the values $\zeta(2n+1)$ of the Riemann ζ-function at odd positive integers. This is, perhaps surprisingly, still unknown to this day, despite the fact that the interest in this problem goes back to Euler himself (see [Ayo74, § 7]). Nevertheless, it is expected that formulas like (0.2) should not hold for $\zeta(2n+1)$, in a very strong sense. More precisely, it is conjectured that, for every $n \in \mathbb{N}$ such that $n \geq 1$, the real numbers $\{\pi, \zeta(3), \zeta(5), \ldots, \zeta(2n+1)\}$ should be algebraically independent over $\mathbb{Q}$. The partial results known towards this conjecture are quite little. Most notably, Roger Apéry mesmerised the audience attending his talk at the Journées arithmétiques in Luminy on the 22nd of June, 1978, by presenting a surprisingly simple proof that the real number $\zeta(3) \in \mathbb{R}$ is irrational. His proof was later clarified by several mathematicians, and its developments through the following year are recounted in Alfred van der Poorten's survey paper [PA79]. Moreover, it is today known that one amongst the four real numbers $\{\zeta(5), \zeta(7), \zeta(9), \zeta(11)\}$ is irrational, by work of Wadim Zudilin (see [Zud04]), and it is also known that the sequence $\{\zeta(2n+1)\}_{n\geq 1}$ contains infinitely many irrational numbers. This was firstly shown by the work of Keith Ball and Tanguy Rivoal (see [BR01]), and it is now known that a big proportion of odd ζ-values is irrational, thanks to the work of Stéphane Fischler, Johannes Sprang and Wadim Zudilin (see [FSZ19]), as well as the recent work [LY20] by Li Lai and Pin Yu.

We point out that, to this day, it is not known whether a single odd ζ-value is transcendental. Thus one may ask why it is in fact reasonable to believe that *all* the odd ζ-values are transcendental, and also algebraically independent amongst themselves. One of the reasons comes from the link between the transcendence of odd zeta values and some deep conjectures in algebraic geometry. To be more precise, for every $n \in \mathbb{N}$ the special value

$$\zeta^*(n) := \lim_{s \to n} \frac{\zeta(s)}{(s-n)^{\mathrm{ord}_{s=n}(\zeta(s))}} \tag{0.3}$$

is known to be a *period*, *i.e.* to be expressible as the integral of an algebraic differential form over an semi-algebraic domain. Then the transcendence of each of the values $\zeta^*(n) = \zeta(n)$ for $n \geq 1$ can be shown to follow from the *period conjecture*, which predicts (as formulated by Maxim Kontsevich and Don Zagier in [KZ01, Conjecture I]) that each equality between periods can be proved using only the elementary rules of calculus (change of variables, linearity and Stokes' theorem). We refer the interested reader to Joseph Ayoub's article [Ayo14b] for a survey of the period conjecture, and to Yves André's survey [And04, § 25.7] of the relations between the transcendence of special values of the ζ function and the period conjecture. These conjectures are nowadays known to be related to multiple ζ-values, which are higher-dimensional analogues of the values of ζ at the positive integers. We refer the interested reader to the forthcoming book [BF] by José Ignacio Burgos Gil and Javier Fresán for an in-depth survey of the theory of multiple ζ-values, with particular focus on its relations to motives.

The conjectural transcendence of odd ζ-values leaves us with the question of finding formulas relating them to other (conjecturally transcendental) numbers of interest (*e.g.* periods having a particularly simple integral representation). In fact, one may wonder whether relations of this kind exist for the special values of more general meromorphic functions $f\colon \mathbb{C} \to \mathbb{C}$, which satisfy a functional equation and admit an Euler product representation analogous to the ones which are known to hold for Riemann's ζ-function. There is a way, at least conjecturally, to produce such kinds of functions from algebraic varieties. More precisely, one can associate to each smooth and proper algebraic variety X defined over a number field κ a plethora of cohomology theories, such as the ℓ-adic bi-graded cohomologies $H_\ell^{i,j}(X) := H^i_{\text{ét}}(X_{\overline{\kappa}}; \mathbb{Q}_\ell(j))$. Since these are defined by base-changing X to the algebraic closure of κ, we see immediately that the Galois group $\mathcal{G}_\kappa := \mathrm{Gal}(\overline{\kappa}/\kappa)$ acts on $H_\ell^{i,j}(X)$. Hence, one can consider the characteristic polynomial of every element $\sigma \in \mathcal{G}_\kappa$ acting on $H_\ell^{i,j}(X)$. These characteristic polynomials can be "assembled together", to create the L-function associated to the i-th cohomology of X. More generally, one can carry out this procedure for every *mixed motive* defined over the number field F, which is an object "cut out", by means of linear algebra, from objects of the form $\underline{H}^i(X)$ associated to smooth and proper varieties X. We refer the interested reader to Chapter 2 for a survey of the theory of motives, and to Section 3.2 for an overview of the construction of the L-function $L(M,s)$ associated to a mixed motive M. These functions are defined as formal Euler products, which converge for every $s \in \mathbb{C}$ with real part $\Re(s) > \sigma_0(M)$ for some real number $\sigma_0(M) \in \mathbb{R}_{>0}$. Moreover, it is conjectured that these L-functions admit a meromorphic continuation to the whole complex plane (see Conjecture 3.3.4) and that this meromorphic continuation satisfies a suitable functional equation (see Conjecture 3.3.6). Finally, deep conjectures of Selberg predict that all the functions having these properties arise as motivic L-functions. We refer the interested reader to Alberto Perelli's survey articles [Per05; Per04] for an introduction to this circle of ideas.

Now, for every L-function $L(M,s)$ associated to a mixed motive $M \in \mathcal{MM}(F;\mathbb{Q})$, which is defined over a number field F and has rational coefficients, one can define the special values

$$L^*(M,n) := \lim_{s\to n} \frac{L(M,s)}{(s-n)^{\mathrm{ord}_{s=n}(L(M,s))}} \in \mathbb{R}$$

associated to any integer $n \in \mathbb{Z}$. These generalise the special values $\zeta^*(n)$ defined in (0.3), which can be obtained by taking $M = \underline{H}^0(\mathrm{Spec}(\mathbb{Q})) \in \mathcal{MM}(\mathbb{Q},\mathbb{Q})$. Then one may ask, in complete analogy to what happens for the special values $\zeta^*(n)$, if the special values $L^*(M,n)$ are irrational, transcendental or even algebraically independent amongst themselves. Moreover, one could ask if these numbers are periods, and if so one could try to find the "simplest" integral representation of such periods. It probably comes as no surprise to the reader, given how little we know already about the special values $\zeta^*(n)$, that even less is known about the irrationality and transcendence of the values $L^*(M,n)$ for a general motive M. For example, the value $L^*(\chi_{-4},2) = L(\chi_{-4},2)$ of the L-function associated to unique non-trivial Dirichlet character $\chi_{-4}\colon (\mathbb{Z}/4\mathbb{Z})^\times \to \mathbb{C}^\times$, equals the famous Catalan constant $G \in \mathbb{R}$, which is not known to be irrational (see Yuri Nesterenko's survey [Nes16]). Nevertheless, if these numbers were periods then their transcendence might be related once again to the *period conjecture*.

Luckily enough, these numbers are indeed known to be periods, in the case of Dirichlet L-functions. This can be seen as an instance of the conjectures of Pierre Deligne (see [Del79]) and Alexander Beilinson (see [Bei84]) on special values of L-functions, which are known to hold for the L-functions associated to Dirichlet characters by the work of Beilinson himself. We refer

the reader to Section 3.3.2 for an introduction to Beilinson's conjecture, and to Jürgen Neukirch's survey [Neu88] for an account of Beilinson's work concerning the Dirichlet L-functions $L(\chi, s)$.

Now, let us come back to the problem of expressing the periods $\zeta^*(n)$ and $L^*(\chi, n)$ in the "simplest form possible". This notion is of course subjective, and depends on the reader's taste for what should be considered a "simple" integral expression. One key example of "simple period" is given by the Mahler measure:

$$m(P) := \int_0^1 \cdots \int_0^1 \log|P(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n})| d\theta_1 \cdots d\theta_n$$

defined by Kurt Mahler (see [Mah62]) for every Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \setminus \{0\}$. One of the first relations between Mahler measures and special values of L-functions comes from the work of Christopher Smyth, who proved in [Smy81] that

$$\begin{aligned} m(x+y+1) &= L'(\chi_{-3}, -1) \\ m(x+y+z+1) &= -14\zeta'(-2) \end{aligned} \tag{0.4}$$

where $\chi_{-3} \colon (\mathbb{Z}/3\mathbb{Z})^\times \to \mathbb{C}^\times$ is the unique non-trivial character (see Theorem 4.2.4). Using the functional equations for $\zeta(s)$ we see for example that Smyth's result gives us one possible integral expression for Apéry's number $\zeta(3)$. We refer the interested reader to the work [Lal06] by Matilde Lalín for a list of identities involving Mahler measures and the special values $\zeta(2n+1)$ for every $n \geq 1$, which generalise Smyth's result. Moreover, we point the reader to Chapter 4 for an introduction to the Mahler measure.

Other special values of L-functions which are known to be periods are given by $L^*(E, n)$ for $n \neq 1$, where E is an elliptic curve. More precisely, if E is an elliptic curve defined over $\mathbb{Q}$ then the modularity theorem (see for instance Bas Edixhoven's survey [Edi02]) shows that the motivic L-function $L(E, s) := L(\underline{H}^1(E), s)$ coincides with the automorphic L-function $L(f, s)$ associated to a newform $f \in S_2(\Gamma_0(N))$. Thus the modularity theorem can be combined with a result of Beilinson (see Christopher Deninger and Anthony Scholl's survey [DS91]) to show that $L^*(E, n)$ is indeed a period. On the other hand, if F is any number field and $E_{/F}$ is an elliptic curve with complex multiplication (see Definition 7.1.5) which satisfies a suitable technical condition (see Definition 7.1.30), then a result of Deninger, proved in the works [Den89] and [Den90], shows that $L^*(E, n)$ is again a period, for $n \neq 1$.

Hence one may wonder whether the special values $L^*(E, n)$ satisfy suitable identities which relate them to simpler periods, for instance to Mahler measures of Laurent polynomials. This seems indeed plausible, at least for the special value $L^*(E, 0)$, thanks to the extensive numerical computations performed by David Boyd during the last decade of the past century, which are contained in [Boy98], and more recently by Hang Liu and Houroung Qin (see [LQ19]). These computations show that, for many Laurent polynomials $P \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$, one should expect a relation of the form

$$\frac{L^*(E_P, 0)}{m(P)} \in \mathbb{Q}^\times \tag{0.5}$$

where E_P is an elliptic curve which appears as a factor of the Jacobian of the curve defined by $\{P = 0\}$. On the other hand, if the Jacobian of the curve defined by P does not have any elliptic factor, then one should not expect a relation like (0.5), as we point out at the end of Section 4.2.

It is now natural to ask what happens for polynomials having multiple variables. First of all, it is often the case that the Mahler measure $m(P)$ of a Laurent polynomial $P \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ in n variables, appears to be related to the special value of some L-function at $s = n$. Finding out

which L-function this is can be tricky. For example, why is it the case that the Dirichlet character χ_{-3} and the Riemann ζ-function make their appearance in Smyth's computations (0.4)? This question was partially answered by Vincent Maillot, during a talk which took place on the 30th of April, 2003 at the Banff International Research Station, in occasion of the meeting *The many aspects of Mahler's measure* (see [Boy+03, § 8]). More precisely, Maillot proved that in certain cases the L-function to be considered comes from the cohomology of the variety $P = P^* = 0$, where $P^*(x_1, \dots, x_n) := P(x_1^{-1}, \dots, x_n^{-1})$ is the *reciprocal* of the polynomial P. In these cases, the polynomial P is said to be *exact*. For example, the polynomial $x + y + 1$ is exact, and the variety $P = P^* = 0$ consists of two points defined over $\mathbb{Q}(\sqrt{-3})$, which shows where the Dirichlet character χ_{-3} comes from. On the other hand, to explain the appearance of $\zeta^*(-2)$ in the second of Smyth's computations (0.4), one needs to introduce the notion of *successive exactness* for a polynomial P. This has been done in the PhD thesis of Matilde Lalín (see [Lal05; Lal07]), mostly for polynomials in two and three variables, using the exactness of suitable differential forms. We devote Chapter 5 to report on joint work in progress with François Brunault, whose aim is to give a notion of k-exactness for polynomials in n-variables, where $0 \leq k \leq n$. We present two candidates for this notion, and we use our approach to provide a strategy of proof for a conjecture concerning the special value $L^*(X_1(15), -1)$ associated to the elliptic modular curve $X_1(15)$.

This is all related to the following general question: is every special value $L^*(M, n)$ of a motivic L-function always related to the Mahler measure of one or more polynomials? While this question is completely out of reach in general, the following theorem, which is the first main result of this thesis, shows that this conjecture holds for the special value $L^*(E, 0) = L'(E, 0)$ of the L-function associated to a CM elliptic curve E which is *also* defined over $\mathbb{Q}$.

Theorem A – Mahler measures and CM elliptic curves (see Theorem 9.2.4)

Let E be an elliptic curve defined over $\mathbb{Q}$ such that $\mathrm{End}(E_{\overline{\mathbb{Q}}}) \cong \mathcal{O}_K$ for some imaginary quadratic field K. Then there exists a polynomial $P \in \mathbb{Z}[x, y]$ such that:

- its zero locus $V_P \hookrightarrow \mathbb{G}_m^2$ is birationally equivalent to E;
- $m(P) = r\, L'(E, 0) + \log|s|$ for two explicit numbers $r \in \mathbb{Q}^\times$ and $s \in \overline{\mathbb{Q}}^\times$ defined in (9.15).

Before moving on, let us point out that the general question concerning the relations between $L^*(M, n)$ and Mahler measures can be seen as parts of even broader speculations concerning the relations between special values of L-functions and different kinds of *heights*. The Mahler measure has indeed been introduced as a height function on polynomials (at least on those with integer coefficients), which satisfies (at least conjecturally) many of the required Diophantine properties of a height function, such as the *Northcott*, *Bogomolov* and *Lehmer* properties. We refer the reader to Chapter 1 for an axiomatic introduction to these different properties. It is then natural to ask whether the special values $L^*(M, n)$ can also be considered as a kind of height, and in particular if they satisfy the aforementioned Diophantine properties. We devote Section 3.4, which is based on joint work in progress with Fabien Pazuki, to the study of these questions.

Let us now go back to the proof of Theorem A, which is contained in Chapter 9. This proof rests on Deninger's foundational work [Den97a], which provides a way of relating the Mahler measure $m(P)$ of a polynomial to periods coming from algebraic geometry, and on

David Rohrlich's work [Roh87], which provides a very explicit result that proves a weak form of Beilinson's conjectures (see Conjecture 3.3.28) for the special values $L^*(E,0)$ associated to a CM elliptic curve E defined over $\mathbb{Q}$. We refer the reader to Section 4.3 for an introduction to Deninger's results, and to Section 7.4 for an exposition of Rohrlich's work.

The aforementioned results of Rohrlich concern pairs of functions $f, g\colon E \to \mathbb{P}^1_{\mathbb{Q}}$ whose zeros and poles are *torsion points*, not necessarily defined over $\mathbb{Q}$. The coordinates of these torsion points can also be used to generate some finite extensions of $\mathbb{Q}$, called *division fields* associated to the elliptic curve E, which are among the most studied families of number fields. More generally, if E is any CM elliptic curve defined over a number field F, which has complex multiplication by an order $\mathcal{O}$ inside an imaginary quadratic field K, one can define for every ideal $I \subseteq \mathcal{O}$ a finite Galois extension $F \subseteq F(E[I])$ generated by the coordinates of those points $P \in E(\overline{F})$ such that $[\alpha]_E(P) = 0$ for every $\alpha \in I$, where $[\alpha]_E\colon E \to E$ denotes the multiplication map associated to α. Then one can form the infinite Galois extension $F \subseteq F(E_{\mathrm{tors}})$ given as the compositum of all the division fields $F(E[I])$, and to this extension one may attach a Galois representation

$$\rho_E\colon \mathrm{Gal}(F(E_{\mathrm{tors}})/F) \hookrightarrow \mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}}) \tag{0.6}$$

where $\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})$ denotes the group of automorphisms of the $\mathcal{O}$-module E_{tors}, defined as $E_{\mathrm{tors}} := E(\overline{F})_{\mathrm{tors}} = \varinjlim_{N\in\mathbb{N}} E[N](\overline{F})$. More generally, one can form the infinite Galois extensions $F \subseteq F(E[p^\infty])$ associated to every rational prime $p \in \mathbb{N}$, which are defined as the compositum of the family of division fields $\{F(E[p^n])\colon n \in \mathbb{N}\}$. The following theorem, proved in collaboration with Francesco Campagna, shows that the family of infinite extensions $\{F(E[p^\infty])\}_p$ with $p \in \mathbb{N}$ varying amongst the rational primes, becomes linearly disjoint over F after removing a finite, explicit sub-family.

Theorem B – Entanglement of CM division fields (see Theorem 8.2.6)

Let F be a number field and $E_{/F}$ an elliptic curve with complex multiplication by an order $\mathcal{O}$ in an imaginary quadratic field $K \subseteq F$. Denote by $\mathfrak{b}_E := \mathfrak{f}_{\mathcal{O}}\,\Delta_F\,N_{F/\mathbb{Q}}(\mathfrak{f}_E)$ the product of the conductor $\mathfrak{f}_{\mathcal{O}} := |\mathcal{O}_K : \mathcal{O}|$ of the order $\mathcal{O}$, the absolute discriminant $\Delta_F \in \mathbb{Z}$ of the number field F and the norm $N_{F/\mathbb{Q}}(\mathfrak{f}_E) := |\mathcal{O}_F/\mathfrak{f}_E|$ of the conductor ideal $\mathfrak{f}_E \subseteq \mathcal{O}_F$.

Then the natural inclusion

$$\mathrm{Gal}(F(E_{\mathrm{tors}})/F) \hookrightarrow \prod_p \mathrm{Gal}(F(E[p^\infty])/F)$$

where the product runs over all rational primes $p \in \mathbb{N}$, induces an isomorphism

$$\mathrm{Gal}(F(E_{\mathrm{tors}})/F) \xrightarrow{\sim} \mathrm{Gal}(F(E[S^\infty])/F) \times \prod_{p\notin S} \mathrm{Gal}(F(E[p^\infty])/F)$$

where $S \subseteq \mathbb{N}$ denotes the finite set of primes dividing $\mathfrak{b}_E$.

It now a natural question to ask whether the finite family of fields $\{F(E[p^\infty])\colon p \mid \mathfrak{b}_E\}$ is linearly disjoint or not. We study this question in Section 8.3 and Section 8.4. On the one hand, we prove in Corollary 8.3.4 that this family is linearly disjoint for every number field F which is an abelian extension of an imaginary quadratic field K, and for every elliptic curve E defined

over F which has complex multiplication by an order $\mathcal{O} \subseteq K$ and does not satisfy a suitable technical condition (see Definition 7.1.30), which was introduced in the foundational work of Goro Shimura and Yutaka Taniyama (see in particular [Shi94, Pages 216-218]). On the other hand, we prove in Theorem 8.4.4 that the finite family of division fields $\{F(E[p^\infty])\colon p \mid \mathfrak{b}_E\}$ is *never* linearly disjoint over F as soon as it contains more than one element, under the condition that $F = K$ is an imaginary quadratic field and $E_{/K}$ is the base-change of an elliptic curve defined over $\mathbb{Q}$, which has potential complex multiplication by an order $\mathcal{O} \subseteq K$ such that $\mathcal{O} \notin \left\{\mathbb{Z}\left[\frac{-1+\sqrt{-3}}{2}\right], \mathbb{Z}[\sqrt{-1}]\right\}$.

Our study of division fields of elliptic curves with complex multiplication led Francesco Campagna and the author also to extend the usual definition of ray class fields to ray class fields relative to orders in number fields. This was already done for orders contained in imaginary quadratic fields in the PhD thesis of Heinz Söhngen [Söh35] using the classical language of class field theory (see also Reinhard Schertz's survey [Sch10]). We generalise this definition to any order inside any number field K, using both the classical and the modern language of class field theory, which involves the group of idèles $\mathbb{A}_K^\times$ (see Definition 6.1.6). We report on our current progress concerning these generalised ray class fields in Chapter 6.

We observe as well that the ray class fields $H_{I,\mathcal{O}}$ associated to imaginary quadratic orders are always contained in the division fields $F(E[I])$, whenever the ideal I is invertible (see Lemma 6.2.7). This was already proved by Söhngen in [Söh35], and we give another proof in Section 7.2, using the language introduced in Chapter 6. This inclusion shows that the division fields $F(E[I])$ can not be "too small", hence that the image of the Galois representation (0.6) cannot be too small either. In particular, we prove in Section 7.3 that the index $[\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})\colon \mathrm{Im}(\rho_E)]$ is finite and explicitly bounded from above. The finiteness of this index is originally due to Max Deuring, and it is the precursor of the celebrated "open image theorem", proved by Jean-Pierre Serre in [Ser71], which states that for an elliptic curve *without* complex multiplication, the index $[\mathrm{Aut}_{\mathbb{Z}}(E_{\mathrm{tors}})\colon \mathrm{Im}(\rho_E)]$ is also finite. Moreover, the aforementioned Corollary 8.3.4 proves that the upper bound for the index $[\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})\colon \mathrm{Im}(\rho_E)]$ provided by Theorem 7.3.1 is optimal, at least for elliptic curves defined over the ring class field $H_{\mathcal{O}}$.

To conclude, let us mention the leitmotiv of this PhD thesis. On the one hand, the world of algebraic geometry, and in particular of motives, gives us a great amount of conjectures to ponder upon, which often rest over a great deal of abstraction (such as a heavy use of category theory). On the other hand, periods and heights, of which the Mahler measure is an instance, give us concretely computable complex (or, more often, real) numbers that measure the complexity, in a suitable sense, of some arithmetic objects, such as algebraic numbers, points on abelian varieties, or even number fields, abelian varieties themselves and so on. A bridge between these two worlds is provided by the realisations of a motive M, which allow one to speak about the periods and the L-function attached to M. The special values of this L-function give us other more or less computable complex numbers, and it is thus a natural question to ask if these can be related to heights of various sort. In this thesis we show how to do this for elliptic curves with complex multiplication, and for one particular special value (at $s = 0$) and one particular height (the Mahler measure). This is possible because of the extra symmetries with which a CM elliptic curve is endowed. These symmetries can be exploited in many different ways, for example to study the division fields of these elliptic curves, or the ray class fields associated to imaginary quadratic orders. Thus we have seen that heights, which are supposed to measure the complexity of an arithmetic object, turn out to be easier to handle when we apply them to objects with extra symmetries, which might appear more complicated at a first sight. This proves yet again the unwavering truth that a first sight is usually taken from a wrong perspective.

Heights and their Diophantine properties

1

> What pleasure lives in height (the shepherd sang)
> In height and cold, the splendour of the hills?
>
> Alfred Tennyson, *The Princess*

The aim of this chapter is to introduce the notion of *height* (or *height function*) in wide generality, and to define the main Diophantine properties of height functions, which are named after Northcott, Bogomolov and Lehmer. The second section of this chapter is then devoted to give examples of heights and to survey what is known about their Diophantine properties.

1.1 An axiomatic approach to heights and their properties

The notion of height is the central cornerstone of modern Diophantine geometry. Height functions were originally meant to be a measure for the size of solutions to Diophantine equations. Since these can be understood as rational points $V(\mathbb{Q})$ on an algebraic variety V defined over the field of rational numbers $\mathbb{Q}$, height functions in this setting can be understood as set-theoretic functions $h\colon V(\overline{\mathbb{Q}}) \to \mathbb{R}$ which measure the complexity of an algebraic point $P \in V(\overline{\mathbb{Q}})$. This height function can then be used to single out the subset $V(F) \subseteq V(\overline{\mathbb{Q}})$ of F-rational points for any given number field $F \subseteq \overline{\mathbb{Q}}$. The most striking example of this approach is Vojta's proof of Mordell's conjecture, which asserts that the set $C(F)$ is finite for every smooth, projective curve C defined over a number field F. One of the ingredients that are part of Vojta's theorem is the fact that, if $C(F) \neq \emptyset$, one can associate to a given rational point $P \in C(F)$ the so called canonical height $\widehat{h}_{C,P}\colon C(\overline{F}) \to \mathbb{R}$, initially introduced for abelian varieties by Néron in [Nér65], which has the property that for every $B \in \mathbb{R}$ the set

$$\{Q \in C(F) : |\widehat{h}_{C,P}(Q)| \leq B\}$$

is finite. We refer the interested reader to [HS00, Part E] for an exposition of Vojta's proof.

The finiteness property enjoyed by the canonical height is called *Northcott's property*, in view of the fact that Northcott showed a similar property for the Weil height of algebraic numbers (see Section 1.2.1), which is closely related to the canonical height. In fact, the first proof of Mordell's conjecture, due to Faltings, employs a similar kind of Northcott property. More precisely, Faltings defined a function $h_{\mathrm{Fal}}\colon \mathcal{A}(\overline{\mathbb{Q}}) \to \mathbb{R}$ on the set of $\overline{\mathbb{Q}}$-isomorphism classes of abelian varieties defined over $\overline{\mathbb{Q}}$, with the property that for every $B_1, B_2 \in \mathbb{R}$ the set

$$\{A \in \mathcal{A}(\overline{\mathbb{Q}})\colon \deg(A) \leq B_1,\ h_{\mathrm{Fal}}(A) \leq B_2\}$$

is finite, where $\deg(A) \in \mathbb{N}$ denotes the minimal degree of a number field over which A can be defined. The interested reader might find a complete account of Faltings's proof in the survey volumes [Fal+92], [Szp85] and [CS86].

The previous examples show that the notion of height is a very general one, and indeed different kinds of heights have been defined for a plethora of different kinds of objects. This leads us to give the following working definition.

Definition 1.1.1 – Height functions

A *height* (or *height function*) on a set S is a function $h\colon S \to \Gamma$ with values in a partially ordered set Γ.

The most common examples of heights land in $\Gamma = \mathbb{R}$, but we introduce this general framework because it makes some definitions cleaner, using the product height (1.1).

The rest of the section is devoted to the description of three properties of heights:

- the *Northcott property*, which asserts that the sets of points of bounded height are finite;
- the *Bogomolov property*, which concerns the lower bound of the set $h(S) \subseteq \Gamma$;
- the *Lehmer property*, which is a Bogomolov property for a slightly modified height.

1.1.1 Northcott property

As we already said in the introduction, an important property of heights $S \to \Gamma$ is that they often allow one to cut the set S into smaller pieces, by limiting the height from above. This property can be axiomatically defined as follows.

Definition 1.1.2 – Northcott property

Let $h\colon S \to \Gamma$ be a height function, and let $\mathbb{S}$ be a collection of subsets of S. Then the height h has:

- the *fibre-wise $\mathbb{S}$-Northcott property* if and only if the fibres of h lie in $\mathbb{S}$;
- the *$\mathbb{S}$-Northcott property* if and only if $\{s \in S \mid h(s) \leq \gamma\} \in \mathbb{S}$ for every $\gamma \in \Gamma$.

If $\mathbb{S}$ consists of the collection of finite subsets of S, we omit it from the notation.

The name *Northcott properties* comes from the fact, already stated in the introduction of this section, that one of the first finiteness results of this type was proved by Northcott for the height of algebraic numbers (see Section 1.2.1). To be precise, this height (as most height functions) does not satisfy a Northcott property by itself, but it does if one also bounds the degree of the algebraic numbers in question. This leads to the following generalisation of Definition 1.1.2.

Definition 1.1.3 – Northcott property for a set of heights

If $\mathbf{h} = \{h_i \colon S \to \Gamma_i\}_{i \in I}$ is a set of height functions we say that $\mathbf{h}$ has one of the properties described in Definition 1.1.2 if and only if the "product height"

$$\begin{aligned} \widetilde{\mathbf{h}} \colon S &\to \prod_{i \in I} \Gamma_i \\ s &\mapsto (h_i(s))_{i \in I} \end{aligned} \tag{1.1}$$

has these properties, where the set $\prod_{i \in I} \Gamma_i$ is endowed with the product order (see [Bou04, Chapter III, § 1.4]).

Before moving on, let us mention the following evident implication

$$\boxed{h \text{ has } \mathbb{S}\text{-Northcott}} + \boxed{\mathbb{S} \text{ is lower-closed}} \Rightarrow \boxed{h \text{ has fibre-wise } \mathbb{S}\text{-Northcott}} \tag{1.2}$$

where $\mathbb{S}$ is called *lower-closed* if for all $Y \subseteq X \subseteq S$ we have that $X \in \mathbb{S} \Rightarrow Y \in \mathbb{S}$. Moreover, if $\mathbb{S}$ is the collection of finite subsets of S then

$$\boxed{h \text{ has fibre-wise Northcott}} + \boxed{h(S) \text{ is upper-finite}} \Rightarrow \boxed{h \text{ has Northcott}}$$

where we say that $X \subseteq \Gamma$ is *upper-finite* if $X_{\leq \gamma} := \{x \in X \mid x \leq \gamma\}$ is finite for all $\gamma \in \Gamma$.

1.1.2 Bogomolov property

Let us now shift to the definition of the Bogomolov property, which concerns the infimum of the set $h(S) \subseteq \Gamma$. This property, which has been widely investigated for the height of algebraic numbers (see the introduction of [CF20] and the references therein), takes his name from the toric version of Bogomolov's conjecture, which has been proved by Zhang in [Zha95a]. Zhang's proof uses the concept of *successive minimum* for the canonical height. This notion can be generalised as follows to arbitrary subsets of a partially ordered set Γ.

Definition 1.1.4 – Successive infima and minima

Let Γ be a partially ordered set and let $k \in \mathbb{N}$. Then a subset $X \subseteq \Gamma$ has *at least k successive infima* (respectively *at least k successive minima*) if:

- X is bounded from below;
- whenever $k \geq 1$, the set X has at least $k-1$ successive infima (resp. minima), and $X \setminus X_{k-1}$ has an infimum (resp. minimum) $\mu_k(X) \in \Gamma$. Here we define $X_0 := \emptyset$ and

$$X_{k-1} := \{x \in X \mid x \leq \mu_{k-1}(X)\} \cup U_{k-1}$$

for any $k \geq 2$, where $U_{k-1} \subseteq \Gamma$ denotes the connected component of $X \cup \{\mu_{k-1}(X)\}$ that contains $\mu_{k-1}(X)$. This connected component is taken with respect to the subspace topology induced on $X \cup \{\mu_{k-1}(X)\}$ by the order topology on Γ.

It is easy to see that $\mu_j(X) \leq \mu_{j+1}(X)$ for any $j \in \mathbb{Z}_{\geq 1}$. Moreover if $\mu_{j+1}(X) = \mu_j(X)$ for some $j \in \mathbb{Z}_{\geq 1}$ then X has at least k successive infima for every $k \in \mathbb{N}$ and $\mu_k(X) = \mu_j(X)$ for every $k \geq j$. This leads to the following definition.

Definition 1.1.5 – Exact number of successive infima/minima

Let Γ be a partially ordered set. Then any subset $X \subseteq \Gamma$ has *exacly k successive infima* (respectively *exacly k successive minima*) for some $k \in \mathbb{N}$ if it has at least k successive infima (resp. minima) and at least one of the following holds:

- X does not have at least $k+1$ successive infima (resp. minima);
- $\mu_{k+1}(X) = \mu_k(X)$.

A related notion is the one of *essential minimum* (see the introduction of [AD03]), which can be generalised as follows.

Definition 1.1.6 – Essential infimum/minimum

Let Γ be a partially ordered set, let $X \subseteq \Gamma$ and let $\mathbb{X}$ be a collection of subsets of X. Write $X_{\leq \gamma} := \{x \in X \mid x \leq \gamma\}$ for every $\gamma \in \Gamma$. Then X has a *$\mathbb{X}$-essential infimum* (resp. *$\mathbb{X}$-essential minimum*) if the set

$$\{\gamma \in \Gamma \mid X_{\leq \gamma} \notin \mathbb{X}\} \subseteq \Gamma$$

has an infimum (resp. a minimum). In this case we denote this element by $\mu_{\mathrm{ess}}(X, \mathbb{X}) \in \overline{\Gamma}$, where $\overline{\Gamma} := \Gamma \sqcup \{+\infty\}$ is the partially ordered set obtained by adjoining to Γ a global maximum $+\infty$. In particular, $\mu_{\mathrm{ess}}(X, \mathbb{X}) = +\infty$ if and only if $X_{\leq \gamma} \in \mathbb{X}$ for every $\gamma \in \Gamma$.

We are now ready to give the definition of Bogomolov property.

Definition 1.1.7 – Bogomolov property

Let $h \colon S \to \Gamma$ be a height function, and let $\mathbb{S}$ be a collection of subsets of S. Then h has:

- the *$\mathbb{S}$-essential Bogomolov property* if the set $h(S) \subseteq \Gamma$ has an $h(\mathbb{S})$-essential infimum, denoted by $\mu_{\mathrm{ess}}(h, \mathbb{S}) \in \overline{\Gamma}$;
- *Bogomolov number* $\mathcal{B}(h) \in \mathbb{N}$ if the set $h(S) \subseteq \Gamma$ has exactly $\mathcal{B}(h)$ successive infima, denoted by $\mu_j(h)$ for $j \in \{1, \ldots, \mathcal{B}(h)\}$;
- the *very weak Bogomolov property* if and only if $\mathcal{B}(h) \geq 0$, *i.e.* if and only if the set $h(S) \subseteq \Gamma$ is bounded from below;
- the *weak Bogomolov property* if and only if $\mathcal{B}(h) \geq 1$ and $\mu_1(h) \in h(S)$, *i.e.* if and only if $h(S)$ has a minimum;
- the *Bogomolov property* if and only if either $|h(S)| = 1$ or $\mathcal{B}(h) \geq 2$ and $\mu_1(h) \in h(S)$, *i.e.* if and only if $h(S)$ has an isolated minimum.

If $\mathbf{h} = \{h_i \colon S \to \Gamma_i\}_{i\in I}$ is a set of height functions we write $\mathcal{B}(\mathbf{h})$ and $\mu_{\mathrm{ess}}(\mathbf{h}, \mathbb{S})$ for the Bogomolov number and the essential infimum of the product height (1.1), and we say that $\mathbf{h}$ has one of the various Bogomolov properties if and only if the product height does.

Clearly one has the chains of implications

$$\boxed{h \text{ has Bogomolov}} \Rightarrow \boxed{h \text{ has weak Bogomolov}} \Rightarrow \boxed{h \text{ has very weak Bogomolov}}$$
$$\boxed{h \text{ has Northcott}} \Rightarrow \boxed{h \text{ has Bogomolov}}.$$

1.1.3 Interlude: examples of successive infima

Before moving to the definition of Lehmer's property, we devote this subsection to the study of examples of successive infima and minima. In particular, we show that our definitions Definition 1.1.4 and Definition 1.1.5 recover the notions of successive infima and minima present in Arakelov geometry, due to Minkowski (for lattices) and Zhang (for heights associated to hermitian line bundles).

Example 1.1.8. Let $\Gamma = \mathbb{R}$. In this case the order topology coincides with the Euclidean topology. Then every set which has at least zero successive infima (*i.e.* is bounded from below) has also has at least n successive infima for every $n \in \mathbb{N}$. Moreover, if $X \subseteq \mathbb{R}$ is a finite union of open intervals $X = \bigcup_{i=1}^{k}(a_i, b_i)$ with $a_1 < b_1 < a_2 < b_2 < \dots$, then it is easy to see that X has exactly k successive infima, with $\mu_i(X) = a_i$ for every $i \in \{1, \dots, k\}$. Finally, if $X \subseteq \mathbb{R}$ is countable then X has exactly $k \in \mathbb{Z}_{\geq 1}$ successive minima if and only if there exists a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $|\{x \in X \mid x \leq x_n,\ \forall n \in \mathbb{N}\}| = k$.

Example 1.1.9 (Minkowski)**.** Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice, and let $g \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be any distance function (see [Cas97, Chapter IV]), *i.e.* any continuous function such that $g(t \cdot \mathbf{x}) = |t|g(\mathbf{x})$ for all $t \in \mathbb{R}$. Then the image of the map

$$\begin{aligned} \Lambda &\to \mathbb{R}_{\geq 0} \times \mathbb{N} \\ \lambda &\mapsto \big(g(\lambda), \dim_{\mathbb{R}}(V_{g,\lambda})\big) \end{aligned} \qquad \text{where} \qquad V_{g,\lambda} := \langle \{x \in \Lambda \mid g(x) \leq g(\lambda)\} \rangle_{\mathbb{R}}$$

has exactly n successive infima, which are given by the pairs $(\mu_j(\Lambda, g), j)$ for some sequence

$$0 < \mu_1(\Lambda, g) \leq \mu_2(\Lambda, g) \leq \cdots \leq \mu_n(\Lambda, g) < +\infty$$

with $\mu_j(\Lambda, g) \in \mathbb{R}_{>0}$ for every $j \in \{1, \dots, n\}$. The numbers $\{\mu_j(\Lambda, g)\}$ are usually called *successive minima* of the function g on the lattice Λ (see [Cas97, Chapter VIII]). However, these numbers are really infima and not minima in general.

Example 1.1.10 (Zhang)**.** Let $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ be an arithmetic variety of dimension d, as defined in [Zha95a], and let $\mathrm{Cl}(X)$ be the set of closed sub-schemes of the generic fibre $X := \mathcal{X}_{\mathbb{Q}}$. Fix $\overline{\mathcal{L}}$ to be a relatively semi-ample hermitian line bundle on $\mathcal{X}$ with ample generic fibre, and let $h_{\overline{\mathcal{L}}} \colon X(\overline{\mathbb{Q}}) \to \mathbb{R}$ be the associated height. Then the image of the map

$$\begin{aligned} \mathrm{Cl}(X) &\to \mathbb{R} \times \mathbb{N} \\ Y &\mapsto \Big(\inf\{h_{\overline{\mathcal{L}}}(x) \mid x \in X(\overline{\mathbb{Q}}) \setminus Y(\overline{\mathbb{Q}})\},\ \dim(Y)\Big) \end{aligned}$$

has exactly $d+1$ successive infima, which are given by pairs $(\mu_j(\mathcal{X},\overline{\mathcal{L}}), j)$ for some sequence

$$\mu_0(\mathcal{X},\overline{\mathcal{L}}) \leq \mu_1(\mathcal{X},\overline{\mathcal{L}}) \leq \cdots \leq \mu_d(\mathcal{X},\overline{\mathcal{L}}) \leq +\infty$$

with $\mu_j(\mathcal{X},\overline{\mathcal{L}}) \in \mathbb{R}$ for every $j \in \{0,\dots,d-1\}$ and $\mu_d(\mathcal{X},\overline{\mathcal{L}}) \in \mathbb{R} \sqcup \{+\infty\}$. It is easy to see that $\mu_d(\mathcal{X},\overline{\mathcal{L}}) = +\infty$ if and only if X is irreducible, and that for every $j \in \{0,\dots,d-1\}$ we have $\mu_j(\mathcal{X},\overline{\mathcal{L}}) = e_{d-j}(\overline{\mathcal{L}})$, where $e_1(\overline{\mathcal{L}}) \geq \cdots \geq e_d(\overline{\mathcal{L}})$ is the sequence defined in [Zha95a, § 5]. We also refer the interested reader to the seminal paper [Gau08], and in particular to [Gau08, § 5.4], for more examples of successive minima.

1.1.4 Lehmer property

As we point out in Section 1.2.1, the height of algebraic numbers has the weak Bogomolov property, because zero the set $h(\overline{\mathbb{Q}})$ has a minimum in zero), but not the Bogomolov property, since $h(\sqrt[n]{2}) \to 0$ for $n \to \infty$. Nevertheless, there is no known sequence $\{\alpha_n\} \subseteq \overline{\mathbb{Q}}$ such that $h(\alpha_n) \neq 0$ for all $n \in \mathbb{N}$, $\lim_{n\to+\infty} h(\alpha_n) = 0$ and $\deg(\alpha_n)$ is bounded. Moreover, it is expected that the function $\alpha \mapsto h(\alpha)\deg(\alpha)$ should have the Bogomolov property, thereby proving that no such sequence $\{\alpha_n\}$ should exist. This expectation is linked to the famous problem posed by Lehmer in [Leh33, § 13], which is discussed in Section 4.1.1. It is now clear why the following property, which generalises the aforementioned conjectural property of the height of algebraic numbers, is called "Lehmer property".

Definition 1.1.11 – Lehmer property

Let $\mathbf{h} = \{h_i : S \to \Gamma_i\}_{i\in I}$ be a set of heights, and let $\alpha : \prod_{i\in I}\Gamma_i \to \Gamma$ be any map of sets, where Γ is a partially ordered set. Then the *Lehmer number* $\mathcal{L}(\mathbf{h},\alpha) \in \mathbb{N}$ is defined to be the Bogomolov number of the height $\alpha \circ \widetilde{\mathbf{h}}$, where $\widetilde{\mathbf{h}}$ denotes the product height (1.1). The successive infima of $\alpha(\widetilde{\mathbf{h}}(S))$ are denoted accordingly by $\mu_j(\mathbf{h},\alpha)$ for $j \in \{1,\dots,\mathcal{L}(\mathbf{h},\alpha)\}$. Moreover, the pair $(\mathbf{h},\alpha)$ has:

- the *very weak Lehmer property* if and only if $\alpha \circ \widetilde{\mathbf{h}}$ has the very weak Bogomolov property;
- the *weak Lehmer property* if and only if $\alpha \circ \widetilde{\mathbf{h}}$ has the weak Bogomolov property;
- the *Lehmer property* if and only if $\alpha \circ \widetilde{\mathbf{h}}$ has the Bogomolov property.

It is easy to observe that one has the following implications

$$\boxed{h' \text{ has very weak Bogomolov}} + \boxed{\alpha \circ \widetilde{\mathbf{h}} \geq h'} \Rightarrow \boxed{(\mathbf{h},\alpha) \text{ has very weak Lehmer}}$$

$$\boxed{h' \text{ has weak Bogomolov}} + \boxed{\alpha \circ \widetilde{\mathbf{h}} \geq h'} \Rightarrow \boxed{(\mathbf{h},\alpha) \text{ has weak Lehmer}}$$

$$\boxed{h' \text{ has Bogomolov}} + \boxed{\alpha \circ \widetilde{\mathbf{h}} \geq h'} \Rightarrow \boxed{(\mathbf{h},\alpha) \text{ has Lehmer}}$$

where $h' : S \to \Gamma$ is any height and $\alpha \circ \widetilde{\mathbf{h}} \geq h'$ means that $\alpha(\widetilde{\mathbf{h}}(s)) \geq h'(s)$ for every $s \in S$.

1.2 Examples of heights

We devote the rest of this chapter to list examples of heights and their properties, and to relate these examples to special values of L-functions. Let us start with the logarithmic Weil height, which was the main inspiration to give the general definitions in Section 1.1.

1.2.1 The height of algebraic numbers

Let $h\colon \overline{\mathbb{Q}} \to \mathbb{R}$ be the absolute logarithmic Weil height (see [BG06, Definition 1.5.4]), and let $\deg\colon \overline{\mathbb{Q}} \to \mathbb{Z}_{\geq 1}$ denote the degree $\deg(\alpha) := [\mathbb{Q}(\alpha)\colon \mathbb{Q}]$. It is immediate to see that h does not have the fibre-wise Northcott property (with respect to the collection of finite subsets of $\overline{\mathbb{Q}}$), for example because $h(\zeta) = 0$ for any root of unity $\zeta \in \overline{\mathbb{Q}}$. Hence h does not have the Northcott property. It is also immediate to see that the same holds for the degree function. However, Northcott's theorem (see [BG06, Theorem 1.6.8]) shows that the set $\mathbf{h} = \{h, \deg\}$ has the Northcott property. Moreover, it is immediate to see that h has the weak Bogomolov property, because $0 \in \mathbb{R}$ is a minimum for $h(\overline{\mathbb{Q}})$, attained exactly at the roots of unity (see [BG06, Theorem 1.5.9]). However, it is easy to see that this minimum is not isolated, because for example $\lim_{n\to+\infty} h(\sqrt[n]{2}) = 0$. Hence $\mathcal{B}(h) = 1$, and h does not have the Bogomolov property. Finally, asking whether the set $\mathbf{h} = \{h, \deg\}$ has the Lehmer property with respect to the function

$$\begin{aligned} \pi\colon \mathbb{R} \times \mathbb{Z}_{\geq 1} &\to \mathbb{R} \\ (x, d) &\mapsto x \cdot d \end{aligned}$$

is equivalent to Lehmer's celebrated problem (see [BG06, § 1.6.15]).

Let us mention some of the recent work concerning Northcott, Bogomolov and Lehmer properties relative to the logarithmic Weil height. First of all, it is known that h has the Bogomolov property when restricted to suitable infinite sub-extensions of $\overline{\mathbb{Q}}$, such as the maximal abelian extension $\mathbb{Q}^{\mathrm{ab}} \subseteq \overline{\mathbb{Q}}$ (see [AZ10]) or the extension obtained by adjoining to $\mathbb{Q}$ the coordinates of torsion points of elliptic curves (see [Hab13]). We refer the interested reader to the introduction of [CF20] for a complete list of references of known results. Moreover, Smyth's theorem [BG06, Theorem 4.4.15] says that $(\mathbf{h}, \pi)$ has the Lehmer property when restricted to the set $S \subseteq \overline{\mathbb{Q}}$ of algebraic numbers which are not Galois-conjugate to their multiplicative inverse. Finally, Dobrowolski's theorem [BG06, Theorem 4.4.1] says that, if we let

$$\begin{aligned} \alpha\colon \mathbb{R} \times \mathbb{Z}_{\geq 1} &\to \mathbb{R} \\ (x, d) &\mapsto x \cdot d \cdot \left(\frac{\log(3d)}{\log\log(3d)} \right)^3 \end{aligned} \tag{1.3}$$

then the pair $(\mathbf{h}, \alpha)$ has Lehmer's property.

1.2.2 Mahler measure

This height can be seen as a multi-dimensional analogue of the function

$$\begin{aligned} \pi \circ \widetilde{\mathbf{h}}\colon \overline{\mathbb{Q}} &\to \mathbb{R} \\ \alpha &\mapsto h(\alpha) \cdot \deg(\alpha) \end{aligned}$$

appearing in Section 1.2.1, and it is one of the protagonists of this work. As such, we devote to it the entire fourth chapter of this thesis. See in particular Definition 4.1.1 for the definition of the Mahler measure, which is a height

$$m\colon \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \dots] \setminus \{0\} \to \mathbb{R}$$

defined for non-zero polynomials with complex coefficients, in any number of variables.

Concerning Diophantine properties of the Mahler measure, it is known that the height m has the weak Bogomolov property if one restricts it to the ring $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \dots]$ of Laurent polynomials with integral coefficients, because for every $P \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \dots]$ one has that $m(P) \geq 0$ and $m(P) = 0$ if and only if P is a product of cyclotomic polynomials evaluated at monomials (see Theorem 4.1.15). Moreover, if we let

$$\begin{aligned} \delta\colon \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \dots] &\to \mathbb{Z}_{\geq 1} \\ P &\mapsto \sum_{i=1}^{+\infty} i \cdot \deg_{x_i}(P) \end{aligned}$$

then the pair (m, δ) has the Northcott property, when restricted to polynomials with integer coefficients. Indeed, this follows from [Mah62], which gives the inequality

$$\exp(m(P)) = \exp(m(\widetilde{P})) \geq 2^{-\sum_{i=1}^{+\infty} \deg_{x_i}(\widetilde{P})} \cdot \sum_{\mathrm{j}} |a_{\mathrm{j}}|$$

where $\{a_{\mathrm{j}}\}_{\mathrm{j}} \subseteq \mathbb{Z}$ are the coefficients of $\widetilde{P} = \sum_{\mathrm{j}} a_{\mathrm{j}} x^{a_{\mathrm{j}}}$ written in multi-index notation.

Finally, let us mention that, for every algebraic number $\alpha \in \overline{\mathbb{Q}}^{\times}$ we have that

$$m(f_\alpha) = h(\alpha) \deg(\alpha)$$

where $f_\alpha \in \mathbb{Z}[t]$ is the integral minimal polynomial of α (see Example 4.1.9).

1.2.3 Canonical height

The Mahler measure of an integral polynomial $P \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ has been related, by work of Maillot, to the *canonical height* of the hypersurface defined by this polynomial. To be more precise, let $V_P \subseteq \mathbb{G}_m^n$ be the zero locus of P, let $\Delta_P \subseteq \mathbb{R}^n$ be the Newton polytope of P (see [GKZ94, Chapter 6]) and let $\overline{V_P} \subseteq \mathbb{P}(\Delta_P)$ denote the closure of V_P inside the projective toric variety $\mathbb{P}(\Delta_P)$ associated to Δ_P. Fix also a family of toric Cartier divisors $\mathbf{D} = \{D_1, \dots, D_n\}$ defined on $\mathbb{P}(\Delta_P)$ which are generated by global sections. Then [Mai00, Proposition 7.2.1] (see also [Gua19, Corollary 6.3]) shows that the canonical height $h_{\mathbf{D}}^{\mathrm{can}}(\overline{V_P})$ can be computed as

$$h_{\mathbf{D}}^{\mathrm{can}}(\overline{V_P}) = \deg_{\mathbf{D}}(\mathbb{P}(\Delta_P)) \cdot (m(P) - \log|\gcd(P)|) \tag{1.4}$$

where $\gcd(P) \in \mathbb{N}$ denotes the greatest common divisor of the coefficients of P and

$$\deg_{\mathbf{D}}(\mathbb{P}(\Delta_P)) := \deg([D_1] \cup \cdots \cup [D_n]) \in \mathbb{Z}$$

denotes the geometric degree. We point out that (1.4) is somehow surprising. Indeed, the Mahler measure of a polynomial can be thought of as a height measuring the complexity of the zero locus V_P. However, one needs to keep in mind that this height does not only depend on V_P, but

on the specific model P that we have chosen. Nevertheless, (1.4) shows that this dependency is not particularly sensitive to changes in P. Moreover, we remark that also the canonical height $h_{\mathrm{D}}^{\mathrm{can}}(\overline{V_P})$ does not depend solely on V_P, because the compactification $\overline{V_P}$ depends on the Newton polytope Δ_P.

We refer the interested reader to [HS00, § B.4] for an introduction to canonical heights on abelian varieties, and to [Gua19, § 3] for an introduction to canonical heights in toric varieties, such as the height $h_{\mathrm{D}}^{\mathrm{can}}$ appearing in (1.4). Other important contributions to the field are given by the works of Zhang (see [Zha95a; Zha95b]), Philippon (see [Phi91; Phi94; Phi95]) and Faltings (see [Fal91] and [Sou92, Chapter III, § 6]), all of whom explore different definitions for the notions of "canonical" heights of sub-varieties (inside $\mathbb{P}^n$, for instance). Numerous Diophantine properties have been proved for these heights in the papers mentioned above. We chose not to describe these properties explicitly. Instead, let us move to another, even more canonical, type of height that can be associated to algebraic varieties.

1.2.4 Faltings's height

Let $\mathcal{A}(\overline{\mathbb{Q}})$ be the set of isomorphism classes of abelian varieties defined over $\overline{\mathbb{Q}}$, and let $h\colon \mathcal{A}(\overline{\mathbb{Q}}) \to \mathbb{R}$ be the stable Faltings height (see [Fal86, Section 3] and [Del85a, Page 27], which use two different normalizations). Then h satisfies the very weak Bogomolov property with respect to the dimension $\dim\colon \mathcal{A}(\overline{\mathbb{Q}}) \to \mathbb{N}$, since one has the lower bound

$$h(A) \geq -\log(\sqrt{2\pi}) \cdot \dim(A)$$

which is due to Bost (see [GR14, Corollary 8.4]). Then [Del85a, Page 29] shows that h has the weak Bogomolov property if we restrict to the set $\mathcal{A}_1(\overline{\mathbb{Q}})$ of $\overline{\mathbb{Q}}$-isomorphism classes of elliptic curves defined over $\overline{\mathbb{Q}}$. Moreover, [Löb17] and [BMR18] show that $h\colon \mathcal{A}_1(\overline{\mathbb{Q}}) \to \mathbb{R}$ has the Bogomolov property *tout court*. It seems reasonable to ask whether h itself satisfies a Bogomolov property with respect to the dimension.

Finally Faltings's theorem [Fal86, Theorem 1], combined with Zahrin's "trick" [Mil86, Remark 16.12], shows that h has the Northcott property with respect to the pair $\mathbf{f} = \{\dim, \deg\}$. This degree function is defined by

$$\begin{aligned} \deg\colon \mathcal{A}(\overline{\mathbb{Q}}) &\to \mathbb{N} \\ A &\mapsto \min\{[F\colon \mathbb{Q}] \mid A \text{ is defined over } F\} \end{aligned}$$

where we say that an abelian variety A defined over a field $\mathbb{L}$ is defined over a sub-field $\mathbb{K}$ if there exists an abelian variety A' defined over $\mathbb{K}$ and such that $A \cong A' \times_{\mathrm{Spec}(\mathbb{K})} \mathrm{Spec}(\mathbb{L})$. Then deg is well defined, because every abelian variety defined over $\overline{\mathbb{Q}}$ can be defined over a number field (see [EGA IV.3, Théorème 8.8.2]). We note that sometimes the *degree* of a polarised abelian variety is defined to be the degree of its polarisation (see [Mil86, § 13]), but this has nothing to do with our function $\deg\colon \mathcal{A}(\overline{\mathbb{Q}}) \to \mathbb{N}$.

We conclude by pointing out that Mocz has recently proved that (if one assumes Artin's and Colmez's conjectures) the function h satisfies Northcott's property with respect to $\emptyset$ if we restrict to the subset of isomorphism classes of abelian varieties with complex multiplication (see [Moc17, Theorem 1.4]).

1.2.5 Conductors of complex Galois representations

The Faltings height of an abelian variety A is a quite difficult invariant to compute. In the case of elliptic curves, an explicit formula for the unstable version of the Faltings height is provided in [Sil86, Proposition 1.1]. Another case in which the Faltings height is conjectured to be explicitly computable is given by abelian varieties with complex multiplication (see Definition 7.1.5). Indeed, the seminal work of Colmez [Col89] predicts that the Faltings height of a CM abelian variety A should be computable using the logarithmic derivatives of some L-functions related to the CM field associated to A. We refer the interested reader to Example 3.4.3 for a brief account of Colmez's conjectural formula.

For now, we only want to point out that Colmez's formula involves the *Artin conductor* $f_\chi \in \mathbb{N}$ of some Artin characters $\chi\colon G_{\mathbb{Q}} \to \mathbb{C}$. By definition $f_\chi := \mathrm{N}(\mathfrak{f}_\rho)$ coincides with the norm of the conductor associated to any complex representation $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})$ such that $\chi = \mathrm{tr}\circ\rho$, where $\mathrm{tr}\colon \mathrm{GL}_n(\mathbb{C}) \to \mathbb{C}$ denotes the trace map. The aim of this subsection is to show that the association $\rho \mapsto \mathfrak{f}_\rho$ behaves almost like a height. In particular it satisfies a Northcott property, at least if we include the Archimedean places in the definition of the conductor.

Let F be a number field, fix $n \in \mathbb{N}$ and let $\mathcal{A}_n(F)$ be the set of cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ (see [IS10, § 1]). Then [Bru06, Corollary 9] shows that the *analytic conductor* $C\colon \mathcal{A}_n(F) \to \mathbb{R}_{\geq 1}$, which is defined in [IS10, Equation (31)], satisfies the Northcott property. In particular, the $n = 1$ case shows that the set of Hecke characters $\psi\colon \mathbb{A}_F^\times \to \mathbb{C}^\times$ with bounded analytic conductor is finite.

Let now $\mathcal{W}_{\mathbb{C}}(F)$ be the set of isomorphism classes of pairs (V, ρ) where V is a finite dimensional complex vector space and $\rho\colon W_F \to \mathrm{GL}(V)$ is a continuous semi-simple representation of the Weil group W_F (see [Tat79, § 1]). Then there is a function $\mathfrak{f}\colon \mathcal{W}_{\mathbb{C}}(F) \to \mathcal{O}_F$ sending each (V, ρ) to its global Artin conductor ideal $\mathfrak{f}_\rho \subseteq \mathcal{O}_F$ (see [Neu99, Chapter VII, § 11]). Moreover, the Archimedean local Langlands correspondence, explained for example in [Kna94], allows one to associate to each $(V, \rho) \in \mathcal{W}_{\mathbb{C}}(F)$ an Archimedean conductor $C_\infty((V, \rho)) \in \mathbb{R}$, defined in exactly the same way as the Archimedean part of the analytic conductor of a cuspidal automorphic form. Then [And+94, Theorem 3.3] can be combined with our previous discussion to show that the function $C\colon \mathcal{W}_{\mathbb{C}}(F) \to \mathbb{R}$ defined as $C((V, \rho)) := \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{f}_\rho) \cdot C_\infty((V, \rho))$ satisfies the Northcott property. Let us observe that:

- one can consider all the number fields at once as follows: if $\mathcal{W}_{\mathbb{C}}$ denotes the set of isomorphism classes of triples (F, V, ρ), where F is a number field and $(V, \rho) \in \mathcal{W}_{\mathbb{C}}(F)$, then [Roh94, Property (a2)] shows that the composite map $C \circ \mathrm{Ind}\colon \mathcal{W}_{\mathbb{C}} \to \mathcal{W}_{\mathbb{C}}(\mathbb{Q}) \to \mathbb{R}$ satisfies the Northcott property, where $\mathrm{Ind}\colon \mathcal{W}_{\mathbb{C}} \to \mathcal{W}_{\mathbb{C}}(\mathbb{Q})$ sends (F, V, ρ) to the induced representation on $W_{\mathbb{Q}} \supseteq W_F$;
- the conductor $\mathfrak{f}_\rho$ is related to L-functions by means of the functional equation (see [Tat79, Theorem 3.5.3]).

1.2.6 Conductors of ℓ-adic representations

This subsection is the analogue of the previous one for representations valued in vector spaces defined over $\mathbb{Q}_\ell$.

Let $\ell \in \mathbb{N}$ be a prime number and let F be a number field. We denote by M_F^0 the set of non-Archimedean places of F, and for every $v \in M_F^0$ we write $\mathrm{Frob}_v \subseteq \mathrm{Gal}(\overline{F}/F)/\mathcal{I}_v$ for the conjugacy class of geometric Frobenius elements relative to v, where $\mathcal{I}_v$ denotes the v-adic inertia subgroup. We define $\mathcal{G}_\ell(F)$ to be the set of isomorphism classes of pairs (V, ρ) where V is a finite dimensional vector space over $\mathbb{Q}_\ell$ and $\rho\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}(V)$ is a continuous semi-simple representation satisfying the following properties:

- the set $S_\rho^{(\mathrm{ram})} \subseteq M_F^0$ of non-Archimedean places at which ρ is ramified is finite;
- the set $S_\rho^{(\mathrm{int})} \subseteq M_F^0$ of non-Archimedean places $v \in M_F^0$ such that $\mathrm{tr}(\rho(\mathrm{Frob}_v)) \in \mathbb{Z}$ has finite complement. Here $\mathrm{tr}\colon \mathrm{GL}(V) \to \mathbb{C}$ denotes the trace.

Let now $(V,\rho) \in \mathcal{G}_\ell(F)$. We set $S_\rho := S_\rho^{(\mathrm{ram})} \cup (M_F^0 \setminus S_\rho^{(\mathrm{int})})$ and we denote by $\mathbf{T}_\rho$ the family of finite sets $T \subseteq M_F^0$ such that $T \cap S_\rho = \emptyset$ and the restriction map

$$\bigcup_{v\in T} \mathrm{Frob}_v \to \mathrm{Gal}(K/F)$$

is surjective for every extension $F \subseteq K$ of number fields which is unramified outside S_ρ and such that $[K\colon F] \leq \ell^{2\dim(V)^2}$. We define two functions $\pi\colon \mathcal{G}_\ell(F) \to \mathbb{N}$ and $\tau\colon \mathcal{G}_\ell(F) \to \mathbb{N}$ as

$$\pi(V,\rho) := \max\{\mathrm{char}(\kappa_v)\colon v \in S_\rho^{(\mathrm{ram})}\}$$
$$\tau(V,\rho) := \min_{T\in\mathbf{T}_\rho} (\max\{|\mathrm{tr}(\rho(\mathrm{Frob}_v))|\colon v \in T\})$$

where κ_v denotes the residue field of F at v. Note in particular that $\mathbf{T}_\rho \neq \emptyset$, as follows from a combination of Chebotarev's density theorem and Hermite's theorem.

Then [Del85b, Théorème 1] shows that the set $\mathbf{h} = \{\dim, \pi, \tau\}$ has the Northcott property. Moreover, the functions π and τ are related to more classical invariants as follows:

- $\pi(V,\rho) \leq C_0(V,\rho)$, where $C_0(V,\rho) := \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{f}_\rho)$ denotes the norm of the conductor ideal $\mathfrak{f}_\rho \subseteq \mathcal{O}_F$ associated to ρ (see for example [Ulm16]). Hence the set $\mathbf{h} = \{\dim, C_0, \tau\}$ has the Northcott property;
- $\tau(V,\rho) \leq \dim(V)\cdot\widetilde{\tau}(V,\rho)$, where $\widetilde{\tau}\colon \mathcal{G}_\ell(F) \to \mathbb{R}$ is the function defined by

$$\widetilde{\tau}(V,\rho) := \min_{T\in\mathbf{T}_\rho} (\max\{|\sigma|\colon \sigma \in \mathrm{Sp}(\rho(\mathrm{Frob}_v))\})$$

  where, for any $f \in \mathrm{End}(V)$, we denote by $\mathrm{Sp}(f)$ the set of its eigenvalues. In particular, if we restrict to the subset $\mathcal{M}_\ell(F) \subseteq \mathcal{G}_\ell(F)$ consisting of those Galois representations that admit a weight filtration with finitely many non-zero graded pieces (see [Jan10, § 2]), then the sets $\{\dim, \pi, w_{\max}\}$ and $\{\dim, C_0, w_{\max}\}$ have the Northcott property, where $w_{\max}\colon \mathcal{M}_\ell(F) \to \mathbb{N}$ sends a representation to the greatest of its weights.

Let us conclude by making the following observations:

- the semi-semplifications of the ℓ-adic étale cohomology groups $H^i_{\text{ét}}(X_{\overline{F}}; \mathbb{Q}_\ell(j))$ associated to a smooth and proper variety X defined over F which has good reduction at all the primes of F lying above ℓ give rise to elements of $\mathcal{M}_\ell(F)$ which are pure of weight $i-2j$. For these Galois representations the set S_ρ equals the set of primes of F which either lie above ℓ or are primes of bad reduction for X. This follows from the smooth and proper base change theorem for étale cohomology, combined with Deligne's proof of the Weil conjectures (see [Jan90, Appendix C]).
- we can consider all the number fields at once, as we did in Section 1.2.5, by defining $\mathcal{G}_\ell$ as the set of isomorphism classes of triples (F,V,ρ) where F is a number field and $(V,\rho) \in \mathcal{G}_\ell(F)$. Then [Roh94, Property (a'2)] implies that the sets $\{\dim \circ\, \mathrm{Ind}, \pi \circ\, \mathrm{Ind}, \tau\}$ and $\{\dim, C_0 \circ\, \mathrm{Ind}, \tau\}$ have the Northcott property. Here $\mathrm{Ind}\colon \mathcal{G}_\ell \to \mathcal{G}_\ell(\mathbb{Q})$ is again the map sending (F,V,ρ) to the representation induced on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \supseteq \mathrm{Gal}(\overline{F}/F)$;

- the conductor $\mathfrak{f}_\rho$ is supposed to be related to the L-function $L(\rho, s)$ by means of the conjectural functional equation (compare with [Tat79, § 4.5]).

1.2.7 Volumes of hyperbolic manifolds

We conclude this roundup of examples by talking about a more geometric example of height, given by the volume of hyperbolic manifolds.

Let $\mathcal{H}$ be the set of isomorphism classes of hyperbolic manifolds of finite volume. Then it is conjectured that the volume $\mathrm{vol}\colon \mathcal{H} \to \mathbb{R}_{\geq 0}$ has the Bogomolov property, and that the minimum is attained at an arithmetic hyperbolic manifold $M \cong \mathfrak{h}_n/\Gamma$, where Γ is an arithmetic subgroup of the isometry group of the hyperbolic space $\mathfrak{h}_n$ (see [BE14]). Moreover, if we restrict to the set $\mathcal{H}^{\mathrm{ar}} \subseteq \mathcal{H}$ of isomorphism classes of arithmetic hyperbolic manifolds, it is conjectured that the set $\mathbf{h} = \{\mathrm{vol}, \dim, \deg\}$ has the Northcott property, where the degree is defined by $\deg(M) := [\mathbb{Q}(\mathrm{tr}(\pi_1(M)^{(2)})) : \mathbb{Q}]$. Here we denote by $\pi_1(M)^{(2)}$ the sub-group generated by the squares, and by $\mathrm{tr}\colon \pi_1(M) \to \mathbb{C}$ the trace map induced from the embedding of $\pi_1(M)$ into the isomorphism group of $\mathfrak{h}_n$. This Northcott property has been proved for three dimensional arithmetic hyperbolic manifolds (see [Jeo14]).

Cohomology theories, motives and regulators

2

What makes life dreary is the want of motive.

George Eliot, *Daniel Deronda*

The aim of this chapter is to review the notion of a *motive*. This was envisioned by Grothendieck as an attempt to gather the properties common to the different cohomology theories which could be defined for algebraic varieties. Most notably, as Serre points out in [Ser91], there are infinitely many ℓ-adic cohomology theories $H^{i,j}_\ell(-)$, one for every rational prime $\ell \in \mathbb{N}$, and it is a challenging question to determine under what circumstances a $\mathbb{Q}_\ell$-linear map $\varphi_\ell : H^{i,j}_\ell(X) \to H^{i,j}_\ell(Y)$ induces a $\mathbb{Q}_{\ell'}$ linear map $\varphi_{\ell'} : H^{i,j}_{\ell'}(X) \to H^{i,j}_{\ell'}(Y)$ for a prime $\ell' \neq \ell$. This is clearly true if $\varphi_\ell = f^*$ for some $f : Y \to X$, and more generally if φ_ℓ is induced by a span $Y \leftarrow Z \to X$ where $Z \to Y$ is proper and of relative dimension zero. These are examples of $\mathbb{Q}_\ell$-linear maps φ_ℓ which are "motivated", *i.e.* which come from the algebraic geometry of the varieties X and Y, and thus have a good reason to extend to ℓ'-adic cohomology theories for $\ell' \neq \ell$. This notion of motivated maps can be encoded in essentially two different ways:

- by the compatibility of φ_ℓ with the various comparison isomorphisms which relate ℓ-adic cohomology to singular cohomology (for varieties defined over sub-fields);
- by keeping only the maps φ_ℓ which come from algebraic correspondences, thereby taking into account the geometry of the varieties in question. Usually one does this by considering the correspondences between X and Y, which are suitable linear combinations of closed sub-varieties of $X \times Y$, modulo an "adequate" equivalence relation (see Definition 2.2.1).

These two notions of "motivated" give rise to two very different notions of "motive": the first type of construction allows one to get abelian categories (which we describe in Section 2.2.2), whereas the second kind of construction allows one to get abelian categories of "pure motives" (see Section 2.2.1), related to smooth and projective varieties, only if one considers algebraic cycles modulo numerical equivalence, which is the coarsest of all adequate equivalence relations. If one wants instead to consider finer equivalence relations (like the ones induced by cohomology theories, which would link the two approaches) one faces immediately some important obstacles, which have been encoded in the form of the "standard conjectures" (see [And04, Chapitre 5]). Moreover, the cohomological approach described in Section 2.2.2 allows one to get an abelian category of "mixed motives", where in particular there are objects associated to each separated scheme of finite type defined over the field we are working with. In contrast, the best one can do to this day with the second approach is to get a triangulated category of "mixed motives" (see Section 2.2.3). Nevertheless, a far reaching program laid down by Beilinson in the foundational papers [Bei87, § 5] and [Bei86b, § 0.3] predicts that the second approach should also lead to an abelian category of mixed motives, and the two approaches should agree. We warn the reader that these two tasks are likely to be extremely difficult: the existence of a "geometric" abelian category of mixed motive would give a positive answer to the standard conjectures (see [Bei12]).

This is also profoundly related to the conjectures concerning the fullness and conservativity of the realisation functors (see for instance [HM17, Proposition 10.2.1]).

Nevertheless, the existing constructions of triangulated categories of mixed motives (due to the work of Voevodsky, Morel, Suslin, Ayoub, Cisinski and Déglise, Robalo, *etc.*) allow one to talk about the motivic cohomology of a scheme X. This is an incredibly rich invariant, which is (conjecturally) linked to the algebraic K-theory of perfect complexes of sheaves on X (see Section 2.3.1) on the one hand, and to complexes of algebraic cycles (see Section 2.3.2) and functions (see Section 2.3.3) on X. Finally, motivic cohomology is suppose to have the role of a "universal cohomology theory", in the sense that every cohomology theory satisfying the axioms that we outline in the next section should receive a map from motivic cohomology. These maps are usually called *regulators*, because they help to tame down, hence to regulate, the wildness of algebraic cycles present in motivic cohomology. We outline their construction and their basic properties in Section 2.4.

2.1 What is a cohomology theory?

The world of algebraic and analytic geometry is a very chaotic one. Algebraic varieties, manifolds and (more generally) topological spaces can be deformed in many different ways, which can become difficult to control. Homology and cohomology theories are a copious source of powerful invariants which allow one to use methods of linear and (co)homological algebra to study the geometric world. The aim of the present section is to recall some of the working definitions for the concept of cohomology theory (due to Weil, Bloch and Ogus, Cisinski and Déglise) and to recall how many familiar cohomology theories (singular cohomology, de Rham cohomology, étale cohomology) fit into this picture.

2.1.1 Axioms for cohomology theories

The question of finding suitable axioms for the concept of homology (or cohomology) is a highly non-trivial one. In algebraic topology, this subject has a very rich history (see [Die09, Chapter IV]), and has led to the following notion (see [AGP02, Definition 12.1.1]).

Definition 2.1.1 – Cohomology theory (for topological spaces)

Let $\mathcal{V}$ denote a sub-category of the category of topological spaces, closed under finite products and such that $\mathbb{R} \in \mathcal{V}$. Let $\mathcal{V}^*$ denote the category of pairs (X, A) of objects of $\mathcal{V}$ such that $A \subseteq X$ is a subspace, and let $\widetilde{\mathcal{V}}^*$ be the category of triples (X, A, B) of objects of $\mathcal{V}$ such that $B \subseteq A \subseteq X$. This category is endowed with two functors $\pi_1, \pi_2 \colon \widetilde{\mathcal{V}}^* \to \mathcal{V}^*$ defined by $\pi_1(X, A, B) := (X, A)$ and $\pi_2(X, A, B) := (A, B)$. Fix an abelian category $\mathcal{A}$ and let $H^\bullet \colon (\mathcal{V}^*)^{\mathrm{op}} \to \mathcal{A}^{\mathbb{Z}}$ be a functor with values in $\mathbb{Z}$-graded objects of $\mathcal{A}$, and $\delta^\bullet \colon H^\bullet \circ \pi_2 \to \sigma \circ H^\bullet \circ \pi_1$ be a natural transformation, where $\sigma \colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $\sigma(\mathbf{A})_i := A_{i+1}$ for every $\mathbf{A} := \{A_i\}_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$.

The pair $(H^\bullet, \delta^\bullet)$ is an $\mathcal{A}$-valued *cohomology theory* if the following conditions are satisfied:

Homotopy invariance — for every $X \in \mathcal{V}$, the projection map $\pi \colon X \times \mathbb{R} \to X$ induces an isomorphism $H^\bullet(\pi) \colon H^\bullet(X) \xrightarrow{\sim} H^\bullet(X \times \mathbb{R})$, where $H^\bullet(Y) := H^\bullet(Y, \emptyset)$ for every $Y \in \mathcal{V}$.

Excision for every $(X, A) \in \mathcal{V}^*$ and every subset $U \subseteq A$ whose closure is contained in the interior of A, the inclusion $j\colon (X\backslash U, A\backslash U) \hookrightarrow (X, A)$ induces an isomorphism $H^\bullet(j)\colon H^\bullet(X, A) \xrightarrow{\sim} H^\bullet(X \setminus U, A \setminus U)$.

Exact Sequence in Cohomology for every $(X, A, B) \in \widetilde{\mathcal{V}}^*$ one has a long exact sequence

$$\cdots \longrightarrow H^i(X, B) \xrightarrow{H^i(\kappa)} H^i(A, B) \xrightarrow{\delta^i} H^{i+1}(X, A) \xrightarrow[H^{i+1}(\iota)]{} H^{i+1}(X, B) \longrightarrow \cdots$$

where $\iota\colon (X, B) \hookrightarrow (X, A)$ and $\kappa\colon (A, B) \hookrightarrow (X, B)$ are the obvious inclusions.

Moreover, $(H^\bullet, \delta^\bullet)$ is said to be *additive* if the following axiom is satisfied:

Additivity $\mathcal{A}$ is closed under products and $H^\bullet$ preserves products, *i.e.* (remember that $H^\bullet$ is contravariant) for every set S and every collection $\{(X_s, A_s)\}_{s\in S} \subseteq \mathcal{V}^*$ the inclusions inside the disjoint union $\iota_s\colon (X_s, A_s) \to \bigsqcup_{s\in S}(X_s, A_s)$ induce an isomorphism

$$\prod_{s\in S} H^\bullet(\iota_s)\colon H^\bullet\left(\bigsqcup_{s\in S} X_s, \bigsqcup_{s\in S} A_s\right) \xrightarrow{\sim} \prod_{s\in S} H^\bullet(X_s, A_s).$$

and $(H^\bullet, \delta^\bullet)$ is said to be *ordinary* if the following axiom is satisfied:

Dimension Axiom $H^i(\{*\}, \emptyset) = 0$ if $i \neq 0$, where $\{*\} \in \mathcal{V}$ is the topological space with only one point.

Remark 2.1.2. Thanks to the homotopy axiom, the functor $H^\bullet\colon (\mathcal{V}^*)^{\mathrm{op}} \to \mathcal{A}^{\mathbb{Z}}$ factors through a functor $hH^\bullet\colon (h\mathcal{V}^*)^{\mathrm{op}} \to \mathcal{A}^{\mathbb{Z}}$, where $h\mathcal{V}^*$ denotes the *homotopy category*, which has the same objects of $\mathcal{V}^*$ but where the morphisms are homotopy classes of maps.

Remark 2.1.3. The axioms for an ordinary cohomology theory (in our terminology) correspond to the axioms laid down by Eilenberg and Steenrod in [ES52, § I.3].

Let us now move to algebraic geometry. In this context the analogue of Definition 2.1.1 can be identified in the concept of a *mixed Weil cohomology* in the sense of [CD19, § 17.2] and [Dre13, § 2.1]. We give here an axiomatic treatment of this notion, following [Pan03, § 2], to emphasise the parallels with Definition 2.1.1.

Definition 2.1.4 – Cohomology theory (algebraic geometry)

Let S be a scheme, let $\mathcal{V}$ be a category of schemes over S and let $\mathcal{V}^*$ be the category whose objects are pairs (X, U) where $X \in \mathcal{V}$ and $U \hookrightarrow X$ is an open immersion in $\mathcal{V}$. Assume that $\mathbb{A}^1_S \in \mathcal{V}$ and that $\mathcal{V}$ is closed under products. Denote by $\widetilde{\mathcal{V}}^*$ the category of triples (X, U, V) where $X \in \mathcal{V}$ and $V \hookrightarrow U \hookrightarrow X$ are open immersions in $\mathcal{V}$. This category is endowed with the functors $\pi_1, \pi_2\colon \widetilde{\mathcal{V}}^* \to \mathcal{V}^*$ defined as $\pi_1(X, U, V) := (X, U)$ and $\pi_2(X, U, V) := (U, V)$. Fix an abelian cateogry $\mathcal{A}$, let $\mathcal{A}^{\mathbb{Z}}$ be the category of $\mathbb{Z}$-

graded objects in $\mathcal{A}$ and let $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be the shift functor defined in Definition 2.1.1. Then a pair $(H^\bullet, \delta^\bullet)$ consisting of a functor $H^\bullet\colon (\mathcal{V}^*)^{\mathrm{op}} \to \mathcal{A}^{\mathbb{Z}}$ together with a natural transformation $\delta^\bullet\colon H^\bullet \circ \pi_2 \to \sigma \circ H^\bullet \circ \pi_1$ is an $\mathcal{A}$-valued *cohomology theory* if the following conditions are satisfied:

Homotopy invariance for every $X \in \mathcal{V}$ the projection map $\pi\colon X \times_S \mathbb{A}^1_S \to X$ induces an isomorphism $H^\bullet(\pi)\colon H^\bullet(X) \xrightarrow{\sim} H^\bullet(X \times_S \mathbb{A}^1_S)$, where $H^\bullet(Y) := H^\bullet(Y, \emptyset)$ for every $Y \in \mathcal{V}$.

Excision every *Nisnevich distinguished* morphism $\varphi\colon (X, U) \to (X', U')$ in $\mathcal{V}^*$ induces an isomorphism $H^\bullet(\varphi)\colon H^\bullet(X', U') \xrightarrow{\sim} H^\bullet(X, U)$. By definition, φ is given by a Cartesian square

$$\begin{array}{ccc} U & \xrightarrow{j} & X \\ \downarrow{\scriptstyle f'} & \lrcorner\ \varphi & \downarrow{\scriptstyle f} \\ U' & \xrightarrow[j']{} & X' \end{array} \tag{2.1}$$

such that the map f is étale and induces an isomorphism

$$f^{-1}((X' \setminus j'(U'))_{\mathrm{red}}) \xrightarrow{\sim} (X' \setminus j'(U'))_{\mathrm{red}}$$

where $(X' \setminus j'(U'))_{\mathrm{red}}$ denotes $X' \setminus j'(U')$, considered as a closed sub-scheme of X' with its reduced sub-scheme structure.

Exact Sequence in Cohomology for every $(X, U, V) \in \widetilde{\mathcal{V}}^*$ we have a long exact sequence

$$\cdots \longrightarrow H^i(X, V) \xrightarrow{H^i(\kappa)} H^i(U, V) \xrightarrow{\delta^i} H^{i+1}(X, U) \xrightarrow[H^{i+1}(\iota)]{} H^{i+1}(X, V) \longrightarrow \cdots \tag{2.2}$$

where $\iota\colon (X, V) \to (X, U)$ and $\kappa\colon (U, V) \to (X, V)$ are the obvious morphisms in $\mathcal{V}^*$.

Suppose now that $\mathcal{A}$ is an abelian tensor category, as defined in [DM82, Definition 1.15], and let $\mathbb{1}_{\mathcal{A}}$ denote the unit object. Then an $\mathcal{A}$-valued cohomology theory $(H^\bullet, \delta^\bullet)$ is said to be *ordinary* if the following axiom is satisfied

Dimension Axiom $S \in \mathcal{V}$, $H^0(S) \cong \mathbb{1}_{\mathcal{A}}$ and $H^i(S) = 0$ for $i > 0$.

and it is said to be *stable* if the following axiom is satisfied

Stability $\mathbb{G}_{m,S} \in \mathcal{V}$, $H^0(\mathbb{G}_{m,S}) \cong H^1(\mathbb{G}_{m,S}) \cong \mathbb{1}_{\mathcal{A}}$ and $H^i(\mathbb{G}_{m,S}) = 0$ for $i > 1$.

where $\mathbb{G}_{m,S} \to S$ denotes the multiplicative group (see [SP, Example 022U]). Moreover, a *cross product* on a cohomology theory $(H^\bullet, \delta^\bullet)$ is a natural transformation

$$\times\colon H^\bullet \otimes H^\bullet \to H^\bullet \circ \mu_S$$

$$\begin{array}{ccccc} H^i(U)\otimes H^j(Y,V) & \xrightarrow{\times} & H^{i+j}(U\times_S Y, U\times_S V) & \xrightarrow[\sim]{H^{i+j}(\iota)^{-1}} & H^{i+j}((U\times_S Y)\cup(X\times_S V), X\times_S V) \\ \downarrow{\scriptstyle \delta^i\otimes \mathrm{Id}} & & & & \downarrow{\scriptstyle \delta^{i+j}} \\ H^{i+1}(X,U)\otimes H^j(Y,V) & & \xrightarrow{\qquad\qquad\times\qquad\qquad} & & H^{i+j+1}(X\times_S Y, (U\times_S Y)\cup(X\times_S V)) \end{array}$$

Figure 2.1.: Compatibility between cross product and boundary in Definition 2.1.4. Here

$$\iota\colon (U\times_S Y, U\times_S V) \to ((U\times_S Y)\cup(X\times_S V), X\times_S V)$$

is the obvious inclusion, which is Nisnevich distinguished.

where the functors $H^\bullet\otimes H^\bullet$ and $\mu_S\colon (\mathcal{V}^*)^2\to\mathcal{V}^*$ are defined by

$$\begin{aligned} H^\bullet\otimes H^\bullet((X,U),(X',U')) &:= H^\bullet(X,U)\otimes H^\bullet(X',U') \\ \mu_S((X,U),(X',U')) &:= (X\times_S X', (X\times_S U')\cup(U\times_S X')) \end{aligned}$$

for every $((X,U),(X',U'))\in(\mathcal{V}^*)^2$. We demand moreover that for every $(X,U),(Y,V)\in\mathcal{V}^*$ the diagram in Figure 2.1 is commutative. Finally, a *mixed Weil cohomology* is a triplet $(H^\bullet,\delta^\bullet,\times)$ consisting of an ordinary and stable cohomology theory $(H^\bullet,\delta^\bullet)$ together with a cross product $\times$ satisfying the following axiom:

Künneth formula the cross product is an isomorphism, when restricted to the category $\mathcal{V}^{(s)}\subseteq\mathcal{V}$ of schemes $X\in\mathcal{V}$ which are smooth over S.

Remark 2.1.5. Observe that the definition of mixed Weil cohomology given in [CD12] and [Dre13] is not entirely axiomatic, as the one we have given here, but assumes that the cohomology comes from an $\mathcal{A}$-valued Nisnevich sheaf of complexes (see also Section 2.1.2).

Remark 2.1.6. Specialising the cross product to the pair $((X,U),(X,U))$ one gets the *cup product*

$$\smile\colon H^\bullet(X,U)^{\otimes 2}\xrightarrow{\times} H^\bullet(X\times_S X,(X\times_S U)\cup(U\times_S X))\xrightarrow{H^\bullet(\Delta)} H^\bullet(X,U)$$

where $\Delta\colon (X,U)\to(X\times_S X, U\times_S U)\hookrightarrow(X\times_S X,(X\times_S U)\cup(U\times_S X))$ is the diagonal map. Vice-versa, the cross product is determined by the cup product as

$$\times\colon H^\bullet(X,U)\otimes H^\bullet(Y,V)\xrightarrow{H^\bullet(\pi_X)\otimes H^\bullet(\pi_Y)} H^\bullet(X\times_S Y,(X\times_S V)\cup(U\times_S Y))^{\otimes 2}\xrightarrow{\smile} H^\bullet(X\times_S Y,(X\times_S V)\cup(U\times_S Y))$$

where $\pi_X\colon (X\times_S Y,(X\times_S V)\cup(U\times_S Y))\to(X,U)$ and $\pi_Y\colon (X\times_S Y,(X\times_S V)\cup(U\times_S Y))\to(Y,V)$ denote the obvious projections. The cup product makes $H^\bullet(X,U)$ into a graded ring, and sometimes (for example in [CD19, § 17.2]) the cross product is called *exterior cup product*.

Remark 2.1.7. The name "mixed Weil cohomology" is related to the fact that every such cohomology theory is expected to give rise to a Weil cohomology theory, in the sense of [And04, Definition 3.3.1.1], when restricted to the sub-category $\mathcal{V}^{(sp)}\subseteq\mathcal{V}$ consisting of those $V\in\mathcal{V}$ which are smooth and projective over S. This is proved in [CD12] when $S=\mathrm{Spec}(\kappa)$ for a perfect field κ, modulo the fact that the cohomology is not known to vanish in negative degrees.

Remark 2.1.8. Sometimes it is useful to have relative cohomology groups $H^i(X,Y)$ defined for a map $Y\to X$ which is not necessarily an open immersion. This is done for example in [Den97b] or [BD99], where relative cohomology is taken with respect to a closed sub-scheme. We believe

that all the axioms laid out in Definition 2.1.4 should carry over to a more general setting (where $\mathcal{V}^*$ is a more general arrow category), except perhaps for the excision axiom.

Remark 2.1.9. Definition 2.1.1 and Definition 2.1.4 admit a common generalisation, where $\mathcal{V}^*$ is taken to be a sub-category of the category of morphisms $\mathrm{Arr}(\mathcal{V})$, where $\mathcal{V}$ is a Grothendieck site (see [SP, Section 03NF]). An example of this can be any category of schemes with one of the many topology which can be defined on them (see *e.g.* [SP, Chapter 020K]). Other examples could be the category of manifolds, or the category of rigid analytic spaces. One of the most general definitions of the notion of cohomology theory, which uses the language of ∞-categories, is given in [Lur17, Definition 1.4.1.6].

Let us conclude this section by mentioning another possible set of axioms for a cohomology theory, which was described by Bloch and Ogus in [BO74, § 2], and was refined by Gillet [Gil81, Definition 1.2], Jannsen [Jan90, § 6] and Levine [Lev98, Chapter V, Definition 1.1.6]. In fact, their setting requires a pair $(H^\bullet, H_\bullet)$ consisting of a cohomology and a homology theory, which should be related by a cap product and by Poincaré duality. We decided to include this set of axioms here, following the exposition given by Jannsen, because it is used in some definitions of the abelian categories of mixed motives given in Definition 2.2.7.

Definition 2.1.10 – Twisted Poincaré duality theory

Let κ be a field and let $\mathcal{V}$ be a full sub-category of the category of schemes of finite type over κ, which contains all the quasi-projective ones (in the sense of [EGA II, Définition 5.3.1]). Denote by $\mathcal{V}_*$ the category with the same objects of $\mathcal{V}$, but only proper morphisms, and by $\mathcal{V}^*$ the category whose objects are closed immersions $Y \hookrightarrow X$ with $Y, X \in \mathcal{V}$, and whose morphisms are Cartesian squares. Let $\mathcal{A}$ be an abelian tensor category, as defined in [DM82, Definition 1.15].

Then a *twisted Poincaré duality theory* with values in $\mathcal{A}$ consists of two families of functors $\{H^{\bullet,j}\colon (\mathcal{V}^*)^{\mathrm{op}} \to \mathcal{A}^{\mathbb{Z}}\}_{j\in\mathbb{Z}}$ and $\{H_{\bullet,j}\colon \mathcal{V}_* \to \mathcal{A}^{\mathbb{Z}}\}_{j\in\mathbb{Z}}$ with values in $\mathbb{Z}$-graded objects in $\mathcal{A}$, satisfying the following axioms:

Exact Sequence in Cohomology if $\alpha\colon Z \hookrightarrow Y$ and $\beta\colon Y \hookrightarrow X$ are closed immersions, for every $j \in \mathbb{Z}$ there exists a long exact sequence

$$\cdots \to H^{i,j}(\beta\circ\alpha) \to H^{i,j}(\beta) \xrightarrow{H^{i,j}(\varphi)} H^{i,j}(\gamma) \to H^{i+1,j}(\beta\circ\alpha) \to \dots$$

where $\gamma\colon Y \setminus \alpha(Z) \hookrightarrow X \setminus \beta(\alpha(Z))$ denotes the closed immersion induced by β, and $\varphi\colon \gamma \to \beta$ is the obvious Cartesian square. Moreover, for every commutative diagram

$$\begin{array}{ccccc} Z & \xhookrightarrow{\alpha} & Y & \xhookrightarrow{\beta} & X \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ Z' & \xhookrightarrow[\alpha']{} & Y' & \xhookrightarrow[\beta']{} & X' \end{array} \qquad (2.3)$$

where the horizontal arrows are closed immersions and the squares are Cartesian, the two exact sequences corresponding to (α, β) and

(α', β') fit in a commutative diagram whose vertical arrows are given by $H^{*,j}(g \circ f)$, $H^{*,j}(g)$ and $H^{*,j}(h)$, where h is the square

$$\begin{array}{ccc} Y \setminus \alpha(Z) & \xrightarrow{\gamma} & X \setminus \beta(\alpha(Z)) \\ \downarrow & \lrcorner\ h & \downarrow \\ Y' \setminus \alpha'(Z') & \xrightarrow[\gamma']{} & X' \setminus \beta'(\alpha'(Z')) \end{array}$$

induced by (2.3), which is evidently Cartesian.

Excision if $\alpha\colon Z \hookrightarrow U$ is a closed immersion and $\beta\colon U \hookrightarrow X$ is an open immersion, the natural map $H^{i,j}(\beta \circ \alpha) \to H^{i,j}(\beta)$ is an isomorphism;

Étale Contravariance for every $i, j \in \mathbb{Z}$ and every étale morphism $f\colon X \to Y$ between two objects $X, Y \in \mathcal{V}$ there exists a map $f^*\colon H_{i,j}(Y) \to H_{i,j}(X)$ such that for every Cartesian square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle\alpha} & \lrcorner & \downarrow{\scriptstyle\beta} \\ Z & \xrightarrow{g} & W \end{array}$$

where α and β are proper and f and g are étale, the square

$$\begin{array}{ccc} H_{i,j}(Y) & \xrightarrow{f^*} & H_{i,j}(X) \\ {\scriptstyle H_{i,j}(\beta)}\downarrow & & \downarrow{\scriptstyle H_{i,j}(\alpha)} \\ H_{i,j}(W) & \xrightarrow{g^*} & H_{i,j}(Z) \end{array}$$

commutes.

Exact Sequence in Homology for every closed immersion $\alpha\colon Z \hookrightarrow X$ with complementary open immersion $\beta\colon X \setminus \alpha(Z) \hookrightarrow X$ there is a long exact sequence

$$\cdots \longrightarrow H_{i,j}(X) \xrightarrow{\beta^*} H_{i,j}(X \setminus \alpha(Z)) \xrightarrow{\partial_{i,j}} H_{i-1,j}(Z) \xrightarrow[H_{i,j}(\alpha)]{} H_{i-1,j}(X) \longrightarrow \cdots$$

and for every proper map $f\colon X \to X'$, the long exact sequences associated to α and to $\alpha'\colon f(Z) \hookrightarrow X'$ fit in a commutative diagram where the vertical arrows are given by $H_{i,j}(f|_Z)$, $H_{i,j}(f)$ and $H_{i,j}(h) \circ \varphi^*$. Here $\varphi\colon X \setminus f^{-1}(f(\alpha(Z))) \hookrightarrow X \setminus \alpha(Z)$ is the obvious immersion and $h\colon X \setminus f^{-1}(f(\alpha(Z))) \to X' \setminus f(Z)$ is the restriction of f.

Cap product for every $a, b, c, d \in \mathbb{Z}$ and every closed immersion $\alpha\colon Z \hookrightarrow X$ there is a pairing

$$H_{a,b}(X) \otimes H^{c,d}(\alpha) \xrightarrow{\frown} H_{a-b,c-d}(Z)$$

such that for every Cartesian square

$$\begin{array}{ccc} Z & \xhookrightarrow{\alpha} & X \\ \downarrow{\scriptstyle f} & \lrcorner\ \varphi & \downarrow{\scriptstyle g} \\ Z' & \xhookrightarrow[\alpha']{} & X' \end{array}$$

and every $\xi \in H_{a,b}(X)$, $\eta \in H^{c,d}(\alpha)$ and $\eta' \in H^{c,d}(\alpha')$ we have that $f^*(\xi \frown \eta) = g^*(\xi) \frown g^*(\eta)$ if f and g are étale, and

$$H_{a,b}(g)(\xi) \frown \eta' = H_{a,b}(f)(\xi \frown H^{c,d}(\varphi)(\eta'))$$

if f and g are proper.

Fundamental Class if $X \in \mathcal{V}$ is irreducible and has dimension d then there exists a morphism $\eta_X \colon \mathbb{1}_{\mathcal{A}} \to H_{2d,d}(X)$, where $\mathbb{1}_{\mathcal{A}} \in \mathcal{A}$ denotes the identity object for the tensor product. Moreover, if $\alpha \colon X \to Y$ is étale then $\alpha^* \circ \eta_Y = \eta_X$, which makes sense because α has relative dimension zero (see [SP, Section 02GH]).

Poincaré duality if $X \in \mathcal{V}$ is smooth, irreducible of dimension d and $\alpha \colon Z \hookrightarrow X$ is a closed immersion, the map

$$\mathbb{1}_{\mathcal{A}} \otimes H^{2d-i,d-j}(\alpha) \xrightarrow{\eta_X \otimes \mathrm{Id}} H_{2d,d}(X) \otimes H^{2d-i,d-j}(\alpha) \twoheadrightarrow H_{i,j}(Z)$$

is an isomorphism. From this we get the Poincaré duality

$$H^{2d-i,d-j}(\alpha) \xrightarrow{\sim} H_{i,j}(Z)$$

using the identification $H^{2d-i,d-j}(\alpha) \xrightarrow{\sim} \mathbb{1}_{\mathcal{A}} \otimes H^{2d-i,d-j}(\alpha)$.

Remark 2.1.11. Usually, if $\alpha \colon Z \hookrightarrow X$ is a closed immersion, the groups $H^{i,j}_Z(X) := H^{i,j}(\alpha)$ are called cohomology groups of X with support on Z.

Remark 2.1.12. Let us explain the relation between Definition 2.1.4 and Definition 2.1.10. First of all, if $H^\bullet$ is a stable cohomology theory in the sense of Definition 2.1.4, we can define a twisted version of it by setting $H^{\bullet,j}(X,U) := H^\bullet(X,U) \otimes \mathbb{1}_{\mathcal{A}}(j)_H$, where the *Tate object* $\mathbb{1}_{\mathcal{A}}(j)_H$ is defined by setting

$$\mathbb{1}_{\mathcal{A}}(j)_H := \begin{cases} (H^1(\mathbb{G}_{m,S})^{\otimes j})^\vee, & \text{if } j \geq 0 \\ H^1(\mathbb{G}_{m,S})^{\otimes(-j)}, & \text{if } j \leq 0 \end{cases}$$

where $A^\vee$ denotes the $\otimes$-dual of an object $A \in \mathcal{A}$ (see [DM82, Page 110]). This dual might not always exist, but in our case it does because $H^\bullet$ is assumed to be stable. Moreover, if $Z \hookrightarrow X$ is a closed immersion we might set $H^{\bullet,j}_Z(X) := H^{\bullet,j}(X,U)$, where $U \hookrightarrow X$ denotes the open complement. This shows that a stable cohomology theory gives rise to functors $H^{\bullet,j}$ on the category of closed immersions, which satisfy the first two axioms of Definition 2.1.10.

2.1.2 Constructing cohomology theories

The aim of this section is to survey some ways in which cohomology theories can be constructed. All the cohomology theories that we describe in Section 2.1.3 can be constructed in one or more of these ways.

Example 2.1.13 (Spectra)**.** Let us start with cohomology theories defined on topological spaces, which were described in Definition 2.1.1. First of all, the mapping cone (see [AGP02, Examples 3.1.2]) defines a functor $C\colon \mathcal{V}^* \to \mathcal{V}_+$, where $\mathcal{V}_+$ denotes the category of pointed spaces (X, x) with $X \in \mathcal{V}$. Then a natural way of constructing a family of functors $\{H^n\colon (\mathcal{V}^*)^{\mathrm{op}} \to \mathrm{Sets}\colon n \in \mathbb{Z}\}$ is by the composition

$$H^n\colon (\mathcal{V}^*)^{\mathrm{op}} \xrightarrow{C^{\mathrm{op}}} (\mathcal{V}_+)^{\mathrm{op}} \xrightarrow{h^{\mathrm{op}}} (h\mathcal{V}_+)^{\mathrm{op}} \xrightarrow{[-,(X_n,x_n)]} \mathrm{Sets} \tag{2.4}$$

where $h\mathcal{V}_+$ denotes the homotopy category of pointed spaces in $\mathcal{V}_+$ and $[-, (X_n, x_n)]$ denotes the representable functor which sends $(Y, y) \in \mathcal{V}_+$ to the set of homotopy classes of maps into (X_n, x_n). Let us write $\widetilde{H}^n := [-, (X_n, x_n)] \circ h^{\mathrm{op}}$, so that $H^n = \widetilde{H}^n \circ C^{\mathrm{op}}$. Then the existence of a natural transformation $\delta^\bullet$ as in Definition 2.1.1 can be encoded in the existence of a natural equivalence $\widetilde{\delta}^\bullet\colon \widetilde{H}^\bullet \xrightarrow{\sim} \sigma \circ \widetilde{H}^\bullet \circ \Sigma^{\mathrm{op}}$, where $\Sigma\colon \mathcal{V}_+ \to \mathcal{V}_+$ denotes the suspension (see [AGP02, § 2.10]). In turn the existence of $\widetilde{\delta}^\bullet$ is equivalent to the existence of pointed homotopy equivalences $(X_n, x_n) \xrightarrow{\sim} \Omega(X_{n+1}, x_{n+1})$, where $\Omega(X_{n+1}, x_{n+1})$ denotes the loop space of (X_{n+1}, x_{n+1}) (see [AGP02, Definition 1.3.9]). One says that such a sequence of pointed topological spaces $\{(X_n, x_n)\}_{n\in\mathbb{Z}}$ together with pointed homotopy equivalences $(X_n, x_n) \xrightarrow{\sim} \Omega(X_{n+1}, x_{n+1})$ forms a *spectrum* (see [AGP02, § 12.3]), and under these circumstances the functors H^n defined by (2.4) form a cohomology theory in the sense of Definition 2.1.1, which is valued in abelian groups. Indeed, for every pointed space $(X, x) \in \mathcal{V}_+$ the two-fold suspension $\Sigma^2(X, x) := \Sigma(\Sigma(X, x))$ is an abelian co-group object in the homotopy category $h\mathcal{V}_+$, with respect to the smash product, and this implies that for every space $(Y, y) \in \mathcal{V}_+$ the set $[(Y, y), \Sigma^2(X, x)]$ has the structure of an abelian group. A fundamental result in homotopy theory, called Brown's representability theorem (see [AGP02, § 12.2]), says that each cohomology theory arises in this way, at least if $\mathcal{V}$ is a category of CW-complexes (see [AGP02, § 5.1]). Moreover, a previous result of Milnor (see [AGP02, Theorem 12.1.19]) asserts that every additive and ordinary cohomology theory can be represented by an Eilenberg-MacLane spectrum $H(G)$ for some abelian group G, and thus that each additive and ordinary cohomology theory coincides with singular cohomology $H^\bullet(-; G)$ with coefficients in G (see Example 2.1.20). This cohomology admits a cup product if and only if G has the structure of a commutative ring with unity.

In the previous example, we have seen a natural way to construct cohomology theories on topological spaces, as functors represented by spectra, and Brown's representability theorem shows that this is essentially the only way in which a reduced cohomology theory $\widetilde{H}^\bullet$ defined on topological spaces can arise. Let us give another possible way of constructing cohomology theories, which is perhaps more familiar, and more amenable to computations.

Example 2.1.14 (Complexes)**.** One way to construct a family of functors $H^\bullet\colon (\mathcal{V}^*)^{\mathrm{op}} \to \mathcal{A}^{\mathbb{Z}}$, where $\mathcal{V}^*$ is *any category* and $\mathcal{A}$ is an abelian category, is to construct a functor

$$\Gamma\colon (\mathcal{V}^*)^{\mathrm{op}} \to \mathrm{C}(\mathcal{A})$$

to the category of $\mathcal{A}$-valued cochain complexes, and then to define $H^\bullet := H^\bullet_{\mathcal{A}} \circ \Gamma$, where $H^\bullet_{\mathcal{A}}\colon \mathrm{C}(\mathcal{A}) \to \mathcal{A}^{\mathbb{Z}}$ denotes the usual cohomology of a cochain complex.

Suppose from now on that $\mathcal{V}^*$ is a sub-category of the category of morphisms $\mathrm{Arr}(\mathcal{V})$ on a given category $\mathcal{V}$. Then for every functor $\Gamma_{\mathcal{V}}\colon \mathcal{V}^{\mathrm{op}} \to \mathrm{C}(\mathcal{A})$, one can define a functor $\Gamma\colon (\mathcal{V}^*)^{\mathrm{op}} \to \mathrm{C}(\mathcal{A})$ by setting $\Gamma(f) := \mathrm{Cone}(\Gamma_V(f))[-1]$ (see [SP, Section 014D]). Using the properties of the cone, it is easy to obtain the natural transformation $\delta^\bullet$ and the long exact sequence (2.2) for the corresponding cohomology theory $H^\bullet\colon (\mathcal{V}^*)^{\mathrm{op}} \to \mathcal{A}^{\mathbb{Z}}$, if $\widetilde{\mathcal{V}}^*$ is defined simply to be the category of triples (X, U, V) where $U \to X$ and $V \to U$ are two objects of $\mathcal{V}^*$. Moreover, $H^\bullet$ satisfies the excision axiom (see Definition 2.1.4) with respect to all the morphisms $\varphi \in \mathrm{Arr}(\mathcal{V}^*)$ corresponding to a square

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \downarrow{\scriptstyle\beta} & \varphi & \downarrow{\scriptstyle\gamma} \\ Z & \xrightarrow[\delta]{} & W \end{array} \tag{2.5}$$

such that the associated square

$$\begin{array}{ccc} \Gamma_{\mathcal{V}}(W) & \xrightarrow{\Gamma_{\mathcal{V}}(\delta)} & \Gamma_{\mathcal{V}}(Z) \\ {\scriptstyle\Gamma_{\mathcal{V}}(\gamma)}\downarrow & \Gamma_{\mathcal{V}}(\varphi) & \downarrow{\scriptstyle\Gamma_{\mathcal{V}}(\beta)} \\ \Gamma_{\mathcal{V}}(Y) & \xrightarrow[\Gamma_{\mathcal{V}}(\alpha)]{} & \Gamma_{\mathcal{V}}(X) \end{array} \tag{2.6}$$

is Cartesian in $\mathrm{C}(\mathcal{A})$. If $\mathcal{A}$ is a tensor category then one can define a cup product (or equivalently a cross product) on $H^\bullet$ by defining a natural transformation $\Gamma \otimes \Gamma \to \Gamma$ which endows $\Gamma(X)$ with the structure of a commutative differential graded algebra (see [SP, Definition 061W]) for every $X \in \mathcal{V}$.

Remark 2.1.15. Since we are only interested in the cohomology of chain complexes, one would like to replace every occurrence of $\mathrm{C}(\mathcal{A})$ appearing in Example 2.1.14 with the homotopy category $h\,\mathrm{C}(\mathcal{A})$. The problem is that, doing this, one loses functoriality of cones (compare with [SP, Lemma 014F]).

Remark 2.1.16. Another approach to define a cohomology theory on $\mathcal{V}$, which is similar to Example 2.1.14 but is more challenging for computational purposes, is to define a complex of sheaves $\Gamma^\bullet_{\mathcal{V}} \in \mathrm{C}(\mathrm{Shv}(\mathcal{V}; \mathcal{A}))$ with respect to some Grothendieck topology defined on $\mathcal{V}$, and then to set $H^\bullet(f) := \mathbb{H}^\bullet(\mathrm{Cone}(\Gamma^\bullet_{\mathcal{V}}(f))[-1])$, where $\mathbb{H}^\bullet$ denotes hyper-cohomology (see [HM17, § 1.4]). Employing this approach usually allows one to specify the functor $\Gamma_{\mathcal{V}}$ only on a sub-category (*e.g.* the category of smooth affine schemes), using the covering properties of the Grothendieck topology. This is the approach adopted in [Gil81] (see also [BKK07, § 1.5]) and [CD12].

Let us conclude with the last approach to define a cohomology theory that we would like to mention, which is intimately related to the theory of motives.

Example 2.1.17 (Extensions)**.** One can construct a cohomology theory $H^\bullet\colon (\mathcal{V}^*)^{\mathrm{op}} \to \mathcal{A}^{\mathbb{Z}}$ with values in an abelian category $\mathcal{A}$, starting from an object $A \in \mathcal{A}$ (to be thought of as the coefficient object of our cohomology theory) and a covariant functor $R\colon \mathcal{V}^* \to \mathcal{A}$, by setting $H^\bullet := \mathrm{Ext}^\bullet(-, A) \circ R^{\mathrm{op}}$, where $\mathrm{Ext}^\bullet(-, A)\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{A}^{\mathbb{Z}}$ denote the usual Ext-groups (see [SP, Section 06XP]). Then the various properties characterising a cohomology theory (homotopy invariance, excision, long exact sequences etc.) can all be restated in terms of properties of the functor R.

Remark 2.1.18. Another way to define a cohomology theory is to define a functor $R\colon \mathcal{V}^* \to \mathcal{T}$ with values in a tensor triangulated category $\mathcal{T}$ (see [MVW06, Appendix 8A]), and then to define $H^n(-;A) := \mathrm{Hom}_{\mathcal{T}}(-, A[n]) \circ R^{\mathrm{op}}$, for any $n \in \mathbb{Z}$. This would in general only define a functor $H^n\colon (\mathcal{V}^*)^{\mathrm{op}} \to \mathrm{Sets}$, but usually one is able to enrich this to an $\mathcal{A}$-valued functor for some abelian category $\mathcal{A}$. A typical example is $\mathcal{A} = \mathcal{T}^\heartsuit$, where $\mathcal{T}^\heartsuit$ denotes the heart of $\mathcal{T}$ with respect to a t-structure defined on $\mathcal{T}$. One can take $\mathcal{T} := D(\mathcal{A})$ to be the derived category of some abelian category $\mathcal{A}$, showing that this approach generalises Example 2.1.17.

Let us conclude by observing that the three constructions given in this section can be understood as examples of the following general construction.

Example 2.1.19 (Stable cohomology)**.** To define a cohomology theory on an ∞-category $\mathcal{V}^*$ endowed with a Grothendieck topology (in the sense of [Lur09, Definition 6.2.2.1]) one can define a functor $R\colon \mathcal{V}^* \to \mathbf{H}$, where $\mathbf{H}$ is an ∞-category, and for every coefficient object $A \in \mathrm{Sp}(\mathbf{H})$ lying in the category of spectrum objects in $\mathbf{H}$ (see [Lur17, § 1.4.2]), one can define the cohomology theory $H^\bullet(-,E) := \pi_0(\mathbf{H}(-;\Omega^\infty(\Sigma^n(E)))) \circ R^{\mathrm{op}}$ given by the groups of connected components of the spaces of maps $R(X) \to \Omega^\infty(\Sigma^n(E))$, where $\Omega^\infty\colon \mathrm{Sp}(\mathbf{H}) \to \mathbf{H}$ is the functor defined in [Lur17, Notation 1.4.2.20] and Σ^n denotes the n-fold iterate of the suspension functor $\Sigma\colon \mathrm{Sp}(\mathbf{H}) \to \mathrm{Sp}(\mathbf{H})$. This makes sense for every $n \in \mathbb{Z}$ because $\mathrm{Sp}(\mathbf{H})$ is stable (see [Lur17, Corollary 1.4.2.17]), and thus Σ is an equivalence (see [Lur17, Page 23]).

2.1.3 Examples of cohomology theories

We devote this section to a brief roundup of examples of cohomology theories. All the constructions that we mention are examples of the general procedures described in Section 2.1.2. Let us start with three cohomology theories coming from the Archimedean world, which are deeply interrelated.

Example 2.1.20 (Singular cohomology)**.** Fix a topological space X endowed with a subspace $A \subseteq X$. The singular cohomology $H^\bullet_{\mathrm{sing}}(X, A; R)$ with coefficients in a ring R can be defined in the following ways:

- as the additive and ordinary cohomology theory induced by the spectrum $H(R)$, as mentioned in Example 2.1.13;
- as the cohomology of the singular cochain complex

$$C^\bullet(X, A; R) := \mathrm{Hom}_R(C_\bullet(X;R)/C_\bullet(A;R), R)$$

  where $C_n(X;R)$ is the free R-module generated by continuous maps $\sigma\colon \Delta_n \to X$, and $C_n(A;R)$ is defined analogously. Here $\Delta_n \subseteq (\mathbb{R}_{\geq 1})^{n+1}$ denotes the standard simplex, defined by the equation $\sum_{i=0}^n t_i = 1$;
- as the cohomology of the direct image with compact supports $j_!(R_{X\setminus A})$ (see [Ive86, Chapter VII, Definition 1.1]), where $j\colon X \setminus A \hookrightarrow X$ is the complementary inclusion to $A \subseteq X$ and $R_{X\setminus A} \in \mathrm{Shv}(X \setminus A; R)$ denotes the constant sheaf associated to R. The usual properties of sheaf cohomology imply that

$$H^n_{\mathrm{sing}}(X, A; R) \cong \mathrm{Ext}^n_{\mathrm{Shv}(X;R)}(R_X, j_!(R_{X\setminus A})) \cong \mathrm{Hom}_{D(\mathrm{Shv}(X;R))}(R_X, j_!(R_{X\setminus A})[n])$$

  which shows that singular cohomology is an example of the construction given in Example 2.1.14 and Remark 2.1.18.

These definitions agree on suitable sub-categories of topological spaces: for example the second and the third definitions are known to agree on topological spaces which are semi-locally contractible, as explained in [Sel16]. Moreover, if (X, A) is the geometric realisation of a pair (X_*, A_*) of simplicial complexes, singular cohomology coincides with the cohomology of the cochain complex

$$C^\bullet(X_*, A_*; R) := \mathrm{Hom}_R(C_\bullet(X_*; R)/C_\bullet(A_*; R), R)$$

where $C_\bullet(X_*; R)$ is the free R-module on the set of n-simplices in X_*, and $C_\bullet(A_*; R)$ is defined analogously.

Example 2.1.21 (de Rham cohomology)**.** The de Rham cohomology theory $H^\bullet_{\mathrm{dr}}(-)$ can be defined for differentiable manifolds, for complex manifolds or for schemes as the hypercohomology of the complex of C^∞, holomorphic or algebraic differentials $\Omega^\bullet$. In order to deal with singular schemes, one of the best choices available is to use the h-topology defined by Voevodsky (see [HM17, Definition 3.2.2]). More precisely, for every morphism of schemes $f\colon A \to X$ one can define (see [HM17, § 3.2]) the relative cohomology group

$$H^\bullet_{\mathrm{dR}}(X, A) := \mathbb{H}^\bullet((\mathrm{Sch}_{/X})_h, \ker(\Omega^\bullet_{h/X} \to f_*(\Omega^\bullet_{h/A}))) \tag{2.7}$$

where $(\mathrm{Sch}_{/X})_h$ denotes the site of schemes of finite type over X, endowed with the h-topology, and $\Omega^\bullet_{h/X}$ denotes the h-sheafification of $\Omega^\bullet_{/X}$, which is usually not a sheaf in the h-topology.

In the case of a differentiable or holomorphic manifold M, the Poincaré lemma (see for example [HM17, Proposition 4.1.3] for the holomorphic case) gives an isomorphism between de Rham and singular cohomology. One can combine this with GAGA theorems to get the period isomorphism

$$\mathrm{per}\colon H^\bullet_{\mathrm{dR}}(X, A) \otimes_\kappa \mathbb{C} \xrightarrow{\sim} H^\bullet_{\mathrm{sing}}(X(\mathbb{C}), A(\mathbb{C}); \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \tag{2.8}$$

defined in [HM17, Definition 5.4.1], where κ is a sub-field of $\mathbb{C}$, $X(\mathbb{C})$ denotes the complex analytification of X, and $A(\mathbb{C})$ is defined analogously.

Example 2.1.22 (Deligne-Beilinson cohomology)**.** Deligne-Beilinson cohomology can be seen as a way to interpolate de Rham and singular cohomology. First of all, let us observe that we can make these two cohomology theories into bi-graded cohomology theories by setting $H^{i,j}_{\mathrm{sing}}(X; \Lambda) := H^i(X; \Lambda(j))$ and $H^{i,j}_{\mathrm{dR}}(X) := F^j(H^i_{\mathrm{dR}}(X))$. Here $\Lambda \subseteq \mathbb{R}$ is any subring, $\Lambda(j) \subseteq \mathbb{C}$ denotes the subgroup $\Lambda(j) := (2\pi\sqrt{-1})^j \cdot \Lambda$ and $F^\bullet$ denotes the Hodge filtration on de Rham cohomology, which comes from the "stupid" filtration (or "filtration bête") obtained by truncating the complex of differentials $\Omega^\bullet$ (see for instance [Del71, § 1.4.7]). Then the Deligne-Beilinson cohomology groups $H^{\bullet,\bullet}_{\mathcal{D}}(X; \Lambda)$ fit into a long exact sequence

$$\cdots \to H^{i,j}_{\mathcal{D}}(X; \Lambda) \to H^{i,j}_{\mathrm{sing}}(X; \Lambda) \to H^i_{\mathrm{dR}}(X)/H^{i,j}_{\mathrm{dR}}(X) \to H^{i+1,j}_{\mathcal{D}}(X; \Lambda) \to \dots$$

which shows how Deligne-Beilinson cohomology interpolates between singular and de Rham cohomology.

As with almost any other cohomology theory, there is a plethora of ways in which Deligne-Beilinson cohomology can be defined. First of all, let X be a smooth variety over $\mathbb{C}$, and let $j\colon X \hookrightarrow \overline{X}$ be a good compactification, by which we mean an open embedding j into a smooth, proper variety $\overline{X} \to \mathrm{Spec}(\mathbb{C})$, such that the complement $D := \overline{X} \setminus j(X)$ is a divisor with normal crossings. Such a compactification always exists, which can be seen, as it is done in [BZ20, § A.3], by combining Nagata's compactification theorem (see [Con07]) together with Hironaka's embedded resolution of singularities (see [BM97, Theorem 1.6] and [BEV05, Theorem 2.4]).

Then one defines a complex $\Omega^\bullet_{\overline{X}}(\log(D))$ of holomorphic forms with logarithmic singularities along D, which is the sub-algebra of the $\mathcal{O}_{\overline{X}}$-algebra $j_*(\Omega^\bullet_X)$ generated by $\Omega^\bullet_{\overline{X}}$ and by the forms $d(f)/f \in j_*(\Omega^1_X)$, where f runs over the local equations of the irreducible components of the normal crossings divisor D. Now, the natural inclusion of the constant sheaf $\Lambda(m) \hookrightarrow \mathcal{O}_X$ induces a map $u_1 \colon Rj_*(\Lambda(m)) \to Rj_*(\Omega^\bullet_X) = j_*(\Omega^\bullet_X)$ in the derived category of quasi-coherent $\mathcal{O}_{\overline{X}}$-modules, where the last equality holds because j is affine (see [SP, Section 0AVV]). Moreover, we denote by $u_2 \colon \Omega^\bullet_{\overline{X}}(\log(D)) \to j_*(\Omega^\bullet_X)$ the natural inclusion. Using this notation, we define the Deligne-Beilinson complex of $(X, \overline{X})$ to be

$$\Lambda(m)_{\mathcal{D}} := \mathrm{Cone}\left(Rj_*(\Lambda(m)) \oplus \Omega^{\leq n}_{\overline{X}}(\log(D)) \xrightarrow{u_1 \oplus (-u_2)} j_*(\Omega^\bullet_X)\right)[-1]$$

where $\Omega^{\leq n}_{\overline{X}}(\log(D))$ denotes again the "stupid" filtration obtained by truncation. The Deligne-Beilinson cohomology of X is then defined as the hypercohomology $H^{i,j}_{\mathcal{D}}(X;\Lambda) := \mathbb{H}^i(\overline{X}; \Lambda(j)_{\mathcal{D}})$. Of course, for this definition to make sense, one needs to show that it does not depend on the good compactification that we have chosen. This is done in [EV88, Lemma 2.8] as follows: first of all, one uses the fact that every two good compactifications $j \colon X \hookrightarrow \overline{X}$ and $j' \colon X \hookrightarrow \overline{X}'$ are linked by a morphism $\tau \colon \overline{X} \to \overline{X}'$ such that $j' = f \circ j$, and then one uses the distinguished triangle

$$\mathbb{H}^n(\overline{X}; \Lambda(m)_{\mathcal{D}}) \to \mathbb{H}^n(\overline{X}; Rj_*(\Lambda(m))) \oplus \mathbb{H}^n(\overline{X}; F^n(\Omega^\bullet_{\overline{X}}(\log(D)))) \to \mathbb{H}^n(\overline{X}; j_*(\Omega^\bullet_X))$$

coming from the definition of $\Lambda(m)_{\mathcal{D}}$, to show that $H^{n,m}_{\mathcal{D}}(X;\Lambda)$ does not depend on the chosen good compactification, because the other two factors do not.

This definition shows that the Deligne-Beilinson cohomology groups carry two pieces of information: the Λ-structure on singular cohomology and the Hodge filtration $F^\bullet$ on de Rham cohomology. Hence for a general ring $\Lambda \subseteq \mathbb{R}$, the definition of Deligne-Beilinson cohomology cannot be substantially simplified, and one has to deal with the hypercohomology of a sheaf. However, if $\Lambda = \mathbb{R}$ there exists a complex of $\mathbb{R}$-vector spaces $\mathcal{D}^\bullet_{\log}(X, j)$ such that $H^{i,j}_{\mathcal{D}}(X;\mathbb{R}) \cong H^i(\mathcal{D}^\bullet_{\log}(X, j))$. This complex was defined by Burgos Gil in [Bur94] (see also [Bur97]): in particular, its definition uses the fact that the category of good compactifications of a variety X is directed (see [Del71, § 3.2.11]) to get rid of the indeterminacy concerning the choice of good compactification at the level of complexes, by taking a direct limit over all of them. Since we do not need this complex in this thesis, we do not give the precise definition. Let us only remark that the association $X \mapsto \mathcal{D}^\bullet_{\log}(X, j)$ gives rise to a Nisnevich sheaf on the site $\mathrm{Sm}_{/\mathbb{C}}$ of smooth complex varieties, which is also $\mathbb{A}^1$-invariant.

Let us mention that Deligne-Beilinson cohomology (with general coefficients $\Lambda \subseteq \mathbb{R}$) can be computed using the formalism outlined in Example 2.1.17, because it can be computed as an extension in the category MHS_Λ of mixed Hodge structures over Λ (see [Bei86b] and [Bur13]). Moreover, Deligne-Beilinson cohomology can be defined for varieties defined over $\mathbb{R}$, using a combination of the action of complex conjugation on the complex points of the variety and on the coefficients (see [EV88, § 2.1] and [BKK07, § 5.7], as well as Section 2.5 for the case of curves). Finally, each of these constructions admits a generalisation to the relative setting (see [EV88, § 4] and [BF12, Definition 1.28]), and to singular varieties by means of simplicial resolutions (see [Bei86b, § 4.1] and [HM17, § 3.3.1]). This generalisation, together with the definition of

Deligne-Beilinson homology (see [Jan88b]), allows one to see that Deligne-Beilinson cohomology is part of a twisted Poincaré duality theory in the sense of Definition 2.1.10.

We continue by mentioning another fundamental example, which is needed in the construction of L-functions.

Example 2.1.23 (ℓ-adic cohomology)**.** Let X be a scheme of finite type over a field κ, and fix an algebraic closure $\overline{\kappa} \supseteq \kappa$, and a rational prime $\ell \in \mathbb{N}$. Then the ℓ-adic cohomology groups of X are defined to be

$$H_\ell^{i,j}(X) := \left(\varprojlim_n H^i((X_{\overline{\kappa}})_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell(j) \tag{2.9}$$

which gives rise to a $\mathbb{Q}_\ell$-linear representation of the absolute Galois group $\mathrm{Gal}(\overline{\kappa}/\kappa)$ endowed with a $\mathbb{Z}_\ell$-linear stable lattice. Here $\mathbb{Q}_\ell(j)$ denotes the vector space $\mathbb{Q}_\ell$ endowed with the action of $\mathrm{Gal}(\overline{\kappa}/\kappa)$ given by the j-th power of the cyclotomic character. One would be tempted to study analogously the groups

$$H^\bullet(Y_{\text{ét}}, \mathbb{Z}_\ell) \to \varprojlim_n H^\bullet(Y_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})$$

associated to a general scheme Y. The first ones are in general very ill-behaved (see [FK88, Chapter I, § 12]), which leads to the definition given in (2.9), whereas the second ones are ill-behaved as soon as the cohomology groups $H^\bullet(Y_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})$ are not finite. This second problem can be overcome using Jannsen's continuous étale cohomology groups (see [Jan88a]), and both these problems have been resolved by Bhatt and Scholze by changing the site $X_{\text{ét}}$ to a bigger site $X_{\text{pro-ét}}$ (see [BS15]). Their construction generalises also work of Ekedahl (see [Eke07]), and allows one to see the construction of continuous ℓ-adic cohomology as an example of the construction outlined in Remark 2.1.18. One can then define a relative version of étale cohomology (or, equivalently, a version of étale cohomology with supports) using either the recipe explained in [SP, Section 09XP] or a definition similar to (2.7).

Remark 2.1.24. There are many more cohomology theories relevant for the non-Archimedean world, among which we mention:

- the *filtered Ogus cohomology* of Chiarellotto, Lazda and Mazzari (see [CLM19]), constructed using crystalline cohomology (compare with [AB05] and [ABB17]);
- the *syntomic cohomology* of Besser (see [Bes00]) , which can be constructed using the formalism of Example 2.1.17 (see [Ban02]) or of Remark 2.1.18 (see [CCM13]) in favorable cases. This cohomology theory has been extended beyond the smooth case by Nekovář and Nizioł ([NN16]), generalising work of Kato (see [Kat94]). Even this new cohomology theory fits in the picture outlined in Example 2.1.17 and Remark 2.1.18, as explained in [DM15], [DN18] and [Niz19];
- the *prismatic cohomology* of Bhatt and Scholze (see [BS19]), which is related to many of the cohomology theories mentioned above by comparison isomorphisms. This cohomology theory is still not proved to be an example of the constructions outlined in Example 2.1.17 and Remark 2.1.18. See nevertheless [Dri20] for a construction of the category which should play the role of the category of coefficients for prismatic cohomology.

2.2 Various categories of motives

The aim of this section is to give a brief review of the various approaches to construct a category of pure and mixed motives. First of all, we recall the notion of pure motives over a

field κ, which goes back to Grohtendieck, following [And04, Chapitre 4]. Then we present the conjectural framework for the category of mixed motives, following Levine's survey [Lev05]. Finally, we present the examples of constructions of candidates for the category of mixed motives, and of its triangulated counterpart.

2.2.1 Pure motives

Let κ be a field, and let $\mathcal{P}(\kappa)$ denote the category of smooth and projective schemes (in the sense of [SP, Definition 01W8]) defined over κ. In particular such schemes are quasi-projective, hence of finite type, over κ. For every commutative ring with unity R, we denote by

$$\mathcal{Z}^\bullet(-)_R \colon \mathcal{P}(\kappa)^{\mathrm{op}} \to \mathrm{Mod}_R$$

the contravariant functor sending a smooth and projective scheme to the R-module of algebraic cycles on X, which can be equivalently described as the free R-module generated by closed, integral subschemes $Z \hookrightarrow X$, or as the free R-module generated by the points $\xi \in X$ (via the correspondence sending a closed, integral sub-scheme to its generic point, and a point to its closure, considered with the reduced sub-scheme structure). This module $\mathcal{Z}^\bullet(X)_R$ admits a natural grading, which can be described as the grading by co-dimension of closed, integral sub-schemes or as the grading by the Krull dimension $\dim(\mathcal{O}_{X,\xi})$ of the stalks of the structure sheaf (see [SP, Lemma 02IZ]). However, the maps $\mathcal{Z}^\bullet(f)_R \colon \mathcal{Z}^\bullet(Y)_R \to \mathcal{Z}^\bullet(X)_R$ associated to a morphism $f \colon X \to Y$, which are induced by the association $Z \mapsto f^{-1}(Z)$ for every $Z \subseteq Y$, do not respect the grading. On the R-module $\mathcal{Z}^\bullet(X)_R$ there is also an R-bilinear *intersection product*, which is defined only on the sub-set of $\mathcal{Z}^\bullet(X)_R \times \mathcal{Z}^\bullet(X)_R$ consisting of pairs of cycles (α, β) which intersect properly, which means that $\mathrm{codim}_X(T) \geq \mathrm{codim}_X(Z) + \mathrm{codim}_X(W)$ for every irreducible component $T \subseteq Z \cap W$ inside each intersection of two closed and integral subschemes $Z, W \subseteq X$ such that $\alpha_Z \neq 0$ and $\beta_W \neq 0$, where $\alpha_Z, \beta_W \in R$ are the multiplicities with which Z and W appear in the cycles $\alpha = \sum_{Z' \subseteq X} \alpha_{Z'}[Z']$ and $\beta_{W' \subseteq X} = \sum \beta_{W'}[W']$. In order to extend this partially defined product, which we denote by $\alpha \cdot \beta$, to the whole group $\mathcal{Z}^\bullet(X)$ we would like to be able to "move" any pair of cycles (α, β) so that it becomes a pair intersecting properly. The types of movement that we allow are captured by the following definition.

Definition 2.2.1 – Adequate equivalence relation (see [And04, Definition 3.1.1.1])

Let κ be a field, and R be a commutative ring with unity. Then an R-linear adequate equivalence relation for κ is given by a family of equivalence relations $\sim$ on the R-modules $\mathcal{Z}^\bullet(X)_R$ for every $X \in \mathcal{P}(\kappa)$, such that:

- $\sim$ is R-linear and respects the grading;
- for every $X \in \mathcal{P}(\kappa)$ and every pair of cycles $\alpha, \beta \in \mathcal{Z}^\bullet(X)_R$, there exists another cycle $\alpha' \in \mathcal{Z}^\bullet(X)_R$ such that $\alpha' \sim \alpha$ and α' intersects β properly;
- for every $X, Y \in \mathcal{P}(\kappa)$ and every pair of cycles $\alpha \in \mathcal{Z}^\bullet(X)_R$ and $\beta \in \mathcal{Z}^\bullet(X \times Y)_R$ such that β intersects properly $\pi_X^{-1}(\alpha)$, one has that $\pi_Y(\beta \cdot \pi_X^{-1}(\alpha)) \sim 0$ inside $\mathcal{Z}^\bullet(Y)_R$. Here $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$ denote the corresponding projection maps.

This shows that for every R-linear adequate equivalence relation $\sim$ one gets a functor $\mathcal{Z}^\bullet_\sim(-)_R \colon \mathcal{P}(\kappa)^{\mathrm{op}} \to \mathrm{Alg}((\mathrm{Mod}_R)^{\mathbb{Z}})$ sending a scheme $X \in \mathcal{P}(\kappa)$ to the graded R-module

$\mathcal{Z}^\bullet_\sim(X)_R := \mathcal{Z}^\bullet(X)_R/\sim$, endowed with the structure of a graded algebra given by the intersection product. This allows one to extend $\mathcal{Z}^\bullet_\sim(-)_R$ to a functor by setting $\mathcal{Z}^\bullet_\sim(f)_R(\alpha) := \pi_X(\Gamma_f^t \cdot \pi_Y^{-1}(\alpha))$ for every $f\colon X \to Y$ and $\alpha \in \mathcal{Z}^\bullet_\sim(Y)_R$, where $\Gamma_f^t := \{(f(x), x)\colon x \in X\} \subseteq Y \times X$. Usually one writes $f^* := \mathcal{Z}^\bullet_\sim(f)_R$. Using the graph of f instead of its transpose Γ_f^t, one can show that there exists a graded map $f_*\colon \mathcal{Z}^\bullet_\sim(X)_R \to \mathcal{Z}^{\dim(f)+\bullet}_\sim(Y)_R$, where $\dim(f)$ denotes the generic dimension of the fibres of f.

This way of defining the morphism $\mathcal{Z}^\bullet_\sim(f)_R$ can be generalised to any *correspondence* between X and Y, not necessarily given by a map $f\colon X \to Y$ (see [MVW06, Lecture 1]). More precisely, for every $X, Y \in \mathcal{P}(\kappa)$ one defines the graded R-module of correspondences from X to Y as

$$\mathrm{Cor}^\bullet_\sim(X, Y)_R := \bigoplus_{U \subseteq X} \mathcal{Z}^{\dim(U)+\bullet}_\sim(U \times Y)_R$$

where $U \subseteq X$ ranges over the connected components of X. Then for every $X, Y, Z \in \mathcal{P}(\kappa)$ one has a composition law

$$\begin{aligned} \mathrm{Cor}^\bullet_\sim(X, Y)_R \otimes_R \mathrm{Cor}^\bullet_\sim(Y, Z)_R &\to \mathrm{Cor}^\bullet_\sim(X, Z)_R \\ (\alpha, \beta) &\mapsto \beta \circ \alpha := (\pi_{X,Z})_*(\pi^*_{X,Y}(\alpha) \cdot \pi^*_{Y,Z}(\beta)) \end{aligned} \tag{2.10}$$

where $\pi_{X,Z}\colon X \times Y \times Z \to X \times Z$, $\pi_{X,Y}\colon X \times Y \times Z \to X \times Y$ and $\pi_{Y,Z}\colon X \times Y \times Z \to Y \times Z$ are the projection maps. In particular, for every map $f\colon X \to Y$ one has a correspondence $\Gamma_f \in \mathrm{Cor}^\bullet_\sim(X, Y)_R$ obtained by summing over all the graphs of the restrictions of f to the connected components of X, and one has that $\Gamma_g \circ \Gamma_f = \Gamma_{g\circ f}$ for every $f\colon X \to Y$ and $g\colon Y \to Z$. Moreover, the composition law makes $\mathrm{Cor}^\bullet_\sim(X, X)_R$ into a graded R-algebra, which in general is not commutative, whose unit is given by the diagonal $\Delta_X \subseteq X \times X$. Moreover, this algebra is endowed with the involution τ^*, where $\tau\colon X \times X \to X \times X$ is the map $\tau(x_1, x_2) := (x_2, x_1)$.

We are now ready to define the category of *pure motives*.

Definition 2.2.2 – Pure motives (see [And04, § 4.1.3])

Let κ be a field, and R be a commutative ring with unit. Then a *pure motive* for the equivalence $\sim$, defined over κ with coefficients in R, is defined to be a triple (X, e, r) where $X \in \mathcal{P}(\kappa)$, $e \in \mathrm{Cor}^0_\sim(X, X)_R$ is an idempotent (*i.e.* $e \circ e = e$) and $r \in \mathbb{Z}$ is an integer. The category of pure motives $\mathcal{M}_\sim(\kappa; R)$ is then defined by setting

$$\mathrm{Hom}_{\mathcal{M}_\sim(\kappa;R)}((X, e, r), (X', e', r')) := e' \circ \mathrm{Cor}^{r'-r}_\sim(X, X')_R \circ e \subseteq \mathrm{Cor}^\bullet(X, X')$$

with the composition law induced by (2.10).

We denote by $\mathfrak{h}_\sim(-; R)\colon \mathcal{P}(\kappa)^{\mathrm{op}} \to \mathcal{M}_\sim(\kappa; R)$ the functor defined as $\mathfrak{h}_\sim(X; R) := (X, \Delta_X, 0)$ on objects, and as

$$\mathfrak{h}_\sim(f; R) := \Gamma_f^t \in \mathrm{Cor}^0_\sim(Y, X) = \mathrm{Hom}_{\mathcal{M}_\sim(\kappa;R)}(\mathfrak{h}_\sim(Y; R), \mathfrak{h}_\sim(X; R))$$

on each morphism $f\colon X \to Y$.

The category $\mathcal{M}_\sim(\kappa; R)$ is endowed with the tensor product given by

$$(X, e, r) \otimes (X', e', r') := (X \times X', e \otimes e', r + r')$$

on objects, and by $f \otimes f' := (e_2 \otimes e_2') \circ (\varphi \otimes \varphi') \circ (e_1 \otimes e_1')$ on morphisms $f\colon (X_1, e_1, r_1) \to (X_2, e_2, r_2)$ and $f'\colon (X_1', e_1', r_1') \to (X_2', e_2', r_2')$, written as $f = e_2 \circ \varphi \circ e_1$ and $f = e_2' \circ \varphi' \circ e_1'$. Here we have used the tensor product of correspondences

$$\otimes\colon \mathrm{Cor}_\sim^\bullet(X, Y)_R \otimes_R \mathrm{Cor}_\sim^\bullet(X', Y')_R \to \mathrm{Cor}_\sim^\bullet(X \times X', Y \times Y')_R$$

which is induced by the product of closed sub-schemes. The identity for this tensor product is the pure motive $\mathbb{1}_{\sim,R} := \mathfrak{h}_\sim(\mathrm{Spec}(\kappa); R)$.

The category $\mathcal{M}_\sim(\kappa; R)$ is also endowed with a direct sum, which is defined as follows. First of all, for every pure motive $M = (X, e, r) \in \mathcal{M}_\sim(\kappa; R)$ and every $n \in \mathbb{Z}$ we write $M(n) := (X, e, r+n)$. Then one can check that $\mathbb{1}_{\sim,R}(-1) \cong (\mathbb{P}^1_\kappa, \mathbb{P}^1_\kappa \times \{x\}, 0)$ for every rational point $x \in \mathbb{P}^1(\kappa)$. Hence for every pure motive $M = (X, e, r) \in \mathcal{M}_\sim(\kappa; R)$ and every $r' \geq r$ one can write

$$M \cong M(r' - r) \otimes \mathbb{1}_{\sim,R}(-1)^{\otimes r'-r} \cong (X \times (\mathbb{P}^1)^{r'-r}, p \otimes e, r')$$

for some idempotent $p \in \mathrm{Cor}_\sim^0((\mathbb{P}^1)^{r'-r}, (\mathbb{P}^1)^{r'-r})$. Then for every pair of motives $M, M' \in \mathcal{M}_\sim(\kappa; R)$ given by $M = (X, e, r)$ and $M' = (X', e', r')$ one defines

$$M \oplus M' \cong (X \times (\mathbb{P}^1)^{r'-r}, p \otimes e, r') \oplus (X', e', r') := \Big((X \times (\mathbb{P}^1)^{r'-r}) \sqcup X', (p \otimes e) \oplus e', r'\Big)$$

if $r' \geq r$, and $(X, e, r) \oplus (X', e', r') := (X', e', r') \oplus (X, e, r)$ otherwise. Here the direct sum of correspondences

$$\oplus\colon \mathrm{Cor}_\sim^\bullet(X, Y) \oplus_R \mathrm{Cor}_\sim^\bullet(X', Y') \to \mathrm{Cor}_\sim^\bullet(X \sqcup X', Y \sqcup Y')$$

is induced from the disjoint union of closed sub-schemes. Using the direct sum, we can define the *dual* of a pure motive $M = (X, e, r) \in \mathcal{M}_\sim(X; R)$ as

$$M^\vee := \bigoplus_{U \subseteq X} (U, \tau^*(e), \dim(U) - r)$$

where the direct sum runs over all the irreducible components $U \subseteq X$, and $\tau\colon X \times X \to X \times X$ denotes, as before, the transposition map $\tau(x_1, x_2) := (x_2, x_1)$. This shows in particular that $\mathfrak{h}_\sim(X; R)^\vee \cong \mathfrak{h}_\sim(X; R)(d)$ if $X \in \mathcal{P}(\kappa)$ is equidimensional of dimension d, which gives rise to a map $\mathrm{Tr}_X^{\sim,R}\colon \mathfrak{h}_\sim(X; R) \to \mathbb{1}_{\sim,R}(-d)$.

The cateogry $\mathcal{M}_\sim(\kappa; R)$ endowed with the operations $\otimes$ and $\oplus$ is thus an R-linear, additive, rigid symmetric tensor category which is pseudo-abelian, *i.e.* such that every $f\colon M \to M$ with $f \circ f = f$ admits a kernel (see [And04, p. 1.1.3.1] for an equivalent definition).

Remark 2.2.3. Typical examples of adequate equivalence relations $\sim$ are given by:

- the *rational equivalence* relation $\sim_{\mathrm{rat}}$, such that $\mathcal{Z}^\bullet_{\mathrm{rat}}(X)_R$ is the quotient of $\mathcal{Z}^\bullet(X)_R$ by the sub-module generated by cycles of the form $[Z \cap (X \times \{0\})] - [Z \cap (X \times \{\infty\})]$, for $Z \subseteq X \times \mathbb{P}^1$ a closed integral sub-scheme which dominates $\mathbb{P}^1$. This is the finest of adequate equivalence relations (see [And04, Lemme 3.2.2.1]), and the corresponding category of motives $\mathrm{CHM}(\kappa; R)$ is called the category of *Chow motives*, because the groups $\mathrm{CH}^\bullet(X; R) := \mathcal{Z}^\bullet_{\mathrm{rat}}(X)_R$ are called Chow groups;
- the *algebraic equivalence* relation $\sim_{\mathrm{alg}}$, such that $\mathcal{Z}^\bullet_{\mathrm{alg}}(X)_R$ is the quotient of $\mathcal{Z}^\bullet(X)_R$ by the sub-module generated by cycles of the form $[Z \cap (X \times \{y_0\})] - [Z \cap (X \times \{y_\infty\})]$, for $Z \subseteq X \times Y$ a closed integral sub-scheme which dominates a connected scheme $Y \in \mathcal{P}(\kappa)$

with two rational points $y_0, y_\infty \in Y(\kappa)$. Note that Y is allowed to vary, but we can assume (using theorems of Bertini type) that Y is a smooth, projective, connected curve;

- the *$\otimes$-nilpotence relation* $\sim_{\otimes\text{-nil}}$, which says that $\alpha \sim_{\otimes\text{-nil}} 0$ for a cycle $\alpha \in \mathcal{Z}^\bullet(X)_R$ if and only if there exists $N \in \mathbb{Z}_{\geq 1}$ such that $\alpha^{\otimes N} \in \mathcal{Z}^\bullet(X^N)_R$ is rationally equivalent to zero. This equivalence relation is coarser than $\sim_{\text{alg}}$ if R is a $\mathbb{Q}$-algebra (see [And04, Proposition 3.2.4.1]);
- the *numerical equivalence* relation $\sim_{\text{num}}$, defined by saying that if $X \in \mathcal{P}(\kappa)$ is irreducible of dimension d and $\alpha \in \mathcal{Z}^j(X)_R$ then $\alpha \sim_{\text{num}} 0$ if and only if $\deg([\alpha]_{\text{rat}} \cdot \beta) = 0$ for every $\beta \in \mathrm{CH}^{d-j}(X)_R$. Here $\deg\colon \mathrm{CH}^d(X)_R \to R$ is defined by setting

$$\deg\left(\sum_\xi n_\xi [\xi]\right) := \sum_\xi n_\xi \cdot [\kappa(\xi) : \kappa]$$

which makes sense because the points $\xi \in X$ are closed. If R is a field, this equivalence relation is the coarsest of all non-trivial adequate equivalence relations.

Let us now recall that the category of pure motives $\mathcal{M}_\sim(\kappa; R)$ has been constructed to be universal with respect to all the Weil cohomology theories, in the sense of the following definition.

Definition 2.2.4 – Weil cohomology (see [And04, § 4.2.4])

Let κ be a field and R be a commutative ring with unity. Fix an additive, R-linear, pseudo-abelian, rigid tensor category $\mathcal{T}$, with identity object $\mathbb{1}_\mathcal{T} \in \mathcal{T}$. Let also $\sim$ be an R-linear adequate congruence relation which is at least as fine as the numerical equivalence. Then a $\mathcal{T}$-valued Weil cohomology theory for $\sim$ is a triple $(H, \mathrm{tr}^H, \mathrm{c})$ such that:

- $H\colon \mathcal{P}(\kappa)^{\text{op}} \to \mathcal{T}$ is a $\otimes$-functor (in the sense of [DM82, Definition 1.8]), with respect to the tensor structure on $\mathcal{P}(\kappa)$ given by the product. In particular the diagonal map $\Delta_X\colon X \to X \times X$ induces a product structure $\smile\colon H(X)^{\otimes 2} \to H(X)$ for every $X \in \mathcal{P}(\kappa)$;
- the structural map $\mathbb{P}^1_\kappa \to \mathrm{Spec}(\kappa)$ and the rational point $\{\infty\}\colon \mathrm{Spec}(\kappa) \to \mathbb{P}^1_\kappa$ induce a decomposition $H^\bullet(\mathbb{P}^1_\kappa) \cong \mathbb{1}_\mathcal{T} \oplus \mathbb{L}$ for some $\otimes$-invertible object $\mathbb{L} \in \mathcal{T}$;
- tr^H is a family of morphisms $\mathrm{tr}^H_X\colon H(X) \to \mathbb{L}^{\oplus \dim(X)}$ defined for every equidimensional $X \in \mathcal{P}(X)$, such that $\mathrm{tr}_{X\times Y}$ is related to $\mathrm{tr}^H_X \otimes \mathrm{tr}_Y$ by the coherence maps pertinent to H. Moreover, we demand that the natural transformation

$$\begin{aligned} \mathrm{Hom}_\mathcal{T}(-, H(X) \otimes \mathbb{L}^{\otimes -d}) &\to \mathrm{Hom}_\mathcal{T}((-) \otimes H(X), \mathbb{1}_\mathcal{T}) \\ f &\mapsto \varphi_X \circ (f \otimes \mathrm{Id}_{H(X)}) \end{aligned}$$

is an isomorphism, for every equidimensional $X \in \mathcal{P}(\kappa)$. Here φ_X is the morphism

$$\varphi_X\colon (H(X) \otimes \mathbb{L}^{\otimes -d}) \otimes H(X) \xrightarrow{\sim} H(X)^{\otimes 2} \otimes \mathbb{L}^{\otimes -d} \xrightarrow{(\mathrm{tr}^H_X \circ \smile) \otimes \mathrm{Id}} \mathbb{L}^{\otimes d} \otimes \mathbb{L}^{\otimes -d} \xrightarrow{\sim} \mathbb{1}_\mathcal{T}$$

where $d := \dim(X)$. In particular, we have an isomorphism $H(X) \otimes \mathbb{L}^{\otimes -d} \cong H(X)^\vee$;

- c is a collection of natural transformations

$$c^r : \mathcal{Z}^r_\sim(-)_R \to \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}_{\mathcal{T}}, H(-) \otimes \mathbb{L}^{\otimes -r})$$

such that $c^r_{X\times Y}$ is identified with $\sum_{i+j=r} c^i_X \otimes c^j_Y$ by the various coherence maps. Moreover, we normalise c^r in such a way that the map

$$\begin{aligned} \mathcal{Z}^{\dim(X)}_\sim(X)_R &\to \mathrm{End}_{\mathcal{T}}(\mathbb{1}_{\mathcal{T}}) \\ \alpha &\mapsto \psi_X \circ c^{\dim(X)}_X(\alpha) \end{aligned}$$

coincides with the degree map $\deg\colon \mathcal{Z}^{\dim(X)}_\sim(X)_R \to R \xrightarrow{\sim} \mathrm{End}_{\mathcal{T}}(\mathbb{1}_{\mathcal{T}})$ for every equidimensional $X \in \mathcal{P}(\kappa)$. Here ψ_X is defined as

$$\psi_X : H(X) \otimes \mathbb{L}^{\otimes -d} \xrightarrow{\mathrm{tr}^H_X \otimes \mathrm{Id}} \mathbb{L}^{\otimes d} \otimes \mathbb{L}^{\otimes -d} \xrightarrow{\sim} \mathbb{1}_{\mathcal{T}}$$

where $d := \dim(X)$. Note also that the degree map is well defined because we demanded that $\sim_{\mathrm{num}}$ is coarser than $\sim$.

We omit the equivalence relation $\sim$ from the notation if $\sim{=}\sim_{\mathrm{rat}}$.

It is now clear from the definition and the construction of the category of pure motives $\mathcal{M}_\sim(\kappa;R)$ that every $\mathcal{T}$-valued Weil cohomology theory H for $\sim$ factors as

$$H : \mathcal{P}(\kappa)^{\mathrm{op}} \xrightarrow{\mathfrak{h}_\sim(-;R)} \mathcal{M}_\sim(\kappa;R) \xrightarrow{\omega_H} \mathcal{T}$$

in such a way that $\mathbb{L} = \omega_H(\mathbb{1}_{\sim,R}(-1))$. Moreover, $\mathrm{tr}^H_X \circ \omega_H = \omega_H \circ \mathrm{tr}^{\sim,R}_X$ and c^r is induced by ω_H, using the fact that $\mathcal{Z}^r_\sim(X)_R \cong \mathrm{Hom}_{\mathcal{M}_\sim(\kappa;R)}(\mathbb{1}_{\sim,R}, \mathfrak{h}_\sim(X;R)(r))$ for every $X \in \mathcal{P}(\kappa)$.

Remark 2.2.5. Every Weil cohomology theory H (with respect to $\sim_{\mathrm{rat}}$) gives rise to a new equivalence relation $\sim_H$, which is coarser than $\sim_{\mathrm{alg}}$ and $\sim_{\otimes\text{-nil}}$, and finer than $\sim_{\mathrm{num}}$. Moreover, one of Grothendieck's standard conjectures predicts that $\sim_{\mathrm{num}}{=}\sim_H$ for every Weil cohomology theory H (see [And04, § 5.4.1]). Voevodsky went further to conjecture a refinement of this, which says that $\sim_{\mathrm{num}}{=}\sim_{\otimes\text{-nil}}$ (see [And04, § 11.5.2]).

We conclude this brief review of pure motives by mentioning that if R is a field the category $\mathcal{M}_\sim(\kappa;R)$ is abelian and semi-simple if and only if $\sim{=}\sim_{\mathrm{num}}$, as proved by Jannsen (see [And04, § 4.5]). Moreover, the standard conjectures of Grothendieck (see [And04, Chapitre 5]) allow one to introduce a Tannakian formalism on this category (by modifying the coherence functor for the symmetry of the tensor product), which gives rise to motivic Galois groups (see [And04, Chapitre 6]) and realisation functors. The latter are conjectured to be full and to have semi-simple image (see [And04, Chapitre 7]).

2.2.2 Abelian categories of mixed motives

We have seen in the previous section that, under the standard conjecture $\sim_{\mathrm{num}}{=}\sim_H$, the functor $\mathfrak{h}_{\mathrm{num}}(-;R)\colon \mathcal{P}(\kappa)^{\mathrm{op}} \to \mathrm{NM}(\kappa;R)$ provides a universal Weil cohomology theory, with the virtue that $\mathrm{NM}(\kappa;R)$ is abelian and semi-simple. The aim of this section is to survey briefly the attempts that have been made to generalise this to mixed Weil cohomology theories (in the sense of Definition 2.1.4), which are defined also for varieties which are not smooth and projective.

The first attempt which can be made to generalise the construction of pure motives to varieties which are not necessarily smooth or projective is to use resolution of singularities, at least in characteristic zero. To do so, let us fix a field κ of characteristic zero. Then for every Cartesian square in $\mathcal{P}(\kappa)$

$$\begin{array}{ccc} E & \hookrightarrow & \widetilde{X} \\ \downarrow & \lrcorner & \downarrow \\ Z & \hookrightarrow & X \end{array} \tag{2.11}$$

where the horizontal maps are closed immersions, and every adequate equivalence relation $\sim$, we have that $[\mathfrak{h}_\sim(X;R)] - [\mathfrak{h}_\sim(Z;R)] = [\mathfrak{h}_\sim(\widetilde{X};R)] - [\mathfrak{h}_\sim(E;R)]$ inside the Grothendieck group $K_0(\mathcal{M}_\sim(\kappa;R))$. This group is defined as the quotient of the free abelian group generated by isomorphism classes $[M]$ of motives $M \in \mathcal{M}_\sim(\kappa;R)$, by the relation $[M \oplus M'] = [M] + [M']$. One can similarly define a group $K_0(\mathcal{S})$ for every category of schemes $\mathcal{S}$ over κ which is closed under disjoint unions. More precisely, this group $K_0(\mathcal{S})$ is given by the quotient of the free abelian group on the isomorphism classes $[X]$ of objects $X \in \mathcal{S}$, by the relation $[X \sqcup Y] = [X] + [Y]$. Hence the inclusion $\mathcal{P}(\kappa) \hookrightarrow \mathcal{V}(\kappa)$ induces a map $K_0(\mathcal{P}(\kappa)) \to K_0(\mathcal{V}(\kappa))$, where $\mathcal{P}(\kappa)$ is the category of smooth and projective schemes over $\mathrm{Spec}(\kappa)$, and $\mathcal{V}(\kappa)$ is the category of reduced and separated schemes of finite type over $\mathrm{Spec}(\kappa)$. Then Bittner's work [Bit04] uses resolution of singularities to show that the map $K_0(\mathcal{P}(\kappa)) \to K_0(\mathcal{V}(\kappa))$ is surjective, and its kernel is the subgroup generated by the elements $[X] - [Z] - ([\widetilde{X}] - [E])$ for every Cartesian square (2.11). Therefore what we have seen implies that the functor $\mathfrak{h}_\sim(-;R)\colon \mathcal{P}(\kappa)^{\mathrm{op}} \to \mathcal{M}_\sim(\kappa;R)$ induces a map of groups $K_0(\mathcal{V}(\kappa)) \to K_0(\mathcal{M}_\sim(\kappa;R))$, showing that we can interpret the Grothendieck group $K_0(\mathcal{M}_\sim(\kappa;R))$ as a first approximation of the category of mixed motives. The existence of this group homomorphism was also proved by Gillet and Soulé [GS96] and by Guillén and Navarro Aznar [GN02], who construct also a contravariant functor $\mathcal{V}(\kappa)^{\mathrm{op}} \to hC^+(\mathcal{M}_\sim(\kappa;R))$ to the homotopy category of the category of $\mathcal{M}_\sim(\kappa;R)$-valued complexes which are bounded below (see in particular [GN02, Théorème 5.10]).

The ideas outlined in the previous paragraph, and namely the usage of resolution of singularities, can also be employed to give a first definition of the abelian category of mixed motives in terms of realisations. This was done by Jannsen, for a field κ which is finitely generated over $\mathbb{Q}$. First of all, he defines a category of *mixed realisations* as follows.

Definition 2.2.6 – The category of mixed realisations (see [Jan90, § 2])

Let κ be a field finitely generated over $\mathbb{Q}$. Then the category of *mixed realisations* $\mathcal{MR}_\kappa$ consists of tuples $H = (H_{dR}, \{H_\ell\}_\ell, \{H_\sigma\}_\sigma, \{I_{\infty,\sigma}\}_\sigma, \{I_{\ell,\widetilde{\sigma}}\}_{\ell,\widetilde{\sigma}})$ such that:

- $\ell \in \mathbb{N}$ ranges over the rational primes;
- σ and $\widetilde{\sigma}$ range over the embeddings $\sigma\colon \kappa \hookrightarrow \mathbb{C}$ and $\widetilde{\sigma}\colon \overline{\kappa} \hookrightarrow \mathbb{C}$;
- $H_{dR} \in (\mathrm{Vec}_\kappa^{\mathrm{f.g.}})^{\mathrm{bi\text{-}fil}}$ is a finite dimensional bi-filtered vector space over κ. This means that H_{dR} is endowed with an exhaustive decreasing filtration $F^\bullet(H_{dR})$ and an exhaustive increasing filtration $W_\bullet(H_{dR})$;
- $H_\ell \in \mathcal{G}_\ell(\kappa)^{\mathrm{fil}}$ is a finite dimensional representation $\rho_{H_\ell}\colon \mathrm{Gal}(\overline{\kappa}/\kappa) \to \mathrm{GL}(H_\ell)$ endowed with an exhaustive, increasing, Galois stable filtration $W_\bullet(H_\ell)$;

- $H_\sigma \in \mathrm{MHS}_{\mathbb{Q}}$ is a mixed Hodge structure defined over $\mathbb{Q}$, in the sense of [Del71, § 2.3]. In other words, H_σ is a finite dimensional $\mathbb{Q}$-vector space endowed with an exhaustive, increasing filtration $W_\bullet(H_\sigma)$ and such that $H_\sigma \otimes_{\mathbb{Q}} \mathbb{C}$ is endowed with an exhaustive decreasing filtration $F^\bullet(H_\sigma \otimes_{\mathbb{Q}} \mathbb{C})$, having the property that the filtrations induced by $F^\bullet$ and by its complex conjugate $\overline{F^\bullet}$ on each graded quotient

$$\mathrm{gr}_n^W(H_\sigma \otimes \mathbb{C}) := W_n(H_\sigma \otimes \mathbb{C})/W_{n-1}(H_\sigma \otimes \mathbb{C})$$

are *n-opposed*, *i.e.* $\mathrm{gr}_F^p(\mathrm{gr}_{\overline{F}}^q(\mathrm{gr}_n^W(H_\sigma \otimes \mathbb{C}))) = 0$ if $p + q \neq n$;

- $I_{\infty,\sigma}\colon H_\sigma \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H_{\mathrm{dR}} \otimes_{\kappa,\sigma} \mathbb{C}$ is an isomorphism, which respects the filtrations $W_\bullet$ and $F^\bullet$ defined on both sides;
- $I_{\ell,\widetilde{\sigma}}\colon H_\sigma \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H_\ell$ is an isomorphism respecting the filtrations $W_\bullet$ defined on both sides, such that for every $g \in \mathrm{Gal}(\overline{\kappa}/\kappa)$ we have that $I_{\ell,\widetilde{\sigma}} = \rho_{H_\ell}(g) \circ I_{\ell,\widetilde{\sigma}\circ g}$. Here we are assuming that $\sigma\colon \kappa \hookrightarrow \mathbb{C}$ is obtained from the restriction of $\widetilde{\sigma}\colon \overline{\kappa} \hookrightarrow \mathbb{C}$.

Jannsen proves that $\mathcal{MR}_\kappa$ is a neutral Tannakian category over $\mathbb{Q}$ (in the sense of [DM82, Definition 2.19]), *i.e.* it is a $\mathbb{Q}$-linear, abelian, rigid tensor category endowed with a $\mathbb{Q}$-linear $\otimes$-functor $\mathcal{MR}_\kappa \to \mathrm{Vec}_{\mathbb{Q}}^{\mathrm{f.g.}}$ which is exact and faithful. In fact, there are as many of these functors as the embeddings $\sigma\colon \kappa \hookrightarrow \mathbb{C}$, given by sending $H \in \mathcal{MR}_\kappa$ to the vector space $H_\sigma \in \mathrm{Vec}_{\mathbb{Q}}^{\mathrm{f.g.}}$.

Now, let $\mathcal{QP}(\kappa)$ denote the category of smooth, quasi-projective schemes of finite type over $\mathrm{Spec}(\kappa)$ (see [SP, Definition 01VW]). Then for every $n \in \mathbb{N}$ there is a functor

$$\begin{aligned}\underline{H}^n\colon \mathcal{QP}(\kappa)^{\mathrm{op}} &\to \mathcal{MR}_\kappa \\ X &\mapsto \left(H^n_{dR}(X), \{H^{n,0}_\ell(X)\}_\ell, \{H^n_{\mathrm{sing}}(X^\sigma(\mathbb{C});\mathbb{Q})\}_\sigma, \{I^n_{\infty,\sigma}(X)\}_\sigma, \{I^n_{\ell,\widetilde{\sigma}}(X)\}_{\ell,\widetilde{\sigma}}\right)\end{aligned}$$

where $X^\sigma := X \times_{\kappa,\sigma} \mathrm{Spec}(\mathbb{C})$, and all the cohomology theories have been defined in Section 2.1.3. Moreover, $I^n_{\infty,\sigma}(X)$ denotes inverse of the period isomorphism (2.8) and

$$I^n_{\ell,\widetilde{\sigma}}(X)\colon H^n_{\mathrm{sing}}(X^\sigma(\mathbb{C});\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H^n_{\mathrm{sing}}(X^\sigma(\mathbb{C});\mathbb{Q}_\ell) \xrightarrow{(\dagger)} H^{n,0}_\ell(X)$$

is given by the change of coefficients $H^n_{\mathrm{sing}}(X^\sigma(\mathbb{C});\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} H^n_{\mathrm{sing}}(X^\sigma(\mathbb{C});\mathbb{Q}_\ell)$ and by

$$(\dagger)\colon H^n_{\mathrm{sing}}(X^\sigma(\mathbb{C});\mathbb{Q}_\ell) \xrightarrow{\sim} \left(\varprojlim_m H^n_{\text{ét}}(X^\sigma;\mathbb{Z}/\ell^m\mathbb{Z})\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \xrightarrow{\sim} H^n_\ell(X)$$

which is the composition of the Artin comparison isomorphism, that depends on σ (see [FK88, Theorem 11.6]), followed by the smooth base change isomorphism, that depends on $\widetilde{\sigma}$ (see [Mil80, Chapter VI, Corollary 4.3]). We note as well that the filtrations appearing on the various cohomology groups can be defined using the fact that X admits a "good compactification" consisting of a smooth and projective variety $\overline{X} \in \mathcal{P}(\kappa)$ and an open immersion $X \hookrightarrow \overline{X}$, whose complement is a divisor with normal crossings and smooth components (see [Jan90, § 3] for details). We are now ready to give the first definition of a candidate for the abelian category of mixed motives.

Definition 2.2.7 – Mixed motives for absolute Hodge cycles (according to Jannsen)

The category of mixed motives for absolute Hodge cycles according to Jannsen $\mathcal{MM}_\kappa^{(J)}$ over a field κ which is finitely generated over $\mathbb{Q}$ is defined as the Tannakian sub-category of $\mathcal{MR}_\kappa$ generated by the union of the images of the functors $\underline{H}^n \colon \mathcal{QP}(\kappa)^{\mathrm{op}} \to \mathcal{MR}_\kappa$ for every $n \in \mathbb{Z}$. We denote by $\underline{H}^n \colon \mathcal{QP}(\kappa)^{\mathrm{op}} \to \mathcal{MM}_\kappa^{(J)}$ the functor induced by $\underline{H}^n$.

By definition the category $\mathcal{MM}_\kappa^{(J)}$ is a $\mathbb{Q}$-linear Tannakian category. The definition also shows that every object $M \in \mathcal{MM}_\kappa^{(J)}$ admits a weight filtration, and the full sub-category $\mathcal{M}_\kappa \subseteq \mathcal{MM}_\kappa^{(J)}$ of semi-simple objects can be identified with the sub-category of *pure objects*, *i.e.* objects which are direct sums of ones with only one non-trivial piece in the weight filtration. Moreover, [Jan90, Theorem 4.4] shows that $\mathcal{M}_\kappa$ is equivalent to the category of absolute Hodge motives defined in [DM82, § 6]. Hence under the standard, Hodge and Tate conjectures, the category $\mathcal{M}_\kappa$ should be equivalent to the category of pure numerical motives $\mathrm{NM}(\kappa;\mathbb{Q})$.

Let us mention also an alternative construction of the category of mixed motives, which is due to Huber (see [Hub95]). The key point here is to consider realisations of complexes of varieties instead of single varieties, in order to be able to define mixed motives associated to non-smooth varieties. More precisely, Huber proves in [Hub95, § 11] and [Hub00, Theorem 2.3.1] that there exists a realisation functor $\underline{R} \colon \mathcal{S}(\kappa)^{\mathrm{op}} \to C_{\mathcal{MR}_\kappa}$, where $\mathcal{S}(\kappa)$ denotes the category of all smooth, reduced, separated schemes of finite type over $\mathrm{Spec}(\kappa)$, and $C_{\mathcal{MR}_\kappa}$ is a category whose objects are triples of complexes in the categories $(\mathrm{Vec}_\kappa^{\mathrm{f.g.}\cdot})^{\text{bi-fil}}$, $\mathcal{G}_\ell(\kappa)^{\mathrm{fil}}$ and $\mathrm{MHS}_\mathbb{Q}$ mentioned in Definition 2.2.6, together with filtered quasi-isomorphisms between them (generalising $I_{\infty,\sigma}$ and $I_{\ell,\widetilde{\sigma}}$), such that the cohomology of these complexes lies in $\mathcal{MR}_\kappa$ and the differentials in these complexes are *strict*, in the sense of [SP, Definition 0123]. This functor has the property that $H^n(\underline{R}(X)) = \underline{H}^n(X)$ for every $X \in \mathcal{QP}(\kappa) \subseteq \mathcal{S}(\kappa)$, which ensures some compatibility between Jannsen's and Huber's constructions. Using the fact that $C_{\mathcal{MR}_\kappa}$ is closed under total complexes (see [Hub00, Lemma 2.2.5]), $\underline{R}$ extends to a functor $\underline{R} \colon C^-(\mathbb{Q}[\mathcal{S}(\kappa)])^{\mathrm{op}} \to C_{\mathcal{MR}_\kappa}$, where $C^-(\mathbb{Q}[\mathcal{S}(\kappa)])$ denotes the category of bounded-above cochain complexes valued in the category $\mathbb{Q}[\mathcal{S}(\kappa)]$ whose objects are the same as those of $\mathcal{S}(\kappa)$ and whose morphisms are defined as the free $\mathbb{Q}$-vector spaces $\mathrm{Hom}_{\mathbb{Q}[\mathcal{S}(\kappa)]}(X,Y) := \mathbb{Q}[\mathrm{Hom}_{\mathcal{S}(\kappa)}(X,Y)]$ generated by morphisms in $\mathcal{S}(\kappa)$. We are now ready to recall Huber's definition of the category of mixed motives for absolute Hodge cycles (see [HM17, Definition 6.3.11]).

Definition 2.2.8 – Mixed motives for absolute Hodge cycles (according to Huber)

The category of mixed motives for absolute Hodge cycles according to Huber $\mathcal{MM}_\kappa^{(H)}$ over a field κ which is finitely generated over $\mathbb{Q}$ is defined as the full abelian, tensor sub-category of $\mathcal{MR}_\kappa$ generated by the images of the functors $\{H^n \circ \underline{R}\}_{n\in\mathbb{Z}}$ and by the dual of $\mathbb{Q}(-1) = H^2(\underline{R}(\mathbb{P}^1))$.

It is known that $\mathcal{MM}_\kappa^{(J)} \subseteq \mathcal{MM}_\kappa^{(H)}$, and that the category of semi-simple objects in $\mathcal{MM}_\kappa^{(H)}$ coincides with the category $\mathcal{M}_\kappa$ of absolute Hodge motives in the sense of [DM82, § 6]. One advantage of Huber's construction is that one can apply [Hub04, Lemma B.5.3] to get a family

of functors $\underline{H}^n \colon \mathcal{V}(\kappa)^{\mathrm{op}} \to \mathcal{MM}_\kappa^{(H)}$, which extends the notion of motive to every *variety* (*i.e.* separated, reduced scheme of finite type) over $\mathrm{Spec}(\kappa)$. There should be a commutative square

$$\begin{array}{ccc} \mathcal{P}(\kappa)^{\mathrm{op}} & \hookrightarrow & \mathcal{V}(\kappa)^{\mathrm{op}} \\ {\scriptstyle \mathfrak{h}_{\mathrm{num}}(-;\mathbb{Q})}\downarrow & ? & \downarrow{\scriptstyle \underline{H}^\bullet(-)} \\ \mathrm{NM}(\kappa;\mathbb{Q}) & \overset{?}{\hookrightarrow} & \mathcal{MM}_\kappa^{(H)} \end{array} \tag{2.12}$$

where $\underline{H}^\bullet(X) := \bigoplus_{n\in\mathbb{Z}} \underline{H}^n(X)$ and the bottom arrow should be given by the conjectural natural equivalence $\mathrm{NM}(\kappa;\mathbb{Q}) \xrightarrow{\sim} \mathcal{M}_\kappa$, followed by the inclusion $\mathcal{M}_\kappa \hookrightarrow \mathcal{MM}_\kappa$. The existence and the commutativity of (2.12) follow of course from a combination of the standard conjectures, together with conjectures of Hodge and Tate type.

Let us finally mention the existence of an entirely new kind of construction, due to Nori, which is still based on realisations. Its definition goes as follows (see [HM17, § 9.1]).

Definition 2.2.9 – Nori motives

Let $\kappa \subseteq \mathbb{C}$ be a sub-field, and let R be a Noetherian ring. Then:

- define $\mathrm{Pairs}_\kappa^{\mathrm{eff}}$ to be the directed graph whose nodes are triples (X, Y, i) where $X \in \mathcal{V}(\kappa)$ is a variety, $Y \hookrightarrow X$ a closed immersion in $\mathcal{V}(\kappa)$ and $i \in \mathbb{N}$. The edges of $\mathrm{Pairs}_\kappa^{\mathrm{eff}}$ are given by maps $f^* \colon (X', Y', i') \to (X, Y, i)$ for every morphism $f \colon X \to X'$ in $\mathcal{V}(\kappa)$, such that $f(Y) \subseteq Y'$, and maps $\partial \colon (Y, Z, i) \to (X, Y, i+1)$ for every chain of closed immersions $Z \hookrightarrow Y \hookrightarrow X$;
- define a category $\mathcal{MM}_{\kappa,R}^{(N),\mathrm{eff}}$ of *effective mixed Nori motives* as a suitable colimit of modules over the rings $\mathrm{End}(H\big|_{\mathcal{D}})$ where $\mathcal{D} \subseteq \mathrm{Pairs}_\kappa^{\mathrm{eff}}$ runs over the finite sub-graphs and $H \colon \mathrm{Pairs}_\kappa^{\mathrm{eff}} \to \mathrm{Mod}_R$ denotes the map of directed graphs which sends (X, Y, i) to the relative cohomology $H^i_{\mathrm{sing}}(X(\mathbb{C}), Y(\mathbb{C}); R)$. For a definition of the endomorphism ring of a morphism of directed graphs, see [HM17, Definition 7.1.8];
- define the category $\mathcal{MM}_{\kappa,R}^{(N)}$ of *mixed Nori motives* by formally inverting the object $\mathbb{1}(-1) := \underline{H}(\mathbb{G}_m, \emptyset, 1)$, where

$$\underline{H} \colon \mathrm{Pairs}_\kappa^{\mathrm{eff}} \to \mathcal{MM}_{\kappa,R}^{(N),\mathrm{eff}}$$

  is the functor coming out of the construction briefly described above (see [HM17, Theorem 7.1.13]).

The striking feature of the category of Nori motives is that of providing a cohomology theory defined on $\mathcal{V}(\kappa)^* := \mathrm{Pairs}_\kappa^{\mathrm{eff}}$ which is *universal* amongst all the cohomology theories comparable with singular cohomology (see [HM17, Theorem 9.1.10]). Moreover, if R is a Dedekind domain then $\mathcal{MM}_{\kappa,R}^{(N)}$ is a neutral Tannakian category over Mod_R (see [HM17, Theorem 9.3.10]), and there exists a triangulated functor (see [HM17, Theorem 9.1.9])

$$\underline{R} \colon hC^b(R[\mathcal{V}(\kappa)])^{\mathrm{op}} \to D^b(\mathcal{MM}_{\kappa,R}^{(N)})$$

such that $\underline{H}^i(X,Y,i) = H^i(\underline{R}(\mathrm{Cone}([Y] \to [X])))$ for every $(X,Y,i) \in \mathcal{V}(\kappa)^*$. Finally, let us mention that there there exists a faithful functor (see [HM17, Proposition 10.1.2])

$$\mathcal{MM}_\kappa^{(N)} \to \mathcal{MM}_\kappa^{(H)} \tag{2.13}$$

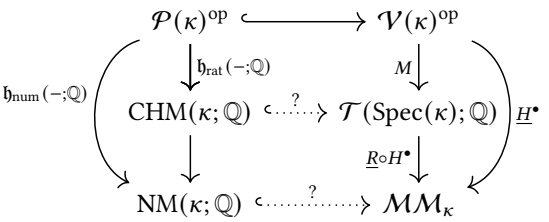
where $\mathcal{MM}_\kappa^{(N)} := \mathcal{MM}_{\kappa,\mathbb{Q}}^{(N)}$, such that every object of $\mathcal{MM}_\kappa^{(H)}$ is a sub-quotient of an object in the image of (2.13). Moreover, each object in $\mathcal{MM}_\kappa^{(N)}$ is endowed with a weight filtration (see [HM17, Theorem 10.2.5]), which is respected by (2.13), and the category of pure objects with respect to this weight filtration is equivalent (see [HM17, Theorem 10.2.7]) to the category of motives constructed by André using motivated cycles (see [And04, § 9.2]). This last category is in turn equivalent to the category of numerical motives $\mathrm{NM}(\kappa;\mathbb{Q})$ assuming the Hodge conjecture, and the functor (2.13) should induce an equivalence between this category and the category $\mathcal{M}_\kappa$ of pure motives for absolute Hodge cycles described above.

2.2.3 Triangulated categories of mixed motives

The constructions of the abelian category of mixed motives $\mathcal{MM}_\kappa$ outlined in Section 2.2.2 might give mixed feelings to the reader. On the one hand, they are abelian, and even Tannakian categories, which allows one to talk about motivic Galois groups. On the other hand, their construction only partially fulfils the program laid down by Beilinson in [Bei87, § 5.10], for multiple reasons. First of all, Beilinson's program should work for every scheme S, whereas the categories $\mathcal{MM}_\kappa$ have been constructed only over a field κ. Secondly, the categories $\mathcal{MM}_\kappa$ are constructed using realisations, which makes it difficult to relate them to algebraic geometry. More precisely, any hope to construct a fully faithful embedding $\mathrm{NM}(\kappa;\mathbb{Q}) \hookrightarrow \mathcal{MM}_\kappa$ rests on some of the most difficult conjectures in algebraic geometry, and there is no hope to have a fully faithful embedding $\mathrm{CHM}(\kappa;\mathbb{Q}) \hookrightarrow \mathcal{MM}_\kappa$, at least if κ is algebraically closed. Indeed, in this case Mumford has shown in [Mum69] that Chow groups are "enormous", usually infinitely generated, hence it would be impossible to gain a fully faithful embedding in a category given by realisations (see also [BS83] for a generalisation of Mumford's result).

The aim of this section is to describe how one could hope to overcome these difficulties, using triangulated categories. More precisely, Deligne noted in a letter to Soulé that it might be easier to construct the conjectural abelian category $\mathcal{MM}(S,\Lambda)$ envisioned by Beilinson (with coefficients in any ring Λ) similarly to how one constructs categories of perverse sheaves (see [BBD82]). First, one should define a triangulated category $\mathcal{T}(S;\Lambda)$ endowed with a functor $M\colon \mathcal{V}(S)^{\mathrm{op}} \to \mathcal{T}(S;\Lambda)$, where $\mathcal{V}(S)$ denotes the category of schemes of finite type over S. Then one should define a t-structure on $\mathcal{T}(S;\Lambda)$ whose heart would give $\mathcal{MM}(S;\Lambda)$. Using this t-structure one would be able to define cohomology functors $H^i\colon \mathcal{T}(S;\Lambda) \to \mathcal{MM}(S;\Lambda)$ given by $H^i(M) := \tau^{\leq 0}(\tau^{\geq 0}(M[i]))$, where $\tau^{\leq 0}$ and $\tau^{\geq 0}$ denote the truncation functors. This would give rise to a functor $H^\bullet\colon \mathcal{T}(S;\Lambda) \to \mathcal{MM}(S;\Lambda)$ defined as the direct sum $H^\bullet(M) := \bigoplus_{i\in\mathbb{Z}} H^i(M)$. Moreover, one should have an equivalence of categories $\underline{R}\colon \mathcal{MM}(\mathrm{Spec}(\kappa);\mathbb{Q}) \xrightarrow{\sim} \mathcal{MM}_\kappa$, where

$\mathcal{MM}_\kappa$ is one of the categories constructed in Section 2.2.2. This equivalence $\underline{R}$ should also fit into a commutative diagram



(2.14)

where the horizontal maps are fully faithful embeddings.

The construction of the triangulated category $\mathcal{T}(S;\Lambda)$ has been essentially achieved in successive steps by Hanamura, Levine, Voevodsky, Morel, Cisinski and Déglise and many others. In particular, we argue in this section that one should take $\mathcal{T}(S;\Lambda) := \mathrm{DM}(S;\Lambda)^{\mathrm{op}}$ to be the opposite of the triangulated category of (homological) mixed motives constructed by Cisinski and Déglise in [CD19], following the work of Morel and Voevodsky (see [MV99]). Let us start with the notion of *geometric motives*, which over an arbitrary base is due to Ivorra (see [CD19, Definition 11.1.10]).

To do so, we need to recall the notion of finite correspondence. Let S be a scheme, $X \in \mathrm{Sch}_S$ a scheme endowed with a morphism $X \to S$ and Λ be a ring. Then the Λ-module of *relative cycles* $\mathcal{Z}^\bullet(X/S)_\Lambda$ is the sub-module of $\mathcal{Z}^\bullet(X)_\Lambda$ consisting of those cycles $\alpha \in \mathcal{Z}^\bullet(X)_\Lambda$ such that the structural morphism $f : X \to S$ sends $\mathrm{Supp}(\alpha)$ to generic points of S. In other words, these are cycles dominant over S. Then one can define $C_0^\bullet(X/S)_\Lambda$ to be the sub-module of $\mathcal{Z}^\bullet(X/S)_\Lambda$ consisting of those cycles which are finite and Λ-universal over S, where the second condition means roughly that the pull-back of these cycles along every point $x : \mathrm{Spec}(\kappa) \to X$ whose image is in the support of S has coefficients in Λ (see [CD19, p. 8.1.47] for the precise definition). Finally, given two schemes $X, Y \in \mathrm{Sch}_S$ one defines the Λ-module of finite correspondences over S to be $C_S(X,Y)_\Lambda := C_0^\bullet(X \times_S Y/X)_\Lambda$. These correspondences admit a composition (see [CD19, Definition 9.1.5]), similar to the one that we outlined in Section 2.2.1 when $S = \mathrm{Spec}(\kappa)$ for some field κ. This allows one to define the category $\mathcal{Sm}_{S,\Lambda}^{\mathrm{cor}}$ as the category whose objects are smooth schemes of finite type over S and whose morphisms are given by the Λ-modules of finite correspondences, with composition given by the product just mentioned. This category admits a functor $\gamma : \mathcal{Sm}_S \to \mathcal{Sm}_{S,\Lambda}^{\mathrm{cor}}$ which is the identity on objects and sends each morphism $f : X \to Y$ to its graph $\Gamma_f \in C_S(X,Y)_\Lambda$. Moreover, $\mathcal{Sm}_{S,\Lambda}^{\mathrm{cor}}$ is a Λ-linear, symmetric tensor category, with tensor product induced by the fibre product of schemes (see [CD19, § 9.2]).

Definition 2.2.10 – Triangulated category of mixed geometric motives

Let S be a scheme and let Λ be a commutative ring with unity. We define the Λ-linear *triangulated category of effective mixed geometric motives* $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S;\Lambda)$ as the pseudo-abelian envelope (see [And04, § 1.1.3.1]) of the quotient of the homotopy category $\mathcal{K}^b(\mathcal{Sm}_{S,\Lambda}^{\mathrm{cor}})$ of bounded chain complexes valued in $\mathcal{Sm}_{S,\Lambda}^{\mathrm{cor}}$, by the triangulated sub-category $\mathcal{V}_S$ generated by complexes of the form:

- $[\mathbb{A}^1_X] \xrightarrow{\gamma(p_X)} [X]$ for every scheme $X \in \mathrm{Sch}_S$ which is smooth and of finite type over S, where $p_X : \mathbb{A}^1_X \to X$ denotes the structural morphism;

- $[U] \xrightarrow{(\gamma(j),-\gamma(f'))} [X] \oplus [U'] \xrightarrow{\gamma(f)+\gamma(j')} [X']$ for every Nisnevich distinguished square (2.1).

Moreover, the Λ-linear *triangulated category of mixed geometric motives* $\mathrm{DM}_{\mathrm{gm}}(S;\Lambda)$ is defined to be the category obtained from $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S;\Lambda)$ by formally inverting the motive $\mathbb{1}(1)$ given by the class of the complex $[\mathbb{P}^1_S] \to [S]$.

Observe that we use chain complexes instead of the usual cochain complexes used so far, because Voevodsky's motives are homological. In particular, there is a covariant functor $M_{\mathrm{gm}}(-/S;\Lambda)\colon \mathcal{S}m_S \to \mathrm{DM}_{\mathrm{gm}}(S;\Lambda)$. To describe this functor, let us unravel the definition of $\mathrm{DM}_{\mathrm{gm}}(S;\Lambda)$. The objects of $\mathrm{DM}_{\mathrm{gm}}(S;\Lambda)$ are of the form (M,r) with $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S;\Lambda)$ and $r \in \mathbb{Z}$. Morphisms between these objects are defined by the formula

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}(S;\Lambda)}((M,r),(M',r')) := \varinjlim_{j \geq -\min(r,r')} \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S;\Lambda)}(M(j+r), M(j+r'))$$

where $M(n) := M \otimes \mathbb{1}(1)^{\otimes n}$ for every $n \in \mathbb{N}$ and every $M \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S;\Lambda)$. It is worth noting that the direct limit stabilises. Furthermore, the objects of $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S;\Lambda)$ are pairs of the form $M = (A,e)$ where $A \in \mathcal{K}^b(\mathcal{S}m_{S,\Lambda}^{\mathrm{cor}})$ and $e \in \mathrm{Hom}_{\mathcal{K}^b(\mathcal{S}m_{S,\Lambda}^{\mathrm{cor}})/\mathcal{V}_S}(A,A)$ is an idempotent (*i.e.* $e \circ e = e$) in the quotient category $\mathcal{K}^b(\mathcal{S}m_{S,\Lambda}^{\mathrm{cor}})/\mathcal{V}_S$. We recall that this quotient category $\mathcal{K}^b(\mathcal{S}m_{S,\Lambda}^{\mathrm{cor}})/\mathcal{V}_S$ has the same objects as $\mathcal{K}^b(\mathcal{S}m_{S,\Lambda}^{\mathrm{cor}})$, but the morphisms are given by

$$\mathrm{Hom}_{\mathcal{K}^b(\mathcal{S}m_{S,\Lambda}^{\mathrm{cor}})/\mathcal{V}_S}(A,B) := \varinjlim_{\widetilde{A} \to A} \mathrm{Hom}_{\mathcal{K}^b(\mathcal{S}m_{S,\Lambda}^{\mathrm{cor}})}(\widetilde{A},B)$$

where the direct limit runs over all the morphisms $\widetilde{A} \to A$ whose cone lies in $\mathcal{V}_S$ (see [SP, Section 05RA]). Finally, morphisms in $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S;\Lambda)$ are given by the following formula

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S;\Lambda)}((A,e),(A',e')) := e' \circ \mathrm{Hom}_{\mathcal{K}^b(\mathcal{S}m_{S,\Lambda}^{\mathrm{cor}})/\mathcal{V}_S}(A,A') \circ e$$

which is analogous to what happens with pure motives (see Section 2.2.1). The functor

$$M_{\mathrm{gm}}(-/S;\Lambda)\colon \mathcal{S}m_S \to \mathrm{DM}_{\mathrm{gm}}(S;\Lambda)$$

can now be defined as the composite of the functor $[-]\colon \mathcal{S}m_S \to \mathcal{K}^b(\mathcal{S}m_{S,\Lambda}^{\mathrm{cor}})$ sending X to the homotopy class of the complex $[X]$ concentrated in degree zero (which was used in Definition 2.2.10), the projection onto the quotient $\mathcal{K}^b(\mathcal{S}m_{S,\Lambda}^{\mathrm{cor}}) \to \mathcal{K}^b(\mathcal{S}m_{S,\Lambda}^{\mathrm{cor}})/\mathcal{V}_S$, the functor $\mathcal{K}^b(\mathcal{S}m_{S,\Lambda}^{\mathrm{cor}})/\mathcal{V}_S \to \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S;\Lambda)$ sending A to (A,Id_A) and the functor

$$\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S;\Lambda) \to \mathrm{DM}_{\mathrm{gm}}(S;\Lambda)$$

sending X to $(X,0)$.

Let us recall some properties of the triangulated category of mixed geometric motives $\mathrm{DM}_{\mathrm{gm}}(S;\Lambda)$. First of all, it is a triangulated category, which agrees with its name. Indeed, the quotient category $\mathcal{K}^b(\mathcal{S}m_{S,\Lambda}^{\mathrm{cor}})/\mathcal{V}_S$ is easily seen to be triangulated, which implies that $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S;\Lambda)$ is triangulated (and pseudo-abelian) by a general result of Balmer and Schlichting (see [BS01]) on pseudo-abelian envelopes. The fact that $\mathrm{DM}_{\mathrm{gm}}(S;\Lambda)$ is triangulated and pseudo-

abelian follows from the fact that it can be written as a 2-colimit of pseudo-abelian, triangulated categories (see [Ivo07, Page 616]). Moreover, for every field κ there exists a commutative square

$$\begin{array}{ccc} \mathcal{P}(\kappa)^{\mathrm{op}} & \hookrightarrow & \mathcal{S}(\kappa)^{\mathrm{op}} \\ \downarrow{\scriptstyle \mathfrak{h}_{\mathrm{rat}}(-;\Lambda)} & & \downarrow{\scriptstyle M_{\mathrm{gm}}(-/\kappa;\Lambda)^{\mathrm{op}}} \\ \mathrm{CHM}(\kappa;\Lambda) & \hookrightarrow & \mathrm{DM}_{\mathrm{gm}}(\kappa;\Lambda)^{\mathrm{op}} \end{array} \tag{2.15}$$

where $\mathrm{DM}_{\mathrm{gm}}(\kappa;\Lambda) := \mathrm{DM}_{\mathrm{gm}}(\mathrm{Spec}(\kappa);\Lambda)$, and the functor on the bottom is a fully faithful embedding, which maps $\mathbb{1}_{\mathrm{rat},\mathbb{Q}}(-1)$ to $\mathbb{1}(1)[2]$ and sends a Chow motive (X, e, r) with $r \geq 0$ to an effective mixed geometric motive (see [And04, § 18.3]). Hence the diagram (2.15) seems to be a first step towards the construction of a diagram like (2.14). It turns out that in order to extend the functor $M_{\mathrm{gm}}(-;\Lambda)$ to singular schemes it is better to work with a bigger category, which we are now going to define.

Let us recall first of all that a Λ-linear presheaf with transfers over a scheme S is a presheaf on the category $\mathcal{S}m^{\mathrm{cor}}_{S,\Lambda}$ of Λ-linear smooth correspondences (see [CD19, Definition 10.1.1]), valued in the category Mod_Λ of modules over Λ. Such a presheaf with transfers $F \in \mathrm{PSh}^{\mathrm{tr}}(S,\Lambda)$ is called a sheaf with transfers, with respect to some Grothendieck topology τ on $\mathcal{S}m_S$, if the composition $F \circ \gamma_S$ is a sheaf with respect to this topology, where $\gamma_S \colon \mathcal{S}m_S \to \mathcal{S}m^{\mathrm{cor}}_{S,\Lambda}$ is the functor sending each object to itself and each morphism $f \colon X \to Y$ to its graph Γ_f. We denote the category of Λ-linear sheaves with transfers by $\mathrm{Sh}^{\mathrm{tr}}_\tau(S,\Lambda)$. Under some technical hypotheses on the Grothendieck topology τ (see [CD19, Definition 10.3.5]), which are satisfied if τ is the étale or the Nisnevich topology (see [CD19, Proposition 10.3.3]), the forgetful functor $\mathrm{Sh}^{\mathrm{tr}}_\tau(S,\Lambda) \to \mathrm{PSh}^{\mathrm{tr}}(S,\Lambda)$ admits a left-adjoint (the τ-sheafification) $\mathrm{PSh}^{\mathrm{tr}}(S,\Lambda) \to \mathrm{Sh}^{\mathrm{tr}}_\tau(S,\Lambda)$, and for every $X \in \mathcal{S}m_S$ one denotes by $\Lambda^{\mathrm{tr}}_S(X)_\tau$ the τ-sheafification of the representable presheaf $\Lambda^{\mathrm{tr}}_S(X) \in \mathrm{PSh}^{\mathrm{tr}}(S,\Lambda)$ defined by $\Lambda^{\mathrm{tr}}_S(X)(Y) := C_S(Y,X)_\Lambda$. This defines a functor $\Lambda_S(-)^{\mathrm{tr}}_\tau \colon \mathcal{S}m_S \to \mathrm{Shv}^{\mathrm{tr}}_\tau(S;\Lambda)$. We note moreover that the category $\mathrm{Shv}^{\mathrm{tr}}_\tau(S;\Lambda)$ is an abelian symmetric tensor category (if we have a τ-sheafification functor), where the tensor structure is induced by the one present on Mod_Λ. Moreover, the identity for the tensor product is given by $\mathbb{1}_{S,\tau} := \Lambda^{\mathrm{tr}}_S(S)_\tau$, and we denote by $\mathbb{1}_{S,\tau}\{1\}$ the cokernel of the map $\Lambda^{\mathrm{tr}}_S(S)_\tau \to \Lambda^{\mathrm{tr}}_S(\mathbb{G}^1_{m,S})_\tau$ induced by the unit $S \to \mathbb{G}^1_{m,S}$. This object allows us to define the category of symmetric Tate spectra $\mathrm{Sp}^{\mathrm{tr}}_\tau(S;\Lambda)$ as follows. First of all, one defines the category of symmetric sequences $\mathrm{Shv}^{\mathrm{tr}}_\tau(S;\Lambda)^{\mathfrak{S}}$ whose objects are sequences $A = (A_n, \rho_n)_{n\in\mathbb{N}}$ of pairs (A_n, ρ_n) where $A_n \in \mathrm{Shv}^{\mathrm{tr}}_\tau(S;\Lambda)$ and $\rho_n \colon \mathfrak{S}_n \to \mathrm{Aut}(A_n)$ is an action of the symmetric group on n-letters $\mathfrak{S}_n$, and whose morphisms are equivariant morphisms. For every symmetric sequence $A = (A_n, \rho_n) \in \mathrm{Shv}^{\mathrm{tr}}_\tau(S;\Lambda)^{\mathfrak{S}}$ and every $n \in \mathbb{N}$, one can define a twist $A\{-n\}$, by considering suitable "fibre products" of the form $\mathfrak{S}_m \times_{\mathfrak{S}_{m-n}} A_{m-n}$ for every $m \geq n$ (see [CD19, Equation 5.3.5.2] for the precise definition). Moreover, one defines the object $\mathbb{1}_{S,\tau}\{*\} \in \mathrm{Shv}^{\mathrm{tr}}_\tau(S;\Lambda)^{\mathfrak{S}}$ to be the sequence of pairs $(\mathbb{1}_{S,\tau}\{n\}, \sigma_n)$ where $\mathbb{1}_{S,\tau}\{n\} := \mathbb{1}_{S,\tau}\{1\}^{\otimes n}$ and $\sigma_n \colon \mathfrak{S}_n \to \mathrm{Aut}(\mathbb{1}_{S,\tau}\{n\})$ denotes the permutation action. This object is a monoid (in the sense of [KS06, Remark 4.3.2]) inside the monoidal category $\mathrm{Shv}^{\mathrm{tr}}_\tau(S;\Lambda)^{\mathfrak{S}}$, where the tensor product is defined by setting $(A_i)_i \otimes (B_j)_j := \left(\bigoplus_{n=i+j}(A_i \otimes B_j)\right)$ for every $A, B \in \mathrm{Shv}^{\mathrm{tr}}_\tau(S;\Lambda)^{\mathfrak{S}}$. Then the category of symmetric Tate spectra $\mathrm{Sp}^{\mathrm{tr}}_\tau(S;\Lambda)$ is finally defined as the full subcategory of $\mathrm{Shv}^{\mathrm{tr}}_\tau(S;\Lambda)^{\mathfrak{S}}$ given by all the objects which are modules over the monoid $\mathbb{1}_{S,\tau}\{*\}$, in the sense of [KS06, Definition 4.3.3]. Note that there is an adjunction $\Sigma^\infty \colon \mathrm{Shv}^{\mathrm{tr}}_\tau(S;\Lambda) \rightleftarrows \mathrm{Sp}^{\mathrm{tr}}_\tau(S;\Lambda) \colon \Omega^\infty$, which is defined in [CD19, Equation 5.3.16.1].

We are now finally ready to define the triangulated categories of mixed motives, in the sense of [CD19, Definition 11.1.1].

Definition 2.2.11 – Triangulated category of mixed motives

Let S be a scheme and let Λ be a commutative ring with unity. Fix a Grothendieck topology τ on the category $\mathcal{S}m_S$ of smooth schemes of finite type over S, which is mildly compatible with transfers (in the sense of [CD19, Definition 10.3.5]). The Λ-linear *triangulated category of effective mixed motives* for the topology τ is defined as the quotient

$$\mathrm{DM}_\tau^{\mathrm{eff}}(S;\Lambda) := D(\mathrm{Shv}_\tau^{\mathrm{tr}}(S;\Lambda))/\mathscr{W}_{S,\tau}^{\mathbb{A}^1}$$

of the derived category of Nisnevich sheaves with transfers modulo the triangulated sub-category $\mathscr{W}_{S,\tau}^{\mathbb{A}^1}$ generated by the complexes of the form

$$\Lambda_S^{\mathrm{tr}}(\mathbb{A}^1_X)_\tau \xrightarrow{\Lambda_S^{\mathrm{tr}}(p)_\tau} \Lambda_S^{\mathrm{tr}}(X)_\tau$$

for every $X \in \mathcal{S}m_S$, where $p\colon \mathbb{A}^1_X \to X$ denotes the canonical projection. Moreover, the Λ-linear *triangulated category of mixed motives* for the topology τ is defined to be the quotient

$$\mathrm{DM}_\tau(S;\Lambda) := D(\mathrm{Sp}_\tau^{\mathrm{tr}}(S;\Lambda))/\mathscr{W}_{S,\tau}^{\mathbb{A}^1,\Omega}$$

where $\mathscr{W}_{S,\tau}^{\mathbb{A}^1,\Omega}$ is the triangulated sub-category generated by the union of $\Sigma^\infty(\mathscr{W}_{S,\tau}^{\mathbb{A}^1})$ together with the complexes of the form $[\Sigma^\infty(\Lambda_S^{\mathrm{tr}}(X)_\tau)]\{-n\} \otimes \widetilde{\mathbb{1}_{S,\tau}\{1\}}$ for every $X \in \mathcal{S}m_S$ and every $n \in \mathbb{N}$. Here $\widetilde{\mathbb{1}_{S,\tau}\{1\}}$ denotes the complex

$$\widetilde{\mathbb{1}_{S,\tau}\{1\}}\colon [\Sigma^\infty(\mathbb{1}_{S,\tau}\{1\})]\{-1\} \to \Sigma^\infty(\mathbb{1}_{S,\tau})$$

induced by the fact that Σ^∞ is a left adjoint. Finally, we write $\mathrm{DM}^{\mathrm{eff}}(S;\Lambda)$ and $\mathrm{DM}(S;\Lambda)$ for the categories associated to the Nisnevich topology.

The main advantage of using sheaves with transfers instead of complexes valued in the category $\mathcal{S}m_{S,\Lambda}^{\mathrm{cor}}$, as we did in Definition 2.2.10, is that now it becomes easy to define a functor $M(-/S;\Lambda)\colon \mathcal{V}(S) \to \mathrm{DM}^{\mathrm{eff}}(S;\Lambda)$ which associates to any scheme of finite type $X \in \mathcal{V}(S)$ its motive $M(X/S;\Lambda)$. To do so, one extends the functor $\Lambda_S(-)_\tau^{\mathrm{tr}}\colon \mathcal{S}m_S \to \mathrm{Shv}_\tau^{\mathrm{tr}}(S;\Lambda)$ to singular schemes, defining $\Lambda_S(X)_\tau^{\mathrm{tr}}$ to be the τ-sheafification of the presheaf with transfers $\Lambda_S^{\mathrm{tr}}(X) \in \mathrm{PSh}^{\mathrm{tr}}(S;\Lambda)$ which associates to each $Y \in \mathcal{S}m_{S,\Lambda}^{\mathrm{cor}}$ the Λ-module of finite correspondences $\Lambda_S^{\mathrm{tr}}(X)(Y) := C_S(Y,X)_\Lambda$. Then one gets a functor

$$M(-/S;\Lambda)\colon \mathcal{V}(S) \to \mathrm{DM}^{\mathrm{eff}}(S,\Lambda) \tag{2.16}$$

taking τ to be the Nisnevich topology. This functor fits in a commutative diagram (see [CD19, Diagram 11.1.12.1])

$$\begin{array}{ccccc}
\mathcal{S}m_S & \xrightarrow{M_{\mathrm{gm}}(-/S;\Lambda)} & \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(S;\Lambda) & \longrightarrow & \mathrm{DM}_{\mathrm{gm}}(S;\Lambda) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{V}(S) & \xrightarrow[M(-/S;\Lambda)]{} & \mathrm{DM}^{\mathrm{eff}}(S;\Lambda) & \xrightarrow[\Sigma^\infty]{} & \mathrm{DM}(S;\Lambda)
\end{array} \tag{2.17}$$

where Σ^∞ is the left adjoint in the adjuction $\Sigma^\infty\colon \mathrm{DM}^{\mathrm{eff}}(S;\Lambda) \rightleftarrows \mathrm{DM}(S;\Lambda)\colon \Omega^\infty$, induced from the corresponding adjunction between sheaves with transfers and symmetric Tate spectra. Moreover, [CD19, Theorem 11.1.13] proves that the horizontal maps appearing in (2.17) are fully faithful embeddings, which identify the category $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(S;\Lambda)$ (respectively, the category $\mathrm{DM}_{\mathrm{gm}}(S;\Lambda)$) with the sub-category of $\mathrm{DM}^{\mathrm{eff}}(S;\Lambda)$ (resp. $\mathrm{DM}(S;\Lambda)$), given by *compact objects*, *i.e.* by those objects X such that $\mathrm{Hom}(X,-)$ commutes with small sums (see [SP, Definition 07LS]). Hence (2.15) can be extended to give the following commutative square

$$\begin{array}{ccc} \mathcal{P}(\kappa)^{\mathrm{op}} & \hookrightarrow & \mathcal{V}(\kappa)^{\mathrm{op}} \\ \downarrow{\scriptstyle \mathfrak{h}_{\mathrm{rat}}(-;\Lambda)} & & \downarrow{\scriptstyle M(-/\kappa;\Lambda)^{\mathrm{op}}} \\ \mathrm{CHM}(\kappa;\Lambda) & \hookrightarrow & \mathrm{DM}(\kappa;\Lambda)^{\mathrm{op}} \end{array}$$

where the horizontal maps are fully faithful embeddings, which should be the top square appearing in (2.14). We note that, if S satisfies a suitable form of resolution of singularities (*e.g.* if $S = \mathrm{Spec}(\kappa)$ where κ is a field of characteristic zero), then the functor

$$M(-/S;\Lambda)\colon \mathcal{V}(S) \to \mathrm{DM}^{\mathrm{eff}}(S,\Lambda)$$

actually factors through the embedding $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(S,\Lambda) \hookrightarrow \mathrm{DM}^{\mathrm{eff}}(S,\Lambda)$, in virtue of [CD19, Corollary 4.4.3]. Thus we could in fact extend our original diagram (2.15), where κ is certainly a field of characteristic zero, by replacing $\mathcal{S}(\kappa)^{\mathrm{op}}$ with $\mathcal{V}(\kappa)^{\mathrm{op}}$. However, it is expected that Huber's or Nori's constructions of $\mathcal{MM}_\kappa$ could generalise to define a category $\mathcal{MM}_S$ over a general base S, and then (until a suitable form of resolution of singularities is proved for S) it might be better to stick with our decision of taking $\mathrm{DM}(S;\Lambda)^{\mathrm{op}}$ to be our candidate for the triangulated category of mixed motives, rather than $\mathrm{DM}_{\mathrm{gm}}(S;\Lambda)^{\mathrm{op}}$. Let us mention that other candidates for the triangulated category of mixed motives can be obtained by applying the same techniques seen in this section to the category $\mathcal{V}(S)^{\mathrm{cor}}_\Lambda$ where the objects are schemes of finite type over S and the morphisms are given by finite correspondences. Doing so, one gets two triangulated categories $\underline{\mathrm{DM}}^{\mathrm{eff}}_\tau(S;\Lambda)$ and $\underline{\mathrm{DM}}_\tau(S;\Lambda)$ which fit in a commutative diagram

$$\begin{array}{ccc} \mathrm{DM}^{\mathrm{eff}}_\tau(S;\Lambda) & \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} & \mathrm{DM}_\tau(S;\Lambda) \\ {\scriptstyle \varphi^*_{\mathrm{eff}}}\uparrow\downarrow{\scriptstyle \varphi^{\mathrm{eff}}_!} & & {\scriptstyle \varphi^*}\uparrow\downarrow{\scriptstyle \varphi_!} \\ \underline{\mathrm{DM}}^{\mathrm{eff}}_\tau(S;\Lambda) & \underset{\underline{\Omega}^\infty}{\overset{\underline{\Sigma}^\infty}{\rightleftarrows}} & \underline{\mathrm{DM}}_\tau(S;\Lambda) \end{array}$$

where $\Sigma^\infty \dashv \Omega^\infty$, $\underline{\Sigma}^\infty \dashv \underline{\Omega}^\infty$, $\varphi^{\mathrm{eff}}_! \dashv (\varphi^{\mathrm{eff}})^*$ and $\varphi_! \dashv \varphi^*$ are adjoints (see [CD19, § 11.1.15]). Moreover, the functors $\varphi_!$ and $\varphi^{\mathrm{eff}}_!$ are fully faithful, and they are compatible with twists (see [CD19, Proposition 11.1.19]).

We conclude this brief review of the various existing candidates for the triangulated category of mixed motives by mentioning the $\mathbb{A}^1$-homotopy categories $\mathrm{DA}^{\mathrm{eff}}_\tau(S;\Lambda)$ and $\mathrm{DA}_\tau(S;\Lambda)$ constructed by Ayoub (see [Ayo07]), which are denoted by $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1,\tau}(S;\Lambda)$ and $\mathrm{D}_{\mathbb{A}^1,\tau}(S;\Lambda)$ in [CD19, Example 5.3.31]. These categories are constructed in the exact same way as $\mathrm{DM}^{\mathrm{eff}}_\tau(S;\Lambda)$ and $\mathrm{DM}_\tau(S;\Lambda)$ (see Definition 2.2.11), by replacing the abelian category $\mathrm{Shv}^{\mathrm{tr}}_\tau(S;\Lambda)$ with the abelian category $\mathrm{Shv}_\tau(S;\Lambda)$ of Λ-linear τ-sheaves on the category $\mathcal{Sm}_S$ of smooth schemes of finite type over S. The functor $\gamma\colon \mathcal{Sm}_S \to \mathcal{Sm}^{\mathrm{cor}}_{S,\Lambda}$ which sends a scheme to itself and a morphism

$f\colon X \to Y$ to its graph Γ_f, induces an adjunction $\gamma^*\colon \mathrm{Shv}_\tau(S;\Lambda) \rightleftarrows \mathrm{Sh}^{\mathrm{tr}}_\tau(S;\Lambda)\colon \gamma_*$, which in turn induces a commutative diagram

$$\begin{array}{ccc} \mathrm{DA}^{\mathrm{eff}}_\tau(S;\Lambda) & \underset{\Omega^\infty_{\mathbb{A}^1}}{\overset{\Sigma^\infty_{\mathbb{A}^1}}{\rightleftarrows}} & \mathrm{DA}_\tau(S;\Lambda) \\ {\scriptstyle \gamma^{\mathrm{eff}}_*}\uparrow\downarrow{\scriptstyle \gamma^*_{\mathrm{eff}}} & & {\scriptstyle \gamma_*}\uparrow\downarrow{\scriptstyle \gamma^*} \\ \mathrm{DM}^{\mathrm{eff}}_\tau(S;\Lambda) & \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} & \mathrm{DM}_\tau(S;\Lambda) \end{array}$$

where $\Sigma^\infty \dashv \Omega^\infty$, $\Sigma^\infty_{\mathbb{A}^1} \dashv \Omega^\infty_{\mathbb{A}^1}$, $\gamma^* \dashv \gamma_*$ and $\gamma^*_{\mathrm{eff}} \dashv \gamma^{\mathrm{eff}}_*$. Moreover, if S is excellent and geometrically unibranch (*e.g.* if $S = \mathrm{Spec}(\kappa)$) then one can combine [CD19, Theorem 16.1.4] and [CD19, Theorem 16.2.13] to see that γ_* gives a fully faithful embedding $\mathrm{DM}(S;\mathbb{Q}) \hookrightarrow \mathrm{DA}(S;\mathbb{Q})$, where as usual $\mathrm{DA}(S;\mathbb{Q}) := \mathrm{DA}_{\mathrm{Nis}}(S;\mathbb{Q})$. Moreover, [CD19, Theorem 16.2.18] shows that one has an equivalence of categories $\mathrm{DM}(S;\mathbb{Q}) \simeq \mathrm{DA}_{\text{ét}}(S;\mathbb{Q})$. This is crucially used in the Section 2.4, since the construction of regulator maps is easier if one works in the $\mathbb{A}^1$-homotopy category. Finally, let us mention that one can work with simplicial sheaves instead of sheaves of Λ-modules, to obtain the category $\mathrm{SH}_\tau(S;\Lambda)$ constructed by Morel and Voevodsky (see [MV99]). There is an adjuction $\mathrm{SH}_\tau(S;\Lambda) \rightleftarrows \mathrm{DA}_\tau(S;\Lambda)$, which becomes an equivalence if Λ is a $\mathbb{Q}$-algebra (see [CD19, p. 5.3.35]). This category admits also an ∞-categorical enhancement (see [Rob15, Definition 2.38]), which can be particularly useful to overcome the non-functoriality of cones.

2.3 Motivic cohomology

In the previous section we have defined various categories of motives, and in particular we have seen that the Beilinson-Deligne program for constructing an abelian category of mixed motives over any base S and with any ring of coefficients Λ would entail the construction of a triangulated category $\mathcal{T}(S;\Lambda)$ together with a functor $M_{\mathcal{T}}(-/S;\Lambda)\colon \mathcal{V}(S)^{\mathrm{op}} \to \mathcal{T}(S;\Lambda)$ which sends a scheme of finite type $X \to S$ to its motive $M_{\mathcal{T}}(X/S;\Lambda)$. This category should also be equipped with a tensor product and with a family of objects $\mathbb{1}_{\mathcal{T},S}(j) \in \mathcal{T}(S;\Lambda)$ such that $\mathbb{1}_{\mathcal{T},S} := \mathbb{1}_{\mathcal{T},S}(0) = M_{\mathcal{T}}(S/S)$ is the unit of the tensor product and $\mathbb{1}_{\mathcal{T}}(j) \otimes \mathbb{1}_{\mathcal{T}}(j') = \mathbb{1}_{\mathcal{T}}(j+j')$ for every $j, j' \in \mathbb{Z}$. Then, to conclude the program, one should define a t-structure on $\mathcal{T}(S;\Lambda)$ whose heart $\mathcal{MM}_S$ would be the abelian category of mixed motives. Now, if we only have the triangulated category $\mathcal{T}(S;\Lambda)$ mentioned above, we can still make sense of the following definition.

Definition 2.3.1 – Motivic cohomology

Let S be a scheme, and let $X \in \mathcal{V}(S)$ be an S-scheme of finite type. Then we define the Λ-linear *motivic cohomology* of S with respect to $\mathcal{T}$ as the modules

$$H^{i,j}_{\mathcal{T}}(S;\Lambda) := \mathrm{Hom}_{\mathcal{T}(S;\Lambda)}(\mathbb{1}_{\mathcal{T}}(j)[i], M_{\mathcal{T}}(S;\Lambda))$$

where $[i]\colon \mathcal{T}(S;\Lambda) \to \mathcal{T}(S;\Lambda)$ denote the shift functors.

In principle, the previous definition of motivic cohomology would depend on the base S. However, the categories $\mathcal{T}(X;\Lambda)$ should satisfy a version of Grothendieck's six-functor formalism (see [CD19, § A.5] for a review), which would in particular imply that for every $f\colon X \to S$ which is separated and of finite type there should be a functor $f^*\colon \mathcal{T}(S;\Lambda) \to \mathcal{T}(X;\Lambda)$ such

that $f^*(\mathbb{1}_{\mathcal{T},S}) = \mathbb{1}_{\mathcal{T},X}$, and this functor should admit a left adjoint $\mathbf{L}f_\#\colon \mathcal{T}(X;\Lambda) \to \mathcal{T}(S;\Lambda)$ which fits in the commutative square

$$\begin{array}{ccc} \mathcal{V}(X)^{\mathrm{op}} & \xrightarrow{M_{\mathcal{T}}(-/X;\Lambda)} & \mathcal{T}(X;\Lambda) \\ \downarrow & & \downarrow \mathbf{L}f_\# \\ \mathcal{V}(S)^{\mathrm{op}} & \xrightarrow[M_{\mathcal{T}}(-/S;\Lambda)]{} & \mathcal{T}(S;\Lambda) \end{array}$$

where the map $\mathcal{V}(X)^{\mathrm{op}} \to \mathcal{V}(S)^{\mathrm{op}}$ is the natural one, which considers a scheme $\pi\colon Y \to X$ as an S-scheme via the composition $f \circ \pi\colon Y \to S$. Using the adjunction property $f_\# \dashv f^*$ one can see that this formalism would imply that Definition 2.3.1 does not depend on the choice of a base scheme S (compare with [CD19, § 11.2.4]).

We have seen in the previous section that the insights of Voevodsky, Morel, Ayoub, Cisinski-Déglise and others have lead to the construction of many candidates for the category $\mathcal{T}(S;\Lambda)$: the category $\mathrm{DM}_{\mathrm{gm}}(S;\Lambda)^{\mathrm{op}}$ of mixed geometric motives, the category $\mathrm{DM}(S;\Lambda)^{\mathrm{op}}$ of Voevodsky motives and the $\mathbb{A}^1$-homotopy category $\mathrm{DA}(S;\Lambda)^{\mathrm{op}}$, which are constructed starting from smooth schemes of finite type over S, and their enlarged versions $\underline{\mathrm{DM}}(S;\Lambda)^{\mathrm{op}}$ and $\underline{\mathrm{DA}}(S;\Lambda)^{\mathrm{op}}$, constructed using separated schemes of finite type over S. To get the formalism described above, and in particular to obtain a functor $\mathbf{L}f_\#$ for every morphism $f\colon X \to S$ which is separated of finite type, the best choice is to take $\mathcal{T}(S;\Lambda) := \underline{\mathrm{DM}}(S;\Lambda)^{\mathrm{op}}$. However, since we have the fully faithful embedding $\varphi_!\colon \mathrm{DM}(S;\Lambda) \hookrightarrow \underline{\mathrm{DM}}(S;\Lambda)$, we can define motivic cohomology to be

$$H^{i,j}_{\mathcal{M}}(X;\Lambda) := \mathrm{Hom}_{\mathrm{DM}(X;\Lambda)}(\mathbb{1}_X, \mathbb{1}_X(j)[i]) = \mathrm{Hom}_{\underline{\mathrm{DM}}(X;\Lambda)}(\underline{\mathbb{1}}_X, \underline{\mathbb{1}}_X(j)[i]) \tag{2.18}$$

which is well defined for every scheme X. This definition agrees with [CD19, Definition 11.2.1] and for every separated morphism of finite type $f\colon X \to S$ one has that

$$H^{i,j}_{\mathcal{M}}(X;\Lambda) = \mathrm{Hom}_{\mathrm{DM}(S;\Lambda)}(\mathbb{1}_S, f_*(f^*(\mathbb{1}_S(j)[i]))) = \mathrm{Hom}_{\underline{\mathrm{DM}}(S;\Lambda)^{\mathrm{op}}}(\underline{\mathbb{1}}_S(j)[i], \underline{M}(X/S;\Lambda))$$

which shows that this definition agrees with Definition 2.3.1 if we take $\mathcal{T}(S;\Lambda) = \underline{\mathrm{DM}}(S;\Lambda)^{\mathrm{op}}$. We recall that the definition of motivic cohomology can be extended to diagrams of S-schemes. More precisely, one can define a category $\mathrm{DM}(D;\Lambda)$ associated to every diagram $D = (\mathcal{I}, \varphi)$, where $\mathcal{I}$ is a small category and $\varphi\colon \mathcal{I} \to \mathrm{Sch}_{/S}$ is a functor (see [CD19, § 3.1]), in such a way that $\mathrm{DM}(S;\Lambda) = \mathrm{DM}(D_S;\Lambda)$ where D_S is the diagram given by the constant functor $\varphi\colon \{*\} \to \mathrm{Sch}$ defined as $\varphi(*) := S$. For every morphism of diagrams $f\colon D \to D'$, which consists of a functor $f\colon \mathcal{I} \to \mathcal{I}'$ and a natural transformation $\varphi \to \varphi' \circ f$, one has a pair of adjoint functors $f^*\colon \mathrm{DM}(D';\Lambda) \rightleftarrows \mathrm{DM}(D;\Lambda)\colon f_*$. Hence one defines the motivic cohomology of every diagram D of S-schemes as

$$H^{i,j}_{\mathcal{M}}(D;\Lambda) := \mathrm{Hom}_{\mathrm{DM}(S;\Lambda)}(\mathbb{1}_S, \pi_*(\pi^*(\mathbb{1}_S(j)[i])))$$

where $\pi\colon D \to D_S$ denotes the structural morphism. Similarly to what we have seen in the case of a single scheme X, one can show that this definition of motivic cohomology does not depend on the base scheme, hence one might take $S = \mathrm{Spec}(\mathbb{Z})$. Moreover, every open immersion $j\colon U \hookrightarrow X$ can be considered as a diagram, and one denotes its motivic cohomology by $H^{n,m}_{\mathcal{M}}(X,U;\Lambda) := H^{n,m}_{\mathcal{M}}(j;\Lambda)$. One clearly has that $H^{n,m}_{\mathcal{M}}(X,\emptyset;\Lambda) = H^{n,m}_{\mathcal{M}}(X;\Lambda)$, and one can check that the motivic cohomology groups $H^{n,0}_{\mathcal{M}}(X,U;\Lambda)$ satisfy all the hypotheses of a mixed

Weil cohomology theory (see Definition 2.1.4) except from the Künneth formula. In particular, for every Nisnevich distinguished square (2.1) one gets a Mayer-Vietoris long exact sequence

$$\cdots \to H^{i,j}_{\mathcal{M}}(X';\Lambda) \xrightarrow{(j')^*\oplus f^*} H^{i,j}_{\mathcal{M}}(U';\Lambda)\oplus H^{i,j}_{\mathcal{M}}(X;\Lambda) \xrightarrow{(f')^*+j^*} H^{i,j}_{\mathcal{M}}(U;\Lambda) \xrightarrow{\partial} H^{i+1,j}_{\mathcal{M}}(X;\Lambda)\to\dots$$

coming from an exact triangle $M(U/S;\Lambda) \to M(U'/S;\Lambda)\oplus M(X/S;\Lambda) \to M(X'/S;\Lambda)$ in $\mathrm{DM}(S;\Lambda)$ (see [CD19, Remark 3.3.6]).

Let us mention that these definitions of motivic cohomology can be generalised by replacing $\mathrm{DM}(S;\Lambda)$ with some other triangulated category of mixed motives. One possible choice is given by taking the category $\mathrm{DM}_{\mathrm{cdh}}(X;\Lambda)$ of Voevodsky motives relative to the cdh-topology, which is obtained as a refinement of the Nisnevich topology by including among the distinguished squares the *abstract blow-ups*, which are Cartesian squares of the form

$$\begin{array}{ccc} E & \xrightarrow{\iota'} & \widetilde{X} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Z & \xhookrightarrow{\iota} & X \end{array} \tag{2.19}$$

where ι is a closed immersion, π is proper and the induced map $\pi^{-1}(X\setminus\iota(Z))\to X\setminus\iota(Z)$ is an isomorphism (see [CD19, Example 2.1.11]). Other options include the $\mathbb{A}^1$-homotopy category $\mathrm{DA}(S;\Lambda)$ and the category of Beilinson motives $\mathrm{DM}_{\mathrm{B}}(S;\Lambda)$, which is defined only when Λ is a $\mathbb{Q}$-algebra (see [CD19, § 14.2] for the precise definition). The three categories mentioned in this paragraph have the advantage that, under some suitable hypotheses, they satisfy all the axioms of the six functors formalism laid down in [CD19, § A.5], and in particular the localisation property.

This property asserts that, if $j\colon U\hookrightarrow X$ is an open immersion with complementary closed immersion $\iota\colon Z\hookrightarrow X$, one should have a distinguished triangle

$$j_!(j^!(M))\to M\to \iota_*(\iota^*(M))\to j_!(j^!(M))[1]$$

for every $M\in\mathcal{T}(X;\Lambda)$, where $\mathcal{T}(-;\Lambda)$ is a triangulated category of mixed homological motives. For $\mathcal{T}(S;\Lambda)=\mathrm{DM}(S;\Lambda)$ the validity of this property is equivalent to a deep conjecture of Voevodsky (see [CD19, Proposition 11.4.7]). Moreover, the triangulated categories $\mathrm{DA}(S;\Lambda)$ and $\mathrm{DM}_{\mathrm{B}}(S;\Lambda_{\mathbb{Q}})$ for a $\mathbb{Q}$-algebra $\Lambda_{\mathbb{Q}}$ always satisfy the localisation property (see [Ayo07, § 4.5.3] and [CD19, Corollary 14.2.11]), and the category $\mathrm{DM}_{\mathrm{cdh}}(S;\Lambda)$ satisfies this property if S is a Noetherian scheme of finite dimension over $\mathrm{Spec}(\kappa)$, where κ is either a field of characteristic zero or a field of characteristic $p>0$ such that $p\in\Lambda^\times$ (see [CD15, Theorem 5.11]). In particular, if $\mathcal{T}$ is one of the three last-mentioned categories and $H^{i,j}_{\mathcal{T}}$ denotes motivic cohomology with respect to this category (see Definition 2.3.1), which can be extended to diagrams as we did above, one has an identification between the relative cohomology $H^{n,m}_{\mathcal{T}}(X,U;\Lambda)$, where $j\colon U\hookrightarrow X$ is an open immersion, and the cohomology groups

$$H^{n,m}_{\mathcal{T},Z}(X;\Lambda) := \mathrm{Hom}_{\mathrm{DM}(X;\Lambda)}(\iota_*(\mathbb{1}_Z),\mathbb{1}_X(m)[n])$$

supported on the closed complement $\iota\colon Z\hookrightarrow X$. Using these functors, [DM15, Corollary 2.2.10] shows that the pair $(H^{\bullet,\bullet}_{\mathcal{T}}, H^{\mathrm{B.M.},\mathcal{T}}_{\bullet,\bullet})$ forms a twisted Poincaré duality theory in the sense of Definition 2.1.10, where for every separated S-scheme $f\colon X\to S$ of finite type we write

$$H^{\mathrm{B.M.},\mathcal{T}}_{i,j}(X/S;\Lambda) := \mathrm{Hom}_{\mathcal{T}(S;\Lambda)}(\mathbb{1}_{S,\mathcal{T}}, f_*(f^!(\mathbb{1}_{S,\mathcal{T}}(-j)[-i])))$$

for the Borel-Moore motivic homology groups (relative to our choice of $\mathcal{T}$) defined in [DM15, § 2.2.1]. Let us mention as well the existence of Mayer-Vietoris long exact sequences

$$\cdots \to H^{i,j}_{\mathcal{T}}(X;\Lambda) \xrightarrow{\iota^*\oplus\pi^*} H^{i,j}_{\mathcal{T}}(Z;\Lambda)\oplus H^{i,j}_{\mathcal{T}}(\widetilde{X};\Lambda) \xrightarrow{(\iota')^*+(\pi')^*} H^{i,j}_{\mathcal{T}}(E;\Lambda) \xrightarrow{\partial} H^{i+1,j}_{\mathcal{T}}(X;\Lambda) \to \dots \tag{2.20}$$

associated to each abstract blow-up (2.19) (see [CD19, Proposition 3.3.10]).

2.3.1 Relations with algebraic K-theory

The aim of the next three sections is to recall some of the conjectural and known ways of computing motivic cohomology. The first one comes from its relation to higher algebraic K-theory. This is an incredibly rich invariant, which can be associated to objects of a very different nature: rings, schemes, topological spaces, adic spaces, C^*-algebras and so on. The general idea is that the K-theory of an object X should capture equivalence classes of objects which live over X. For example, when X is a ring one looks at the category Mod_R of modules over that ring, and the equivalence relation is given by exact sequences in this category. For topological spaces, and schemes, the role of $\mathrm{Mod}(R)$ is played by the category $\mathrm{Vec}(X)$ of vector bundles over X, or the category $\mathrm{Perf}(X)$ of perfect complexes on X with globally finite Tor-amplitude (see [TT90, Definition 3.1]). These categories can be treated as Waldhausen categories, *i.e.* categories with a collection of cofibrations and weak equivalences. Using cofibrations, one can build a simplicial Waldhausen category $S_*(C)$ out of any Waldhausen category C, and then one defines the K-theory space $K(C) := \Omega|w(S_*(C))|$ to be the loop space of the geometric realisation of the simplicial sub-category of $S_*(C)$ whose morphisms are weak equivalences. We refer the interested reader to [Wei13, Chapter IV, § 8] for an exposition of Waldhausen's work, and to [BGT13, § 7.1] for a general construction in the context of ∞-categories, which allows one to prove a suitable universal property of algebraic K-theory. Then, one defines the algebraic K-theory groups of C as $K_n(C) := \pi_n(K(C))$, and one defines in this way the K-theory groups of a ring R or a scheme X by taking $C = \mathrm{Mod}(R)$ to be the category of modules or $C = \mathrm{Perf}(X)$ to be the category of perfect complexes on X having globally finite Tor-amplitude.

Computing these algebraic K-theory groups of schemes or rings has been notoriously difficult. We mention, among the very few cases in which these K-groups are completely known, the K-theory of a finite field $\mathbb{F}_q$ with q elements, given by $K_0(\mathbb{F}_q) = \mathbb{Z}$, $K_{2n}(\mathbb{F}_q) = 0$ and

$$K_{2n-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^n - 1)$$

for every $n \in \mathbb{Z}_{\geq 1}$ (see [Wei13, Chapter IV, Corollary 1.13]), and the relative K-theory groups $K_\bullet(A, I)$, where I is a nilpotent ideal (see [HM97], [AGH09], [HN20] and [Spe20] among others). The situation becomes much better if we consider the rational K-groups $K_\bullet(-)_\mathbb{Q} := K_\bullet(-)\otimes_\mathbb{Z}\mathbb{Q}$. The main computations which are known concern the K-theory of fields, in particular of a number field F. In this case Borel proved that $K_0(F)_\mathbb{Q} \cong \mathbb{Q}$, $K_{2n}(F)_\mathbb{Q} = 0$, $K_{4n-1}(F)_\mathbb{Q} \cong \mathbb{Q}^{r_2}$ and $K_{4n-3}(F)_\mathbb{Q} \cong \mathbb{Q}^{r_1+r_2}$ for every $n \in \mathbb{Z}_{\geq 1}$ (see [Bur02, Theorem 9.9]), where r_1 denotes the number of real embeddings of F and r_2 denotes the number of conjugate pairs of complex embeddings of F which are not real. As we see in next chapter, further work of Borel shows that these K-groups are also related to the special values of the Dedekind ζ-function $\zeta_F(s)$.

In order to see this result of Borel as a confirmation of Beilinson's conjectures on special values of L-functions in the case of number fields, one needs to relate the K-theory groups to motivic cohomology. To this end, let us recall that the direct sum $K_\bullet(X) := \bigoplus_n K_n(X)$ of the K-theory groups $K_n(X)$ associated to a scheme X has the structure of a graded ring (with respect to the derived $\otimes$-product of perfect complexes, as explained in [TT90, § 3.15]) and also

the structure of a graded λ-ring. This means that there exist operations $\lambda^k\colon K_\bullet(X)\to K_\bullet(X)$ for $k\in\mathbb{N}$, which are compatible with the grading, such that $\lambda^0(x)=1$, $\lambda^1(x)=x$ and

$$\lambda^k(x+y)=\sum_{i+j=k}\lambda^i(x)\lambda^j(y)$$

for every $x,y\in K_\bullet(X)$. Moreover, the map $\lambda_t\colon K_\bullet(X)\to\mathbb{W}(K_\bullet(X))$ to the ring of Witt vectors $\mathbb{W}(K_\bullet(X)):=1+K_\bullet(X)[\![t]\!]$ sending $x\in K_\bullet(X)$ to $\sum_k\lambda^k(x)t^k$ is a homomorphism of λ-rings. We refer the reader to [Wei13, Chapter II, § 4] for a detailed exposition of λ-rings and Witt vectors (see also [Bor11, § 1] for a modern account), and to [Lev97, § 5] for the proof of the existence of a λ-ring structure on the K-theory of a scheme, which essentially comes from the existence of exterior powers of vector bundles (at least in the case when X admits an ample family). We note also that these operations can be easily defined on $K_0(X)$ (which is also a λ-ring) using its explicit presentation, and then they can be extended to $K_n(X)$ if X is regular using [Rio10, Theorem 1.1.1]. In particular, the projection map $K_\bullet(X)\to K_0(X)$ is a morphism of λ-rings, and composing it with the map $K_0(X)\to H^0(X;\mathbb{Z})$ induced by taking the rank of a vector bundle one gets a map of λ-rings $\varepsilon\colon K_\bullet(X)\to H^0(X;\mathbb{Z})$ which is a section of the inclusion $H^0(X;\mathbb{Z})\subseteq K_0(X)\subseteq K_\bullet(X)$ and is an augmentation in the sense of [Wei13, Definition 4.2.1]. In any case, the λ-ring structure allows one to define the γ-operations $\gamma^k\colon K_\bullet(X)\to K_\bullet(X)$, by the formula $\gamma^k(x)=\lambda^k(x+k-1)$, and the Adams operations $\Psi^k\colon K_\bullet(X)\to K_\bullet(X)$ by the equality

$$\sum_{k=0}^{+\infty}\Psi^k(x)t^k=\varepsilon(x)-t\cdot\frac{d}{dt}\log(\lambda_{-t}(x))$$

which amounts to the inductive formulas $\Psi^k(x)=\sum_{j=1}^{k-1}(-1)^{j+1}\lambda^j(x)\Psi^{k-j}(x)-(-1)^k k\lambda^k(x)$ normalised by the initial conditions $\Psi^0(x)=\varepsilon(x)$, $\Psi^1(x)=x$ and $\Psi^2(x)=x^2-2\lambda^2(x)$. The γ-operations allow one to define a decreasing filtration $\mathrm{F}^\bullet_\gamma$ of ideals on $K_\bullet(X)$, by setting $\mathrm{F}^0_\gamma(K_\bullet(X)):=K_\bullet(X)$, $\mathrm{F}^1_\gamma(X):=\ker(\varepsilon)$ and

$$\mathrm{F}^n_\gamma(K_\bullet(X)):=\left\langle\left\{\gamma^{a_1}(x_1)\cdots\gamma^{a_m}(x_m)\middle|x_1,\dots,x_m\in\mathrm{F}^1_\gamma(X),\ \sum_{j=1}^m a_j\geq n\right\}\right\rangle$$

for every $n\geq 2$. Essentially by definition of the category of Beilinson motives $\mathrm{DM}_{\mathrm{B}}(S;\Lambda)$, one has a decomposition of the form

$$H^{i,j}_{\mathrm{B}}(X;\Lambda)\cong\mathrm{gr}^j_\gamma\left(K_{2j-i}(X)\otimes_{\mathbb{Z}}\Lambda\right)\tag{2.21}$$

which holds for every regular scheme X and every $\mathbb{Q}$-algebra Λ. Moreover, for every $n\in\mathbb{Z}_{\geq 1}$ one has also the decompositions

$$K_n(X)_{\mathbb{Q}}\cong\bigoplus_{m=1}^{+\infty}\mathrm{gr}^m_\gamma(K_n(X)_{\mathbb{Q}})\cong\bigoplus_{m=1}^{+\infty}H^{2m-n,n}_{\mathrm{B}}(X;\mathbb{Q})\tag{2.22}$$

and the various graded pieces $\mathrm{gr}^j_\gamma(K_n(X)_{\mathbb{Q}})$ can be identified with the k^j-th eigenspace for the Adams operation $\Psi^k\colon K_n(X)\to K_n(X)$ for any (or all) $k>1$. We refer the reader to [CD19, Corollary 14.2.14] for a proof of the identification (2.21), and to [Lev94, Lemma 2.1] for a proof of the decomposition (2.22).

2.3.2 Computing motivic cohomology: higher Chow groups

As we have seen in the previous section, the relations between motivic cohomology and K-theory groups allow one to use computations of the latter to understand more about the former. Nevertheless, this is arguably a very minor improvement in computability, since the definition of algebraic K-theory is also very abstract (as is the definition of motivic cohomology) and difficult to compute with. Moreover, any relation between K-theory and motivic cohomology holds only rationally, even in the case of regular schemes. The aim of this section is to briefly recall the theory of higher Chow groups, introduced by Bloch (see [Blo86a]) and developed by Totaro (see [Tot92]) and Levine (see [Lev94]). We present here the cubical theory for smooth schemes over an affine base $S = \mathrm{Spec}(R)$, where R is a field or more generally a Dedekind domain (see [Gei04]), following [Tot92]. This theory be extended to schemes of finite type (not necessarily smooth) over any base S by considering a complex of sheaves instead of a single complex of modules (see [Lev05, Chapter II, § 2.5]).

Let $S = \mathrm{Spec}(R)$ be the spectrum of a field or a Dedekind domain, and let $X \to S$ be a smooth S-scheme of finite type. Let $\square := \mathbb{P}^1 \setminus \{1\}$ and let $\mathbf{t} = (t_1, \dots, t_n)$ denote the coordinates on $\square^n$. Then the *faces* of $\square^n$ are the closed sub-schemes given by $t_i = 0$ or by $t_i = \infty$ for some $i \in \{1, \dots, n\}$, and one denotes by $\partial\square^n \subseteq \square^n$ the divisor given by the sum of the faces. Fixing a commutative ring with unity Λ, one defines $c^{i,j}(X/S;\Lambda)$ to be the free Λ-module generated by closed sub-schemes of $X \times_S \square^j$ having codimension i which meet $X \times_S \partial\square^j$ properly. Moreover, one defines $d^{i,j}(X/S;\Lambda) \subseteq c^{i,j}(X/S;\Lambda)$ to be the sub-module generated by the inverse images $\sigma_1^*(Z), \dots, \sigma_j^*(Z)$, where $Z \subseteq X \times_S \square^{j-1}$ runs over all the closed sub-schemes of codimension i meeting $X \times_S \partial\square^{j-1}$ properly, and for every $k \in \{1, \dots, j\}$ we let $\sigma_k \colon X \times_S \square^j \to X \times_S \square^{j-1}$ denote the map $\sigma_k(x, \mathbf{t}) = (x, (t_1, \dots, \hat{t_k}, \dots, t_j))$. Finally, one can put all the Λ-modules $z^{i,j}(X/S;\Lambda) := c^{i,j}(X/S;\Lambda)/d^{i,j}(X/S;\Lambda)$ into a homological complex $z^{i,\bullet}(X/S;\Lambda)$, whose differentials are induced by the maps

$$\begin{aligned} \partial_{i,j} \colon c^{i,j}(X/S;\Lambda) &\to c^{i,j-1}(X/S;\Lambda) \\ x &\mapsto \sum_{k=1}^{j} (-1)^k \cdot \left((\delta^{\infty}_{j,k})^*(x) - (\delta^{0}_{j,k})^*(x) \right) \end{aligned} \tag{2.23}$$

where $\delta^{\alpha}_{j,k} \colon X \times \square^{j-1} \to X \times \square^j$ is defined as $\delta^{\alpha}_{j,k}(x, \mathbf{t}) := (x, (t_1, \dots, t_{k-1}, \alpha, t_k, \dots, t_{j-1}))$ for every $\alpha \in \square$. Then the higher Chow groups of X over S are defined to be the homology groups $\mathrm{CH}^{i,j}(X/S;\Lambda) := H_j(z^{i,\bullet}(X/S;\Lambda))$. As we said in the previous paragraph, one can check that $z^{i,\bullet}(-;\Lambda)$ is a complex of sheaves for the étale topology (see [Gei04, Lemma 3.1] for a proof using the simplicial version of this complex), and one can define higher Chow groups for any scheme X as the hypercohomology groups of this complex of sheaves (see [Lev05, Chapter II, § 2.5] for details).

Now, it is evident that higher Chow groups give rise to a theory which is much more suitable for explicit computations than motivic cohomology (defined as groups of morphisms in a derived category) or algebraic K-theory. Nevertheless, one of the main results of Voevodsky is that, for every scheme X which is smooth, separated and of finite type over $S = \mathrm{Spec}(\kappa)$ where κ is a perfect field, one has that (see [CD19, Example 11.2.3])

$$H^{i,j}_{\mathcal{M}}(X;\Lambda) \cong \mathrm{CH}^{j,2j-i}(X;\Lambda) \tag{2.24}$$

which holds for any commutative ring with unity Λ (because it holds for $\Lambda = \mathbb{Z}$). Moreover, [Lev97, Corollary 8.2] shows that if X is also quasi-projective of dimension d then

$$\mathrm{CH}^{i,j}(X;\Lambda) \cong \mathrm{gr}^i_\gamma(K_j(X)) \otimes_{\mathbb{Z}} \Lambda$$

for every Λ such that $(d+j-1)! \in \Lambda^\times$. Finally, there is another version of motivic cohomology

$$H^{i,j}_{\mathcal{S}}(X;\Lambda) := \mathrm{Hom}_{\mathrm{SH}(X)}(\mathbb{1}_X, \Sigma^{i,j}(\mathbf{M}\Lambda_X))$$

defined using the spectra $\mathbf{M}\Lambda_X := f^*(\mathbf{M}\Lambda_{\mathrm{Spec}(\mathbb{Z})})$ obtained by pulling back along the structural morphism $f\colon X \to \mathrm{Spec}(\mathbb{Z})$ the spectrum $\mathbf{M}\Lambda_{\mathrm{Spec}(\mathbb{Z})} \in \mathrm{SH}(\mathrm{Spec}(\mathbb{Z}))$ introduced by Spitzweck (see [Spi18]). This new version of motivic cohomology, which should coincide with $H^{i,j}_{\mathrm{cdh}}$ under Voevodsky's conjecture (see [CD19, Remark 11.4.8]) has the property that

$$H^{i,j}_{\mathcal{S}}(X;\Lambda) \cong \mathrm{CH}^{j,2j-i}(X;\Lambda)$$

for every X which is smooth over $S = \mathrm{Spec}(R)$, where R is a Dedekind domain. This should be particularly promising for arithmetic applications: indeed every smooth and proper variety X over a number field K can be "spread out" to a smooth and proper $\mathcal{X}$ over the Dedekind domain $O_K[1/N]$ for some $N \in \mathbb{Z}_{\geq 1}$ (see [Poo17, Theorem 3.2.1]).

We conclude this section by recalling that the Λ-modules $z^{i,j}(X;\Lambda)$ can be replaced with suitable complexes of equidimensional cycles, introduced by Suslin (see [Sus00]).

2.3.3 Computing motivic cohomology: polylogarithmic motivic complexes

As we have seen in the previous section, higher Chow groups provide an explicit family of cochain complexes $\mathcal{Z}^{\bullet,j}(X;\Lambda) := z^{j,2j-\bullet}(X;\Lambda)$ of Λ-modules, whose i-th cohomology group should compute Λ-linear motivic cohomology $H^{i,j}_{\mathcal{M}}(X;\Lambda)$. The aim of this section is to introduce another family of cochain complexes $\mathcal{B}^{\bullet,j}(X;\Lambda)$, which are called Bloch group complexes or polylogarithmic motivic complexes, which should also compute the motivic cohomology of a scheme X, at least under suitable regularity assumptions. These complexes were introduced by Goncharov in [Gon95b] and their relation with the complexes $\mathcal{Z}^{\bullet,j}(X;\Lambda)$ should be thought of as analogous to the relation that elapses between Beilinson's and Zagier's conjectures on special values of L-functions. Let us say right from the start that, while the definition of these complexes is only conjectural and they exist only for regular schemes X, they are still a very interesting and completely explicit candidate for motivic cohomology. Moreover, as Goncharov claims in the introduction of [Gon05], these complexes are the smallest ones which can compute motivic cohomology. In particular, their cohomology $H^{i,j}_{\mathcal{B}}(X;\Lambda) := H^i(\mathcal{B}^{\bullet,j}(X;\Lambda))$ vanishes by definition for $i > j + \dim(X)$ for every smooth variety X over a field. The analogous statement for motivic cohomology is true, but not at all obvious (see [MVW06, Theorem 3.6]).

To define the complex $\mathcal{B}^{\bullet,j}(X;\Lambda)$ one starts by defining the Bloch groups $\mathcal{B}_n(F;\Lambda)$ associated to a field F and to an integer $n \in \mathbb{Z}$. These are defined as $\mathcal{B}_n(F;\Lambda) := 0$ if $n \leq 0$ and as $\mathcal{B}_n(F;\Lambda) := \Lambda[\mathbb{P}^1(F)]/\mathcal{R}_n(F;\Lambda)$ if $n \geq 1$. The sub-modules $\mathcal{R}_n(F;\Lambda) \subseteq \Lambda[\mathbb{P}^1(F)]$ are defined as

$$\begin{aligned}
\mathcal{R}_1(F;\Lambda) &:= \left\langle \{0\}, \{\infty\}, \{x\cdot y\} - \{x\} - \{y\} \;\middle|\; x,y \in F^\times \right\rangle_\Lambda \\
\mathcal{R}_n(F;\Lambda) &:= \left\langle \{0\}, \{\infty\}, \{f(1)\} - \{f(0)\} \;\middle|\; f \in \mathbb{P}^1(F(t)),\ \delta^{1,n}(\{f\}) = 0 \right\rangle_\Lambda
\end{aligned} \tag{2.25}$$

where $\{x\} \in \Lambda[\mathbb{P}^1(F)]$ denotes the generator of the free Λ-module $\Lambda[\mathbb{P}^1(F)]$ corresponding to $x \in \mathbb{P}^1(F)$. Moreover, for every $j > i \geq 1$ and any field K, we denote by $\delta^{i,j}$ the maps

$$\Lambda[\mathbb{P}^1(K)] \otimes_\Lambda \bigwedge^{i-1} K_\Lambda^\times \xrightarrow{\delta^{i,j}} \begin{cases} \bigwedge^j K_\Lambda^\times, \text{ if } i = j-1 \\ \mathcal{B}_{j-i}(K;\Lambda) \otimes_\Lambda \bigwedge^i K_\Lambda^\times, \text{ if } i < j-1 \end{cases}$$

$$\{x\} \otimes y \mapsto \begin{cases} (1-x) \wedge x \wedge y, & \text{if } i = j-1 \text{ and } x \in \mathbb{P}^1(K) \setminus \{0,1,\infty\} \\ \{x\}_{j-i} \otimes (x \wedge y), & \text{if } i < j-1 \text{ and } x \in \mathbb{P}^1(K) \setminus \{0,1,\infty\} \\ 0, & \text{if } x \in \{0,1,\infty\} \end{cases}$$

where $K_\Lambda^\times := K^\times \otimes_{\mathbb{Z}} \Lambda$ and $\{x\}_{j-i} \in \mathcal{B}_{j-i}(K;\Lambda)$ denotes the class of $\{x\} \in \Lambda[\mathbb{P}^1(K)]$ inside the quotient $\mathcal{B}_{j-i}(K;\Lambda)$. Note that the maps $\delta^{1,n}$ appearing in (2.25) are well defined because the module $\mathcal{B}_{n-1}(K;\Lambda)$ has been constructed in the previous inductive steps. Taking $K = F(t)$ allows one to define the modules $\mathcal{R}_n(F;\Lambda)$ and $\mathcal{B}_n(F;\Lambda)$, as we have seen above.

We are now ready to define the complexes $\mathcal{B}^{\bullet,j}(X;\Lambda)$ for $X = \mathrm{Spec}(F)$. First of all, one sets

$$\mathcal{B}^{i,j}(F;\Lambda) := \begin{cases} \mathcal{B}_{j-(i-1)}(F;\Lambda) \otimes_\Lambda \bigwedge^{i-1} F_\Lambda^\times, \text{ if } i \neq j \\ \bigwedge^j F_\Lambda^\times, \text{ if } i = j \end{cases}$$

which implies in particular that $\mathcal{B}^{i,j}(F;\Lambda) \neq 0$ only if $1 \leq i \leq j$. This agrees with the conventions $\mathcal{B}_r(F;\Lambda) = \bigwedge^s F = 0$ for every $r \leq 0$ and $s \leq -1$. The complex $\mathcal{B}^{\bullet,j}(F;\Lambda)$ is then defined as

$$\cdots \to 0 \to \mathcal{B}^{1,j}(F;\Lambda) \xrightarrow{\delta^{1,j}} \ldots \xrightarrow{\delta^{i-1,j}} \mathcal{B}^{i,j}(F;\Lambda) \xrightarrow{\delta^{i,j}} \ldots \xrightarrow{\delta^{j-1,j}} \mathcal{B}^{j,j}(F;\Lambda) \to 0 \to \ldots$$

which makes sense because $\delta^{i,j}(\mathcal{R}_{j-(i-1)}(K;\Lambda) \otimes \bigwedge^{i-1} K_\Lambda^\times) = 0$ for every field K (see [Gon95b, Lemma 1.16]).

Now, in order to be able to give the (conjectural) definition of $\mathcal{B}^{\bullet,j}(X;\Lambda)$ one needs to introduce the residue maps $\partial_v^{\bullet,j} : \mathcal{B}^{\bullet,j}(K;\Lambda) \to \mathcal{B}^{\bullet,j-1}(\kappa_v;\Lambda)[-1]$ associated to every discretely valued field (K,v) with residue field κ_v. These maps are defined as $\partial_v^{i,j} := s_v^{j-(i-1)} \otimes \theta_v^{i-1}$ when $1 \leq i < j$, and $\partial_v^{j,j} := \theta_v^j$. Here, for any $m \in \mathbb{N}$ the map $s_v^m : \mathcal{B}_m(K;\Lambda) \to \mathcal{B}_m(\kappa_v;\Lambda)$ is simply defined as $s_v^m(\{x\}_m) := \{\overline{x}\}_m$ for every $x \in \mathbb{P}^1(K)$, where $\overline{x} \in \mathbb{P}^1(\kappa_v)$ denotes the reduction of $x \in \mathbb{P}^1(K)$. In particular $\overline{x} = \infty$ if and only if $x = \infty$ or $x \in K$ and $v(x) < 0$. Moreover, θ_v^m is defined as

$$\begin{aligned} \bigwedge^m K_\Lambda^\times &\xrightarrow{\theta_v^m} \bigwedge^{m-1}(\kappa_v^\times)_\Lambda \\ y_1 \wedge \cdots \wedge y_m &\mapsto \sum_{i=1}^m (-1)^{i-1} \cdot v(y_i) \cdot \Big(\mathrm{res}_\pi(y_1) \wedge \cdots \wedge \widehat{\mathrm{res}_\pi(y_i)} \wedge \cdots \wedge \mathrm{res}_\pi(y_m)\Big) \end{aligned}$$

where $\pi \in K^\times$ is any uniformiser (*i.e.* $v(\pi) = 1$) and for any $y \in K^\times$ we write $\mathrm{res}_\pi(y) \in \kappa_v^\times$ for the reduction of the unit $y/\pi^{v(y)}$. It is now easy to see that, for every field K, there is a map

$$\frac{\mathcal{B}^{\bullet,j}(K(t);\Lambda)}{\mathcal{B}^{\bullet,j}(K;\Lambda)} \xrightarrow{\oplus_{\mathfrak{p}} \partial_{\mathfrak{p}}^{\bullet,j}} \bigoplus_{\mathfrak{p}} \mathcal{B}^{\bullet,j}\left(\frac{K[t]}{\mathfrak{p}};\Lambda\right)[-1] \tag{2.26}$$

where the sum runs over all the non-zero prime ideals $\mathfrak{p} \subseteq K[t]$. Then Goncharov makes the following conjecture, which is inspired by Milnor's theorem for algebraic K-theory (see [Wei13, Chapter III, Theorem 7.4]).

Conjecture 2.3.2 – Goncharov's homotopy invariance conjecture

For every field K the map (2.26) is a quasi-isomorphism of cochain complexes. In particular, for every finite extension $F \subseteq F'$ of fields, there exists a norm map

$$\mathrm{N}^{\bullet,j}_{F'/F} \colon \mathcal{B}^{\bullet,j}(F';\Lambda) \dashrightarrow \mathcal{B}^{\bullet,j}(F;\Lambda)$$

in the derived category $D(\mathrm{Mod}_\Lambda)$, uniquely determined by the property that

$$\partial^{\bullet,j}_{\infty} = -\sum_{\mathfrak{p}} \mathrm{N}^{\bullet,j}_{\mathfrak{p}} \circ \partial^{\bullet,j}_{\mathfrak{p}}$$

where for any non-zero prime ideal $\mathfrak{p} \subseteq K[t]$ the symbol $\mathrm{N}^{\bullet,j}_{\mathfrak{p}}$ denotes the norm relative to the finite extension $K \subseteq K[t]/\mathfrak{p}$.

Let us now move on to give the definition of $\mathcal{B}^{\bullet,j}(X;\Lambda)$ for any regular scheme X, assuming Conjecture 2.3.2. First of all, one defines the Λ-modules

$$\widetilde{\mathcal{B}}^{a,b,j}(X;\Lambda) := \bigoplus_{x\in X^{(a)}} \mathcal{B}^{b,j-a}(\kappa(x);\Lambda)[-a] = \bigoplus_{x\in X^{(a)}} \mathcal{B}_{j-(b-1)}(\kappa(x);\Lambda) \otimes_\Lambda \bigwedge^{b-(a+1)} \kappa(x)^\times_\Lambda$$

where $\kappa(x)$ denotes the residue field of a point $x \in X$ and $X^{(a)} \subseteq X$ denotes the set of points having codimension a. These modules should give rise to a family double complexes $\widetilde{\mathcal{B}}^{\bullet,\bullet,j}(X;\Lambda)$. The vertical differentials $\delta^{a,b,j} : \widetilde{\mathcal{B}}^{a,b,j}(X;\Lambda) \to \widetilde{\mathcal{B}}^{a,b+1,j}(X;\Lambda)$ in this family of double complexes are defined unconditionally as $\delta^{a,b,j} := \bigoplus_{x\in X^{(a)}} \delta^{b-a,j-a}_x$, where $\delta^{\bullet,j-a}_x$ are the differentials of the complex $\mathcal{B}^{\bullet,j-a}(\kappa(x);\Lambda)$. Then Goncharov assumes Conjecture 2.3.2 to define the horizontal differentials $\partial^{a,b,j} : \widetilde{\mathcal{B}}^{a,b,j}(X;\Lambda) \to \widetilde{\mathcal{B}}^{a+1,b,j}(X;\Lambda)$ by setting

$$\partial^{a,\bullet,j} = \bigoplus_{x\in X^{(a)}} \bigoplus_{y\in \mathrm{Cl}(x)} \partial^{\bullet,j-a}_{x,y}[-a]$$

where $\mathrm{Cl}(x)$ denotes the closure of x inside X. Here, $\partial^{\bullet,n}_{x,y}$ denotes, for every $n \in \mathbb{Z}_{\geq 2}$, a map of complexes $\partial^{\bullet,n}_{x,y} \colon \mathcal{B}^{\bullet,n}(\kappa(x);\Lambda) \dashrightarrow \mathcal{B}^{\bullet,n-1}(\kappa(y);\Lambda)[-1]$ which is defined for every pair of points $x, y \in X$ such that $y \in \mathrm{Cl}(x)$ as

$$\mathcal{B}^{\bullet,n}(\kappa(x);\Lambda) \xrightarrow{\bigoplus_{m=1}^{M} \partial^{\bullet,n}_{v_m}} \bigoplus_{m=1}^{M} \mathcal{B}^{\bullet,n-1}(\kappa(x)_m;\Lambda)[-1] \xrightarrow{\bigoplus_{m=1}^{M} \mathrm{N}^{\bullet,n-1}_{\kappa(x)_m/\kappa(y)}} \mathcal{B}^{\bullet,n-1}(\kappa(y))[-1]$$

where $v_1, \dots, v_M$ are the discrete valuations of $\kappa(x)$ which are trivial on $\kappa(y) \subseteq \kappa(x)$, and $\kappa(x)_1, \dots, \kappa(x)_M$ are their residue fields. Finally, having the double complex $\widetilde{\mathcal{B}}^{\bullet,\bullet,j}(X;\Lambda)$ at our disposal, we define $\mathcal{B}^{\bullet,j}(X;\Lambda)$ to be its total complex, and we write $H^{i,j}_{\mathcal{B}}(X;\Lambda) := H^i(\mathcal{B}^{\bullet,j}(X;\Lambda))$ for the cohomology of this cochain complex.

Remark 2.3.3. A brutal way to make unconditional the definition of the horizontal differentials in the double complex $\widetilde{\mathcal{B}}^{\bullet,\bullet,j}(X;\Lambda)$ would be to set $\partial^{\bullet,j}_{x,y} = 0$ unless there is exactly one valuation v of $\kappa(x)$ which is trivial on $\kappa(y)$, and $\kappa(x)_v = \kappa(y)$. This happens when $y \in \mathrm{Cl}(x)^{\mathrm{reg}}$ is a regular point of $\mathrm{Cl}(x)$. Hence one could try to use embedded resolution of singularities and theorems

of Bertini type to relate this new (well defined) complex to the conjecturally defined complex $\mathcal{B}^{\bullet,j}(X;\Lambda)$, when X is regular, separated and of finite type over a field F of characteristic zero.

Remark 2.3.4. The definition of $\mathcal{B}^{\bullet,j}(X;\Lambda)$ gives in general only a complex which is well-defined in the derived category $\mathrm{D}(\mathrm{Mod}_\Lambda)$. However, one could hope for the norm maps $\mathrm{N}^{\bullet,j}_{F'/F}$ to be defined as maps of complexes, even if the equality $\partial^{\bullet,j}_{\infty} = -\sum_{\mathfrak{p}} \mathrm{N}^{\bullet,j}_{\mathfrak{p}} \circ \partial^{\bullet,j}_{\mathfrak{p}}$ probably does not hold on the nose, but only up to quasi-isomorphism.

Remark 2.3.5. We note that the complexes $\mathcal{B}^{\bullet,j}(X;\Lambda)$ are defined unconditionally if X has dimension at most one over a field, or if $j \leq 3$, because in both cases the norm maps do not appear in the horizontal differentials of $\widetilde{\mathcal{B}}^{\bullet,\bullet,j}(X;\Lambda)$.

To conclude we remark that de Jeu has constructed in [DeJ95] another family of cochain complexes, the so called *wedge complexes* $\mathcal{W}^{\bullet,j}(X;\Lambda)$, which should compute motivic cohomology. These complexes are slightly more complicated to define than $\mathcal{B}^{\bullet,j}(X;\Lambda)$, but have the advantage of being directly related to the graded pieces of algebraic K-theory (see [DeJ95, Theorem 3.15]).

2.3.4 Computing motivic cohomology: low degrees

The aim of this section is to use some of the comparison isomorphisms and of the ideas introduced in the previous sections to compute some motivic cohomology groups in low degrees. First of all, let us recall the following computation, which deals with the groups $H^{i,0}$ and $H^{i,1}$, computed in the category of effective motives $\mathrm{DM}^{\mathrm{eff}}(X;\Lambda)$.

Proposition 2.3.6 – Motivic cohomology with twists 0 and 1

For every scheme X and every commutative ring with unity Λ one has that

$$H^{i,0}_{\mathrm{DM}^{\mathrm{eff}}}(X;\Lambda) := \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(X;\Lambda)}(M(X;\lambda), M(X;\Lambda)[i]) \cong \begin{cases} \Lambda^{\pi_0(X)}, & \text{if } i = 0 \\ 0, & \text{otherwise} \end{cases}$$

where $M(X;\Lambda) := \Lambda^{\mathrm{tr}}_X(X)_{\mathrm{Nis}}$, as we defined in (2.16). Moreover, if X is regular we have

$$H^{i,1}_{\mathrm{DM}^{\mathrm{eff}}}(X;\Lambda) := \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(X;\Lambda)}(M(X;\lambda), M(X;\Lambda)(1)[i]) \cong \begin{cases} O^\times(X) \otimes_{\mathbb{Z}} \Lambda, & \text{if } i = 1 \\ \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \Lambda, & \text{if } i = 2 \\ 0, & \text{otherwise} \end{cases}$$

and finally, if X is smooth over a field, the same computations hold true for the motivic cohomology groups $H^{i,j}_{\mathcal{M}}(X)$ and not only for their effective versions.

We refer the reader to [CD19, Theorem 11.2.14] and [MVW06, Corollary 4.2] for the proofs of these statements, and we remark that, for a smooth variety over a field, Proposition 2.3.6 can be deduced from the isomorphisms (2.24). For instance, the isomorphism $\mathrm{CH}^{1,1}(X;\Lambda) \cong O^\times(X) \otimes_{\mathbb{Z}} \Lambda$ is easily induced by the map which sends a function $f \in O^\times(X)$ to 0 if $f \equiv 1$ and to its graph $\Gamma_f := \{(x,t) \in X \times \square \mid f(x) = t\}$ otherwise. Note in particular that $\partial_{1,1}(\Gamma_f) = 0$ precisely because $f \in O^\times(X)$, *i.e.* f has no zeros nor poles.

Let us now recall what happens in the first range of indices not covered by Proposition 2.3.6, namely for the groups $H^{2,2}_{\mathcal{M}}(X;\Lambda)$. In this case we can be completely explicit when X is a regular curve over a number field, and $\Lambda = \mathbb{Q}$ (or, more generally, a $\mathbb{Q}$-algebra).

Proposition 2.3.7 – Motivic cohomology for curves over number fields

Let X be a regular and connected curve over a number field κ. Then we have the isomorphism

$$H^{2,2}_{\mathcal{M}}(X;\mathbb{Q}) \cong \ker\left(\begin{array}{rl} \dfrac{(\kappa(X)^\times \otimes_{\mathbb{Z}} \mathbb{Q})^{\otimes 2}}{\langle h \otimes (1-h) \colon h \in \kappa(X)^\times \setminus \{1\}\rangle} & \xrightarrow{\partial} \displaystyle\bigoplus_{x\in|X|} \kappa(x)^\times \otimes_{\mathbb{Z}} \mathbb{Q} \\ \{f,g\} & \mapsto \displaystyle\bigoplus_{x\in|X|} \partial_x(\{f,g\}) \end{array}\right) \tag{2.27}$$

where $\{f,g\}$ denotes the class of $f \otimes g$ in the quotient of $(\kappa(X)^\times \otimes_{\mathbb{Z}} \mathbb{Q})^{\otimes 2}$ appearing in (2.27), and

$$\partial_x(\{f,g\}) \coloneqq (-1)^{\operatorname{ord}_x(f)\operatorname{ord}_x(g)} \left.\frac{f^{\operatorname{ord}_x(g)}}{g^{\operatorname{ord}_x(f)}}\right|_x \tag{2.28}$$

for every closed point $x \in |X|$

Proof. The previous proposition can be proved in different ways, using the different techniques to compute motivic cohomology that we have recalled in the previous sections. For instance, let us start by saying that $H^{2,2}_{\mathcal{M}}(X;\mathbb{Q}) \cong H^{2,2}_{\mathrm{B}}(X;\mathbb{Q})$, since $S = \operatorname{Spec}(\kappa)$, or even $S = X$, are excellent and geometrically unibranch (see [CD19, Theorem 16.1.4]). Hence in particular we can use the isomorphism (2.21), together with Matsumoto's theorem on K_2 of a field (see [Wei13, Chapter II, Theorem 6.1]) and the localisation sequence for K-theory (see [Wei13, Chapter V, § 6.12]) to conclude.

However, a more intrinsic way to prove (2.3.7) is to use directly the localisation sequence for motivic cohomology, coming from the fact that motivic cohomology with rational coefficients is part of a twisted Poincaré duality theory in the sense of Definition 2.1.10. In particular, for every finite set of closed points $Y \subseteq |X|$ we get an exact sequence

$$0 \to H^{2,2}_{\mathcal{M}}(X) \to H^{2,2}_{\mathcal{M}}(X \setminus Y) \xrightarrow{\delta} H^{1,1}_{\mathcal{M}}(Y) \to H^{3,2}_{\mathcal{M}}(X) \to H^{3,2}_{\mathcal{M}}(X \setminus Y) \to 0 \tag{2.29}$$

where $H^{\bullet,\bullet}_{\mathcal{M}}(-)$ denotes motivic cohomology with rational coefficients. This comes from the localisation sequence relative to the closed immersion $Y \hookrightarrow X$ together with the fact that

$$H^{i,j}_{\mathcal{M}}(Y) \cong \bigoplus_{y\in|Y|} H^{i,j}_{\mathcal{M}}(\operatorname{Spec}(\kappa(y))) \tag{2.30}$$

for every $i, j \in \mathbb{Z}$ because motivic cohomology commutes with disjoint unions (as a very particular case of Nisnevich descent), which shows that $H^{i,j}_{\mathcal{M}}(Y) = 0$ if $2 \mid i$, as a consequence of

Borel's theorem on the K-theory of number fields (see Section 2.3.1). Now, we can let Y grow in the exact sequence (2.30). Combining this with the identification

$$H^{1,1}_{\mathcal{M}}(Y) \overset{(2.30)}{\cong} \bigoplus_{y\in|Y|} H^{1,1}_{\mathcal{M}}(\mathrm{Spec}(\kappa(y))) \cong \bigoplus_{y\in|Y|} \kappa(y)^{\times}\otimes_{\mathbb{Z}}\mathbb{Q} \tag{2.31}$$

provided by Proposition 2.3.6, we get an exact sequence

$$0 \to H^{2,2}_{\mathcal{M}}(X) \to H^{2,2}_{\mathcal{M}}(\xi_X) \xrightarrow{\delta} \bigoplus_{x\in|X|} \kappa(x)^{\times}\otimes_{\mathbb{Z}}\mathbb{Q} \to \dots$$

where $\xi_X \in X$ denotes the generic point. We can now get the isomorphism (2.27) using the identification

$$H^{2,2}_{\mathcal{M}}(\xi_X) \cong H^{2,2}_{\mathcal{M}}(\mathrm{Spec}(\kappa(X))) \cong \frac{(\kappa(X)^{\times}\otimes_{\mathbb{Z}}\mathbb{Q})^{\otimes 2}}{\langle x\otimes(1-x) : x\notin\{0,1\}\rangle}$$

provided by Matsumoto's theorem. □

Remark 2.3.8. Proposition 2.3.7 is extensively used in Chapter 9, to explicitly construct elements in the motivic cohomology group $H^{2,2}_{\mathcal{M}}(E)$ associated to a CM elliptic curve E defined over $\mathbb{Q}$. Proposition 2.3.7 is also used in Section 4.4.1, to prove that Boyd's conjectures for a given two-variable polynomial families follow from Beilinson's ones.

From now until the end of this section, we use the notation $H^{\bullet,\bullet}_{\mathcal{M}}$ to denote motivic cohomology with rational coefficients, as we did in the proof of Proposition 2.3.7.

Remark 2.3.9. We note that for every finite extension of fields $F \subseteq F'$ there exists a norm map

$$\mathrm{N}_{F'/F} \colon H^{\bullet,j}_{\mathcal{M}}(F') \to H^{\bullet,j}_{\mathcal{M}}(F)$$

where $H^{\bullet,j}_{\mathcal{M}}(F') := H^{\bullet,j}_{\mathcal{M}}(\mathrm{Spec}(F'))$ and $H^{\bullet,j}_{\mathcal{M}}(F) := H^{\bullet,j}_{\mathcal{M}}(\mathrm{Spec}(F))$. This norm map is given by the push-forward along the morphism $\mathrm{Spec}(F') \to \mathrm{Spec}(F)$ induced by the inclusion $F \subseteq F'$.

Before moving on, we record one interesting feature of the localisation sequence (2.29).

Lemma 2.3.10 – Weil's reciprocity law

Let X be a regular, connected, projective curve defined over a number field κ, and let $Y \hookrightarrow X$ be a closed finite sub-scheme. Then for every $j \in \mathbb{N}$ we have that

$$\sum_{y\in Y} \mathrm{N}^{1,j-1}_{\kappa(y)/\kappa} \circ \partial^{2,j}_{y} = 0 \tag{2.32}$$

where $\partial^{2,j}_{y} \colon H^{2,j}_{\mathcal{M}}(X\setminus Y) \to H^{1,j-1}_{\mathcal{M}}(\kappa(y))$ is the map appearing in the localisation sequence

$$\cdots \to H^{2,j}_{\mathcal{M}}(X) \to H^{2,j}_{\mathcal{M}}(X\setminus Y) \xrightarrow{\oplus_y \partial^{2,j}_y} \bigoplus_y H^{1,j-1}_{\mathcal{M}}(\kappa(y)) \xrightarrow{\oplus_y (\iota_y)_*} H^{3,j}_{\mathcal{M}}(X) \to \dots \tag{2.33}$$

Proof. Observe that $(\iota_y)_*\colon H^{1,j-1}_{\mathcal{M}}(\kappa(y)) \to H^{3,j}_{\mathcal{M}}(X)$ is the push-forward map associated to the inclusion $\iota_y\colon \operatorname{Spec}(\kappa(y)) \to X$ giving rise to the point $y \in X$. This fact gives us the equality

$$\pi_* \circ (\iota_y)_* = \mathrm{N}_{\kappa(y)/\kappa} \tag{2.34}$$

by the functoriality of push-forwards, where $\pi\colon X \to \operatorname{Spec}(\kappa)$ is the structural morphism. Finally (2.34) gives the equality (2.32) using the fact that the localisation sequence (2.33) is exact. □

Remark 2.3.11. We note that the previous proof works only in the smooth projective case because the existence of π_* is not guaranteed otherwise (see [CD19, § 11.3.4]).

Remark 2.3.12. The usual Weil reciprocity law deals with the motivic cohomology (or the K-theory) of the function field of the curve X (see for instance [Wei13, § 6.12.1]). One gets this version of the Weil reciprocity law by letting Y grow and taking the limit of the equality (2.32).

To conclude this section, we show how to use what we just recalled to construct elements in the motivic cohomology of a curve.

Proposition 2.3.13 – Constructing motivic cohomology classes on a curve

Let X be a regular, connected, projective curve over a number field κ and let $Y \subseteq X(\overline{\kappa})$ be a finite set of closed points. Assume that there exists $y_0 \in X(\kappa)$ such that $y - y_0 \in (J_X)(\overline{\kappa})_{\mathrm{tors}}$ for every $y \in Y$, where J_X denotes the Jacobian of X. Then the natural restriction map $H^{2,2}_{\mathcal{M}}(X) \to H^{2,2}_{\mathcal{M}}(X \setminus Y)$ admits a natural retraction $H^{2,2}_{\mathcal{M}}(X \setminus Y) \twoheadrightarrow H^{2,2}_{\mathcal{M}}(X)$.

Proof. Let $\kappa' \supseteq \kappa$ be a finite Galois extension, such that all the points of Y are κ'-rational. Then the identification (2.31) gives the isomorphism

$$H^{1,1}_{\mathcal{M}}(Y_{\kappa'}) \cong \bigoplus_{y \in |Y|} (\kappa')^\times \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[Y] \otimes_{\mathbb{Z}} (\kappa')^\times \tag{2.35}$$

where $\mathbb{Q}[Y]$ denotes the group of divisors with rational coefficients which are supported on Y. Now the exact sequence (2.29) induces a short exact sequence

$$0 \to H^{2,2}_{\mathcal{M}}(X_{\kappa'}) \to H^{2,2}_{\mathcal{M}}((X \setminus Y)_{\kappa'}) \xrightarrow{\delta'} \operatorname{Im}(\delta') \to 0$$

and using Lemma 2.3.10 we can see that, under the isomorphism (2.35), we have that

$$\operatorname{Im}(\delta') \subseteq \mathbb{Q}[Y]^0 \otimes_{\mathbb{Z}} (\kappa')^\times$$

where $\mathbb{Q}[Y]^0 \subseteq \mathbb{Q}[Y]$ denotes the $\mathbb{Q}$-vector space of divisors of degree zero. Moreover, we also have that $\operatorname{Im}(\delta') = \mathbb{Q}[Y]^0 \otimes_{\mathbb{Z}} (\kappa')^\times$, because δ' fits into the commutative diagram

$$\begin{array}{ccc} H^{2,2}_{\mathcal{M}}((X \setminus Y)_{\kappa'}) & \xrightarrow{\quad\delta'\quad} & \mathbb{Q}[Y]^0 \otimes_{\mathbb{Z}} (\kappa')^\times \\ \cup\uparrow & & \uparrow{\scriptstyle \operatorname{div} \otimes \mathrm{Id}_{\kappa'}} \\ H^{1,1}_{\mathcal{M}}((X \setminus Y)_{\kappa'}) \otimes H^{1,1}_{\mathcal{M}}(\operatorname{Spec}(\kappa')) & \xrightarrow{\sim} & (O^\times((X \setminus Y)_{\kappa'}) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Z}} (\kappa')^\times \end{array}$$

and the divisor map $(\mathcal{O}^\times((X \setminus Y)_{\kappa'}) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Z}} (\kappa')^\times \xrightarrow{\mathrm{div} \otimes \mathrm{Id}_{\kappa'}} \mathbb{Q}[Y]^0 \otimes_{\mathbb{Z}} (\kappa')^\times$ is surjective. This follows from the fact that we are taking rational coefficients, together with the assumption that there exists a point $y_0 \in Y$ such that $y - y_0 \in (J_X(\overline{\kappa}))_{\mathrm{tors}}$ for every $y \in Y$.

We have shown that

$$H^{2,2}_{\mathcal{M}}((X \setminus Y)_{\kappa'}) \cong H^{2,2}_{\mathcal{M}}(X_{\kappa'}) \oplus \{\mathcal{O}^\times((X \setminus Y)_{\kappa'}), (\kappa')^\times\}$$

where $\{\mathcal{O}^\times((X\setminus Y)_{\kappa'}), (\kappa')^\times\} \subseteq H^{2,2}_{\mathcal{M}}((X\setminus Y)_{\kappa'})$ denotes the subspace of symbols $\{f, c\} = \{f\} \cup \{c\}$ where $f \in \mathcal{O}^\times((X \setminus Y)_{\kappa'})$ and $c \in (\kappa')^\times$ is a constant. To conclude we can use Galois descent for motivic cohomology (see [CD19, Theorem 14.3.4]) to get an isomorphism

$$H^{2,2}_{\mathcal{M}}(X \setminus Y) \cong H^{2,2}_{\mathcal{M}}(X) \oplus \psi_*(\{\mathcal{O}^\times((X \setminus Y)_{\kappa'}), (\kappa')^\times\}) \tag{2.36}$$

where $\psi\colon (X \setminus Y)_{\kappa'} \to X \setminus Y$ denotes the Galois covering induced by base change. Then, the retraction $H^{2,2}_{\mathcal{M}}(X \setminus Y) \twoheadrightarrow H^{2,2}_{\mathcal{M}}(X)$ is simply given by the projection onto the first factor in the decomposition (2.36). □

We can now use the retraction $H^{2,2}_{\mathcal{M}}(X \setminus Y) \twoheadrightarrow H^{2,2}_{\mathcal{M}}(X)$ given by Proposition 2.3.13 to get a map

$$\begin{array}{ccccccc} \mathcal{O}^\times(X \setminus Y)^{\otimes 2} \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{\sim} & H^{1,1}_{\mathcal{M}}(X \setminus Y) & \xrightarrow{\cup} & H^{2,2}_{\mathcal{M}}(X \setminus Y) & \twoheadrightarrow & H^{2,2}_{\mathcal{M}}(X) \\ f \otimes g & \longmapsto & & & & & \eta_{f,g} \end{array} \tag{2.37}$$

which can be used to construct elements in motivic cohomology. In the case of elliptic curves, (2.37) can be made more explicit, as the following example, due to Bloch (see [Blo00, Proposition 10.1.1]), shows.

Example 2.3.14 (Bloch's trick)**.** Let E be an elliptic curve defined over a number field κ. Fix two functions $f, g\colon E \to \mathbb{P}^1$, and let $S_{f,g} \subseteq E(\overline{\kappa})$ denote the set of their zeros and poles. Suppose that $S_{f,g} \subseteq E_{\mathrm{tors}}$, where $E_{\mathrm{tors}} := E(\overline{\kappa})_{\mathrm{tors}}$ denotes the set of torsion points of E defined over the algebraic closure $\overline{\kappa}$, and suppose as well that both f and g have the origin $0 \in E$ as their unique pole. Then we have that

$$\eta_{f,g} := n_{f,g}\,\{f, g\} - \sum_{x \in S_{f,g} \setminus \{0\}} \{\partial_x(\{f, g\}), \varphi^{(x)}_{f,g}\} \tag{2.38}$$

where $n_{f,g} := |\mathrm{lcm}\{\mathrm{ord}_z(f), \mathrm{ord}_z(g) \mid z \in S_{f,g} \setminus \{0\}\}| \in \mathbb{Z}_{\geq 1}$. Moreover, ∂_x denotes the map defined in Proposition 2.3.7, and for every $x \in S_{f,g}$ the notation $\varphi^{(x)}_{f,g}$ stands for any function $\varphi^{(x)}_{f,g}\colon E \to \mathbb{P}^1$ defined over κ such that $\mathrm{div}(\varphi^{(x)}_{f,g}) = n_{f,g} \cdot ((x) - (0))$.

It is now easy to see that η is bilinear, alternating and invariant by scaling, *i.e.*

$$\eta_{fg,h} = \eta_{f,h} + \eta_{g,h}, \quad \text{and} \quad \eta_{f,g} = -\eta_{g,f} \quad \text{and} \quad \eta_{c,f} = 0$$

for every $f, g \in E \to \mathbb{P}^1_\kappa$ and $c \in \kappa$. This shows that we have an alternating, bilinear pairing

$$\begin{array}{rcl} [\,,\,]_{\mathcal{M}}\colon \bigwedge^2 \mathbb{Q}[E(\overline{\kappa})_{\mathrm{tors}}]^{0,\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} & \to & H^{2,2}_{\mathcal{M}}(E) \\ D_1 \wedge D_2 & \mapsto & \eta_{f_1,f_2} \end{array} \tag{2.39}$$

where $f_1, f_2 \colon E \to \mathbb{P}^1$ are any two functions such that $\mathrm{div}(f_j) = \mathrm{ord}(D_j)\, D_j$. Here, we define the order $\mathrm{ord}(D) \in \mathbb{N}$ of a divisor $D \in \mathbb{Q}[E(\overline{\kappa})_{\mathrm{tors}}]^0$ to be the smallest natural number $n \in \mathbb{N}$ such that $n \cdot D$ is a principal divisor. Finally, we observe that

$$[D_1, D_2]_{\mathcal{M}} = \mathrm{ord}(D_1)\ \mathrm{ord}(D_2)\ [D_1, D_2]_{\mathcal{A}}$$

for every $D_1, D_2 \in \mathbb{Q}[E(\overline{\mathbb{Q}})_{\mathrm{tors}}]^{0,\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$, where $[\,,\,]_{\mathcal{A}}$ is the pairing defined in [DW88, Theorem 5.1].

Remark 2.3.15. The map (2.37) has a natural generalisation in the context of polylogarithmic motivic complexes, at least conjecturally. More precisely, every pair of functions $f, g \in \kappa(X)^\times$ defines an element $\{f\}_{n-1} \otimes g \in \mathcal{B}^{2,n}(X;\mathbb{Q})$. For this element to define a cohomology class $\{f,g\}_n \in H^{2,n}_{\mathcal{B}}(X;\mathbb{Q})$ it is necessary and sufficient to show that

$$\delta^{2,n}(\{f\}_{n-1} \otimes g) = \partial^{2,n}(\{f\}_{n-1} \otimes g) = 0$$

as it is easily seen by specialising the construction of the complex $\mathcal{B}^{\bullet,n}(X;\mathbb{Q})$ to the case of smooth curves. Let now $Y \subseteq X(\overline{\kappa})$ be the set of zeros and poles of f and g, and assume that there exists $y_0 \in X(\kappa)$ such that $y - y_0 \in J_X(\overline{\kappa})_{\mathrm{tors}}$ for every $y \in Y$. Then one can subtract from $\{f\}_{n-1} \otimes g$ some elements of the form $\{a\}_{n-1} \otimes h$, where $a \in \overline{\kappa}$ is a constant and $h \in \kappa(X)^\times$, to get a new element $\eta^{(n)}_{f,g} \in \mathcal{B}_{n-1}(X) \otimes \kappa(X)^\times$ such that $\partial^{2,n}(\eta^{(n)}_{f,g}) = 0$. The proof uses the same ideas appearing in the proof of Proposition 2.3.13 (which is recovered as the case $n = 2$), together with the conjectural formula

$$\sum_{x \in X(\overline{\kappa})} \partial^{2,n}_x(\{f\}_n \otimes g) = 0 \in \mathcal{B}_{n-1}(K')$$

which is the equivalent of Weil's reciprocity formula for the polylogarithmic motivic complexes (see [Rud18] for partial results towards the validity of this formula).

2.4 Regulators

We have seen in the previous sections that the motivic cohomology $H^{\bullet,\bullet}_{\mathcal{M}}(X;\Lambda)$ of a scheme X with coefficients in a ring Λ is an incredibly rich invariant, whose definition requires a conspicuous amount of setup. Our lack of knowledge about motivic cohomology is also extremely tantalising: on the one hand it is quite difficult to construct motivic cohomology classes, with the notable exception of $H^{1,1}_{\mathcal{M}}(X;\Lambda) \cong \mathcal{O}^\times(X) \otimes_{\mathbb{Z}} \Lambda$, and on the other hand it is completely out of reach (as of today) to prove that motivic cohomology groups are finitely generated, with the notable exception of the theorems of Borel. It is therefore natural to attempt to relate the motivic cohomology groups of a scheme X to more computable invariants, given for instance by the cohomology theories described in Section 2.1. These relations have the form of *regulator maps*

$$r_? \colon H^{\bullet,\bullet}_{\mathcal{M}}(-;\Lambda) \to H^{\bullet,\bullet}_?(-;\Lambda)$$

between Λ-linear motivic cohomology and some other Λ-linear bi-graded cohomology theory satisfying the axioms of Section 2.1. Regulators can be constructed in one of the following ways:

- from *realisation functors* $R_? \colon \mathrm{DM}(S;\Lambda) \to \mathcal{T}_?$, where $\mathcal{T}$ is a triangulated category with Tate twists, whose homomorphisms compute the cohomology theory $H^{\bullet,\bullet}_?$ (see Remark 2.1.18);

- from *unit maps of ring spectra*, whenever $H_?^{\bullet,\bullet}$ is represented by a spectrum $\mathbb{E}_? \in \mathrm{DA}(S;\Lambda)$. More precisely, suppose that there exists a monoid object $\mathbb{E}_? \in \mathrm{DA}(S;\Lambda)$ (usually called a *ring spectrum*) and a functorial family of isomorphisms

$$H_?^{i,j}(X) \cong \mathrm{Hom}_{\mathrm{DA}(S;\Lambda)}(M_{\mathbb{A}^1}(X/S;\Lambda), \mathbb{E}_?(j)[i])$$

for every $X \in \mathcal{S}m_S$, where $M_{\mathbb{A}^1}(X/S,-)\colon \mathcal{S}m_S \to \mathrm{DA}(S;\Lambda)$ denotes the functor which sends an S-scheme to its motive in the $\mathbb{A}^1$-homotopy category $\mathrm{DA}(S;\Lambda)$ (see Section 2.2.3). Then the unit map $\eta\colon \mathbb{1}_S \to \mathbb{E}_?$ induces maps $\mathbb{1}_S(j)[i] \to \mathbb{E}_?(j)[i]$. Applying the covariant functor $\mathrm{Hom}_{\mathrm{DA}(S;\lambda)}(M_{\mathbb{A}^1}(X/S;\Lambda),-)$ to these maps, one gets regulator maps

$$r_X^{\mathbb{E}_?}\colon H^{i,j}_{\mathcal{M},\mathbb{A}^1}(X;\Lambda) \to H_?^{i,j}(X;\Lambda)$$

where $H^{i,j}_{\mathcal{M},\mathbb{A}^1}(X;\Lambda) := \mathrm{Hom}_{\mathrm{DA}(S;\Lambda)}(M_{\mathbb{A}^1}(X/S;\Lambda), \mathbb{1}_S(j)[i])$ denotes motivic cohomology computed in the $\mathbb{A}^1$-homotopy category. We recall that this coincides with the motivic cohomology $H^{i,j}_{\mathcal{M}}(X;\Lambda)$ computed in $\mathrm{DM}(S;\Lambda)$ if S is excellent and geometrically unibranch (*e.g.* $S = \mathrm{Spec}(\kappa)$ for a field κ) and Λ is a $\mathbb{Q}$-algebra;

- as a *Chern character*, using the fact that motivic cohomology is related to K-theory (see Section 2.3.1). More precisely, the Chern character is a family of natural transformations $\mathrm{ch}_{i,j}\colon K_{2j-i}(-) \to H_?^{i,j}(-)$, from which we get a regulator map using the isomorphism (2.21);
- as *cycle class maps* (sometimes called *Abel-Jacobi maps*), using the relation between motivic cohomology and higher Chow groups (see Section 2.3.2). More precisely, the regulator is induced by maps of complexes of sheaves $\mathcal{Z}^{\bullet,j} \to E_j^?$, where $E_j^?$ are complexes of sheaves on $\mathcal{S}m_S$ such that $H_?^{i,j}(X) \cong H^i(E_j^?(X))$ for every $X \in \mathcal{S}m_S$;
- as *higher polylogarithms*, using the conjectural relations between motivic cohomology and the cohomology of polylogarithmic motivic complexes (see Section 2.3.3). More precisely, the regulator is induced by a map of complexes of sheaves $\mathcal{B}^{\bullet,j} \to E_j^?$, where again $E_j^?$ are complexes of sheaves on $\mathcal{S}m_S$ such that $H_?^{i,j}(X) \cong H^i(E_j(X))$ for every $X \in \mathcal{S}m_S$.

As it is probably evident to the reader, the first three approaches are the most general, and they are the best to ensure the naturality properties of regulator maps, whereas the last two approaches are best for explicit computations.

The first approach, which constructs regulators from realisation functors, has been pursued by a great number of people. Let us mention the works of Huber (see [Hub00] and [Hub04]), Ayoub (see [Ayo14a]), Lecomte and Wach (see [LW09] and [LW13]) and Ivorra (see [Ivo07], [Ivo10] and [Ivo16]). We do not use this approach in this thesis, except from mentioning it in passing in Remark 2.4.7.

Remark 2.4.1. Suppose that a bi-graded cohomology theory $H_?^{\bullet,\bullet}$ is represented by a ring spectrum $\mathbb{E}_? \in \mathrm{DA}(S;\Lambda)$, *i.e.* suppose that we can apply the second approach to the construction of a regulator map. Then we can define $\mathcal{T}_?$ to be the triangulated category of $\mathbb{E}_?$-modules inside $\mathrm{DA}(S;\Lambda)$, and we can define a realisation functor $R_?\colon \mathrm{DA}(S;\Lambda) \to \mathcal{T}_?$ by setting $R_?(M) := M \otimes \mathbb{E}_?$ for every $M \in \mathrm{DA}(S;\Lambda)$. This shows that each time a cohomology theory is represented by a motivic spectrum, it can also be obtained as homomorphisms in a category of realisations. Of course, the construction of the category $\mathcal{T}_?$ is of a non explicit nature, which is in stark contrast to the usual categories of coefficients, like the category of mixed Hodge structures (or mixed Hodge modules), Ekhedal's category for ℓ-adic sheaves and so on.

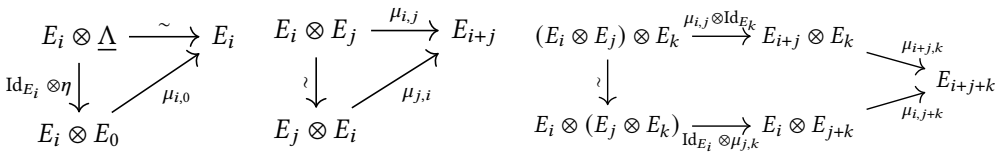
The previous remark shows that the first approach to construct regulator maps is somehow a special case of the second. The following result, due to Déglise and Mazzari (see [DM15, Proposition 1.4.10]), allows one to represent a Λ-linear cohomology theory $H_?^{\bullet,\bullet}$ by a motivic spectrum $\mathbb{E}_? \in \mathrm{DA}(S;\Lambda)$, as long as this cohomology theory is defined by a family $E_j^?$ of Λ-linear Nisnevich sheaves on $\mathcal{S}m_S$, via the formula $H_?^{i,j}(X) := H^i(E_j^?(X))$.

Theorem 2.4.2 – Motivic spectra associated to cohomology theories

Let Λ be a $\mathbb{Q}$-algebra, and let $(E_j)_{j\in\mathbb{N}} \subseteq C(\mathrm{Sh}_{\mathrm{Nis}}(S;\Lambda))$ be a family of complexes of Nisnevich sheaves $E_j : \mathcal{S}m_S^{\mathrm{op}} \to \mathrm{Mod}_\Lambda$ on the category of smooth schemes of finite type over a Noetherian, finite dimensional base S. Suppose that:

- the cohomology of each complex E_j is $\mathbb{A}^1$-homotopy invariant, *i.e.* for every scheme $X \in \mathcal{S}m_S$ the canonical map $\mathbb{A}^1_X \to X$ induces isomorphisms
$$H^i(E_j(X)) \xrightarrow{\sim} H^i(E_j(\mathbb{A}^1_X))$$
for every $i, j \in \mathbb{Z}$;
- there exists a unit map $\eta : \underline{\Lambda} \to E_0$ (where $\underline{\Lambda} \in C(\mathrm{PSh}(S;\Lambda))$ denotes the constant presheaf concentrated in degree zero) and a family of product maps $\mu_{i,j} : E_i \otimes E_j \to E_{i+j}$ for every $i, j \in \mathbb{N}$, such that the diagrams appearing in Figure 2.2 commute;
- there exists a map $c : \Lambda(\mathbb{G}_{m,S}) \to E_1[1]$, where $\Lambda(\mathbb{G}_{m,S}) \in \mathrm{PSh}(S;\Lambda)$ denotes the presheaf given by $\Lambda(\mathbb{G}_{m,S})(X) := \mathrm{Hom}(X, \mathbb{G}_{m,S}) \otimes_{\mathbb{Z}} \Lambda$. Moreover, for every $X \in \mathcal{S}m_S$ and every $i, j \in \mathbb{N}$ the map
$$H^i(E_j(X)) \xrightarrow{\cdot\times\overline{c}} H^{i+1}(E_{j+1}(X \times_S \mathbb{G}_{m,S})) \twoheadrightarrow \mathrm{Coker}(\pi^*)$$
is an isomorphism, where $\overline{c} := c(\mathrm{Id}_{\mathbb{G}_{m,S}}) \in H^1(E_1(\mathbb{G}_m))$, $\pi : X \times_S \mathbb{G}_{m,S} \to X$ is the canonical projection and $\times$ denotes the exterior product
$$H^i(E_j(X)) \otimes H^{i'}(E_{j'}(X')) \xrightarrow{\times} H^{i+i'}(E_{j+j'}(X \times_S X'))$$
induced by the maps $\mu_{i,j}$.

Then the collection $(E_j)_{j\in\mathbb{N}}$ gives rise to a spectrum $\widetilde{\mathbb{E}} \in \mathrm{Sp}_{\mathrm{Nis}}(S;\Lambda)$, and we have isomorphisms
$$H^i(E_j(X)) \cong \mathrm{Hom}_{\mathrm{DA}(S;\Lambda)}(M_{\mathbb{A}^1}(X/S;\Lambda), \mathbb{E}(j)[i])$$
for every $i, j \in \mathbb{N}$ and every $X \in \mathcal{S}m_S$, where $\mathbb{E} \in \mathrm{DA}(S;\Lambda)$ denotes the motive corresponding to $\widetilde{\mathbb{E}}$. This isomorphism is compatible with products and functorial in X, and the construction of $\mathbb{E}$ is functorial in the families $\{(E_j), \eta, \mu_{i,j}\}$ and in the choice of c. Here $M_{\mathbb{A}^1}(-/S;\Lambda) : \mathcal{S}m_S \to \mathrm{DA}(S;\Lambda)$ denotes the functor
$$M_{\mathbb{A}^1}(X) := \gamma_*(M(X/S;\Lambda)) = \Sigma^\infty(\Lambda(X))$$
sending a scheme to its motive (see Section 2.2.3).



(a) Unit **(b)** Commutativity **(c)** Associativity

Figure 2.2.: Compatibility diagrams needed in Theorem 2.4.2 to construct a spectrum out of a family of sheaves. In these diagrams all the isomorphisms indicated with $\sim$ are the natural commutativity and associativity constraints of the category $C(PSh(\mathcal{Sm}_S; \Lambda))$

Remark 2.4.3. The commutativity of the diagrams appearing in Figure 2.2 can be relaxed by asking that the diagrams "Unit" and "Associativity" are commutative up to homotopy, or that all three are commutative up to coherent homotopy (see [BNT18]).

One can apply Theorem 2.4.2 to essentially all the cohomology theories mentioned in Section 2.1.3. Let us mention a few examples explicitly.

Example 2.4.4 (Betti cohomology)**.** Take $S = \mathrm{Spec}(K)$ with $K \subseteq \mathbb{C}$ and $\Lambda \subseteq \mathbb{R}$, one can define $E_j^{\mathbb{B}}(X; \Lambda) := C^\bullet(X(\mathbb{C}); (2\pi i)^j \cdot \Lambda)$ to be the singular cochain complex. Then Theorem 2.4.2 gives $\mathbb{B} \in \mathrm{DA}(K; \Lambda)$ such that

$$\mathrm{Hom}_{\mathrm{DA}(K;\Lambda)}(M_{\mathbb{A}^1}(X), \mathbb{B}(m)[n]) \cong H^n_{\mathrm{sing}}(X(\mathbb{C}); (2\pi i)^m \Lambda)$$

for every X of finite type over $\mathbb{C}$. This gives rise to a Betti regulator map

$$r_X^{\mathbb{B}} \colon H^{i,j}_{\mathcal{M},\mathbb{A}^1}(X; \Lambda) \to H^{i,j}_{\mathbb{B}}(X; \Lambda)$$

for every X of finite type over $\mathbb{C}$.

Example 2.4.5 (filtered de Rham cohomology)**.** Take S to be a scheme and $\Lambda = H^0(S; \mathcal{O}_S)$. Assume that, for every $X \in \mathcal{Sm}_S$, the category of good compactifications $X \hookrightarrow \overline{X}$ (see Example 2.1.22) is directed and functorial, by which we mean that for every $f \colon X \to Y$ and every pair of good compactifications $j_X \colon X \hookrightarrow \overline{X}$ and $j_Y \colon Y \hookrightarrow \overline{Y}$ there exists $\overline{f} \colon \overline{X} \to \overline{Y}$ such that $\overline{f} \circ j_X = j_Y \circ f$. As it is shown in [Del71], these conditions are satisfied if S has resolutions of singularities, for instance if $S = \mathrm{Spec}(K)$ where K is a field of characteristic zero. We define

$$E_j^{\mathrm{d}\mathbb{R}}(X) := \varinjlim_{X \hookrightarrow \overline{X}} H^0(\overline{X}, \widetilde{\mathrm{Gdm}}(\Omega^{\leq j}_{\overline{X}/S}(\log(\overline{X} \setminus X))))$$

where $\Omega^{\leq j}_{\overline{X}/S}(\log(\overline{X} \setminus X))$ denotes the truncation of the sheaf of differentials with logarithmic singularities (see again Example 2.1.22), and $\widetilde{\mathrm{Gdm}}(\mathcal{F})$ denotes the Thom-Sullivan normalisation of the Godement resolution of a sheaf $\mathcal{F}$, which is described in [DM15, § 3.1.3]. Then Theorem 2.4.2 gives a spectrum $\mathrm{d}\mathbb{R} \in \mathrm{DA}(S; \Lambda)$ such that

$$\mathrm{Hom}_{\mathrm{DA}(S;\Lambda)}(M_{\mathbb{A}^1}(X), \mathrm{d}\mathbb{R}(j)[i]) \cong F^j(H^i_{\mathrm{dR}}(X/S))$$

for every $X \in \mathcal{Sm}_S$, where F^j denotes the Hodge filtration. This gives rise to regulator maps

$$r_X^{\mathrm{d}\mathbb{R}} \colon H^{i,j}_{\mathcal{M},\mathbb{A}^1}(X; \Lambda) \to H^{i,j}_{\mathrm{d}\mathbb{R}}(X; \Lambda)$$

for every $X \in \mathcal{S}m_S$.

Example 2.4.6 (Deligne-Beilinson cohomology)**.** Take $S = \mathrm{Spec}(K)$ with $K \subseteq \mathbb{C}$, and let $\Lambda = \mathbb{R}$. Then one can take $E_j^{\mathbb{DB}}(X) := \mathcal{D}^\bullet_{\log}(X_{\mathbb{C}}, j)$ to be the complex defined by Burgos Gil in [Bur94] (see Example 2.1.22), and Theorem 2.4.2 gives a spectrum $\mathbb{DB}_{\mathbb{C}} \in \mathrm{DA}(K;\mathbb{R})$ such that

$$\mathrm{Hom}_{\mathrm{DA}(K;\Lambda)}(M_{\mathbb{A}^1}(X), \mathbb{DB}_{\mathbb{C}}(j)[i]) \cong H^{i,j}_{\mathcal{D}}(X_{\mathbb{C}};\mathbb{R})$$

for every X which is smooth and of finite type over K. This gives rise to regulator maps

$$r_X^{\mathbb{DB}_{\mathbb{C}}} : H^{i,j}_{\mathcal{M},\mathbb{A}^1}(X;\mathbb{R}) \to H^{i,j}_{\mathcal{D}}(X_{\mathbb{C}};\mathbb{R})$$

for every X which is smooth and of finite type over K. A similar construction can be applied to real Deligne-Beilinson cohomology (by taking complex conjugation into account), and gives rise to a spectrum $\mathbb{DB}_{\mathbb{R}} \in \mathrm{DA}(K;\mathbb{R})$ for every sub-field $K \subseteq \mathbb{R}$. In this way we get regulator maps

$$r_X^{\mathbb{DB}_{\mathbb{R}}} : H^{i,j}_{\mathcal{M},\mathbb{A}^1}(X;\mathbb{R}) \to H^{i,j}_{\mathcal{D}}(X_{\mathbb{R}};\mathbb{R})$$

for every X which is smooth and of finite type over a field $K \subseteq \mathbb{R}$. For more details, we refer the reader to [BZ20, § A.3].

Remark 2.4.7. Since Example 2.4.6 uses the complexes constructed by Burgos Gil, we get a regulator map only with real coefficients. One way to get regulator maps with general coefficients is to use the realisation approach. This is explained in [Hub00, Corollary 2.3.5], [LW13] and [Ivo16].

Example 2.4.8 (ℓ-adic cohomology)**.** Take $S = \mathrm{Spec}(\kappa)$ for some field κ with a fixed algebraic closure $\overline{\kappa}$, and fix $\Lambda = \mathbb{Q}_\ell$ for some rational prime $\ell \in \mathbb{N}$. Then one can define

$$E_0^{\acute{\mathbb{E}}\mathrm{T}_\ell}(X) := \varinjlim_{\underline{\mathcal{X}} \twoheadrightarrow X_{\overline{\kappa}}} \varinjlim_t \left(\varprojlim_n \varinjlim_m \Omega^\bullet(\pi_0(\check{C}(\underline{\mathcal{X}}^{\mathrm{tot}}_m / X_{\overline{\kappa}})), \mathbb{Z}/\ell^n\mathbb{Z}) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \tag{2.40}$$

where $\underline{\mathcal{X}} \twoheadrightarrow X_{\overline{\kappa}}$ runs over all the non-empty finite families of étale fundamental systems. In other words, $\underline{\mathcal{X}} = \{\mathcal{X}^{(1)}, \dots, \mathcal{X}^{(k)}\}$ and

$$\mathcal{X}^{(j)} := \left[\cdots \twoheadrightarrow \mathcal{X}^{(j)}_{n+1} \twoheadrightarrow \mathcal{X}^{(j)}_n \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{X}^{(j)}_1 \twoheadrightarrow \mathcal{X}^{(j)}_0 = X_{\overline{\kappa}} \right]$$

is a sequence of surjective étale coverings $\mathcal{X}^{(j)}_n \twoheadrightarrow X_{\overline{\kappa}}$ such that every surjective étale covering $Y \twoheadrightarrow X_{\overline{\kappa}}$ factors through a surjective étale covering $Y \twoheadrightarrow \mathcal{X}^{(j)}_n$ for some $n \in \mathbb{N}$. Moreover, for every étale covering $Y \to X_{\overline{\kappa}}$ one defines the Čech simplicial scheme

$$\check{C}(Y/X_{\overline{\kappa}})_i := \underbrace{Y \times_{X_{\overline{\kappa}}} \cdots \times_{X_{\overline{\kappa}}} Y}_{i \text{ times}}$$

and for every $\underline{X} \twoheadrightarrow X_{\overline{\kappa}}$ one defines $\underline{\mathcal{X}}^{\mathrm{tot}}$ to be the simplicial scheme whose m-th component $\underline{\mathcal{X}}^{\mathrm{tot}}_m$ is the fibre product

$$\underline{\mathcal{X}}^{\mathrm{tot}}_m := \mathcal{X}^{(1)}_m \times_{X_{\overline{\kappa}}} \cdots \times_{X_{\overline{\kappa}}} \mathcal{X}^{(k)}_m$$

where $\underline{X} = \{\mathcal{X}^{(1)}, \dots, \mathcal{X}^{(k)}\}$. Finally, for every scheme Y and every ring R one denotes by $\Omega^\bullet(Y, R)$ the "$R\{t\}$-de Rham complex" defined in [Del85b, § 5.1.4]. Here $R\{t\}$ denotes the

divided power polynomial algebra on R (see [SP, Section 07H4]), and the injective limit $\varinjlim_t$ appearing in (2.40) is taken with respect to the t-grading on $\Omega^\bullet(Y,R)$ (see [Del80, Page 238]).

One now defines $E_j^{\acute{\mathbb{E}}\mathbb{T}_\ell}(X) := E_0^{\acute{\mathbb{E}}\mathbb{T}_\ell}(X) \otimes E_0^{\acute{\mathbb{E}}\mathbb{T}_\ell}(\mathbb{G}_m)[-j]$ for every $j \in \mathbb{N}$, and Theorem 2.4.2 gives a spectrum $\acute{\mathbb{E}}\mathbb{T}_\ell \in \mathrm{DA}(S;\mathbb{Q}_\ell)$ with the property that

$$\mathrm{Hom}(M_{\mathbb{A}^1}(X), \acute{\mathbb{E}}\mathbb{T}_\ell(j)[i]) \cong H_\ell^{i,j}(X)$$

for every X which is smooth and of finite type over κ.

Example 2.4.9 (Syntomic cohomology)**.** Take $S = \mathrm{Spec}(O_K)$ where K is a finite extension of $\mathbb{Q}_p$ for some prime $p \in \mathbb{N}$, and take $\Lambda = \mathbb{Q}_p$. Then Déglise and Mazzari start from Theorem 2.4.2 to construct a spectrum $\mathbb{SYN} \in \mathrm{DA}(O_K;\mathbb{Q}_p)$ which represents syntomic cohomology (see Remark 2.1.24). In particular, we get regulator maps

$$r_X^{\mathbb{SYN}} : H^{i,j}_{\mathcal{M},\mathbb{A}^1}(X;\mathbb{Q}_p) \to H^{i,j}_{\mathrm{syn}}(X;\mathbb{Q}_p)$$

for every X which is smooth and of finite type over O_K.

Remark 2.4.10. Syntomic cohomology is generally believed to be the non-Archimedean analogue of Deligne-Beilinson cohomology, and this analogy can be made precise in multiple ways. For example, they both arise as "absolute Hodge cohomologies", *i.e.* as extensions in the categories of mixed Hodge structures over the given complete local field K. This is the usual category of mixed Hodge structures if $K \in \{\mathbb{R},\mathbb{C}\}$. On the other hand, [DN18, § 2.6] suggests that Fontaine's category of admissible, filtered $(\varphi, N, \mathrm{Gal}(\overline{K}/K))$-modules can be regarded as a category of mixed Hodge structures if K is a non-Archimedean local field. Moreover, as Deligne-Beilinson cohomology is expected to be related to (Archimedean) L-functions by the Beilinson conjecture (see Conjecture 3.3.18), syntomic cohomology is expected to be related to p-adic L-functions.

For these reasons, we believe that it is not unreasonable to introduce the following notation: for every number field F and every place $v \in M_F$ we denote by

$$r_X^v : H^{i,j}_{\mathcal{M},\mathbb{A}^1}(X;\mathbb{Q}_{p_v}) \to H^{i,j}_{\mathcal{M},\mathbb{A}^1}(X_{F_v};\mathbb{Q}_{p_v}) \to H^{i,j}_{\mathcal{AH},v}(X_{F_v};\mathbb{Q}_{p_v}) \tag{2.41}$$

the regulator map associated either to syntomic cohomology (when v is non-Archimedean) or to Deligne-Beilinson cohomology (when v is Archimedean). More precisely, the first map appearing in (2.41) is induced by base-change to F_v, and the absolute Hodge cohomology groups are defined by

$$H^{i,j}_{\mathcal{AH},v}(X_{F_v};\mathbb{Q}_{p_v}) := \begin{cases} H^{i,j}_{\mathrm{syn}}(X_{F_v};\mathbb{Q}_{p_v}), & \text{if } p_v < +\infty \\ H^{i,j}_{\mathcal{D}}(X_{F_v};\mathbb{R}), & \text{if } p_v = +\infty \end{cases}$$

where $p_v \in M_{\mathbb{Q}}$ denotes the place lying under v. With this notation in mind, the second map in (2.41) is either the syntomic regulator introduced in Example 2.4.9 or the Deligne-Beilinson regulator introduced in Example 2.4.6.

Let us conclude this section with a brief review of the other three approaches to construct regulator maps:

- Gillet has proved in [Gil81] that one can construct a Chern character associated to each cohomology theory which can be defined as the Zariski hypercohomology of a graded family of complexes of Zariski sheaves. This result may be regarded as a K-theoretic analogue of Theorem 2.4.2, and has been generalised in [BKK07].

- explicit cycle class maps for Deligne-Beilinson cohomology have been constructed by Bloch (see [Blo86b, § 4]) at the level of cohomology groups, and by Goncharov (see [Gon95a, § 5.3] and [Gon05, Theorem-Construction 2.3]) at the level of complexes. Goncharov's construction has been refined by multiple authors: Kerr's thesis gave an integral version of Goncharov's construction (see [Ker03, § 2.4.1] and [KLM06]), and the thesis of Fan refined this construction to the étale hypercohomology of the sheaves given by Bloch's complexes (see [Fan15]). Moreover, Burgos Gil and Feliu [BF12] have replaced the Deligne complex of currents used by Goncharov with the complexes $\mathcal{D}^{\bullet}_{\log}(X, j)$ (see Example 2.1.22) and Bloch's simplicial techniques with Totaro's/Levine's cubical ones, to give another construction of the cycle class map at the level of complexes, which was shown to be compatible with Beilinson's regulator in [BFT11];
- finally, Goncharov has constructed a regulator map for the polylogarithmic motivic complexes $\mathcal{B}^{\bullet,j}(X;\mathbb{Z})$ in [Gon02].

2.5 Deligne-Beilinson cohomology of curves over the reals

The aim of this section is to give a more explicit description of Deligne-Beilinson cohomology (with real coefficients) for a smooth algebraic curve X defined over $\mathbb{R}$, which is used in Chapter 9. In particular, we only need the groups $H^{1,1}_{\mathcal{D}}(X;\mathbb{R})$ and $H^{2,2}_{\mathcal{D}}(X;\mathbb{R})$ for a smooth algebraic curve X defined over $\mathbb{R}$. Hence it is sufficient to recall how to compute the Deligne-Beilinson cohomology groups $H^{n,n}_{\mathcal{D}}(X;\mathbb{R})$ for a smooth variety X defined over $\mathbb{R}$ or $\mathbb{C}$. In order to do so, we follow [Nek94, §7.3], which is a special case of [BKK07, Definition 5.50] (see also Example 2.1.22).

Let us start by introducing the following notation:

- an analytic space Y over $\mathbb{R}$ can be seen as a pair (X, F_∞) where X is a complex analytic space and $F_\infty\colon X \to X$ is an anti-holomorphic involution (see [Tog67, Teorema 14]). Moreover, a sheaf $\mathcal{S}$ on Y can also be seen as a pair $(\mathcal{T}, \sigma)$ where $\mathcal{T}$ is a sheaf on X and $\sigma\colon F_\infty^*(\mathcal{T}) \to \mathcal{T}$ is an isomorphism whose inverse is $F_\infty^*(\sigma)$;
- for every algebraic variety X over $\mathbb{C}$ we denote by $X(\mathbb{C})$ the usual complex analytification, given by the set of complex points endowed with the complex analytic topology. If Y is an algebraic variety over $\mathbb{R}$ we denote by Y^{an} the real analytic space $(Y_{\mathbb{C}}(\mathbb{C}), F_\infty)$ where F_∞ is complex conjugation (on points);
- for every subgroup $A \subseteq \mathbb{C}$ and every $j \in \mathbb{Z}$ we write $A(j) := (2\pi i)^j A \subseteq \mathbb{C}$ and we denote by $\pi_j\colon \mathbb{C} \to \mathbb{R}(j)$ the projection map given by $\pi_j(z) := (z + (-1)^j \overline{z})/2$. If X is a complex analytic space we denote by $\underline{A(j)}$ the constant sheaf with value $A(j)$, and if $Y = (X, F_\infty)$ is a real analytic space we denote by $A(j)$ the pair $(\underline{A(j)}, \overline{(\,)})$, where

$$\overline{(\,)}\colon F_\infty^*(\underline{A(j)}) = \underline{A(j)} \to \underline{A(j)}$$

  denotes complex conjugation (on coefficients);
- for every smooth complex analytic space X we denote by $\mathcal{A}^{\bullet,j}(X)$ the complex of smooth differential forms with values in $\mathbb{R}(j)$. If Y is a smooth real analytic space given by the pair (X, F_∞) we write $\mathcal{A}^{\bullet,j}(Y) := \mathcal{A}^{\bullet,j}(X)^{\overline{F_\infty^*}}$ where $\overline{(\,)}$ denotes again the action of complex conjugation on the coefficients of the differential forms. If X is an algebraic variety over $\mathbb{C}$ (respectively, over $\mathbb{R}$) we write $\mathcal{A}^{\bullet,j}(X) := \mathcal{A}^{\bullet,j}(X(\mathbb{C}))$ (resp. $\mathcal{A}^{\bullet,j}(X) := \mathcal{A}^{\bullet,j}(X^{\mathrm{an}})$);

- a *good compactification* of a morphism $f\colon X \to Y$ of schemes (or analytic spaces) is a factorisation $f = p \circ j$ where $j\colon X \hookrightarrow Z$ is an open immersion, $p\colon Z \to Y$ is proper and $Z \setminus j(X)$ is a divisor with normal crossings. Moreover, if $f\colon X \to Y$ is smooth we assume that $p\colon Z \to Y$ is also smooth. When $Y = \operatorname{Spec}(\kappa)$ and κ is a field of characteristic zero, we always have a good compactification, and any two good compactifications are dominated by a third one (see [Del71, §3.2.II]). When X is a smooth curve over a field, then a good compactification is simply a smooth, proper curve $\overline{X}$ with an open immersion $j\colon X \hookrightarrow \overline{X}$ such that $\overline{X} \setminus j(X)$ is finite;
- if $\iota\colon D \hookrightarrow Z$ is a divisor with normal crossings on Z, and $j\colon Z \setminus D \hookrightarrow Z$ is the complementary open immersion, we denote by $\Omega^\bullet_Z\langle D\rangle \subseteq j_*(\Omega^\bullet_{Z\setminus D})$ the complex of sheaves of differential forms with logarithmic singularities along D (see [SP, Definition 0FUA]). This makes sense for schemes and also for analytic spaces. The global sections $\Omega^\bullet_Z\langle D\rangle(\overline{X}) \subseteq \Omega^\bullet_{Z\setminus D}(Z \setminus D)$ can be interpreted as (algebraic, smooth or holomorphic) differential forms on $Z \setminus D$ which have at worst logarithmic singularities "at infinity";
- for every smooth variety X defined over $\mathbb{C}$ and any good compactification $X \hookrightarrow \overline{X}$ we define the complex

$$\mathcal{F}^\bullet(X \hookrightarrow \overline{X}) := \Omega^\bullet_{\overline{X}(\mathbb{C})}\langle(\overline{X} \setminus X)(\mathbb{C})\rangle(\overline{X}(\mathbb{C}))$$

  which, up to quasi-isomorphism, is independent from the choice of a good compactification (see [BKK07, Theorem 5.46]). For this reason, we usually abuse notation and write $\mathcal{F}^\bullet(X) := \mathcal{F}^\bullet(X \hookrightarrow \overline{X})$;
- if X is a smooth variety defined over $\mathbb{R}$ and $X \hookrightarrow \overline{X}$ is a good compactification we define the complex

$$\mathcal{F}^\bullet(X \hookrightarrow \overline{X}) := \mathcal{F}^\bullet\left(X_{\mathbb{C}} \hookrightarrow \overline{X}_{\mathbb{C}}\right)^{\overline{F^*_\infty}}$$

  and we abuse again notation, denoting it by $\mathcal{F}^\bullet(X)$.

Using this notation, we can introduce the following explicit way of computing motivic cohomology in terms of differential forms, following [Nek94, §7.3] (see also [CLJ19, § 4.1]).

Proposition 2.5.1 – A simple description of Deligne-Beilinson cohomology

Let X be a smooth algebraic variety defined over $\mathbb{R}$ or $\mathbb{C}$. Then one can compute the Deligne-Beilinson cohomology groups $H^{n,n}_{\mathcal{D}}(X;\mathbb{R})$ as

$$H^{n,n}_{\mathcal{D}}(X;\mathbb{R}) \cong \frac{\{(\omega,\eta) \in \mathcal{A}^{n-1,n-1}(X) \oplus \mathcal{F}^n(X \hookrightarrow \overline{X})\colon d(\omega) = \pi_{n-1}(\eta)\}}{d(\mathcal{A}^{n-2,n-1}(X))}$$

where $X \hookrightarrow \overline{X}$ denotes any good compactification. In particular, we have that

$$H^{n+1,n+1}_{\mathcal{D}}(X;\mathbb{R}) \cong H^n(X;\mathbb{R}(n))$$

for every n-dimensional variety X.

Remark 2.5.2. We have an explicit description (see [EW99, § 3, 10]) of the cup product

$$\begin{aligned} H_{\mathcal{D}}^{n,n}(X) \otimes H_{\mathcal{D}}^{m,m}(X) &\to H_{\mathcal{D}}^{n+m,n+m}(X) \\ [(\omega_1, \eta_1)] \otimes [(\omega_2, \eta_2)] &\mapsto [(\omega_1 \wedge \pi_m(\eta_2) + (-1)^m \pi_n(\eta_1) \wedge \omega_2, \eta_1 \wedge \eta_2)] \end{aligned}$$

and of Beilinson's regulator map

$$\begin{aligned} r_X^\infty \colon O^\times(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_{\mathcal{M}}^{1,1}(X) &\to H_{\mathcal{D}}^{1,1}(X) \\ f \otimes 1 &\mapsto [(\log|f|, d\log(f))] \end{aligned}$$

which gives us the equality $r_X^\infty(\{f,g\}) = [(\log|f| d\arg(g) - \log|g| d\arg(f), 0)]$ for every pair of functions $f, g \in O(X)^\times$. Here $\{f,g\} \in H_{\mathcal{M}}^{2,2}(X)$ denotes the cup product of the two motivic cohomology classes $\{f\}, \{g\} \in H_{\mathcal{M}}^{1,1}(X) \cong O(X)^\times \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark 2.5.3. For every n-dimensional smooth algebraic variety X over $\mathbb{R}$ or $\mathbb{C}$ we have an integration pairing

$$\begin{aligned} \langle\, , \rangle \colon \mathcal{F}^j(X) \otimes H_j(X(\mathbb{C}); \mathbb{R}) &\to \mathbb{C} \\ \omega \otimes \gamma &\mapsto \langle \omega, \gamma \rangle := \int_\gamma \omega \end{aligned} \tag{2.42}$$

between differential forms and singular homology classes. If X is proper then there is another integration pairing

$$\begin{aligned} \mathcal{A}^{n,j}(X) \otimes \mathcal{A}^{n,j}(X) &\to \mathbb{R} \\ \alpha \otimes \beta &\mapsto \frac{1}{(\pi i)^j} \int_{X_{\mathbb{C}}(\mathbb{C})} \alpha \wedge \beta \end{aligned}$$

between differential forms, which is related to the first one by Poincaré duality (see [Bos92, § A.2.5]).

Let now Y be a smooth curve over $\mathbb{C}$, let $c \in \mathbb{C}^\times$ and let $f \in O(Y)^\times$. We can use the explicit descriptions provided by Remark 2.5.2 to compute the pairing of the regulator of the symbol $\{c, f\} \in H_{\mathcal{M}}^{2,2}(Y)$ with a homology class $\mathfrak{c} \in H_1(Y; \mathbb{Z})$. To make this precise, let us recall some elements from the theory of Riemann surfaces, following [Bos92, Appendix A].

Remark 2.5.4. Let X be a complex compact Riemann surface of genus g. Then the first singular homology group $H_1(X; \mathbb{Z})$ supports an intersection pairing $\# \colon H_1(X; \mathbb{Z})^{\otimes 2} \to \mathbb{Z}$ which is bilinear and anti-symmetric. Moreover, $H_1(X; \mathbb{Z}) \cong \mathbb{Z}^{2g}$, where $g \in \mathbb{N}$ denotes the genus of X, and there exists a $\mathbb{Z}$-basis $\{\alpha_i, \beta_j\}_{i,j=1,\dots,g} \subseteq H_1(X; \mathbb{Z})$ which is *symplectic, i.e.* for every $i, j \in \{1, \dots, g\}$ we have that

$$\alpha_i \# \alpha_i = \beta_j \# \beta_j = 0 \qquad \text{and} \qquad \alpha_i \# \beta_j = \delta_{i,j}$$

where $\delta_{i,j} \in \{0,1\}$ denotes Kronecker's symbol (*i.e.* $\delta_{i,j} = 1$ if $i = j$, and $\delta_{i,j} = 0$ otherwise).

Now, let $S \subseteq X$ be a finite set of points and let $\iota \colon X \setminus S \hookrightarrow X$ denote the canonical inclusion. Then for every symplectic basis $\{\alpha_i, \beta_j\} \subseteq H_1(X; \mathbb{Z})$ and every point $x \in X \setminus S$ there exist smooth loops $\{a_i, b_j \colon [0,1] \to X \setminus S\}_{i,j=1,\dots,g}$ such that:

- $a_i(0) = b_j(0) = a_i(1) = b_j(1) = x$ for every $i, j \in \{1, \dots, g\}$;
- $a_i(]0,1[) \cap b_j(]0,1[) = \emptyset$ for every $i, j \in \{1, \dots, g\}$;
- $a_i\big|_{[0,1[}$ and $b_j\big|_{[0,1[}$ are injective for every $i, j \in \{1, \dots, g\}$;

- the vectors $\{a_i'(0), b_j'(0), a_i'(1), b_j'(1)\}_{i,j\in 1,\dots,g} \subseteq T_x(X)$ are pairwise non-collinear;
- the loops $\iota \circ a_i$ and $\iota \circ b_j$ are representatives of the homology classes $\alpha_i, \beta_j \in H_1(X;\mathbb{Z})$.

We commit a slight abuse of notation, and denote by $\alpha_i, \beta_j \in H_1(X(\mathbb{C}) \setminus S;\mathbb{Z})$ the classes associated to the loops $a_i, b_j \colon [0,1] \to X \setminus S$.

Now, observe that the loops a_i, b_j correspond to a *canonical dissection* (Δ, φ) of X with $S \subseteq \varphi(\Delta^\circ)$. More precisely, for every choice of $\{a_i, b_j\}$ as above there exists a polygon $\Delta \subseteq \mathbb{R}^2$ with $4g$ edges, an open $U \subseteq \mathbb{R}^2$ such that $\Delta \subseteq U$ and a surjective smooth map $\varphi \colon U \twoheadrightarrow X$ such that $\varphi\big|_{\Delta^\circ}$ is a diffeomorphism onto $X \setminus C$ where

$$C := \bigcup_i a_i([0,1]) \cup \bigcup_j b_j([0,1])$$

is the union of all the loops given by a_i and b_j. Each loop a_i or b_j corresponds to precisely two edges of Δ under φ, which are glued together with the same orientation (see [Bos92, Figure 23]).

To conclude, observe that for every $s \in S$ we can define a loop $c_s \colon [0,1] \to \Delta \setminus \varphi^{-1}(S) \to X \setminus S$, where the map $[0,1] \to \Delta \setminus \varphi^{-1}(S)$ is a small circle around $\varphi^{-1}(s)$ connected to one vertex of Δ by a straight line. Let $\gamma_s \in H_1(X \setminus S;\mathbb{Z})$ be the singular cohomology class associated to c_s, which does not depend on the choice of the small circle c_s if all the circles $\{c_s\}_{s\in S}$ are pairwise disjoint and oriented coherently. Then we have an exact sequence

$$\begin{aligned} 0 \to \mathbb{Z} \to \mathbb{Z}^S \to H_1(X(\mathbb{C}) \setminus S;\mathbb{Z}) \to H_1(X(\mathbb{C});\mathbb{Z}) \to 0 \\ \{m_s\}_{s\in S} \mapsto \sum_{s\in S} m_s \gamma_s \end{aligned} \tag{2.43}$$

where the map $\mathbb{Z} \to \mathbb{Z}^S$ is the diagonal one. In particular, for every $s_0 \in S$ the set

$$\{\alpha_i, \beta_j, \gamma_s \mid i,j \in \{1,\dots,g\},\ s \in S \setminus \{s_0\}\}$$

is a basis of $H_1(X \setminus S;\mathbb{Z})$. This can be easily shown using the Mayer-Vietoris exact sequence (see [Spa95, § 4.6]).

Let us now use Remark 2.5.4 to compute the pairing that we announced.

Proposition 2.5.5 – Regulator pairings on a punctured curve

Let X be a smooth, proper algebraic curve over $\mathbb{C}$ of genus g, and let $S \subseteq X(\mathbb{C})$ be a finite set of points. Let (Δ, φ) be a canonical dissection of $X(\mathbb{C})$ such that $S \subseteq \varphi(\Delta^\circ)$ and let $\alpha_i, \beta_j, \gamma_s$ be the homology classes associated to (Δ, φ). Then we have that

$$\langle r^\infty_{X\setminus S}(\{c,f\}), \alpha_i \rangle = \langle r^\infty_{X\setminus S}(\{c,f\}), \beta_j \rangle = 0 \tag{2.44}$$

$$\langle r^\infty_{X\setminus S}(\{c,f\}), \gamma_s \rangle = \log|\partial_s(\{c,f\})| = \mathrm{ord}_s(f)\ \log|c| \tag{2.45}$$

for every $c \in \mathbb{C}$, every $f \in \mathbb{C}(X)$ such that $S_f \subseteq S$, every $i,j \in \{1,\dots,g\}$ and every $s \in S$.

Proof. The computation (2.44) follows from the fact that $\int_{X(\mathbb{C})} d(\log(f)) \wedge \alpha = 0$, whereas (2.45) is an application of *Jensen's formula*, as explained in [Rod99, Page 25]. □

To conclude this section let us introduce some notations concerning the cohomology of elliptic curves defined over the reals.

Notation 2.5.6. Let E be an elliptic curve defined over $\mathbb{R}$. We introduce the following notation:

- $E(\mathbb{R})^0 \subseteq E(\mathbb{R})$ denotes the connected component of the identity;
- $\omega_E \in \mathcal{F}^1(E)$ the unique differential form such that $\int_{E(\mathbb{R})^0} \omega_E = 1$. We clearly have that $\omega_E \in H^1(E^{\mathrm{an}}; \mathbb{Q}(1))$, because $H_1(E^{\mathrm{an}}; \mathbb{Q})$ is generated by the homology class of $E(\mathbb{R})^0$;
- $H_1(E(\mathbb{C}); \mathbb{Q})^- \subseteq H_1(E(\mathbb{C}); \mathbb{Q})$ denotes the subspace of homology classes which are anti-invariant by complex conjugation;
- $\gamma_E \in H_1(E(\mathbb{C}); \mathbb{Q})^-$ denotes the Poincaré dual of ω_E.

L-functions and their special values

3

The function of freedom is to free someone else.

Toni Morrison,
Commencement speech at Barnard College, 1979

As we have seen in the previous chapter, the category of algebraic varieties is a very chaotic one. It is in particular very difficult to describe the sub-objects of a product of two algebraic varieties, and we have seen how one can use the closed sub-varieties of $X \times Y$ to form the graded module of correspondences $C^\bullet(X, Y)_\Lambda$, and from this a category of Λ-linear motives, which is an attempt to linearise the category of varieties.

In order to understand even better algebraic varieties, one could attempt to associate to their cohomology groups an invariant which is more computable. If we consider the ℓ-adic cohomology of algebraic varieties, then L-functions provide such an invariant, which allows one to put together into an analytic object the data concerning the Galois action on the ℓ-adic cohomology groups. This idea rests on Chebotarëv's density theorem (see [Neu99, Chapter VII, Theorem 13.4]). More precisely, let $\rho\colon \mathcal{G}_F \to G$ be a continuous group homomorphism from the absolute Galois group $\mathcal{G}_F := \mathrm{Gal}(\overline{F}/F)$ to a topological group G, and assume that $\rho(\mathcal{I}_v) = 1$ for all non-Archimedean places $v \in M_F^0 \setminus S$, where $S \subseteq M_F^0$ is finite and $\mathcal{I}_v \subseteq \mathcal{G}_{F_v} \subseteq \mathcal{G}_F$ denotes the inertia group of the absolute Galois group $\mathcal{G}_{F_v} := \mathrm{Gal}(\overline{F_v}/F_v)$ of the v-adic completion F_v of F. Then, for every $v \in M_F^0$ the group $\mathcal{G}_v/\mathcal{I}_v \cong \mathrm{Gal}(\overline{\kappa_v}/\kappa_v)$ is generated by the geometric Frobenius $f_{F_v} := \Phi_v^{-[\kappa_v\colon \mathbb{F}_p]}$, where $\Phi_{F_v}\colon (\overline{F_v})_0 \to (\overline{F_v})_0$ is the automorphism of the maximal unramified extension $F_v \subseteq (\overline{F_v})_0$ which lifts the automorphism $\phi_v\colon \overline{\kappa_v} \to \overline{\kappa_v}$ of the algebraic closure of the residue field κ_v, defined by setting $\phi_v(x) := x^p$, where $p \in \mathbb{N}$ is the characteristic of κ_v. Now, Chebotarëv's density theorem implies that our continuous group homomorphism $\rho\colon \mathcal{G}_F \to G$ is determined by the values $\{\rho(f_{F_v})\colon v \in M_F^0 \setminus S\}$, which are well defined because $\rho(\mathcal{I}_v) = 1$ for all $v \in M_F^0 \setminus S$. In the case when $G = \mathrm{GL}(V)$ for some topological module V, one can observe that the characteristic polynomials of $\rho(f_{F_v})$ determine ρ up to semi-semplification. The L-function associated to ρ is a way of putting together all these characteristic polynomials.

L-functions have been the cornerstone of numerous developments in number theory since the introduction of Riemann's ζ-function. First of all, L-functions conjecturally provide a factorisation of the Hasse-Weil ζ-function associated to integral models $\mathcal{X}$ of algebraic varieties X defined over a number field F. Henceforth the study of the location of zeros of L-functions allows one to obtain some information about the asymptotics of the number of points $\mathcal{X}(O_F/\mathfrak{p}^n)$ as $n \to +\infty$, for any prime ideal $\mathfrak{p} \subseteq O_F$.

Moreover, L-functions are at the centre of two far-reaching sets of conjectures: the Langlands program, whose aim is to relate algebraic varieties (and motives) to automorphic representations, and the Deninger program, whose aim is to relate algebraic varieties (and motives) to dynamical systems. One expects to be able to attach L-functions both to motives and to automorphic forms/dynamical systems, and the Langlands and Deninger programs may be considered as

a "lift", at the level of the objects themselves, of the correspondences that are apparent at the level of L-functions. We do not touch on these topics in this chapter, and we refer the interested reader to [Clo90] and [Den98] for further details.

Finally, the values and the derivatives of L-functions of algebraic varieties at the integers play a pivotal role. On the one hand, they are conjectured to be related to the regulators which were introduced in Section 2.4, and on the other hand it is known by work of Deninger that one can recover an L-function from its values at the integers (see Section 3.1). The study of these special values was initiated by the work of Euler on $\zeta(2) = \pi^2/6$, and continued by the work of Dirichlet on the evaluation of the special value $\zeta_K^*(1)$ for every imaginary quadratic field K, which was extended by Hecke to any number field. Birch and Swinnerton-Dyer then investigated the special values $L^*(E, 1)$ of L-functions associated to elliptic curves, and their conjectures were extended by Tate to higher dimensional abelian varieties. The revolutionary work of Beilinson then introduced a framework for the study of the special values $L^*(M, n)$ of any motive M at any integer $n \in \mathbb{Z}$. The conjectural relations studied by Beilinson hold up to a non-zero rational number (or, more generally, up to an element of $K^\times$, where K is the number field over which the motive M is defined), but the subsequent work of Bloch and Kato, further precised by Fontaine and Perrin-Riou, gives a conjecture which predicts the special values $L^*(M, n)$ up to sign (or, more generally, up to an element of $O_K^\times$). The work on these conjectures is still ongoing to this day: we mention among others the equivariant generalisation of the conjecture of Bloch and Kato carried out by Burns and Flach, the work of Flach and Morin on arithmetic schemes and the work of Braunling, which gives a more categorical formulation of the conjecture of Burns and Flach. We give precise statements and references for these conjectures in Section 3.3.2.

If these conjectures on special values were true, these values could be considered as a form of height, because they would be related to regulators. In particular, it is interesting to study Diophantine properties of special values of L-functions, such as the Northcott, Bogomolov and Lehmer properties defined in Section 1.1. We devote Section 3.4, which is based on joint work in progress with Fabien Pazuki, to show some initial examples of relations between heights and special values of L-functions, and some Diophantine properties satisfied (at least conjecturally) by the latter.

3.1 Dirichlet series and their special values

The aim of this section is to describe a set of functions $f\colon \mathbb{C} \to \mathbb{C}$ which can be recovered by their values at almost all positive integers. This set contains all the holomorphic functions that can be expressed as a *Dirichlet series* $f(s) = \sum_{n \gg -\infty} a_n/n^s$, and thus in particular all the L-functions that are usually considered in number theory. This can be seen as a motivation for the conjectures on special values of L-functions, which are outlined in Section 3.3.2. Indeed, a combination of the results in this section with the conjectures in question shows that one can think about the motivic L-function $L(M, s)$ as a set of arithmetic invariants associated to M, corresponding to the special values $\{L^*(M, n)\colon n \in \mathbb{Z}\}$.

Let us start by defining the ambient space which contains the sequences of special values of our functions. This is a $\mathbb{C}$-algebra $\widetilde{\mathcal{A}}$ given by

$$\widetilde{\mathcal{A}} := \{(n_0, \mathbf{a}) \mid n_0 \in \mathbb{Z},\ \mathbf{a} = (a_n)_{n \geq n_0} \in \mathbb{C}^{\mathbb{Z}_{\geq n_0}}\}/\sim_{\widetilde{\mathcal{A}}} \tag{3.1}$$

where the equivalence relation $\sim_{\widetilde{\mathcal{A}}}$ is defined by setting $\mathbf{a} \sim_{\widetilde{\mathcal{A}}} \mathbf{b}$ if and only if there exists $k_0 \in \mathbb{Z}$ such that $a_k = b_k$ for every $k \geq k_0$.

By analogy, we define an ambient space of functions $\widetilde{\mathcal{F}}$ which contains all the functions that can be recovered from their special values. This is the $\mathbb{C}$-vector space

$$\widetilde{\mathcal{F}} := \{(\beta, f) \mid \beta \in \mathbb{R},\ f\colon \mathfrak{R}_\beta \to \mathbb{C} \text{ is holomorphic}\}/\sim_{\widetilde{\mathcal{F}}}$$

where $\mathfrak{R}_\beta := \{z \in \mathbb{C} \mid \Re(z) > \beta\}$ and $\sim_{\widetilde{\mathcal{F}}}$ is defined by setting $(\beta_1, f_1) \sim_{\widetilde{\mathcal{F}}} (\beta_2, f_2)$ if and only if there exists $\gamma \in \mathbb{R}$ such that $\gamma \geq \max(\beta_1, \beta_2)$ and $f(z) = g(z)$ for every $z \in \mathfrak{R}_\gamma$. Using the evident compatibility between $\sim_{\widetilde{\mathcal{A}}}$ and $\sim_{\widetilde{\mathcal{F}}}$ we get a map

$$\begin{aligned} S\colon \widetilde{\mathcal{F}} &\to \widetilde{\mathcal{A}} \\ [(\beta, f)] &\mapsto \left[\left(n_0, (f(n))_{n \geq n_0}\right)\right] \end{aligned} \tag{3.2}$$

where $n_0 := \min\{n \in \mathbb{Z} \mid n > \beta\}$. We should think about S as the map sending a function to its sequence of "special" values $f(n) \in \mathbb{C}$ at the integers.

Remark 3.1.1. As we point out later in this chapter, for every meromorphic function $\phi\colon \mathbb{C} \to \mathbb{C}$ and every $s_0 \in \mathbb{C}$ one usually defines the *special value* as

$$\phi^*(s_0) := \lim_{s \to s_0} \frac{\phi(s)}{(s - s_0)^{\operatorname{ord}_{s=s_0}(\phi)}} \in \mathbb{C}^\times$$

where $\operatorname{ord}_{s=s_0}(\phi) \in \mathbb{Z}$ is the unique integer $n \in \mathbb{Z}$ such that $\lim_{s \to s_0} (s - s_0)^{-n} \phi(s) \in \mathbb{C}^\times$.

The relation between this definition and the map S comes from the following observation. Let $(\beta, f) \in \widetilde{\mathcal{F}}$ be a pair with the property that there exist $\beta' > \beta$ and a collection of holomorphic functions $\{f_j\colon \mathfrak{R}_{\beta'} \to \mathbb{C}\}_{j \in J}$ indexed over some set $J \subseteq \mathbb{N}$ such that f can be expressed as an Euler product

$$f(z) = \prod_{j \in J} f_j(z)^{-1} \tag{3.3}$$

for every $z \in \mathfrak{R}_{\beta'}$. Then $f^*(n) = f(n)$ for every $n \in \mathbb{Z}$ such that $n > \beta'$, because f is holomorphic in $\mathfrak{R}_\beta \supseteq \mathfrak{R}_{\beta'}$ and (3.3) shows that if $f(z) = 0$ then $\Re(z) \leq \beta'$.

Let us now see how to define a partial right inverse to S. In order to do so, we define $\mathcal{P}$ to be the $\mathbb{C}$-vector space

$$\mathcal{P} := \left\{ (U, \psi, \alpha) \;\middle|\; \begin{array}{l} U \subseteq \mathbb{C} \text{ is open, } \mathbb{R}_{\leq 0} \subseteq U \\ \psi\colon U \setminus \{0\} \to \mathbb{C} \text{ is holomorphic} \\ \alpha \in \mathbb{R} \text{ and } |\psi(z)| = O(|z|^\alpha) \text{ as } z \to -\infty \end{array} \right\} / \sim_{\mathcal{P}}$$

where $\sim_{\mathcal{P}}$ is the equivalence relation defined by saying that $(U_1, \psi_1, \alpha_1) \sim_{\mathcal{P}} (U_2, \psi_2, \alpha_2)$ if and only if there exists $V \subseteq \mathbb{C}$ open such that $0 \in V \subseteq U_1 \cap U_2$ and there exists a Laurent polynomial $P \in \mathbb{C}[z^{\pm 1}]$ such that $\psi_1(z) - \psi_2(z) = P(z)$ for every $z \in V \setminus \{0\}$. Now, we can restrict our set of admissible sequences of special values as follows.

Definition 3.1.2 – Admissible sequences of special values

Let $\widetilde{\mathcal{A}}$ be the set defined in (3.1). We define $\mathcal{A}' \subseteq \widetilde{\mathcal{A}}$ to be the set of those $\mathbf{a} \in \widetilde{\mathcal{A}}$ such that:

- there exists a neighbourhood of the origin $U' \subseteq \mathbb{C}$ such that the Laurent series $\sum_{n=-\infty}^{+\infty} a_n z^n$ converges for every $z \in U' \setminus \{0\}$;
- there exist an open $U \subseteq \mathbb{C}$ and a holomorphic function $\psi_{\mathbf{a}} \colon U \setminus \{0\} \to \mathbb{C}$ such that $\mathbb{R}_{\leq 0} \cup U' \subseteq U$ and
$$\psi_{\mathbf{a}}(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$$
for every $z \in U' \setminus \{0\}$;
- there exists $\alpha \in \mathbb{R}$ such that $|\psi(z)| = O(|z|^{\alpha})$ as $z \to -\infty$.

Moreover, we define $\mathcal{F}' := S^{-1}(\mathcal{A}')$.

Note that $\mathcal{A}'$ is well defined, because if $\mathbf{a} \sim_{\widetilde{\mathcal{A}}} \mathbf{b}$ then $\sum_{n=-\infty}^{+\infty} a_n z^n - \sum_{n=-\infty}^{+\infty} b_n z^n \in \mathbb{C}[x^{\pm 1}]$, hence one of the two Laurent series converges if and only if the other converges (away from zero) and one of them can be analytically continued if and only if the other can be analytically continued. Moroever, the compatibility between $\sim_{\widetilde{\mathcal{A}}}$ and $\sim_{\mathcal{P}}$ gives us a well-defined map

$$\begin{aligned} \Psi \colon \mathcal{A}' &\to \mathcal{P} \\ [\mathbf{a}] &\mapsto [(U, \psi_{\mathbf{a}}, \alpha)] \end{aligned}$$

where $U \subseteq \mathbb{C}$ and $\alpha \in \mathbb{R}$ are as in Definition 3.1.2. The next theorem (see [Den00, Theorem 2.1]) shows that the special value map $S \colon \mathcal{F}' \to \mathcal{A}'$ admits an explicitly defined, $\mathbb{C}$-linear right inverse.

Theorem 3.1.3 – Recovering a function from its special values

Let $\mathcal{A}'$ and $\mathcal{F}'$ be the $\mathbb{C}$-vector spaces defined in Definition 3.1.2, and let $S \colon \mathcal{F}' \to \mathcal{A}'$ be the special value map defined in (3.2). Then the map

$$\begin{aligned} I \colon \mathcal{P} &\to \widetilde{\mathcal{F}} \\ [(U, \psi, \alpha)] &\mapsto \left[\begin{pmatrix} I(\psi) \colon \mathfrak{R}_{\alpha} \to \mathbb{C} \\ \alpha, \qquad\qquad s \mapsto \dfrac{1}{2\pi i} \displaystyle\int_{\gamma} z^{-s} \psi(z) \frac{dz}{z} \end{pmatrix} \right] \end{aligned}$$

is well defined. Moreover, $I(\Psi(\mathcal{A}')) \subseteq \mathcal{F}'$ and $S \circ I \circ \Psi = \mathrm{Id}_{\mathcal{A}'}$. Here $\gamma \subseteq U$ denotes any loop which starts from $-\infty$, goes around the origin counterclockwise and returns to $-\infty$ (see Figure 3.1). Thus, if we define $\mathcal{F} := I(\Psi(\mathcal{A}')) \subseteq \mathcal{F}'$ and $\mathcal{A} := \mathcal{A}'$ we have a $\mathbb{C}$-linear isomorphism $S \colon \mathcal{F} \xrightarrow{\sim} \mathcal{A}$ whose $\mathbb{C}$-linear inverse is $I \circ \Psi$.

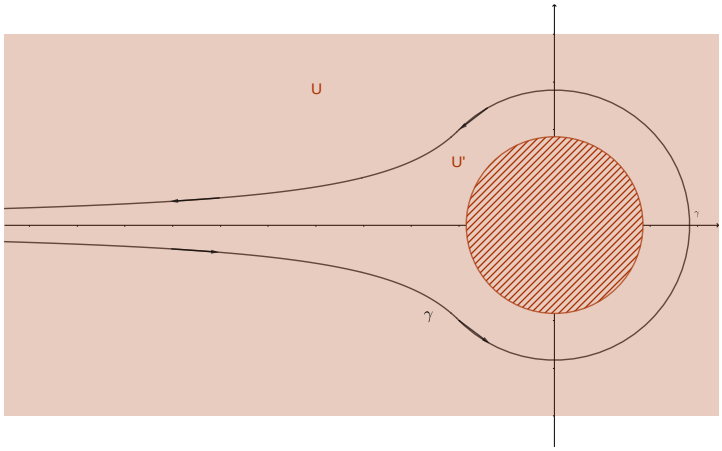


Figure 3.1.: The figure depicts two opens $U' \subseteq U \subseteq \mathbb{C}$ as the ones featured in Definition 3.1.2, and a path $\gamma \subseteq U$ as the one featured in Theorem 3.1.3

Proof. First of all, for every $[(U, \psi, \alpha)] \in \mathcal{P}$ there exists $\beta \in \mathbb{R}$ such that

$$\left|\int_\gamma z^{-s}\psi(z)\frac{dz}{z}\right| \leq \left|\int_{\gamma\cap\mathfrak{R}_\beta} z^{-s}\psi(z)\frac{dz}{z}\right| + \int_{\gamma\setminus\mathfrak{R}_\beta} \frac{dz}{|z|^{1+\Re(s)-\alpha}}$$

and the residue theorem [SS03, Chapter 3, Theorem 2.1] shows that for every rational function $f \in \mathbb{C}(z)$ which does not have any pole in $\mathbb{R}_{\leq 0}$ we have the formula

$$\frac{1}{2\pi i}\int_{\gamma'} z^{-s}f(z)\frac{dz}{z} = -\sum_{z_0\in\mathbb{C}\setminus\mathbb{R}_{\leq 0}} \mathrm{Res}_{z=z_0}(f(z)\cdot z^{1-s})$$

for every $s \in \mathfrak{R}_d$, where $d = \deg(f)$ is the degree of the rational function $f(z)$ and $\gamma' \subseteq \mathbb{C}$ denotes any path which starts from $-\infty$, goes around the origin counterclockwise, returns to $+\infty$ and avoids the poles of f in its interior. The combination of these two computations shows that $I(\psi)\colon \mathfrak{R}_\alpha \to \mathbb{C}$ is well defined (*i.e.* the definition is compatible with $\sim_{\mathcal{P}}$) and holomorphic.

Another application of the residue theorem shows that $I(\Psi(\mathbf{a}))(n) = a_n$ for every $\mathbf{a} \in \mathcal{A}'$ and every $n > \alpha$, where $\alpha \in \mathbb{R}$ is taken as in Definition 3.1.2. This shows immediately that $I(\Psi(\mathcal{A}')) \subseteq \mathcal{F}'$ and in fact that $S \circ I \circ \Psi = \mathrm{Id}_{\mathcal{A}'}$. □

Example 3.1.4. Let us mention a few examples of the values of the function $I \circ \Psi\colon \mathcal{A} \to \mathcal{F}$:

- if $\lambda \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ then $\mathbf{a} := [(\lambda^{-n})_{n\geq 0}] \in \mathcal{A}$ and

$$I(\Psi(\mathbf{a})) = \lambda^{-s} := \exp(-(\log|\lambda| + \arg(\lambda)\cdot i)\cdot s)$$

  where the argument is normalised by $\arg(\lambda) \in (-\pi, \pi]$;

- if $\mathbf{a} := (1/n!)_{n\geq 0}$ then $\mathbf{a} \in \mathcal{A}$ and

$$I(\Psi(\mathbf{a}))(s) = I(e^z)(s) = \frac{1}{2\pi i}\int_\gamma \frac{e^z}{z^{s+1}}dz = \frac{1}{\Gamma(s+1)}$$

where the last equality follows from [WW96, § 12.22];

- if we take $\mathbf{a} := (-B_{n+1}(a)/(n+1)!)_{n\geq -1}$ where $a \in \mathbb{R}$, $0 < a \leq 1$ and $B_n(x) \in \mathbb{Q}[x]$ denotes the n-th Bernoully polynomial, then $\mathbf{a} \in \mathcal{A}$ and we have that

$$I(\Psi(\mathbf{a}))(s) = I\left(\frac{e^{az}}{1-e^z}\right)(s) = \frac{\zeta(-s,a)}{\Gamma(s+1)}$$

where $\zeta(x,y) := \sum_{n=0}^{+\infty}(x+y)^{-n}$ denotes Hurwitz's zeta function (see [WW96, § 13.13]).

This shows that our interpolation procedure explained in Theorem 3.1.3 recovers the most commonly known interpolation procedures: the function λ^{-s} defined as a function which interpolates the well defined values λ^{-n}, and the Γ-function as an interpolation of factorials.

Theorem 3.1.3 shows that any holomorphic function $f\colon \mathfrak{R}_\beta \to \mathbb{C}$ such that $[(\beta, f)] \in \mathcal{F}$ can be recovered from any subset of its special values having the form $\{f^*(n) \mid n \geq n_0\}$ for some $n_0 \in \mathbb{Z}$. However, the definition of $\mathcal{F}$ is somehow implicit, and it is difficult in general to decide whether or not $[(\beta, f)] \in \mathcal{F}$. The next theorem, which is also due to Deninger (see [Den00, Theorem 3.2]), shows that a certain interpolation formula of Hardy and Ramanujan allows one to provide an explicit class of functions $\mathcal{F}_H$ such that $\mathcal{F}_H \subseteq \mathcal{F}$.

Theorem 3.1.5 – The Hardy-Ramanujan class

Let $\mathcal{F}$ denote the class of functions defined in Theorem 3.1.3, and let

$$\mathcal{F}_H := \left\{ (\sigma_0, A, B, \varphi, f) \;\middle|\; \begin{array}{l} \sigma_0, A, B \in \mathbb{R},\ B < \pi \\ f\colon \mathfrak{R}_{\sigma_0} \to \mathbb{C} \text{ is holomorphic} \\ \varphi\colon \mathbb{R} \to \mathbb{R}_{>0},\ \varphi \in L^1(\mathbb{R}),\ \lim\limits_{t\to\pm\infty} \varphi(t) = 0 \\ |f(s)| \leq \varphi(\Im(s)) \cdot e^{A\Re(s)+B|\Im(s)|},\ \forall s \in \mathfrak{R}_{\sigma_0} \end{array} \right\} / {\sim_{\mathcal{F}_H}}$$

where $(\sigma_0, A, B, \varphi, f) \sim_{\mathcal{F}_H} (\sigma_0', A', B', \varphi', f')$ if and only if there exists $\sigma_1 \in \mathbb{R}$ such that $\sigma_1 \geq \max(\sigma_0, \sigma_0')$ and $f(z) = f'(z)$ for every $z \in \mathfrak{R}_{\sigma_1}$. Then the map

$$\begin{aligned} \iota\colon \mathcal{F}_H &\to \widetilde{\mathcal{F}} \\ [(\sigma_0, A, B, \varphi, f)] &\mapsto [(\sigma_0, f)] \end{aligned}$$

is well defined and injective. Moreover, $\iota(\mathcal{F}_H) \subseteq \mathcal{F}$.

Proof. It is clear from the definitions of $\sim_{\mathcal{F}_H}$ and $\sim_{\widetilde{\mathcal{F}}}$ that ι is well defined and injective, hence we only have to check that $\iota(\mathcal{F}_H) \subseteq \mathcal{F}$. To do this, fix some element $[(\sigma_0, A, B, C, f)] \in \mathcal{F}_H$ and let $n_0 := \min\{n \in \mathbb{Z} \mid n > \beta\}$. Then the power series $\sum_{n=n_0}^{+\infty} f(n) \cdot z^n$ converges absolutely in the punctured disc $0 < |x| < e^{-A}$. Fix now $r, L \in \mathbb{R}$ such that $n_0 - 1 < r < n_0 < L$, and for every $R \in \mathbb{R}_{>0}$ consider the contour $\gamma(L,R) = \gamma_1(L,R) + \cdots + \gamma_4(L,R)$ given by the boundary of the square $[r, L] \times [-R, R]$ oriented counterclockwise (see Figure 3.2). Then the residue theorem shows that

$$\frac{1}{2\pi i}\int_{\gamma(L,R)} \frac{\pi}{\sin(\pi s)} \cdot f(s)(-x)^s ds = \sum_{n_0 \leq n < L} f(n)x^n \tag{3.4}$$

and for $L \in \mathbb{R}$ fixed we have that

$$\lim_{R\to+\infty} \int_{\gamma_2(L,R)} \frac{\pi}{\sin(\pi s)} \cdot f(s)(-x)^s ds = \lim_{R\to+\infty} \int_{\gamma_4(L,R)} \frac{\pi}{\sin(\pi s)} \cdot f(s)(-x)^s ds = 0 \tag{3.5}$$

as follows easily from the bounds $|f(s)| \leq \varphi(\Im(s)) \cdot e^{A\Re(s)+B|\Im(s)|}$ and $|\pi/\sin(\pi s)| \leq C_1 \cdot e^{-\pi|\Im(s)|}$ which hold for every $s \in \gamma(L,R)$. Moreover, the same estimates show that

$$\lim_{L\to+\infty} \lim_{R\to+\infty} \int_{\gamma_3(L,R)} \frac{\pi}{\sin(\pi s)} \cdot f(s)(-x)^s ds = 0 \tag{3.6}$$

for every $x \in \mathbb{R}$ such that $-e^{-A} < x < 0$. Indeed, one has that

$$\left| \int_{-\infty}^{+\infty} \frac{\pi}{\sin(\pi(L+it))} f(L+it)(-x)^{L+it} \right| \leq C_1 \int_{-\infty}^{+\infty} e^{-\pi|t|} \varphi(t) e^{AL+\pi|t|} \cdot e^{L\log(-x)} dt =$$
$$= C_1\, e^{L(A+\log(-x))} \int_{-\infty}^{+\infty} \varphi(t) dt$$

which allows one to show (3.6) because $\varphi \in L^1(\mathbb{R})$ and $\lim_{L\to+\infty} e^{L(A+\log(-x))} = 0$ for every $x \in \mathbb{R}$ such that $-e^{-A} < x < 0$. Thus combining (3.4), (3.5) and (3.6) one gets that

$$\sum_{n=n_0}^{+\infty} f(n)x^n = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\pi}{\sin(\pi(r+it))} \cdot f(r+it)(-x)^{r+it} dt \tag{3.7}$$

for every $r \in \mathbb{R}$ such that $n_0 - 1 < r < n_0$, and every $x \in \mathbb{R}$ such that $-e^{-A} < x < 0$.

Now, one can easily see that the integral on the right hand side of (3.7) converges for every $x \in \mathbb{C}$ such that $|\arg(-x)| < \pi - B$. Indeed

$$\left| \int_{-\infty}^{+\infty} \frac{\pi}{\sin(\pi(r+it))} \cdot f(r+it)(-x)^{r+it} dt \right| \leq C_1 \cdot \int_{-\infty}^{+\infty} e^{-\pi|t|} e^{Ar+B|t|} |x|^r e^{\arg(-x)|t|} dt =$$
$$= C_1 e^{Ar} |x|^r \int_{-\infty}^{+\infty} e^{(B-\pi+\arg(-x))|t|} < +\infty \tag{3.8}$$

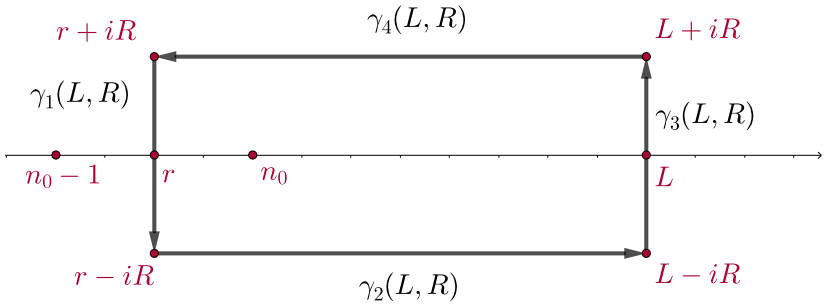


Figure 3.2.: The integration contour $\gamma(L,R)$ featured in Theorem 3.1.5

using the usual estimates $|f(s)| \leq \varphi(\Im(s)) \cdot e^{A\Re(s)+B|\Im(s)|}$ and $|\pi/\sin(\pi s)| \leq C_1 \cdot e^{-\pi|\Im(s)|}$. Observe that (3.8) shows that $(f(n))_{n\geq n_0} \in \mathcal{A}$. Indeed the power series $\sum_{n=n_0}^{+\infty} f(n)x^n$ converges in the punctured neighbourhood $0 < |x| < e^{-A}$, and (3.8) shows that it can be analytically continued to a function $\psi_f \colon U \setminus \{0\} \to \mathbb{C}$, where $U := \{x \in \mathbb{C} \mid |\arg(-x)| < \pi - B\}$. Evidently U is open, $\mathbb{R}_{\leq 0} \subseteq U$ and (3.8) shows that $\psi_f(z) = O(|z|^r)$ for $z \to -\infty$.

The previous paragraph proves that $\iota(\mathcal{F}_H) \subseteq \mathcal{F}'$. We show now that $\iota(\mathcal{F}_H) \subseteq \mathcal{F}$. To do so, we prove that $I(\psi_f)(s) = f(s)$ for every $s \in \mathfrak{R}_{\sigma_0}$ and every $[(\sigma_0, A, B, \varphi, f)] \in \mathcal{F}_H$. First of all, we observe that the function $G(x) := -\psi_f(-x)/x^{n_0}$ is given by the inverse Mellin transform

$$G(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} g(s)x^{-s}ds$$

with $\sigma := n_0 - r > 0$ and $g(s) := \pi/\sin(\pi(s-n_0)) \cdot f(n_0 - s)$. Hence, applying Mellin's inversion formula (see [Igu78, Chapter 1, Theorem 3.1]) we get that

$$f(s) = -\frac{\sin(\pi s)}{\pi} \int_0^{+\infty} t^{-s-1}\psi_f(-t)dt$$

for every $s \in \mathbb{C}$ such that $\sigma_0 < \mathrm{Re}(s) < n_0$. On the other hand, we have the holomorphic function $I(\psi_f) \colon \mathfrak{R}_{\sigma_0} \to \mathbb{C}$, which can be expressed as

$$\begin{aligned} I(\psi_f)(s) &= \frac{1}{2\pi i} \lim_{\varepsilon\to 0^+} \int_\varepsilon^{+\infty} (e^{-i\pi s} - e^{i\pi s})x^{-s-1}\psi_f(-x)\,dx + \oint_{|z|=\varepsilon} z^{-s-1}\psi(z)dz = \\ &= -\frac{\sin(\pi s)}{\pi} \int_0^{+\infty} x^{-s-1}\psi_f(-x) = f(s) \end{aligned} \tag{3.9}$$

for every $s \in \mathbb{C}$ such that $\sigma_0 < \Re(s) < n_0$. The last equality follows from the equality $e^{-i\pi s} - e^{i\pi s} = -2i\sin(\pi s)$ and the inequality

$$\left| \frac{1}{2\pi i} \oint_{|z|=\varepsilon} z^{-s-1}\psi(z)dz \right| \leq C_2 \cdot \varepsilon^{n_0-\Re(s)}$$

which holds because $|\psi_f(z)| = |\sum_{n=n_0}^{+\infty} f(n)z^n| = O(|z|^{n_0})$ as $|z| \to 0$. Hence we have proved in (3.9) that $f(s) = I(\psi_f)(s)$ for every $s \in \mathbb{C}$ such that $\sigma_0 < \Re(s) < n_0$, and we can conclude by analytic continuation that $f(s) = I(\psi_f)(s)$ for every $s \in \mathfrak{R}_{\sigma_0}$, which is what we wanted to prove. □

We conclude this section by showing that Dirichlet series (and thus L-functions) belong to $\mathcal{F}_H$. To do so we define $\mathcal{F}_H^0 \subseteq \mathcal{F}_H$ to be the $\mathbb{C}$-linear subspace generated by those $[(\sigma_0, A, B, \varphi, f)] \in \mathcal{F}_H$ with $B = 0$. Fix now $k_0 \in \mathbb{Z}$ and a sequence $\lambda = (\lambda_k) \in (\mathbb{R}_{>0})^{\mathbb{Z}_{\geq k_0}}$ such that $\lim_{n\to+\infty} \lambda(n) = +\infty$. Then if a generalised Dirichlet series

$$f(s) = \sum_{k=k_0}^{+\infty} \frac{a_k}{\lambda_k^s}$$

has a finite abscissa of absolute convergence $\sigma_0 \in \mathbb{R}$, one has that $[(\sigma_1, 0, 0, \varphi, f)] \in \mathcal{F}_H^0$ for every $\sigma_1 > \sigma_0$, where

$$\varphi(t) \equiv \sum_{k=k_0}^{+\infty} \frac{|a_k|}{\lambda_k^{\sigma_1}}$$

is a constant function. This shows that a Dirichlet series can be reconstructed from its special values.

Remark 3.1.6. Let us also mention that one can recover some *completed* Dirichlet series from their special special values. By this we mean functions of the form

$$\widehat{f}(s) = \omega^{-s} \cdot \left(\prod_{n \in \mathbb{Z}} \Gamma_{\mathbb{R}}(s+n)^{\alpha_n} \right) \cdot \left(\prod_{m \in \mathbb{Z}} \Gamma_{\mathbb{C}}(s+m)^{\beta_m} \right) \cdot \sum_{k=1}^{+\infty} \frac{a_k}{k^s}$$

where $\omega \in \mathbb{R}^\times$ and $(\alpha_n), (\beta_m) \in \mathbb{N}^{\mathbb{Z}}$ are two sequences with finite support. Here the two Γ-factors $\Gamma_{\mathbb{C}}(s) := (2\pi)^{-s}\Gamma(s)$ and $\Gamma_{\mathbb{R}}(s) := (\pi^{-s/2}/\sqrt{2}) \cdot \Gamma(s/2)$ are normalised according to Deninger (see Remark 3.2.8). Now, in order to recover $\widehat{f}(s)$ from its values at the integers one shows that $1/\widehat{f}(s) \in \mathcal{F}_H$, by the following steps:

- the reciprocal of a (classical) non-zero Dirichlet series $f(s) = \sum_{k=1}^{+\infty} a_k k^{-s}$ with finite abscissa of absolute convergence is again a Dirichlet series with finite abscissa of absolute convergence. Hence $1/f(s) \in \mathcal{F}_H^0$;
- $\Gamma_{\mathbb{R}}(s+n)^{-1}, \Gamma_{\mathbb{C}}(s+n)^{-1} \in \mathcal{F}_H$ for every $n \in \mathbb{Z}$. This is a consequence of Stirling's formula, and we refer the reader to [Den00, Proposition 4.1] for further details;
- $\omega^{-s}, (2\pi)^s, \sqrt{2}\pi^{s/2} \in \mathcal{F}_H^0$, as it is easy to see.

3.2 Constructing the motivic L-functions

Fix two number fields F and E. The aim of this section is to recall the procedure which associates to a (mixed) motive $M \in \mathcal{MM}(F;E)$ an L-function $L(M,s) \in \mathfrak{R}_\sigma \to (E \otimes_{\mathbb{Q}} \mathbb{C})$, which is conjectured to have a meromorphic continuation to the whole complex plane (see Conjecture 3.3.4), and to satisfy a functional equation (see Conjecture 3.3.6). The validity of these conjectures would imply, using what we have developed in the previous section, that $L(E,s)$ is determined either by the sequence $\{L(M,n) : n \in \mathbb{Z},\ n \geq n_0\}$ for any $n_0 \in \mathbb{Z}$ such that $n_0 > \sigma$, or by the sequence $\{L^*(M,n) : n \in \mathbb{Z},\ n \leq n_0\}$ for any $n_0 \in \mathbb{Z}$. Here $L^*(M,n) \in (E \otimes \mathbb{C})^\times$ denotes the special value

$$L^*(M,n) := \lim_{s \to n} \frac{L(M,s)}{(s-n)^{\mathrm{ord}_{s=n}(L(M,s))}}$$

which we defined in Remark 3.1.1. Then we describe in Section 3.3.2 how these special values $L^*(M,n)$ are supposed to be connected to regulators and to other arithmetic invariants associated to M. Finally, Section 3.3.3 is dedicated to the study of some specific examples of the validity of these conjectures concerning special values.

3.2.1 Preliminaries

If we want to define the L-function associated to a mixed motive $M \in \mathcal{MM}(F;E)$ defined over a number field F with coefficients in a number field E, we run immediately into a problem, because the abelian category $\mathcal{MM}(F;E)$ is not unequivocally defined. We content ourselves

with using one of the tentative definitions given by Jannsen, Huber and Nori, which we recalled in Section 2.2.2. To be more precise, the categories $\mathcal{MM}_F^{(J)}$ and $\mathcal{MM}_F^{(H)}$ defined by Jannsen and Huber are $\mathbb{Q}$-linear, but they can be turned into E-linear categories $\mathcal{MM}^{(J)}(F;E)$ and $\mathcal{MM}^{(H)}(F;E)$ using the following general procedure.

Proposition 3.2.1 – Additive categories with coefficients (see [Del79, § 2.1])

Let $\mathcal{A}$ be $\mathbb{Q}$-linear category, *i.e.* an additive category such that for every $X, Y \in \mathcal{A}$ the homomorphism group $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ is actually a $\mathbb{Q}$-vector space. Suppose also that $\mathcal{A}$ is pseudo-abelian and fix a number field E. Consider the following two categories:

- the category $\mathrm{Mod}_E(\mathcal{A})$ with objects (X,ρ) where $X \in \mathcal{A}$ and $\rho\colon E \to \mathrm{End}(X)$ is a map of $\mathbb{Q}$-algebras with unity. The morphisms $(X,\rho_X) \to (Y,\rho_Y)$ are those maps $f \in \mathrm{Hom}_{\mathcal{A}}(X,Y)$ such that $f \circ \rho_X(e) = \rho_Y(e) \circ f$ for all $e \in E$;
- the category $(\mathcal{A} \otimes_{\mathbb{Q}} E)^{\natural}$, which is the pseudo-abelian envelope (see [And04, § 1.1.3.1]) of the category $\mathcal{A} \otimes_{\mathbb{Q}} E$ whose objects are the same as $\mathcal{A}$ and whose morphisms are defined by $\mathrm{Hom}_{\mathcal{A}\otimes E}(X,Y) := \mathrm{Hom}_{\mathcal{A}}(X,Y) \otimes_{\mathbb{Q}} E$.

For every $X \in \mathcal{A}$ we denote by $(X_E, \rho_{X_E}) \in \mathrm{Mod}_E(\mathcal{A})$ the object characterised up to isomorphism by the fact that

$$\mathrm{Hom}_{\mathrm{Mod}_E(\mathcal{A})}((X_E,\rho_{X_E}),(Y,\rho_Y)) = \mathrm{Hom}_{\mathbb{Q}}(E, \mathrm{Hom}_{\mathcal{A}}(X,Y))$$

for every $(Y,\rho_Y) \in \mathrm{Mod}_E(\mathcal{A})$. Then the association $X \mapsto X_E$ gives rise to an equivalence of categories $(\mathcal{A} \otimes_{\mathbb{Q}} E)^{\natural} \xrightarrow{\sim} \mathrm{Mod}_E(\mathcal{A})$. Moreover, if $\mathcal{A}$ is abelian then $\mathrm{Mod}_E(\mathcal{A})$ and $(\mathcal{A} \otimes_{\mathbb{Q}} E)^{\natural}$ are abelian.

The categories $\mathcal{MM}(F;E)$ that we consider in this section are obtained by applying Proposition 3.2.1 to Huber's category $\mathcal{MM}_F^{(H)}$. They are expected to be equivalent to Nori's category $\mathcal{MM}_{F,E}^{(N)}$ which was defined in Section 2.2.2.

Remark 3.2.2. The reason why one wants to consider categories $\mathcal{MM}(F;E)$ of motives with coefficients is that certain L-functions cannot be obtained with $\mathbb{Q}$-coefficients, The most important example of this is given by Artin's L-functions.

On the other hand, for every finite extension $F \subseteq F'$ we have an adjunction

$$\pi^*\colon \mathcal{MM}(F;E) \rightleftarrows \mathcal{MM}(F';E)\colon \pi_*$$

where $\pi\colon \mathrm{Spec}(F') \to \mathrm{Spec}(F)$ denotes the structural morphism. More precisely, π^* is the motivic analogue of the schematic base-change $X \mapsto X \times_F \mathrm{Spec}(F')$ and π_* is the motivic analogue of the schematic Weil restriction $X \mapsto \mathrm{N}_{F'/F}(X)$ (see [BLR90, § 7.6]). All the conjectures that we mention in this section are compatible with base-change and the Weil restriction, hence the reader might assume $F = \mathbb{Q}$ in what follows.

As we have seen in Section 2.2.2 and Section 2.4, all the abelian categories of mixed motives are supposed to be endowed with realisation functions, and Jannsen's and Huber's categories are in fact constructed starting from the categories of mixed realisations. In particular, for every rational prime $\ell \in \mathbb{N}$ the category $\mathcal{MM}(F;E)$ is endowed with a realisation functor

$$R_\ell\colon \mathcal{MM}(F;E) \to \mathrm{Rep}_{\mathrm{cont}}(\mathrm{Gal}(\overline{F}/F); E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$$

into the category of continuous Galois representations $\rho\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}(V)$, where V is a finitely generated module over $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Note in particular that, using the isomorphism $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong \prod_{\lambda|\ell} E_\lambda$, where the product runs over all the places of E lying above ℓ, one can view ρ as the collection of the λ-adic representations $\rho \otimes E_\lambda\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}(V \otimes E_\lambda)$.

Notation 3.2.3. From now until the end of the chapter, we denote by $\mathcal{G}_\kappa := \mathrm{Gal}(\overline{\kappa}/\kappa)$ the absolute Galois group of a field κ.

Remark 3.2.4 (Weil-Deligne representations and independence of ℓ). The Galois representations $\{R_\ell(M)\}_\ell$, or at least their semi-semplifications, are supposed to be independent of ℓ. A good way to state this is to use the discrete representations of the Weil-Deligne group. More precisely, for every prime ℓ and every place v of F we have a commutative square

$$\begin{array}{ccc} \mathcal{MM}(F;E) & \xrightarrow{R_\ell} & \mathrm{Rep}_{\mathrm{cont}}(\mathcal{G}_F; E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \\ \downarrow & & \downarrow \\ \mathcal{MM}(F_v;E) & \xrightarrow{R_\ell} & \mathrm{Rep}_{\mathrm{cont}}(\mathcal{G}_{F_v}; E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \end{array} \tag{3.10}$$

where F_v denotes the v-adic completion of F, and $\mathcal{MM}(F_v;E)$ denotes the abelian category of mixed motives over F_v with coefficients in E. This can be constructed by fixing an embedding $\iota\colon F_v \hookrightarrow \mathbb{C}$ and using Jannsen's, Huber's or Nori's formalism, although this construction should not depend on the choice of ι. Nevertheless, one expects to be able to attach at the bottom of the commutative square (3.10) the triangle

$$\begin{array}{ccc} \mathcal{MM}(F_v;E) & \xrightarrow{R_\ell} & \mathrm{Rep}_{\mathrm{cont}}(\mathcal{G}_{F_v}; E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \\ & \overset{R_{\mathcal{WD}}}{\searrow} & \downarrow \underline{\mathcal{WD}} \\ & & \mathrm{Rep}(\mathcal{WD}_{F_v}; E \otimes_{\mathbb{Q}} \mathbb{C}) \end{array} \tag{3.11}$$

which is supposed to commute only up to natural isomorphism. Here $\mathcal{WD}_{F_v}$ is the Weil-Deligne group of F_v (see [Tat79, Definition 4.1.2]), and $\mathrm{Rep}(\mathcal{WD}_{F_v}; E \otimes_{\mathbb{Q}} \mathbb{C})$ denotes the category of discrete representations $\mathcal{WD}_{F_v} \to \mathrm{GL}(V)$ where V is a finite dimensional module over $E \otimes_{\mathbb{Q}} \mathbb{C}$. Moreover, the functor $R_{\mathcal{WD}}\colon \mathcal{MM}(F_v;E) \dashrightarrow \mathrm{Rep}(\mathcal{WD}_{F_v}; E \otimes_{\mathbb{Q}} \mathbb{C})$ should not depend on any auxiliary data (*e.g.* the prime ℓ or an embedding $F_v \hookrightarrow \mathbb{C}$). Finally the functor

$$\underline{\mathcal{WD}}\colon \mathrm{Rep}_{\mathrm{cont}}(\mathcal{G}_{F_v}; E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \to \mathrm{Rep}(\mathcal{WD}_{F_v}; E \otimes_{\mathbb{Q}} \mathbb{C})$$

is the one described in [Fon94, § 2.3.7], composed with a suitable base-change to $\mathbb{C}$. This functor depends in general on some choices, which is the reason why the triangle (3.11) is supposed to commute only up to natural isomorphism.

An interesting consequence of the expectations outlined in Remark 3.2.4 would be that, for every $M \in \mathcal{MM}(F;E)$ and every $\sigma \in \mathcal{G}_F$ the characteristic polynomial

$$\det(1 - T \cdot \sigma \mid R_\ell(M)) \in (E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)[T]$$

should be independent of ℓ. This can either be assumed in the construction of the L-function $L(M,s)$, or the latter can be constructed using purely p-adic methods, as we recount in the following sections.

3.2.2 non-Archimedean local L-factors

Let us now turn to the construction of the motivic L-function, following [Fon92, § 3] and [Den94]. We start by recalling the construction of the L-function associated to a continuous λ-adic Galois representation $\rho\colon \mathcal{G}_K \to \mathrm{GL}(V)$, where K is a finite extension of $\mathbb{Q}_p$ for some rational prime $p \in \mathbb{N}$ and V is a finite dimensional vector space over the λ-adic completion E_λ of a number field E, associated to some non-Archimedean place $\lambda \in M_E^0$.

To give the precise definition of $L(\rho, s)$ we need to introduce a fair amount of notation. First of all, we let $\ell \in \mathbb{N}$ be the rational prime lying below λ, and we write

$$\underline{D}(\rho) := \begin{cases} V^{\mathcal{I}_K}, & \text{if } \ell \neq p \\ (B_{\mathrm{crys}}(\mathcal{O}_{\overline{K}}) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_K}, & \text{if } \ell = p \end{cases} \tag{3.12}$$

where $\mathcal{G}_K$ denotes the absolute Galois group of K and $\mathcal{I}_K \subseteq \mathcal{G}_K$ denotes its inertia sub-group, given by all the elements which induce the trivial map on the residue field κ. Hence if $\ell \neq p$ we see that $\underline{D}(\rho)$ is a finite dimensional vector space over $E_{\lambda,p} := E_\lambda$, given by the elements of V which are invariant under the action of the inertia group $\mathcal{I}_K$. On the other hand, if $\ell = p$ then $\underline{D}(\rho)$ is a free module over $E_{\lambda,p} := K_0 \otimes_{\mathbb{Q}_p} E_\lambda$, where $K_0 \subseteq K$ denotes the maximal sub-field which is unramified over $\mathbb{Q}_p$. Indeed, this follows from the fact that the crystalline period ring $B_{\mathrm{crys}}(\mathcal{O}_{\overline{K}})$ is an algebra over $\overline{K}_0 \subseteq \overline{K}$, which is again defined as the maximal sub-field of $\overline{K}$ unramified over $\mathbb{Q}_p$ (and *not* as the algebraic closure of K_0). Equivalently, $\overline{K}_0$ can be described as the field of fractions of the ring of p-typical Witt vectors $W_{p^\infty}(\overline{\kappa_v})$ of the algebraic closure of the residue field κ_v of K.

Remark 3.2.5. Let us briefly recall the definition of Fontaine's period ring functor B_{crys}, following [Car19, § 3.2]. This functor depends on the choice of a prime p, that we fix.

First of all, we define the p-adic tilt of a ring R to be $R^\flat := \varprojlim_\phi R/p$, where $\phi\colon R/p \to R/p$ denotes the Frobenius map $\phi(x) := x^p$. Then one defines $A_{\mathrm{inf}}(R) := W_{p^\infty}(R^\flat)$ to be the ring of p-typical Witt vectors on $R^\flat$. If R is p-adically complete then $R^\flat$ is a perfect $\mathbb{F}_p$-algebra and we get a map $\theta_R\colon A_{\mathrm{inf}}(R) \to R$, uniquely defined by the commutative square

$$\begin{array}{ccc} A_{\mathrm{inf}}(R) & \twoheadrightarrow & R^\flat \\ \downarrow{\scriptstyle \theta_R} & & \downarrow \\ R & \twoheadrightarrow & R/p \end{array}$$

where $R \twoheadrightarrow R/p$ is the canonical projection, $R^\flat \twoheadrightarrow R/p$ is the projection onto the first factor and $A_{\mathrm{inf}}(R) \twoheadrightarrow R^\flat$ is also the projection onto the first factor of an inverse limit, namely

$$A_{\mathrm{inf}}(R) := W_{p^\infty}(R^\flat) := \varprojlim W_{p^n}(R^\flat)$$

where $W_p(R^\flat) = R^\flat$ by definition.

Now, let us recall that a ring R is called *perfectoid* (with respect to the prime p) if:

- R is p-adically complete;
- the Frobenius map $\phi\colon R/p \to R/p$ is surjective;
- there exists $\omega \in R$ such that $p \cdot R = \omega^p \cdot R$;
- $\ker(\theta_R)$ is principal.

If R is perfectoid of characteristic zero, one can define the following *period rings* associated to R:

- $A_{\mathrm{crys}}(R)$, which is the p-adic completion of the $A_{\mathrm{inf}}(R)$-algebra generated by the elements $\xi^n/n! \in A_{\mathrm{inf}}(R)[1/p]$, where $\xi \in A_{\mathrm{inf}}(R)$ is any generator of $\ker(\theta_R)$;
- $B^+_{\mathrm{crys}}(R) := A_{\mathrm{crys}}(R)[1/p]$ which is the ring obtained by inverting p in $A_{\mathrm{crys}}(R)$;
- $B^+_{\mathrm{dR}}(R) := B^+_{\mathrm{crys}}(R)^{\wedge_\xi}$ which is the ξ-adic completion of $B^+_{\mathrm{crys}}(R)$;
- $B_{\mathrm{dR}}(R) := B^+_{\mathrm{dR}}(R)[1/\xi]$ which is the ring obtained by inverting ξ in $B^+_{\mathrm{dR}}(R)$.

One sees immediately that the previous definitions do not depend on the choice of ξ. Suppose finally that $\{x \in R \mid x^{p^n} = 1\} \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$ for every $n \in \mathbb{N}$, *i.e.* that R contains all the p-power roots of unity. We can choose a compatible system of primitive roots $\varepsilon = (\dots, \varepsilon_2, \varepsilon_1, \varepsilon_0)$, where $\varepsilon_j \in R$ is a primitive p^j-th root of unity (hence $\varepsilon_0 = 1$) and $\varepsilon_j^p = \varepsilon_{j-1}$. This induces an element $\varepsilon \in R^\flat$, and we can consider its image $[\varepsilon] \in A_{\mathrm{inf}}(R)$, where $[\cdot]\colon R^\flat \to A_{\mathrm{inf}}(R)$ denotes the Teichmüller map. Then it is easy to see that the formal power series

$$\log([\varepsilon]) := -\sum_{n=1}^{+\infty} \frac{(1-[\varepsilon])^n}{n}$$

converges in $A_{\mathrm{crys}}(R)$. Ultimately, one defines the last period ring:

- $B_{\mathrm{crys}}(R) := B^+_{\mathrm{crys}}(R)[1/\log([\varepsilon])]$, which is the ring obtained by inverting the element $\log([\varepsilon]) \in A_{\mathrm{crys}}(R) \subseteq B^+_{\mathrm{crys}}(R)$ in $B^+_{\mathrm{crys}}(R)$.

It is straightforward to check that this definition does not depend on the choice of ε.

The main examples of perfectoid rings R which contain all p-power roots of unity are given by $R = \mathcal{O}_{\mathbb{C}_p}$ and $R = \mathcal{O}_{\overline{K}}$ for any K which is a finite extension of $\mathbb{Q}_p$. Let us concentrate on this second case. Unravelling the definitions we see that $B_{\mathrm{dR}}(\mathcal{O}_{\overline{K}})$ is a $\overline{K}$-algebra and that $B_{\mathrm{crys}}(\mathcal{O}_{\overline{K}}) \subseteq B_{\mathrm{dR}}(\mathcal{O}_{\overline{K}})$ is a $\overline{K}_0$ sub-algebra. Moreover, $B_{\mathrm{dR}}(\mathcal{O}_{\overline{K}})$ is endowed with a decreasing filtration and with an action of $\mathcal{G}_K$, which induces an action on $B_{\mathrm{crys}}(\mathcal{O}_{\overline{K}}) \subseteq B_{\mathrm{dR}}(\mathcal{O}_{\overline{K}})$. Finally, $B_{\mathrm{crys}}(\mathcal{O}_{\overline{K}})$ is endowed with a map of abelian groups $\varphi\colon B_{\mathrm{crys}}(\mathcal{O}_{\overline{K}}) \to B_{\mathrm{crys}}(\mathcal{O}_{\overline{K}})$ which commutes with the action of $\mathcal{G}_K$ and satisfies the equality $\varphi(ay) = \Phi_K(a) \cdot \varphi(y)$ for every $x \in B_{\mathrm{crys}}(\mathcal{O}_{\overline{K}})$ and $a \in \overline{K}_0$. Here $\Phi_K\colon \overline{K}_0 \to \overline{K}_0$ is the unique field automorphism which lifts the map $\phi_\kappa\colon \overline{\kappa} \to \overline{\kappa}$ defined on the algebraic closure of the residue field κ by setting $\phi_\kappa(x) := x^p$.

Now, let us observe that $\underline{D}(\rho)$, which we defined in (3.12) as a free, finitely generated module over $E_{\lambda,p}$, is endowed with a $E_{\lambda,p}$-linear endomorphism $f_\rho\colon \underline{D}(\rho) \to \underline{D}(\rho)$. Indeed, if $\ell \neq p$ then we see that $\underline{D}(\rho) := V^{\mathcal{I}_K}$ is endowed with an action of

$$\mathcal{G}_K/\mathcal{I}_K \cong \mathrm{Gal}(\overline{K}_0/K_0) \cong \mathrm{Gal}(\overline{\kappa}/\kappa)$$

and this group is topologically generated by the geometric Frobenius

$$f_K := \Phi_K^{-[\kappa\colon \mathbb{F}_p]} \tag{3.13}$$

where $\Phi_K\colon \overline{K}_0 \to \overline{K}_0$ is the map defined at the end of Remark 3.2.5. Thus in the case $\ell \neq p$ one simply defines f_ρ to be the map induced by the action of f_K over $\underline{D}(\rho)$. On the other hand, if $\ell = p$ the map $f_\rho\colon (B_{\mathrm{crys}}(\mathcal{O}_{\overline{K}}) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_K} \to (B_{\mathrm{crys}}(\mathcal{O}_{\overline{K}}) \otimes_{\mathbb{Q}_p} V)^{\mathcal{G}_K}$ is induced by the map

$\varphi^{[\kappa:\mathbb{F}_p]} \otimes \mathrm{Id}_V$, where $\varphi\colon B_{\mathrm{crys}}(\mathcal{O}_{\overline{K}}) \to B_{\mathrm{crys}}(\mathcal{O}_{\overline{K}})$ is the map mentioned in Remark 3.2.5. We can finally define the L-function

$$L(\rho, s)_{E_\lambda} := \det(1 - |\kappa|^{-s} \cdot f_\rho \mid \underline{D}(\rho)) \in E_\lambda[|\kappa|^{-s}]$$

which is the characteristic polynomial of f_ρ evaluated at $|\kappa|^{-s}$. The notation $\det(- \mid \underline{D}(\rho))$ is used simply to stress the fact that we are taking the determinant of an endomorphism of $\underline{D}(\rho)$. Observe moreover that if $\ell = p$ then $L(\rho, s)$ has coefficients in $(E_\lambda)_0 \subseteq E_\lambda$ (see [FP94, Chapitre I, Remarque 1.3.3, (ii)]).

Finally, let ρ be an ℓ-adic Galois representation of K with coefficients in E, by which we mean a continuous group homomorphism $\rho\colon : \mathcal{G}_K \to \mathrm{GL}(V)$ where K is a finite extension of $\mathbb{Q}_p$ for some rational prime $p \in \mathbb{N}$ and V is a free, finitely generated module over $E_\ell := E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Then we can still define an L-function

$$L(\rho, s)_{E_\ell} = (L(\rho, s)_{E_\lambda})_\lambda \in E_\ell[|\kappa|^{-s}]$$

using the isomorphism $E_\ell := E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong \prod_\lambda E_\lambda$, where λ runs over all the places lying over the rational prime $\ell \in \mathbb{N}$. Then we can make the following conjecture, which is related the questions of independence of ℓ that we explored in Remark 3.2.4.

Conjecture 3.2.6 – Coefficients of the non-Archimedean L-factors

Let $p \in \mathbb{N}$ be a rational prime and K a finite extension of $\mathbb{Q}_p$. Fix another rational prime $\ell \in \mathbb{N}$ and a number field E, and let V be a free, finitely generated module over $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Then for every continuous group homomorphism $\rho\colon : \mathcal{G}_K \to \mathrm{GL}(V)$ we have that

$$L(\rho, s)_{E_\ell} \in E[|\kappa|^{-s}]$$

where κ denotes the residue field of K.

Under the validity of Conjecture 3.2.6, one can associate a complex L-function $L_{\mathbb{C}}(\rho, s)$ to every ℓ-adic Galois representation $\rho\colon \mathcal{G}_K \to \mathrm{GL}(V)$ with coefficients in a number field E. Indeed, one simply sets

$$L(\rho, s)_{\mathbb{C}} := (\sigma(L(\rho, s)_{E_\ell}))_\sigma \in (E \otimes_{\mathbb{Q}} \mathbb{C})[|\kappa|^{-s}] \tag{3.14}$$

where σ runs over the embeddings $\sigma\colon E \hookrightarrow \mathbb{C}$, using the isomorphism $E \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^{\mathrm{Hom}(E,\mathbb{C})}$.

3.2.3 Archimedean local L-factors

Before going back to the definition of the L-function of a motive $M \in \mathcal{MM}(F; E)$ let us give a sort of Archimedean analogue of the previous paragraphs. More precisely, we are going to define the L-function $L(H, s)$ associated to a mixed Hodge structure $H \in \mathrm{MHS}(K; E)$ defined over an Archimedean local field K and having coefficients in a number field E. First of all, we recall the definition of the category $\mathrm{MHS}(K; E)$, which we implicitly used in Definition 2.2.7.

Definition 3.2.7 – Mixed Hodge structures

Let K be an Archimedean local field (*i.e.* $K \in \{\mathbb{R}, \mathbb{C}\}$). Then the category $\mathrm{MHS}(K;\mathbb{Q})$ is defined as follows:

- $\mathrm{MHS}(\mathbb{C};\mathbb{Q})$ is the category of triples $\underline{H}_{/\mathbb{C}} = (H, W_\bullet(H), F^\bullet(H_\mathbb{C}))$ where H is a finite dimensional $\mathbb{Q}$-vector space endowed with an increasing filtration (called *weight filtration*) $W_\bullet(H)$ such that there exist $i_0, i_1 \in \mathbb{Z}$ with $W_i(H) = 0$ for $i \leq i_0$ and $W_i(H) = H$ for $i \geq i_1$, and $F^\bullet(H_\mathbb{C})$ is a decreasing filtration (called *Hodge filtration*) on $H_\mathbb{C} := H \otimes_\mathbb{Q} \mathbb{C}$ such that the three filtrations $\{W_\bullet, F^\bullet, \overline{F}^\bullet\}$ formed by the weight filtration, the Hodge filtration and its complex conjugate are opposed, which means that $\mathrm{gr}_F^p(\mathrm{gr}_{\overline{F}}^q(\mathrm{gr}_n^W(H_\mathbb{C}))) = 0$ if $p + q \neq n$;
- $\mathrm{MHS}(\mathbb{R};\mathbb{Q})$ is the category of pairs $\underline{H}_{/\mathbb{R}} = (\underline{H}_{/\mathbb{C}}, \rho)$ where $\underline{H}_{/\mathbb{C}} \in \mathrm{MHS}(\mathbb{C};\mathbb{Q})$ and $\rho\colon \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \mathrm{Aut}(\underline{H}_{/\mathbb{C}})$ is an action of complex conjugation on $\underline{H}_{/\mathbb{C}}$. This amounts to a direct sum decomposition $H = H^+ \oplus H^-$ at the level of $\mathbb{Q}$-vector spaces, which is compatible with the weight filtration and such that the Hodge filtration on $H_\mathbb{C}$ is induced from a filtration defined over $(H_\mathbb{C})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} := H^+ \oplus i \cdot H^-$, considered as a real vector space.

Finally, let E be a number field, and $K \in \{\mathbb{R}, \mathbb{C}\}$ be an Archimedean local field. Then the category $\mathrm{MHS}(K;E)$ of rational mixed Hodge structures over K with coefficients in E is defined as $\mathrm{MHS}(K;E) := \mathrm{Mod}_E(\mathrm{MHS}(K;\mathbb{Q})) \simeq (\mathrm{MHS}(K;\mathbb{Q}) \otimes_\mathbb{Q} E)^\natural$, using Proposition 3.2.1.

Now, let us define the local Archimedean L-function associated to a mixed Hodge structure $\underline{H}_{/K} \in \mathrm{MHS}(K;E)$, where K is an Archimedean local field and E is a number field. First of all, let us observe that $H_\mathbb{C}$ is a free module over the ring $E \otimes_\mathbb{Q} \mathbb{C} \cong \mathbb{C}^{\mathrm{Hom}(E,\mathbb{C})}$. Moreover, $H_\mathbb{C}$ supports the decreasing filtration $\gamma^\bullet(H_\mathbb{C})$, given by

$$\gamma^n(H_\mathbb{C}) := F^n(H_\mathbb{C}) \cap \overline{F}^n(H_\mathbb{C}) = (F^n(H_\mathbb{C}) \cap H) \otimes \mathbb{C}$$

and the sub-spaces $\gamma^n(H_\mathbb{C}) \subseteq H_\mathbb{C}$ are modules over $(E \otimes \mathbb{C})$, which are also invariant under the action $\rho\colon \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \mathrm{Aut}(\underline{H}_{/\mathbb{C}})$ if $K = \mathbb{R}$. This allows one to define, for every $j \in \mathbb{Z}$ and every $\sigma \in \mathrm{Hom}(E, \mathbb{C})$ the number

$$n_{j,\sigma}(\underline{H}_{/\mathbb{C}}) := \dim_\mathbb{C}(\mathrm{gr}_\gamma^j(H_\mathbb{C}) \otimes_{E\otimes\mathbb{C},\sigma} \mathbb{C})$$

associated to every mixed Hodge structure $\underline{H}_{/\mathbb{C}} \in \mathrm{MHS}(\mathbb{C};E)$ defined over $K = \mathbb{C}$. In a similar fashion, one can define for every $j \in \mathbb{Z}$, every $\sigma \in \mathrm{Hom}(E, \mathbb{C})$ and every $\varepsilon \in \{\pm 1\}$ a number

$$n_{j,\sigma}(\underline{H}_{/\mathbb{R}})^\varepsilon := \dim_\mathbb{R}\left((\mathrm{gr}_\gamma^j(H_\mathbb{C}) \otimes_{E\otimes\mathbb{C},\sigma} \mathbb{C})^{F_\infty = \varepsilon}\right)$$

associated to every mixed Hodge structure $\underline{H}_{/\mathbb{R}} \in \mathrm{MHS}(\mathbb{R};E)$ defined over $K = \mathbb{R}$. Here $F_\infty\colon \mathbb{C} \to \mathbb{C}$ denotes complex conjugation, *i.e.* the unique non-trivial element $F_\infty \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

This being said, we can define the local L-function associated to each mixed Hodge structure $\underline{H}_{/K} \in \mathrm{MHS}(K;E)$ as the function

$$L(\underline{H}_{/K}, s)_\mathbb{C} = (L(\underline{H}_{/K}, s)_\sigma)_{\sigma\in\mathrm{Hom}(E,\mathbb{C})} : \mathbb{C} \to (E \otimes \mathbb{C}) \tag{3.15}$$

with components $L(\underline{H}_{/K}, s)_\sigma : \mathbb{C} \to \mathbb{C}$ which are given by

$$L(\underline{H}_{/K}, s)_\sigma := \begin{cases} \prod_{j\in\mathbb{Z}} \Gamma_{\mathbb{R}}(s-j+\varepsilon_j)^{n^+_{j,\sigma}(\underline{H}_{/\mathbb{R}})} \cdot \Gamma_{\mathbb{R}}(s-j+(1-\varepsilon_j))^{n^-_{j,\sigma}(\underline{H}_{/\mathbb{C}})}, & \text{if } K=\mathbb{R} \\ \prod_{j\in\mathbb{Z}} \Gamma_{\mathbb{C}}(s-j)^{n_{j,\sigma}(\underline{H}_{/\mathbb{C}})}, & \text{if } K=\mathbb{C} \end{cases}$$

where, for every $j \in \mathbb{Z}$, the number $\varepsilon_j \in \{0,1\}$ is defined by the congruence $\varepsilon_j \equiv j (2)$.

Remark 3.2.8 (Gamma factors). We remark that in this thesis we have decided to consider the Γ-factors $\Gamma_{\mathbb{C}}(s) := (2\pi)^{-s}\Gamma(s)$ and $\Gamma_{\mathbb{R}}(s) := (\pi^{-s/2}/\sqrt{2}) \cdot \Gamma(s/2)$ as normalised by Deninger (see [Den91]). As explained in [FP94, Remarque 1.2.6], one may replace $(\Gamma_{\mathbb{R}}(s), \Gamma_{\mathbb{C}}(s))$ by $(a\Gamma_{\mathbb{R}}(s), a^2\Gamma_{\mathbb{C}}(s))$ for every $a \in \mathbb{C}^\times$, without changing the good properties of the pair $(\Gamma_{\mathbb{R}}, \Gamma_{\mathbb{C}})$. Choosing $a = \sqrt{2}$ one gets the Γ-factors used by Deligne in [Del79, § 5.3].

3.2.4 The global L-function

Let us finally turn to the definition of the L-functions associated to a mixed motive defined over a number field F with coefficients in another number field E.

Definition 3.2.9 – The L-function of a mixed motive

Let F and E be two number fields, and let $\mathcal{MM}(F;E)$ be the category of mixed motives defined over F with coefficients in E (see Section 3.2.1). Fix a finite set of places $S \subseteq M_F$ of F, and a motive $X \in \mathcal{MM}(F;E)$.

For every non-Archimedean place $v \in M_F^0 \setminus S$ lying above a rational prime $p \in \mathbb{N}$ we let $R_p(X) \in \mathrm{Rep}_{\mathrm{cont}}(\mathcal{G}_F, E \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ denote the p-adic realisation, and $R_v(X)$ denote the restriction of $R_p(X)$ to the absolute Galois group $\mathcal{G}_{F_v} \subseteq \mathcal{G}_F$. Moreover, for every Archimedean place $v \in M_F^\infty \setminus S$ we let $R_v(X) \in \mathrm{MHS}(F_v; E)$ denote the mixed Hodge structure coming from the rational Betti realisation of X. Observe that $R_v(X)$ has still coefficients in E, despite the fact that we are taking the rational Betti realisation, because X does.

Now, assuming Conjecture 3.2.6 for the family $\{R_v(X) : v \in M_F^0 \setminus S\}$, we define the L-function $L_S(X,s)$ as a formal Euler product

$$L_S(X,s) := \prod_{v\in M_F\setminus S} L(R_v(X), s)_{\mathbb{C}} \tag{3.16}$$

where the local L-factors appearing in the product are the ones defined in (3.14) and (3.15).

We introduce the notation $L(X,s) := L_{M_F^\infty}(X,s)$ and $\widehat{L}(X,s) := L_\emptyset(X,s)$.

Remark 3.2.10. Let us explain the relation between Definition 3.2.9 and the more common ℓ-adic definition of motivic L-functions. Let F, E be two number fields, and recall that for every non-Archimedean place $\lambda \in M_E^0$ one can define the λ-adic realisation $R_\lambda(X)$ of a motive $X \in \mathcal{MM}(F;E)$, which is a λ-adic representation of the global Galois group $\mathcal{G}_F$. Hence for every non-Archimedean place $v \in M_F^0$ one can consider the restriction $R_\lambda(X)_v$ of $R_\lambda(X)$ to the local Galois group $\mathcal{G}_{F_v}$. Fix a finite set of places $S \subseteq M_F$, and assume that $L(R_\lambda(X)_v, s)_{E_\lambda} \in E[|\kappa_v|^{-s}]$ for every $v \in M_F^0 \setminus S$, where κ_v denotes the residue field of F_v.

Then one can consider the formal Euler product

$$\prod_{v \in M_F \setminus S} L(R_\lambda(X)_v, s)_{\mathbb{C}} \tag{3.17}$$

where, for every non-Archimedean place $v \in M_F^0 \setminus S$ we define $L(R_\lambda(X)_v, s)_{\mathbb{C}}$ as in (3.14), and for every Archimedean $v \in M_F^\infty \setminus S$ we define $R_\lambda(X)_v := R_v(X)$ as in Definition 3.2.9. The various conjectures on the independence of ℓ imply that the Euler product (3.17) should coincide with $L_S(X, s)$ as defined in Definition 3.2.9. In particular, the product (3.17) should not depend on the non-Archimedean place $\lambda \in M_E^0$. This is particularly useful because it would be possible to choose λ as a place of "good reduction" for the motive X, which makes many computations easier. On the other hand, Definition 3.2.9 has the advantage of not depending on the choice of an auxiliary place $\lambda \in M_E^0$.

Remark 3.2.11. If $X \in \mathcal{MM}(F; E)$ and $j \in \mathbb{Z}$ then the L-function of the Tate twist $X(j)$ satisfies $L_S(X(j), s) = L_S(X, j + s)$.

Remark 3.2.12. Let F and E be two number fields. To every mixed motive $M \in \mathcal{MM}(F; E)$, endowed with its weight filtration $W_\bullet(M)$, one can associate the semi-simplification

$$M^{\mathrm{ss}} := \bigoplus_{w \in \mathbb{Z}} \mathrm{gr}_w^W(M) \in \mathcal{MM}(F; E)^{\mathrm{ss}}$$

which could be identified with a numerical motive under the conjectural equivalence of categories $\mathrm{NM}(F; E) \simeq \mathcal{MM}(F; E)^{\mathrm{ss}}$. Then the various semi-simplicity conjectures for the ℓ-adic realisations of M imply that there is a strong relation between $L_S(M, s)$ and $L_S(M^{\mathrm{ss}}, s)$. Hence it would be nice to have an alternative to Definition 3.2.9, which would reflect more the mixed nature of M. Some attempts towards this program may be found in the recent work of Brown (in particular [Bro19b] and [Bro19a]). Moreover, the recent "derived" approach to ζ-functions of Campbell, Wolfson and Zakharevich (see [CWZ19]) and Campbell, Lind, Malkiewich, Ponto and Zakharevich (see [Cam+20]) might shed some light on the "correct" definition of $L(M, s)$ which captures the mixed nature of M.

3.3 Conjectures on motivic L-functions

The aim of this section is to state the main conjectures concerning motivic L-functions. As we have seen already in Section 3.2, the very construction of these L-functions is made possible only assuming a conjecture. namely Conjecture 3.2.6. This conjecture is known for the Galois representation $R_v(M)$ associated to every mixed motive $M \in \mathcal{MM}(F; E)$ of the form $M = \underline{H}^j(X)$, where X is a smooth and proper variety defined over F with good reduction at the place $v \in M_F^0$. Alternatively, one may neglect Conjecture 3.2.6 and choose instead a family of embeddings $(E_\lambda)_0 \hookrightarrow \mathbb{C}$ for every place $\lambda \in M_E^0$. This allows one to define unconditionally an L-function, and Conjecture 3.2.6 becomes more or less equivalent to the fact that this L-function is independent from the choice of these embeddings.

3.3.1 Convergence, meromorphic continuation and functional equations

Up until now, we have defined motivic L-functions as formal Euler products (3.16). This implies in particular that the L-function $L_S(M,s)$ associated to a mixed motive $M \in \mathcal{MM}(F;E)$ and a finite set of places $S \subseteq M_F$ can be seen as a formal Dirichlet series

$$L_S(M,s) = \sum_{n=1}^{+\infty} \frac{a_n(M,S)}{n^s} \tag{3.18}$$

having coefficients $a_n(M,S) \in E \otimes_{\mathbb{Q}} \mathbb{C}$. Hence we can associate to $L_S(M,s)$ an abscissa of convergence $\sigma_0(M,S) \in \mathbb{R} \cup \{\pm\infty\}$ (see [HR64, Chapter II, § 6]). This can be computed by the explicit formula $\sigma_0(M,S) := \max_{\iota \in \mathrm{Hom}(E,\mathbb{C})} \{\sigma_0(M,S)_\iota\}$, where

$$\sigma_0(M,S)_\iota := \begin{cases} \limsup\limits_{n\to+\infty} \dfrac{\log\left|\sum_{k=n+1}^{+\infty} a_k(M,S)_\iota\right|}{\log(n+1)}, & \text{if } \displaystyle\sum_{k=1}^{+\infty} a_k(M,S)_\iota \text{ converges} \\ \limsup\limits_{n\to+\infty} \dfrac{\log\left|\sum_{k=1}^{n} a_k(M,S)_\iota\right|}{\log(n)}, & \text{otherwise} \end{cases}$$

and the series (3.18) is known to converge for every $s \in \mathbb{C}$ such that $\Re(s) > \sigma_0(M,S)$, *i.e.* for every $s \in \mathfrak{R}_{\sigma_0(M,S)}$ in the notation of Section 3.1. This shows that one may regard the formal Euler product (3.16) as a holomorphic function $L_S(M,s) : \mathfrak{R}_{\sigma_0(M,S)} \to E \otimes_{\mathbb{Q}} \mathbb{C}$, and makes appealing the following conjecture.

Conjecture 3.3.1 – Convergence of motivic L-functions

Let F and E be two number fields, and let $S \subseteq M_F$ be a finite set of places. Then for every mixed motive $M \in \mathcal{MM}(F;E)$ we have that $\mathfrak{R}_{\sigma_0(M,S)} \neq \emptyset$, *i.e.* $\sigma_0(M,S) < +\infty$.

Remark 3.3.2. The work of Deligne on the Weil conjectures implies that Conjecture 3.3.1 holds for all motives $M \in \mathcal{MM}(F;E)$ of the form $M = \underline{H}^j(X)$ where X is a smooth and proper variety defined over F, under the assumption that the finite set of places $S \subseteq M_F$ contains the places of bad reduction of X. Moreover, in this case one has that $\sigma_0(\underline{H}^j(X),S) = j/2+1$.

Remark 3.3.3. Conjecture 3.3.1 is compatible with extensions. More precisely, if

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence in $\mathcal{MM}(F;E)$ and Conjecture 3.3.1 holds for M' and M'' then Conjecture 3.3.1 holds for M.

In particular, if one takes $\mathcal{MM}(F;E)$ to be Jannsen's abelian category of mixed motives (see Definition 2.2.7) then each $M \in \mathcal{MM}(F;E)$ is an iterated extension of motives of the form $\underline{H}^j(X)$ where X is smooth and proper over F. Hence Remark 3.3.2 shows that Conjecture 3.3.1 holds for every $M \in \mathcal{MM}(F;E)$, if S contains the finite set of places where all the smooth and proper varieties appearing in the extensions may have bad reduction. In particular, S contains

the set (conjectured to be empty) of places where M is not L-admissible, in the sense of [FP94, Chapitre III, § 2.1.5]. Moreover, one has that $\sigma_0(M,S) = w_{\max}(M)/2 + 1$, where

$$w_{\max}(M) := \max\{w \in \mathbb{Z} \mid \mathrm{gr}_w^W(M) \neq 0\}$$

denotes the maximum weight appearing in the weight filtration $W_\bullet(M)$ of M.

As we have seen, Conjecture 3.3.1 allows one to see the formal Euler product defining a motivic L-function as an actual holomorphic function $L_S(M,s)\colon \mathfrak{R}_{\sigma_0(M,S)} \to (E \otimes_{\mathbb{Q}} \mathbb{C})$. The theory of the Riemann ζ-function shows that a great amount of interesting information (*e.g.* the prime number theorem) can be derived from knowing that an L-function admits a meromorphic continuation to the whole complex plane. Moreover, the theory of automorphic L-functions (*e.g.* L-functions associated to modular forms) shows that often this meromorphic continuation is actually entire. This is summarised in the next conjecture.

Conjecture 3.3.4 – Meromorphic continuation of motivic L-functions

Let F and E be two number fields, and let $S \subseteq M_F$ be a finite set of places. Then for every motive $M \in \mathcal{MM}(F;E)$ we have that

$$L_S(M,s) = \frac{L_S^{(1)}(M,s)}{L_S^{(2)}(M,s)} \quad \text{for all } s \in \mathfrak{R}_{\sigma_0(M,s)}$$

where $L_S^{(1)}(M,s)\colon \mathbb{C} \to (E \otimes_{\mathbb{Q}} \mathbb{C})$ is an entire function of order of growth equal to one (see [SS03, Chapter 5, § 2]) and $L_S^{(2)}(M,s) \in (E \otimes_{\mathbb{Q}} \mathbb{C})[s]$ is a polynomial whose components $L_S^{(2)}(M,s)_\iota \in \mathbb{C}[s]$ associated to each embedding $\iota \in \mathrm{Hom}(E,\mathbb{C})$ have zeros in $\mathbb{Z}$.

Remark 3.3.5. We observe that there exist L-functions of automorphic origin which have poles that are not rational integers. Most notably, if $\tau \in \mathbb{R}$ and $\chi_\tau\colon \mathbb{A}_F^\times \to \mathbb{C}^\times$ is the character given by $\chi_\tau(s) := \|s\|^{-i\tau}$, the L-function $L(\chi,s)$ has two poles at $s = i\tau$ and $s = 1 + i\tau$ (see [RV99, Theorem 7-19]).

Essentially all the cases when Conjecture 3.3.4 is known are given by motivic L-functions which can be related to automorphic ones. The two most notable examples of this are the L-functions $L(\psi,s)$ associated to algebraic Hecke characters $\psi\colon \mathbb{A}_F^\times \to E^\times$ (see Remark 7.1.18) and the L-functions $L(f,s)$ associated to modular forms $f = \sum_{n=1}^{+\infty} a_n q^n \in S_k(\Gamma)$ (see [Sch90, § 1.2.4] for the construction of the motive $M(f) \in \mathcal{MM}(\mathbb{Q};\mathbb{Q}(\{a_n\}_{n\geq 1}))$ corresponding to f). In these two cases one can prove Conjecture 3.3.4 by appealing to harmonic analysis on two different kinds of objects: for Hecke characters one uses the Fourier transform on the locally compact abelian group $\mathbb{A}_F^\times$ (see [RV99, § 7]), and for modular forms one uses the Mellin transform for the locally compact abelian group $\mathbb{R}_{>0}$ (see [DS05, § 5.10]). Using the inversion theorems coming from harmonic analysis, one is able to prove that the completed L-functions $\widehat{L}(\psi,s)$ and $\widehat{L}(f,s)$ satisfy a functional equation. This initial evidence leads to conjecture that a similar functional equation might hold in general.

Conjecture 3.3.6 – Functional equation of motivic L-functions

Let F and E be two number fields. Then, for every motive $M \in \mathcal{MM}(F;E)$ there should exist two complex numbers $a(M), b(M) \in E \otimes_{\mathbb{Q}} \mathbb{C}$ such that $a(M) \neq 0$ and

$$\widehat{L}(M,s) = \varepsilon(M,s) \cdot \widehat{L}(M^\vee, 1-s) \tag{3.19}$$

where $\varepsilon(M,s) := a(M) \cdot e^{b(M)s}$ and $M^\vee \in \mathcal{MM}(F;E)$ is the dual (with respect to the tensor product) of M.

Remark 3.3.7. The ε-factor $\varepsilon(M,s)$ admits also a decomposition in an Euler product, as the L-function $\widehat{L}(M,s)$ itself. More precisely, we have a decomposition

$$\varepsilon(M,s) = \prod_{v \in M_F} \varepsilon_{\psi_v,\mu_v}(\underline{\mathcal{WD}}(R_v(M)), s)$$

where $\psi_v \colon F_v \to \mathbb{C}^\times$ and μ_v are the local components, for each place $v \in M_F$, of a continuous group homomorphism $\psi \colon \mathbb{A}_F/F \to \mathbb{C}^\times$ and of a Haar measure μ on $\mathbb{A}_F/F$. Here ψ can be any character, satisfying only the condition that $\psi_v(x) = \exp(2\pi i \operatorname{Tr}_{F_v/\mathbb{R}}(x))$ for every Archimedean place $v \in M_F^\infty$ and every $x \in F_v$. Moreover, μ is the unique Haar measure on $\mathbb{A}_F/F$ such that $\mu(\mathbb{A}_F/F) = 1$. The local components μ_v are Haar measures for the additive groups $(F_v, +)$. We assume that $\mu_v = [F_v \colon \mathbb{R}] \cdot \mu_{F_v}$ for every Archimedean place $v \in M_F^\infty$, where μ_{F_v} is the Lesbesgue measure for $F_v \in \{\mathbb{R}, \mathbb{C}\}$. Finally, the local factors $\varepsilon_{\psi_v,\mu_v}(\underline{\mathcal{WD}}(R_v(M)), s)$ associated to the Weil-Deligne representation $\underline{\mathcal{WD}}(R_v(M))$ (see Remark 3.2.4) are the ones defined in [Roh94, § 11], whose general definition uses Brauer's induction theorem (see [Roh94, § 2, Corollary 2]) to reduce to explicit formulas for characters (see [Roh94, Equation 11.3]).

Example 3.3.8. If $M = M(\psi)$ for some algebraic Hecke character $\psi \colon \mathbb{A}_F^\times \to E^\times$, Conjecture 3.3.6 holds with $a(M(\psi)) = W(\psi)\sqrt{|\Delta_F \operatorname{N}_{F/\mathbb{Q}}(\mathfrak{f}_\psi)|}$ and $b(M(\psi)) = -\log||\Delta_F|\operatorname{N}_{F/\mathbb{Q}}(\mathfrak{f}_\psi)|$. Here $W(\psi) = (W(\psi)_\iota)_{\iota \in \operatorname{Hom}(E,\mathbb{C})} \in (E \otimes \mathbb{C})^\times$ is a number whose components have absolute value $|W(\psi)_\iota| = 1$, and $\mathfrak{f}_\psi \subseteq \mathcal{O}_F$ denotes the conductor of the Hecke character ψ. Both of them are defined in terms of local constributions associated to every place $v \in M_F$. Finally, $\Delta_F \in \mathbb{Z}$ denotes the absolute discriminant of the number field F.

Example 3.3.9. If $M = M(f)$ for some cuspidal modular form $f \in S_k(\Gamma_1(N))$ then Conjecture 3.3.6 holds with $a(M(f)) = W(f) \cdot \sqrt{N}^k$ and $b(M(f)) = -\log(N)$. Here again

$$W(f) = (W(f)_\iota)_{\iota \in \operatorname{Hom}(E,\mathbb{C})} \in (E \otimes_{\mathbb{Q}} \mathbb{C})^\times$$

is a number whose components have absolute value $|W(f)_\iota| = 1$.

Remark 3.3.10. If $M = \underline{H}^w(X)$ for some smooth and proper variety X defined over F, then $M^\vee \cong M(w)$, where $M(w)$ denotes the w-th Tate twist of M. Hence, applying Remark 3.2.11 we see that the functional equation (3.19) becomes $\widehat{L}(\underline{H}^w(X), s) = \varepsilon(\underline{H}^w(X), s)\widehat{L}(\underline{H}^w(X), w+1-s)$. This holds more generally for motives $M \in \mathcal{MM}(F;E)$ having a single non-zero weight in their weight filtration, which is in particular the case for the motive $M(f)$ mentioned in the previous example. In this case the only non-zero weight of $M(f)$ is given precisely by $w(M(f)) = k-1$, where $k \in \mathbb{Z}$ is the weight of $f \in S_k(\Gamma_1(N))$.

Remark 3.3.11. Conjecture 3.3.4 and Conjecture 3.3.6 are both compatible with short exact sequences. More precisely, for every short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{MM}(F;E)$ we have that if Conjecture 3.3.4 and Conjecture 3.3.6 hold for two out of three of $\{M, M', M''\}$ then they hold for the third.

3.3.2 Special values of motivic L-functions

The aim of this section is to recall some of the conjectures which aim at describing the special values $L^*(M,s)$ of the L-function associated to a motive in terms of some arithmetic invariants associated to it. To define these invariants, one needs the notion of *f-cohomology*, as introduced by Beilinson and Bloch and Kato. To do so, we follow the exposition of [Sch12, § 6.1], which in turn is inspired by [FP94, Chapitre II, § 1.3].

Definition 3.3.12 – f-cohomology

Let F and E be two number fields, and fix a motive $M \in \mathcal{MM}(F;E)$. Then one defines the f-cohomology groups $H_f^{i,j}(M/F)$ as the subgroups $H_f^{i,j}(M/F) \subseteq H_{\mathcal{M}}^{i,j}(M/F)$ given by the Cartesian square

$$\begin{array}{ccc} H_f^{i,j}(M/F) & \hookrightarrow & H_{\mathcal{M}}^{i,j}(M/F) \\ \downarrow & \lrcorner & \downarrow \prod_\ell r_M^\ell \\ \prod_\ell H_f^{i,j}(F, M_\ell) & \hookrightarrow & \prod_\ell H^{i,j}(F, M_\ell) \end{array}$$

where the products run over all the rational primes $\ell \in \mathbb{N}$. Here $H_{\mathcal{M}}^{i,j}(M/F)$ denotes motivic cohomology with rational coefficients, M_ℓ denotes the ℓ-adic realisation of M and $H^{i,j}(F, M_\ell)$ denotes the i-th Galois cohomology group of $M_\ell(j)$ with respect to the global Galois group $\mathcal{G}_F$ (see for instance [Ser02, Chapter II, § 1.1]). Moreover, the maps r_M^ℓ denote the ℓ-adic regulators, induced by the realisation functors $M \mapsto M_\ell$ (see also Example 2.4.8), and the subgroups $H_f^{i,j}(F, M_\ell) \subseteq H^{i,j}(F, M_\ell)$ are defined by the Cartesian squares

$$\begin{array}{ccc} H_f^{i,j}(F, M_\ell) & \hookrightarrow & H^{i,j}(F, M_\ell) \\ \downarrow & \lrcorner & \downarrow \\ \prod_{v \in M_F^0} H_f^{i,j}(F_v, M_\ell) & \hookrightarrow & \prod_{v \in M_F^0} H^{i,j}(F_v, M_\ell) \end{array} \tag{3.20}$$

where the products run over all the non-Archimedean places $v \in M_F^0$. The vertical maps appearing in (3.20) are induced by the restriction of the Galois representations M_ℓ to the sub-groups $\mathcal{G}_{F_v} \subseteq \mathcal{G}_F$, and again $H^{i,j}(F_v, M_\ell)$ denotes the i-th Galois cohomology group

of $M_\ell(j)$ with respect to the local Galois group $\mathcal{G}_{F_v}$. Finally, the groups $H_f^{i,j}(F_v, M_\ell)$ are defined by

$$H_f^{i,j}(F_v, M_\ell) := \begin{cases} H^{0,j}(F_v, M_\ell), & \text{if } i = 0 \\ \{\underline{W} \in H^1(F_v, M_\ell(j)) \mid \underline{D}(W) \to \underline{D}(E_\lambda) \text{ is surjective}\}, & \text{if } i = 1 \\ 0, & \text{otherwise} \end{cases}$$

where $E_{\ell,v}$ denotes the trivial representation of $\mathcal{G}_{F_v}$ with coefficients in $E_\ell := E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Here $\underline{W} \in H^1(F_v, M_\ell(j)) = \mathrm{Ext}^1(E_\ell, M_{\ell,v}(j))$ is the class of an extension

$$0 \to M_\ell(j) \to W \to E_{\ell,v} \to 0$$

and $\underline{D}(W) \to \underline{D}(E_\lambda)$ denotes the map obtained by applying the functor $\underline{D}$ defined in (3.12) to the surjection $W \twoheadrightarrow E_{\ell,v}$.

Remark 3.3.13. The f-cohomology groups $H_f^{i,j}(M/F)$ can be defined in another, more geometrical way, by extending M to a motive over the ring of integers $\mathcal{O}_F$. More precisely, if $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_F)$ is smooth, proper and flat, with generic fibre $X := \mathcal{X}_F$, one defines

$$H_f^{i,j}(\underline{M}(X/F)) := \mathrm{Im}(H_{\mathcal{M}}^{i,j}(\mathcal{X}) \to H_{\mathcal{M}}^{i,j}(X))$$

where $H_{\mathcal{M}}^{i,j}$ denotes motivic cohomology with rational coefficients. It can be shown that $H_f^{i,j}(\underline{M}(X/F))$ depends indeed only on the generic fibre $X = \mathcal{X}_F$, and not on the model $\mathcal{X}$. Moreover, Scholl has extended the association $X \mapsto H_f^{i,j}(\underline{M}(X/F))$ to a functor $M \mapsto H_f^{i,j}(M)$ from the category of Chow motives (see [Sch00]), and Scholbach proved in [Sch12] that this extension essentially coincides with the definition that we have given in Definition 3.3.12 for the abelian category of mixed motives. Scholbach's result is proved assuming deep conjectures on motives, such as the existence of the motivic t-structure, whose validity is necessary to give a new definition of f-cohomology, inspired by perverse sheaves, to which the other two definitions are then compared. Finally, new unconditional definitions for the f-cohomology of a Chow motive over a general base have been given by Wildeshaus (see [Wil12, Remark 1.11]) and Bondarko (see [Bon14, Remark 3.8]) using the motivic weight structure introduced by the latter.

The f-cohomology groups $H_f^{i,j}(M/F)$ defined in Definition 3.3.12 are global objects, and in particular E-vector spaces. The following Definition 3.3.14 defines a class of motives $M \in \mathcal{MM}(F; E)$ for which there is a strong relation between the global f-cohomology groups $H_f^{i,j}(M/F)$ and the local f-cohomology groups of their realisations, which were used in Definition 3.3.12.

Definition 3.3.14 – f-admissibility

Let F and E be two number fields. Then a motive $M \in \mathcal{MM}(F; E)$ is called f-admissible if:

- for every prime $\ell \in \mathbb{N}$ and every $i, j \in \mathbb{Z}$ the natural map

$$H_f^{i,j}(M/F) \otimes_E E_\ell \to H_f^{i,j}(F, M_\ell)$$

is an isomorphism, where $E_\ell := E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$;

- whenever $H_f^{0,1}(M^\vee/F) = H_f^{1,1}(M^\vee/F) = 0$ the natural map

$$H_f^{i,j}(M/F) \otimes_E E_\infty \to \bigoplus_{v \in M_F^\infty} H^i(F_v, M(j)_v \otimes_E E_\infty)$$

is an isomorphism, for every $i, j \in \mathbb{Z}$. Here $E_\infty := E \otimes_{\mathbb{Q}} \mathbb{R}$.

Remark 3.3.15. The "natural maps" appearing in Definition 3.3.14 are induced from the realisation functors. Henceforth, they can be seen as analogous to the regulators for the ℓ-adic and Deligne-Beilinson cohomology that were defined in Section 2.4.

The following conjecture predicts that the class of f-admissible motives coincides with the whole category $\mathcal{MM}(F;E)$.

Conjecture 3.3.16 – Every motive is f-admissible

Let F and E be two number fields. Then every $M \in \mathcal{MM}(F;E)$ is f-admissible, in the sense of Definition 3.3.14.

Remark 3.3.17. If our motive $M \in \mathcal{MM}(F;E)$ is the "mixed realisation" of a geometrically defined motive $\widetilde{M} \in \mathrm{DM}(F;E)$, we can define $H_f^{i,j}(M/F)$ starting from the motivic cohomology $H_{\mathcal{M}}^{i,j}(\widetilde{M}/F)$ computed in $\mathrm{DM}(F;E)$. If we do so, Conjecture 3.3.16 turns out to be incredibly difficult to prove. Indeed, Conjecture 3.3.16 is strongly related to the Hodge and Tate conjectures, and to the conservativity of the realisation functors, which all seem out of reach at the moment. Moreover, for every f-admissible motive $M \in \mathcal{MM}(F;E)$ we have that $\dim_E(H_f^{i,j}(M/F)) < +\infty$. This is surely expected, but it is not known outside of Artin-Tate motives, where it follows from the work of Borel.

Let us finally move towards the conjectures relating special values of L-functions to the arithmetic invariants of the motive M. Using the f-cohomology one can define the one-dimensional E-vector space

$$L_f(M) := \det(H_{\mathcal{M}}^{0,0}(M/F)) \otimes (\det(H_f^{1,0}(M/F)))^\vee$$

associated to every motive $M \in \mathcal{MM}(F;E)$ defined over a number field F with coefficients in another number field E. Here det denotes the determinant line $\det(V) := \bigwedge^{\dim(V)} V$ for every finite dimensional vector space V. Then one defines the *fundamental line* associated to $M \in \mathcal{MM}(F;E)$ as

$$\Delta_f(M) := L_f(M) \otimes L_f(M^\vee(1)) \otimes \det(M_{\mathrm{dR}}/F^0(M_{\mathrm{dR}})) \otimes \det\left(\bigoplus_{v \in M_F^\infty} H^0(F_v, M_v)\right)^\vee$$

where M_{dR} denotes the de Rham realisation of $M \in \mathcal{MM}$, endowed with the Hodge filtration $F^\bullet(M_{\mathrm{dR}})$, and for every Archimedean place $v \in M_F^\infty$ we denote by $M_v \in \mathrm{MHS}(F_v;E)$ the v-adic Betti realisation of M. Thus M_v is endowed with an action of the Galois group $\mathrm{Gal}(\mathbb{C}/F_v)$, and it makes sense to write $H^0(F_v, M_v)$ for the corresponding group cohomology, given by the invariants $H^0(F_v, M_v) := (M_v)^{\mathrm{Gal}(\mathbb{C}/F_v)}$.

Now, supposing that M is f-admissible in the sense of Definition 3.3.14, we can construct a family of norms $\|\cdot\|_\lambda : \Delta_f(M) \otimes_E E_\lambda \to E_\lambda$ associated to the places $\lambda \in M_E$. More precisely, for every Archimedean place $\lambda \in M_E^\infty$ one has a map

$$\alpha_{M_\lambda} : \left(\bigoplus_{v \in M_F^\infty} H^0(F_v, M_v) \otimes_E E_\lambda \right) \to (M_{\mathrm{dR}}/F^0(M_{\mathrm{dR}})) \otimes_E E_\lambda$$

obtained by composing the change of coefficients

$$\left(\bigoplus_{v \in M_F^\infty} H^0(F_v, M_v) \otimes_E E_\lambda \right) \to \left(\bigoplus_{v \in M_F^\infty} H^0(F_v, M_v \otimes_{E,\lambda} \mathbb{C}) \right)$$

with the period map induced by (2.8)

$$\mathrm{per}_M : \left(\bigoplus_{v \in M_F^\infty} H^0(F_v, M_v \otimes_{E,\lambda} \mathbb{C}) \right) \to M_{\mathrm{dR}} \otimes_E E_\lambda$$

and then with the projection $M_{\mathrm{dR}} \otimes_E E_\lambda \twoheadrightarrow (M_{\mathrm{dR}}/F^0(M_{\mathrm{dR}})) \otimes E_\lambda$. The admissibility of M implies that the sequence

$$0 \to H_f^{0,0}(M)_\lambda \to \ker(\alpha_{M_\lambda}) \to H_f^{1,1}(M^\vee)_\lambda^\vee \to H_f^{1,0}(M)_\lambda \to \mathrm{coker}(\alpha_{M_\lambda}) \to H_f^{0,1}(M^\vee)_\lambda \to 0$$

is exact, where $H_f^{i,j}(X)_\lambda := H_f^{i,j}(X/F) \otimes_E E_\lambda$ for every $X \in \mathcal{MM}(F;E)$. This induces an isomorphism

$$(L_f(M) \otimes_E L_f(M^\vee(1))) \otimes_E E_\lambda \xrightarrow{\sim} \det(\ker(\alpha_{E_\lambda})) \otimes_{E_\lambda} \det(\mathrm{coker}(\alpha_{E_\lambda}))^\vee \tag{3.21}$$

and analogously the tautological exact sequence

$$0 \to \ker(\alpha_{M_\lambda}) \to \left(\bigoplus_{v \in M_F^\infty} H^0(F_v, M_v) \otimes_E E_\lambda \right) \xrightarrow{\alpha_{M_\lambda}} (M_{\mathrm{dR}}/F^0(M_{\mathrm{dR}})) \otimes_E E_\lambda \to \mathrm{coker}(\alpha_{E_\lambda}) \to 0$$

induces an isomorphism

$$\left(\det(M_{\mathrm{dR}}/F^0(M_{\mathrm{dR}})) \otimes \det\left(\bigoplus_{v \in M_F^\infty} H^0(F_v, M_v) \right)^\vee \right) \otimes_E E_\lambda \xrightarrow{\sim} \det(\ker(\alpha_{E_\lambda}))^\vee \otimes_{E_\lambda} \det(\mathrm{coker}(\alpha_{E_\lambda})) \tag{3.22}$$

for every motive $M \in \mathcal{MM}(F;E)$. Putting together the two isomorphisms (3.21) and (3.22) we get an isomorphism

$$\|\cdot\|_\lambda : \Delta_f(M) \otimes_E E_\lambda \to E_\lambda$$

which is defined for every mixed motive $M \in \mathcal{MM}(F;E)$ that is f-admissible, and for every Archimedean place $\lambda \in M_E^\infty$. This allows us already to enounce Beilinson's conjecture on special values of L-functions, using the isomorphism

$$\|\cdot\|_\infty : \Delta_f(M) \otimes_E E_\infty \xrightarrow{\sim} E_\infty := E \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\lambda \in M_E^\infty} E_\lambda$$

obtained by gluing together the various isomorphisms $\|\cdot\|_\lambda$.

Conjecture 3.3.18 – Beilinson's conjecture on special values of L-functions

Let F and E be two number fields. Then, for every f-admissible motive $M \in \mathcal{MM}(F;E)$, there exists a (necessarily unique) element $\mathcal{L}^*(M,0) \in \Delta_f(M) \setminus \{0\}$ such that

$$L^*(M,0) \cdot \|\mathcal{L}^*(M,0) \otimes 1\|_\infty = 1$$

where $L^*(M,0) \in E_\infty^\times$ denotes the special value of the L-function defined in Definition 3.2.9.

Remark 3.3.19. Let us mention that Conjecture 3.3.18 can be expressed in another, perhaps cleaner form when $M = \underline{H}^i(X)(n)$ for some smooth and projective variety X defined over F, and some pair of integers $i, n \in \mathbb{N}$ such that n lies in the region of absolute convergence for the L-function $L(\underline{H}^i(X), s)$, *i.e.* such that $n > i/2+1$. More precisely, we still have the f-cohomology group $H_f^{i,j}(X) \subseteq H_{\mathcal{M}}^{i,j}(X)$, which can be defined geometrically (see Remark 3.3.13). Then we have Beilinson's regulator map

$$r_X^\infty : H_{\mathcal{M}}^{\bullet,\bullet}(X) \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow H_{\mathcal{D}}^{\bullet,\bullet}(X;\mathbb{R})$$

whose target is Deligne-Beilinson cohomology (see Example 2.4.6). Then Conjecture 3.3.18 for the motive $M = \underline{H}^i(X)(n) \in \mathcal{MM}(F;\mathbb{Q})$ is equivalent to the following two assertions:

- Beilinson's regulator r_X^∞ induces an isomorphism

$$r_X^\infty : H_f^{i+1,n}(X) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} H_{\mathcal{D}}^{i+1,n}(X;\mathbb{R});$$

- we have that

$$\frac{\det(r_X^\infty)}{L^*(\underline{H}^i(X),n)} \in \mathbb{Q}^\times$$

where $\det(r_X^\infty)$ is taken with respect to any two bases of the $\mathbb{Q}$-vector spaces $H_f^{i+1,n}(X)$ and $H_{\mathcal{D}}^{i+1,n}(X;\mathbb{Q})$.

Finally, the other norm maps $\|\cdot\|_\lambda : \Delta_f(M) \otimes_E E_\lambda \to E_\lambda$ are defined by using a suitable integral structure of $\Delta_f(M)$. Since we do not need them in what follows, we content ourselves with using them to give the statement of the conjecture of Bloch and Kato, as stated by Fontaine and Perrin-Riou (see [FP94, Chapitre III, § 4.5]). We refer the interested reader to [FP94] for the detailed definition of the norms $\|\cdot\|_\lambda$ associated to non-Archimedean places $\lambda \in M_E^0$.

Conjecture 3.3.20 – Bloch-Kato's conjecture on special values of L-functions

Let F and E be two number fields. Then, for every f-admissible motive $M \in \mathcal{MM}(F;E)$, Conjecture 3.3.18 holds and we have that

$$\|\mathcal{L}^*(M,0)\otimes 1\|_\lambda = 1$$

for every non-Archimedean place $\lambda \in M_F^0$.

Remark 3.3.21. We observe that Conjecture 3.3.18 determines $L^*(M,0)$ only up to an element of $E^\times$. Indeed, even if we know that $\mathcal{L}^*(E,0)$ is necessarily unique, the only thing that is sure from Conjecture 3.3.18 is that $\mathcal{L}^*(E,0) \neq 0$. This determines $L^*(M,0)$ up to an element of $E^\times$ since $\Delta_f(M)$ is a one dimensional vector space over E. On the other hand, as we have said, Conjecture 3.3.20 says essentially that $\mathcal{L}^*(M,0)$ belongs to a suitable integral O_E-sub-module of $\Delta_f(M)$. Hence this determines $L^*(M,0)$ up to an element of $O_E^\times$.

Remark 3.3.22. The conjectures of Beilinson and Bloch-Kato are usually stated only for the special values of a motivic L-function $L(M,s)$ at $s=0$. However, since $L^*(M,n) = L^*(M(n),0)$, the conjectures immediately generalise to the special values at every integer $n \in \mathbb{Z}$. In particular, one can define an element $\mathcal{L}^*(M,n) \in \Delta_f(M(n))$, which is expected to satisfy

$$L^*(M,n)\cdot\|\mathcal{L}^*(M,n)\otimes 1\|_\infty = 1$$

and $\|\mathcal{L}^*(M,n)\otimes 1\|_\lambda = 1$ for every $\lambda \in M_E^0$. If the L-function $L(M,s)$ satisfies the functional equation predicted in Conjecture 3.3.6, it is natural to expect that the Beilinson and Bloch-Kato conjectures for the special value $L^*(M,n)$ are equivalent to the corresponding ones for the special value $L^*(M^\vee, 1-n)$. This is known to be true for the Beilinson conjecture (see [FP94, Chapitre III, Remarque 4.4.4.(iv)], which refers to Deligne's computation [Del79, Théorème 5.6]), but it is not known in general for the Bloch-Kato conjecture (see [FP94, § 4.5.4]).

Remark 3.3.23. In the case when $F = E = \mathbb{Q}$ the sign ambiguity in the determination of $L^*(M,0) \in \mathbb{R}$ can be deduced by the orders of $L(M,s)$ at positive integers. More precisely, $L^*(M,0) > 0$ if and only if $\sum_{n>0} \mathrm{ord}_{s=n}(L(M,s))$ is even. In general, the determination of $L^*(M,0)$ on the nose remains a challenging problem.

Remark 3.3.24. Using the fact that every mixed motive $M \in \mathcal{MM}(F;\mathbb{Q})$ should be given by finitely many successive extensions of motives of the form $\underline{H}^i(X)(n)$ for a regular scheme X, it should be able to compute the f-cohomology groups $H_f^{i,j}(M)$ using the polylogarithmic motivic complexes described in Section 2.3.3. If one does so, Conjecture 3.3.20 becomes intimately related to Zagier's polylogarithmic conjecture (see [Zag86], and the survey [ZG00]), which is still open even for Dedekind ζ-functions. We refer the reader to [Gon95b] and [DeJ95] for the general picture, and to [GR18] for recent progress in the case of the special value $\zeta_F^*(4)$.

Let us mention perhaps the most challenging part of the conjectures of Bloch and Kato: the one concerning the order of vanishing of L-functions.

Conjecture 3.3.25 – Orders of vanishing of motivic L-functions

Let F and E be two number fields, and let $M \in \mathcal{MM}(F;E)$ be an f-admissible motive. Then we have that

$$\mathrm{ord}_{s=n}(L(M,s)) = \dim_E(H_f^{1,1-n}(M^\vee)) - \dim_E(H_f^{0,n+1}(M))$$

for every $n \in \mathbb{Z}$.

To conclude, we briefly survey some recent developments on the conjectures of Beilinson and Bloch-Kato:

- Burns and Flach have formulated a version of the conjecture of Bloch and Kato for motives endowed with the action of a semisimple $\mathbb{Q}$-algebra A, which might be non-commutative (see [BF01, § 4.3]). The main difficulty in doing this lies in the definition of a suitable analogue of the fundamental line for equivariant motives. This analogue is given by a suitable relative algebraic K-group $K_0(\mathfrak{U},\mathbb{R})$, where $\mathfrak{U} \subseteq A$ is an order. Then one constructs two elements inside $K_0(\mathfrak{U},\mathbb{R})$: one coming from the L-value $L^*(M,0)$, and another coming from the f-cohomology of the motive M. The conjecture of Burns and Flach states then that these two elements should be equal. Observe that here M is an equivariant motive, and the definition of the L-function $L(M,s)$ takes this into account;
- Braunling has given a new interpretation of the relative K-group $K_0(\mathfrak{U},\mathbb{R})$ appearing in the conjecture of Burns and Flach (see [Bra20]). More precisely, this group is proved to be isomorphic to the first K-group $K_1(\mathrm{LCA}^*_{\mathfrak{U}})$ of a suitable sub-category $\mathrm{LCA}^*_{\mathfrak{U}} \subseteq \mathrm{LCA}_{\mathfrak{U}}$ of the category $\mathrm{LCA}_{\mathfrak{U}}$ of locally compacy topological right modules over $\mathfrak{U}$. Moreover, Braunling has also proposed a new version of the conjecture of Burns and Flach (see [Bra19]) which uses a group of non-commutative idèles associated to the algebra A instead of the relative K-group $K_0(\mathfrak{U},\mathbb{R})$;
- Flach and Morin have provided yet another interpretation of the conjecture of Bloch and Kato (see [FM18]). More precisely, their conjecture concerns the special values of the ζ-function

$$\zeta(\mathcal{X},s) := \prod_{x\in|\mathcal{X}|} \frac{1}{1-|\kappa(x)|^{-s}}$$

  associated to a scheme of finite type $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ which is proper and regular. Here the product runs over all the closed points of $\mathcal{X}$ and $\kappa(x)$ denotes the residue field of a closed point $x \in |\mathcal{X}|$, which is a finite field. This ζ-function is known to factor, up to finitely many bad primes, into the product of the L-functions of the motives $\underline{H}^w(X)$, where $X := \mathcal{X}_{\mathbb{Q}}$ is the generic fibre of $\mathcal{X}$ (see [Den94, Equation 1.4]). This allows one to relate the conjecture of Flach and Morin to the conjecture of Bloch and Kato, which is done in [FM18, § 5.6].

3.3.3 Two special cases of the Bloch-Kato conjecture

The last section was heavily charged with far reaching conjectures, which might seem too unrealistic to believe. Nevertheless, the somehow abstract conjectures of Beilinson and Bloch-Kato have been inspired by more concrete questions and results. The first one, in the long series of identities which appear to be related to the Beilinson and Bloch-Kato conjectures, is the analytic class number formula for the Dedekind ζ-function $\zeta_F(s)$ associated to a number field F.

This can be easily seen to coincide with the L-function of the motive $\underline{H}^0(F) \in \mathcal{MM}(F;\mathbb{Q})$, and the following formula (3.23) can be shown to be equivalent to Conjecture 3.3.20 for the motive $\underline{H}^0(F)$ (see [HK03, § 2.3]).

Theorem 3.3.26 – Analytic class number formula

Let F be a number field, and let ζ_F denote its Dedekind zeta function. Then one has that

$$\mathrm{ord}_{s=0}(\zeta_F(s)) = \mathrm{rk}(\mathcal{O}_F^\times) \quad \text{and} \quad \zeta_F^*(0) = -\frac{|\mathrm{Pic}(\mathcal{O}_F)|}{|(\mathcal{O}_F^\times)_{\mathrm{tors}}|} \cdot R_F \tag{3.23}$$

where $R_F := \det(\log|\sigma_i(\gamma_j)|)_{i,j=1,\dots,\mathrm{rk}(\mathcal{O}_F^\times)}$ denotes the unit regulator of F and $\mathrm{Pic}(\mathcal{O}_F)$ denotes the class group of $\mathcal{O}_F$, which is a finite group (see [Neu99, Chapter I, Theorem 6.3]). We recall as well that the unit group $\mathcal{O}_F^\times$ is a finitely generated abelian group of rank $\mathrm{rk}(\mathcal{O}_F^\times) = r_1(F) + r_2(F) - 1$ (see [Neu99, Chapter I, Theorem 7.4]).

We observe that, in the case studied in Theorem 3.3.26, the f-cohomology groups $H_f^{i,j}(\underline{H}^0(F))$ involved are rather simple, and they amount essentially to considering the group of units $\mathcal{O}_F^\times$ and the class group $\mathrm{Pic}(\mathcal{O}_F)$ appearing in (3.23). On the other hand, motivic cohomology groups associated to higher dimensional objects are much more difficult to compute. In particular, they are not known to be finitely generated, even if this is predicted by Conjecture 3.3.16. Nevertheless, if $M = \underline{H}^0(F)(n)$ for some $n \in \mathbb{Z}$ the f-cohomology groups $H_f^{i,j}(M)$ are known to be finitely generated from the work of Borel (see [Sou10] for a survey). Moreover, the conjecture of Bloch and Kato (and ever the equivariant analogue of Burns and Flach) are completely known for all the special values $\zeta_F^*(n)$, as soon as F is an abelian extension of $\mathbb{Q}$. We refer the reader to [Ngu15, § 9.5] for a survey of the proof, a detailed account of which can be obtained by combining the three papers [BN02], [BG03] and [Fla11].

Let us now turn to higher dimensional motives. Similarly to what happened in the case of the analytic class number formula (3.23), one can often make explicit the Bloch-Kato conjecture in the case of a special value in the critical strip, because in this case the f-cohomology groups are more explicitly computable. The most famous instance of this phenomenon is given by the special value at $s = 1$ of the L-function $L(A, s)$ associated to an abelian variety A defined over a number field F. In this case, the Bloch-Kato conjecture for the motive $\underline{H}^1(A)(1) \in \mathcal{MM}(F;\mathbb{Q})$ can be shown to be equivalent to the following conjecture of Tate (see [Tat66]), which generalises the famous Birch and Swinnerton-Dyer conjecture for elliptic curves. To state this conjecture, we need to introduce the following notation associated to an abelian variety A defined over a number field F:

- $\mathcal{A} \to \mathrm{Spec}(\mathcal{O}_F)$ is the *Néron model* of A, which is a smooth and separated (but usually not proper) model of A over $\mathcal{O}_F$, uniquely characterised by the fact that the map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{X}, \mathcal{A}) &\to \mathrm{Hom}_F(\mathcal{X}_F, A) \\ f &\mapsto f \times_{\mathcal{O}_F} F \end{aligned}$$

  is a bijection for every smooth and separated scheme $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_F)$;

- $\eta_{\mathcal{A}} \in \omega_{\mathcal{A}/\mathcal{O}_F}$ is a generator of the canonical bundle $\omega_{\mathcal{A}/\mathcal{O}_F} := \bigwedge^g \Omega^1_{\mathcal{A}/\mathcal{O}_F}$, where we set $g := \dim(A/F) = \dim(\mathcal{A}/\mathcal{O}_F)$;

- $\Omega_A := \int_{A(\mathbb{R})} |\eta_{\mathcal{A}}| \in \mathbb{R}_{>0}$ is called the *real period* of A;
- for every non-Archimedean place $v \in M_F^0$, we let $c_v(A) := |\pi_0(\mathcal{A}_{O_{F_v}})| \in \mathbb{N}$ denote the number of connected components of the Néron model of A_{F_v};
- we let $Ш(A/F)$ denote the Tate-Shavarevich group

$$Ш(A/F) := \bigcap_{v \in M_F^0} \ker\left(H^1(F, A) \to H^1(F_v, A_{F_v})\right)$$

  where $H^1(F, A)$ denotes Galois cohomology with respect to the global absolute Galois group $\mathcal{G}_F := \mathrm{Gal}(\overline{F}/F)$ and $H^1(F_v, A_{F_v})$ denotes Galois cohomology with respect to the local absolute Galois group $\mathcal{G}_{F_v} := \mathrm{Gal}(\overline{F_v}/F_v)$ relative to the v-adic completion of F;
- $A^\vee$ is the dual of the abelian variety A, which is another abelian variety whose points are in bijective correspondence with the equivalence classes of line bundles of degree zero over A (see [HS00, Theorem A.7.3.4]);
- $\mathcal{P} \in \mathrm{Pic}(A \times A^\vee)$ denotes the Poincaré divisor class (see again [HS00, Theorem A.7.3.4]);
- $A(F)$ denotes the set of points of A which are defined over F. This is an abelian group because A is an abelian variety, and it is finitely generated by the Mordell-Weil theorem (see [HS00, Theorem C.0.1]);
- $\hat{h}_{\mathcal{P}} \colon A(\overline{F}) \times A^\vee(\overline{F}) \to \mathbb{R}$ denotes the Néron-Tate height associated to the Poincaré divisor class (see [HS00, Theorem B.5.6]);
- $R_{A,F} \in \mathbb{R}_{>0}$ denotes the regulator of A over F, which is defined as the determinant

$$R_{A,F} := \left|\det\left(\hat{h}_{\mathcal{P}}(P_i, \check{P}_j)\right)_{i,j=1,\dots,r_{A,F}}\right| \tag{3.24}$$

  where $\{P_i\}_{i=1}^{r_{A,F}} \subseteq A(F)$ and $\{\check{P}_j\}_{j=1}^{r_{A,F}} \subseteq A^\vee(F)$ denote any sets of points such that the set $\{P_i \otimes 1\}_{i=1}^{r_{A,F}} \subseteq A(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a basis of the $\mathbb{Q}$-vector space $A(F) \otimes_{\mathbb{Z}} \mathbb{Q}$, which has dimension $r_{A,F} := \mathrm{rk}(A(F))$, and the set $\{\check{P}_j \otimes 1\}_{j=1}^{r_{A,F}} \subseteq A^\vee(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a dual basis.

We are finally ready to give the statement of the conjecture of Tate (see [Tat66]) which generalises to higher dimensional abelian varieties the conjecture of Birch and Swinnerton-Dyer.

Conjecture 3.3.27 – Birch and Swinnerton-Dyer, Tate

Let F be a number field and let A be an abelian variety defined over F. Then we have that:

- Conjecture 3.3.4 and Conjecture 3.3.6 hold for the L-function $L(A, s) := L(\underline{H}^1(A), s)$;
- $\mathrm{ord}_{s=1}(L(A, s)) = \mathrm{rk}(A(F))$, where $A(F)$ denotes the abelian group of points of A which are defined over F. This group is finitely generated by the Mordell-Weil theorem (see [HS00, Theorem C.0.1]);
- the Tate-Shafarevich group $Ш(A/F)$ is finite;

- we have the equality

$$L^*(A,1) = \left(\frac{|Ш(A/F)| \cdot \prod_{v \in M_F^0} c_v(A)}{|A(F)_{\mathrm{tors}}| \cdot |A^\vee(F)_{\mathrm{tors}}|} \right) \cdot R_{A/F} \cdot \Omega_A \tag{3.25}$$

where $L^*(A,1)$ denotes the special value of the L-function $L(A,s)$ at $s=1$.

As we already pointed out, the validity of Conjecture 3.3.27 is equivalent to the combined validity of Conjecture 3.3.4, Conjecture 3.3.6, Conjecture 3.3.20 and Conjecture 3.3.25 for the motive $M = \underline{H}^1(A) \in \mathcal{MM}(F;\mathbb{Q})$. Extensive evidence in favour of Conjecture 3.3.27 is known when $F=\mathbb{Q}$ and A is an elliptic curve such that $\mathrm{rk}(A(\mathbb{Q})) \in \{0,1\}$. In this case, Conjecture 3.3.4 and Conjecture 3.3.6 are known from the modularity theorem (see [Edi02] for a survey), and Conjecture 3.3.25 is known by the work of Gross-Zagier (see [Coa86] for a survey), Kolyvagin and Rubin (see [Per90] for a survey). Moreover, the p-part of the group $Ш(A/\mathbb{Q})$ is known to be finite in a variety of cases, and when this is known the formula (3.25) can be proved by a numerical computation (see for example [Cre11]). We refer the reader to [BHM20] and [Bom20] for some recent developments on the numerical verification of Conjecture 3.3.27 in the case when $F=\mathbb{Q}$ and $A = \mathrm{Jac}(C)$ is the Jacobian of a curve $C_{/\mathbb{Q}}$ having genus $g(C) \geq 2$.

3.3.4 Evidence towards the conjecture of Bloch and Kato

We conclude this section with a brief survey of the known evidence towards the validity of Conjecture 3.3.20 for the special values of a motivic L-function $L^*(M,s)$ at integers $n \in \mathbb{Z}$ that lie in the region of absolute convergence. On the one hand, these special values are easier to treat (especially numerically) than the special values which lie inside or at the boundaries of the critical strip. On the other hand, the f-cohomology groups appearing in the fundamental line $\Delta_f(M(n))$ are in general much harder to compute than the ones when n lies in the region of absolute convergence.

First of all, as we mentioned already in the previous section, the Bloch-Kato conjecture is known for the special values $\zeta_F^*(n)$ of Dedekind ζ-functions associated to number fields F which are abelian over $\mathbb{Q}$. Moreover, the Bloch-Kato conjecture is also known for the L-functions $L(\chi,s)$ associated to Dirichlet characters, which appear in the factorisations

$$\zeta_{\mathbb{Q}(\mu_N)}(s) = \left(\prod_{\mathfrak{p} | N\mathbb{Z}[\mu_N]} \frac{1}{1 - |\mathrm{N}_{\mathbb{Q}(\mu_N)/\mathbb{Q}}(\mathfrak{p})|^{-s}} \right) \left(\prod_{\chi} L(\chi,s) \right)$$

where $\mu_N \subseteq \overline{\mathbb{Q}}$ denotes the group of N-th roots of unity, and $\mathbb{Z}[\mu_N]$ denotes the ring of integers of the N-th cyclotomic field $\mathbb{Q}(\mu_N)$. This follows again from the work of Burns and Greither [BG03], Burns and Flach [BF06] and Flach [Fla11]. We point out that a different proof of the Bloch-Kato conjecture for Dirichlet L-functions can be obtained using the works [HK99] and [HK03] of Huber and Kings, which provide a different approach to the construction of the required motivic cohomology classes (see also [Hub15] and [Kin15] for a survey). In all these cases, one is also able to prove Conjecture 3.3.25 thanks to the work of Borel.

Moving away from the zero dimensional cases provided by Dirichlet characters and Dedekind ζ-functions, we immediately face the tantalising problem posed by the computation of the rank of the f-cohomology groups $H_f^{i,j}(M)$. These groups are conjectured to be finitely generated, but this is not known for any motive which is not an Artin-Tate motive, apart from some peculiar

values of i and j. For instance, given an abelian variety A defined over a number field F we have that $H_f^{1,1}(\underline{H}^1(A)) \cong A(F) \otimes_{\mathbb{Z}} \mathbb{Q}$, as explained in [Kin11, Example 1.21].

Nevertheless, it is often possible to prove a weak version of the Bloch-Kato conjecture, even without knowing the finite generation of the f-cohomology groups. This is the content of the following conjecture.

Conjecture 3.3.28 – Weak form of the Beilinson and Bloch-Kato conjectures

Let F and E be number fields. Then for every motive $M \in \mathcal{MM}(F;E)$ and every $i, j \in \mathbb{Z}$ there exists a finitely generated sub-space $\widetilde{H}_f^{i,j}(M) \subseteq H_f^{i,j}(M)$ such that

$$\operatorname{ord}_{s=n}(L(M,s)) = \dim_E(\widetilde{H}_f^{1,1-n}(M^\vee)) - \dim_E(\widetilde{H}_f^{0,n+1}(M))$$

for every $n \in \mathbb{Z}$. Moreover, for every motive $M \in \mathcal{MM}(F;E)$ and every integer $n \in \mathbb{Z}$ there exists an element $\widetilde{\mathcal{L}}^*(M,n) \in \widetilde{\Delta}_f(M(n))$ such that

$$L^*(M,n) \cdot \|\widetilde{\mathcal{L}}^*(M,n) \otimes 1\|_\infty = 1 \quad \text{and} \quad \|\widetilde{\mathcal{L}}^*(M,n) \otimes 1\|_\lambda = 1,\ \forall \lambda \in M_E^0$$

where $\widetilde{\Delta}_f(M(n))$ is the weak motivic fundamental line. This is defined as

$$\widetilde{\Delta}_f(N) := \widetilde{L}_f(N) \otimes \widetilde{L}_f(N^\vee(1)) \otimes \det(N_{\mathrm{dR}}/F^0(N_{\mathrm{dR}})) \otimes \det\left(\bigoplus_{v \in M_F^\infty} H^0(F_v, N_v)\right)^\vee$$

for every motive $N \in \mathcal{MM}(F;E)$, where $\widetilde{L}_f(N) := \det_E(\widetilde{H}_f^{0,0}(N)) \otimes \det_E(\widetilde{H}_f^{1,0}(N))^\vee$. Finally, the norms

$$\|\cdot\|_\infty : \widetilde{\Delta}_f(M(n)) \otimes_E E_\infty \to E_\infty := E \otimes_{\mathbb{Q}} \mathbb{R}$$
$$\|\cdot\|_\lambda : \widetilde{\Delta}_f(M(n)) \otimes_E E_\lambda \to E_\lambda$$

are induced by the inclusion $\widetilde{\Delta}_f(M(n)) \subseteq \Delta_f(M(n))$, and for every non-Archimedean place $\lambda \in M_F^0$ the non-Archimedean norm $\|\cdot\|_\lambda$ is supposed to be an isomorphism.

Remark 3.3.29. If the motive M is of the form $M = \underline{H}^i(X)(n) \in \mathcal{MM}(F;\mathbb{Q})$ for some smooth and projective variety X defined over a number field F, and for some pair of integers $i, n \in \mathbb{N}$ such that $n > i/2 + 1$, we can combine Conjecture 3.3.28 with Remark 3.3.19 to get a weak form of Beilinson's conjecture. More precisely, there should exist a sub-space $\widetilde{H}_f^{i+1,n}(X) \subseteq H_f^{i+1,n}(X)$ such that Beilinson's regulator induces an isomorphism

$$\widetilde{r}_X^\infty : \widetilde{H}_f^{i+1,n}(X) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} H_{\mathcal{D}}^{i+1,n}(X;\mathbb{R})$$

and we should have that

$$\frac{\det(\widetilde{r}_X^\infty)}{L^*(\underline{H}^i(X),n)} \in \mathbb{Q}^\times$$

where the determinant is taken with respect to any $\mathbb{Q}$-basis of $\widetilde{H}_f^{i+1,n}(X)$ and $H_{\mathcal{D}}^{i+1,n}(X;\mathbb{Q})$.

This weaker form of Conjecture 3.3.20 is known to hold in a number of cases. First of all, some results are known when M is a motive which is automorphic in nature, *i.e.* a motive cut off from the motive of a Shimura variety (see [Lem17] and [CLJ19]). Moreover, the work of Deninger (see [Den89] and [Den90], as well as the surveys [Den88] and [Den97b]) and Kings (see [Kin01]) proves Conjecture 3.3.28 for all the special values of the form $L^*(E,n)$, where $n \in \mathbb{Z}$ is any integer and $E_{/F}$ is an elliptic curve with complex multiplication by the ring of integers O_K of an imaginary quadratic field K. We refer the reader to Chapter 7 for a survey of the theory of complex multiplication, and in particular to Section 7.4 for a more detailed survey of the proof of Conjecture 3.3.28 for the special value $L^*(E,2)$.

3.4 Diophantine properties of special values of L-functions

The aim of this section, which is based on joint work in progress with Fabien Pazuki, is to show that special values of L-functions can be interpreted as heights. To be more precise, let us fix an integer $n \in \mathbb{Z}$ and two number fields F and E. Then, assuming Conjecture 3.3.4, we can consider the function

$$\begin{aligned} \mathcal{MM}(F;E) &\to \mathbb{R} \\ M &\mapsto |L^*(M,n)| \end{aligned} \tag{3.26}$$

sending a motive to the special value of its L-function at $s = n$. We actually take the absolute value of $L^*(M,n) \in E_\infty := E \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\lambda \in M_E^\infty} E_\lambda$, defined as the product of all the absolute values of the components. As we outline in Section 3.4.1, there are numerous examples of relations between special values of L-functions and heights of various sorts. Thus it is natural to ask whether the map (3.26), perhaps restricted to a subset $S \subseteq \mathcal{MM}(F;E)$, satisfies any of the Diophantine properties outlined in Section 1.1. It is interesting, in particular, to see for which subsets $S \subseteq \mathcal{MM}(F;E)$ the restriction of the map (3.26) to S satisfies the Northcott property. We prove in Section 3.4.2 that, fixing $n = 0$, the first example of such a subset $S \subseteq \mathcal{MM}(F;E)$ is given by taking $F = E = \mathbb{Q}$ and $S := \{\underline{H}^0(\mathrm{Spec}(K)) \mid [K\colon \mathbb{Q}] < +\infty\}$ to be the set of motives whose L-functions coincide with the Dedekind ζ-functions $\zeta_K(s)$ associated to a number field K. Moreover, we discuss in Section 3.4.3 to what extent, assuming numerous conjectures and taking $n = 1$, one can take S to be the set of pure motives $\underline{H}^1(A)$ associated to an abelian variety A. Finally, Section 3.4.4 shows that one can take S to be the set of all pure motives of a given weight $w \in \mathbb{Z}$, under the condition that $n < -w/2$ and under the assumption that our L-functions satisfy the expected functional equation (see Conjecture 3.3.6).

Let us point out that properties similar to the ones we discuss here have already been studied in the case of automorphic L-functions by Sarnak, Shin and Templier in [SST16]. Moreover, we devote Section 3.4.5 to explore a possible connection between special values of motivic L-functions and motivic heights, in the sense of Kato (see [Kat18]).

3.4.1 Some relations between heights and special values

This section contains a few examples of relations between special values of L-functions and heights of various sorts, which motivate the search for Diophantine properties of special values of L-functions.

Example 3.4.1 (logarithmic Weil height)**.** The results of [AV16], combined with the class number formula (3.23), show that for every number field F with unit rank $r_F := \dim_{\mathbb{Q}}(O_F^\times \otimes_{\mathbb{Z}} \mathbb{Q})$ there exists a basis $\{\gamma_1, \dots, \gamma_{r_F}\} \subseteq O_F^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\{\gamma_1, \dots, \gamma_{r_F}\} \subseteq O_F^\times$ and

$$\frac{h_F \cdot d_F^{r_F} \cdot (2r_F)!}{2 \cdot w_F \cdot (r_F!)^4} \prod_{i=1}^{r_F} h(\gamma_i) \leq |\zeta_F^*(0)| \leq \frac{h_F \cdot d_F^{r_F}}{w_F} \cdot \prod_{i=1}^{r_F} h(\gamma_i) \tag{3.27}$$

which shows that the special value $\zeta_F^*(0)$ of the Dedekind ζ-function associated to a number field F is commensurable to a product of Weil heights (see Section 1.2.1). The constants appearing in (3.27) are given by $d_F := [F\colon \mathbb{Q}]$, $h_F := |\mathrm{Pic}(O_F)|$ and $w_F := |(O_F^\times)_{\mathrm{tors}}|$.

Example 3.4.2 (Mahler measures)**.** As we illustrate in the next chapter, the study of relations between Mahler measures and special values of L-functions is very active and rich. Most of the conjectural relations of this kind involve a family of Laurent polynomials $P_k \in \mathbb{Z}[k][x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, for which it is conjectured that

$$\frac{L^*(\underline{H}^{n-1}(V_{P_k}), 0)}{m(P_k)} \in \mathbb{Q}^\times \tag{3.28}$$

where V_{P_k} denotes the zero locus of P_k inside $\mathbb{G}_m^n$ (see Question 4.2.9). Moreover, in most cases it seems that the ratio appearing in (3.28) is actually an integer. If this was the case, we see immediately that the Northcott property for the family of special values $|L^*(\underline{H}^{n-1}(V_{P_k}), 0)|$ would follow from the Northcott property of the Mahler measure, as the function $\delta(P_k)$ is constant in k (see Section 1.2.2).

Example 3.4.3 (Faltings's height)**.** The stable Faltings's height $h\colon \mathcal{A}(\overline{\mathbb{Q}}) \to \mathbb{R}$ (see Section 1.2.4) is expected to be related to L-functions by Colmez's conjecture [Col93, Conjecture 0.4] which predicts the relation

$$-\ h(A) \stackrel{?}{=} \sum_{\chi} m_{(E,\Phi)}(\chi) \left(\frac{L'(\chi, 0)}{L(\chi, 0)} + \log(f_\chi) \right) \tag{3.29}$$

where (E, Φ) is the CM-type of A and the sum runs over all the Artin characters

$$\chi\colon G_{\mathbb{Q}} \to \mathbb{C}$$

whose value on complex conjugation $c \in G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ equals $\chi(c) = -1$. This implies in particular that $L(\chi, 0) \in \mathbb{C}^\times$. Moreover, $f_\chi \in \mathbb{N}$ denotes the Artin conductor of χ and the family of rational numbers $\{m_{(E,\Phi)}(\chi)\}_\chi \subseteq \mathbb{Q}$ is defined by the equality

$$\frac{1}{[G_{\mathbb{Q}}\colon\ \mathrm{Stab}(\Phi)]} \sum_{\sigma \in G_{\mathbb{Q}}/\mathrm{Stab}(\Phi)} |\Phi \cap \sigma \circ \Phi| = \sum_{\chi} m_{(E,\Phi)}(\chi) \cdot \chi(\sigma)$$

which holds for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In particular, $m_{(E,\Phi)}(\chi) = 0$ for all but finitely many Artin characters.

We recall that Colmez's conjecture has recently been proved to hold *on average*. We refer the interested reader to the original work of Yuan and Zhang for an analytic proof (see [YZ18]), and of Andreatta, Goren, Howard and Madapusi Pera for a more geometric proof (see [And+18]), as well as to Howard's survey [How18].

Example 3.4.4 (Conductors)**.** We have seen in Section 1.2.5 and Section 1.2.6 that conductors of complex and ℓ-adic Galois representations can be seen as examples of heights. These conductors are related to special values of L-functions by means of the functional equation associated to a given Galois representation, as we recall in Section 3.4.4.

Example 3.4.5 (Volumes of hyperbolic manifolds)**.** The relations of hyperbolic volumes (see Section 1.2.7) with special values of L-functions comes from the fact that for every number field F we have that

$$\zeta_F^*(-1) \sim_{\mathbb{Q}^\times} \mathrm{vol}\left(\frac{\mathfrak{h}_3^{r_2(F)}}{\Gamma}\right)$$

where Γ is a finite-index and torsion-free subgroup of the group $O^{(1)} \subseteq O$ of units having norm one in some order $O \subseteq B$ in a totally definite quaternion algebra $B \neq \mathrm{Mat}_{2\times 2}(K)$ defined over K (see [Vig80, Example IV.1.5]).

3.4.2 Special values at the boundary of the critical strip: Dedekind ζ-functions

The aim of this section is to provide the first example of an infinite family of special values of L-functions which satisfies the Northcott property. This is given by the special values $\{\zeta_F^*(0)\}$ at $s = 0$ of the Dedekind ζ-functions associated to number fields F.

Theorem 3.4.6 – Northcott property for Dedekind ζ-values at $s = 0$

Let S be the set of isomorphism classes of number fields. Then for every $B \in \mathbb{R}_{>0}$ the set

$$\{[F] \in S : |\zeta_F^*(0)| \leq B\}$$

is finite.

Proof. Let $B > 0$ be a real number. Our aim is to prove that $\zeta_F^*(0) \leq B$ implies that Δ_F is bounded above, which allows us to conclude by Hermite's discriminant theorem (see [Neu99, Theorem III.2.16]).

By the class number formula (3.23), if $\zeta_F^*(0) \leq B$, then we have

$$\frac{h_F R_F}{w_F} \leq B. \tag{3.30}$$

The proof proceeds with two steps: first we prove that inequality (3.30) implies an upper bound on R_F. This leads to finiteness, except possibly for CM fields. The second step is proving finiteness of CM fields with $\zeta_F^*(0)$ bounded from above.

Step 1: Observe first of all that for every number field F of degree $d_F := [F : \mathbb{Q}]$ we have that $\varphi(w_F) \leq d_F$, where φ is Euler's totient function. Indeed $\mathbb{Q}(\mu_{w_F}) \subseteq F$, where $\mu_n \subseteq \overline{\mathbb{Q}}$ denotes the group of n-th roots of unity. Then one can use the easy estimate $2 \cdot \varphi(n) \geq \sqrt{n}$ to get that $w_F \leq 4\, d_F^2$. Now we can use the fact that for every number field F we have that $R_F \geq c_1 \cdot c_2^{d_F}$ where $c_1 = (11.5)^{-39}$ and $c_2 = 1.15$, as it was proved by Zimmert in [Zim81, Satz 3] (see also [Sko93] for a simpler proof and [FS99] for a more general statement). This surely implies the weaker inequality $w_F \leq c_3 \sqrt{R_F}$ for some absolute constant $c_3 \in \mathbb{R}_{>0}$. Going back to (3.30) and

using $h_F \geq 1$, we see that $\zeta_F^*(0) \leq B$ implies that $R_F \leq (c_3 \cdot B)^2$. By [Paz14, Theorem 1.1] we obtain that the set

$$\{[F] \in S \mid \zeta_F^*(0) \leq B\} \setminus \{\text{CM fields}\}$$

is finite. We recall briefly the argument here for completeness. Observe that Zimmert's inequality $R_F \geq c_1 \cdot c_2^{d_F}$ implies that number fields with regulator bounded from above have degree bounded from above. Then we can use [Fri89, Theorem C], providing us with the inequality

$$R_F \geq \frac{c_4}{d_F^{2d_F}} \cdot \left(\log\left(\frac{\Delta_F}{d_F^{d_F}}\right)\right)^{r_1(F)+r_2(F)-1-r_0(F)} \tag{3.31}$$

where $c_4 \in \mathbb{R}_{>0}$ is an absolute constant and

$$r_0(F) := \max\{r_1(L) + r_2(L) - 1 \mid L \subsetneq F\}$$

is the biggest unit rank of proper sub-fields of F. This gives a useful upper bound on the discriminant if and only if F is not a CM field. Indeed, we always have $r_0(F) \leq r_1(F) + r_2(F) - 1$ and the equality $r_0(F) = r_1(F) + r_2(F) - 1$ is satisfied if and only if F is a CM field (see [Paz14, Proposition 3.7]). The final step is Hermite's discriminant theorem, which shows that the discriminant has the Northcott property.

Step 2: Let now S_{CM} be the set of isomorphism classes of CM fields. We want to prove that

$$\{[F] \in S_{\mathrm{CM}} \mid \zeta_F^*(0) \leq B\} \tag{3.32}$$

is finite. To do so observe that for a CM field F of degree $d_F := [F\colon \mathbb{Q}]$ and with maximal real subfield denoted F^+, we have that $R_F \geq 2^{\frac{d_F}{2}-1} R_{F^+}$ (see [Paz14, Proposition 3.7]), and thus any upper bound on $\zeta_F^*(0)$ entails an upper bound on R_{F^+}. This implies by [Paz14, Theorem 1.1] that if F is an element of the set given in (3.32), then F^+ belongs to a finite set of isomorphism classes of totally real fields. Hence to conclude we can assume that F^+ is fixed. Then any upper bound on $\zeta_F^*(0)$ implies an upper bound on h_F, which in turn implies the finiteness of the set given in (3.32) by results of Siegel and Stark. To be more precise, when $F^+ = \mathbb{Q}$ (hence F is an imaginary quadratic field), Siegel proved the following: for any fixed $\varepsilon > 0$ there exists a constant $c_5(\varepsilon) > 0$, such that for any imaginary quadratic field F

$$h_F \geq c_5(\varepsilon) \cdot \Delta_F^{\frac{1}{2}-\varepsilon}.$$

This implies that the set of isomorphism classes of imaginary quadratic fields of class number bounded from above is finite by Hermite's theorem (see [Gol74] for a short and elegant proof of Siegel's result). If $F^+ \neq \mathbb{Q}$ we can use a result of Stark, who proved the following: for any fixed $\varepsilon > 0$ there exists a constant $c_6(\varepsilon) > 0$, such that for any CM field F of degree $d_F \geq 4$,

$$h_F \geq \frac{c_6(\varepsilon)^{d_F}}{d_F\, g(F^+)} \cdot \left(\frac{|\Delta_F|}{|\Delta_{F^+}|^2}\right)^{\frac{1}{2}-\frac{1}{d_F}} \cdot |\Delta_F|^{\frac{1}{2}-\frac{2}{d_F}-\varepsilon}$$

(see [Sta74, Theorem 2]) where for every number field κ we set $g(\kappa) = 1$ if there is a tower $\mathbb{Q} = \kappa_0 \subseteq \cdots \subseteq \kappa_n = \kappa$ such that $\kappa_i \subseteq \kappa_{i+1}$ is Galois, and $g(\kappa) = [\kappa\colon \mathbb{Q}]!$ otherwise. If we fix F^+ then this inequality gives us immediately an upper bound for the discriminant of F over $\mathbb{Q}$ depending on h_F (recall $d_F \geq 4$), and thus implies the finiteness of isomorphism classes of

CM fields with $F^+ \neq \mathbb{Q}$ fixed and h_F bounded, again by Hermite's theorem. Putting everything together, we have proved that the set given in (3.32) is finite and thus we can conclude. □

3.4.3 Special values inside the critical strip: abelian varieties

In this section, let us look at the Northcott properties of the special values at the integer $s = 1$ of the L-functions $L(A, s) := L(\underline{H}^1(A), s)$ associated to abelian varieties A defined over a number field F. We note that these are much more difficult to prove than the Northcott properties for the special values $\zeta_F^*(0)$ which we considered in the previous section. First of all, if we want to follow the strategy that we used in the previous section, we should relate the special value $L^*(A, 1)$ to some regulator determinant. This relation was given by the class number formula Theorem 3.3.26 in the case of the special value $\zeta_F^*(0)$ studied in the previous section, and was thus unconditional. On the other hand, $L^*(A, 1)$ is related to a regulator determinant only by the conjectural equality (3.25), re-written here as

$$L^*(A,1) \stackrel{?}{=} \left(\frac{\prod_{v \in M_F^0} c_v(A)}{|A(F)_{\mathrm{tors}}| \cdot |A^\vee(F)_{\mathrm{tors}}|} \right) \cdot \frac{|Ш(A/F)| \cdot R_{A/F}}{\Omega_A^{-1}} \tag{3.33}$$

which is the subject of the celebrated conjecture by Birch and Swinnerton-Dyer (see Conjecture 3.3.27).

Now, the first step in the proof of Theorem 3.4.6 was observing that the quantity $|(O_F^\times)_{\mathrm{tors}}|$ appearing in the class number formula Theorem 3.3.26 is clearly bounded from above by a polynomial in the degree $[F\colon \mathbb{Q}]$ of the number field F. An analogous statement for abelian varieties is the content of the following, widely believed conjecture.

Conjecture 3.4.7 – Torsion conjecture

For every number field F and every $g \in \mathbb{N}_{\geq 1}$ there exists a natural number $c(g, F) \in \mathbb{N}$ such that $|A(F)_{\mathrm{tors}}| \leq c(g, F)$ for all g-dimensional abelian varieties A defined over F.

We recall that, in the case of elliptic curves, Conjecture 3.4.7 is proved to be true, thanks to work of Merel (see [Mer96]). Moreover, the prime number theorem shows easily that

$$|A(F)_{\mathrm{tors}}| \cdot |A^\vee(F)_{\mathrm{tors}}| \ll (\log|\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{f}_A)|)^{4\dim(A)}$$

as explained in [Hin07, Lemma 3.6].

Now, observing that the Tamagawa numbers $c_v(A)$ are integers, we see that any upper bound for the quantity $|L^*(A, 1)|$ entails an upper bound for the quantity

$$\frac{|Ш(A/F)| \cdot R_{A/F}}{\Omega_A^{-1}} \tag{3.34}$$

if one assumes the validity of the formula (3.33), which is a version of the Birch and Swinnerton-Dyer conjecture. Since our goal is to study Northcott properties for the quantity $|L^*|$, it would be useful to compare the quantity (3.34) to other quantities for which a Northcott property is already known to hold. The best candidates for this are the stable Faltings height $h_{\mathrm{st}}(A)$ and the norm of the conductor ideal $\mathfrak{f}_A$ of the abelian variety A.

This is exactly the same strategy which was achieved in the proof of Theorem 3.4.6, where the quantity $h_F \cdot R_F$ was compared to the quantity $|\Delta_F|$, which satisfies the Northcott property thanks to Hermite's theorem. However, there is one fundamental difference between the proof of Theorem 3.4.6 and the current discussion: both the numerator and the denominator of the ratio (3.34) are comparable to something satisfying a Northcott property, at least conjecturally. First of all, one has that

$$H(A) \ll \Omega_A^{-1} \ll H(A)(\log(H(A)))^{\dim(A)/2}$$

as shown in [Hin07, Lemma 3.7]. Secondly, [Hin07, Conjecture 5.5] predicts that

$$H(A)^{1-\varepsilon} \ll |Ш(A/F)| \cdot R_{A/F} \ll H(A)^{1+\varepsilon} \tag{3.35}$$

which would be analogous to the Brauer-Siegel theorem that holds for the regulator of number fields, and was used in the proof of Theorem 3.4.6. Hindry proves in [Hin07, Proposition 5.6] that (3.35) holds if one assumes a suitable generalisation of Szpiro's conjecture (see [Hin07, Conjecture 3.4]) as well as the validity of the following inequalities

$$|\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{f}_A)|^{-\varepsilon} \overset{?}{\ll} |L^*(A,1)| \overset{?}{\ll} |\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{f}_A)|^{\varepsilon} \tag{3.36}$$

for every $\varepsilon \in \mathbb{R}_{>0}$.

The previous discussion shows that it is necessary to gain further evidence in order to be able to prove a Northcott property for the special value $L^*(A,1)$ associated to abelian varieties. In particular, the two quantities $|Ш(A/F)| \cdot R_{A/F}$ and Ω_A^{-1} appearing at the numerator and denominator of the ratio (3.34), appear to have the same order of magnitude, at least conjecturally. It is henceforth necessary to study better these quantities, to understand in which sorts of infinite families of abelian varieties one can expect that the quantity $L^*(A,1)$ satisfies a Northcott property. Two final remarks are in order:

- the validity of the inequalities (3.36) has been questioned by Watkins in [Wat08, § 4.5];
- in the case of elliptic curves, one knows from [AHP18] that the following inequality holds

$$\frac{R_{E/F}}{|E(F)_{\mathrm{tors}}| \cdot |E^\vee(F)_{\mathrm{tors}}|} \gg h^{\frac{r_{E/F}-4}{3}} \cdot (\log(3 \cdot h))^{\frac{2 \cdot r_{E/F}+2}{3}}$$

  where $h := \max\{1, h(j(E))\}$ is a quantity comparable with the stable Faltings height (see for instance [Paz18, Lemma 3.2]), and $r_{E/F} := \mathrm{rk}(E(F))$. This inequality shows that a part of the right hand side of (3.33) can indeed be related to some height, even if this relation is too weak to conclude that (under the Birch and Swinnerton-Dyer conjecture) the special value $L^*(E,1)$ satisfies a Northcott property.

3.4.4 Special values outside the critical strip: Weil's conjectures and the functional equation

The aim of this section is to show how to get a Northcott property for special values of L-functions at the left of the critical strip using the conjectural functional equation.

Proposition 3.4.8 – Northcott properties at the left of the critical strip

Let $F \neq \mathbb{Q}$ and E be two number fields, and fix $w \in \mathbb{Z}$. Then for every $B_1, B_2 \in \mathbb{R}_{\geq 0}$ and every $n \in \mathbb{Z}$ such that $n < w/2$, the set

$$S := \{M \in \mathcal{MM}(F;E) \mid M \cong \mathrm{gr}_w^{\mathcal{W}}(M),\ |L^*(M,n)| < B_1,\ \mathfrak{h}_\infty(M) < B_2\}/\sim_{\mathrm{iso}}$$

is finite, under the assumption of Conjecture 3.3.4 and Conjecture 3.3.6. Here $\mathfrak{h}_\infty(M) \in \mathbb{N}$ is defined as

$$\mathfrak{h}_\infty(M) := \max_{\substack{j\in\mathbb{Z}\\ \sigma\in\mathrm{Hom}(E,\mathbb{C})}} \left(\{n_{j,\sigma}(R_v(M)) \mid v \in M_F^{\mathbb{C}}\} \cup \{n^{\varepsilon}_{j,\sigma}(R_v(M)) \mid \varepsilon \in \{\pm\},\ v \in M_F^{\mathbb{R}}\}\right)$$

where $M_F^{\mathbb{C}} := \{v \in M_F^{\infty} : F_v \cong \mathbb{C}\}$ and $M_F^{\mathbb{R}}$ is defined analogously. We recall that the various numbers $n_{j,\sigma}(\underline{H}_{/\mathbb{C}})$ and $n^{\pm}_{j,\sigma}(\underline{H}_{/\mathbb{R}})$ associated to a Hodge structure $\underline{H} \in \mathrm{MHS}(K;E)$ defined over $K \in \{\mathbb{R}, \mathbb{C}\}$ were introduced in Section 3.2.3.

Proof. Applying the functional equation (3.19) we see that the inequality $|L^*(M,n)| < B$ is equivalent to

$$|\varepsilon(M,n)| \leq B \cdot |L^*(M^\vee, 1-n)|^{-1} \cdot \frac{|L_\infty^*(M,n)|}{|L_\infty^*(M^\vee,1-n)|} \tag{3.37}$$

where $L_\infty := \prod_{v\in M_F^\infty} L(R_v(M),s)_{\mathbb{C}}$ denotes the Archimedean part of the completed L-function $\widehat{L}(M,s)$. Since $\mathfrak{h}_\infty(M)$ is bounded from above, we see from the definition of the Archimedean component of the L-function that there exists $B_3 \in \mathbb{R}_{\geq 0}$ (depending on B_2) such that

$$M \in S \Rightarrow \frac{|L_\infty^*(M,n)|}{|L_\infty^*(M^\vee,1-n)|} \leq B_3$$

which can be combined with (3.37) to get that

$$|\varepsilon(M,n)| \leq (B \cdot B_3) \cdot |L^*(M^\vee,1-n)|^{-1} \tag{3.38}$$

for every $M \in S$.

Now, the assumption that $M \cong \mathrm{gr}_w^{\mathcal{W}}(M)$, *i.e.* that M is pure of weight w, implies that for every non-Archimedean place $v \in M_F^0$ the absolute values of the roots of the polynomial $f_{R_v(M)}$ attached to the local Galois representation $R_v(M)$ (see Section 3.2.2) are bounded by a function depending only on w, which is equal to $|\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{p}_v)|^{w/2}$ for almost all places $v \in M_F^0$. Moreover, Remark 3.3.10 shows that

$$|L^*(M^\vee,1-n)| = |L^*(M, w+1-n)|$$

and if we combine this with the previous observation we see that

$$M \in S \Rightarrow |L^*(M^\vee,1-n)| \geq B_4 \tag{3.39}$$

for some $B_4 \in \mathbb{R}$, depending only on w and n. Hence, putting together (3.38) and (3.39) we see that $|\varepsilon(M,n)| \leq B_5$ for every $M \in S$, where $B_5 := B \cdot B_3/B_4$.

To conclude, it is sufficient to recall that for every $M \in S$ we have

$$|\varepsilon(M,n)| = |\Delta_F|^{\frac{w+1}{2}\dim(R_\ell(M))} \cdot |\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{f}_{R_\ell(M)})|^{\frac{w+1}{2}-n} \leq B_5$$

where $\ell \in \mathbb{N}$ is any prime such that M has good reduction at every place of F lying above ℓ (see [Roh94, § 12, Corollary]). Thus we see that, since $F \neq \mathbb{Q}$, both the dimension and the norm of the conductor of the Galois representations $R_\ell(M)$ is bounded from above. Hence we can apply the Northcott property for the conductor that we have seen in Section 1.2.6 to see that there are only finitely many $R_\ell(M)$, up to isomorphism. Since $\mathcal{MM}(F;E)$ is one of the categories of mixed motives defined in Section 2.2.2, we see that every motive is determined by its realisations, and therefore we have also finitely many elements in S. □

Remark 3.4.9. The proof of Proposition 3.4.8 shows that Proposition 3.4.8 can be extended to $F = \mathbb{Q}$, if we simply add the dimension of the ℓ-adic realisations of M to the bounded functions. However, it is possible that in fact Proposition 3.4.8 would hold without changes even for $F = \mathbb{Q}$. To show this, one would need to show that the set of isomorphism classes of pure ℓ-adic Galois representations defined over $\mathbb{Q}$ which have their conductor bounded and their weight fixed is finite, without bounding the dimension as we did in Section 1.2.6. Such a statement is even not known for weight one Galois representation. However, the fact that there are no abelian varieties of any dimension $d \geq 1$ which are unramified over $\mathbb{Q}$, as was proved by Fontaine in [Fon85], might be seen as evidence for these kinds of statements.

3.4.5 Connections with motivic heights

The aim of this short section is to briefly describe a possible connection between the Diophantine properties (and in particular the Northcott property) described in Section 3.4, and the motivic heights defined by Kato in [Kat18]. Our driving question is the following.

Question 3.4.10 – Special values of L-functions and motivic heights

Let F and E be two number fields. Fix $h\colon \mathcal{MM}(F;E) \to \mathbb{R}$ to be one of the height functions defined in [Kat18], *e.g.* $h = h_{*,\diamond}$ (see Example 3.4.11). Let $M \in \mathcal{MM}(F;E)$ be a motive and $n \in \mathbb{Z}$ be an integer, for which the special value $L^*(M,n)$ is defined. Does there exist another motive M_n such that

$$|L^*(M,n)| = h(M_n)$$

or maybe a finite family of motives $\mathbf{M}_n$, in the guise of a square matrix, such that $|L^*(M,n)| = \det(h(\mathbf{M}_n))$?

First of all, let us briefly review the definition of Kato's heights, and then let us describe one possible strategy to answer Question 3.4.10. Kato's heights are defined in [Kat18] by the formula

$$\begin{aligned} h_\psi\colon \mathcal{M}_F &\to \mathbb{R} \\ M &\mapsto \sum_{d\in\mathbb{N}} h_d^{\Psi_d}(M) \end{aligned} \tag{3.40}$$

which depends on a family of functions $\psi = \{\psi_d\}_{d\in\mathbb{N}}$. These functions are given by

$$\psi_0\colon \mathcal{M}_F \times \mathbb{Z}^2 \to \mathbb{R}$$

and $\psi_d \colon \mathcal{M}_F \times \mathbb{Z} \to \mathbb{R}$ for $d \geq 1$.

The idea is that the complexity measured by the heights h_d^{ψ} increases with respect to $d \in \mathbb{N}$. In particular, $h_0^{\psi}(M)$ measures the complexity of the graded pieces $\mathrm{gr}_{\mathcal{W}}^w(M)$ similarly to how Faltings's height measures the complexity of an abelian variety. On the other hand, the higher heights $h_d^{\psi}(M)$ measure how distant the motive M is from being isomorphic to $\bigoplus_{w\in\mathbb{Z}} \mathrm{gr}_{\mathcal{W}}^w(M)$ by measuring the complexity of monodromy at the different primes. In particular, if

$$M \cong \bigoplus_{w\in\mathbb{Z}} \mathrm{gr}_{\mathcal{W}}^w(M)$$

then $h_{w,d}^{\psi}(M) = 0$ for every $d \geq 1$. Finally, the functions φ and ψ are supposed to be "simple" functions that serve merely as coefficients in the linear combinations (3.40) and (3.41). See Example 3.4.11 for some examples of functions φ and ψ.

Let us get into the definition of $h_0^{\psi_0}$, which is given by

$$h_0^{\psi}(M) := \sum_{(w,r)\in\mathbb{Z}^2} \psi_0(M; w, r) \cdot \widehat{\deg}\left(\mathcal{L}_r(\mathrm{gr}_{\mathcal{W}}^w(M)), \{|\cdot|_v\}_{v\in\Omega_F}\right). \tag{3.41}$$

where $\mathcal{L}_r(\mathrm{gr}_{\mathcal{W}}^w(M))$ is a one dimensional F-vector space endowed with an absolute value $|\cdot|_v$ for every place $v \in \Omega_F$. For every such vector space $(V, \{|\cdot|_v\}_{v\in\Omega_F})$ we define the "Arakelov degree"

$$\widehat{\deg}(V, \{|\cdot|_v\}_{v\in\Omega_F}) := \sum_{v\in\Omega_F} \log|x|_v \qquad \text{for any generator} \quad x \in V$$

which is well defined because V is one dimensional and the absolute values of F are normalized to satisfy the product formula $\prod_{v\in\Omega_F}|x|_v = 1$. In our specific case, for every $r \in \mathbb{Z}$ and every pure motive N of weight w_N we define

$$\mathcal{L}_r(N) := \det(\mathrm{gr}_{\mathcal{H}}^r(N_{\mathrm{dR}})) \otimes_F \det(\mathrm{gr}_{\mathcal{H}}^{w_N - r}(N_{\mathrm{dR}}))$$

and one uses Hodge theory and p-adic Hodge theory to define the different absolute values. We refer the interested reader to [Kat18, § 1.4] for further details.

Now we can turn to the definition of $h_1^{\psi_1}$, which is given by

$$h_1^{\psi_1}(M) := \sum_{w\in\mathbb{Z}} \psi_1(M; w) \cdot \left(\sum_{v\in\Omega_F} \log\left(\|\alpha_{w,1}(M)\|_{1,v}^{M,w} \right) \right)$$

where $\alpha_{w,1}(M) \in \mathrm{Ext}^1_{\mathcal{M}_F}(\mathrm{gr}_{\mathcal{W}}^w(M), \mathrm{gr}_{\mathcal{W}}^{w-1}(M))$ is the class of the extension

$$0 \to \mathrm{gr}_{\mathcal{W}}^{w-1}(M) \to \frac{\mathrm{Fil}_{\mathcal{W}}^w(M)}{\mathrm{Fil}_{\mathcal{W}}^{w-2}(M)} \to \mathrm{gr}_{\mathcal{W}}^w(M) \to 0$$

and the absolute values $\|\cdot\|_{M,w,1,v} \colon \mathrm{Ext}^1_{\mathcal{M}_F}(\mathrm{gr}_{\mathcal{W}}^w(M), \mathrm{gr}_{\mathcal{W}}^{w-1}(M)) \to \mathbb{R}_{\geq 0}$ are defined using Beilinson's height pairing. Indeed, we can identify

$$\mathrm{Ext}^1_{\mathcal{M}_F}(\mathrm{gr}_{\mathcal{W}}^w(M), \mathrm{gr}_{\mathcal{W}}^{w-1}(M)) = \mathrm{Ext}^1_{\mathcal{M}_F}(\mathbb{Q}, M^{(w)})$$

where $M^{(w)} := (\mathrm{gr}^w_{\mathcal{W}}(M))^\vee \otimes \mathrm{gr}^{w-1}_{\mathcal{W}}(M)$. If we assume that for every $w \in \mathbb{Z}$ the graded quotient $\mathrm{gr}^w_{\mathcal{W}}(M)$ is polarized then we have a polarization $p_{M^w} \colon M^{(w)} \to (M^{(w)})^\vee(1)$ that we can use to define the absolute values

$$\begin{aligned} \|\cdot\|_{M,w,1,v} \colon \mathrm{Ext}^1_{\mathcal{M}_F}(\mathbb{Q}, M^{(w)}) &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \sqrt{\langle x, p_{M^{(w)}}(x)\rangle_{\mathcal{B},M^{(w)},v}} \end{aligned}$$

using the local components of Beilinson's height pairing for the motive M^w. We refer the interested reader to [Kat18, § 1.7.1] for more details.

Finally, let us define $h_d^{\psi_d}$ for $d \geq 2$. We have again a local decomposition

$$h_d^{\psi_d}(M) := \sum_{w\in\mathbb{Z}} \psi_d(M;w) \cdot \left(\sum_{v\in\Omega_F} \log\left(\|\alpha_{w,d,v}(M)\|_{M,w,d,v} \right) \right)$$

but now the elements of which we take the absolute value vary with the place $v \in \Omega_F$. These elements $\alpha_{w,d,v}(M)$ are defined using the monodromy theorem of Grothendieck and the (conjectural) weight monodromy filtration that arises from the relation between the weight filtration and the monodromy operator. We refer the interested reader to [Kat18, § 1.7.3] for more details.

Example 3.4.11. Let us list some choices for φ, which allow us to recover all the examples of heights defined in [Kat18].

First of all, we can choose $\psi_d = 0$ for $d \geq 1$. Moreover, we can choose

$$\psi_0(M;w,r) := f(M;w) \cdot g(\mathrm{gr}^w_{\mathcal{W}}(M);r)$$

for two functions $f \colon \mathcal{M}_F \times \mathbb{Z} \to \mathbb{R}$ and $g \colon \mathcal{M}_F^{\mathrm{pure}} \times \mathbb{Z} \to \mathbb{R}$. Possible choices for f and g are given by

$$\begin{aligned} f(M;w) &:= 1 & g(N;r) &:= 1 \\ f(M;w) &:= \begin{cases} 1, & \text{if } w = w_0 \\ 0, & \text{otherwise} \end{cases} & g(N;r) &:= \sum_{j<r} \dim_F(\mathrm{gr}^j_{\mathcal{H}}(N_{\mathrm{dR}})) \\ f(M;w) &:= \dim_F(\mathrm{gr}^w_{\mathcal{W}}(M_{\mathrm{dR}})) & g(N;r) &:= \dim_F(\mathrm{gr}^j_{\mathcal{H}}(N_{\mathrm{dR}})) \end{aligned}$$

and these examples (in any combination of choices from the first and the second column) allow us to recover the functions h_* and $h_\clubsuit$ defined by Kato. Observe that [Kat18] defines these heights only for pure motives: the first choice of g paired with any of the first two choices of f gives us h_* when restricted to pure motives of fixed weight w_0.

On the other hand, the function $h_\spadesuit$ can be recovered by taking $\psi_d = 0$ for $d \geq 1$ and

$$\psi_0(M;w,r) := \sum_{\substack{i<w \\ j<r}} \varphi_{i,j}(M) - \sum_{\substack{i>w \\ j>r}} \varphi_{i,j}(M) + \sum_{\substack{j<r \\ j\neq w-r}} \varphi_{w,j}(M) + \begin{cases} \varphi_{w,r}(M) - (-1)^w, & \text{if } r > w/2, \\ 0, & \text{otherwise} \end{cases}$$

where $\varphi(M;i,j) := \dim_F(\mathrm{gr}^j_{\mathcal{H}}(\mathrm{gr}^i_{\mathcal{W}}(M_{\mathrm{dR}})))$.

Finally, we can of course consider many choices for the functions $\psi_d(M; w, r)$ for $d \geq 1$. In particular, if we set $\psi_d \equiv 1$ for every $d \in \mathbb{N}_{\geq 1}$ then we get the height $h_{*,\diamond}$ mentioned in Question 3.4.10.

As we have briefly seen, Kato's height functions may seem rather involved at the first sight. However, the complexity of their definitions sparks even more hope that they might be the right invariant to capture all the intrinsic complexity inherent to the category of mixed motives. In particular, one could ask whether Kato's heights are the right ones to answer Question 3.4.10.

Let us conclude this section by briefly mentioning our strategy to construct the motive M_n (or the matrix of motives $\mathbf{M}_n$) appearing in Question 3.4.10, at least in the case $F = E = \mathbb{Q}$. Moreover, let us assume that $M \in \mathcal{MM}(\mathbb{Q}/\mathbb{Z}; \mathbb{Q})$, where $\mathcal{MM}(\mathbb{Q}/\mathbb{Z}; \mathbb{Q})$ denotes the category of "mixed motives over $\mathbb{Z}$" defined by Scholl in [Sch91, Page 376]. This is the full subcategory of $\mathcal{MM}(\mathbb{Q}; \mathbb{Q})$, consisting of those motives M such that the weight filtration on the ℓ-adic realisation $R_\ell(M)$ splits over $\mathbb{Q}_p^{\mathrm{nr}}$ (the maximal unramified extension of $\mathbb{Q}_p$) for every pair of distinct primes $\ell, p \in \mathbb{N}$. In particular, $\mathcal{MM}(\mathbb{Q}/\mathbb{Z}; \mathbb{Q})$ contains all the pure motives.

Now, if $M \in \mathcal{MM}(\mathbb{Q}/\mathbb{Z}; \mathbb{Q})$ and $n \in \mathbb{Z}$ then we can use a construction of Scholl (see [Sch94]) to get a new motive $M_{(n)} \in \mathcal{MM}(\mathbb{Q}; \mathbb{Q})$ such that

$$L(M_{(n)}, s) = L(M, s)\zeta(s - n)^a \zeta(s - n + 1)^b \tag{3.42}$$

and n is a *critical value* for the L-function $L(M_{(n)}, s)$. This means that

$$\mathrm{Hom}(\mathbb{1}, M_{(n)}(n)) = \mathrm{Ext}^1(\mathbb{1}, M_{(n)}(n)) = \mathrm{Hom}(M_{(n)}(n), \mathbb{1}) = \mathrm{Ext}^1(M_{(n)}(n), \mathbb{1}) = 0 \tag{3.43}$$

where the homomorphism and extension groups are taken in the category $\mathcal{MM}(\mathbb{Q}/\mathbb{Z}; \mathbb{Q})$. Hence we have in particular that $L(M_{(n)}, n) \neq 0$, and the conjectures of Bloch and Kato imply that the special value $L^*(M_{(n)}, n) = L(M_{(n)}, n)$, which is related to $L^*(M, n)$ by (3.42), can be computed using the determinant of a matrix whose entries are given by Beilinson's height pairing. Thus it is not unreasonable to expect that, for some choice of motivic height $h\colon \mathcal{MM}(\mathbb{Q}; \mathbb{Q}) \to \mathbb{R}$, one has that $h(M_{(n)})$ is related (or even equal) to $L^*(M, n)$. This will be the subject of future investigations.

An introduction to the Mahler measure

4

Measure what is measurable,
and make measurable what is not so.

Galileo Galilei,
Dialogue Concerning the Two Chief World Systems

The present chapter contains a brief survey of the theory surrounding the Mahler measure of polynomials. This "measure" gives a number $M(f) \in \mathbb{R}_{\geq 0}$ that can be associated to a vast class of complex-valued functions $f\colon \mathbb{T}^n \to \mathbb{C}$ defined on the real torus

$$\mathbb{T}^n := \{\mathbf{z} = (z_1, \dots, z_n) \in (\mathbb{C}^\times)^n \mid |z_1| = \cdots = |z_n| = 1\}.$$

This class of functions comprises Laurent polynomials $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, for which one knows that $M(P) \neq 0$ as soon as $P \neq 0$. This allows one to define the *logarithmic Mahler measure* $m(P) := \log(M(P))$. Since in this thesis we are mainly interested in the logarithmic version, we deserve the term *Mahler measure* for $m(P)$, and call $M(P)$ by the name of *exponential Mahler measure*.

The exponential Mahler measure $M(P)$ for one-variable monic polynomials $P \in \mathbb{Z}[t]$ was introduced by Lehmer (see [Leh33, Theorem 16]), using the formula

$$M(P) := \prod_{\substack{\alpha \in \mathbb{C} \\ P(\alpha)=0}} \max(1, |\alpha|) \tag{4.1}$$

which was used to show that $M(P)$ computes the growth rate of the sequence of integers

$$\Delta_n(P) := \prod_{\substack{\alpha \in \mathbb{C} \\ P(\alpha)=0}} \alpha^n - 1 \in \mathbb{Z}$$

which were introduced by Pierce (see [Pie16]) as a generalisation of Mersenne's sequence $2^n - 1 = \Delta_n(t-2)$. Note in particular that $M(P) \geq 1$ for every $P \in \mathbb{Z}[t] \setminus \{0\}$. Exactly like Mersenne's numbers, any integer of the form $\Delta_n(P)$ is easier to factor than a randomly chosen one. In particular, Lehmer pointed out in [Leh33, § 13] that the smaller $M(P)$ was, the more primes there seemed to be in the sequence $\{\Delta_n(P)\}_{n\in\mathbb{N}}$. This lead him to ask whether the height function

$$M\colon \mathbb{Z}[t] \setminus \{0\} \to \mathbb{R} \tag{4.2}$$

has the *Bogomolov property* (see Definition 1.1.7), *i.e.* whether the set $S_1 := M(\mathbb{Z}[t] \setminus \{0\}) \subseteq \mathbb{R}$ has an isolated minimum. It is known that $\min(S_1) = 1$ (see Theorem 4.1.15), but Lehmer's general question remains unanswered, despite the numerous attempts and partial results, which

we describe in Section 4.1.1. Clearly, the Bogomolov property holds for the function (4.2) if and only if it holds for the function $m\colon \mathbb{Z}[t] \setminus \{0\} \to \mathbb{R}$.

One of the most interesting attempts towards the solution of Lehmer's problem has been given by Boyd (see [Boy81b]). More precisely, studying Lehmer's question entails a deeper understanding of the numbers $\alpha \in \mathbb{R}$ that arise as limits of sequences of Mahler measures, *i.e.* of the derived set $S_1^{(1)} \subseteq \mathbb{R}_{\geq 1}$. A result of Boyd and Lawton shows that $S_n \subseteq S_1^{(1)}$ for every $n \in \mathbb{Z}_{\geq 1}$, where $S_n := M(\mathbb{Z}[x_1, \dots, x_n] \setminus \{0\}) \subseteq \mathbb{R}_{\geq 1}$. This gives rise to a nested, increasing chain of sets

$$S_1 \subseteq \dots S_n \subseteq S_{n+1} \subseteq \cdots \subseteq S_1^{(1)} \subseteq \overline{S_1} \subseteq \mathbb{R}_{\geq 1}$$

where $\overline{S_1}$ denotes the closure of S_1. Now, Boyd conjectures in [Boy81b, Conjecture 1] that the set $S_\infty := \varinjlim_n S_n \subseteq \mathbb{R}_{\geq 1}$ is closed. Since S_∞ is countable, this would easily imply Lehmer's conjecture (see Lemma 4.1.19).

The theorem of Boyd and Lawton, together Boyd's prediction that S_∞ is closed, generated an increasing interest in the study of Mahler measures of polynomials in multiple variables. The pioneering work [Smy81] by Smyth showed that these real numbers could be surprisingly related to special values of L-functions. More precisely, Smyth computed the two formulas

$$m(x + y + 1) = L^*(\chi_{-3}, -1)$$
$$m(x + y + z + 1) = -14\zeta^*(-2)$$

where $\chi_{-3}\colon (\mathbb{Z}/3\mathbb{Z})^\times \to \{\pm 1\}$ is the unique non-trivial Dirichlet character modulo 3, associated to the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$, and ζ denotes Riemann's ζ-function. These results, further enriched by the thesis of Ray [Ray87], prompted Boyd to start an intensive numerical investigation concerning the relations between Mahler measures of polynomials in multiple variables and special values of L-functions. This led to the foundational paper [Boy98], which contains an incredible amount of predictions and numerical computations relating special values of L-functions arising from elliptic curves to polynomials in two variables. These conjectural identities are still largely unproved today, despite the fact that the modular methods of Rogers and Zudilin (see [RZ12], [RZ14] and [Zud14]) and Brunault (see [Bru16b] and [BZ20, Chapter 10]) have almost reduced the task of proving some of these identities to a purely algorithmic one.

The last years of the twentieth century did not only see the publication of Boyd's numerical computations, but they also featured the appearance of two theoretical papers attempting at explaining them. First of all, we mention Rodriguez Villegas's paper [Rod99], which focuses on the fact that most identities conjectured by Boyd come in families of polynomials $P_k \in \mathbb{Z}[x, y]$, where $P_k(x, y) \in \mathbb{Z}[k][x, y]$. Thus the Mahler measure $m(P_k)$ can be studied as a function of k, and for suitable families this function can be related to Eisenstein-Kronecker series, which are intimately linked to modular forms. Moreover, this approach of studying Mahler measures in families has also proved incredibly useful in proving relations of the form $m(P_k) = m(Q_k)$ for two families of polynomials $P_k, Q_k \in \mathbb{Z}[x, y]$. We do not deal with these kinds of problems in this thesis, despite their great interest, and we refer the interested reader to the papers by Bertin-Zudilin (see [BZ16; BZ17] and the survey [BZ20, Chapter 5]) and Lalín and Wu (see [LW18] and [LW20]) for two different methods that allow one to prove relations between the Mahler measures of different families of polynomials.

The second (or rather the first, in chronological order) of the innovative works that appeared towards the end of the twentieth century consists in Deninger's paper [Den97a]. In this work, Deninger proves that the seemingly transcendental integral defining the Mahler measure can be converted into an integral related to algebraic geometry. More precisely, for a suitable class of

Laurent polynomials $P \in \overline{\mathbb{Q}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ there exists a mixed motive $M_P \in \mathcal{MM}(F; \mathbb{Q})$, where F is the number field generated by the coefficients of P, such that the Mahler measure $m(P)$ appears as a period of M_P (see Section 4.3). This class of polynomials has been enlarged by numerous subsequent work, among which we cite the paper [BD99] by Besser and Deninger, and Bornhorn's thesis (see [Bor99; Bor15]).

Deninger's work can also be used to relate Boyd's conjectural links between L-values and Mahler measures to the conjectures of Beilinson and Bloch-Kato that we surveyed in Section 3.3.2. This has been fully done by Bornhorn in his PhD thesis for the family $P_k(x, y) := t_1 + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + k$ (see Theorem 4.4.1). We dedicate Section 4.4.1 to give a similar proof for the family

$$P_k(x, y) = t_1 + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + \frac{t_1}{t_2} + \frac{t_2}{t_1} + k.$$

Finally, Section 4.4.2 is devoted to the computation of the Mahler measure of two families providing a "canonical model" for elliptic curves: a certain Weierstraß family, and the Edwards family.

4.1 Definition and Diophantine properties

The aim of this section is to introduce the theory of the Mahler measure, by giving its definition and explaining its relation to the height of algebraic number, by means of Jensen's formula. Moreover, we outline the main questions of Diophantine nature related to the Mahler measure.

First of all, let us define the Mahler measure m and its exponential variant M, which were introduced by Mahler in [Mah62].

Definition 4.1.1 – Mahler measure

Let $n \in \mathbb{N}$ be an integer. Then the *Mahler measure* is the functional

$$m \colon \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \to \mathbb{R} \cup \{-\infty\}$$

defined on the ring $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ of Laurent polynomials as

$$m(P) := \int_{\mathbb{T}^n} \log|P| \, d\mu_{\mathbb{T}^n} \tag{4.3}$$

where $\mathbb{T}^n := \{\mathbf{z} = (z_1, \dots, z_n) \in (\mathbb{C}^\times)^n \mid |z_1| = \cdots = |z_n| = 1\}$ is the n-dimensional real-analytic torus and $\mu_{\mathbb{T}^n}$ is the unique Haar probability measure on $\mathbb{T}^n$. Moreover, we define the *exponential Mahler measure* $M \colon \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \to \mathbb{R}_{\geq 0}$ as $M(P) := \exp(m(P))$. Finally, we define the *plus-Mahler measures* as the positive functionals $m^+ \colon \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \to \mathbb{R}_{\geq 0}$ and $M^+ \colon \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \to \mathbb{R}_{\geq 1}$ given by

$$m^+(P) := \int_{\mathbb{T}^n} \log^+|P| \, d\mu_{\mathbb{T}^n} \quad \text{and} \quad M^+(P) := \exp(m^+(P))$$

where $\log^+ \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined by setting $\log^+(0) := 0$ and $\log^+(t) := \max(0, \log(t))$ for every $t > 0$.

Example 4.1.2. We regard $\mathbb{T}^0$ as a point, hence the Mahler measure of a constant $\alpha \in \mathbb{C}$ is defined to be $m(\alpha) := \log(\alpha)$. Analogously, one has $m^+(\alpha) = \log^+|\alpha|$, and the exponential Mahler measures are given by $M(\alpha) = |\alpha|$ and $M^+(\alpha) = \max(1, |\alpha|)$.

Remark 4.1.3. It is easy to prove by induction on the number of variables $n \in \mathbb{Z}_{\geq 1}$ that for every fixed non-zero Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\}$ we have that $\mu_{\mathbb{T}^n}(V_P(\mathbb{C}) \cap \mathbb{T}^n) = 0$, where $V_P \hookrightarrow \mathbb{G}^n_{m,\mathbb{C}}$ is the closed sub-variety given by the zero locus of P. Actually, one can easily prove the estimate (see [EW99, Lemma 3.8])

$$\mu_{\mathbb{T}^n}(\{\mathbf{z} \in \mathbb{T}^n \mid |P(\mathbf{z})| < \varepsilon\}) \leq C_P \cdot \varepsilon^{\delta(P)} \tag{4.4}$$

where $C_P \in \mathbb{R}_{>0}$ depends on the coefficients of P and $\delta(P) \in \mathbb{R}_{>0}$ depends on the degree of P. This estimate (4.4) implies in particular that $m(P) = -\infty$ if and only if $P = 0$.

Remark 4.1.4. The inclusions $\iota_n \colon \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \hookrightarrow \mathbb{C}[x_1^{\pm 1}, \dots, x_{n+1}^{\pm 1}]$ are compatible with the Mahler measure, *i.e.* $m(\iota_n(P)) = m(P)$ for every Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. This allows one to see the Mahler measure as a functional

$$m \colon H^0(\mathbb{G}^\infty_{m,\mathbb{C}}, \mathcal{O}_{\mathbb{G}^\infty_{m,\mathbb{C}}}) \cong \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \dots] \to \mathbb{R} \cup \{-\infty\}$$

where $\mathbb{G}^\infty_{m,\mathbb{C}}$ denotes the inverse limit of the complex algebraic tori $\mathbb{G}^n_{m,\mathbb{C}}$ along the projections which forget the last coordinate.

Remark 4.1.5. The Mahler measure can be defined in a far more general setting. More precisely, let (X, μ_X) be a probability space and let $L^0(X)$ denote the complex vector space of measurable functions $f \colon X \to \mathbb{C}$, quotiented by the sub-space given by those functions which are zero almost everywhere. Then one can define the L^p-spaces

$$L^p(X) := \left\{ f \in L^0(X) \;\middle|\; \|f\|_p := \left(\int_X |f|^p \, d\mu_X \right)^{1/p} < +\infty \right\}$$

for every non-negative real number $p \in \mathbb{R}_{\geq 0}$. These are complex vector spaces, endowed with a function

$$\|\cdot\|_p \colon L^p(X) \to \mathbb{R}_{\geq 0}$$

for every $p > 0$. This function is a norm if $p \geq 1$, but only a quasi-norm in general. Now, since X is supposed to be a probability space we have that $L^p(X) \subseteq L^q(X)$ for every $p \geq q \geq 0$. Using this, one can define the *exponential Mahler measure* (relative to X) as the functional

$$\begin{aligned} M \colon L^{>0}(X) &\to \mathbb{R}_{\geq 0} \\ f &\mapsto \lim_{p \to 0} \|f\|_p \end{aligned} \tag{4.5}$$

on the complex vector space

$$L^{>0}(X) := \varinjlim_{p \to 0^+} L^p(X) = \bigcup_{p > 0} L^p(X) \subseteq L^0(X)$$

given by those measurable functions $f \colon X \to \mathbb{C}$ such that $\|f\|_p < +\infty$ for some $p > 0$. It is easy to see that

$$M(f) = \exp\left(\lim_{p \to 0^+} \frac{1}{p} \log\left(\int_X |f|^p \, d\mu_X \right) \right) = \exp\left(\int_X \log|f| \, d\mu_X \right)$$

where we set $\exp(-\infty) := 0$. This shows that the functional M defined in (4.5) coincides with the exponential Mahler measure defined in Definition 4.1.1, if we take X to be the probability space $\mathbb{G}^n_{m,\mathbb{C}}$ with the unique Haar probability measure $\mu_{\mathbb{T}^n}$. Hence it makes sense to define the functional $m\colon L^{>0}(X) \to \mathbb{R} \cup \{-\infty\}$ by setting

$$m(f) := \int_X \log|f|\, d\mu_X$$

so that $M(f) = \exp(m(f))$. As we did in Definition 4.1.1, we define the plus-Mahler measure $m^+\colon L^{>0}(X) \to \mathbb{R}_{\geq 0}$ as

$$m^+(f) := \int_X \log^+|f|\, d\mu_X$$

and we denote by $M^+\colon L^{>0}(X) \to \mathbb{R}_{\geq 1}$ the exponential analogue $M^+(f) := \exp(m^+(f))$.

Remark 4.1.6. There are various other generalisations of the Mahler measure, which are not studied in this thesis. Some of them are in fact particular instances of the general framework developed in Remark 4.1.5, such as:

- the *Mahler measure for arbitrari tori* studied by Lalín and Mittal in [LM18], which is obtained by taking $X = \mathbb{T}_{\mathbf{a}}$ for some $\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{R}_{>0})^n$, where

  $$\mathbb{T}_{\mathbf{a}} := \{\mathbf{z} \in (\mathbb{C}^\times)^n \mid |z_1| = a_1, \dots, |z_n| = a_n\}$$

  is the real analytic n-torus associated to $\mathbf{a}$;
- the *Mahler measure for compact abelian groups* studied by Lind in [Lin05], who takes $X = G$ to be a compact abelian group (with the unique probability Haar measure), and considers only functions $f \in \mathbb{Z}[\widehat{G}]$, where $\widehat{G} := \mathrm{Hom}(G, \mathbb{T}^1)$ is the Pontryagin dual of G;
- the *elliptic Mahler measure* introduced by Everest and Fhlathúin in [EF96] (see also [EW99, § 6.3] for the definition in several variables). This elliptic Mahler measure is essentially defined by taking $X = E(\mathbb{C})^n$ for some elliptic curve E, endowed with the unique probability Haar measure.

Let us mention also some other generalisations of the notion of Mahler measure, such as:

- the *higher Mahler measure* introduced by Kurokawa, Lalín and Ochiai in [KLO08]. Using the general framework outlined in Remark 4.1.5, this can be defined for every probability space X and every $k \in \mathbb{N}$ as the functional

  $$\begin{aligned} m_k\colon L^{>0}(X)^k &\to \mathbb{R} \cup \{-\infty\} \\ (f_1, \dots, f_k) &\mapsto \int_{X^k} \log|f_1| \cdots \log|f_k|\, d\mu_{X^k} \end{aligned}$$

  where μ_{X^k} denotes the product probability measure;
- the *metric Mahler measure* of Dubickas and Smyth [DS01] (see also [Sam14] for a generalisation), which satisfies the triangle inequality and hence is more amenable to topological considerations;
- the *q-Mahler measure* and the *crystal Mahler measure* introduced by Kurokawa in [Kur04].

Remark 4.1.7. As it is mentioned in the preface of [BZ20], the Mahler measure has connections with "practically every other part of mathematics". Let us mention some of these connections, which do not otherwise appear in this thesis:

- Mahler measures can arise as entropies of dynamical systems. More precisely, for every $n \in \mathbb{N}$ and every Laurent polynomial $P \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ we let $\mathcal{O}_P$ denote the sheaf on $\mathbb{G}_{m,\mathbb{Z}}^n$ corresponding to the zero locus of P, *i.e.* to the quotient $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/(P)$ seen as a module over $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then we have an action of $\mathbb{Z}^n$ on $H^0(\mathbb{G}_{m,\mathbb{Z}}^n, \mathcal{O}_P)$, induced by the action $\rho\colon \mathbb{Z}^n \to \mathrm{Aut}(\mathbb{G}_{m,\mathbb{Z}}^n)$ given by $\rho(\mathbf{a})(x_1, \dots, x_n) := (x_1^{a_1}, \dots, x_n^{a_n})$. Now, if we give to $H^0(\mathbb{G}_{m,\mathbb{Z}}^n, \mathcal{O}_P)$ the discrete topology we see that the Pontryagin dual $H^0(\mathbb{G}_{m,\mathbb{Z}}^n, \mathcal{O}_P)^\wedge := \mathrm{Hom}(H^0(\mathbb{G}_{m,\mathbb{Z}}^n, \mathcal{O}_P), \mathbb{T}^1)$ is a compact abelian group endowed with an action of $\mathbb{Z}^n$. Then a theorem of Lind, Schmidt and Ward says that the entropy of this action is precisely given by the Mahler measure $m(P)$ (see [Sch95, Theorem 18.1]);
- the previous connections between Mahler measures and entropies have been extensively explored by Deninger. In particular, Mahler measures and more general heights have been related to the entropy of certain actions of suitable amenable groups, and to Fuglede-Kadison determinants of certain operators on the algebras associated to the action (see [Den12] for a survey). This work of Deninger is part of the bigger program which aims to relate arithmetic schemes $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ to dynamical systems, in a way which would respect the ζ-functions definable on both sides. Under such a program, the relations between regulators and entropies outlined in [Den12] would become relations between the values of the corresponding ζ-functions, assuming the validity of the conjectures described in Section 3.3.2;
- the work of Breuillard gives another relation between the Mahler measure and the theory of amenable groups, with particular emphasis on the problem of Lehmer (see Question 4.1.14). We refer the interested reader to the survey [Bre14];
- finally, the work of Borwein, Straub, Wan and Zudilin (see [Bor+12] and [SZ20]) shows that the Mahler measures $\mu(n) := m(x_1 + \cdots + x_n)$ can be computed as the derivative in $s = 0$ of the moment function $W_n(s)$ associated to a uniform random walk of n steps. This allows one to compute explicit hypergeometric formulas for the values $\mu(n)$, which are much more amenable to computation.

The first remarkable property of the Mahler measure is a consequence of Jensen's formula (see [SS03, § 5.1] and [BZ20, Proposition 1.4]), which allows one to reduce the number of variables in the integral computing $m(P)$.

Proposition 4.1.8 – Jensen's formula

For every $n \in \mathbb{N}$ let $\varphi_n\colon L^{>0}(\mathbb{T}^n) \to L^{>0}(\mathbb{T}^{n+1})$ be the functional sending $f\colon \mathbb{T}^n \to \mathbb{C}$ to the function $\varphi_n(f)\colon \mathbb{T}^{n+1} \to \mathbb{C}$ given by $\varphi_n(f)(x_1, \dots, x_{n+1}) := x_{n+1} - f(x_1, \dots, x_n)$. Then we have that $m \circ \varphi_n = m^+$, *i.e.*

$$m(x_{n+1} - f(x_1, \dots, x_n)) = m^+(f(x_1, \dots, x_n))$$

for every $f \in L^{>0}(\mathbb{T}^n)$.

Let us show how to use Proposition 4.1.8 to reduce the computation of the Mahler measure $m(P)$ of a polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, t^{\pm 1}]$ in $n+1$ variables to the computation of some n-variable integrals. First of all, it is clear from Definition 4.1.1 that $m(P \cdot Q) = m(P) + m(Q)$ for every pair of Laurent polynomials $P, Q \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Hence one can factor each Laurent polynomial $P \in \mathbb{C}[\mathbf{x}, t]$ as $P = \alpha_0(\mathbf{x}) \cdot (t - \alpha_1(\mathbf{x})) \cdots (t - \alpha_d(\mathbf{x}))$, where $\alpha_0(\mathbf{x}) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a Laurent polynomial and $\alpha_1, \cdots, \alpha_d$ are algebraic functions in the variables $\mathbf{x} = (x_1, \dots, x_n)$. Now, applying Proposition 4.1.8 to this factorisation we see that

$$m(P) = m(\alpha_0) + \sum_{j=1}^{d} m(t - \alpha_j(\mathbf{x})) = m(\alpha_0) + \sum_{j=1}^{d} m^+(\alpha_j) \tag{4.6}$$

which allows one to reduce the computation of the Mahler measure of a polynomial P in $n+1$ variables to the computation of the plus-Mahler measures of algebraic functions in n variables.

Example 4.1.9. Applying the factorisation (4.6) to a polynomial $P \in \mathbb{C}[t]$ in one variable, we get the formula

$$m(P) = \log|a_0| + \sum_{\substack{\alpha \in \mathbb{C} \\ P(\alpha)=0}} \log^+|\alpha| \tag{4.7}$$

where $a_0 \in \mathbb{C}^\times$ denotes the leading coefficient of P. We observe that (4.7) is the logarithmic analogue of Lehmer's formula (4.1). In particular, (4.7) shows that for every algebraic number $\alpha \in \overline{\mathbb{Q}}^\times$ we have the following relation between the Mahler measure and the absolute logarithmic Weil height $h(\alpha) \in \mathbb{R}$ (see Section 1.2.1):

$$m(f_\alpha(t)) = \deg(\alpha) \cdot h(\alpha)$$

where $f_\alpha(t) = a_d t^d + \cdots + a_0 \in \mathbb{Z}[t]$ is the integral minimal polynomial of α, defined as the unique irreducible integral polynomial satisfying the two conditions $\gcd(a_0, \dots, a_d) = 1$ and $a_d > 0$, such that $f_\alpha(\alpha) = 0$.

4.1.1 Small values of the Mahler measure

Let $P \in \mathbb{C}[t]$ be a monic polynomial. The identity (4.1) shows immediately that

$$\lim_{n \to +\infty} \left| \frac{\Delta_{n+1}(P)}{\Delta_n(P)} \right| = M(P) \tag{4.8}$$

where $\{\Delta_n(P)\}_{n \in \mathbb{N}} \subseteq \mathbb{C}$ is the sequence of complex numbers defined by the formula

$$\Delta_n(P) := \prod_{\substack{\alpha \in \mathbb{C} \\ P(\alpha)=0}} \alpha^n - 1 \in \mathbb{C}.$$

When $P \in \mathbb{Z}[t]$ is a monic polynomial with integer coefficients, Galois theory shows that $\Delta_n(P) \in \mathbb{Z}$ for every $n \in \mathbb{N}$, and one has that

$$\frac{\Delta_{nm}(P)}{\Delta_m(P)} = \prod_{\substack{\alpha \in \mathbb{C} \\ P(\alpha)=0}} \left(\sum_{j=0}^{n-1} \alpha^{mj} \right) \in \mathbb{Z}$$

for every $n, m \in \mathbb{N}$. These numbers are sometimes called *cyclotomic integers*, because they have a factorisation which resembles the one of cyclotomic polynomials. Pierce and Lehmer used this sequence of integers to find new large prime numbers of the form $\Delta_p(P)/\Delta_1(P)$, where $p \in \mathbb{N}$ is a small rational prime. Two famous examples of this include the prime numbers

$$\begin{aligned} \Delta_{113}(P_1) &= 63088004325217 \\ \Delta_{127}(P_1) &= 3233514251032733 \end{aligned} \tag{4.9}$$

associated to the polynomial $P_1(t) \in \mathbb{Z}[t]$ defined as $P_1(t) := t^3 - t - 1$. Note that these numbers can indeed be primes because $\Delta_1(P_1) = 1$. We also invite the reader to observe that, in the beginning of the twentieth century, the largest prime number known was the Mersenne prime

$$\Delta_{127}(t-2) = 170141183460469231731687303715884105727$$

discovered by Lucas in 1876 (see [Luc76]). Thus the number of digits of the primes discovered by Lehmer and Pierce was not incredibly distant from the world record at the time, and the two numbers $\Delta_{113}(P_1)$ and $\Delta_{127}(P_1)$ had the advantage of not being Mersenne primes.

Lehmer observed in [Leh33] that the prime counting function

$$\pi(Q; x) := |\{p \leq x \mid \Delta_p(Q)/\Delta_1(Q) \text{ is prime}\}| \tag{4.10}$$

seems to grow faster as soon as the Mahler measure $m(Q)$ is small. This has one notable exception, given by the following proposition (see [BZ20, Exercise 1.7]).

Proposition 4.1.10 – Cyclotomic integers and reciprocal polynomials

Let $P \in \mathbb{Z}[t]$ be a non-zero, monic polynomial and suppose that $P(t) = t^d P(1/t)$ where $d = \deg(P)$. Then for every $n, m \in \mathbb{N}$ such that $m \mid n$ the ratio $|\Delta_n(P)/\Delta_m(P)| \in \mathbb{Z}$ is a square if $n \equiv m\ (2)$.

Proof. It is clearly sufficient to prove that $|\Delta_{2n}(P)/\Delta_2(P)|$ and $|\Delta_{2n+1}(P)/\Delta_1(P)|$ are squares for every $n \in \mathbb{N}$. We can also assume that P is irreducible, since clearly $\Delta_n(Q_1 \cdot Q_2) = \Delta_n(Q_1) \cdot \Delta_n(Q_2)$ for every pair of polynomials $Q_1, Q_2 \in \mathbb{Z}[t]$. If P is irreducible, it is not difficult to see that d is even, because $P(t) = t^d P(1/t)$. Moreover, we can order the roots $\alpha_1, \ldots, \alpha_d$ of P in such a way that $\alpha_{2i} = \alpha_{2i-1}^{-1}$ for every $i \geq 1$. Doing so, we see that

$$\frac{\Delta_{2n}(P)}{\Delta_2(P)} = \prod_{i=1}^{d/2} \frac{(\alpha_{2i-1}^{2n} - 1) \cdot (\alpha_{2i}^{2n} - 1)}{(\alpha_{2i-1}^{2} - 1) \cdot (\alpha_{2i}^{2} - 1)} = \prod_{i=1}^{d/2} \left(\alpha_{2i-1}^{n-1} \sum_{j=0}^{n-1} \alpha_{2i}^{2j} \right)^2 \tag{4.11}$$

$$\frac{\Delta_{2n+1}(P)}{\Delta_1(P)} = \prod_{i=1}^{d/2} \left(\frac{\alpha_{2i-1}^{2n+1} - 1}{\alpha_{2i-1} - 1} \right) \cdot \left(\frac{\alpha_{2i}^{2n+1} - 1}{\alpha_{2i} - 1} \right) = \prod_{i=1}^{d/2} \left(1 + \sum_{j=1}^{n} \alpha_{2i-1}^{j} + \alpha_{2i}^{j} \right)^2 \tag{4.12}$$

which allows us to conclude, because the expressions (4.11) and (4.12) are clearly rational integers thanks to Galois theory. □

The class of polynomials appearing in Proposition 4.1.10 deserves a name of its own.

Definition 4.1.11 – Reciprocal polynomial

Let $n \in \mathbb{N}$. We define two involutions $P \mapsto P^*$ and $P \mapsto P^\dagger$ on the ring $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ of Laurent polynomials, by setting

$$P^*(x_1, \dots, x_n) := P(x_1^{-1}, \dots, x_n^{-1})$$
$$P^\dagger(x_1, \dots, x_n) := \overline{P(\overline{x_1}^{-1}, \dots, \overline{x_n}^{-1})}$$

where $z \mapsto \overline{z}$ denotes complex conjugation. We say that a given Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is *self-reciprocal* (respectively *conjugate self-reciprocal*) if P/P^* (resp. $P/P^\dagger$) is a monomial.

Remark 4.1.12. Sometimes the polynomial P^* is called the *reciprocal polynomial* of P. However, if $P \in \mathbb{C}[t]$ one usually calls by this name the polynomial $t^d \cdot P^*(t)$, where d denotes the degree of P.

Hence, going back to the counting function $\pi(Q; x)$ defined in (4.10), we see that $\pi(Q; x) = 0$ for every $x \in \mathbb{R}$ as soon as Q is self-reciprocal. On the other hand, one can observe experimentally that $\pi(t^3 - t - 1; x)$ seems to grow faster than $\pi(t - 2; x)$, and this reflects Lehmer's prediction because $m(t^3 - t - 1) = 0.281199\cdots < 0.693147\cdots = m(t - 2)$. In fact, the choice of the polynomial $t^3 - t - 1$ is not a coincidence. This is shown by the following theorem, proved in the PhD thesis of Smyth (see [Smy71] and [BZ20, Theorem 2.1]).

Theorem 4.1.13 – Bogomolov's property for non-reciprocal polynomials

The Mahler measure function $P \mapsto m(P)$ has the Bogomolov property, in the sense of Definition 1.1.7, when restricted to the set of non-zero integral polynomials $P \in \mathbb{Z}[t]$ such that $P(t) \neq t^{\deg(P)} P(1/t)$. More precisely, for every non-zero, irreducible polynomial $P \in \mathbb{Z}[t]$ we have that

$$m(P) < m(P_1) \Rightarrow P(t) = t^{\deg(P)} P(1/t)$$

where $P_1(t) := t^3 - t - 1$.

Thus, combining Proposition 4.1.10 with Theorem 4.1.13 and Lehmer's prediction about the functions $\pi(Q; x)$ defined in (4.8), one sees that the most efficient way to obtain primes of the form $\Delta_p(Q)/\Delta_1(Q)$ is to consider the sequence $\Delta_p(P_1)$, which was already studied by Lehmer as we have seen in (4.9). However, Proposition 4.1.10 suggests that it might be interesting to consider the sequence of integers $\sqrt{|\Delta_p(Q)/\Delta_1(Q)|}$ associated to a self-reciprocal polynomial $Q \in \mathbb{Z}[t]$. As Lehmer himself points out at the end of his paper [Leh33], the amateur Belgian mathematician Poulet was able to compute that

$$\sqrt{|\Delta_{379}(P_0)|} \stackrel{?}{=} 1794327140357 \tag{4.13}$$

is a prime number, where $P_0 \in \mathbb{Z}[t]$ denotes the self-reciprocal polynomial

$$P_0(t) = t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1$$

which is known today as Lehmer's polynomial. We note, thanks to an observation of Burgos Gil, that the equality (4.13) cited by Lehmer appears to be wrong. Indeed, the correct value (which nowadays can be computed easily using any computer algebra system) is

$$\sqrt{|\Delta_{379}(P_0)|} = 37098890596487$$

which is also a prime number.

It might look surprising that Poulet was able to compute the sequence $\sqrt{\Delta_p(P_0)}$ up to the prime index $p = 379$ (even if his computation turned out to be wrong). However, we notice that this sequence grows quite slowly, because $\sqrt{M(P_0)} = 1.084564\cdots$ is far less than the Mahler measure $M(P_1) = 1.324717\cdots$. In fact, we also have that $m(P_0) = 0.162357\cdots < 0.281199\cdots = m(P_1)$, and Lehmer pointed out that $m(P_0)$ was the smallest Mahler measure that he could find, after extensive research. Hence, he proposed the following question (see [Leh33, § 13]).

Question 4.1.14 – Lehmer's problem

Weak Lehmer problem Is it true that the Mahler measure function $m\colon \mathbb{Z}[t] \setminus \{0\} \to \mathbb{R}$ has the Bogomolov property, in the sense of Definition 1.1.7?

Lehmer's problem Is it true that

$$m(P) \geq m(P_0) = m(t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1)$$

for every $P \in \mathbb{Z}[t] \setminus \{0\}$ such that $m(P) \neq 0$?

It might not be clear immediately why a positive answer to the second question implies a positive answer to the first question. This is indeed the case, because the function $m\colon \mathbb{Z}[t] \setminus \{0\} \to \mathbb{R}$ satisfies a weak Bogomolov property, in the sense of Definition 1.1.7. Indeed, using (4.7) we see that $m(P) \geq 0$ for every $P \in \mathbb{Z}[t]$. In fact, this inequality extends to several variables by induction, because Jensen's formula (4.6) shows that $m(P) \geq m(a_0)$ for every $P \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, where $a_0 \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$ is the leading coefficient of P in the variable x_n.

The following theorem, which is due independently to Lawton (see [Law77]), Boyd (see [Boy81a]) and Smyth (see [Smy81]), shows that we can completely characterise the set of polynomials with integer coefficients that achieve the minimal Mahler measure.

Theorem 4.1.15 – Kronecker's theorem

Let $m\colon \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \to \mathbb{R} \cup \{-\infty\}$ denote the Mahler measure (see Definition 4.1.1). Then $m(P) = -\infty$ if and only if $P = 0$ and $m(P) \geq 0$ for every $P \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \setminus \{0\}$. Moreover, for every $P \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ we have that $m(P) = 0$ if and only if there exists $N \in \mathbb{N}$ and a finite sequence of n-tuples $\{\mathbf{a}_j = (a_{j,1}, \ldots, a_{j,n})\}_{j=1}^N \subseteq \mathbb{Z}^n$ such that

$$P = \prod_{j=0}^{N} \Phi_j (x_1^{a_{j,1}} \cdots x_n^{a_{j,n}})^{b_j}$$

where $\Phi_0(t) := t \in \mathbb{Z}[t]$ and for every $j \geq 1$ we denote by $\Phi_j(t) \in \mathbb{Z}[t]$ the j-th cyclotomic polynomial.

Let us conclude this section by mentioning some partial steps towards a positive solution to Lehmer's problem, in addition to the aforementioned Theorem 4.1.13:

- it is known that every irreducible polynomial $P \in \mathbb{Z}[t]$ of degree $d := \deg(P) \geq 2$ satisfies

$$m(P) > \frac{1}{4}\left(\frac{\log\log(d)}{\log(d)}\right)^3 \tag{4.14}$$

  which was proved by Voutier (see [Vou96]), following work of Dobrowolski (see [Dob79]). The inequality (4.14) was already implicitly mentioned in Section 1.2.1, to show that the pair (h, α) consisting of the absolute logarithmic Weil height $h\colon \overline{\mathbb{Q}}^{\times} \to \mathbb{R}$ and the function (1.3) satisfies Lehmer's property;

- Lehmer's problem is known for the families of polynomials

$$\mathcal{D}_m := \left\{ \sum_{j=0}^{d} a_j x^j \in \mathbb{Z}[t] \;\middle|\; a_j \equiv 1(m),\ \forall 0 \leq j \leq m \right\}$$

  as was shown by Borwein, Dobrowolski and Mossinghoff in [BDM07];

- the Schinzel-Zassenhaus conjecture, which can be regarded as a strengthening of Dobrowolski's bound, is now known thanks to the recent work of Dimitrov (see [Dim19]). In particular, it is now known that for every monic, irreducible polynomial $P \in \mathbb{Z}[t]$ of degree $d := \deg(P) > 1$ we have that

$$\max\{|\alpha| \,:\, \alpha \in \mathbb{C},\ P(\alpha) = 0\} \geq 2^{1/4d}$$

  which shows that also the single terms appearing in the formula (4.1) cannot be too small.

4.1.2 Limits of Mahler measures

As we have seen in the previous section, Lehmer's Question 4.1.14 remains tantalisingly unanswered to this day. However, the following conjecture of Boyd provides a very interesting strategy to attack the weak version of Lehmer's problem.

Conjecture 4.1.16 – Boyd's conjecture on Lehmer's problem

For every $n \in \mathbb{N}$, define $S_n := m(\mathbb{Z}[x_1, \ldots, x_n] \setminus \{0\}) \subseteq \mathbb{R}_{\geq 0}$ to be the set of Mahler measures of non-zero integral polynomials in n-variables. Then the set

$$S_\infty := \varinjlim_{n \to +\infty} S_n \subseteq \mathbb{R}_{\geq 0} \tag{4.15}$$

is closed. Here the direct limit denotes a nested union, because $S_n \subseteq S_{n+1}$ for every $n \in \mathbb{N}$.

Remark 4.1.17. A fundamental partial result towards a complete proof of Conjecture 4.1.16 has been recently shown by Smyth in [Smy18]. More precisely, for every matrix $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$ and every Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ we define

$$P_A(x_1, \dots, x_m) := P(x_1^{a_{1,1}} \cdots x_m^{a_{m,1}}, \dots, x_1^{a_{1,n}} \cdots x_m^{a_{m,n}}) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_m^{\pm 1}]$$

and we set $\mathcal{M}(P) := \varinjlim_m \mathcal{M}_m(P)$ where $\mathcal{M}_m(P) := \{m(P_A) \mid A \in \mathrm{Mat}_{m\times n}(\mathbb{Z})\} \setminus \{-\infty\}$. Then Smyth proves that $\mathcal{M}(P) \subseteq \mathbb{R}$ is closed, for every Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and that if $0 \in \mathcal{M}(P)$ and $P \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \mathbb{Z}$ then 0 is an isolated point of $\mathcal{M}(P)$. In other words, the Mahler measure function $m\colon \mathbb{Z}[x_1^{\pm 1}, \dots] \setminus \{0\} \to \mathbb{R}_{\geq 0}$ satisfies the Bogomolov property (in the sense of Definition 1.1.7) when restricted to each set of the form

$$\mathcal{P}(P) := \varinjlim_{m\in\mathbb{Z}_{\geq 1}} \{P_A \mid A \in \mathrm{Mat}_{m\times n}(\mathbb{Z})\} \setminus \{0\}$$

such that $0 \in \mathcal{P}(P)$, where $P \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\}$ is a fixed polynomial. Finally, Smyth shows that the set S_∞ appearing in (4.15) can be written as the nested union

$$S_\infty := \varinjlim_{n\to+\infty} \mathcal{M}(F^{(n)})$$

where $F^{(n)} \in \mathbb{Z}[x_1, \dots, x_{2n}]$ denotes the polynomial $F^{(n)} := \sum_{j=1}^{2n} (-1)^j x_j$.

In order to prove that Conjecture 4.1.16 implies indeed a positive answer to the weak Lehmer problem (as stated in Question 4.1.14) we need the following theorem, which is originally due to Lawton (see [Law83]), and has been revisited recently by Dimitrov and Habegger in [DH19, Appendix A].

Theorem 4.1.18 – Multivariate Mahler measures as limits of univariate Mahler measures

Let $P \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\}$ and let $k := |\{\mathbf{j} \in \mathbb{Z}^n \mid a_{\mathbf{j}}(P) \neq 0\}|$, where $a_{\mathbf{j}}(P) \in \mathbb{Z}$ denotes the $\mathbf{j}$-th coefficient of $P(\mathbf{x}) = \sum_{\mathbf{j}} a_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}$ written in multi-index notation. Suppose that $k \geq 2$, and fix $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\rho(\mathbf{a}) := \inf \left\{ \|\mathbf{b}\|_\infty \;\middle|\; \mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}^n \setminus \{0\}, \sum_{j=1}^{n} a_j b_j = 0 \right\} > \deg(P)$$

where $\|\mathbf{b}\|_\infty := \max_{j=1}^n |b_j|$. Then we have that

$$m(P(t^{a_1}, \dots, t^{a_n})) - m(P) \leq C(n,k) \cdot \left(\frac{\deg(P)^{16n^2}}{\rho(\mathbf{a})^{1/16(k-1)}} \right)$$

for some function $C\colon \mathbb{N}^2 \to \mathbb{R}$. In particular, we have that

$$\lim_{\rho(\mathbf{a})\to+\infty} m(P(t^{a_1}, \dots, t^{a_n})) = m(P)$$

which shows that the Mahler measure of any non-zero polynomial $P \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ can be approximated by the Mahler measures of univariate, integral polynomials.

We do not give the details of the proof of Theorem 4.1.18, which are already greatly exposed in [DH19, Appendix A]. Instead, let us use Theorem 4.1.18 to show that Conjecture 4.1.16 gives a positive answer to the weak Lehmer problem. Indeed, Theorem 4.1.18 shows that $S_\infty \subseteq \overline{S_1}$, and if Conjecture 4.1.16 was true we would have that $\overline{S_1} = S_\infty$. On the other hand, the following lemma shows that if the weak Lehmer problem has a negative answer we have that $\overline{S_1} = \mathbb{R}_{\geq 0}$, which would lead to a contradiction because the set S_∞ is countable.

Lemma 4.1.19 – Closures of semi-groups

Let $S \subseteq \mathbb{R}_{\geq 0}$ be a subset such that $n \cdot S \subseteq S$ for every $n \in \mathbb{Z}_{\geq 1}$. If $0 \in S^{(1)}$ is a limit point of S we have that $\overline{S} = [0, +\infty)$

Proof. For every $\alpha \in \mathbb{R}$, let $\{\alpha\} := \alpha - \lfloor \alpha \rfloor \in [0, 1)$ denote the fractional part of α. Then it is immediate to see that

$$\lim_{n \to +\infty} \left\{ \frac{\alpha}{\varepsilon_n} \right\} \varepsilon_n = 0$$

for every sequence $\{\varepsilon_n\} \subseteq \mathbb{R}$ such that $\varepsilon_n \to 0$ as $n \to +\infty$. This implies that

$$\lim_{n \to +\infty} \left\lfloor \frac{\alpha}{\varepsilon_n} \right\rfloor \varepsilon_n = \alpha \tag{4.16}$$

for every $\alpha \in \mathbb{R}$. Now, by assumption there exists a sequence $\{\varepsilon_n\}_{n \in \mathbb{N}} \subseteq S$ such that $\varepsilon_n \to 0$ as $n \to +\infty$, because $0 \in S^{(1)}$. Moreover, since $N \cdot S \subseteq S$ for every integer $N \geq 1$, we have that $\lfloor \alpha / \varepsilon_n \rfloor \cdot \varepsilon_n \in S$ for every $\alpha \in \mathbb{R}_{>0}$. Hence (4.16) implies that $\alpha \in S^{(1)}$ for every $\alpha \in \mathbb{R}_{\geq 0}$. This shows that $S^{(1)} = \mathbb{R}_{\geq 0}$, and we can conclude by recalling that $S^{(1)} \subseteq \overline{S_1} \subseteq \mathbb{R}_{\geq 0}$. □

To conclude this section, we remark that Conjecture 4.1.16 shows that, even to understand the Diophantine properties of the Mahler measure of polynomials in one variable, one is naturally led to study Mahler measures of polynomials in multiple variables.

4.2 Mahler measures and special values of L-functions: an historical introduction

The aim of this section is to briefly recall the history of the conjectural links between the Mahler measure and special values of L-functions. The first example of this relation occurs already in one variable (see [BZ20, § 1.4]).

Example 4.2.1. Let $P_k^{\pm}(t) := t^2 + kt \pm 1 \in \mathbb{Z}[t]$, for $k \in \mathbb{N}$ such that $k \geq 3$. Then, using Lehmer's formula (4.7) we see that

$$m(P_k^{\pm}(t)) = \log \left(\frac{k + \sqrt{k^2 \pm 4}}{2} \right)$$

and combining this with the analytic class number formula (see Theorem 3.3.26) we see that

$$\frac{\zeta^*_{\mathbb{Q}(\varepsilon_k^{\pm})}(0)}{m(P_k^{\pm}(t))} = -\frac{|\mathrm{Pic}(\mathbb{Z}[\varepsilon_k])|}{2|\mathbb{Z}[\varepsilon_k^{\pm}]^{\times} : (\varepsilon_k^{\pm})^{\mathbb{Z}}|} \in \mathbb{Q}^{\times}$$

where $\varepsilon_k^{\pm} := (k + \sqrt{k^2 \pm 4})/2$.

Remark 4.2.2. It would be interesting to generalise Example 4.2.1 to number fields of higher degree. The first natural number fields F to study are those for which $\mathrm{rk}(O_F^{\times}) = 1$. Apart from real quadratic fields, which are the subject of Example 4.2.1, these include cubic fields with one real and one complex place, and totally imaginary quartic fields. In the first case one can use [SS73, Theorem 1.1] (see also [BZ20, Page 11]) to prove that for every $k \in \mathbb{Z}_{\geq 1}$ one has

$$\frac{\zeta^*_{F_k}(0)}{m(x^3 - kx^2 - 1)} = -\frac{|\mathrm{Pic}(O_{F_k})|}{2|O_{F_k}^{\times} : (\alpha_k)^{\mathbb{Z}}|} \in \mathbb{Q}^{\times}$$

where $\alpha_k \in \mathbb{R}_{>1}$ is the unique real root of the polynomial $Q_k(t) := t^3 - kt^2 - 1 \in \mathbb{Z}[t]$, and $F_k := \mathbb{Q}[t]/(Q_k)$. For quartic fields, one can use for example the explicit family provided by [BW19, Theorem 1.1] to prove similar kinds of identities.

Remark 4.2.3. Let us mention that it would be very appealing to prove relations of the form

$$\frac{\zeta^*_F(0)}{m(\mathbf{P})} \in \mathbb{Q}^{\times} \tag{4.17}$$

for number fields F having unit rank $r := \mathrm{rk}(O_K^{\times}) > 1$. Here $\mathbf{P} = (P_{i,j}) \in \mathrm{Mat}_{r\times r}(\mathbb{Z}[t])$ would be a matrix of polynomials $P_{i,j} \in \mathbb{Z}[t]$ such that $F \cong \mathbb{Q}[t]/(P_{i,j})$ for every $i, j \in \{1, \ldots, r\}$, and $m(\mathbf{P}) := |\det(m(P_{i,j}))| \in \mathbb{R}_{>0}$ would denote the absolute value of the determinant of the matrix whose entries are given by the Mahler measures of the polynomials appearing in $\mathbf{P}$. We believe that proving a formula like (4.17) should be possible using Minkowski's theorem on units (see [AV17]), at least for number fields F which are Galois over $\mathbb{Q}$. Indeed, this theorem allows one to express the matrix computing the unit regulator of F as a circulant matrix (see [Dav79]), and this makes it easier to relate the regulator determinant to a determinant of Mahler measures. This will be the subject of future investigations.

Historically speaking, the next two examples of identities between Mahler measures and special values of L-functions were proved by Smyth in [Smy81]. As it is stated in the introduction of that paper, Smyth was indeed inspired by Boyd's work [Boy81a] (and in particular by Conjecture 4.1.16) to gain a better understanding of Mahler measures of polynomials in multiple variables.

Theorem 4.2.4 – Smyth's computations

Let $\chi_{-3}\colon (\mathbb{Z}/3\mathbb{Z})^\times \to \{\pm 1\}$ be the unique non-trivial Dirichlet character modulo 3, *i.e.* the Dirichlet character associated to the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$. Then we have:

$$m(x_1 + x_2 + 1) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1) \tag{4.18}$$

$$m(x_1 + x_2 + x_3 + 1) = \frac{7}{2\pi^2}\zeta(3) = -14\zeta'(-2) \tag{4.19}$$

where $\zeta(s)$ denotes Riemann's ζ-function.

Proof. We follow [Boy81a, Appendix 1] (see also [BZ20, § 3.3]). First of all, Jensen's formula (4.6) shows that $m(x_1 + x_2 + 1) = m^+(1 + t)$, and we can compute

$$m^+(1+t) = \Re\left(\int_{-1/3}^{1/3} \log(1 + e^{2\pi i t})dt\right) = \Re\left(\int_{-1/3}^{1/3} \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}e^{2\pi n t}}{n} dt\right)$$

which can be combined with the identity

$$\int_{-1/3}^{1/3} e^{2\pi i n t} dt = \frac{1}{n\pi}\sin\left(\frac{2\pi n}{3}\right) = \frac{\sqrt{3}}{2n\pi}\chi_{-3}(n)$$

to conclude that

$$m^+(1+t) = \frac{\sqrt{3}}{2\pi}\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}\chi_{-3}(n)}{n^2} = \frac{\sqrt{3}}{2\pi}\left(\sum_{n=1}^{+\infty}\frac{\chi_{-3}(n)}{n^2} - 2\sum_{n=1}^{+\infty}\frac{\chi_{-3}(2n)}{(2n)^2}\right) = \frac{3\sqrt{3}}{\pi}L(\chi_{-3}, 2)$$

where the last equality follows from the fact that $\chi_{-3}(2n) = \chi_{-3}(2)\chi_{-3}(n) = -\chi_{-3}(n)$.

To prove (4.19), observe that $m(x_1 + x_2 + x_3 + 1) = m(X_1 + X_2(1 + X_3) + 1)$, by the change of variables $(X_1, X_2, X_3) = (x_1, x_2, x_2/x_3)$. Using again Jensen's formula (4.6) one sees that $m(X_1 + X_2(1 + X_3) + 1) = m(\max(|1 + t_1|, |1 + t_2|))$, and we can compute

$$\begin{aligned}
m(\max(|1+t_1|,|1+t_2|)) &= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \max(\log|1+e^{it_1}|, \log|1+e^{it_2}|)dt_1dt_2 = \\
&= \frac{1}{\pi^2}\left(\int_{0\le t_2<t_2\le\pi}\log|1+e^{it_1}|dt_1dt_2 + \int_{0\le t_1<t_2\le\pi}\log|1+e^{it_2}|dt_1dt_2\right) = \\
&= \frac{2}{\pi^2}\int_0^\pi (\pi-\theta)\log|1+e^{i\theta}|d\theta = \\
&= \frac{2}{\pi^2}m(1+t) - \frac{2}{\pi^2}\int_0^\pi \theta\log|1+e^{i\theta}|d\theta = \\
&= -\frac{2}{\pi^2}\int_0^\pi \theta\log|1+e^{i\theta}|d\theta
\end{aligned}$$

where the last equality follows from Theorem 4.1.15. Now we can again use the power series for the logarithm to get

$$\int_0^\pi \theta \log|1+e^{i\theta}|d\theta = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \Re\left(\int_0^\pi \theta e^{in\theta} d\theta\right) = -2\sum_{k=0}^{+\infty} \frac{1}{(2k+1)^3}$$

where the last equality follows from the computation

$$\int_0^\pi te^{\alpha t}dt = \frac{e^{\pi\alpha}}{\alpha} - \int_0^\pi \frac{e^{\alpha t}}{\alpha}dt = \frac{e^{\pi\alpha}(\alpha-1)+1}{\alpha^2}$$

obtained integrating by parts. Hence we can see that

$$m(x_1+x_2+x_3+1) = \frac{4}{\pi^2}\sum_{k=0}^{+\infty}\frac{1}{(2k+1)^3} = \frac{4}{\pi^2}\left(\sum_{n=1}^{+\infty}\frac{1}{n^3} - \sum_{n=1}^{+\infty}\frac{1}{(2n)^3}\right) = \frac{7}{2\pi^2}\zeta(3)$$

which concludes the proof. □

Remark 4.2.5. The identity $m(x_1+x_2+1) = L'(\chi_{-3},-1)$ is only the first example of identities of the form

$$\frac{L'(\chi_F,-1)}{m(P_F)} \in \mathbb{Q}^\times \tag{4.20}$$

where χ_F denotes the quadratic character associated to the imaginary quadratic field F, and $P_F \in \mathbb{Z}[x_1,x_2]$ is a polynomial with integer coefficients. Identities of this kind are known in a finite number of special cases (see Table A.4), and Chinburg proved in [Chi84] that for every imaginary quadratic field F there exist two polynomials $P_F, Q_F \in \mathbb{Z}[x_1,x_2]$ such that

$$\frac{L'(\chi_F,-1)}{m(P_F)-m(Q_F)} \in \mathbb{Q}^\times$$

but in general one has $m(Q_F) \neq 0$ (in Chinburg's construction). Unfortunately Chinburg's proof never appeared in the literature, and we were not able to find an account of his construction elsewhere. Nevertheless, in the same unpublished manuscript [Chi84], Chinburg conjectures that identities of the kind (4.20) should hold for every imaginary quadratic field F. This conjecture was our original inspiration for the work which appears in Chapter 9. Indeed, as we have mentioned in Section 3.3.4, the special values of the L-functions $L(\chi_F,s)$ are amongst the few families of examples for which the conjectures of Beilinson (see Conjecture 3.3.18) and Bloch-Kato (see Conjecture 3.3.20) are completely known. Using in particular the constructions of Huber and Kings (see [HK99] and [HK03]), which are related to modular curves, and the constructions appearing in Chapter 5, we think that it should be possible to tackle new cases of Chinburg's conjecture. This will be the subject of future work. Instead, Chapter 9 focuses on the special values $L^*(E,2)$ associated to elliptic curves E with complex multiplication, for which a weak form of Beilinson's conjecture is also known to hold (see Section 7.4).

Remark 4.2.6. We observe that the polynomials x_1+x_2+1 and $x_1+x_2+x_3+1$ appearing in Theorem 4.2.4 are part of the family $P_n := x_1+\cdots+x_n+1$, whose Mahler measures $m(P_n)$ are sometimes called *linear Mahler measures*. We recall that Rodriguez Villegas used the insight of Maillot, which is investigated in Chapter 5, to conjecture that $m(P_4)$ and $m(P_5)$ should be

related to the special values of certain explicitly defined modular forms of weight three and four, respectively (see [Boy+03, § 8] and [BZ20, § 6.2]).

Continuing with our historical introduction, the next crucial step to be mentioned consists in Boyd's extensive numerical computations, published in [Boy98]. These numerical computations concerned certain explicitly given families of Laurent polynomials $P_k(x,y) \in \mathbb{Z}[k][x^{\pm 1}, y^{\pm 1}]$, whose zero locus $V_{P_k} \subseteq \mathbb{G}^2_{m,\mathbb{Z}}$ is birationally equivalent to an elliptic curve E_k for almost all values of $k \in \mathbb{Z}$. The first example of such a family is given by

$$P_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + k \tag{4.21}$$

whose zero locus is birationally equivalent to the elliptic curve given by the Weierstraß equation

$$E_k \colon y^2 = x^3 + (k^2 - 8)x^2 + 16x \tag{4.22}$$

for every $k \in \mathbb{Z} \setminus \{0, \pm 4\}$. Boyd computed numerically that for every $k \in \mathbb{Z} \setminus \{0, \pm 4\}$ such that $|k| \leq 40$ the ratio

$$\alpha\left(P_k\right) := \frac{L^*(E_k, 0)}{m(P_k)}$$

seemed to be a rational number, and in fact an integer for every $|k| \notin \{2^3, 2^4, 2^5, 3, 5, 12, 15\}$. Here $L^*(E,0) = L'(E,0)$ denotes the special value of the L-function of E at $s = 0$. Note that in this case Conjecture 3.3.4 and Conjecture 3.3.6 hold for the L-function $L(E,s) := L(\underline{H}^1(E), s)$ thanks to the modularity theorem. We recall that $m(P_{-k}(x,y)) = m(P_k(-x,y)) = m(P_k(x,y))$, hence it is sufficient to study only positive values of k. Moreover, the Weierstraß equation (4.22) shows that E_k is defined over $\mathbb{Q}$ as soon as $k^2 \in \mathbb{Q}$, which originated some interest also in the ratios $\alpha(P_{\sqrt{k}})$ for $k \in \mathbb{Z}$. Other families of polynomials that were studied by Boyd are described in Appendix A.1. For now, we content ourselves with remarking that most of these families were *tempered*, in the sense of the following definition.

Definition 4.2.7 – Tempered polynomial

Let $P \in R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial with coefficients in a ring R, and let $V_P \hookrightarrow \mathbb{G}^n_{m,R}$ denote its zero locus inside the split algebraic n-torus $\mathbb{G}^n_{m,R}$ defined over R. Fix an embedding $\iota \colon V_P \hookrightarrow \overline{V_P}$ of V_P inside a proper R-scheme $\overline{V_P}$. Then P is said to be *tempered with respect to* ι if there exists $\eta_P \in H^{n,n}_{\mathcal{M}}(\overline{V_P})$ such that

$$\iota^*(\eta_P) = \{x_1, \dots, x_n\} \in H^{n,n}_{\mathcal{M}}(V_P)$$

where $\{x_1, \dots, x_n\} := \{x_1\} \cup \dots \cup \{x_n\}$ denotes the cup product of all the motivic cohomology classes $\{x_i\} \in H^{1,1}_{\mathcal{M}}(V_P) \cong \mathcal{O}(V_P)^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ induced by the coordinate functions $x_i \colon V_P \hookrightarrow \mathbb{G}^n_{m,R} \twoheadrightarrow \mathbb{G}^1_{m,R}$. Here $H^{\bullet,\bullet}_{\mathcal{M}}$ denotes motivic cohomology with rational coefficients (see Definition 2.3.1).

Remark 4.2.8. Let $F \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$ be an irreducible Laurent polynomial in two variables, and denote by $\Delta_F \subseteq \mathbb{R}^2$ its *Newton polygon*, which is defined to be the convex hull of the set of indices $\mathbf{j} \in \mathbb{Z}^2$ such that $a_{\mathbf{j}}(F) \neq 0$, where $\{a_{\mathbf{j}}(F) \colon \mathbf{j} \in \mathbb{Z}^2\} \subseteq \mathbb{C}$ denote the coefficients of the polynomial $F(x_1, x_2) = \sum_{\mathbf{j}} a_{\mathbf{j}}(F)\mathbf{x}^{\mathbf{j}}$ written in multi-index notation. Fix a smooth, projective

curve X defined over $\mathbb{C}$ such that $\mathbb{C}(X) \cong \mathrm{Frac}(\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]/(F))$. Then [Coo+94, § 3] shows that, supposing that $a_0(F) \neq 0$, one can associate to every side σ of Δ_F a point $P_\sigma \in X(\mathbb{C})$ and a power series $f_\sigma(t) \in \mathbb{C}[\![t]\!]$ such that, if we set

$$x_1(t) := t^{p(\sigma)} \quad \text{and} \quad x_2(t) := t^{q(\sigma)} f_\sigma(t)$$

we obtain a local parametrisation of X around P_σ. Here $p(\sigma) \in \mathbb{Z}$ and $q(\sigma) \in \mathbb{N}$ are such that $(p(\sigma) : q(\sigma)) \in \mathbb{P}^1(\mathbb{Z})$ is the slope of the side σ. Moreover, the same paper [Coo+94] shows that for every point $P \in X(\mathbb{C})$ which is either a zero or a pole of the coordinate functions x_1 and x_2, there exists a face σ of Δ_F such that $P = P_\sigma$. Finally it is known that the tame symbol $\partial_{P_\sigma}(\{x, y\}) \in \mathbb{C}^\times$ (see Equation (2.28)) satisfies

$$\partial_{P_\sigma}(\{x, y\}) \in \langle \pm \mathcal{R}(F_\sigma) \rangle_{\mathbb{Z}} \tag{4.23}$$

where $\mathcal{R}(F_\sigma) \subseteq \mathbb{C}^\times$ denotes the set of roots of the *face polynomial* $F_\sigma(t) := \sum_{j=0}^{d(\sigma)} a_{\sigma(j)}(F) t^j$. Here $\sigma(0), \dots, \sigma(d(\sigma)) \in \mathbb{Z}^2$ denote the points of σ having integral coordinates, ordered by reading the faces of Δ_F counterclockwise.

Now, combining Proposition 2.3.7 with (4.23) and the fact that each zero or pole of the coordinate functions on X can be expressed as a point of the form P_σ, we see that our Laurent polynomial $F \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$ is tempered (with respect to any compactification of V_P) if and only if $m(F_\sigma) = 0$ for every face σ of the Newton polygon Δ_F. This can be regarded already as an interesting example of the relationships intercurrent between Mahler measures and motives. More relations of this kind are investigated in the next section. Moreover, this gives us a practical way to compute whether a polynomial is tempered. We refer the reader to [Rod99, § 8] for a nice introduction to tempered polynomials, and to [DK11, Theorem 3.1] for a generalisation of (4.23) to several variables.

The extensive computations of Boyd lead naturally to wonder for which Laurent polynomials $P \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ one might expect a link between the Mahler measure $m(P)$ and an L-value. This is to this day a very open question. Even the condition of being tempered (see Definition 4.2.7), which appears like a natural one, does not seem to be necessary to get interesting links between Mahler measures and special values of L-functions. This has been the subject of extensive investigation in recent years (see [LSZ16],[LM18], [MS19], [Gia20]), and the results contained in Chapter 9 (see also [Pen20]) give another class of polynomials which are usually not tempered, whose Mahler measure is related to special values of L-functions. Nevertheless, we can give a precise formulation of these types of questions, for polynomials in any number of variables.

Question 4.2.9 – Relations of Boyd type

Let $P \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial such that $m(P) \neq 0$, and denote by $V_P \hookrightarrow \mathbb{G}_{m,\mathbb{Q}}^n$ the zero locus of P. We ask the following two questions:

- suppose that Conjecture 3.3.4 holds for the L-function $L(\underline{H}^{n-1}(V_P), s)$. Under which conditions on the polynomial P is it true that the ratio

$$\frac{L^*(\underline{H}^{n-1}(V_P), 0)}{m(P)} \in \mathbb{R}^\times$$

is a rational number, or even an integer?

- fix an open embedding $\iota\colon V_P \hookrightarrow \overline{V_P}$ of V_P inside a proper $\mathbb{Q}$-scheme $\overline{V_P}$, and a desingularisation $\pi\colon \widetilde{V_P} \twoheadrightarrow \overline{V_P}$. Under which conditions on the triple (P, ι, π) is it true that the ratio
$$\frac{L^*(\underline{H}^{n-1}(\widetilde{V_P}), 0)}{m(P)} \in \mathbb{R}^\times$$
is a rational number, or even an integer?

More generally, one can ask the same questions replacing the motives $\underline{H}^{n-1}(V_P)$ and $\underline{H}^{n-1}(\widetilde{V_P})$ with suitable sub-motives.

We remark that Question 4.2.9 starts from a polynomial P and asks whether or not its Mahler measure is linked to the special value of some L-function related to P. The inverse approach leads to the following question.

Question 4.2.10 – Inverse problems of Boyd type

Let $M \in \mathcal{MM}(\mathbb{Q}; \mathbb{Q})$ be a mixed motive over $\mathbb{Q}$. Under which conditions does there exist a polynomial $P \in \mathbb{Q}[x_1, \dots, x_n]$ such that $m(P) \in \mathbb{R}^\times$ and the quotient
$$\frac{L^*(M, 0)}{m(P)} \in \mathbb{R}^\times$$
is rational? Moreover, does there exist such a polynomial with the property that the motive M can be identified with a sub-motive of $\underline{H}^{n-1}(V_P)$ or $\underline{H}^{n-1}(\widetilde{V_P})$, where $V_P \hookrightarrow \mathbb{G}_m^n$ is the zero locus of P and $\widetilde{V_P}$ is some desingularisation of some compactification of V_P?

Finally, one can ask if the identities appearing in Question 4.2.9 can be "deformed" in a suitable way. For example, one can ask whether these identities are sensitive to *twists*, *i.e.* isomorphisms up to a finite extension. This is made precise in the following question.

Question 4.2.11 – Twisting identities of Boyd type

Let $P \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial which answers affirmatively Question 4.2.9. In particular, let us assume that $m(P) \in \mathbb{R}^\times$ and
$$\frac{L^*(\underline{H}^{n-1}(\widetilde{V_P}), 0)}{m(P)} \in \mathbb{Q}^\times$$
where $\widetilde{V_P}$ denotes some desingularisation of some compactification of the zero locus $V_P \hookrightarrow \mathbb{G}_m^n$. Let $M \in \mathcal{MM}(\mathbb{Q}; \mathbb{Q})$ be a motive which is a twist of $\underline{H}^{n-1}(\widetilde{V_P})$, *i.e.* such that $M_{/F} \cong \underline{H}^{n-1}(\widetilde{V_P}/F)$ for some number field F. Does there exist a Laurent polynomial $Q \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ such that $L^*(M, 0)/m(Q) \in \mathbb{Q}^\times$? Is there an algorithmic way to construct Q starting from P?

We conclude this section with a series of remarks:

- Boyd's computations considered mainly families of polynomials whose Newton polygon is *reflexive*, *i.e.* it contains only one interior point with integer coordinates (see Appendix A.2). A notable exception is provided by the families studied in [Boy98, § 3], which consist of polynomials defining curves of genus two whose Jacobian splits into the product of two elliptic curves. We refer the interested reader to [LQ19] for an extensive list of families of polynomials defining curves of genus 2 and 3, whose Jacobian has a one-dimensional factor. Some of the results conjectured by Boyd and Liu and Qin have been proved in recent years, especially in the work of Bertin and Zudilin (see [BZ16; BZ17]) and Lalín and Wu (see [LW18; LW20]);
- Question 4.2.9 has been answered in the positive for some polynomials in three variables such that V_P is (birationally equivalent to) a $K3$-surface. Moreover, variants of Question 4.2.9 have been proved for polynomials in three variables, by replacing $\underline{H}^2(V_P)$ with a suitable one-dimensional sub-motive. We cite in particular the work of Bertin and collaborators (see [Ber08; Ber10; Ber+13]), the paper [PRS14] by Papanikolas, Rogers and Samart and the article [BN18] by Brunault and Neururer;
- we see in the next chapter that Question 4.2.9 has a negative answer for a certain class of polynomials, which satisfy suitable *exactness conditions*. In particular, the next chapter contains the outline of a framework which allows one to generalise Question 4.2.9 to these kinds of exact polynomials.

4.3 Mahler measures, motives and regulators

The aim of this section is to present the work of Deninger [Den97a], later refined by the work of Besser and Deninger [BD99] and Bornhorn (see [Bor99] and [Bor15]), which relates the Mahler measure of a polynomial to regulators and periods of mixed motives. The main idea is the following: one can use Jensen's formula to change the domain of integration for the Mahler measure, in such a way that the resulting differential form, albeit not closed, can be easily modified to a closed form without changing the value of the integral in question. The first step towards this is provided by the following proposition (see [Den97a, Proposition 3.3]).

Proposition 4.3.1 – Deninger's integral

Let $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\}$ be a non-zero Laurent polynomial, such that:

D.1 if we write $P(x_1, \dots, x_n) = \sum_{i=i_0}^{\deg_{x_n}(P)} a_i(x_1, \dots, x_{n-1}) x_n^i$ with $a_i \in \mathbb{C}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$ and $P^* := a_{i_0} \neq 0$ then $i_0 = 0$ and $\{\mathbf{z} \in \mathbb{T}^{n-1} \mid a_{i_0}(\mathbf{z}) = 0\} = \emptyset$;

D.2 if we let $V_P \hookrightarrow \mathbb{G}_{m,\mathbb{C}}^n$ denote the zero locus of P, and γ_P denote the sub-space

$$\gamma_P := \{\mathbf{z} \in (\mathbb{C}^\times)^n \mid |z_1| = \cdots = |z_{n-1}| = 1,\ |z_n| \leq 1\} \cap V_P(\mathbb{C}) \tag{4.24}$$

then $\gamma_P \cap V_P^{\mathrm{sing}}(\mathbb{C}) = \emptyset$. We orient γ_P using the canonical orientation coming from the real analytic torus $\mathbb{T}^n$.

Then we have that

$$m(P) = m(P^*) - \frac{1}{(2\pi i)^{n-1}} \int_{\gamma_P} \eta_n$$

where η_n is the smooth differential form on $\mathbb{G}_{m,\mathbb{C}}^n$ defined by

$$\eta_n := \sum_{j=1}^{n} \frac{(-1)^{n+j}}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\, \log|x_{\sigma(1)}| \frac{d\overline{x}_{\sigma(2)}}{\overline{x}_{\sigma(2)}} \wedge \cdots \wedge \frac{d\overline{x}_{\sigma(j)}}{\overline{x}_{\sigma(j)}} \wedge \frac{dx_{\sigma(j+1)}}{x_{\sigma(j+1)}} \wedge \cdots \wedge \frac{dx_{\sigma(n)}}{x_{\sigma(n)}} \tag{4.25}$$

where $\mathfrak{S}_n$ denotes the symmetric group on n letters. The differential form η_n is closed on $V_P^{\mathrm{reg}} \hookrightarrow \mathbb{G}_{m,\mathbb{C}}^n$.

Proof. Jensen's formula, as stated in [SS03, Chapter 5, Theorem 1.1], implies that, for every $\mathbf{z}' \in \mathbb{T}^{n-1}$ such that $P^*(\mathbf{z}') \neq 0$ we have that

$$\int_{\mathbb{T}^1} \log|P(\mathbf{z}', \alpha)| d\mu_{\mathbb{T}^1}(\alpha) = \log|P^*(\mathbf{z}')| - \sum_{\substack{\alpha \in D(0;1)^\circ \setminus \{0\} \\ P(\mathbf{z}', \alpha) = 0}} \log|\alpha| \tag{4.26}$$

where $D(0;1) := \{z \in \mathbb{C} \mid |z| \leq 1\}$. This implies that

$$\begin{aligned} m(P) &= \int_{\mathbb{T}^{n-1}} \left(\int_{\mathbb{T}^1} \log|P(\mathbf{x}', x_n)| d\mu_{\mathbb{T}^1}(x_n) \right) d\mu_{\mathbb{T}^{n-1}}(\mathbf{x}') = \\ &= \int_{\mathbb{T}^{n-1}} \log|P^*(\mathbf{x}')| - \sum_{\substack{x_n \in D(0;1)^\circ \setminus \{0\} \\ P(\mathbf{x}', x_n) = 0}} \log|x_n| d\mu_{\mathbb{T}^{n-1}}(\mathbf{x}') = \\ &= m(P^*) - \frac{1}{(2\pi i)^{n-1}} \int_{\gamma_P} \log|x_n| \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{n-1}}{x_{n-1}} \end{aligned}$$

because $\mu_{\mathbb{T}^{n-1}}(\{\mathbf{z}' \in \mathbb{T}^{n-1} \mid P^*(\mathbf{z}') = 0\}) = 0$ (see Remark 4.1.3). Now we can observe that

$$m(P) = m(P^*) - \frac{1}{(2\pi i)^{n-1}} \int_{\gamma_P} \log|x_n| \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{n-1}}{x_{n-1}} = m(P^*) - \frac{1}{(2\pi i)^{n-1}} \int_{\gamma_P} \eta_P$$

because the differential form

$$\log|x_n| \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{n-1}}{x_{n-1}}$$

coincides with η_n on γ_P. Indeed, we have that

$$\eta_n\big|_{\gamma_P} = \sum_{j=1}^{n} \frac{(-1)^{n+j}}{n!} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1) = n}} \operatorname{sgn}(\sigma) \log|x_n| \frac{d\overline{x}_{\sigma(2)}}{\overline{x}_{\sigma(2)}} \wedge \cdots \wedge \frac{d\overline{x}_{\sigma(j)}}{\overline{x}_{\sigma(j)}} \wedge \frac{dx_{\sigma(j+1)}}{x_{\sigma(j+1)}} \wedge \cdots \wedge \frac{dx_{\sigma(n)}}{x_{\sigma(n)}} \tag{4.27}$$

$$= \frac{(-1)^{n-1}}{(n-1)!} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1) = n}} \operatorname{sgn}(\sigma) \log|x_n| \frac{dx_{\sigma(2)}}{x_{\sigma(2)}} \wedge \cdots \wedge \frac{dx_{\sigma(n)}}{x_{\sigma(n)}} \tag{4.28}$$

$$= \log|x_n| \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{n-1}}{x_{n-1}} \tag{4.29}$$

where (4.27) follows from the fact that $\log|x_j| = 0$ on γ_P for every $j \in \{1, \dots, n-1\}$, because $|x_1| = \cdots = |x_{n-1}| = 1$ on γ_P. Moreover, (4.28) follows from the equality

$$\left.\frac{d\overline{z}}{\overline{z}}\right|_{\mathbb{T}^1} = \frac{d(z^{-1})}{z^{-1}} = -\frac{dz}{z}$$

and (4.29) follows from the alternating property of the wedge product, which gives

$$\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{n-1}}{x_{n-1}} = (-1)^{n-1} \operatorname{sgn}(\sigma) \frac{dx_{\sigma(2)}}{x_{\sigma(2)}} \wedge \cdots \wedge \frac{dx_{\sigma(n)}}{x_{\sigma(n)}}$$

for every $\sigma \in \mathfrak{S}_n$ such that $\sigma(1) = n$. Finally, we easily see that

$$d(\eta_n) = \mathfrak{R}_n \left(d\log(x_1) \wedge \cdots \wedge d\log(x_n) \right)$$

where $\mathfrak{R}_{2m+1}(z) := \mathfrak{R}(z)$ and $\mathfrak{R}_{2m}(z) = \mathfrak{I}(z)$ for every $m \in \mathbb{N}$. This implies, using the Cauchy-Riemann equations for P, that η_n is closed when restricted to V_P^{reg}. □

It is now worth reflecting upon the two conditions D.1 and D.2 which appear in Proposition 4.3.1. First of all, the following lemma shows that one can always modify a polynomial, without changing its Mahler measure, in such a way that it satisfies D.1 (see [BD99, Fact 2.1]).

Lemma 4.3.2 – Introducing constant terms

Let R be a ring and let $P \in R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then there exist $\gamma \in \mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$ and $a_0 \in R \setminus \{0\}$ such that

$$P_\gamma - a_0 \in x_n \cdot R[x_1, \dots, x_n]$$

where $P_\gamma := (\gamma * P)$ denotes the γ-image of P under the action of $\mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$ on $R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, which is given by

$$(\mathbf{v} \rtimes A) * P := (x_1^{v_1} \cdots x_n^{v_n}) \cdot P(x_1^{a_{1,1}} \cdots x_n^{a_{n,1}}, \dots, x_1^{a_{1,n}} \cdots x_n^{a_{n,n}}) \tag{4.30}$$

for every vector $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{Z}^n$ and every matrix $A = (a_{i,j}) \in \mathrm{GL}_n(\mathbb{Z})$. In particular, if $R \subseteq \mathbb{C}$ then P_γ satisfies the condition D.1 appearing in Proposition 4.3.1, and we have that $m(P_\gamma) = m(P)$.

Proof. We proceed by induction on $n \in \mathbb{N}$. If $n = 0$ there is nothing to prove, and if $n = 1$ there exists $v_1 \in \mathbb{Z}$ such that $x_1^{v_1} P(x_1) = \sum_{j=0}^{d} a_j x_1^j$ with $a_0 \neq 0$. Hence we can take $A = 1 \in \mathrm{GL}_1(\mathbb{Z})$ and $\mathbf{v} = (v_1) \in \mathbb{Z}$. Now, assume that $n \geq 2$. Then there exists $\mathbf{w} \in \mathbb{Z}^n$ such that

$$(x_1^{w_1} \cdots x_n^{w_n}) \cdot P = P_1(x_2, \dots, x_n) + x_1 Q_1(x_1, \dots, x_n) \tag{4.31}$$

where $P_1 \in R[x_2, \dots, x_n]$ and $Q_1 \in R[x_1, \dots, x_n]$. If we set $\gamma' \in \mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$ to be the element defined as $\gamma' := \mathbf{w} \rtimes B$, where

$$B = \begin{pmatrix} 1 & & \\ & \ddots & \\ 1 & \dots & 1 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Z})$$

we see from (4.31) that

$$P_{\gamma'} = P_2(x_2, \dots, x_n) + x_n Q_2(x_1, \dots, x_n)$$

where $P_2 := P_1(x_2 x_n, \cdots, x_{n-1} x_n, x_n)$ and $Q_2 := x_1 Q_1(x_1 x_n, \dots, x_{n-1} x_n, x_n)$. Now, by induction we know that there exist $\gamma'' \in \mathbb{Z}^{n-1} \rtimes \mathrm{GL}_{n-1}(\mathbb{Z})$ and $a_0 \in R \setminus \{0\}$ such that

$$(P_2)_{\gamma''} - a_0 \in x_n R[x_2, \dots, x_n]$$

which shows that $P_\gamma - a_0 \in x_n R[x_1, \dots, x_n]$ if we take $\gamma := \gamma' \cdot \iota_n(\gamma'')$. Here ι_n denotes the embedding

$$\iota_n \colon \mathbb{Z}^{n-1} \rtimes \mathrm{GL}_{n-1}(\mathbb{Z}) \hookrightarrow \mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$$
$$\mathbf{v} \rtimes A \mapsto (0, \mathbf{v}) \rtimes \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}$$

which makes $\mathbb{Z}^{n-1} \rtimes \mathrm{GL}_{n-1}(\mathbb{Z})$ act on the last $n-1$ coordinates of $R[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. To conclude, P_γ clearly satisfies D.1 with $P^* = a_0$ (a constant). Moreover, $m(P_\gamma) = m(P)$, because $m(x_1^{v_1} \cdots x_n^{v_n}) = 0$ for every $\mathbf{v} \in \mathbb{Z}^n$ (compare with Theorem 4.1.15) and $m(P_A) = m(P)$ for every $A \in \mathrm{GL}_n(\mathbb{Z})$, as one can observe by performing a change of variables in the integral (4.3). □

Getting rid of the second restriction D.2 appearing in Proposition 4.3.1 is more difficult. This can be done when $n = 2$ as long as $V_P(\mathbb{C})^{\mathrm{sing}} \cap \mathbb{T}^2 = \emptyset$, as the following result of Bornhorn shows (see [Bor99, Lemma 5.2.8] and [Bor15, Lemma 1.7]).

Lemma 4.3.3 – Eliminating singularities in Deninger's cycle

Let $P \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}] \setminus \{0\}$ and denote by $V_P \hookrightarrow \mathbb{G}_{m,\mathbb{C}}^2$ the zero locus of P. Suppose that $V_P(\mathbb{C})^{\mathrm{sing}} \cap \mathbb{T}^2 = \emptyset$ and that there exists $a_0 \in \mathbb{C}^\times$ such that $P - a_0 \in x_2 \mathbb{C}[x_1, x_2]$ (which can always be achieved by Lemma 4.3.2). Then there exists a matrix $A \in \mathrm{GL}_2(\mathbb{Z})$ such that, if we set $Q := P_A$ we have that

$$Q - a_0 \in x_2 \mathbb{C}[x_1, x_2] \quad \text{and} \quad \gamma_Q \cap V_Q^{\mathrm{sing}}(\mathbb{C}) = \emptyset$$

where γ_Q is the cycle defined in (4.24).

Proof. Write $P = a_0 + \sum_{j=1}^d a_j(x_1) x_2^j$ and take $m \in \mathbb{N}$ such that

$$m + 1 > \max\left\{ \{\deg(a_j)\}_{j=1}^d \cup \left\{ \left| \frac{\log|\alpha_2|}{\log|\alpha_1|} \right| \ : \ (\alpha_1, \alpha_2) \in V_P(\mathbb{C})^{\mathrm{sing}}, \ |\alpha_1| \neq 1 \right\} \right\} \tag{4.32}$$

which surely exists because the set $V_P(\mathbb{C})^{\text{sing}}$ is finite. Then we can take $A \in \mathrm{GL}_2(\mathbb{Z})$ to be the matrix

$$A := \begin{pmatrix} -1 & m \\ -1 & m+1 \end{pmatrix}$$

which is surely invertible since $\det(A) = -1$. Moreover, we have that $Q - a_0 \in x_2\mathbb{C}[x_1, x_2]$ because we have chosen $m > \deg(a_j)$ for every $j \in \{1, \dots, d\}$. Finally, let us show that $V_Q(\mathbb{C})^{\text{sing}} \cap \gamma_Q = \emptyset$. Indeed, if by contradiction $(\beta_1, \beta_2) \in V_Q(\mathbb{C})^{\text{sing}} \cap \gamma_Q$ then $|\beta_1| = 1$ and $(\alpha_1, \alpha_2) := ((\beta_1\beta_2)^{-1}, \beta_1^m \beta_2^{m+1}) \in V_P(\mathbb{C})^{\text{sing}}$. This implies that

$$\left| \frac{\log|\alpha_2|}{\log|\alpha_1|} \right| = m + 1$$

which contradicts (4.32) unless $|\alpha_1| = 1$. But in this case we would have that

$$(\alpha_1, \alpha_2) \in V_P(\mathbb{C})^{\text{sing}} \cap \mathbb{T}^2$$

which contradicts the hypothesis $V_P(\mathbb{C})^{\text{sing}} \cap \mathbb{T}^2 = \emptyset$, and thus we conclude that it was absurd to suppose that $V_Q(\mathbb{C})^{\text{sing}} \cap \gamma_Q \neq \emptyset$. □

Let us now use Proposition 4.3.1 to achieve the main result of this section, which relates Mahler measures and regulators. This result has been proved by Deninger in [Den97a, Theorem 3.4].

Theorem 4.3.4 – Mahler measures and regulators

Let $K \subseteq \mathbb{C}$ be a field, and $P \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\}$ be a non-zero Laurent polynomial. Suppose that at least one of the following holds:

- the condition D.2 of Proposition 4.3.1 is satisfied;
- $n = 2$, and $\mathbb{T}^2 \cap V_P(\mathbb{C})^{\text{sing}} = \emptyset$.

Then there exist a sub-set $\Delta_P \subseteq V_P(\mathbb{C})^{\text{reg}}$, a relative homology class

$$\alpha_P \in H_{n-1}^{\text{sing}}(V_P(\mathbb{C})^{\text{reg}}, \Delta_P; \mathbb{Z}(1-n))$$

and a number $a_0 \in K^\times$ such that

$$m(P) = \log|a_0| + \langle r^{\infty}_{V_P^{\text{reg}}}(\{x_1, \dots, x_n\}), \alpha_P \rangle_{\text{per}} \tag{4.33}$$

where $\langle \cdot, \cdot \rangle_{\text{per}}$ denotes the period pairing induced by (2.8), and $r^{\infty}_{V_P^{\text{reg}}}$ denotes Beilinson's regulator (see Example 2.4.6) applied to the motivic cohomology class

$$\{x_1, \dots, x_n\} := \{x_1\} \cup \dots \cup \{x_n\} \in H^{n,n}_{\mathcal{M}}(V_P^{\text{reg}}; \mathbb{Q})$$

where $\{x_1\}, \dots, \{x_n\} \in H^{1,1}_{\mathcal{M}}(V_P^{\text{reg}}; \mathbb{Q}) \cong \mathcal{O}(V_P^{\text{reg}})^\times \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. Combining Lemma 4.3.2 and Lemma 4.3.3 we see that there exist $\beta \in \mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$ and $a_0 \in K^\times$ such that $Q - a_0 \in x_n K[x_1, \dots, x_n]$ and $\gamma_Q \cap V_Q(\mathbb{C})^{\mathrm{sing}} = \emptyset$, where $Q := P_\beta$. In particular, applying Proposition 4.3.1 we see that

$$m(P) = m(Q) = \log|a_0| - \frac{1}{(2\pi i)^{n-1}} \int_{\gamma_Q} \eta_n$$

where γ_Q is defined as in (4.24) and η_n is the differential form defined in (4.25).

Now, observe that $H^{n,n}_{\mathcal{D}}(V_Q^{\mathrm{reg}}; \mathbb{R}) \cong H^{n-1,n-1}_{\mathrm{dR}}(V_Q^{\mathrm{reg}}; \mathbb{R})$ because $\dim(V_Q^{\mathrm{reg}}) = n - 1$. Moreover, using the description of the cup-product provided by Remark 2.5.2 one easily sees (by induction on $n \in \mathbb{N}$) that the the cohomology class $r^\infty_{V_Q^{\mathrm{reg}}}(\{x_1, \dots, x_n\}) \in H^{n-1,n-1}_{\mathrm{dR}}(V_Q^{\mathrm{reg}}; \mathbb{R})$ is represented by the restriction of η_n to V_Q^{reg}, which is a closed form as we pointed out in Proposition 4.3.1. Hence we get that $r^\infty_{V_Q^{\mathrm{reg}}}(\{x_1, \dots, x_n\})$ defines a relative cohomology class

$$r^\infty_{V_Q^{\mathrm{reg}}}(\{x_1, \dots, x_n\}) \in H^{n-1,n-1}_{\mathrm{dR}}(V_Q^{\mathrm{reg}}, \mathbb{T}^n; \mathbb{R}) \subseteq H^{n-1,n-1}_{\mathrm{dR}}(V_Q^{\mathrm{reg}}, \partial\gamma_Q; \mathbb{R})$$

because η_n vanishes (as a differential form) on the real torus $\mathbb{T}^n \supseteq V_Q(\mathbb{C}) \cap \mathbb{T}^n \supseteq \partial\gamma_Q$. Finally, we observe that $\gamma_Q \subseteq (\mathbb{C}^\times)^n$ is a semi-algebraic set (see [HM17, § 2.6]). Hence we can use the triangulation theorem for semi-algebraic sets (see [Hir75]) to get a relative homology class $[\gamma_Q] \in H^{\mathrm{sing}}_{n-1}(V_Q(\mathbb{C})^{\mathrm{reg}}, \partial\gamma_Q; \mathbb{Z})$ such that

$$\langle r^\infty_{V_Q^{\mathrm{reg}}}(\{x_1, \dots, x_n\}), [\gamma_Q] \otimes (2\pi i)^{1-n} \rangle_{\mathrm{per}} = \frac{1}{(2\pi i)^{n-1}} \int_{\gamma_Q} \eta_n.$$

We conclude by setting $\Delta_P := \phi_\beta(\partial\gamma_Q)$ and $\alpha_P := \phi^\beta_*(-[\gamma_Q])$, where $\phi_\beta \colon \mathbb{G}^n_{m,\mathbb{C}} \xrightarrow{\sim} \mathbb{G}^n_{m,\mathbb{C}}$ is the isomorphism induced by β which sends V_Q to V_P. □

Remark 4.3.5. Note that, in the statement and proof of Theorem 4.3.4, we are using implicitly the comparison between algebraic de Rham cohomology and analytic de Rham cohomology. Indeed, the subset $\Delta_P \subseteq V_P(\mathbb{C})^{\mathrm{reg}}$ featured in the theorem cannot be obtained as the complex points of a sub-variety. However, there is a way around this, which is implicitly stated by Deninger in [Den97a, Page 274]. More precisely, Deninger writes:

> "In general if one wishes to interpret the formula in Proposition 3.3 in terms of Deligne cohomology or even K-theory, it will be necessary to replace ∂A by an algebraic variety. Possibly some complexification will do..."

Indeed, one can proceed as follows. Let $\mathbb{S}^n := \mathrm{N}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}^n_{m,\mathbb{C}})$ denote the n-dimensional Deligne torus (see [Mil13, § 5]), where $\mathrm{N}_{\mathbb{C}/\mathbb{R}}$ denotes Weil's restriction of scalars (see [BLR90, § 7.6]). We recall that $\mathbb{S}^n$ is a scheme over $\mathbb{R}$, such that $\mathbb{S}^n(\mathbb{R}) = \mathbb{G}^n_{m,\mathbb{C}}(\mathbb{C}) = (\mathbb{C}^\times)^n$. Moreover, there exists a scheme-theoretic map $\tau \colon \mathbb{S}^n \to \mathbb{G}^n_{m,\mathbb{R}}$ which on real points is given by $z \to z\bar{z}$. Hence we have that $\ker(\tau)(\mathbb{R}) = \mathbb{T}^n$, and in particular for every Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ we can identify the set $\mathbb{T}^n \cap V_P(\mathbb{C})$ with the real points of the sub-scheme $A_P := \ker(\tau) \cap \mathrm{N}_{\mathbb{C}/\mathbb{R}}(V_P) \hookrightarrow \mathbb{S}^n$. Proceeding similarly to what we have done in Theorem 4.3.4 one is able to write

$$m(P) = \log|a_0| + \langle r^\infty_{V^{\mathrm{reg}}_{P,\mathbb{R}}}(\{x_1, y_1, \dots, x_n, y_n\}), \alpha_{P,\mathbb{R}} \rangle_{\mathrm{per}}$$

where $V_{P,\mathbb{R}} := \mathrm{N}_{\mathbb{C}/\mathbb{R}}(V_P) \hookrightarrow \mathbb{S}^n$ and $x_1, y_1, \dots, x_n, y_n \in \mathcal{O}(\mathbb{S}^n)^\times$ are the coordinate functions. Now, the homology class $\alpha_{P,\mathbb{R}}$ is relative to $\Delta_{P,\mathbb{R}} := A_P(\mathbb{R})$, and hence we can view the differential form $r^{\infty}_{V^{\mathrm{reg}}_{P,\mathbb{R}}}(\{x_1, y_1, \dots, x_n, y_n\})$ as an element of a relative algebraic de Rham cohomology group. Thus if one believes in the injectivity of the Beilinson regulator map associated to V_P (when $P \in \overline{\mathbb{Q}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$) one is able to see $\{x_1, y_1, \dots, x_n, y_n\}$ as a relative motivic cohomology class. The details of this construction, as well as applications of this point of view, will be the subject of future research.

Remark 4.3.6. Deninger uses Theorem 4.3.4 to prove that, under the same assumptions on the polynomial $P \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the Mahler measure $m(P)$ appears as the period of a mixed motive $M_P \in \mathcal{MM}(K; \mathbb{Q})$. Let us recall briefly its construction. First of all, recall that the general linear group $\mathrm{GL}_n(\mathbb{Z})$ acts on $\mathbb{G}^n_{m,\mathbb{C}}$ via the action

$$A * (x_1, \dots, x_n) := (x_1^{a_{1,1}} \cdots x_n^{a_{n,1}}, \dots, x_1^{a_{1,n}} \cdots x_n^{a_{n,n}})$$

which was already used in Lemma 4.3.2. Then for every finite subgroup $\Gamma \subseteq \mathrm{GL}_n(\mathbb{Z})$ one associates to each sub-scheme $X \hookrightarrow \mathbb{G}^n_{m,K}$ the following two schemes

$$X^{\sqcup\Gamma} := \bigsqcup_{\gamma \in \Gamma} X^\gamma \to \mathbb{G}^n_{m,K}$$
$$X^{\cup\Gamma} := \bigcup_{\gamma \in \Gamma} X^\gamma \hookrightarrow \mathbb{G}^n_{m,K}$$

which can both be thought of as a sort of "completion" of X along the action of Γ. Using these schemes, one can define four motives over K with coefficients in any ring Λ such that $|\Gamma| \in \Lambda^\times$. This can be done in any of the abelian categories $\mathcal{MM}(K; \Lambda)$ defined in Section 2.2.2, by setting:

$$\begin{aligned} M^{\sqcup}_{P,\Gamma} &:= e_\Gamma\left(\underline{H}^{n,n}(\mathbb{G}^n_{m,K}, V_P^{\sqcup\Gamma})\right) & M^{\sqcup,\mathrm{reg}}_{P,\Gamma} &:= e_\Gamma\left(\underline{H}^{n,n}(\mathbb{G}^n_{m,K}, (V_P^{\mathrm{reg}})^{\sqcup\Gamma})\right) \\ M^{\cup}_{P,\Gamma} &:= e_\Gamma\left(\underline{H}^{n,n}(\mathbb{G}^n_{m,K}, V_P^{\cup\Gamma})\right) & M^{\cup,\mathrm{reg}}_{P,\Gamma} &:= e_\Gamma\left(\underline{H}^{n,n}(\mathbb{G}^n_{m,K}, (V_P^{\mathrm{reg}})^{\cup\Gamma})\right) \end{aligned} \tag{4.34}$$

where $e_\Gamma \in \Lambda[\Gamma]$ denotes the element of the group algebra $\Lambda[\Gamma]$ defined by:

$$e_\Gamma := \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \det(\gamma)[\gamma^{-1}] \in \Lambda[\Gamma]$$

which is an idempotent, and acts as a projector on any motive endowed with an action of Γ. Now, for every subgroup $\Gamma \subseteq \mathrm{GL}_n(\mathbb{Z})$ such that $e_\Gamma(\underline{H}^{n-1,n}(\mathbb{G}^n_{m,K})) = 0$, Deninger observes that the relative cohomology exact sequence (see Definition 2.1.4):

$$\cdots \to \underline{H}^{n-1,n}(\mathbb{G}^n_{m,K}) \to \underline{H}^{n-1,n}(V_P^{\sqcup\Gamma}) \to \underline{H}^{n,n}(\mathbb{G}^n_{m,K}, V_P^{\sqcup\Gamma}) \to \underline{H}^{n,n}(\mathbb{G}^n_{m,K}) \to \underline{H}^{n,n}(V_P^{\sqcup\Gamma}) \to \dots$$

shows that $M^{\sqcup}_{P,\Gamma} \in \mathrm{Ext}^1(\Lambda(0), \underline{H}^{n-1,n}(V_P))$. Indeed, $\underline{H}^{n,n}(V_P^{\sqcup\Gamma}) = 0$ because $V_P^{\sqcup\Gamma}$ is affine and $\dim(V_P^{\sqcup\Gamma}) = n - 1$. Moreover, $e_\Gamma(\underline{H}^{n-1,n}(\mathbb{G}^n_{m,K})) = 0$ by assumption, and it is easy to see that $\underline{H}^{n,n}(\mathbb{G}^n_{m,K}) \cong \Lambda(0)$, with trivial Γ-action. Finally, there is a canonical isomorphism

$$\underline{H}^{n-1,n}(V_P) \cong e_\Gamma(\underline{H}^{n-1,n}(V_P^{\sqcup\Gamma}))$$

which comes from the definition of $V_P^{\sqcup\Gamma}$. One can show similar results for the other three motives mentioned in (4.34). Moreover, Deninger shows that, under the assumptions of Theorem 4.3.4, the Mahler measure $m(P)$ appears in the image of the period pairing

$$(M_{P,\Gamma}^{\vee}(1))_{\mathbb{B}}^{+} \times F^{0}(M_{P,\Gamma})_{\mathrm{d}\mathbb{R}} \longrightarrow \mathbb{R}$$

where $M_{P,\Gamma}$ is any of the motives defined in (4.34), and $\Gamma \subseteq \mathrm{GL}_n(\mathbb{Z})$ is any finite subgroup such that $e_\Gamma(\underline{H}^{n-1,n}(\mathbb{G}_{m,K}^n)) = 0$.

In particular, one can use as $\Gamma \subseteq \mathrm{GL}_n(\mathbb{Z})$ any finite subgroup which contains the diagonal matrix $-\mathrm{Id}_n \in \mathrm{GL}_n(\mathbb{Z})$. Indeed, Künneth's formula shows that

$$\underline{H}^{n-1}(\mathbb{G}_m^n) \cong \bigoplus_{i=1}^{n} \Lambda(1-n) \tag{4.35}$$

and any matrix $\gamma \in \mathrm{GL}_n(\mathbb{Z})$ acts on the i-th component in the sum (4.35) as multiplication by $\det(\gamma^{(i)}) \in \{\pm 1\}$, where $\gamma^{(i)} \in \mathrm{GL}_{n-1}(\mathbb{Z})$ denotes the $(n-1)$-minor obtained by removing the i-th row and column. Since $\det(-\mathrm{Id}_n) = (-1)^n = -(-1)^{n-1} = -\det(\gamma^{(i)})$ for every $i \in \{1, \dots, n\}$, we see that the projector $[\mathrm{Id}_n] + (-1)^n[-\mathrm{Id}_n]$ acts as the zero map on $\underline{H}^{n-1}(\mathbb{G}_m^n)$. We observe that Maillot's trick, which is the subject of the following chapter, is strongly related to the previous discussion, specialised to the choice of subgroup $\Gamma \subseteq \mathrm{GL}_n(\mathbb{Z})$ given by $\Gamma := \{\mathrm{Id}_n, -\mathrm{Id}_n\}$.

It would be very interesting to generalise Deninger's construction to every Laurent polynomial $P \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ defined over a number field $K \subseteq \mathbb{C}$. This would give us a more or less explicit way to construct a motive $M_P \in \mathcal{MM}(K; \mathbb{Q})$ such that $m(P)$ appears as a period of M_P, with the property that M_P is "as small as possible". Approaching this question would undoubtedly require a generalisation of Theorem 4.3.4, of the kind discussed in Remark 4.3.5.

4.4 Some explicit computations

The aim of this final section of the current chapter is to show two different kinds of examples of explicit computations related to the techniques outlined in Section 4.3. First of all, we show how Theorem 4.3.4 can be used to relate special values of L-functions to Mahler measures, assuming Beilinson's conjecture (see Conjecture 3.3.18). Then, we use the methods developed by Rodriguez Villegas in [Rod99] to compute explicit expressions for the Mahler measure of polynomials in families.

4.4.1 Mahler measures and Beilinson's conjecture

We start by recalling the theorem which inspired this whole section, which is due to Deninger (see [Den97a, Page 274]) and Bornhorn (see [Bor99, Satz 5.3.8] and [Bor15, Theorem 2.2]).

Theorem 4.4.1 – Beilinson's conjecture and Deninger's family

Let $P_k(t_1, t_2) := t_1 + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + k \in \mathbb{Z}[k][t_1^{\pm 1}, t_2^{\pm 1}]$ be Deninger's family defined in (4.21). Fix $k \in \mathbb{C}$ such that $k^2 \in \mathbb{Z} \setminus \{0, \pm 4\}$. Then Beilinson's conjecture (see Conjecture 3.3.18) for the motive $\underline{H}^1(E_k)$ implies that

$$\frac{L'(E_k, 0)}{m(P_k)} \in \mathbb{Q}^{\times}$$

where E_k denotes the elliptic curve defined by the affine Weierstraß equation

$$E_k : y^2 - kxy = x^3 - 2x^2 + x$$

which is birationally equivalent to the zero locus of P_k, and isomorphic over $\mathbb{Q}$ to the elliptic curve defined in (4.22).

Let us review the strategy used in the proof of Theorem 4.4.1, before applying it to another example. The conjectural f-admissibility of the motive $\underline{H}^1(E_k)$ (see Conjecture 3.3.16) implies that $\dim(H_f^{2,2}(E_k)) = 1$, and Beilinson's conjectures imply that for every $c_k \in H_{1,1}^{\mathrm{sing}}(E_k(\mathbb{C}); \mathbb{Z})$ there exists $\beta_k \in H_f^{2,2}(E_k)$ such that $L'(E_k, 0) = \langle r_{E_k}^{\infty}(\beta_k), c_k \rangle_{\mathrm{per}}$, where $r_{E_k}^{\infty}$ denotes Beilinson's regulator (see Example 2.4.6). Since P_k is reciprocal, one can use Theorem 4.3.4 to show that there exists a class $\alpha_k \in H_{n-1,1-n}^{\mathrm{sing}}(V_{P_k}(\mathbb{C})^{\mathrm{reg}}; \mathbb{Z})$ such that

$$m(P_k) = \langle r_{V_{P_k}}^{\infty}(\{x, y\}), \alpha_k \rangle_{\mathrm{per}} \tag{4.36}$$

because in this case we can take $a_0 = 1$ in (4.33). Combining this with the prediction

$$\dim(H_f^{2,2}(E_k)) \stackrel{?}{=} 1$$

it is sufficient to show that there exists a class $\omega_k \in H_f^{2,2}(E_k)$ such that $\varphi_k(\omega_k) = \{x, y\}$, where φ_k denotes the restriction map

$$\varphi_k : H_f^{2,2}(E_k) \to H_{\mathcal{M}}^{2,2}(E_k) \to H_{\mathcal{M}}^{2,2}(V_{P_k}^{\mathrm{reg}})$$

induced by the inclusion $V_k^{\mathrm{reg}} \hookrightarrow E_k$. Since we can view $x, y \in \mathbb{Q}(E_k)$, the existence of ω_k can be proved using the following result of Schappacher-Scholl [SS91, Proposition 3.2] (compare with Proposition 2.3.7 and [DJZ06, Remark 3.8]).

Theorem 4.4.2 – Computing f-cohomology for elliptic curves

For every elliptic curve E defined over a number field κ we have $H_f^{2,2}(E) \cong \ker(\partial)$, where

$$\partial : \frac{(\kappa(E)^{\times} \otimes_{\mathbb{Z}} \mathbb{Q})^{\otimes 2}}{\langle h \otimes (1-h) : h \in \kappa(E)^{\times} \setminus \{1\} \rangle} \to \left(\bigoplus_{x \in |E|} \kappa(x)^{\times} \otimes_{\mathbb{Z}} \mathbb{Q} \right) \oplus \left(\bigoplus_{\mathfrak{p} \in S_E} \mathbb{Q} \right)$$

is a family of residue maps, indexed over the set of closed points $x \in |E|$ and the set S_E of maximal ideals $\mathfrak{p} \subseteq \mathcal{O}_{\kappa}$ at which E has split multiplicative reduction. More precisely, the components ∂_x at closed points $x \in |E|$ are given by the formula

$$\partial_x(\{f, g\}) := (-1)^{\mathrm{ord}_x(f)\,\mathrm{ord}_x(g)} \left. \frac{f^{\mathrm{ord}_x(g)}}{g^{\mathrm{ord}_x(f)}} \right|_x$$

which already appeared in (2.28). Here $f, g \in \kappa(E)^\times$ is any pair of functions, and

$$\{f,g\} \in \frac{(\kappa(E)^\times \otimes_{\mathbb{Z}} \mathbb{Q})^{\otimes 2}}{\langle h \otimes (1-h) : h \in \kappa(E)^\times \setminus \{1\}\rangle}$$

denotes the class of the tensor $f \otimes g$ inside the quotient.

Let now $\mathcal{E} \to \mathrm{Spec}(O_\kappa)$ be the minimal regular model of E (see [Liu02, § 9.3.3]), and denote by $\mathcal{E}_\mathfrak{p}$ its fibres at different primes $\mathfrak{p} \subseteq O_\kappa$. Then for every $\mathfrak{p} \in S_E$ and every pair of functions $f, g \in \kappa(E)^\times$ such that $S_{f,g} \subseteq \mathcal{E}_\mathfrak{p}^{\mathrm{reg}}$, where $S_{f,g}$ denotes the set of zeros and poles of f and g, we have that

$$\partial_\mathfrak{p}(\{f,g\}) := \pm \frac{1}{3N_\mathfrak{p}(E)} \sum_{u,v \in \mathbb{Z}/N_\mathfrak{p}(E)} \delta_{f,\mathfrak{p}}(u)\, \delta_{g,\mathfrak{p}}(u+v)\, B_3\left(\left\{\frac{v}{N_\mathfrak{p}(E)}\right\}\right) \tag{4.37}$$

where:

- $N_\mathfrak{p}(E) \in \mathbb{N}$ denotes the number of connected components of $\mathcal{E}_\mathfrak{p}$, which is a Néron polygon (see [Sil94, Chapter IV, Theorem 8.2]) because we are assuming that E has split multiplicative reduction at $\mathfrak{p}$. We note that $N_\mathfrak{p}(E_k) = \mathrm{ord}_\mathfrak{p}(\Delta_\mathfrak{p}(E_k))$ where $\Delta_\mathfrak{p}(E_k) \in$ denotes the minimal discriminant of the base change of E_k to the $\mathfrak{p}$-adic completion of κ (see [Sil09, Chapter VII, § 1]);
- for every function $h \in \kappa(E)^\times$ with divisor $\mathrm{div}(h) = \sum_x \mathrm{ord}_x(h)[x]$ whose support S_h is contained in the smooth part of the fibre $\mathcal{E}_\mathfrak{p}$ for some $\mathfrak{p} \in S_E$, we write

$$\begin{aligned} \delta_{h,\mathfrak{p}} : \mathbb{Z}/N_\mathfrak{p}(E) &\to \mathbb{Z} \\ n &\mapsto \sum_x \mathrm{ord}_x(h)\delta_{x,\mathfrak{p}}(n) \end{aligned}$$

  where $\delta_{x,\mathfrak{p}}(n) = 1$ if and only if x lies in the n-th component of the Néron polygon $\mathcal{E}_\mathfrak{p}$, and $\delta_{x,\mathfrak{p}}(n) = 0$ otherwise;
- $B_3(t) := t^3 - \frac{3}{2}t^2 + \frac{1}{2}t$ is the third Bernoulli polynomial;
- $\{v/N_\mathfrak{p}(E)\}$ denotes the unique representative of the quotient $v/N_\mathfrak{p}(E) \in \mathbb{Q}/\mathbb{Z}$ such that $0 \le \{v/N_\mathfrak{p}(E)\} < 1$.

Thus if one proves that $\{x, y\} \in \ker(\partial)$ then one can combine Theorem 4.4.2 with (4.36) to obtain Theorem 4.4.1. Let us see what this entails in practice, by studying a different family of polynomials appearing in Boyd's numerical investigations (see [Boy98, Table 2] and Appendix A.1).

Theorem 4.4.3 – Beilinson's conjecture and a polynomial family

Let $P_k(t_1, t_2) \in \mathbb{Z}[k][t_1^{\pm 1}, t_2^{\pm 1}]$ denote the family of polynomials

$$P_k(t_1,t_2) = t_1 + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + \frac{t_1}{t_2} + \frac{t_2}{t_1} + k$$

with zero locus $V_{P_k} \hookrightarrow \mathbb{G}_m^2$. Denote by E_k the curve defined by the Weierstraß equation

$$E_k : y^2 + kxy - 2y = (x-1)^3 \tag{4.38}$$

which is birationally equivalent (over $\mathbb{Q}$) to V_{P_k}. Then the validity of Beilinson's conjecture (see Conjecture 3.3.18) for the motive $\underline{H}^1(E_k)$ implies that

$$\frac{L'(E_k, 0)}{m(P_k)} \in \mathbb{Q}^\times \tag{4.39}$$

for every $k \in \mathbb{Z} \setminus \{-6, 2, 3\}$.

Proof. The polynomial P_k is evidently tempered (see Remark 4.2.8) and reciprocal, hence we can apply [Bor15, Corollary 1.9] (see also [LW20, Equation 9]) to see that there exists a homology class $\alpha_k \in H_{1,-1}^{\mathrm{sing}}(V_{P_k}(\mathbb{C})^{\mathrm{reg}}; \mathbb{Z})$ such that

$$m(P_k) = \langle r^\infty_{V_{P_k}}(\{t_1, t_2\}), \alpha_k \rangle_{\mathrm{per}} \tag{4.40}$$

for every $k \in \mathbb{Z} \setminus \{-6, 2, 3\}$. We note in passing that if $k \in \{-6, 2, 3\}$ then $\mathbb{T}^2 \cap V_P(\mathbb{C})^{\mathrm{sing}} \neq \emptyset$, and the Weierstraß equation (4.38) is singular. One can indeed show that

$$\begin{aligned} m(P_{-6}) &= 5\, L'(\chi_{-3}, -1) \\ m(P_2) &= m(x+y) + m(y+1) + m(x+1) = 0 \\ m(P_3) &= m(x+y+1) + m(x+y+xy) = 2\, L'(\chi_{-3}, -1) \end{aligned} \tag{4.41}$$

where the first result is due to Rodriguez Villegas (see [Boy98, Page 54]), the second is easy to show (see also Theorem 4.1.15) and the third follows from the computations of Smyth (see Theorem 4.2.4). Observe now that we have a birational identification

$$\begin{aligned} E_k &\dashrightarrow V_{P_k} \\ (x, y) &\mapsto \left(\frac{x+y-1}{x+k-3}, \frac{(1-k)x - y + 1}{x+k-3} \right) \end{aligned} \tag{4.42}$$

whose inverse is given by

$$\begin{aligned} V_{P_k} &\dashrightarrow E_k \\ (t_1, t_2) &\mapsto \left(\frac{(t_1+t_2)(3-k)}{t_1+t_2+k-2}, \frac{(k-2)(t_2+1+(k-2)t_1)}{t_1+t_2+k-2} \right) \end{aligned} \tag{4.43}$$

where the coordinates (t_1, t_2) on V_{P_k} are induced by P_k and the coordinates (x, y) on E_k are given by the Weierstraß equation (4.38). Hence to conclude the proof it is sufficient to show that for every prime $p \in S_{E_k}$ we have that $\partial_p(\{f, g\}) = 0$, where ∂_p is the map defined in (4.37), and $f, g \in \mathbb{Q}(E)^\times$ denote the functions

$$f := \frac{x+y-1}{x+k-3} \quad \text{and} \quad g := \frac{(1-k)x - y + 1}{x+k-3}$$

appearing in (4.42). To do so, we use the explicit formula (4.37), and in particular we show that for each prime $p \in S_{E_k}$ and every $n \in \mathbb{Z}/N_p(E_k)$ we have that $\delta_{f,p}(n) = \delta_{g,p}(n) = 0$. This is done by studying the order of the images of points $x \in S_{f,g}$ in the Néron component group

$$\Phi_p(E_k) := \frac{E(\mathbb{Q}_p)}{E_0(\mathbb{Q}_p)} \cong \frac{\mathbb{Z}}{N_p(E_k)}$$

where $E_0(\mathbb{Q}_p) \subseteq E(\mathbb{Q}_p)$ denotes the subgroup of points with non-singular reduction. More precisely, we use the fact that

$$\mathrm{ord}_{\Phi_p(E_k)}(x) = \mathrm{ord}_{\Phi_p(E_k)}(y) \Rightarrow \delta_{x,p}(n) = \delta_{y,p}(n) \tag{4.44}$$

for every $p \in S_{E_k}$ and every pair of points $x, y \in E(\mathbb{Q})$. This is combined with the explicit form of the divisors of f and g, which is given by

$$\begin{aligned} \mathrm{div}(f) &= (4Q) + (3Q) - (Q) - (0) \\ \mathrm{div}(g) &= (2Q) + (3Q) - (5Q) - (0) = [-1]^*(\mathrm{div}(f)) \end{aligned} \tag{4.45}$$

where $[-1] \colon E_k \to E_k$ is the inversion map, and $Q := (3-k, (k-2)^2) \in E_k(\mathbb{Q})[6] = E_k(\mathbb{Q})_{\mathrm{tors}}$ denotes the generator of the torsion subgroup of $E_k(\mathbb{Q})$, whose multiples are given by

$$\begin{array}{lll} Q = (3-k, (k-2)^2) & 2Q = (1, 2-k) & 3Q = (0, 1) \\ 4Q = (1, 0) & 5Q = (3-k, k-2) & 6Q = 0. \end{array}$$

Let us dive into the details of the proof. First of all, we define two polynomials:

$$\begin{aligned} c_4(k) &= k^4 - 24k^2 + 48k \\ \Delta(k) &= (k+6)(k-3)^2(k-2)^3 \end{aligned}$$

which are the c_4-invariant and the discriminant of the Weierstraß equation (4.38) (see for example [Sil09, Chapter III, § 1]). Then we know by [Sil09, Chapter VII, Proposition 5.1] that $p \in S_{E_k}$ if and only if $p \in \mathbb{N}$ is a rational prime such that $p \mid \Delta(k)$ and $p \nmid c_4(k)$. Suppose that $p \geq 3$ and $p \mid (k-2)$. Then $p \mid \Delta(k)$ and $p \nmid k(k-3)$, which implies that $p \nmid c_4(k)$ because $c_4(k) = k(k^3 - 24(k-2))$. This shows that $p \in S_{E_k}$ and also that the Weierstraß equation (4.38) is minimal at the prime p (see [Sil09, Chapter VII, Remark 1.1]). Thus $N_p(E_k) = \mathrm{ord}_p(\Delta(k)) = 3$, and the reduction of E_k modulo p is given by the curve $y^2 + 2xy - 2y = (x-1)^3$, which is singular exactly at the points $\overline{2Q} = \overline{4Q} = (1, 0)$. This shows that

$$\mathrm{ord}_{\Phi_p(E_k)}(mQ) = \begin{cases} 1, & \text{if } m \equiv 0, 3 (6) \\ 3, & \text{if } m \equiv 1, 2, 4, 5 (6) \end{cases}$$

which in turn implies, as we have seen, that $\delta_{Q,p}(n) = \delta_{2Q,p}(n) = \delta_{4Q,p}(n) = \delta_{5Q,p}(n)$ and $\delta_{3Q,p}(n) = \delta_{0,p}(n)$ for every $n \in \mathbb{Z}/N_p(E_k)$. Finally, we see that

$$\begin{aligned} \delta_{f,p}(n) &= (\delta_{4Q,p}(n) - \delta_{Q,p}(n)) + (\delta_{3Q,p}(n) - \delta_{0,p}(n)) = 0 \\ \delta_{g,p}(n) &= (\delta_{2Q,p}(n) - \delta_{5Q,p}(n)) + (\delta_{3Q,p}(n) - \delta_{0,p}(n)) = 0 \end{aligned}$$

for every $n \in \mathbb{Z}/N_p(E_k)$, where $p \in \mathbb{N}$ is any rational prime such that $p \geq 3$ and $p \mid (k-2)$. We can proceed in a similar way to analyse the other primes $p \mid \Delta(k)$, dividing our discussion in the following cases:

- $p \neq 3$ and $p \mid (k-3)$. Then $p \nmid c_4(k)$ because $c_4(k) = 9+(k-3)(k^3+3k^2-15k+3)$, which shows again that $p \in S_{E_k}$ and the Weierstraß equation (4.38) is minimal. Then $N_p(E_k) = 2$ and we have that
$$\operatorname{ord}_{\Phi_p(E_k)}(mQ) = \begin{cases} 1, & \text{if } m \equiv 0, 2, 4 (6) \\ 2, & \text{if } m \equiv 1, 3, 5 (6) \end{cases}$$
which can be used in combination with (4.44) to see that
$$\begin{aligned} \delta_{f,p}(n) &= (\delta_{4Q,p}(n) - \delta_{0,p}(n)) + (\delta_{3Q,p}(n) - \delta_{Q,p}(n)) = 0 \\ \delta_{g,p}(n) &= (\delta_{2Q,p}(n) - \delta_{0,p}(n)) + (\delta_{3Q,p}(n) - \delta_{5Q,p}(n)) = 0 \end{aligned}$$
for every $n \in \mathbb{Z}/N_p(E_k)$;
- $p \geq 5$ and $p \mid (k+6)$. In this case $p \nmid c_4(k)$ because $c_4(k) = 144+(k+6)(k^3-6k^2+12k-24)$, hence $p \in S_{E_k}$ and the Weierstraß equation (4.38) is minimal. Thus $N_p(E_k) = 1$, which immediately shows that $\delta_{f,p}(n) = \delta_{g,p}(n) = 0$;
- $p = 2$ and $2 \mid k$. Then E_k has additive reduction at $p = 2$, which implies that $2 \nmid S_{E_k}$, unless either $2^4 \mid (k-2)$ or $2^4 \mid k+6$. If $2^4 \mid (k-2)$ then the Weierstraß equation (4.38) is not minimal at $p = 2$. However, it is easy to see that a 2-minimal Weierstraß equation for E_k is given by
$$E_k : y^2 + \frac{k}{2}xy + \frac{k-2}{8}y = x^3 \tag{4.46}$$
and this new equation shows that $2 \in S_{E_k}$ whenever $2^4 \mid (k-2)$. In this case we see that $N_2(E_k) = 3$, and (4.46) reduces modulo 2 to the curve $y^2 + xy = x^3$, which is singular at the point $(0, 0)$. This implies that
$$\operatorname{ord}_{\Phi_2(E_k)}(mQ) = \begin{cases} 1, & \text{if } m \equiv 0, 3 (6) \\ 3, & \text{if } m \equiv 1, 2, 4, 5 (6) \end{cases}$$
which shows once again that $\delta_{f,2}(n) = \delta_{g,2}(n) = 0$ for every $n \in \mathbb{Z}/N_2(E_k)$. On the other hand, if $2^4 \mid (k+6)$ then the 2-minimal Weierstraß equation for E_k is given by
$$y^2 + \frac{k}{2}xy - \frac{3}{8}(k+6)y = x^3 - 3x^2 + \frac{k+6}{2}x - \frac{3}{8}(k+6)$$
which shows that $N_2(E_k) = 1$ whenever $2^4 \mid (k+6)$. Therefore $\delta_{f,2}(n) = \delta_{g,2}(n) = 0$ for every $n \in \mathbb{Z}/N_2(E_k)$;
- if $p = 3$ and $3 \mid k$, then $9 \mid k^3 + 3k^2 - 15k = (c_4(k) - 9)/(k-3)$, which shows that $\operatorname{ord}_3(c_4(k)) = 2$. Thus the original Weierstraß equation (4.38) is minimal at $p = 3$, and the reduction is additive.

We have shown, as we anticipated, that for every $k \in \mathbb{Z} \setminus \{-6, 2, 3\}$ and every prime $p \in S_{E_k}$ at which the elliptic curve E_k has split multiplicative reduction, we have that $\delta_{f,p}(n) = \delta_{g,p}(n)$ for all $n \in \mathbb{Z}/N_p(E_k)$. Hence using the explicit formula (4.37) we see that $\partial_p(\{f, g\}) = 0$ for every $p \in S_{E_k}$. Since the polynomial P is tempered we also have that $\partial_P(\{f, g\}) = 0$ for every $P \in E_k(\mathbb{Q})$.

Hence combining these two things together with Theorem 4.4.2 we see that $\{f,g\} \in H^{2,2}_f(E_k)$ for every $k \in \mathbb{Z} \setminus \{-6,2,3\}$. Now, Beilinson's conjecture (see Conjecture 3.3.18) implies that there exists an element $\beta_k \in H^{2,2}_f(E_k)$ such that

$$L'(E_k,0) = \langle r^{\infty}_{E_k}(\omega_k), (\iota_k)_*(\alpha_k)\rangle_{\mathrm{per}} \tag{4.47}$$

where $\alpha_k \in H^{\mathrm{sing}}_{1,-1}(V_{P_k}(\mathbb{C})^{\mathrm{reg}};\mathbb{Z})$ is the homology class appearing in (4.40), and $\iota_k \colon V^{\mathrm{reg}}_{P_k} \hookrightarrow E_k$ is the natural embedding, coming from the birational map (4.43). Moreover, what we have shown implies that there exists $\xi_k \in H^{2,2}_f(E_k)$ such that $\iota_k^*(\xi_k) = \{t_1,t_2\}$. Hence we can use (4.40) and (4.47) to get

$$\frac{L'(E_k,0)}{m(P)} = \frac{\langle r^{\infty}_{E_k}(\omega_k), (\iota_k)_*(\alpha_k)\rangle_{\mathrm{per}}}{\langle r^{\infty}_{V^{\mathrm{reg}}_{P_k}}(\{t_1,t_2\}), \alpha_k\rangle_{\mathrm{per}}} = \frac{\langle r^{\infty}_{E_k}(\omega_k), (\iota_k)_*(\alpha_k)\rangle_{\mathrm{per}}}{\langle r^{\infty}_{E_k}(\xi_k), (\iota_k)_*(\alpha_k)\rangle_{\mathrm{per}}}$$

which allows us to conclude that (4.39) holds, using the conjectural admissibility of the motive $\underline{H}^1(E_k)$. □

Remark 4.4.4. Methods similar to the ones outlined in this section have been used to construct specific planar models of elliptic curves, which would give rise to f-cohomology classes in the group $H^{2,2}_f(E_k)$. One particular example of this is given by the polynomial family

$$P_k(x,y) := y^3 - (3x+3k)y^2 + (3x^2+6kx-1)y - (x^3-x^2-x+1)$$

which arises from work of Schappacher and Nekovář, and has been studied in Rolshausen's PhD thesis (see [RS98, § 5.2] and [Rol96, Chapitre IV]).

Remark 4.4.5. As already pointed out by Boyd in [Boy98, Page 12], it would be interesting to generalise the two theorems Theorem 4.4.1 and Theorem 4.4.3 by showing that the ratios $L'(E_k,0)/m(P_k)$ are not only rational but also integral for all but finitely many $k \in \mathbb{Z}$. To do this one probably needs to assume the conjecture of Bloch and Kato (see Conjecture 3.3.20), and then compute the p-adic norms $\|\xi_k\|_p$ of the element $\xi_k \in H^{2,2}_f(E_k)$ such that $\iota_k^*(\xi_k) = \{t_1,t_2\}$.

4.4.2 Weierstraß and Edwards models of elliptic curves

The aim of this section is to show that the Mahler measures of naturally occurring families of polynomials can be completely unrelated to special values of L-functions. The first example of this is given by the following observation, which is due to Smyth (see [Smy81, Theorem 1]).

Proposition 4.4.6 – Mahler measures of Weierstraß forms

For every $k \in \mathbb{C}$ such that $|k| \geq 2$ we have that $m(y^2 - x^3 - k) = \log|k|$.

Proof. It is sufficient to apply the change of variables $(x,y) \mapsto (x,1/y)$ and the formula (4.6), which imply that

$$m(y^2 - x^3 - k) = \log|k| + m^+(\alpha_+(x)) + m^+(\alpha_-(x))$$

where $\alpha_\pm(x) := \pm(x^3+k)^{-1/2}$. Since $|k| \geq 2$ we have that $|\alpha_\pm(x)| \leq 1$ whenever $|x| = 1$. Thus, the definition of m^+ shows that $m^+(\alpha_\pm) = 0$, and we can conclude. □

Remark 4.4.7. Despite what one may be lead to believe from Proposition 4.4.6, the Mahler measures of Weierstraß forms can be related to L-values of elliptic curves. Some examples of these kinds of relations are collected in Table A.3.

Now, another canonical model for elliptic curves has been introduced by Edwards in [Edw07], as we now recall.

Proposition 4.4.8 – The existence of Edwards model

Let E be an elliptic curve defined over a field κ such that $\mathrm{char}(\kappa) \neq 2$. Then there exists a finite extension $\kappa' \supseteq \kappa$ and a parameter $k \in \mathrm{P}^1(\kappa') \setminus \{0, 1, \infty\}$ such that E is birationally equialent to the zero locus of the polynomial

$$P_k(x, y) := x^2 + y^2 - kx^2y^2 - 1 \tag{4.48}$$

which is commonly known as *Edwards polynomial*.

Proof. The zero locus of P_k is birational to the curve defined by the Weierstraß equation

$$\frac{1}{1-k}Y^2 = X^3 + \left(2\,\frac{1+k}{1-k}\right)X^2 + X$$

by the substitution $x = 2X/Y$ and $y = (X-1)/(X+1)$ (which is invertible if and only if $\mathrm{char}(\kappa) \neq 2$). Hence if $\kappa' := \kappa(\sqrt{1-k})$ we see that the Edwards curve $P_k = 0$ is birational over κ' to a Weierstraß curve of the form $Y^2 = X^3 + AX^2 + BX + C$.

Vice-versa, if $\mathrm{char}(\kappa) \neq 2$ then every general Weierstraß form can be reduced to

$$Y^2 = X^3 + AX^2 + BX + C$$

for some $A, B, C \in \kappa$ (see [Sil09, Page 42]) and then to $Y^2 = X^3 + \alpha X^2 + X$ for some $\alpha \in \overline{\kappa}$. To see this we can write $X^3 + AX^2 + BX + C = (X - e_1)(X - e_2)(X - e_3)$ for some $e_1, e_2, e_3 \in \overline{\kappa}$ and then use the substitution $X = \sqrt{(e_1 - e_2)(e_1 - e_3)}\,X' + e_1$ and $Y = ((e_1 - e_2)(e_1 - e_3))^{3/4}\,Y'$ to get an equation of the form $(Y')^2 = (X')^3 + \alpha\,(X')^2 + X'$ where $\alpha = \sqrt{(e_1 - e_2)(e_1 - e_3)}\,(2e_1 - e_2 - e_3)$. Now, if we start already from an equation of the form $Y^2 = X^3 + \alpha X^2 + X$ for some $\alpha \in \overline{\kappa}$ and we apply the substitution $X = (1+y)/(1-y)$ and $Y = (2+2y)/(x-xy)$, which is the inverse to $x = 2X/Y$ and $y = (X-1)/(X+1)$, we get the curve

$$\left(1 + \frac{\alpha}{2}\right)x^2 + \left(1 - \frac{\alpha}{2}\right)x^2y^2 = 2(1 - y^2)$$

and thus if we set $\kappa' := \kappa(e_1, \alpha, \sqrt{\alpha+2}, \sqrt[4]{(e_1 - e_2)(e_1 - e_3)})$ we get that our Weierstraß curve is birational over κ' to the Edwards curve given by $P_k = 0$ with $k = \frac{2-\alpha}{2+\alpha}$. □

Now, we would like to compute the Mahler measure of the Edwards forms given by (4.48). To do this, we use the following result of Rodriguez Villegas (see [Rod99, § 11]), which allows one to express Mahler measures of families of polynomials depending on a parameter k as a power series in k with rational coefficients.

Theorem 4.4.9 – Mahler measures expansions

Let $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial with no constant term. Then for every $k \in \mathbb{C}$ such that $|k| \geq \max_{\mathbf{z} \in \mathbb{T}^n} |P(\mathbf{z})|$ we have that

$$m(P(x_1, \dots, x_n) - k) = \Re\left(\log(k) - \sum_{m=1}^{+\infty} \frac{[P^m]_0}{m} k^{-m}\right) \tag{4.49}$$

where $[P^m]_0 \in \mathbb{C}$ denotes the constant coefficient of the polynomial P^m. If we write the polynomial P as $P(x_1, \dots, x_n) = \sum_{j=1}^{N_P} c_j \, \mathbf{x}^{\mathbf{a}_j}$ for some coefficients $c_j \in \mathbb{C}^\times$ and some exponents $\mathbf{a}_j \in \mathbb{Z}^n$, then the constant terms $[P^m]_0$ can be computed explicitly as:

$$[P^m]_0 = \sum_{\substack{\mathbf{w} \in \mathbb{N}^{N_P} \\ \sigma(\mathbf{w}) = m \\ \mathbf{w} \in \ker(\Xi_P)}} \frac{m!}{w_1! \cdots w_{N_P}!} \cdot c_1^{w_1} \cdots c_{N_P}^{w_{N_P}} \tag{4.50}$$

where $\Xi_P := (\mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_{N_P}) \in \mathrm{Mat}_{n \times N_P}(\mathbb{Z})$ and $\sigma(\mathbf{w}) := w_1 + \cdots + w_{N_P}$.

Proof. First of all, we observe that

$$\int_{\mathbb{T}^n} \mathbf{x}^{\mathbf{a}} d\mu_{\mathbb{T}^n} = \begin{cases} 1, & \text{if } \mathbf{a} = 0 \\ 0, & \text{otherwise} \end{cases}$$

for every $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$. Here we write $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} \cdots x_n^{a_n}$, as we did in the statement of the theorem. Hence we get that

$$\Re\left(\log(k) - \sum_{m=1}^{+\infty} \frac{[P^m]_0}{m} k^{-m}\right) = \log|k| + \int_{\mathbb{T}^n} \Re\left(-\sum_{m=1}^{+\infty} \frac{(P(\mathbf{x})/k)^m}{m}\right) d\mu_{\mathbb{T}^n} = (\dagger)$$

and using the Taylor expansion $\log(1-t) = -\sum_{m=1}^{+\infty} \frac{t^m}{m}$ with $t = P(\mathbf{x})/k$ we see that

$$(\dagger) = \log|k| + \int_{\mathbb{T}^n} \log\left|1 - \frac{P(\mathbf{x})}{k}\right| d\mu_{\mathbb{T}^n} = \int_{\mathbb{T}^n} \log|P(\mathbf{x}) - k| \, d\mu_{\mathbb{T}^n} = m(P(\mathbf{x}) - k)$$

which shows (4.49). Observe that our usage of the Taylor expansion of $\log(1-x)$ is admissible because we assumed that $|k| \geq \max_{\mathbf{z} \in \mathbb{T}^n} |P(\mathbf{z})|$. To conclude, it is sufficient to point out that (4.50) is an easy consequence of the multinomial theorem (see [AS64, § 24.1.2]). □

We can finally use Theorem 4.4.9 to see that the Mahler measure of almost all Edwards polynomials with integer coefficients is very easy to compute.

Proposition 4.4.10 – Mahler measures of Edwards polynomials

For every $k \in \mathbb{C}$ such that $|k| \geq 3$ we have that $m(x^2 + y^2 - kx^2y^2 - 1) = \log|k|$.

Proof. Let $P_k(x,y) := x^2 + y^2 - kx^2y^2 - 1$, and observe that $P_k = x^2y^2(Q(x,y) - k)$ where

$$Q(x,y) = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{x^2y^2}$$

which implies that $m(P_k) = m(Q(x,y)-k)$. Hence we can apply Theorem 4.4.9 to the polynomial Q. Since we have that $N_Q = 3$ and

$$\Xi_Q = \begin{pmatrix} -2 & 0 & -2 \\ 0 & -2 & -2 \end{pmatrix}$$

we see that $\ker(\Xi_Q) \cap \mathbb{Z}^{N_P} = \mathbb{Z} \cdot (1,1,-1)$. This shows that $\ker(\Xi_Q) \cap \mathbb{N}^{N_P} = 0$, and combining this with (4.50) we see that $[Q^m]_0 = 0$ for every $m \in \mathbb{Z}_{\geq 1}$. Finally, (4.49) gives us the equality $m(P_k) = m(Q-k) = \log|k|$ for every $k \in \mathbb{C}^\times$ such that

$$|k| \geq \max_{(z_1,z_2)\in\mathbb{T}^2} |Q(z_1,z_2)| = 3$$

where $\max_{(z_1,z_2)\in\mathbb{T}^2}|Q(z_1,z_2)| = 3$ because $Q(1,1) = 3$ and $|Q(z_1,z_2)| \leq 3$ by the triangle inequality. □

Mahler measures of exact polynomials

5

> The finest of all the devil's tricks was persuading you that he doesn't exist.
>
> Charles Baudelaire, *Paris Spleen*

This aim of this chapter, which is based on joint work in progress with François Brunault, is to give an outline of some possible ways to explain geometrically the identities between Mahler measures and special values of L-functions which escape the framework of Question 4.2.9. Many of these different identities, as we discuss in this chapter, can be explained using a remarkable idea due to Maillot (who in fact dates this idea back to Darboux), concerning the intersection between the zero locus of a Laurent polynomial $P(x_1, \dots, x_n) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and the zero locus of its conjugate reciprocal $P^\dagger$ (see Definition 4.1.11). Unfortunately, Maillot's own work never appeared in print, but was only exposed in a talk on the 30th of April, 2003, at the Banff International Research Station, a short report of which is fortunately available online (see [Boy+03, § 8]). Nevertheless, Maillot's ideas were pursued in the theses of Condon (see [Con04, Chapter 5]) and Lalín (see [Lal05, Chapter 5] and [Lal07]), where it was shown how to obtain identities relating the Mahler measure of polynomials in three or more variables, and zero-dimensional L-functions (such as the Riemann ζ-function and Dirichlet L-functions). Later on, Lalín investigated as well the problem of finding a three-variable polynomial whose Mahler measure is related to the special value $L^*(E, -1)$ for some elliptic curve E. This question was raised by Rodriguez-Villegas after Maillot's talk, and the candidate polynomial

$$P := z - (1 - x)(1 - y) \in \mathbb{Z}[x, y, z] \tag{5.1}$$

was proposed. Boyd then checked numerically that

$$m(z - (1 - x)(1 - y)) \stackrel{?}{=} -2L^*(X_1(15), -1) \tag{5.2}$$

where $X_1(15)$ is the modular curve relative to the congruence subgroup $\Gamma_1(15) \subseteq \mathrm{SL}_2(\mathbb{Z})$. This modular curve is an elliptic curve, which can be defined for example by the Weierstraß equation $y^2 + xy + y = x^3 + x^2$ (see [LMFDB, Elliptic Curve 15.a7]). More importantly, $X_1(15)$ is also birationally equivalent to the curve defined by the equation

$$(1 - x)(1 - y) = \left(1 - \frac{1}{1 - x}\right)\left(1 - \frac{1}{1 - y}\right)$$

which is precisely the *Maillot variety* $P = P^* = 0$ associated to the polynomial P appearing in (5.1). Lalín then went on to prove that, if Beilinson's conjecture (see Conjecture 3.3.18) holds for the motive $\underline{H}^1(X_1(15))(-1)$, and in particular if this motive is f-admissible in the

sense of Definition 3.3.14, then the identity (5.2) holds true, up to a rational number, *i.e.* $L^*(X_1(15),-1)/m(P) \in \mathbb{Q}^\times$ (see [Lal15, Theorem 2 and § 4.1]). We illustrate at the end of this chapter how the cohomological methods developed to explain Maillot's trick can be used to approach the identity (5.2).

5.1 Maillot's trick, exactness of polynomials and Smyth's results

The aim of this section is to introduce the surprisingly simple observation supporting Maillot's "trick" to compute the Mahler measure of polynomials which escape the framework of Question 4.2.9. This is connected with the notion of *exactness* of a polynomial, which we also review. We focus in particular on the two identities $m(x+y+1) = L'(\chi_{-3},-1)$ and $m(x+y+z+1) = -14\zeta'(-2)$, proved by Smyth in [Smy81] (see also Theorem 4.2.4), which are our guiding examples throughout this chapter.

Just as a differential form is said to be exact if and only if the cohomology class it represents vanishes, so a Laurent polynomial $P \in R[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ with coefficients in a ring R is said to be *exact* if the motivic cohomology class $\{x_1,\dots,x_n\} \in H^{n,n}_{\mathcal{M}}(V_P)$ vanishes, where $V_P \hookrightarrow \mathbb{G}_m^n$ is the zero locus of P and $\{x_1,\dots,x_n\} := \{x_1\}\cup\cdots\cup\{x_n\}$ denotes the cup product of the motivic cohomology classes $\{x_1\},\dots,\{x_n\} \in H^{1,1}_{\mathcal{M}}(V_P) \cong \mathcal{O}(V_P)^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ (see Proposition 2.3.6). Suppose now that $P \in \mathbb{C}[x_1^{\pm 1},\dots,x_n^{\pm 1}]\setminus\{0\}$ is an exact polynomial, such that the Deninger cycle $\gamma_P \subseteq (\mathbb{C}^\times)^n$ defined in (4.24) has no boundary, and suppose that P satisfies the hypotheses of Theorem 4.3.4. Then the formula (4.33) shows that $m(P) = \log|a_0|$, where $a_0 \in \mathbb{C}^\times$ is the number appearing in Lemma 4.3.2. In particular, if $P \in R[x_1^{\pm 1},\dots,x_n^{\pm 1}]\setminus\{0\}$ for some ring $R \subseteq \mathbb{C}$ then $a_0 \in R$, hence we see that $m(P)$ is rather uninteresting. We note in passing that, in order to have the equality $m(P) = \log|a_0|$, it is sufficient to have that the Deligne-Beilinson cohomology class $r_{V_P^{\mathrm{reg}}}(\{x_1,\dots,x_n\}) \in H^{n,n}_{\mathcal{D}}(V_P^{\mathrm{reg}})$ vanishes, where V_P^{reg} denotes the regular locus of V_P and $r_{V_P^{\mathrm{reg}}}$ denotes the Deligne-Beilinson regulator (see Example 2.4.6).

Despite these initial comments, there are in fact many examples of exact polynomials which are known to have interesting Mahler measures. In particular, either the Deninger cycle γ_P associated to these polynomials has a boundary, or these polynomials do not satisfy the hypotheses of Theorem 4.3.4. The first examples of polynomials of this kind are in fact given by the linear forms $L_2(x,y) = x+y+1$ and $L_3(x,y,z) = x+y+z+1$ studied by Smyth (see Theorem 4.2.4). These two polynomials are exact because the varieties defined by them are clearly rational, hence their motivic cohomology groups $H^{2,2}_{\mathcal{M}}(V_{L_2})$ and $H^{3,3}_{\mathcal{M}}(V_{L_2})$ vanish altogether, as it follows from the $\mathbb{A}^1$-invariance of motivic cohomology and Borel's theorem on the K-theory of number fields (see Section 2.3.1). Nevertheless, it is easy to see that the two Deninger cycles γ_{L_2} and γ_{L_2} have a boundary, and it is indeed this boundary which is interesting. For example, the boundary of γ_{L_2} is given by the points $(\zeta_6,-1-\zeta_6)$ and $(\overline{\zeta_6},-1-\overline{\zeta_6})$, where $\zeta_6 := e^{\pi i/3} = (1+\sqrt{-3})/2$ is a primitive sixth root of unity. In particular, in this case we can view $\partial\gamma_{L_2}$ as a sub-scheme of V_{L_2}. Hence it makes sense to consider the relative long exact sequence in motivic cohomology

$$\cdots \longrightarrow H^{1,2}_{\mathcal{M}}(\partial\gamma_{L_2}) \xrightarrow{\delta} H^{2,2}_{\mathcal{M}}(V_{L_2},\partial\gamma_{L_2}) \longrightarrow H^{2,2}_{\mathcal{M}}(V_{L_2}) = 0 \longrightarrow \dots$$

from which we see that there exists a motivic cohomology class $\alpha_{L_2} \in H^{1,2}_{\mathcal{M}}(\partial\gamma_{L_2})$ such that $\delta(\alpha_{L_2}) = \{x, y\}$. Now, we can apply Theorem 4.3.4 (whose hypotheses are satisfied by L_2) to get:

$$m(L_2) = \langle r^{\infty}_{V_{L_2}}(\{x, y\}), \gamma_{L_2}\rangle_{\mathrm{per}} = \langle r^{\infty}_{\partial\gamma_{L_2}}(\alpha_{L_2}), \partial\gamma_{L_2}\rangle_{\mathrm{per}}$$

where the last equality follows from Stokes's theorem. Clearly the period pairing is simply given by the evaluation of a function at the points of $\partial\gamma_{L_2}$. This function, it turns out, is given by the Bloch-Wigner dilogarithm (see [Zag07, Chapter I, § 3]), and this allows one to compute that $m(L_2) = L'(\chi_{-3}, -1)$, as was shown by Smyth (see Theorem 4.2.4) with a different, analytically flavoured proof. We refer the interested reader to [Lal06, § 4] for the details of this new proof of Smyth's result. The key point to remember, which is also useful in our discussion, is that the restriction of the differential form η_2 (see Proposition 4.3.1) to V_{L_2} is exact, and a primitive is given by the Bloch-Wigner dilogarithm. Thus the regulator $r^{\infty}_{\partial\gamma_{L_2}}(\alpha_{L_2})$ is simply given by the restriction of the Bloch-Wigner dilogarithm function to $\partial\gamma_{L_2}$. We refer the interested reader to Vandervelde's work [Van08] for a formula which computes the Mahler measure of every genus zero polynomial $P(x, y) \in \mathbb{C}[x, y]$, which is automatically exact, and to Guilloux's and Marché's work [GM18], which computes a general formula for the Mahler measure of any exact two-variable polynomial, in terms of the primitive of the differential form η_2.

Moving on to the three-variable polynomial $L_3 = x + y + z + 1$, we see that $\partial\gamma_{L_3}$ is not zero-dimensional, but it is a closed path inside $(\mathbb{C}^{\times})^3$. It is at this point that Maillot's insight, which is epitomised in the next result, kicks in.

Proposition 5.1.1 – Maillot's trick

Let $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\}$ be a Laurent polynomial, and let $\gamma_P \subseteq (\mathbb{C}^{\times})^n$ be Deninger's cycle, which was defined in (4.24). Then we have that

$$\partial\gamma_P \subseteq V_P(\mathbb{C}) \cap \mathbb{T}^n \subseteq V_P(\mathbb{C}) \cap V_{P^\dagger}(\mathbb{C})$$

where $P^\dagger(x_1, \dots, x_n) := \overline{P(\overline{x}_1^{-1}, \dots, \overline{x}_n^{-1})}$ (see Definition 4.1.11).

Proof. Take $\mathbf{z} = (z_1, \dots, z_n) \in \partial\gamma_P$. Then $\mathbf{z} \in \mathbb{T}^n$, hence $z_j^{-1} = \overline{z}_j$ for every $j \in \{1, \dots, n\}$. This implies that

$$P^\dagger(\mathbf{z}) = \overline{P(\overline{z}_1^{-1}, \dots, \overline{z}_n^{-1})} = \overline{P(z_1, \dots, z_n)} = 0$$

which allows us to conclude. □

Remark 5.1.2. If $P \in \mathbb{C}[\mathbf{x}^{\pm 1}] \setminus \{0\}$ is written in multi-index notation as $P(\mathbf{x}) = \sum_{\mathbf{j}} a_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}$, then $P^\dagger(\mathbf{x}) = \sum_{\mathbf{j}} \overline{a_{\mathbf{j}}} \mathbf{x}^{-\mathbf{j}}$. Hence $P^\dagger = P^* := P(x_1^{-1}, \dots, x_n^{-1})$ for every polynomial $P \in \mathbb{R}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Now, the reason why Maillot's insight is so important is that one can consider cohomology and homology groups relative to $W_P := V_P \cap V_{P^\dagger}$ rather than $\partial\gamma_P \subseteq W_P$. More precisely, we clearly have that

$$\gamma_P \in H^{\mathrm{sing}}_{n-1,1-n}(V_P(\mathbb{C}), \partial\gamma_P) \subseteq H^{\mathrm{sing}}_{n-1,1-n}(V_P(\mathbb{C}), W_P)$$

and moreover the relative cohomology long exact sequence

$$\cdots \longrightarrow H^{n-1,n}_{\mathcal{D}}(W_P) \xrightarrow{\delta} H^{n,n}_{\mathcal{D}}(V_P, W_P) \longrightarrow H^{n,n}_{\mathcal{D}}(V_P) \longrightarrow H^{n,n}_{\mathcal{D}}(W_P) = 0 \longrightarrow \dots \tag{5.3}$$

shows that there exists a relative cohomology class $\omega_P \in H^{n,n}_{\mathcal{D}}(V_P, W_P)$ which maps to the cohomology class $r^{\infty}_{V_P}(\{x_1, \dots, x_n\}) \in H^{n,n}_{\mathcal{D}}(V_P)$. We note that here all the various varieties may be singular, and the Deligne-Beilinson cohomology groups are defined using the spectrum $\mathbb{DB}$ constructed in Example 2.4.6. Thus we see in particular that $H^{n,n}_{\mathcal{D}}(W_P) = H^{n-1,n-1}_{\mathrm{dR}}(W_P) = 0$, as follows from the fact that W_P is affine, together with resolution of singularities. We also observe that the class ω_P is not unique. Indeed, each cohomology class

$$\xi \in H^{n,n}_{\mathcal{D}}(V_P, W_P) = H^{n-1,n-1}_{\mathrm{dR}}(V_P, W_P)$$

is represented, at least in the case when both V_P and W_P are smooth, by a pair (α, β) consisting of an $(n-1)$-form α on V_P and a $(n-2)$-form β on W_P having the property that $\alpha\big|_{W_P} = d\beta$. Hence we see that ω_P can be represented by the pair $(\eta_n, \widetilde{\omega}_P)$, where $\widetilde{\omega}_P$ is any primitive of the restriction of η_n to W_P. In any case, we can see from Theorem 4.3.4 that

$$m(P) = \log|a_0| + \langle \omega_P, \gamma_P \rangle_{\mathrm{per}}$$

where now $\langle \cdot, \cdot \rangle_{\mathrm{per}}$ denotes the period pairing

$$\langle \cdot, \cdot \rangle_{\mathrm{per}} \colon H^{n,n}_{\mathcal{D}}(V_P, W_P) \times H^{\mathrm{sing}}_{n-1,1-n}(V_P(\mathbb{C}), W_P(\mathbb{C})) \to \mathbb{R}$$

given on cohomology and homology groups relative to W_P. Therefore, if the polynomial P is exact, we can use again the relative cohomology long exact sequence (5.3) to see that there exists a cohomology class $\alpha_P \in H^{n-1,n}_{\mathcal{D}}(W_P)$ such that $\delta(\alpha_P) = \omega_P$. Thus, one can use again Stokes's theorem, as we did in the previous paragraph for the two-variable polynomial $L_2(x, y) = x + y + 1$, to obtain the equality

$$m(P) = \log|a_0| + \langle \alpha_P, \partial\gamma_P \rangle_{\mathrm{per}}$$

which shows that the Mahler measure $m(P)$ is now related to an integral over the variety W_P, which is $(n-2)$-dimensional. To be more precise, we observe that W_P could be singular, in which case the pairing $\langle \cdot, \cdot \rangle_{\mathrm{per}}$ is not given by a single integral.

Going back to Smyth's three-variable polynomial $L_3(x, y, z) := x + y + z + 1$, we see that Proposition 5.1.1 allows one to relate $m(L_3)$ to a specific pairing over the "Maillot variety" $W_{L_3} := \{L_3 = L_3^* = 0\}$. We observe that the variety L_3 corresponds to the curve given by the equation

$$(x + y + 1)\left(\frac{1}{x} + \frac{1}{y} + 1\right) = 1$$

which turns out to be equivalent to the equation $(x + y)(x + 1)(y + 1) = 0$. Thus the variety $W_{L_3} \subseteq \mathbb{G}^3_m$ is the union of three lines, disposed in a triangle whose vertices are the points $W^{\mathrm{sing}}_{L_3} := \{(-1, 1, -1), (1, -1, -1), (-1, -1, 1)\}$, which form the singular locus of W_{L_3}. Since W_{L_3} is singular, we can consider its desingularisation $\widetilde{W}_{L_3}$, which consists of three disjoint rational lines, defined over $\mathbb{Q}$. Now, we see that $H^{2,3}_{\mathcal{M}}(\widetilde{W}_{L_3}) = H^{2,3}_{\mathcal{M}}(W^{\mathrm{sing}}_{L_3}) = 0$ because these motivic cohomology groups are related to the K-theory group $K_{2\cdot 3-2}(\mathbb{Q}) = K_4(\mathbb{Q})$, which is a torsion group (see Section 2.3.1). Analogously, and every more easily, one can show that the Deligne-Beilinson cohomology groups $H^{2,3}_{\mathcal{D}}(\widetilde{W}_{L_3})$ and $H^{2,3}_{\mathcal{D}}(W^{\mathrm{sing}}_{L_3})$ also vanish. Hence we can use the Mayer-Vietoris long exact sequence (see (2.20) for the motivic analogue):

$$\cdots \to H^{1,3}_{\mathcal{D}}(E_{L_3}) \to H^{2,3}_{\mathcal{D}}(W_{L_3}) \to H^{2,3}_{\mathcal{D}}(\widetilde{W}_{L_3}) \oplus H^{2,3}_{\mathcal{D}}(W^{\mathrm{sing}}_{L_3}) = 0 \to \dots$$

associated to the abstract blow-up square

$$\begin{array}{ccc} E_{L_3} & \hookrightarrow & \widetilde{W}_{L_3} \\ \downarrow & & \downarrow \\ W_{L_3}^{\mathrm{sing}} & \hookrightarrow & W_{L_3} \end{array}$$

to see that there exists $\beta_{L_3} \in H^{1,3}_{\mathcal{D}}(E_{L_3})$ which maps to α_{L_3} via the map $H^{1,3}_{\mathcal{D}}(E_{L_3}) \to H^{2,3}_{\mathcal{D}}(W_{L_3})$. This fact leads one to say that the polynomial L_3 is 2-*exact*. Indeed, the fact that L_3 was exact in the first place allowed us to choose a primitive of the restriction of η_3 to V_{L_3}, and this lead to the cohomology class $\alpha_P \in H^{2,3}_{\mathcal{D}}(W_{L_3})$. Now analogously, we see that the geometry of W_{L_3} (and in particular the fact that it is the union of three lines) allows us to take a primitive of α_P when "restricted" to the desingularisation $\widetilde{W}_P$ and to the singular locus W_P^{sing}. Taking a suitable difference of these primitives, and restricting it to the exceptional locus E_{L_3} leads to the cohomology class $\beta_{L_3} \in H^{1,3}_{\mathcal{D}}(E_{L_3})$. To conclude, one can see that Stokes's theorem gives a compatibility between the period pairing $\langle \cdot, \cdot \rangle_{\mathrm{per}}$ and Mayer-Vietoris long exact sequences, which can be used to show that $m(P)$ is related to a suitable "integral" of β_{L_3}. As in the case of L_2, this integral is just given by the evaluation of a suitable function on the six points which make up E_{L_3}. One can show that this function is given by the trilogarithm $\mathcal{L}_3$ (see [Lal06, Equation 8]), and thus one gets the link between the ζ-value $\zeta^*(-2)$ and the Mahler measure $m(L_3)$, which was proved in Theorem 4.2.4 using different, more analytic methods. We refer the interested reader to [Lal06, Theorem 8] for the detailed computation which shows that it is indeed $\mathcal{L}_3$ the right function to evaluate on the points forming E_{L_3}, and to [Lal05, § 5.4.1] for a complete proof of Smyth's theorem using these techniques. Finally, we remark that cohomological considerations similar to the ones appearing in this section appeared in the PhD thesis of Standfest (see [Sta01]).

5.2 Two approaches towards successive exactness

We have seen in the previous section how one can give a cohomological proof of Smyth's identities (see Theorem 4.2.4). This proof involves the notion of exactness of a polynomial, and we have seen that the polynomial $x + y + z + 1$ is 2-exact. This concept of *successive exactness* was firstly studied by Lalín in her PhD thesis. Her work initially focused on 2-exact polynomials in three variables (see [Lal05, § 5.2] and [Lal06, § 4]), and then laid down the bases for studying $(n-1)$-exact polynomials in n-variables, for $n \geq 4$ (see [Lal05, § 5.5, 5.8] and [Lal06, § 5, 6]).

The aim of this section is to introduce two conjectural geometric ways of saying when a polynomial $P \in \overline{\mathbb{Q}}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is k-exact, for some natural number $k \in \mathbb{N}$. In particular, one would always have that $0 \leq k \leq n$, and it is reasonable to expect that $k = n$ if and only if $m(P) = 0$. Moreover, usually one expects $m(P)$ to be related to the special value of some L-function at $s = n$, and this L-function should be associated to the cohomology of an $(n-k-1)$-dimensional variety. We see however in Section 5.4 that there are examples of proved identities which fail to meet this expectation.

5.2.1 The first approach: successive desingularisations

Fix throughout this section a Laurent polynomial $P \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with coefficients in a number field $K \subseteq \mathbb{C}$. Suppose that there exists $a_0 \in K^\times$ such that $P - a_0 \in x_n K[x_1, \ldots, x_n]$,

hence in particular that P has no denominators. This can be always achieved without changing the Mahler measure of P, as was shown in Lemma 4.3.2. Finally, we fix a triangulated category of mixed motives $\mathcal{T}$, in the sense of Section 2.2.3, which satisfies *cdh-descent*. For example we could take $\mathcal{T}$ to be the category of Beilinson motives $\mathrm{DM}_{\mathbb{B}}(K;\mathbb{Q})$, or $\mathcal{T}$ to be Ayoub's $\mathbb{A}^1$-homotopy category $\mathrm{DA}(K;\Lambda)$ for some ring Λ.

The first idea to define successive exactness for P is to take successive desingularisations of the complex affine variety $V_1 := V_P \cup V_{P^\dagger}$, where $P^\dagger$ denotes the conjugate reciprocal of P (see Definition 4.1.11) and $V_P, V_{P^\dagger} \hookrightarrow \widetilde{V}_0$ denote the corresponding (reduced) zero sub-schemes of $\widetilde{V}_0 := \mathbb{G}_m^n$. The embedding $\widetilde{\iota_0} : V_1 \hookrightarrow \widetilde{V}_0$ fits in a diagram of complex varieties

$$\begin{array}{ccccccccc}
V_N & \longrightarrow & \cdots & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\cdots & \longrightarrow & V_{j+1} & \xhookrightarrow{\widetilde{\iota_j}} & \widetilde{V}_j & & & & \\
 & & \downarrow{\scriptstyle p_{j+1}} & & \downarrow{\scriptstyle \pi_j} & & & & \\
 & & W_{j+1} & \xhookrightarrow{\iota_j} & V_j & \longrightarrow & \cdots & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \cdots & \longrightarrow & V_3 & \xhookrightarrow{\widetilde{\iota_2}} & \widetilde{V}_2 \\
 & & & & & & \downarrow{\scriptstyle p_3} & & \downarrow{\scriptstyle \pi_2} \\
 & & & & & & W_3 & \xhookrightarrow{\iota_2} & V_2 \xhookrightarrow{\widetilde{\iota_1}} \widetilde{V}_1 \\
 & & & & & & & & \downarrow{\scriptstyle p_2} \quad \downarrow{\scriptstyle \pi_1} \\
 & & & & & & & & W_2 \xhookrightarrow{\iota_1} V_1 \xhookrightarrow{\widetilde{\iota_0}} \widetilde{V}_0
\end{array} \tag{5.4}$$

where $W_{j+1} := V_j^{\mathrm{sing}}$ and $\widetilde{V}_j$ is always smooth. Moreover, we can demand that each square is an *abstract blow-up*, by which we mean that each square is Cartesian, π_j is proper and induces an isomorphism $\pi_j^{-1}(V_j^{\mathrm{reg}}) \to V_j^{\mathrm{reg}}$. If we demand this, then there is a preferred way of constructing such a diagram, where the only ambiguity lies in the choice of $\widetilde{V}_1$. More precisely, we can take $\widetilde{V}_1$ to be an embedded resolution of singularities of V_1, in such a way that $V_2 \hookrightarrow \widetilde{V}_1$ is a divisor with simple normal crossings. Then $V_2 = \bigcup_{i=1}^r D_r$ and $W_3 = V_2^{\mathrm{sing}} = \bigcup_{i\neq j}(D_i \cap D_j)$. We can hence take $\widetilde{V}_2 := \bigsqcup_{i=1}^r D_r$, which implies that $V_3 = \bigsqcup_{i=1}^r \bigcup_{j\neq i}(D_i \cap D_j)$. We go on by taking

$$\begin{aligned}
V_j &:= \bigsqcup_{\substack{\mathbf{a}\in\{1,\dots,r\}^{j-2}\\ \#\sigma(\mathbf{a})=j-2}} \bigcup_{\substack{I\subseteq\{1,\dots,r\}\setminus\sigma(\mathbf{a})\\ \#I=1}} D_{\sigma(\mathbf{a})\cup I'} \\
\widetilde{V}_j &:= \bigsqcup_{\substack{\mathbf{b}\in\{1,\dots,r\}^{j-1}\\ \#\sigma(b)=j-1}} D_{\sigma(b)} = \bigsqcup_{\substack{\mathbf{a}\in\{1,\dots,r\}^{j-2}\\ \#\sigma(\mathbf{a})=j-2}} \bigsqcup_{\substack{I\subseteq\{1,\dots,r\}\setminus\sigma(\mathbf{a})\\ \#I=1}} D_{\sigma(\mathbf{a})\cup I} \\
W_{j+1} &:= \bigsqcup_{\substack{\mathbf{a}\in\{1,\dots,r\}^{j-2}\\ \#\sigma(\mathbf{a})=j-2}} \bigcup_{\substack{J\subseteq\{1,\dots,r\}\setminus\sigma(\mathbf{a})\\ \#J=2}} D_{\sigma(\mathbf{a})\cup J}
\end{aligned}$$

where for every subset $I \subseteq \{1,\dots,r\}$ we define $D_I := \bigcap_{i\in I} D_i$. Here, we use the notation $\sigma(\mathbf{a}) \subseteq \{1,\dots,r\}$ for the set of elements of a tuple $\mathbf{a}$. In particular, the conditions on the

cardinality $\#\sigma(\mathbf{a})$ simply mean that we allow only tuples without repetitions. We observe that this explicit way of defining the diagram (5.4) shows that $N \leq n$, because $V_2 \hookrightarrow \widetilde{V}_1$ is a divisor with normal crossings.

We are now ready to construct suitable cohomology classes which allow us to give a tentative definition of the notion of successive exactness. To do so, fix a motivic ring spectrum $\mathbb{E} \in \mathcal{T}$ (*i.e.* a monoid object in the category $\mathcal{T}$, in the sense of [KS06, Remark 4.3.2]). As we have seen in Section 2.4, examples of these kinds of objects are given by the motivic cohomology spectrum $\mathbb{1}_{\mathcal{T}}$ and (for suitable categories $\mathcal{T}$) by the Deligne-Beilinson cohomology spectrum $\mathbb{DB}$. Since $\mathcal{T}$ satisfies cdh-descent, we have one Mayer-Vietoris sequence

$$\begin{array}{c} \cdots \longrightarrow H^{n-j}(W_{j+1};\mathbb{E}) \oplus H^{n-j}(\widetilde{V}_j;\mathbb{E}) \xrightarrow{p_{j+1}^* - \widetilde{\iota_j}^*} H^{n-j}(V_{j+1};\mathbb{E}) \\ \xrightarrow{\delta} \\ H^{n-(j-1)}(V_j;\mathbb{E}) \xrightarrow{\iota_j^* \oplus \pi_j^*} H^{n-(j-1)}(W_{j+1};\mathbb{E}) \oplus H^{n-(j-1)}(\widetilde{V}_j;\mathbb{E}) \longrightarrow \cdots \end{array} \tag{5.5}$$

associated to each abstract blow-up appearing in the diagram (5.4). Here the $\mathbb{E}$-cohomologies are defined as $H^i(X;\mathbb{E}) := \mathrm{Hom}_{\mathcal{T}}(X, \mathbb{E}[i])$, so that $H^{i,j}_{\mathcal{M}}(X) \cong H^i(X; \mathbb{1}_{\mathcal{T}}(j))$ and analogously for Deligne-Beilinson cohomology. Now, we have a class $\widetilde{\alpha_0} \in H^n(\widetilde{V}_0;\mathbb{E}(n))$ given by the regulator of $\{x_1, \ldots, x_n\} \in H^{n,n}_{\mathcal{M}}(\mathbb{G}_m^n)$ with respect to $\mathbb{E}$ (see Section 2.4). We can then say that a polynomial P is 1-*exact* (with respect to the cohomology theory given by $\mathbb{E}$) if

$$\pi_1^*(\alpha_1) = \iota_1^*(\alpha_1) = 0$$

where $\alpha_1 := \widetilde{\iota_0}(\widetilde{\alpha_0})$. Then the Mayer-Vietoris long exact sequence (5.5) can be used to construct a class $\alpha_2 \in H^{n-1}(V_2;\mathbb{E}(n))$ such that $\delta(\alpha_2) = \alpha_1$, up to an ambiguity coming from the cohomology groups $H^{n-1}(\widetilde{V}_1;\mathbb{E}(n)) \oplus H^{(n-1)}(W_2;\mathbb{E}(n))$.

Now, we have two possible definitions of the notion of k-exactness, for $k \geq 2$:

- we can say that the polynomial is 2-exact if $\pi_2^*(\alpha_2) = \iota_2^*(\alpha_2) = 0$, and this would allow us to construct a class $\alpha_3 \in H^{n-2}(V_3;\mathbb{E})$ such that $\delta(\alpha_3) = \alpha_2$. There are two problems with this approach:
  1. a priori, being 2-exact depends on the ambiguity in the definition of α_2;
  2. there is again an ambiguity in the definition of α_3.

  Nevertheless, we can move upwards in the diagram at each step, and define a notion of k-exactness in this way. More precisely, we say that P is k-*exact* if it is $(k-1)$-exact and $\pi_k^*(\alpha_k) = \iota_k^*(\alpha_k) = 0$. This definition is probably not well posed in general, since it depends on the ambiguities in the choice of the spectrum $\mathbb{E}$, of the diagram (5.4) and of the cohomology classes $\alpha_2, \ldots, \alpha_k$. We note that $k \leq N \leq n$;
- we can use the Mayer-Vietoris spectral sequence for the simple normal crossings divisor $V_2 \hookrightarrow \widetilde{V}_1$ to get a descending filtration $\mathrm{Fil}^\bullet_{\mathrm{M.V.}}$ on the cohomology group $H^{n-1}(V_2;\mathbb{E})$. Then we say that our polynomial P is k-exact (for $k \geq 2$) if it is 1-exact and we have that $\alpha_1 \in \mathrm{Fil}^{k-1}_{\mathrm{M.V.}}(H^{n-1}(V_2;\mathbb{E}))$. This definition has the advantage of being canonical, and again we necessarily have that $k \leq N \leq n$.

Let us turn to the original problem of relating the Mahler measure $m(P)$ to special values of L-functions. Our objective is to obtain a relation similar to (4.33), which allows one to compare $m(P)$ with some regulator integrals. We fix $\mathbb{E} = \mathbb{DB}$ to be the Deligne-Beilinson spectrum, until the end of this section. Suppose first of all that P is 0-exact, which is in fact an empty

assumption. Then we have the cohomology class $\alpha_1 \in H^{n,n}_{\mathcal{D}}(V_1)$, which is given by the regulator of $\{x_1, \dots, x_n\} \in H^{n,n}_{\mathcal{M}}(V_1)$. Now, we expect to be able to generalise Theorem 4.3.4 and get an identity of the form

$$m(P) \stackrel{?}{=} \log|a_0| + \frac{1}{2}\langle \alpha_1, \gamma_1 \rangle_{\mathrm{per}} \tag{5.6}$$

where $\alpha_1 := \widetilde{\iota_0}^*(\widetilde{\alpha_0}) \in H^{n,n}_{\mathcal{D}}(V_1)$ and

$$\gamma_1 := (2\pi i)^{1-n} \otimes [\{(x_1, \dots, x_n) \in (\mathbb{C}^\times)^n \mid |x_1| = \cdots = |x_{n-1}| = 1\} \cap V_1(\mathbb{C})] \in H^{\mathrm{sing}}_{n-1,1-n}(V_1(\mathbb{C}))$$

denotes a "symmetrised" version of Deninger's cycle defined in (4.24), which can be equivalently defined as $\gamma_1 = \gamma_P \cup \gamma_{P^\dagger} = \gamma_P \cup \gamma_{P^*}$. The pairing appearing in (5.6) is given by the period isomorphism

$$\langle \cdot, \cdot \rangle_{\mathrm{per}} \colon H^{n,n}_{\mathcal{D}}(V_1) \otimes H^{\mathrm{sing}}_{n-1,1-n}(V_1(\mathbb{C})) \to \mathbb{C}$$

using the identification $H^{n,n}_{\mathcal{D}}(V_1) = H^{n-1,n-1}_{\mathrm{dR}}(V_1)$ provided by the fact that V_1 is n-dimensional. We note in particular that V_1 is generally singular, but this is not a problem as we are taking all the cohomology theories to be defined by applying the motivic formalism (see [HM17, § 5.4, 5.5] for a related discussion). We expect that the cohomological methods employed in the proof of [BD99, Proposition 2.2] should be a key tool to prove the equality (5.6).

Suppose now that the polynomial P is 1-exact, so that we can find $\alpha_2 \in H^{n-1,n}_{\mathcal{D}}(V_2)$ such that $\alpha_1 = \delta(\alpha_2)$. Then Stokes's theorem shows that (5.6) becomes

$$m(P) \stackrel{?}{=} \log|a_0| + \frac{1}{2}\langle \alpha_2, \gamma_2 \rangle_{\mathrm{per}} \tag{5.7}$$

where $\gamma_2 \in H^{\mathrm{sing}}_{n-2,1-n}(V_2)$ is defined as $\gamma_2 := \partial(\gamma_1)$, using the boundary map ∂ in the sequence

$$\cdots \longrightarrow H^{\mathrm{sing}}_{n-(j-1),1-n}(W_{j+1}) \oplus H^{\mathrm{sing}}_{n-(j-1),1-n}(\widetilde{V}_j) \xrightarrow{(\iota_j)_* - (\pi_j)_*} H^{\mathrm{sing}}_{n-(j-1),1-n}(V_j) \xrightarrow{\partial} H^{\mathrm{sing}}_{n-j,1-n}(V_{j+1}) \xrightarrow{(p_{j+1})_* \oplus (\widetilde{\iota_j})_*} H^{\mathrm{sing}}_{n-j,1-n}(W_{j+1}) \oplus H^{\mathrm{sing}}_{n-j,1-n}(\widetilde{V}_j) \longrightarrow \cdots$$

which is the Mayer-Vietoris long exact sequence for singular homology. The period pairing appearing in (5.7) is given by

$$\langle \cdot \rangle_{\mathrm{per}} \colon H^{n-1,n}_{\mathcal{D}}(V_2) \otimes H^{\mathrm{sing}}_{n-2,1-n}(V_2(\mathbb{C})) \to \mathbb{C}$$

which is again well defined due to the identification $H^{n-1,n}_{\mathcal{D}}(V_2) \cong H^{n-2,n-1}_{\mathrm{dR}}(V_2)$, which holds because V_2 is $(n-1)$-dimensional. We observe that the number $\langle \alpha_2, \gamma_2 \rangle_{\mathrm{per}}$ does not depend on the ambiguities in the definition of α_2.

It is now clear how the pattern should continue for k-exact polynomials. More precisely, if P is k-exact, an iterated application of Stokes's theorem, together with the initial formula (5.6), shows that

$$m(P) \stackrel{?}{=} \log|a_0| + \frac{1}{2}\langle \alpha_{k+1}, \gamma_{k+1} \rangle_{\mathrm{per}} \tag{5.8}$$

where $\alpha_{k+1} \in H^{n-(k+1),n}_{\mathcal{D}}(V_k)$ is any Deligne-Beilinson cohomology class such that $\delta(\alpha_{k+1}) = \alpha_k$, and $\gamma_{k+1} \in H^{\mathrm{sing}}_{n-k,1-n}(V_k(\mathbb{C}))$ is defined by induction as $\gamma_{k+1} = \partial(\gamma_k)$.

We conclude this section with an intriguing observation about the relations between the exactness of a polynomial P and the geometry of the complex points of V_P. Let us say that a polynomial $P \in K[x_1, \dots, x_n]$, satisfying the hypotheses mentioned at the beginning of this section, has *exactness index* $e(P) \in \mathbb{N}$ if P is $e(P)$-exact and not $(e(P)+1)$-exact. On the other hand, let us say that P has *closedness number* $c(P) \in \mathbb{N}$ if $\gamma_{c(P)} \neq 0$ and $\gamma_{c(P)+1} = 0$. Then, if $m(P) \neq \log|a_0|$, and we believe that (5.8) should hold, it is natural to ask the following question.

Question 5.2.1 – Are polynomials more closed than exact?

Let $P \in K[x_1, \dots, x_n]$ be a polynomial, defined over a number field $K \subseteq \mathbb{C}$, such that $m(P) \notin \log|K^\times|$. Is it true that $e(P) < c(P)$?

We observe that $c(P) \leq r \leq n$, where r is the number of components of V_2 as a simple normal crossings divisor inside $\widetilde{V}_1$. Moreover, Question 5.2.1 has a positive answer for $n = 2$ and $e(P) = 1$, as was shown by Guilloux and Marché in [GM18, Theorem 2.10].

5.2.2 The second approach: relative cohomology

The aim of this section is to describe briefly a second possible approach to the definition of successive exactness, which has the advantage of being completely canonical. We fix the same notation that we used in Section 5.2.1: $P \in a_0 + x_n K[x_1, \dots, x_n]$ is a polynomial with coefficients in a number field $K \subseteq \mathbb{C}$, and $\mathbb{E}$ is a motivic spectrum, *i.e.* an object in a fixed triangulated category of mixed motives $\mathcal{T}$ which has cdh descent. In addition to the hypotheses made in Section 5.2.1, we assume as well that $\mathbb{E}$ is constructed applying Theorem 2.4.2 to a family of complexes of Nisnevich sheaves $\{E_j^\bullet : j \in \mathbb{N}\}$ which are *pseudo-flasque*, in the sense of [BKK07, Definition 5.26]. The key property of pseudo-flasque complexes of sheaves is that, for any scheme X, one can compute the hypercohomology of the complex as the cohomology of its global sections (see [BKK07, Proposition 5.27]). The key example for us is once again the Deligne-Beilinson cohomology spectrum $\mathbb{E} = \mathbb{DB}$. Indeed, Deligne-Beilinson cohomology can be defined from the complexes of sheaves $E_j^\bullet := \mathcal{D}^\bullet_{\log}(-, j)$ introduced by Burgos Gil in [Bur94] (see Example 2.1.22 and Example 2.4.6). The complexes $\mathcal{D}^\bullet_{\log}(-, j)$ turn out to be pseudo-flasque, as it is proved by Burgos Gil, Kramer and Kühn in [BKK07, Proposition 5.29].

Let us now go back to the notion of exactness. First of all, we take again

$$V_1 := V_P \cup V_{P^\dagger} \hookrightarrow \widetilde{V}_0 := \mathbb{G}_m^n$$

to be the sub-variety of $\mathbb{G}_m^n$ defined by the equation $P = P^\dagger = 0$, where $P^\dagger$ denotes the conjugate reciprocal of P (see Definition 4.1.11). As before, we fix a desingularisation $\widetilde{V}_1 \twoheadrightarrow V_1$, which fits in an abstract blow-up square

$$\begin{array}{ccc} V_2 & \xhookrightarrow{\widetilde{\iota_1}} & \widetilde{V}_1 \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi_1} \\ W_2 & \xhookrightarrow{\iota_1} & V_1 \end{array} \tag{5.9}$$

where $W_2 := V_1^{\mathrm{sing}}$ and $V_2 \hookrightarrow \widetilde{V}_1$ is a divisor with simple normal crossings. Once again, we let $\widetilde{\alpha_0} \in H^n(\widetilde{V}_0; \mathbb{E}(n))$ denote the $\mathbb{E}$-regulator of $\{x_1, \dots, x_n\} \in H^{n,n}_{\mathcal{M}}(\widetilde{V}_0)$, and we denote by $\alpha_1 \in H^n(V_1; \mathbb{E}(n))$ the restriction of α_0 along $\widetilde{\iota_0}$.

From now on, the new approach differs from the old one. First of all, one notices that W_2 is an affine, $(n-2)$-dimensional variety, and as such it is plausible to expect that $H^n(W_2;\mathbb{E}(n)) = 0$ (compare with [Mil80, Theorem 7.2] and [SP, Proposition 0F0V]). This is certainly true for Deligne-Beilinson cohomology, by comparing it with de Rham cohomology. In any case, if indeed we assume that $H^n(W_2;\mathbb{E}(n)) = 0$, then we can use the long exact sequence in relative cohomology

$$\cdots \longrightarrow H^{n-1}(W_2;\mathbb{E}(n)) \longrightarrow H^n(V_1,W_2;\mathbb{E}(n)) \longrightarrow H^n(V_1;\mathbb{E}(n)) \longrightarrow H^n(W_2;\mathbb{E}(n)) = 0 \longrightarrow \dots$$

to lift the cohomology class $\alpha_1 \in H^n(V_1;\mathbb{E}(n))$ to a relative class $\alpha_{\mathrm{rel}} \in H^n(V_1,W_2;\mathbb{E}(n))$. Let us point out that this lifting is not unique, with the ambiguity coming from $H^{n-1}(W_2;\mathbb{E}(n))$. We can also observe now that since the square (5.9) is an abstract blow-up, the corresponding restriction map

$$H^n(V_1,W_2;\mathbb{E}(n)) \longrightarrow H^n(\widetilde{V}_1,V_2;\mathbb{E}(n))$$

is an isomorphism. Hence α_{rel} can be restricted to become a class $\omega_{\mathrm{rel}} \in H^n(\widetilde{V}_1,V_2;\mathbb{E}(n))$, without losing any information.

Now, we can use our assumption that $\mathbb{E}$ comes from a family of pseudo-flasque complexes of sheaves $E_j^\bullet$. Indeed, this assumption allows one to compute the relative cohomology group $H^n(\widetilde{V}_1,V_2;\mathbb{E}(n))$ as the cohomology of an explicit complex. More precisely, let us write

$$V_2 = D_1 \cup \cdots \cup D_r$$

as a union of its components, and let us set $D^0 := \widetilde{V}_1$ and

$$D^p := \bigsqcup_{\substack{I\subset\{1,\dots,r\}\\ |I|=p}} \bigcap_{i\in I} D_i$$

for every integer $p \in \mathbb{Z}_{\geq 1}$. Then the cohomology groups $E_n^q(D^p)$ can be arranged in a double complex $E_n^\bullet(D^\bullet)$, whose differentials are induced by the ones of E_n in the vertical direction, and by an alternating sum of restriction maps in the horizontal one. Then the i-th cohomology of the total complex $\mathrm{Tot}(E_n^\bullet(D^\bullet))$ computes indeed the relative cohomology group $H^i(\widetilde{V}_1,V_2;\mathbb{E})$. We refer the reader interested in the details, and in particular in the explicit shape of the differentials of the double complex $E_n^\bullet(D^\bullet)$, to [BF, Construction 2.72], which focuses on algebraic de Rham cohomology.

The use of computing the relative cohomology group $H^n(\widetilde{V}_1,V_2;\mathbb{E}_n)$ as the cohomology of the total complex of a double complex is that one can use the spectral sequence of the double complex (see [SP, Section 012X]) to obtain a decreasing filtration $\mathrm{Fil}^\bullet_{\mathrm{rel}}$ on the cohomology group $H^n(\widetilde{V}_1,V_2;\mathbb{E}(n))$. In particular, $\mathrm{Fil}^0_{\mathrm{rel}}(H^n(\widetilde{V}_1,V_2;\mathbb{E}(n))) = H^n(\widetilde{V}_1,V_2;\mathbb{E}(n))$. Using the explicit representation of the relative cohomology group $H^n(\widetilde{V}_1,V_2;\mathbb{E}(n))$ that comes out of this approach, one sees that it makes a lot of sense to say that the polynomial P is *k-exact* (with respect to $\mathbb{E}$) for some $k \in \mathbb{N}$, if the cohomology class $\omega_{\mathrm{rel}} \in H^n(\widetilde{V}_1,V_2;\mathbb{E}(n))$ lies in the subspace $\mathrm{Fil}^k_{\mathrm{rel}}(H^n(\widetilde{V}_1,V_2;\mathbb{E}(n)))$. This definition, as we said in the introduction, is almost canonical: the only ambiguity comes from the definition of ω, *i.e.* from the cohomology group $H^{n-1}(W_2;\mathbb{E}(n))$.

To conclude, let us observe that using this framework it is also much easier to describe the connection to the Mahler measure. To do this, we fix once again $\mathbb{E} = \mathbb{D}\mathbb{B}$ to be the Deligne-Beilinson cohomology spectrum. Then, the Deninger cycle defined in (4.24) can be seen as a relative homology class inside $H^{\mathrm{sing}}_{n-1,1-n}(V_1(\mathbb{C}),W_2(\mathbb{C}))$. Moreover, one can lift this class through

the abstract blow-up (5.9), to get a relative homology class $\gamma_{\mathrm{rel}} \in H^{\mathrm{sing}}_{n-1,1-n}(\widetilde{V_1}(\mathbb{C}), V_2(\mathbb{C}))$. Now, Theorem 4.3.4 should admit the following generalisation

$$m(P) = \log|a_0| + \langle \omega_{\mathrm{rel}}, \gamma_{\mathrm{rel}} \rangle_{\mathrm{per}}$$

where $\langle \cdot, \cdot \rangle_{\mathrm{per}} \colon H^{n,n}_{\mathcal{D}}(\widetilde{V_1}, V_2) \otimes H^{\mathrm{sing}}_{n-1,1-n}(\widetilde{V_1}(\mathbb{C}), V_2(\mathbb{C})) \to \mathbb{C}$ denotes a relative version of the period pairing. Let us note that, using the explicit description of the relative cohomology group $H^{n,n}_{\mathcal{D}}(\widetilde{V_1}(\mathbb{C}), V_2(\mathbb{C}))$, one should be able to express the number $\langle \omega_{\mathrm{rel}}, \gamma_{\mathrm{rel}} \rangle_{\mathrm{per}}$ as an alternating sum of integrals defined over the smooth varieties D^p. Finally, we observe that Question 5.2.1 has clearly an analogue in this setting. Indeed, we can say that P has *exactness index* $e(P) \in \mathbb{N}$ if $\omega_{\mathrm{rel}} \in \mathrm{Fil}^{e(P)}_{\mathrm{rel}} \setminus \mathrm{Fil}^{e(P)+1}_{\mathrm{rel}}$, and we can say that the polynomial P has *closedness number* $c(P) \in \mathbb{N}$ if $\gamma_{\mathrm{rel}} \in \mathrm{Fil}^{\mathrm{rel}}_{n-c(P)} \setminus \mathrm{Fil}^{\mathrm{rel}}_{n-(c(P)+1)}$, where $\mathrm{Fil}^{\mathrm{rel}}_{\bullet}$ denotes the increasing filtration induced on the homology group $H_{n-1,1-n}(\widetilde{V_1}(\mathbb{C}), V_2(\mathbb{C}))$.

5.3 An explicit computation for $X_1(15)$

The aim of this section is to review an explicit application of the ideas outlined in Section 5.2. More precisely, we are concerned with the identity (5.2), which was shown to hold up to a rational number in [Lal15]. The key point here is that this identity involves the L-function associated to $X_1(15)$, which is both an elliptic curve and a modular curve. Thus one could use modular techniques to construct elements in the motivic cohomology group $H^{2,3}_{\mathcal{M}}(X_1(15))$, whose regulators should be related to the L-value $L^*(X_1(15), -1)$. This was done by Beilinson in [Bei86a], although his construction is somehow implicit. Beilinson's construction has then been made more explicit in the work of Brunault (see [Bru07] and [Bru17]) for the special values at $s = 2$, using pairs of Siegel units. Suitably chosen triplets of modular units can be used to explicitly construct elements inside $H^{2,3}_{\mathcal{M}}(X_1(15))$ using the polylogarithmic motivic complexes that we introduced in Section 2.3.3, which have been proved to be related to $H^{2,3}_{\mathcal{M}}(X_1(15))$ by the work of Goncharov (see [Gon96]).

Suppose now that we have indeed constructed an explicit element $\mu \in H^{2,3}_{\mathcal{M}}(X_1(15))$ and an explicit homology class $\varepsilon \in H^{\mathrm{sing}}_{1,2}(X_1(15)(\mathbb{C}))$ such that $\langle \mu, \varepsilon \rangle_{\mathrm{per}} = L^*(X_1(15), -1)$. Then, in order to prove (5.2), one can try to apply the techniques outlined in Section 5.2 to construct another explicit motivic cohomology element whose regulator is related to the Mahler measure. Since both elements are explicitly defined, it is then not unreasonable to look for an explicit comparison between the two at the level of motivic cohomology. More precisely, following the approach outlined in Section 5.2.1, we would like to construct a motivic cohomology class $\alpha_2 \in H^{2,3}_{\mathcal{M}}(X_1(15))$ whose regulator is related to the Mahler measure $m(P)$.

Let us see how to construct such a class explicitly and unconditionally. To do so, we use the comparison between motivic cohomology and higher Chow groups that was outlined in Section 2.3.2. This gives us an explicit sheaf of cochain complexes of $\mathbb{Q}$-vector spaces $\mathcal{Z}^{\bullet,3}$, whose cohomology computes the motivic cohomology groups $H^{\bullet,3}_{\mathcal{M}}$. In particular, the motivic cohomology class $\{x, y, z\} \in H^{3,3}_{\mathcal{M}}(\mathbb{G}^3_m)$ given by the cup product of the three classes $\{x\}, \{y\}, \{z\} \in H^{1,1}_{\mathcal{M}}(\mathbb{G}^3_m) \cong \mathcal{O}^{\times}(\mathbb{G}^3_m) \otimes_{\mathbb{Z}} \mathbb{Q}$ is represented in the group $\mathcal{Z}^{3,3}(\mathbb{G}^3_m)$ by the class of the graph $\Gamma_{x,y,z} \subseteq \mathbb{G}^3_m \times \square^3$ of these three functions, where $\square := \mathbb{P}^1 \setminus \{1\}$. In general, the elements of $\mathcal{Z}^{i,3}(X)$ are represented by sub-varieties of codimension three inside $X \times \square^{6-i}$.

Now, let us recall that $V_1 \hookrightarrow \mathbb{G}^3_m$ is given by the union of the two smooth varieties V_P and V_{P^*} (notice that $P^{\dagger} = P^*$ because P has real coefficients), which intersect transversely. Hence

$W_2 := V_1^{\mathrm{sing}}$ is just given by the intersection $W_2 = V_P \cap V_{P^*}$, and the desingularisation $\widetilde{V_1} \twoheadrightarrow V_1$ is given by the disjoint union $\widetilde{V_1} = V_P \sqcup V_{P^*}$. Since V_P and V_{P^*} are both rational and defined over the number field $\mathbb{Q}$, we see that $H^{3,3}_{\mathcal{M}}(\widetilde{V_1}) = 0$. This is enough to show that P is 1-exact (with respect to motivic cohomology), because the restriction map $\iota_1^* \colon H^{3,3}_{\mathcal{M}}(\mathbb{G}_m^3) \to H^{3,3}_{\mathcal{M}}(W_2)$ factors through $H^{3,3}_{\mathcal{M}}(V_P) = 0$. Thus, we can indeed aim at constructing a motivic cohomology class $\alpha_2 \in H^{2,3}_{\mathcal{M}}(V_2)$ such that $\delta(\alpha_2) = \alpha_1$. Using the explicit description of $H^{2,3}_{\mathcal{M}}(V_2)$ as the second cohomology of the complex $\mathcal{Z}^{\bullet,3}(V_2)$, one sees that a particularly good way of representing α_2 is by taking the difference of two primitives (in the complex $\mathcal{Z}^{\bullet,3}$) of the restriction of the cycle $\Gamma_{x,y,z} \subseteq \mathbb{G}_m^3 \times \square^3$ to the subvarieties $V_P, V_{P^\dagger} \hookrightarrow \mathbb{G}_m^3$. This leads to a closed cycle η representing α_2. Doing this concretely in our situation amounts to the following explicit computation.

Proposition 5.3.1 – An explicit Maillot cycle

Let $P(x,y,z) := z - (1-x)(1-y)$, and let $V_P, V_{P^*} \hookrightarrow \mathbb{G}_m^3$ be the zero loci of P and its reciprocal $P^*(x,y,z) := P(x^{-1}, y^{-1}, z^{-1})$. We denote also by $W := V_P \cap V_{P^*}$ the intersection of these two sub-schemes.

Let $\square := \mathbb{P}^1 \setminus \{1\}$, whose coordinate is denoted by t. Using this notation, we can introduce the closed sub-scheme

$$\eta_1 := \left\{ \left(x, y, t, \frac{(1-x)t - (1-x)(1-y)}{t - (1-x)(1-y)} \right) \right\} \hookrightarrow W \times \square^4$$

given by those pairs $(P, (t_1, \dots, t_4)) \in W \times \square^4$ such that $t_1 = x(P)$ and $t_2 = y(P)$, as well as

$$t_4 = \frac{(1-x)t - (1-x)(1-y)}{t - (1-x)(1-y)} \tag{5.10}$$

where $t := t_3$. Using a similar notation, we define the following closed immersions:

$$\begin{aligned}
\eta_2 &= \left\{ \left(x, t, 1-t, 1 - \frac{y}{t} \right) \right\} \hookrightarrow W \times \square^4 \\
\eta_3 &= \left\{ \left(t, y, 1-t, 1 - \frac{x}{t} \right) \right\} \hookrightarrow W \times \square^4 \\
\eta_4 &= \left\{ \left(x, y, t, \frac{(1-(1-x)^{-1})t - (1-(1-x)^{-1})(1-(1-y)^{-1})}{t - (1-(1-x)^{-1})(1-(1-y)^{-1})} \right) \right\} \hookrightarrow W \times \square^4 \\
\eta_5 &= \left\{ \left(x, \frac{1}{t}, \frac{1}{1-t}, 1 - \frac{1}{yt} \right) \right\} \hookrightarrow W \times \square^4 \\
\eta_6 &= \left\{ \left(\frac{1}{t}, y, \frac{1}{1-t}, 1 - \frac{1}{xt} \right) \right\} \hookrightarrow W \times \square^4
\end{aligned} \tag{5.11}$$

which are the constituents of the cycle $\eta := \eta_1 - \eta_2 - \eta_3 - (\eta_4 - \eta_5 - \eta_6) \in \mathcal{Z}^{2,3}(W)$.

Then the cycle η has the property that $\delta([\eta]) = \alpha_1$, where $\alpha_1 \in H^{3,3}_{\mathcal{M}}(V_P \cup V_{P^*})$ denotes the cup product of the coordinate function symbols $\{x\}, \{y\}, \{z\} \in H^{1,1}_{\mathcal{M}}(V_P \cup V_{P^*})$, and

$[\eta] \in H^{2,3}_{\mathcal{M}}(W)$ denotes the class of η in the cohomology of the complex $\mathcal{Z}^{\bullet,3}(W)$. More precisely, we have that

$$\partial(\eta_1 - \eta_2 - \eta_3) = \{(x,y,z)\}\big|_W = \partial(\eta_4 - \eta_5 - \eta_6) \tag{5.12}$$

where $\partial\colon \mathcal{Z}^{2,3}(W) \to \mathcal{Z}^{3,3}(W)$ denotes the differential of the cochain complex $\mathcal{Z}^{\bullet,3}(W)$.

Proof. Using the explicit shape of the differential ∂ introduced in (2.23) we see that

$$\begin{aligned}
\partial(\eta_1 - \eta_2 - \eta_3) = {} & (\eta_1\big|_{x=0} - \eta_1\big|_{x=\infty}) + (\eta_1\big|_{y=\infty} - \eta_1\big|_{y=0}) \\
& + (\eta_1\big|_{t=0} - \eta_1\big|_{t=\infty}) + (\eta_1\big|_{f=\infty} - \eta_1\big|_{f=0}) \\
& + (\eta_2\big|_{x=\infty} - \eta_2\big|_{x=0}) + (\eta_2\big|_{t=0} - \eta_2\big|_{t=\infty}) \\
& + (\eta_2\big|_{1-t=\infty} - \eta_2\big|_{1-t=0}) + (\eta_2\big|_{1-y/t=0} - \eta_2\big|_{1-y/t=\infty}) \\
& + (\eta_3\big|_{t=\infty} - \eta_3\big|_{t=0}) + (\eta_3\big|_{y=0} - \eta_3\big|_{y=\infty}) \\
& + (\eta_3\big|_{1-t=\infty} - \eta_3\big|_{1-t=0}) + (\eta_3\big|_{1-x/t=0} - \eta_3\big|_{1-x/t=\infty}) = (\dagger)
\end{aligned}$$

where $f\colon W \times \square \to \mathbb{P}^1$ denotes the function

$$f(x,y,z,t) := \frac{(1-x)t - (1-x)(1-y)}{t - (1-x)(1-y)}$$

already introduced in (5.10). Observe now that the following equations

$$\begin{aligned}
(1-x)(1-y) &= (1-(1-x)^{-1})(1-(1-y)^{-1}) \\
xy &= (1-x)^2(1-y)^2 \\
(1-x)^{-1}(1-y)^{-1} &= (1-x^{-1})(1-y^{-1})
\end{aligned} \tag{5.13}$$

hold on W. Moreover, since $W \hookrightarrow \mathbb{G}_m^3$ we have that

$$W \cap \{x=0\} = W \cap \{y=0\} = W \cap \{x=\infty\} = W \cap \{y=\infty\} = \emptyset \tag{5.14}$$

and we also have that $W \cap \{x=1\} = W \cap \{y=1\} = \emptyset$, which follows immediately from a combination of (5.13) with (5.14). Finally, we know that all the coordinates in all the cycles are $\neq 1$, because by definition $\square := \mathbb{P}^1 \setminus \{1\}$. Thus we see that the expression $(\dagger)$ can be hugely simplified, to give

$$\begin{aligned}
(\dagger) = {} & -\eta_1\big|_{t=\infty} + \eta_1\big|_{f=\infty} - \eta_1\big|_{f=0} \\
& + \eta_2\big|_{1-y/t=0} + \eta_3\big|_{1-x/t=0} = \\
& = -\{(x,y,1-x)\} + \{(x,y,(1-x)(1-y))\} - \{(x,y,1-y)\} \\
& + \{(x,y,1-y)\} + \{(x,y,1-x)\} = \\
& = \{(x,y,(1-x)(1-y))\} = \{(x,y,z)\}
\end{aligned}$$

using (5.13). This shows the first equality appearing in (5.12).

The proof of the second equality appearing in (5.12) is essentially analogous, but we give it for the sake of completeness. First of all, we see that

$$\begin{aligned}\partial(\eta_4-\eta_5-\eta_6) &= (\eta_4\big|_{x=0}-\eta_4\big|_{x=\infty})+(\eta_4\big|_{y=\infty}-\eta_4\big|_{y=0})\\ &+(\eta_4\big|_{t=0}-\eta_4\big|_{t=\infty})+(\eta_4\big|_{g=\infty}-\eta_4\big|_{g=0})\\ &+(\eta_5\big|_{x=\infty}-\eta_5\big|_{x=0})+(\eta_5\big|_{t^{-1}=0}-\eta_5\big|_{t^{-1}=\infty})\\ &+(\eta_5\big|_{(1-t)^{-1}=\infty}-\eta_5\big|_{(1-t)^{-1}=0})+(\eta_5\big|_{h=0}-\eta_5\big|_{h=\infty})\\ &+(\eta_6\big|_{t^{-1}=\infty}-\eta_6\big|_{t^{-1}=0})+(\eta_6\big|_{y=0}-\eta_6\big|_{y=\infty})\\ &+(\eta_6\big|_{(1-t)^{-1}=\infty}-\eta_6\big|_{(1-t)^{-1}=0})+(\eta_6\big|_{j=0}-\eta_6\big|_{j=\infty})=(\heartsuit)\end{aligned}$$

where the function $g\colon W\times\square\to\mathbb{P}^1$ is given by the expression

$$g(x,y,z,t):=\frac{(1-(1-x)^{-1})t-(1-(1-x)^{-1})(1-(1-y)^{-1})}{t-(1-(1-x)^{-1})(1-(1-y)^{-1})}$$

and the functions $h, j\colon W\times\square\to\mathbb{P}^1$ are defined as

$$h(x,y,z,t):=1-\frac{1}{yt}\quad\text{and}\quad j(x,y,z,k):=1-\frac{1}{xt}.$$

Note that all these expressions already appear in the definition of the varieties η_4, η_5 and η_6 (see Equation (5.11)). As before, we can apply (5.13) and (5.14) to simplify enormously the expression $(\heartsuit)$, and we get

$$\begin{aligned}(\heartsuit) &= -\eta_4\big|_{t=\infty}+\eta_4\big|_{g=\infty}-\eta_4\big|_{g=0}+\eta_5\big|_{h=0}+\eta_6\big|_{j=0}=\\ &=-\{(x,y,1-(1-x)^{-1})\}-\{(x,y,1-(1-y)^{-1})\}\\ &+\{(x,y,(1-(1-x)^{-1})(1-(1-y)^{-1}))\}\\ &+\{(x,y,(1-y^{-1})^{-1})\}+\{(x,y,(1-x^{-1}))^{-1}\}=\\ &=\{(x,y,(1-(1-x)^{-1})(1-(1-y)^{-1}))\}=\{(x,y,(1-x)(1-y))\}=\{(x,y,z)\}\end{aligned}$$

using (5.13) together the fact that $1-(1-x)^{-1}=(1-x^{-1})^{-1}$ and $1-(1-y)^{-1}=(1-y^{-1})^{-1}$. This concludes the proof. □

Remark 5.3.2. Explicit computations similar to the ones carried out in Proposition 5.3.1 can be found in the works of Zhao [Zha07] and Petras [Pet09].

Now, the next step is to show that the class $[\eta]\in H^{2,3}_{\mathcal{M}}(V_2)$ constructed in Proposition 5.3.1 is actually the restriction of a motivic cohomology class $\alpha_2\in H^{2,3}_{\mathcal{M}}(X_1(15))$ defined on the modular curve $X_1(15)$. This follows from the fact that the motivic cohomology class $\alpha_1\in H^{3,3}_{\mathcal{M}}(V_P)$ has the same property, since P is tempered. However, the details remain to be fully worked out.

Finally, one has to relate $\alpha_2\in H^{2,3}_{\mathcal{M}}(X_1(15))$ to the class $\mu\in H^{2,3}_{\mathcal{M}}(X_1(15))$ mentioned in the introduction. One possible approach is to use strongly the explicit nature of the two classes, and try to relate them for instance in the polylogarithmic motivic group $\mathcal{B}^{2,3}(X_1(15))$, using some computationally expensive linear algebra. This will be the subject of future investigations.

5.4 A catalogue of identity types

We conclude this chapter by mentioning a possible expansion of the ideas described so far. More precisely, we have already seen that Maillot's trick (see Proposition 5.1.1) can be crucially used to explain relations between Mahler measures and special values of L-functions which go beyond the framework of Question 4.2.9. It is therefore useful to take a step back, and analyse the plethora of relations between Mahler measures and special values of L-functions that have been proved or conjectured to hold. We are particularly interested in a qualitative study, which highlights the common features unifying different identities. Let us mention a few of these types, which we were able to encounter in the literature:

- we have relations of the form

$$\frac{L^*(\underline{H}^{n-1}(V_P),0)}{m(P)} \in \mathbb{Q}^\times \quad \text{or} \quad \frac{L^*(\underline{H}^{n-1}(\widetilde{V_P}),0)}{m(P)} \in \mathbb{Q}^\times \tag{5.15}$$

  for some Laurent polynomials $P \in \mathbb{Q}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ (see Question 4.2.9). Here V_P is the zero locus of P inside $\mathbb{G}_m^n$, and $\widetilde{V_P}$ is a desingularisation of a good compactification of V_P. Usually, when relations like (5.15) hold one has that $\widetilde{V_P}$ is a Calabi-Yau variety (*e.g.* an elliptic curve, or a $K3$ surface), and the polynomial P is tempered (see Definition 4.2.7). We refer the interested reader to Section 4.2 for a history of the subject, focusing on polynomials in two variables;
- sometimes, relations of the form (5.15) hold after replacing $\underline{H}^{n-1}(V_P)$ with a suitable sub-motive. This can be the case, for example, when $P \in \mathbb{Q}[x^{\pm 1},y^{\pm 1}]$ is a two-variable polynomial giving a (possibly singular) plane model of a smooth and proper curve $\widetilde{V_P}$ of genus $g(\widetilde{V_P}) \geq 2$, whose Jacobian $\mathrm{Jac}(\widetilde{V_P})$ has a one-dimensional factor in its Poincaré decomposition (see Theorem 7.1.1). We refer the reader to the work of Bertin and Zudilin [BZ16; BZ17] and Lalín and Wu [LW18; LW20] for proved examples of these identities, and to the work of Liu and Qin [LQ19] for infinite families of conjectural ones;
- the Mahler measure of a polynomial is sometimes related to zero-dimensional L-functions. Examples of this type of relations include Smyth's results (see Theorem 4.2.4), as well as the infinite families of results proved by Lalín in [Lal06], and some results proved in Condon's thesis (see [Con04, Chapter 2]) and in work of Rogers (see [Rog06]);
- more generally, the Mahler measure of a polynomial in n variables is sometimes numerically related to an L-function associated to an object whose dimension is strictly less than $n-1$. Typical examples of this include the three-variable relations numerically discovered by Boyd in the talk [Boy06], which were investigated further in the work of Lalín [Lal15], as well as the relations involving the family of linear polynomials $P_n(x_1,\dots,x_n) := x_1+\cdots+x_n+1$, which were examined by Rodriguez-Villegas following Maillot's talk in Banff (see [Boy+03, § 8] and [BZ20, § 6.2]);
- some Mahler measures simply evaluate to be logarithms of algebraic numbers, as we have seen for example in Section 4.4.2.
- finally, sometimes one sees linear combinations of the previous types appearing. Identities of this kind have been investigated already in Boyd's seminal paper (see for instance [Boy98, Page 76]), and then in the thesis of Bornhorn (see [Bor99, § 5.6] and [Bor15, § 4]).

Remark 5.4.1. We observe that often one expects the Mahler measure of a polynomial in n variables to be related to L-values at $s = n$. However, the equality

$$m((1-x_1)(1-x_2)-(1-x_3)(1-x_4)) = \frac{9}{2\pi^2}\zeta(3) = -18\zeta'(-2)$$

computed by D'Andrea and Lalín (see [DL07, Theorem 7]), shows that this expectation fails to be true in general.

Now, it would be extremely interesting (in the author's opinion) to find *a complete list* of these types, *i.e.* a complete classification of the types of identities that may occur between Mahler measures and special values of L-functions. We imagine that such a *type* would be given in the form of a natural number $r \in \mathbb{N}$ and triple $(\mathbf{n}, \mathbf{w}, \mathbf{d}) \in (\mathbb{Z}^r)^2 \times \mathbb{N}^r$, where r represents the number of special values appearing in the relation, $\mathbf{n}$ represents the set of integers at which these special values are taken, and the two vectors $\mathbf{w}$ and $\mathbf{d}$ represent respectively the weights and the dimensions of the motives involved.

The next step would be to device an *algorithm* which takes as input a Laurent polynomial $P \in \overline{\mathbb{Q}}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and outputs the type $(r, \mathbf{n}, \mathbf{w}, \mathbf{d})$ associated to this polynomial P. We note in passing that such a type might not be unique, and that in order to have only a finite list of types associated to each polynomial one should not count different identities which are "trivially equal", such as the trivial relation

$$m(P) = L^*(M, 0) = L^*(M, 0) + L^*(N, -1) - L^*(N, -1)$$

or identities coming from functional equations. How to make this precise still remains a challenging open question.

Finally, we remark that this problem is fundamentally related to the problem of associating to each polynomial $P \in \overline{\mathbb{Q}}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ a motive M_P which has $m(P)$ as a period (see Remark 4.3.6). More precisely, a possible approach to understand the combinatorics of identity types would be to find suitable ways to decompose the motive M_P, according to the "motivic Meccano" briefly described by Serre in [Ser91, Page 339].

Ray class fields for orders

6

The moon swam back,
its rays all silvered,
and time and again the darkness
would be broken.

Pablo Neruda, *It is born*

The aim of this chapter is to introduce the notion of *ray class field* associated to data relative to an order $\mathcal{O} \subseteq F$ inside a number field F. More precisely, we describe how to construct ray class fields $H_{\mathfrak{m}}$ associated to a generalised module $\mathfrak{m} := (\mathcal{O}, I, \mathfrak{m}_\infty)$, where $\mathcal{O} \subseteq \mathcal{O}_F$ is an order, $I \subseteq \mathcal{O}$ is a non-zero ideal and $\mathfrak{m}_\infty \subseteq M_F^\infty$ is a collection of Archimedean places. This generalises the usual notion of ray class field, as we point out in Remark 6.2.13. We define $H_{\mathfrak{m}}$ using the idelic approach to class field theory (see Definition 6.2.11). This definition is then related to the classical language of class field theory in Theorem 6.2.17 and Remark 6.2.18.

This chapter is based on joint work in progress with Francesco Campagna, part of which appeared in [CP20, Appendix A]. The current aim of this project is to develop the theory of ray class field for orders, which is outlined in this chapter, and to generalise to this context some results already present in the literature for the usual notion of ray class field. Examples of this include the analysis of the ramification behaviour in these ray class fields (including a computation of the different and discriminant), and the computational results appearing in [CS08]. These results are in all likelihood known to the experts, but to the authors' knowledge their proofs have never been collected in a single place. Our work, of which this chapter represents the first version, aims at filling this gap.

We point out that the original interest of the authors in ray class fields for orders arose from the aim of giving an adelic proof of Theorem 7.2.5. This result, which is well known for elliptic curves having complex multiplication by the maximal order $\mathcal{O}_K$ of an imaginary quadratic field K (see for example [Sil94, Chapter II, Theorem 5.6]), was originally proved by Söhngen in his PhD thesis (see [Söh35] and [Sch10, Theorem 6.2.3]), using the classical language of class field theory.

Despite the fact that our original interest was only confined to orders in imaginary quadratic fields, there are at least two reasons to engage with the development of a general theory of ray class fields for orders:

- Theorem 7.2.5 has been generalised to abelian varieties of each dimension by Shimura and Taniyama (see [ST61, Main Theorem 3]), still using the classical language of class field theory employed by Söhngen. We point out that in this more general case it is not true anymore that the division fields associated to an abelian variety A with complex multiplication by an order $\mathcal{O}$ inside a CM field K contain some ray class fields (relative to this order $\mathcal{O}$), because one has to take into account the *reflex norm* associated to the CM type induced on K by A. Nevertheless, developing a general, adelic theory of ray class fields for orders is a key step towards providing a completely adelic proof of the aforementioned result of Shimura and Taniyama;

- the ring class field of an order O, which is a generalisation of the Hilbert class field associated to a number field F, has been studying recently in two works by Lv and Deng (see [LD15]) and by Yi and Lv (see [YL18]). They show how to apply this study to find criteria for the solubility of equations of the form $\mathrm{N}_{F/\mathbb{Q},\mathcal{B}}(\mathbf{x}) = y$, where $\mathbf{x} = (x_1, \dots, x_d)$ is a vector of indeterminates having length equal to the degree $d = [F\colon \mathbb{Q}]$ of a number field F, and $\mathrm{N}_{F/\mathbb{Q},\mathcal{B}}(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$ denotes the polynomial corresponding to the field norm $\mathrm{N}_{F/\mathbb{Q}}\colon F^\times \to \mathbb{Q}^\times$, computed with respect to a given integral basis $\mathcal{B}$. As we point out in Remark 6.2.14, our ray class fields for orders generalise the ring class fields defined by Lv and Deng, and may provide further insight to the study of different kinds of Diophantine equations.

Let us conclude this introduction by outlining the contents of this chapter: Section 6.1.1 provides the necessary background on the set of lattices contained in a given number field F, which is endowed with a natural action of the idèle group $\mathbb{A}_F^\times$ (see Section 6.1.2). This group is also the source of the global Artin map $[\cdot, F]\colon \mathbb{A}_F^\times \twoheadrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F)$. This map, which is a surjective group homomorphism, allows one to describe the abelian extensions of F in terms of subgroups of the group of idèles $\mathbb{A}_F^\times$, as we recall in Section 6.1.3. Moreover, Section 6.2 contains the heart of this chapter, consisting of the idelic definition of ray class fields for orders, which is related to the classical language of class field theory in Theorem 6.2.17. Finally, Section 6.2 contains also various computations of the Galois group of the extension $F \subseteq H_{\mathfrak{m}}$ and of suitable sub-extensions.

6.1 Lattices, idèles and class field theory

As we stated in the introduction, the aim of this section is to collect some background material on lattices in number fields and the group of idèles, together with the relations between them and the idelic version of class field theory.

6.1.1 Lattices

This short section collects some definitions and crucial properties of lattices in number fields. Let us start by giving the following general definition.

Definition 6.1.1 – Lattices

Let κ be a field of characteristic zero, endowed with a sub-ring $R \subseteq \kappa$, and let V be a vector space over κ. An *R-lattice* is a free R-module $\Lambda \subseteq V$ such that $V = \Lambda \otimes_R \kappa$. We denote by $\mathcal{L}(V; R)$ the set of R-lattices inside the vector space V, and by $\mathcal{L}(V)$ the set of $\mathbb{Z}$-lattices.

Remark 6.1.2. There are at least two other notions of lattices present in the literature, namely:

- an additive subgroup $\Lambda \subseteq E$ inside an $\mathbb{R}$-vector space E endowed with an inner product $\langle \cdot, \cdot \rangle\colon E \times E \to \mathbb{R}$ is called a *lattice* if the metric $d\colon E \times E \to \mathbb{R}$ defined as

$$d(x, y) := \sqrt{\langle x - y, x - y \rangle}$$

  induces the discrete topology on Λ;
- a pair (Λ, φ) consisting of an abelian group Λ endowed with a symmetric bilinear form $\varphi\colon \Lambda \otimes \Lambda \to \mathbb{R}$.

These two notions coincide as soon as Λ has finite rank (see [Len08, § 2]). We note that a lattice $\Lambda \subseteq E$ inside an Euclidean space E is not required to satisfy $\mathrm{rk}_{\mathbb{Z}}(\Lambda) = \dim_{\mathbb{R}}(E)$, whereas we have $\mathrm{rk}_R(\Lambda) = \dim_\kappa(V)$ for every $\Lambda \subseteq V$ which is a lattice in the sense of Definition 6.1.1. Despite this difference, the aforementioned two notions of lattice coincide otherwise with the one given in Definition 6.1.1, when V is finite dimensional and we take $\kappa = \mathbb{R}$ and $R = \mathbb{Z}$.

Fix a field κ of characteristic zero, a sub-ring $R \subseteq \kappa$ and two κ-vector spaces V and V'. Let us introduce a few operations that can be performed on the R-lattices contained in V and V':

Direct sum There is a map $\oplus\colon \mathcal{L}(V;R) \times \mathcal{L}(V';R) \to \mathcal{L}(V \oplus V';R)$ which sends a pair of lattices $\Lambda \subseteq V$ and $\Lambda' \subseteq V'$ to their direct sum $\Lambda \oplus \Lambda' \subseteq V \oplus V'$ (as a R-module). This is again an R-lattice, generated by the direct sum $\mathcal{B} \oplus \mathcal{B}' := \{b \oplus b' : b \in \mathcal{B}, b' \in \mathcal{B}'\}$ of the bases $\mathcal{B} \subseteq V$ and $\mathcal{B}' \subseteq V'$ which generate Λ and Λ'.

Tensor product There is a map $\otimes\colon \mathcal{L}(V;R) \times \mathcal{L}(V';R) \to \mathcal{L}(V \otimes V';R)$ taking a pair of lattices $\Lambda \subseteq V$ and $\Lambda' \subseteq V'$ to their tensor product $\Lambda \otimes_R \Lambda'$ (as R-modules), which coincides with the R-lattice generated inside $V \otimes_\kappa V'$ by the tensor product $\mathcal{B} \otimes \mathcal{B}' := \{b \otimes b' : b \in \mathcal{B}, b' \in \mathcal{B}'\}$ of the bases $\mathcal{B} \subseteq V$ and $\mathcal{B}' \subseteq V'$ which generate Λ and Λ'.

Homomorphisms There is a function $\mathrm{Hom}\colon \mathcal{L}(V;R) \times \mathcal{L}(V';R) \to \mathcal{L}(\mathrm{Hom}_\kappa(V,V');R)$ mapping a pair of R-lattices $\Lambda \subseteq V$ and $\Lambda' \subseteq V'$ to $\mathrm{Hom}_R(\Lambda,\Lambda')$. Fix two bases $\mathcal{B} \subseteq V$ and $\mathcal{B}' \subseteq V'$ which generate Λ and Λ'. Then the isomorphism

$$\mathrm{Hom}_R(\Lambda,\Lambda') \cong \{f \in \mathrm{Hom}_\kappa(V,V') \mid f(\Lambda) \subseteq \Lambda'\} \subseteq \mathrm{Hom}_\kappa(V,V')$$

shows that the R-module $\mathrm{Hom}_R(\Lambda,\Lambda')$ can be identified with the lattice generated inside $\mathrm{Hom}_\kappa(V,V')$ by the basis

$$\mathrm{Hom}(\mathcal{B},\mathcal{B}') := \{f_{v,v'} : v \in \mathcal{B}, v' \in \mathcal{B}'\}$$

where $f_{v,v'}\colon V \to V'$ is defined by setting $f_{v,v'}(v) := v'$ and $f_{v,v'}(w) := 0$ for every $w \in \mathcal{B} \setminus \{v\}$.

Dual lattice As a special case of the previous one, there is a map $\mathcal{L}(V) \to \mathcal{L}(V^\vee)$ sending a lattice $\Lambda \subseteq V$ to its dual $\Lambda^\vee \subseteq V^\vee$.

Base-change Consider a commutative square

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ \kappa & \longrightarrow & \kappa' \end{array}$$

where κ and κ' are two fields of characteristic zero, endowed with sub-rings $R \subseteq \kappa$ and $R' \subseteq \kappa'$. For every κ-vector space V we have a base-change map $\mathcal{L}(V;R) \to \mathcal{L}(V \otimes_\kappa \kappa';R')$ which sends a R-lattice $\Lambda \subseteq V$ to the R'-lattice $\Lambda_{R'} := \Lambda \otimes_R R'$.

We now restrict our attention to finitely generated vector spaces V, and to sub-rings $R \subseteq \kappa$ such that $\kappa = \mathrm{Frac}(R)$. This is due to the following easy result.

Proposition 6.1.3 – Intersections and images of lattices

Let κ be a field of characteristic zero, endowed with a sub-ring $R \subseteq \kappa$ such that $\kappa = \mathrm{Frac}(R)$. Fix a finite dimensional κ-vector space V. Then any pair of R-lattices $\Lambda_1, \Lambda_2 \in \mathcal{L}(V;R)$ is commensurable, *i.e.* there exist $\alpha, \beta \in \kappa^\times$ such that $\alpha\Lambda_2 \subseteq \Lambda_1 \subseteq \beta\Lambda_2$.

Suppose now that R is a principal ideal domain. Then:

- for every subspace $W \subseteq V$ and every R-lattice $\Lambda \in \mathcal{L}(V;R)$ the intersection $\Lambda \cap W$ is a R-lattice inside W;
- for every surjective map $f\colon V \twoheadrightarrow V'$ and every R-lattice $\Lambda \in \mathcal{L}(V;R)$, the image $f(\Lambda) \subseteq V'$ is again a R-lattice.

Proof. To prove the first assertion we can assume without loss of generality that $V = \kappa^n$ and $\Lambda_1 = R^n$. Then $\Lambda_2 = M \cdot R^n$ for some $M \in \mathrm{GL}_n(\kappa)$, hence there exists $\alpha \in R$ such that $\alpha M \in \mathrm{Mat}_n(R)$, which implies that $\alpha\Lambda_2 \subseteq \Lambda_1$. We can conclude, swapping Λ_1 and Λ_2 in the previous discussion, that there exists $\beta \in \kappa$ such that $\beta^{-1} \in R$ and $\Lambda_1 \subseteq \beta\Lambda_2$.

Suppose now that R is a principal ideal domain. Then:

- to prove the first point in the list we can assume that $V = \kappa^n$ and $W = \kappa^m$ for some $m \leq n$, where $\kappa^m \hookrightarrow \kappa^n$ is the inclusion of the first m coordinates. Then $\Lambda \cap \kappa^m$ is a torsion-free sub-module of κ^m, which is free because R is a principal ideal domain (see for instance [SP, Lemma 0AUW]). Moreover, we observe that $\mathrm{rk}_R(\Lambda \cap \kappa^m) = m$ because there exists $\alpha \in R$ such that $\alpha \cdot R^m \subseteq \Lambda$, hence

$$\alpha \cdot \mathbb{Z}^m = \alpha \cdot R^n \cap \kappa^m \subseteq \Lambda \cap \kappa^m$$

  which shows that $\mathrm{rk}_R(\Lambda \cap \kappa^m) \geq m$. This allows us to conclude because $\mathrm{rk}_R(M) \leq \dim_\kappa(V)$ for every free R sub-module $M \subseteq \kappa$;
- to prove the second point we proceed analogously. More precisely, $f(\Lambda) \subseteq V'$ is a free module over R (again by [SP, Lemma 0AUW]) and contains a κ-basis of V (because f is surjective). This is enough to conclude that $f(\Lambda) \in \mathcal{L}(V';R)$, as before.

□

Fix now a principal ideal domain of characteristic zero R and let $\kappa := \mathrm{Frac}(R)$. We can use Proposition 6.1.3 to define two new kinds of operations on R-lattices contained in a finite dimensional κ-vector space V, endowed with a κ-bilinear pairing $\psi\colon V \otimes V \to V$:

Internal sum there is a map $+\colon \mathcal{L}(V;R) \times \mathcal{L}(V;R) \to \mathcal{L}(V;R)$ sending a pair of R-lattices $\Lambda_1, \Lambda_2 \subseteq V$ to their internal sum $\Lambda_1 + \Lambda_2 \subseteq V$ (as R-modules). This is again a free R-module by [SP, Lemma 0AUW], and it is a lattice because it obviously contains a basis.

Internal product if ψ is surjective we can define a map $\cdot_\psi\colon \mathcal{L}(V;R) \times \mathcal{L}(V;R) \to \mathcal{L}(V;R)$ by setting $\Lambda_1 \cdot_\psi \Lambda_2 := \psi(\Lambda_1 \otimes_{\mathbb{Z}} \Lambda_2)$, which is a lattice in virtue of Proposition 6.1.3.

Quotient if the map $\widetilde{\psi}\colon V \to \mathrm{End}(V)$ induced by ψ is injective, we can define a quotient operation $(\cdot\colon\cdot)_\psi\colon \mathcal{L}(V;R) \times \mathcal{L}(V;R) \to \mathcal{L}(V;R)$ by setting

$$(\Lambda_1 : \Lambda_2)_\psi := \widetilde{\psi}^{-1}\left(\mathrm{Hom}(\Lambda_2, \Lambda_1) \cap \widetilde{\psi}(V)\right)$$

for any pair of R-lattices $\Lambda_1, \Lambda_2 \subseteq V$. We observe that $(\Lambda_1 \colon \Lambda_2)_\psi$ is again a lattice thanks to Proposition 6.1.3.

To conclude, we want to apply the previous discussion to number fields. Indeed, every number field F is a finite dimensional $\mathbb{Q}$-vector space, endowed with a $\mathbb{Q}$-bilinear map $\psi \colon F \otimes_{\mathbb{Q}} F \to F$ given by $\psi(x, y) := x \cdot y$. We see immediately that ψ is surjective (because F is a unital ring) and that $\widetilde{\psi} \colon F \to \mathrm{End}_{\mathbb{Q}}(F)$ is injective (because F is an integral domain). Hence the set of $\mathbb{Z}$-lattices $\mathcal{L}(F)$ supports the following three operations:

$$\begin{aligned} \Lambda_1 + \Lambda_2 &:= \{a + b \colon a \in \Lambda_1,\ b \in \Lambda_2\} \subseteq F \\ \Lambda_1 \cdot \Lambda_2 &:= \Lambda_1 \cdot_\psi \Lambda_2 = \left\{ \sum_{i=1}^{r} a_i b_i \mid a_i \in \Lambda_1, b_i \in \Lambda_2 \right\} \subseteq F \\ (\Lambda_1 \colon \Lambda_2) &:= (\Lambda_1 \colon \Lambda_2)_\psi = \{x \in F \mid x\Lambda_2 \subseteq \Lambda_1\} \subseteq F \end{aligned}$$

which satisfy the natural associativity and distributivity properties. Moreover, we observe that for every lattice $\Lambda \subseteq F$ the quotient $\mathcal{O}_\Lambda := (\Lambda \colon \Lambda) \subseteq \mathcal{O}_F$ is an order, in the sense of Definition 6.2.1. The lattice Λ then becomes a fractional ideal for $\mathcal{O}_\Lambda$, which is invertible if F is an imaginary quadratic field (see [Cox13, Proposition 7.4]). Finally, we conclude this section by recalling a result about the behaviour of lattices under localisation and completion (see [Lan87, Chapter 8, § 1]).

Definition 6.1.4 – Localisation and completion of lattices

Let R be an integral domain of characteristic zero, endowed with a prime ideal $\mathfrak{p} \subseteq R$, and denote by $R_{(\mathfrak{p})} \subseteq R_{\mathfrak{p}}$ the localisation and the completion of R at the prime $\mathfrak{p}$. Let $K := \mathrm{Frac}(R)$ and fix V to be a vector space over K. Then for every prime $\mathfrak{p} \subseteq R$ we write $V_{\mathfrak{p}} := V \otimes_K \mathrm{Frac}(R_{\mathfrak{p}})$ and we associate to every lattice $\Lambda \in \mathcal{L}(V; R)$ two lattices

$$\Lambda_{(\mathfrak{p})} := \Lambda \otimes_R R_{(\mathfrak{p})} \qquad \text{and} \qquad \Lambda_{\mathfrak{p}} := \Lambda \otimes_R R_{\mathfrak{p}} = \Lambda_{(\mathfrak{p})} \otimes_{R_{(\mathfrak{p})}} R_{\mathfrak{p}}$$

which are the $\mathfrak{p}$-*localisation* $\Lambda_{(\mathfrak{p})} \in \mathcal{L}(V; R_{(\mathfrak{p})})$ and the $\mathfrak{p}$-*completion* $\Lambda_{\mathfrak{p}} \in \mathcal{L}(V_{\mathfrak{p}}; R_{\mathfrak{p}})$ of Λ.

Lemma 6.1.5 – Behaviour of lattices under localisation and completion

Let R be a domain of characteristic zero, and let V be a vector space over $K := \mathrm{Frac}(R)$. Then for every pair of lattices $\Lambda, \Lambda' \in \mathcal{L}(V; R)$ we have that:

$$\Lambda \subseteq \Lambda' \Leftrightarrow \Lambda_{(\mathfrak{p})} \subseteq \Lambda'_{(\mathfrak{p})},\ \forall \mathfrak{p} \in \mathrm{Spec}(R) \Leftrightarrow \Lambda_{\mathfrak{p}} \subseteq \Lambda'_{\mathfrak{p}},\ \forall \mathfrak{p} \in \mathrm{Spec}(R) \tag{6.1}$$

and for every lattice $\Lambda \in \mathcal{L}(V; R)$ we have that:

$$\Lambda = \bigcap_{\mathfrak{p}} \Lambda_{(\mathfrak{p})} \quad \text{and} \quad \frac{V}{\Lambda} \cong \bigoplus_{\mathfrak{p}} \frac{V}{\Lambda_{(\mathfrak{p})}} \cong \bigoplus_{\mathfrak{p}} \frac{V_{\mathfrak{p}}}{\Lambda_{\mathfrak{p}}} \tag{6.2}$$

where $\mathfrak{p}$ runs over all the primes of R.

Proof. It is immediate to see that for every prime $\mathfrak{p} \subseteq R$ we have that

$$\Lambda_{(\mathfrak{p})} \subseteq \Lambda'_{(\mathfrak{p})} \Leftrightarrow \Lambda_{\mathfrak{p}} \subseteq \Lambda'_{\mathfrak{p}}$$

because $\Lambda_{(\mathfrak{p})} = \Lambda_{\mathfrak{p}} \cap V \subseteq V_{\mathfrak{p}}$ and $\Lambda_{\mathfrak{p}} = \Lambda_{(\mathfrak{p})} \otimes_{R_{(\mathfrak{p})}} R_{\mathfrak{p}}$. Hence showing the implication

$$\Lambda_{(\mathfrak{p})} \subseteq \Lambda'_{(\mathfrak{p})},\ \forall \mathfrak{p} \in \mathrm{Spec}(R) \Rightarrow \Lambda \subseteq \Lambda' \tag{6.3}$$

is sufficient to prove (6.1). Since $\Lambda \subseteq \Lambda_{(\mathfrak{p})}$ for every prime $\mathfrak{p} \subseteq R$, we see that for every $x \in \Lambda$ there exist two collections of elements $\{x_{\mathfrak{p}}, r_{\mathfrak{p}} : \mathfrak{p} \in \mathrm{Spec}(R)\}$ such that for every prime $\mathfrak{p} \subseteq R$ we can write $x = x_{\mathfrak{p}}/r_{\mathfrak{p}}$ with $x_{\mathfrak{p}} \in \Lambda'$ and $r_{\mathfrak{p}} \in R \setminus \mathfrak{p}$. This last condition implies in particular that there exists a sequence of elements $\{s_{\mathfrak{p}}\}_{\mathfrak{p}} \subseteq R$, with $s_{\mathfrak{p}} = 0$ for all but finitely many primes $\mathfrak{p} \subseteq R$, such that $\sum_{\mathfrak{p}} r_{\mathfrak{p}} s_{\mathfrak{p}} = 1$. Thus we see immediately that

$$x = x \cdot 1 = \sum_{\mathfrak{p}} s_{\mathfrak{p}}(x r_{\mathfrak{p}}) = \sum_{\mathfrak{p}} s_{\mathfrak{p}} x_{\mathfrak{p}} \in \Lambda'$$

which shows that $x \in \Lambda'$, and allows us to conclude the proof of (6.3).

Take now $x \in V$ such that $x \in \Lambda_{(\mathfrak{p})}$ for every prime ideal $\mathfrak{p} \subseteq R$. Then we can choose again $\{x_{\mathfrak{p}}, r_{\mathfrak{p}}, s_{\mathfrak{p}} : \mathfrak{p} \subseteq \mathrm{Spec}(R)\}$ such that we can write $x = x_{\mathfrak{p}}/r_{\mathfrak{p}}$ and $1 = \sum_{\mathfrak{p}} r_{\mathfrak{p}} s_{\mathfrak{p}}$, with $x_{\mathfrak{p}} \in \Lambda$ and $r_{\mathfrak{p}}, s_{\mathfrak{p}} \in R$ having the property that $s_{\mathfrak{p}} \in R \setminus \mathfrak{p}$ and $s_{\mathfrak{p}} = 0$ for all but finitely many primes $\mathfrak{p} \subseteq R$. Thus once again we have that $x = \sum_{\mathfrak{p}} s_{\mathfrak{p}} x_{\mathfrak{p}} \in \Lambda$, which shows that

$$\Lambda = \bigcap_{\mathfrak{p}} \Lambda_{(\mathfrak{p})} \tag{6.4}$$

because $\Lambda \subseteq \Lambda_{(\mathfrak{p})}$ for every prime $\mathfrak{p} \subseteq R$. Finally, (6.4) shows that

$$\frac{V}{\Lambda} \cong \bigoplus_{\mathfrak{p}} \frac{V}{\Lambda_{(\mathfrak{p})}}$$

and, since $\Lambda_{(\mathfrak{p})} = \Lambda_{\mathfrak{p}} \cap V \subseteq V_{\mathfrak{p}}$ for every prime $\mathfrak{p} \subseteq R$, we have that

$$\bigoplus_{\mathfrak{p}} \frac{V}{\Lambda_{(\mathfrak{p})}} \cong \bigoplus_{\mathfrak{p}} \frac{V_{\mathfrak{p}}}{\Lambda_{\mathfrak{p}}}$$

which allows us to conclude. □

6.1.2 Idelic multiplication on lattices

The aim of this section is to describe the action of the group of idèles $\mathbb{A}_F^\times$ on the set of lattices $\mathcal{L}(F)$ contained in a number field F. First of all, let us recall the definition of the adèle ring $\mathbb{A}_F$, and of its group of units $\mathbb{A}_F^\times$.

Definition 6.1.6 – Adèles and idèles

Let F be a number field. Then we define the *adèle ring* $\mathbb{A}_F$ as the restricted product

$$\mathbb{A}_F := \prod_{w \in M_F}{}' F_w = \left\{ s = (s_w)_{w \in M_F} \in \prod_{w \in M_F} F_w \,\middle|\, s_w \in \mathcal{O}_{F_w} \text{ for almost all } w \in M_F^0 \right\}.$$

where M_F denotes the set of places of F, containing the set $M_F^0 \subseteq M_F$ of non-Archimedean places. We endow $\mathbb{A}_F$ with the structure of a topological ring: the sum and product operations are defined component-wise, and the topology is generated by the open sets of the form $\prod_{w \in M_F} U_w$ where $U_w \subseteq F_w$ is open and $U_w = \mathcal{O}_{F_w}$ for all but finitely many $w \in M_F^0$. Finally, we define the *idèle group* $\mathbb{A}_F^\times$ as the group of units in the adèle ring $\mathbb{A}_F$.

In order to describe the action of the idèle group $\mathbb{A}_F^\times$ on the set of lattices $\mathcal{L}(F)$, we need to use the fact that adèles are compatible with base-change, as it is shown by the following proposition (see [Neu99, Page 371]).

Proposition 6.1.7 – Adèles and base-change

Let $K \subseteq F$ be an extension of number fields. Then there is a natural isomorphism of topological rings

$$\mathbb{A}_K \otimes_K F \xrightarrow{\sim} \mathbb{A}_F \tag{6.5}$$

induced by the natural inclusion $\mathbb{A}_K \hookrightarrow \mathbb{A}_F$ and the diagonal embedding $F \hookrightarrow \mathbb{A}_F$.

Proof. Use the fact that

$$K_v \otimes_K F \xrightarrow{\sim} \prod_{w \mid v} F_w$$

which holds for every place $v \in M_K$ (see [Neu99, Chapter II, Proposition 8.3]), together with the fact that tensor products distribute over restricted products. □

We are finally ready to define the action of the idèle group $\mathbb{A}_F^\times$ on the set $\mathcal{L}(F)$ (see [Lan87, Chapter 8, § 1]).

Proposition 6.1.8 – The idelic action on lattices

Let F be a number field. Then for every $\mathbb{Z}$-lattice $\Lambda \subseteq F$ and every idèle $s \in \mathbb{A}_F^\times$ there exists a unique lattice $s \cdot \Lambda \subseteq F$ such that for every prime $p \in \mathbb{N}$ we have that

$$(s \cdot \Lambda) \otimes_{\mathbb{Z}} \mathbb{Z}_p = s_p \cdot \Lambda_p \subseteq F_p$$

where $F_p := F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $s_p \in F_p^\times$ is the p-adic component of s, coming from the isomorphism (6.5) with $K = \mathbb{Q}$.

Proof. Fix an idèle $s \in \mathbb{A}_F^\times$ and a lattice $\Lambda \subseteq F$. To show the existence of the lattice $s \cdot \Lambda$, we observe that $s_p \cdot \Lambda_p \in \mathcal{L}(F; \mathbb{Z}_{(p)})$ for every prime $p \in \mathbb{N}$. This shows that

$$((s_p \cdot \Lambda_p) \cap F) \otimes_{\mathbb{Z}} \mathbb{Z}_q = \begin{cases} s_p \cdot \Lambda_p, & \text{if } p = q \\ F_q, & \text{if } p \neq q \end{cases}$$

for every pair of primes $p, q \in \mathbb{N}$, which implies that

$$\left(\bigcap_p \left((s_p \cdot \Lambda_p) \cap F \right) \right) \otimes_{\mathbb{Z}} \mathbb{Z}_q = (s_q \cdot \Lambda_q) \cap F$$

for every pair of primes $p, q \in \mathbb{N}$. Hence the lattice $s \cdot \Lambda \subseteq F$ can be taken to be

$$s \cdot \Lambda := \bigcap_p \left((s_p \cdot \Lambda_p) \cap F \right)$$

and this defines $s \cdot \Lambda$ uniquely thanks to Lemma 6.1.5. □

The map $(s, \Lambda) \mapsto s \cdot \Lambda$ defines an action of $\mathbb{A}_F^\times$ on $\mathcal{L}(F)$, such that $(s \cdot \Lambda_1) \cdot \Lambda_2 = s \cdot (\Lambda_1 \cdot \Lambda_2)$ for every idèle $s \in \mathbb{A}_F^\times$ and every pair of lattices $\Lambda_1, \Lambda_2 \in \mathcal{L}(F)$. Furthermore, for every idèle $s \in \mathbb{A}_F^\times$ and every lattice $\Lambda \in \mathcal{L}(F)$ we have a *multiplication by s* map $F/\Lambda \xrightarrow{s\cdot} F/(s \cdot \Lambda)$, defined by means of the following commutative diagram

$$\begin{array}{ccc} \dfrac{F}{\Lambda} & \xrightarrow{\quad s\cdot \quad} & \dfrac{F}{s \cdot \Lambda} \\ \wr \downarrow & & \wr \downarrow \\ \displaystyle\bigoplus_{p \in M_{\mathbb{Q}}^0} \dfrac{F_p}{\Lambda_p} & \xrightarrow{(s_p \cdot)_p} & \displaystyle\bigoplus_{p \in M_{\mathbb{Q}}^0} \dfrac{F_p}{s_p \Lambda_p} \end{array} \tag{6.6}$$

where the bottom map is given by $(x_p)_p \mapsto (s_p\, x_p)_p$ and the vertical maps are the isomorphisms given by (6.2).

6.1.3 The global Artin map

The aim of this short section is to briefly recall some of the main properties of the global Artin map. We give no proofs, referring the reader to [Neu99, Chapter VI] for a complete account of global class field theory, and to [Poo12] for a summary of the statements.

First of all, let us recall that for every finite extension of number fields $K \subseteq F$ and every place $w \in M_F$ lying above a place $v \in M_K$ there exists a commutative diagram

$$\begin{array}{ccccc} F^\times & \xhookrightarrow{\iota_F} & \mathbb{A}_F^\times & \xhookleftarrow{\iota_{F_w}} & F_w^\times \\ \downarrow{\scriptstyle \mathrm{N}_{F/K}} & & \downarrow{\scriptstyle \mathrm{N}_{F/K}} & & \downarrow{\scriptstyle \mathrm{N}_{F_w/K_v}} \\ K^\times & \xhookrightarrow{\iota_K} & \mathbb{A}_K^\times & \xhookleftarrow{\iota_{K_v}} & K_v^\times \end{array} \tag{6.7}$$

where $\iota_K \colon K^\times \hookrightarrow \mathbb{A}_K^\times$ and $\iota_F \colon F^\times \hookrightarrow \mathbb{A}_F^\times$ denote the diagonal embeddings. On the other hand, $\iota_{K_w} \colon K_w^\times \hookrightarrow \mathbb{A}_K^\times$ denotes the inclusion obtained by the identification

$$K_w^\times \cong \{s \in \mathbb{A}_K^\times \mid s_{w'} = 1,\ \forall w' \in M_K \setminus \{w\}\}$$

and $\iota_{F_v} \colon F_v^\times \hookrightarrow \mathbb{A}_F^\times$ is defined analogously. Finally, the homomorphism

$$\mathrm{N}_{F/K} \colon \mathbb{A}_F^\times \to \mathbb{A}_K^\times$$

is the so-called *idelic norm map* (see [Neu99, Chapter VI, § 2]).

Let us now state the main theorem of global class field theory, which contains in itself the definition of the global Artin map. In what follows, we assume that K is embedded in an algebraically closed field Ω, and we denote by $K^{\mathrm{ab}} \subseteq \Omega$ the maximal sub-field of Ω which is an abelian extension of K. Of course, the isomorphism class of this field does not depend on Ω nor on the embedding $K \hookrightarrow \Omega$.

Theorem 6.1.9 – Main theorem of global class field theory

For every number field K there exists a unique surjective, continuous group homomorphism $[\cdot, K] \colon \mathbb{A}_K^\times \twoheadrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$ with the following properties:

- the kernel of $[\cdot, K]$ equals the topological closure of the subgroup $K^\times \cdot K_\infty^+ \subseteq \mathbb{A}_K^\times$, where $K_\infty^+ \subseteq K_\infty^\times$ denotes the connected component of the identity in the group of units of the topological ring $K_\infty := K \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{v \in M_K^\infty} K_v$. The kernel of $[\cdot, K]$ is also equal to the inverse image of the connected component of the identity of the topological group $\mathbb{A}_K^\times / K^\times$ under the quotient map $\mathbb{A}_K^\times \twoheadrightarrow \mathbb{A}_K^\times / K^\times$;
- for every finite abelian extension $K \subseteq F$, the following square

$$\begin{array}{ccc} \mathbb{A}_F^\times & \xrightarrow{[\cdot, F]} & \mathrm{Gal}(F^{\mathrm{ab}}/F) \\ \downarrow{\scriptstyle \mathrm{N}_{F/K}} & & \downarrow{\scriptstyle r_{F/K}} \\ \mathbb{A}_K^\times & \xrightarrow{[\cdot, K]} & \mathrm{Gal}(K^{\mathrm{ab}}/K) \end{array} \tag{6.8}$$

  commutes, where $r_{F/K} \colon \mathrm{Gal}(F^{\mathrm{ab}}/F) \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ denotes the restriction map. In particular, the map $[\cdot, K]$ induces an isomorphism

$$[\cdot, F/K] \colon \frac{\mathbb{A}_K^\times}{K^\times \cdot \mathrm{N}_{F/K}(\mathbb{A}_F^\times)} \xrightarrow{\sim} \mathrm{Gal}(F/K) \tag{6.9}$$

  for every finite abelian extension $K \subseteq F$;
- for every non-Archimdean place $v \in M_K^0$ lying above a prime $p \in \mathbb{N}$ we have that $[\iota_{K_v}(K_v^\times), K] \subseteq \mathrm{Gal}(K_v^{\mathrm{ab}}/K_v)$. Moreover, for every uniformiser $\pi \in K_v^\times$ the homomorphism $[\iota_{K_v}(\pi), K] \in \mathrm{Gal}(K_v^{\mathrm{ab}}/K_v)$ acts on the maximal unramified extension $K_v \subseteq (\overline{K_v})_0$ (which is pro-cyclic, hence abelian) as the arithmetic Frobenius element $f_{K_v}^{-1}$ (see (3.13) for the definition of the geometric Frobenius f_{K_v}).

Definition 6.1.10 – Global Artin map

Let K be a number field. Then the *global Artin map* is the unique surjective, continuous group homomorphism

$$[\cdot, K] : \mathbb{A}_K^\times \twoheadrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

satisfying the properties stated in Theorem 6.1.9.

Remark 6.1.11. Our formulation of the main theorem of global class field theory may appear a little different from the ordinary. It can be obtained by combining the main theorem of local class field theory (see [Neu99, Chapter V, Theorem 1.3]) and the main theorem of global class field theory (see [Neu99, Chapter VI, Theorem 6.1]). We refer in particular to [Mil20, Theorem 1.13] for a proof of the uniqueness of the local Artin map.

6.2 The notion of ray class fields for orders

The aim of this section, as we stated in the introduction of the chapter, is to study the notion of ray class field for an order O inside a number field F. We introduce them using the idelic language of class field theory, which we recalled in Theorem 6.1.9. We then show in Theorem 6.2.17 how to relate this to a definition coming from the classical language of class field theory.

The material present in this section is based on joint work in progress with Francesco Campagna. We think that much of this material is probably known to the experts, but we were unable to find it explained in any suitable reference. In particular, our definition Definition 6.2.11 has not appeared elsewhere in this generality. Nevertheless, it can be seen as a generalisation of the notions introduced by Söhngen and by Lv-Deng and Yi-Lv, as pointed out in Remark 6.2.14.

6.2.1 Number rings and orders

We start this section by recalling the notion of number ring and order, following [Ste08, § 2].

Definition 6.2.1 – Number rings and orders

A *number ring* R is a domain whose field of fractions $\mathrm{Frac}(R)$ is a number field. A number ring $O \subseteq K$ inside a number field K is called an *order in* K if $K = \mathrm{Frac}(O)$ and O is finitely generated as an abelian group.

Example 6.2.2. The basic example of order inside a number field K is given by the ring of integers $O_K \subseteq K$, consisting of all the elements $\alpha \in K$ which are integral over $\mathbb{Z}$, *i.e.* such that there exists a monic polynomial $f(t) \in \mathbb{Z}[t]$ with $f(\alpha) = 0$. We refer the reader to [Neu99, Chapter I, Theorem 3.1] for a proof of the fact that $O_K \subseteq K$ is indeed an order in K.

Example 6.2.3. Further examples of number rings are given by the rings of the form

$$R = \mathbb{Z}[\alpha_1, \dots, \alpha_n]$$

with $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$. We note that the ring $\mathbb{Z}[\alpha_1, \dots, \alpha_n]$ is an order inside the number field $\mathbb{Q}(\alpha_1, \dots, \alpha_n)$ if and only if $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Z}}$ (see [Neu99, Chapter I, Proposition 2.2]).

One of the reasons why orders are more amenable to computations than general number rings is given by the fact that they are lattices in the corresponding number field, as the following result shows (see [Ste08, Theorem 2.2]).

Proposition 6.2.4 – Orders as lattices

Let $R \subseteq K$ be a number ring inside a number field $K = \mathrm{Frac}(R)$. Then R is an order in K if and only if $R \subseteq \mathcal{O}_K$ and the index $[\mathcal{O}_K : R]$ is finite. Moreover, every ideal $I \subseteq \mathcal{O}$ inside an order $\mathcal{O} \subseteq K$ is finitely generated and every non-zero prime ideal $\mathfrak{p} \subseteq \mathcal{O}$ is maximal, *i.e.* every order $\mathcal{O}$ is a one dimensional, Noetherian integral domain. Finally, for every $\Lambda \subseteq K$ which is a fractional ideal for an order $\mathcal{O} \subseteq K$ (*i.e.* a non-zero finitely generated $\mathcal{O}$ sub-module of K) we have that $\Lambda \in \mathcal{L}(K)$, *i.e.* Λ is a $\mathbb{Z}$-lattice in K (see Definition 6.1.1).

Proof. Since $\mathcal{O}_K \subseteq K$ is an order in K we have that any ring $R \subseteq \mathcal{O}_K$ is finitely generated as an abelian group, and if the index $[\mathcal{O}_K : R]$ is finite we have that $K = \mathrm{Frac}(R)$. For the converse, we observe that Example 6.2.3 implies that for any $\mathcal{O} \subseteq K$ which is an order for K we have that $\mathcal{O} \subseteq \mathcal{O}_K$. Since $K = \mathrm{Frac}(\mathcal{O})$ we have that $\mathcal{O}$ and $\mathcal{O}_K$ are finitely generated abelian groups of the same rank. This shows as well that every ideal $I \subseteq \mathcal{O}$ is a finitely generated abelian group, hence a finitely generated $\mathcal{O}$-module, which implies that $\mathcal{O}$ is a Noetherian integral domain. Moroever, for every ideal $I \subseteq \mathcal{O}$ we have that $n \cdot \mathcal{O} \subseteq I \subseteq \mathcal{O}$ for every $n \in I \cap \mathbb{Z}$. Hence if we take $I \neq (0)$ we see that I and $\mathcal{O}$ are finitely generated abelian groups of the same rank, which implies that for every non-zero ideal $I \subseteq \mathcal{O}$ we have that I is a lattice in K and $\mathcal{O}/I$ is a finite ring. This applies in particular to every non-zero prime ideal $\mathfrak{p} \subseteq \mathcal{O}$, which implies that $\mathfrak{p}$ is maximal because the finite integral domain $\mathcal{O}/\mathfrak{p}$ is a field, thanks to Wedderburn's little theorem (see for instance [Coh03, Theorem 7.8.6]). To conclude, it is sufficient to observe that for every fractional $\mathcal{O}$-ideal $\Lambda \subseteq K$ there exists $n \in \mathbb{Z}$ such that $n \neq 0$ and $n\Lambda \subseteq \mathcal{O}$ is an ideal. Thus $n\Lambda$ and Λ are both $\mathbb{Z}$-lattices inside K. □

To conclude, let us recall the notion of conductor of an order $\mathcal{O} \subseteq K$.

Definition 6.2.5 – Conductor of an order

Let K be a number field and let $\mathcal{O} \subseteq K$ be an order in K. Then the *conductor* of $\mathcal{O}$ is:

$$\mathfrak{f}_{\mathcal{O}} := (\mathcal{O} : \mathcal{O}_K) = \{\alpha \in \mathcal{O}_K \mid \alpha\mathcal{O}_K \subseteq \mathcal{O}\} \subseteq \mathcal{O}$$

which is the biggest ideal of $\mathcal{O}_K$ to be contained in $\mathcal{O}$.

Remark 6.2.6. Let $\mathcal{O}$ be an order inside a number field F. Then it is immediate to see that $|\mathcal{O}_F : \mathcal{O}| \cdot \mathcal{O}_F \subseteq \mathcal{O}$, which implies that the conductor ideal $\mathfrak{f}_{\mathcal{O}} \subseteq \mathcal{O}_F$ divides the principal ideal $|\mathcal{O}_F : \mathcal{O}| \cdot \mathcal{O}_F$ generated by the index $|\mathcal{O}_F : \mathcal{O}| \in \mathbb{N}$.

The conductor $\mathfrak{f}_{\mathcal{O}}$ of an order is important in view of the following result concerning the invertibility of ideals.

Lemma 6.2.7 – The conductor and invertible ideals

Let $\mathcal{O}$ be an order inside a number field K. Then an ideal $I \subseteq \mathcal{O}$ is invertible if and only if $I \cdot (\mathcal{O} : I) = \mathcal{O}$. This happens if $I + \mathfrak{f}_{\mathcal{O}} = \mathcal{O}$, *i.e.* every ideal which is coprime to the conductor $\mathfrak{f}_{\mathcal{O}} \subseteq \mathcal{O}$ is invertible, and the converse holds if I is a prime ideal. Moreover, the map

$$\begin{aligned} \{J \subseteq \mathcal{O}_K \mid J + \mathfrak{f}_{\mathcal{O}} = \mathcal{O}_K\} &\to \{I \subseteq \mathcal{O} \mid I + \mathfrak{f}_{\mathcal{O}} = \mathcal{O}\} \\ J &\mapsto J \cap \mathcal{O} \end{aligned}$$

is a bijection, whose inverse is given by $I \mapsto I \cdot \mathcal{O}_K$. These maps are multiplicative, *i.e.* $(J \cdot J') \cap \mathcal{O} = (J \cap \mathcal{O}) \cdot (J' \cap \mathcal{O})$ and $(I \cdot I')\mathcal{O}_K = (I \cdot \mathcal{O}_K) \cdot (I' \cdot \mathcal{O}_K)$ for every two pairs of ideals $J, J' \subseteq \mathcal{O}$ and $I, I' \subseteq \mathcal{O}_K$ such that $J + \mathfrak{f}_{\mathcal{O}} = J' + \mathfrak{f}_{\mathcal{O}} = \mathcal{O}$ and $I + \mathfrak{f}_{\mathcal{O}} = I' + \mathfrak{f}_{\mathcal{O}} = \mathcal{O}_K$. Finally, for every ideal $J \subseteq \mathcal{O}_K$ such that $J \cap \mathcal{O}$ is invertible, the natural map

$$\frac{\mathcal{O}}{J \cap \mathcal{O}} \to \frac{\mathcal{O}_K}{J} \tag{6.10}$$

is an isomorphism.

Proof. The first statement is the content of [Con19, Lemma 3.2] and the second can be obtained by combining [Neu99, Chapter I, Proposition 12.10] and [Con19, Theorem 3.6]. The fact that the maps $J \mapsto J \cap \mathcal{O}$ and $I \mapsto I \cdot \mathcal{O}_K$ establish a bijection between the sets of ideals of $\mathcal{O}$ and $\mathcal{O}_K$ which are coprime with $\mathfrak{f}_{\mathcal{O}}$ is the content of [Con19, Theorem 3.6]. Finally, [Con19, Theorem 3.12] proves that (6.10) is an isomorphism for every ideal $J \subseteq \mathcal{O}_K$ such that $J \cap \mathcal{O}$ is invertible. □

Example 6.2.8 (Conductors of imaginary quadratic orders)**.** Let $\mathcal{O}$ be an order inside an imaginary quadratic field K. Then it can be shown that the conductor $\mathfrak{f}_{\mathcal{O}} \subseteq \mathcal{O}_K$ equals the ideal generated inside $\mathcal{O}_K$ by the integer $|\mathcal{O}_K : \mathcal{O}| \in \mathbb{N}$ (see [Con19, Example 2.1]). Hence, we usually abuse notation and denote by $\mathfrak{f}_{\mathcal{O}}$ both the index $|\mathcal{O}_K : \mathcal{O}|$ and the conductor ideal $\mathfrak{f}_{\mathcal{O}} \subseteq \mathcal{O}_K$ generated by this index. Moreover, let us observe that for every natural number $f \in \mathbb{N}$ and every number field K there exists a unique order $\mathcal{O}_f \subseteq \mathcal{O}_K$ whose conductor equals $f \cdot \mathcal{O}_K$. Indeed, $\mathcal{O}_f$ is given by the ring $\mathcal{O}_f := \mathbb{Z} + f\mathcal{O}_K$.

6.2.2 Definition of ray class fields for orders

We are now ready to define the ray class fields relative to an order $\mathcal{O} \subseteq \mathcal{O}_F$ inside a number field F. First of all, we package the data needed to give the definition of such a ray class field in the following concept.

Definition 6.2.9 – Generalised modules

Let F be a number field. Then a *generalised module* for F is a triple

$$\mathfrak{m} = (\mathcal{O}, I, \mathfrak{m}_\infty)$$

where $\mathcal{O} \subseteq F$ is an order, $I \subseteq \mathcal{O}$ is a non-zero ideal and $\mathfrak{m}_\infty \subseteq M_F^\infty$ is a collection of Archimedean places of F such that $F_v \cong \mathbb{R}$ for every $v \in \mathfrak{m}_\infty$.

Remark 6.2.10. Usually one defines a *module* $\widetilde{\mathfrak{m}}$ for a number field F as a formal $\mathbb{N}$-linear combination $\widetilde{\mathfrak{m}} = \sum_v \mathrm{ord}_v(\widetilde{\mathfrak{m}})[v] \in \mathbb{N}[M_F]$ of places of F, such that $\mathrm{ord}_v(\widetilde{\mathfrak{m}}) = 0$ for all but finitely many $v \in M_F$ and $\mathrm{ord}_v(\widetilde{\mathfrak{m}}) \in \{0,1\}$ if $v \in M_F^\infty$ (see for example [Neu99, Page 363]). Then the triple

$$\mathfrak{m} = \left(O_F, \prod_{v \in M_F^0} \mathfrak{p}_v^{\mathrm{ord}_v(\widetilde{\mathfrak{m}})}, \{v \in M_F^\infty \mid \mathrm{ord}_v(\widetilde{\mathfrak{m}}) \neq 0\} \right)$$

defines a generalised module for F, in the sense of Definition 6.2.9. Moreover, the map $\widetilde{\mathfrak{m}} \mapsto \mathfrak{m}$ induces a bijection between the set of modules $\widetilde{\mathfrak{m}}$ (in the classical sense) and the set of generalised modules $\mathfrak{m} = (O, I, \mathfrak{m}_\infty)$ such that $O = O_F$.

Let us now introduce the ray class field $H_\mathfrak{m}$ associated to a generalised module $\mathfrak{m}$.

Definition 6.2.11 – Ray class fields for orders

Let F be a number field and $\mathfrak{m} = (O, I, \mathfrak{m}_\infty)$ be a generalised module for F. Then we define the *ray class field of F modulo $\mathfrak{m}$* as

$$H_\mathfrak{m} := (F^{\mathrm{ab}})^{[U_\mathfrak{m}, F]} \subseteq F^{\mathrm{ab}} \tag{6.11}$$

where $[\cdot, F] \colon \mathbb{A}_F^\times \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$ is the *global Artin map* and $U_\mathfrak{m} \subseteq \mathbb{A}_F^\times$ is the subgroup

$$U_\mathfrak{m} := \left\{ s \in \mathbb{A}_F^\times \;\middle|\; \begin{array}{ll} s_p \in \left(O_p^\times \cap (1 + I \cdot O_p)\right) & \text{for all rational primes } p \in \mathbb{N} \\ s_v \in \mathbb{R}_{>0}, & \text{for all places } v \in \mathfrak{m}_\infty \end{array} \right\} \tag{6.12}$$

where $s_p \in F_p^\times := (F \otimes_\mathbb{Q} \mathbb{Q}_p)^\times$ denotes the p-adic component of an idèle $s \in \mathbb{A}_F^\times$, which is defined using (6.5) with $K = \mathbb{Q}$, and

$$O_p := \varprojlim_{n \in \mathbb{N}} \frac{O}{p^n O} \cong O \otimes_\mathbb{Z} \mathbb{Z}_p \subseteq F_p := F \otimes_\mathbb{Q} \mathbb{Q}_p \tag{6.13}$$

denotes the completion of O with respect to the ideal $p\,O$ (see also Definition 6.1.4).

When $\mathfrak{m}_\infty = \emptyset$ we write $U_{I,O} := U_\mathfrak{m}$, and if $I = N \cdot O$ for some $N \in \mathbb{Z}$ we denote $U_{I,O}$ by $U_{N,O}$, and we write $U_O := U_{1,O}$. The corresponding ray class fields are denoted by $H_{I,O}, H_{N,O}$ and H_O, respectively.

Remark 6.2.12. Definition 6.2.11 is easily generalised to all number rings. More precisely, one can define a generalised module $\mathfrak{m}$ to be a triple $\mathfrak{m} = (R, I, \mathfrak{m}_\infty)$ where R is a number ring, $I \subseteq R$ is an ideal and $\mathfrak{m}_\infty$ is a set of Archimedean places of $F := \mathrm{Frac}(R)$, with the property that $F_v \cong \mathbb{R}$ for every $v \in \mathfrak{m}_\infty$. Then one can define

$$V_\mathfrak{m} := \left\{ s \in \mathbb{A}_F^\times \;\middle|\; \begin{array}{ll} s_p \in \left(O_p^\times \cap (1 + I \cdot O_p)\right) & \text{for all rational primes } p \in \mathbb{N} \\ s_v \in \mathbb{R}_{>0}, & \text{for all places } v \in \mathfrak{m}_\infty \end{array} \right\}$$

where $R_p := R \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and again $F = \mathrm{Frac}(R)$. Finally, the ray class field for the generalised module $\mathfrak{m}$ is defined to be $H_{\mathfrak{m}} := (F^{\mathrm{ab}})^{[V_{\mathfrak{m}},F]}$, and use again the notation $H_{I,R}, H_{N,R}$ and H_R, defined exactly as in the case of orders.

We decided to focus on the theory for orders in this chapter, because it is less technical to develop. For instance, already for the ring class field H_R associated to a number ring R, showing that $\mathrm{Gal}(H_R/\mathrm{Frac}(R)) \cong \mathrm{Pic}(R)$ takes up half of the proof of [YL18, Theorem 4.2].

Remark 6.2.13. When $\mathcal{O} = \mathcal{O}_F$ is the ring of integers, the ray class fields $H_{\mathfrak{m}}$ coincide with the usual ray class fields of F, which are defined for example in [Neu99, Chapter VI, Definition 6.2]). This is evident from the definitions, using the bijection $\widetilde{\mathfrak{m}} \mapsto \mathfrak{m}$ described in Remark 6.2.10, and the isomorphism

$$\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \prod_{\mathfrak{p}|p} \mathcal{O}_{F_{\mathfrak{p}}}$$

which holds for every rational prime $p \in \mathbb{N}$ (see also (6.18) for the analogous decomposition in the case of a general order).

Remark 6.2.14. When $F = K$ is an imaginary quadratic field, the ray class fields $H_{I,\mathcal{O}}$ have been defined by Söhngen in [Söh35]. His work is exposed in great detail by Schertz in [Sch10, §3.3], and if $I = N \cdot \mathcal{O}$ for some $N \in \mathbb{N}$ the construction of $H_{I,\mathcal{O}} = H_{N,\mathcal{O}}$ has been reformulated by Stevenhagen in [Ste01, § 4], using an idelic language. Finally, the ring class fields $H_{\mathcal{O}}$ have been studied for general number fields F by Lv and Deng in [LD15] and by Yi and Lv in [YL18], who treated also the case of general number rings.

Remark 6.2.15. For every generalised module $\mathfrak{m} = (\mathcal{O}, I, \mathfrak{m}_\infty)$ we have that $U_{\mathfrak{m}} \subseteq U_{I,\mathcal{O}}$, which implies that $H_{\mathfrak{m}} \supseteq H_{I,\mathcal{O}}$. Moreover, for every pair of ideals $I \subseteq J \subseteq \mathcal{O}$ we have that $U_{I,\mathcal{O}} \subseteq U_{J,\mathcal{O}}$, which implies that $H_{I,\mathcal{O}} \supseteq H_{J,\mathcal{O}}$. In particular, $H_{\mathcal{O}} \subseteq H_{I,\mathcal{O}}$ for every ideal $I \subseteq \mathcal{O}$. Similarly, for every pair of orders $\mathcal{O}_1 \subseteq \mathcal{O}_2 \subseteq F$ and every ideal $I \subseteq \mathcal{O}_1$ we have that $U_{I,\mathcal{O}_1} \subseteq U_{I\cdot\mathcal{O}_2,\mathcal{O}_2}$, which gives the containment $H_{I,\mathcal{O}_1} \supseteq H_{I\cdot\mathcal{O}_2,\mathcal{O}_2}$. This generalises Deuring's *Anordnungssatz*, as explained for example in [Ste01, Page 169]. In particular for every generalised module $\mathfrak{m} = (\mathcal{O}, I, \mathfrak{m}_\infty)$ we have the following diagram of inclusions

$$\begin{array}{ccccccc}
 & & & & H_{\mathfrak{m}} & & \\
 & & & & \cup & & \\
 & & H_{I\cdot\mathcal{O}_F,\mathcal{O}_F} & \subseteq & H_{I,\mathcal{O}} & \subseteq & H_{I\cdot\mathfrak{f}_{\mathcal{O}}\cdot\mathcal{O}_F,\mathcal{O}_F} \\
 & & \cup & & \cup & & \cup \\
F & \subseteq & H_{\mathcal{O}_F} & \subseteq & H_{\mathcal{O}} & \subseteq & H_{\mathfrak{f}_{\mathcal{O}},\mathcal{O}_F}
\end{array}$$

where $\mathfrak{f}_{\mathcal{O}} \subseteq \mathcal{O}$ is the *conductor* of $\mathcal{O}$ (see Definition 6.2.5). This shows, applying [Neu99, Chapter VI, Corollary 6.6], that the extension $F \subseteq H_{I,\mathcal{O}}$ is unramified outside the set of primes dividing $I \cdot \mathfrak{f}_{\mathcal{O}} \cdot \mathcal{O}_F$.

6.2.3 Galois groups of ray class fields for orders

The aim of this final section, which constitutes the technical part of this chapter, is to describe the Galois groups of the abelian extensions $F \subseteq H_{\mathfrak{m}}$. First of all, we have the following idelic description, which follows essentially from Theorem 6.1.9 and Definition 6.2.11.

Lemma 6.2.16 – Galois groups of ray class fields, in terms of idèles

Let $\mathfrak{m} = (\mathcal{O}, I, \mathfrak{m}_\infty)$ be a generalised module, relative to the number field $F := \mathrm{Frac}(\mathcal{O})$. Then $F^\times \cdot U_\mathfrak{m} \subseteq \mathbb{A}_F^\times$ is a closed subgroup of finite index and one has

$$F^\times \cdot F_\infty^+ \subseteq \ker([\cdot, F]) \subseteq F^\times \cdot U_\mathfrak{m} = F^\times \cdot \mathrm{N}_{H_\mathfrak{m}/F}(\mathbb{A}_{H_\mathfrak{m}}^\times)$$

where $\mathrm{N}_{H_\mathfrak{m}/F} \colon \mathbb{A}_{H_\mathfrak{m}}^\times \to \mathbb{A}_F^\times$ denotes the idelic norm map, and $F_\infty^+ \subseteq F_\infty^\times$ is the connected component of the identity in the group of units of the topological ring $F_\infty := F \otimes_\mathbb{Q} \mathbb{R}$. Moreover, there is an isomorphism

$$\mathrm{Gal}(H_\mathfrak{m}/F) \cong \frac{\mathbb{A}_F^\times}{F^\times \cdot U_\mathfrak{m}} \tag{6.14}$$

induced by the global Artin map $[\cdot, F] \colon \mathbb{A}_F^\times \twoheadrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F)$.

Proof. Let $\widetilde{\mathfrak{m}} := (\mathcal{O}_F, I \cdot \mathfrak{f}_\mathcal{O} \cdot \mathcal{O}_F, \mathfrak{m}_\infty)$, so that $U_\mathfrak{m} \supseteq U_{\widetilde{\mathfrak{m}}}$. This implies that the subgroup $F^\times \cdot U_\mathfrak{m} \subseteq \mathbb{A}_F^\times$ is closed and has finite index, thanks to [Neu99, Chapter VI, Proposition 1.8]. Moreover, we have by definition that $F_\infty^+ \subseteq U_\mathfrak{m}$, so the inclusions

$$F^\times \cdot F_\infty^+ \subseteq \ker([\cdot, F]) \subseteq F^\times \cdot U_\mathfrak{m}$$

follow from the fact that $F^\times \cdot U_{I,\mathcal{O}}$ is closed in $\mathbb{A}_F^\times$ and $\ker([\cdot, F])$ is the closure of $F^\times \cdot F_\infty^+$ inside $\mathbb{A}_F^\times$, as explained in Theorem 6.1.9. The isomorphism (6.9) now gives (6.14) and shows that $F^\times \cdot \mathrm{N}_{H_\mathfrak{m}/F}(\mathbb{A}_{H_\mathfrak{m}}^\times) \subseteq \mathbb{A}_F^\times$ is also a closed subgroup of finite index containing the kernel of the Artin map and fixing precisely the field $H_\mathfrak{m}$. Then by Galois theory we must have $F^\times \cdot U_\mathfrak{m} = F^\times \cdot \mathrm{N}_{H_\mathfrak{m}/F}(\mathbb{A}_{H_\mathfrak{m}}^\times)$, and this concludes the proof. □

The next step in our description of the Galois group $\mathrm{Gal}(H_\mathfrak{m}/F)$ is to relate it to suitable sub-quotients of the group $\mathcal{I}_\mathcal{O}$ of invertible ideals $\mathfrak{a} \subseteq \mathcal{O}$. This is achieved by the following result, which extends [YL18, Theorem 4.2], where it is shown that $\mathrm{Gal}(H_R/R) \cong \mathrm{Pic}(R)$ for every number ring R, to general ray class fields for orders.

Theorem 6.2.17 – Galois groups of ray class fields, in terms of ideals

Let $\mathfrak{m} = (\mathcal{O}, I, \mathfrak{m}_\infty)$ be a generalised module, relative to the number field $F = \mathrm{Frac}(\mathcal{O})$. Let $\mathcal{I}_\mathfrak{m}$ be the group of invertible ideals $\mathfrak{a} \subseteq \mathcal{O}$ such that $\mathfrak{a} + I = \mathcal{O}$, and let $\mathcal{P}_\mathfrak{m} \subseteq \mathcal{I}_\mathfrak{m}$ be the subgroup of principal ideals generated by elements $\alpha \in \mathcal{O}$ such that $\alpha \equiv 1 (I)$ and $\iota_v(\alpha) > 0$ for every $v \in \mathfrak{m}_\infty$, where $\iota_v \colon F \hookrightarrow F_v$ denotes the canonical embedding of F inside its completion $F_v \cong \mathbb{R}$. Then there is an isomorphism

$$\mathrm{Gal}(H_\mathfrak{m}/F) \cong \frac{\mathcal{I}_\mathfrak{m}}{\mathcal{P}_\mathfrak{m}}$$

which shows in particular that $\mathrm{Gal}(H_\mathcal{O}/F) \cong \mathrm{Pic}(\mathcal{O})$ for every order $\mathcal{O} \subseteq F$.

Proof. First of all, let $\mathcal{J}_\mathfrak{m}$ be the group of fractional $\mathcal{O}$-ideals $\mathfrak{a}_1 \cdot \mathfrak{a}_2^{-1} \subseteq F$, where $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathcal{I}_\mathfrak{m}$, and let $\mathcal{Q}_\mathfrak{m} \subseteq \mathcal{J}_\mathfrak{m}$ be the subgroup of principal fractional $\mathcal{O}$-ideals $(\alpha_1/\alpha_2) \cdot \mathcal{O}$, where $\alpha_1, \alpha_2 \in \mathcal{O}$

are elements such that $\alpha_1 O + I = \alpha_2 O + I = O$ and $\alpha_1 \equiv \alpha_2 (I)$. Then the natural inclusion $\iota\colon \mathcal{I}_{\mathfrak{m}} \hookrightarrow \mathcal{J}_{\mathfrak{m}}$ induces an isomorphism

$$\iota\colon \frac{\mathcal{I}_{\mathfrak{m}}}{\mathcal{P}_{\mathfrak{m}}} \xrightarrow{\sim} \frac{\mathcal{J}_{\mathfrak{m}}}{\mathcal{Q}_{\mathfrak{m}}} \tag{6.15}$$

as follows from clearing denominators. Indeed, if $\mathfrak{a} = \mathfrak{a}_1 \cdot \mathfrak{a}_2^{-1} \in \mathcal{J}_{\mathfrak{m}}$ and $\mathfrak{a}_2 = (\alpha_1, \dots, \alpha_n)$ then $\alpha \cdot \mathfrak{a} \subseteq O$, where $\alpha := \alpha_1 \cdots \alpha_n$. Since $\mathfrak{a}_2 + I = O$ we have that α becomes a unit modulo I, hence there exists $\beta \in O$ such that $\alpha\beta \equiv 1 (I)$, which shows that the class of $\mathfrak{a}$ in the quotient $\mathcal{J}_{\mathfrak{m}}/\mathcal{Q}_{\mathfrak{m}}$ coincides with the class of $(\alpha\beta) \cdot \mathfrak{a}$, where $(\alpha\beta) \cdot \mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}$. This shows that the map (6.15) is surjective. Since it is also naturally injective, we see that it is an isomorphism.

Now, let $\widetilde{\mathfrak{m}} := (O_F, I \cdot \mathfrak{f}_O \cdot O_F, \mathfrak{m}_\infty)$, where $\mathfrak{f}_O \subseteq O$ denotes the conductor ideal of O. Then there is a map $\varphi\colon \mathcal{J}_{\widetilde{\mathfrak{m}}} \to \mathcal{J}_{\mathfrak{m}}$ induced by the map $\mathcal{I}_{\widetilde{\mathfrak{m}}} \to \mathcal{I}_{\mathfrak{m}}$ given by $\mathfrak{a} \mapsto \mathfrak{a} \cap O$. Moreover, for every $\mathfrak{a} \in \mathcal{I}_{\widetilde{\mathfrak{m}}}$ we have that $\mathfrak{a} = (\mathfrak{a} \cap O) \cdot O_F$ because $\mathfrak{a}$ is coprime with $\mathfrak{f}_O$, which shows that $\varphi^{-1}(\mathcal{Q}_{\mathfrak{m}})$ is the subgroup of all principal fractional ideals $(\alpha_1/\alpha_2) \cdot O_F$ which are generated by quotients of elements $\alpha_1, \alpha_2 \in O$ such that $\alpha_1 O + I\mathfrak{f}_O = \alpha_2 O + I\mathfrak{f}_O = O$ and $\alpha_1 \equiv \alpha_2 (I)$.

We can now proceed as in [YL18, Theorem 2.9] to show that for every $\mathfrak{a} \in \mathcal{I}_{\mathfrak{m}}$ there exists $\alpha \in O$ such that $\alpha O \in \mathcal{P}_{\mathfrak{m}}$ and $\mathfrak{a} O_{(\mathfrak{p})} = \alpha O_{(\mathfrak{p})}$ for every prime $\mathfrak{p} \subseteq O$ such that $\mathfrak{p} \supseteq \mathfrak{f}_O$. Indeed, let $\mathfrak{p}_1, \dots, \mathfrak{p}_n \subseteq O$ be all the prime ideals containing $\mathfrak{f}_O$ and let $O_{(\mathfrak{p}_i)}$ be the localisation of O at $\mathfrak{p}_i$. Observe that these ideals are all maximal, and in particular pairwise coprime, because O is one-dimensional (see Proposition 6.2.4). Then one can use the Chinese remainder theorem combined with [Neu99, Chapter I, Proposition 12.4] and [Bou89, Chapter II, § 2.6, Proposition 15] to show that for every $i \in \{1, \dots, n\}$ there exist $\alpha_i, \beta_i \in O$ and $e_i \in \mathbb{Z}_{\geq 1}$ such that

$$\mathfrak{p}_i^{e_i} O_{(\mathfrak{p}_i)} \subseteq \mathfrak{a} O_{(\mathfrak{p}_i)} = \beta_i O_{(\mathfrak{p}_i)} \not\subseteq \mathfrak{p}_i^{e_i} O_{(\mathfrak{p}_i)} \qquad \text{and} \qquad \begin{aligned} &\alpha_i \equiv \beta_i (\mathfrak{p}_i^{e_i+1}) \\ &\alpha_i \equiv 1 (\mathfrak{p}_j),\ \forall j \neq i \\ &\alpha_i \equiv 1 (I) \end{aligned}$$

and in particular one can take $\alpha_i = \beta_i = e_i = 1$ whenever $\mathfrak{p}_i \supseteq I$, since $\mathfrak{a} + I = O$. Under these assumptions we can set $\alpha := \alpha_1 \cdots \alpha_n$, because $\alpha \equiv 1 (I)$ and

$$\mathfrak{a} O_{(\mathfrak{p}_i)} = \alpha_i O_{(\mathfrak{p}_i)} = \alpha_i \left(\prod_{j \neq i} \alpha_j O_{(\mathfrak{p}_i)} \right) = \alpha O_{(\mathfrak{p}_i)}$$

for every $i \in \{1, \dots, n\}$. Now, since $\alpha^{-1}\mathfrak{a} \subseteq O_{(\mathfrak{p}_i)}$ for every $i \in \{1, \dots, n\}$ we see that $\alpha^{-1}\mathfrak{a} = \gamma^{-1}\mathfrak{b}$ for some $\mathfrak{b} \subseteq O$ and some $\gamma \in O$ such that $\mathfrak{b} + I\mathfrak{f}_O = \gamma O + I\mathfrak{f}_O = O$ and $\gamma \equiv 1 (I)$. Hence the map $\varphi\colon \mathcal{J}_{\widetilde{\mathfrak{m}}} \to \mathcal{J}_{\mathfrak{m}}$ induces an isomorphism

$$\varphi\colon \frac{\mathcal{J}_{\widetilde{\mathfrak{m}}}}{\mathcal{R}_{\mathfrak{m}}} \xrightarrow{\sim} \frac{\mathcal{J}_{\mathfrak{m}}}{\mathcal{Q}_{\mathfrak{m}}}$$

where $\mathcal{R}_{\mathfrak{m}} := \varphi^{-1}(\mathcal{Q}_{\mathfrak{m}})$ is, as we said above, the set of all principal fractional ideals $(\alpha_1/\alpha_2) \cdot O_F$ which are generated by quotients of elements $\alpha_1, \alpha_2 \in O$ such that $\alpha_1 O + I\mathfrak{f}_O = \alpha_2 O + I\mathfrak{f}_O = O$ and $\alpha_1 \equiv \alpha_2 (I)$.

Now, let $\mathbb{A}_{F,\mathfrak{m}}^\times \subseteq \mathbb{A}_F^\times$ be the subgroup given by those idèles $s \in \mathbb{A}_F^\times$ such that $s_p \in O_p^\times \cap (1 + IO_p)$ for every rational prime $p \in \mathbb{N}$ dividing $\mathrm{N}_{F/\mathbb{Q}}(I \cdot \mathfrak{f}_O \cdot O_F)$, and $\iota_v(s_v) > 0$ for every real place $v \in \mathfrak{m}_\infty$. Observe that $\mathbb{A}_F^\times = \mathbb{A}_{F,\mathfrak{m}}^\times \cdot F^\times$. Indeed, let $I \cdot \mathfrak{f}_O \cdot O_F = \mathfrak{P}_1^{a_1} \cdots \mathfrak{P}_r^{a_r}$ be the factorisation of the ideal $I \cdot \mathfrak{f}_O \cdot O_F \subseteq O_F$ into prime powers, and fix any element $\varepsilon \in \mathbb{R}_{>0}$ such that for every

$j \in \{1, \dots, r\}$ the ball of radius ε centred at the origin of $F_{\mathfrak{P}_j}$ is contained in $\mathfrak{P}_j^{a_j} \cdot O_{F_{\mathfrak{P}_j}}$. Then for every $j \in \{1, \dots, r\}$ and every idèle $s \in \mathbb{A}_F^\times$ we see that there exists $t_j \in F$ such that

$$|t_j - s_{\mathfrak{P}_j}^{-1}|_{\mathfrak{P}_j} < |s_{\mathfrak{P}_j}^{-1}|_{\mathfrak{P}_j} \cdot (\varepsilon/2)$$

because $F \subseteq F_{\mathfrak{P}_j}$ is dense, Moreover, the approximation theorem [Neu99, Chapter II, Theorem 3.4] shows that there exists $x \in F^\times$ such that $|x - t_j|_{\mathfrak{P}_j} < |s_{\mathfrak{P}_j}^{-1}|_{\mathfrak{P}_j} \cdot (\varepsilon/2)$. Hence we get

$$|(x \cdot s)_{\mathfrak{P}_j} - 1|_{\mathfrak{P}_j} \leq |s_{\mathfrak{P}_j}|_{\mathfrak{P}_j} \cdot \left(|x - t_j|_{\mathfrak{P}_j} + |t_j - s_{\mathfrak{P}_j}^{-1}|_{\mathfrak{P}_j}\right) < \varepsilon$$

which implies that $(x \cdot s)_{\mathfrak{P}_j} \in 1 + \mathfrak{P}_j^{a_j} O_{F_{\mathfrak{P}_j}}$ for every $j \in \{1, \dots, r\}$. This allows us to conclude that $\mathbb{A}_{F,\widetilde{\mathfrak{m}}}^\times \cdot F^\times = \mathbb{A}_F^\times$, where $\mathbb{A}_{F,\widetilde{\mathfrak{m}}}^\times \subseteq \mathbb{A}_F^\times$ is defined analogously to $\mathbb{A}_{F,\mathfrak{m}}^\times$ using the modulus $\widetilde{\mathfrak{m}} = (O_F, I \cdot \mathfrak{f}_O \cdot O_F, \mathfrak{m}_\infty)$. Since it is easy to see that $\mathbb{A}_{F,\widetilde{\mathfrak{m}}}^\times \subseteq \mathbb{A}_{F,\mathfrak{m}}^\times$ we get that $\mathbb{A}_F^\times = \mathbb{A}_{F,\mathfrak{m}}^\times \cdot F^\times$, as we wanted.

We can now observe that there is a well defined group homomorphism

$$\psi \colon \mathbb{A}_{F,\mathfrak{m}}^\times \to \mathcal{J}_{\widetilde{\mathfrak{m}}}$$

which sends an idèle $s \in \mathbb{A}_F^\times$ to the fractional ideal $(s) := \prod_{\mathfrak{p} \subseteq O_F} \mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(s)} \subseteq F$. Indeed for any $s \in \mathbb{A}_{F,\mathfrak{m}}^\times$ and every $j \in \{1, \dots, r\}$ we have that $\mathrm{ord}_{\mathfrak{P}_j}(s) = 0$ because

$$s_{p_j} = (s_\mathcal{P})_{\mathcal{P} | p_j O_F} \in O_p^\times \subseteq \prod_{\mathcal{P} | p_j O_F} O_{F_\mathcal{P}}^\times$$

where $p_j \in \mathbb{N}$ denotes the rational prime lying under $\mathfrak{P}_j$. Now, ψ is surjective because for every fractional ideal $J = \prod_\mathfrak{p} \mathfrak{p}^{a_\mathfrak{p}}$ we have $J = (s)$ with $s_\mathfrak{p} := \pi_\mathfrak{p}^{a_\mathfrak{p}}$ for some uniformiser $\pi_\mathfrak{p} \in F_\mathfrak{p}$. Moreover, we have that $\psi^{-1}(\mathcal{R}_\mathfrak{m}) = U_\mathfrak{m} \cdot F^\mathfrak{m}$ where $F^\mathfrak{m} := \mathbb{A}_{F,\mathfrak{m}}^\times \cap F^\times$. Indeed let $s \in \mathbb{A}_{F,\mathfrak{m}}^\times$ and suppose that $(s) = \alpha \cdot O_F$ where $\alpha = \alpha_1/\alpha_2$ for some $\alpha_1, \alpha_2 \in O$ such that

$$\alpha_1 O + I\mathfrak{f}_O = \alpha_2 O + I\mathfrak{f}_O = O$$

and $\alpha_1 \equiv \alpha_2 (I)$. Then $\alpha_1, \alpha_2 \in O_p^\times$ for every rational prime $p \in \mathbb{N}$ dividing $\mathrm{N}_{F/\mathbb{Q}}(I \cdot \mathfrak{f}_O \cdot O_F)$, which shows that $\alpha \in F^\mathfrak{m}$. Moreover, since $(s) = \alpha O_F$ we see that $s_p \cdot O_{F,p} = \alpha O_{F,p}$ for every rational prime $p \in \mathbb{N}$, which implies that $s_p \cdot O_p = \alpha \cdot O_p$ after noticing that α_1 and α_2 are coprime with $\mathfrak{f}_O$. Hence we get $\alpha^{-1} \cdot s \in (U_O \cap \mathbb{A}_{F,\mathfrak{m}}^\times) = U_\mathfrak{m}$, and thus $s \in \alpha \cdot U_\mathfrak{m} \subseteq U_\mathfrak{m} \cdot F^\mathfrak{m}$, as we wanted to prove. This shows that ι, φ and ψ induce an isomorphism

$$\frac{\mathbb{A}_F^\times}{F^\times \cdot U_\mathfrak{m}} = \frac{F^\times \cdot \mathbb{A}_{F,\mathfrak{m}}^\times}{F^\times \cdot U_\mathfrak{m}} \cong \frac{F^\times \cdot \mathbb{A}_{F,\mathfrak{m}}^\times / F^\times}{F^\times \cdot U_\mathfrak{m} / F^\times} \cong \frac{\mathbb{A}_{F,\mathfrak{m}}^\times}{U_\mathfrak{m} \cdot F^\mathfrak{m}} \xrightarrow[\psi]{\sim} \frac{\mathcal{J}_{\widetilde{\mathfrak{m}}}}{\mathcal{R}_\mathfrak{m}} \xrightarrow[\varphi]{\sim} \frac{\mathcal{J}_\mathfrak{m}}{\mathcal{Q}_\mathfrak{m}} \xrightarrow[\iota^{-1}]{\sim} \frac{\mathcal{I}_\mathfrak{m}}{\mathcal{P}_\mathfrak{m}}$$

which allows us to conclude using the isomorphism (6.14). □

Remark 6.2.18. The isomorphism $\mathrm{Gal}(H_\mathfrak{m}/F) \cong \mathcal{J}_{\widetilde{\mathfrak{m}}}/\mathcal{R}_\mathfrak{m}$ can be used to define the ray class fields $H_\mathfrak{m}$ using the classical language of class field theory. More precisely, we know from Remark 6.2.10 that $\widetilde{\mathfrak{m}} = (O_F, I \cdot \mathfrak{f}_O \cdot O_F, \mathfrak{m}_\infty)$ can be thought of as a classical module for the number field F. Then the classical version of global class field theory (see for example [Neu99,

Chapter VI, Corollary 7.2]) shows that for every finite abelian extension $F \subseteq L$ there exists a classical module $\mathfrak{n} = (\mathcal{O}_F, J, \mathfrak{n}_\infty)$ such that

$$\mathrm{Gal}(L/F) \cong \frac{\mathcal{J}_\mathfrak{n}}{\mathfrak{R}_{L/F,\mathfrak{n}}}$$

where $\mathfrak{R}_{L/F,\mathfrak{n}}$ is a subgroup containing the "ray" $\mathcal{Q}_\mathfrak{n}$. This can be reversed, to show that for every classical module $\mathfrak{n}$ and every subgroup $\mathcal{R} \subseteq \mathcal{J}_\mathfrak{n}$ such that $\mathcal{Q}_\mathfrak{n} \subseteq \mathcal{R}$ there exists a finite abelian extension $F \subseteq L$ such that $\mathcal{R} = \mathfrak{R}_{L/F,\mathfrak{n}}$. Hence we can define $H_\mathfrak{m}$ precisely in this way, by taking $\mathfrak{n} = \widetilde{\mathfrak{n}}$ and $\mathcal{R} := \mathcal{R}_\mathfrak{m}$.

Example 6.2.19. If F is an imaginary quadratic field, we can use Remark 6.2.18 to retrieve the classical definition of ray class fields for imaginary quadratic orders appearing in the PhD thesis of Söhngen [Söh35] (see also [Sch10, §3.3]).

The previous results shows that we can split the abelian extension $F \subseteq H_\mathfrak{m}$ into the subextension $F \subseteq H_\mathcal{O}$ given by the ring class field $H_\mathcal{O}$, which depends only on the order $\mathcal{O}$ and has Galois group $\mathrm{Gal}(H_\mathcal{O}/F) \cong \mathrm{Pic}(\mathcal{O})$, and the upper part $H_\mathcal{O} \subseteq H_\mathfrak{m}$. The following result, which concludes this chapter, computes the Galois group of this upper part.

Theorem 6.2.20 – Galois groups of ray class fields over the ring class field

Let $\mathfrak{m} = (\mathcal{O}, I, \mathfrak{m}_\infty)$ be a generalised module, relative to the number field $F := \mathrm{Frac}(\mathcal{O})$. Then we have the isomorphism:

$$\mathrm{Gal}(H_\mathfrak{m}/H_\mathcal{O}) \cong \frac{(\mathcal{O}/I)^\times}{\pi_I^\times(\mathcal{O}^\times_{\mathfrak{m}_\infty})}$$

where $\pi_I^\times \colon \mathcal{O}^\times \to (\mathcal{O}/I)^\times$ is the map induced by the projection $\pi_I \colon \mathcal{O} \twoheadrightarrow \mathcal{O}/I$, and $\mathcal{O}^\times_{\mathfrak{m}_\infty} \subseteq \mathcal{O}^\times$ denotes the subgroup given by those units $\alpha \in \mathcal{O}^\times$ such that $\iota_v(\alpha) > 0$ for every $v \in \mathfrak{m}_\infty$.

Proof. First of all, we see that

$$\begin{aligned}
\mathrm{Gal}(H_\mathfrak{m}/H_\mathcal{O}) = \ker\left(\mathrm{Gal}(H_\mathfrak{m}/F) \twoheadrightarrow \mathrm{Gal}(H_\mathcal{O}/F)\right) &\overset{(a)}{\cong} \ker\left(\frac{\mathbb{A}_F^\times}{F^\times \cdot U_\mathfrak{m}} \twoheadrightarrow \frac{\mathbb{A}_F^\times}{F^\times \cdot U_\mathcal{O}}\right) \cong \\
&\cong \frac{F^\times \cdot U_\mathcal{O}}{F^\times \cdot U_\mathfrak{m}} \cong \frac{F^\times \cdot U_\mathcal{O}/F^\times}{F^\times \cdot U_\mathfrak{m}/F^\times} \overset{(b)}{\cong} \frac{U_\mathcal{O}/(F^\times \cap U_\mathcal{O})}{(U_\mathfrak{m} \cdot (F^\times \cap U_\mathcal{O}))/(F^\times \cap U_\mathcal{O})} \cong \\
&\cong \frac{U_\mathcal{O}}{U_\mathfrak{m} \cdot (F^\times \cap U_\mathcal{O})} \overset{(c)}{=} \frac{U_\mathcal{O}}{U_\mathfrak{m} \cdot \mathcal{O}^\times}
\end{aligned}$$

where (a) comes from Lemma 6.2.16, (b) holds because $U_\mathfrak{m} \subseteq U_\mathcal{O}$ and (c) follows from the fact that $F^\times \cap U_\mathcal{O} = \mathcal{O}^\times$.

Now, observe that $F_\infty^\times \subseteq U_\mathcal{O}$, where $F_\infty := F \otimes_\mathbb{Q} \mathbb{R} \cong \prod_{w|\infty} F_w \hookrightarrow \mathbb{A}_F$. Moreover, we have

$$\mathfrak{c}_\mathcal{O} \colon \frac{U_\mathcal{O}}{F_\infty^\times} \cong \prod_{p \in \mathbb{N}} \mathcal{O}_p^\times \cong \prod_{p \in \mathbb{N}} \varprojlim_{n \in \mathbb{N}} \left(\frac{\mathcal{O}}{p^n\mathcal{O}}\right)^\times \cong \varprojlim_{N \in \mathbb{Z}_{\geq 1}} \left(\frac{\mathcal{O}}{N\mathcal{O}}\right)^\times \cong \widehat{\mathcal{O}}^\times \tag{6.16}$$

where the products run over the rational primes $p \in \mathbb{N}$, and O_p is the ring defined in (6.13). In the chain of isomorphisms (6.16) the ring $\widehat{O}$ is the profinite completion of O, *i.e.*

$$\widehat{O} := \varprojlim_{N \in \mathbb{Z}_{\geq 1}} \frac{O}{NO} \cong \prod_{p \in \mathbb{N}} O_p \cong \prod_{\mathfrak{p} \subseteq O} O_{\mathfrak{p}} \tag{6.17}$$

where the second product runs over all the non-zero prime ideals $\mathfrak{p} \subseteq O$ and $O_{\mathfrak{p}} := \varprojlim_{n \in \mathbb{N}} O/\mathfrak{p}^n$ is the completion of O at the prime $\mathfrak{p}$. The second isomorphism appearing in (6.17) can be obtained by applying [Eis95, Corollary 7.6] to $R = \mathbb{Z}_p$ and $A = O_p$. This gives the decomposition

$$O_p \cong \prod_{\mathfrak{p} \supseteq p} O_{\mathfrak{p}} \tag{6.18}$$

where the product runs over all primes $\mathfrak{p} \subseteq O$ lying above p.

Under the isomorphism (6.16) the subgroup $U_{I,O}/F_\infty^\times \subseteq U_O/F_\infty^\times \cong \widehat{O}^\times$ is identified with the kernel of the map $\widehat{\pi}_I^{\,\times} \colon \widehat{O}^\times \to (\widehat{O}/I\widehat{O})^\times$ induced by the projection $\widehat{\pi}_I \colon \widehat{O} \twoheadrightarrow \widehat{O}/I\widehat{O}$. Using this, one sees that the subgroup $(U_{\mathfrak{m}} \cdot O^\times \cdot F_\infty^\times)/F_\infty^\times \subseteq U_O/F_\infty^\times \cong \widehat{O}^\times$ is identified with $\ker(\widehat{\pi}_I^{\,\times}) \cdot O_{\mathfrak{m}_\infty}^\times$. Hence we get that

$$\mathrm{Gal}(H_{\mathfrak{m}}/H_O) \cong \frac{U_O}{U_{\mathfrak{m}} \cdot O^\times} \cong \frac{U_O/F_\infty^\times}{(U_{\mathfrak{m}} \cdot O^\times \cdot F_\infty^\times)/F_\infty^\times} \cong \frac{\widehat{O}^\times}{\ker(\widehat{\pi}_I^{\,\times}) \cdot O_{\mathfrak{m}_\infty}^\times} \cong \frac{(\widehat{O}/I\widehat{O})^\times}{\widehat{\pi}_I^{\,\times}(O_{\mathfrak{m}_\infty}^\times)}$$

because $\widehat{\pi}_I^{\,\times}$ is surjective. This surjectivity is shown by the factorisation

$$\begin{array}{ccc} \widehat{O}^\times & \xrightarrow{\widehat{\pi}_I^{\,\times}} & \left(\widehat{O}/I\widehat{O}\right)^\times \\ & \searrow \quad \nearrow & \\ & \prod_{\mathfrak{p} \supseteq I} O_{\mathfrak{p}}^\times & \end{array}$$

where the first map $\widehat{O}^\times \twoheadrightarrow \prod_{\mathfrak{p} \supseteq I} O_{\mathfrak{p}}^\times$ is surjective as follows from (6.17), and the second map

$$\prod_{\mathfrak{p} \supseteq I} O_{\mathfrak{p}}^\times \twoheadrightarrow \prod_{\mathfrak{p} \supseteq I} \left(\frac{O_{\mathfrak{p}}}{IO_{\mathfrak{p}}}\right)^\times \cong \left(\frac{\widehat{O}}{I\widehat{O}}\right)^\times$$

is surjective by [Che20, Corollary 2.3], which can be applied since the ring $\prod_{\mathfrak{p} \supseteq I} O_{\mathfrak{p}}$ has finitely many maximal ideals.

To finish our proof we need to show the isomorphism

$$\frac{(\widehat{O}/I\widehat{O})^\times}{\widehat{\pi}_I^{\,\times}(O_{\mathfrak{m}_\infty}^\times)} \cong \frac{(O/I)^\times}{\pi_I^\times(O_{\mathfrak{m}_\infty}^\times)}.$$

To do this recall that π_I and $\widehat{\pi}_I$ are related by the commutative diagram

$$\begin{array}{ccccc} \mathcal{O} & \xrightarrow{\pi_I} & \mathcal{O}/I & \xrightarrow{\gamma} & \prod_{\mathfrak{p}\supseteq I} \frac{\mathcal{O}_{(\mathfrak{p})}}{I\mathcal{O}_{(\mathfrak{p})}} \\ \downarrow & & \downarrow & & \downarrow \beta \\ \widehat{\mathcal{O}} & \xrightarrow{\widehat{\pi}_I} & \widehat{\mathcal{O}}/I\widehat{\mathcal{O}} & \xrightarrow[\sim]{\alpha} & \prod_{\mathfrak{p}\supseteq I} \frac{\mathcal{O}_{\mathfrak{p}}}{I\mathcal{O}_{\mathfrak{p}}} \end{array}$$

where α is the isomorphism coming from the decomposition (6.17), and β and γ are the maps induced by the natural inclusions $\mathcal{O} \subseteq \mathcal{O}_{(\mathfrak{p})} \subseteq \mathcal{O}_{\mathfrak{p}}$. Moreover the products run over all the prime ideals $\mathfrak{p} \subseteq \mathcal{O}$ such that $\mathfrak{p} \supseteq I$, and $\mathcal{O}_{(\mathfrak{p})}$ denotes the localisation of $\mathcal{O}$ at the prime $\mathfrak{p}$.

Hence to conclude it is sufficient to observe that γ is an isomorphism by [Neu99, Chapter I, Proposition 12.3], and β is an isomorphism because $\mathcal{O}$ is a one-dimensional Noetherian domain (see [Neu99, Chapter I, Proposition 12.2]). More explicitly, for any prime $\mathfrak{p} \subseteq \mathcal{O}$ such that $\mathfrak{p} \supseteq I$ we have that $\mathfrak{p} \cdot \mathcal{O}_{(\mathfrak{p})} = \sqrt{I \cdot \mathcal{O}_{(\mathfrak{p})}}$ because $\mathcal{O}_{(\mathfrak{p})}$ is a one-dimensional local ring. Hence [Bou89, Chapter II, § 2.6, Proposition 15] shows that $\mathcal{O}_{(\mathfrak{p})}/I\mathcal{O}_{(\mathfrak{p})}$ is complete with respect to $\mathfrak{p}\mathcal{O}_{(\mathfrak{p})}$. Thus we can conclude that $\mathcal{O}_{(\mathfrak{p})}/I\mathcal{O}_{(\mathfrak{p})}$ is isomorphic to $\mathcal{O}_{\mathfrak{p}}/I\mathcal{O}_{\mathfrak{p}}$ using the exactness of completion, which holds because $\mathcal{O}_{(\mathfrak{p})}$ is Noetherian (see [Eis95, Lemma 7.15]). □

The theory of complex multiplication

7

> The more complex the mind,
> the greater the need for the simplicity of play.
>
> James T. Kirk,
> *Star Trek: The Original Series*

The aim of this chapter is to introduce the main points of the theory of complex multiplication (often abbreviated by CM) which are needed in what follows. This theory is incredibly rich, and it has been understood for a long time that objects with complex multiplication form a fertile testing ground for numerous conjectures in arithmetic geometry. These include the Mumford-Tate conjecture (see for instance [FC20, Page 4] for a motivic version of this conjecture) and Beilinson's conjectures for the special values of L-functions (see Conjecture 3.3.18). More precisely, the Mumford-Tate conjecture is known by work of Pohlmann (see [Poh68]) for every abelian variety with complex multiplication, and the weak form of Beilinson's conjecture (see Conjecture 3.3.28) is known in the following cases:

- for the special value $L^*(A, 1)$ associated to any abelian variety A with complex multiplication, thanks to work of Blasius and Harder (see [HS85]), which was later revisited in Colmez's thesis (see [Col89]);
- for the special values $L^*(E, n)$ with $n \neq 1$, associated to any elliptic curve E with complex multiplication that satisfies Shimura's condition (see Definition 7.1.30), thanks to the work of Deninger (see [Den89] and [Den90]).

All these proofs and conjectures use heavily the structure of torsion points on CM abelian varieties, and the fact that the Galois representation induced by these is far better understood in the CM case than in the general case. This is exemplified by the main theorem of complex multiplication (see Theorem 7.1.25) which is part of to the groundbreaking work of Shimura and Taniyama [ST61], and describes the Galois action on torsion points in terms of class field theory and the global Artin map.

Let us review the contents of this chapter. First of all, we devote Section 7.1 to defining abelian varieties with complex multiplication, and to giving the statement of the main theorem of complex multiplication, of which we present a partial proof in the case of elliptic curves. The two central sections of the chapter are then based on joint work with Francesco Campagna. First of all, the aim of Section 7.2 is to present a proof of the third main theorem of complex multiplication (see [ST61, Page 142]) for elliptic curves with complex multiplication by general orders. While the corresponding result for curves with CM by maximal orders is classical (see [Sil94, Chapter II, Theorem 5.6]), the corresponding result for elliptic curves having complex multiplication by non-maximal orders is not equally well documented. This result can of course be derived as a special case of the result of Shimura and Taniyama, but it is in fact due to Söhngen [Söh35]. Both proofs, as well as the pedagogical account of Söhngen's proof given by Schertz in

[Sch10, Theorem 6.2.3], use the classical language of class field theory, whereas Stevenhagen's account [Ste01, § 4] of Söhngen's proof uses an idelic language, but focuses only on the case of N-torsion points, for some integer $N \in \mathbb{N}$, instead of the more general case of I-torsion points for some ideal $I \subseteq \mathcal{O}$. Therefore we thought it meaningful to include a completely idelic proof of the general case, based on the notions of ray class fields for orders that we introduced in the previous chapter. Secondly, Section 7.3 contains the proof of an optimal bound for the index of the image of the Galois representation attached to the torsion points of an elliptic curve with complex multiplication. This partially generalises work of Lombardo (see [Lom17]) and Bourdon and Clark (see [BC20]). Finally, Section 7.4 is dedicated to recalling the results of Deninger on Beilinson's conjecture for elliptic curves with complex multiplication at non-critical integers. We focus in particular on the integer $s = 2$, which is the subject of a theorem of Rohrlich (see Theorem 7.4.5), which generalises earlier work of Bloch (see [Blo00, Theorem 11.2.1]).

7.1 Abelian varieties with complex multiplication

It is a common, general theme of mathematics to study the symmetries of objects. Most notably, it was precisely to study symmetries of roots of polynomials that the notion of group was envisioned by Galois. Since then, it has become apparent that the theory underlying objects whose number of symmetries is different from the average one is both richer and easier to develop than the general one. Going back to the solutions of polynomial equations, it is no mystery (as we reviewed in Section 6.1.3), that class field theory, which studies abelian extensions of number fields, results in a theory far more vivid than the general theory of Galois extensions of number fields.

The theory of abelian varieties is no exception to this rule, but here the situation is somehow reversed: the more symmetries an abelian variety possesses, the richer the theory. To be more precise, symmetries of an abelian variety A are given by the elements of its endomorphism ring $\mathrm{End}(A)$. In order to study this ring, we can employ the following result, which allows us to split an abelian variety into its simple constituents (see [CCO14, Theorem 1.2.1.3]).

Theorem 7.1.1 – Poincaré reducibility theorem

Let A be a non-zero abelian variety defined over a field κ. Then there exists an isogeny

$$A \longrightarrow A_1^{e_1} \times \cdots \times A_n^{e_n} \tag{7.1}$$

where $e_1, \ldots, e_n \in \mathbb{N}$ and $\{A_1, \ldots, A_n\}$ are pairwise non-isogenous *simple* abelian varieties, *i.e.* abelian varieties which do not have any non-trivial abelian sub-varieties.

Using Theorem 7.1.1 we see that the endomorphism $\mathbb{Q}$-algebra $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is semi-simple. More precisely, each algebra $\mathrm{End}^0(A_i)$ is a simple division $\mathbb{Q}$-algebra, which means that $\mathrm{End}^0(A_i)$ does not have any non-trivial two-sided ideal and for every pair $\alpha, \gamma \in \mathrm{End}^0(A_i)$ there exists a unique pair $\beta, \delta \in \mathrm{End}^0(A_i)$ such that $\alpha = \beta\gamma = \gamma\delta$. Moreover, the endomorphism algebra $\mathrm{End}^0(A)$ decomposes as

$$\mathrm{End}^0(A) \cong \prod_{i=1}^{n} \mathrm{Mat}_{e_i \times e_i}(\mathrm{End}^0(A_i))$$

because End^0 is invariant under isogeny.

Since the algebra $\mathrm{End}^0(A)$ is semi-simple, we can measure how big it is by means of its *reduced degree*, which is defined using the following lemma on simple algebras (see [Coh03, Theorem 5.4.6]).

Lemma 7.1.2 – Centres of finite dimensional simple algebras

Let F be a field and let M be a finite dimensional simple algebra over F. Then the centre $Z(M) \subseteq M$ is a field, which is a finite extension of F. Moreover, the index $[M\colon Z(M)]$ is finite, and is the square of an integer.

Definition 7.1.3 – Reduced degree

Let F be a field and M be a finite dimensional semi-simple algebra over F. Then its *reduced degree* is defined as

$$[M\colon F]^{\mathrm{red}} := \sum_{i=1}^{n} [M_i\colon F]^{\mathrm{red}} := \sum_{i=1}^{n} \sqrt{[M_i\colon Z(M_i)]}\,[Z(M_i)\colon F] \in \mathbb{N}$$

where $M = M_1 \times \cdots \times M_n$ is the decomposition of M as a product of simple algebras.

As we said, the reduced degree of the $\mathbb{Q}$-algebra $\mathrm{End}^0(A)$ is a good way of measuring how many endomorphisms A has. This is also expressed by the following result, which shows that the reduced degree is bounded in terms of the dimension of the abelian variety A (see [CCO14, Theorem 1.3.1.1]).

Theorem 7.1.4 – Reduced degree and étale sub-algebras

Let A be an abelian variety over a field κ. Then we have that

$$[\mathrm{End}^0(A)\colon \mathbb{Q}]^{\mathrm{red}} = \max\{[M\colon \mathbb{Q}] \mid M \subseteq \mathrm{End}^0(A) \text{ is étale over } \mathbb{Q}\}$$

and $[\mathrm{End}^0(A)\colon \mathbb{Q}]^{\mathrm{red}} \leq 2\dim(A)$.

As we mentioned in the beginning, CM abelian varieties are those abelian varieties which have more symmetries than usual. Since we have seen that the reduced degree is a way of measuring how many endomorphisms an abelian variety has, it is not surprising to give the following definition.

Definition 7.1.5 – Abelian varieties with complex multiplication

Let A be an abelian variety defined over a field κ. Then A has *complex multiplication* if

$$[\mathrm{End}^0(A)\colon \mathbb{Q}]^{\mathrm{red}} = 2\dim(A)$$

and A has *potential complex multiplication* if there exists a finite extension of fields $\kappa \subseteq \kappa'$ such that the base-change $A_{/\kappa'}$ has complex multiplication. We call *CM abelian varieties* the abelian varieties with potential complex multiplication.

Definition 7.1.5 has the virtue of being an intrinsic definition, which does not depend on anything but the abelian variety A. On the other hand, we may observe that every polarisation $A \to A^\vee$ induces a positive involution on $\mathrm{End}^0(A)$, and thus on the single factors $\mathrm{End}^0(A_i)$ coming from the decomposition (7.1). Hence one can use Albert's classification of division $\mathbb{Q}$-algebras endowed with a positive involution, to describe the endomorphism algebra of an abelian variety with complex multiplication as follows (see [CCO14, § 1.3.6]).

Theorem 7.1.6 – Endomorphisms of abelian varieties with CM

Let A be an abelian variety with complex multiplication defined over a field κ. Fixing a decomposition of A into its simple isogeny factors (see Equation (7.1)) we have that

$$\mathrm{End}^0(A) \cong \prod_{i=1}^{n} \mathrm{Mat}_{e_i \times e_i}(K_i) \tag{7.2}$$

where the $\mathbb{Q}$-algebras K_i are:

- non-split quaternion algebras if $\mathrm{char}(\kappa) > 0$;
- number fields, which are totally imaginary quadratic extensions of a totally real number field $K_i^+ \subseteq K_i$, if $\mathrm{char}(\kappa) = 0$.

The class of number fields appearing in Theorem 7.1.6 deserves a special name.

Definition 7.1.7 – CM fields, CM algebras, CM types and CM pairs

A *CM field* K is a number field such that there exists a sub-field $K^+ \subseteq K$ with the property that $[K\colon K^+] = 2$ and $[K_w\colon K_v^+] = 2$ for every Archimedean place $w \in M_K^\infty$ lying above the place $v \in M_{K^+}^\infty$. In other words, K is a totally imaginary quadratic extension of a totally real number field K^+.

A *CM algebra* E is a product of CM fields $E = K_1 \times \cdots \times K_n$, and a *CM type* for E is a collection $\Phi \subseteq \mathrm{Hom}(E, \mathbb{C})$ such that $\Phi \cap \overline{\Phi} = \emptyset$ and $\Phi \cup \overline{\Phi} = \mathrm{Hom}(E, \mathbb{C})$, where the elements of $\overline{\Phi} \subseteq \mathrm{Hom}(E, \mathbb{C})$ are obtained by composing the elements of Φ with complex conjugation.

Finally, a *CM pair* is a pair (E, Φ) where E is a CM algebra and Φ is a CM type for E.

Remark 7.1.8. CM fields can equivalently be defined as number fields K endowed with an automorphism $\sigma\colon K \to K$ such that $\Phi_\infty \circ \iota = \iota \circ \sigma$ for every embedding $\iota\colon K \hookrightarrow \mathbb{C}$, where $\Phi_\infty\colon \mathbb{C} \to \mathbb{C}$ denotes complex conjugation.

Example 7.1.9. CM fields of degree $[K\colon \mathbb{Q}] = 2$ are precisely the imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d})$ for some square-free $d \in \mathbb{N}$. Hence for every elliptic curve E defined over a field κ of characteristic zero we have that either $\mathrm{End}^0(E) \cong \mathbb{Q}$ or $\mathrm{End}^0(E) \cong K$ for some imaginary quadratic field K.

Let us now consider the problem of determining when some CM abelian variety A defined over a field κ of characteristic zero has all its complex multiplications defined over κ. In order to give a complete answer to this question, we need to introduce the *reflex* of a CM pair (E, Φ), using the following result (see [ST61, § 8.3]).

Proposition 7.1.10 – Fields generated by traces and norms

Let K be a CM field, and $\Phi \subseteq \mathrm{Hom}(K, \mathbb{C})$ be a CM type for K. Then we have that $\mathbb{Q}(\mathrm{tr}_\Phi(K)) = \mathbb{Q}(\mathrm{N}_\Phi(K^\times))$, where

$$\begin{aligned} \mathrm{tr}_\Phi\colon K &\to \mathbb{C} \\ x &\mapsto \sum_{\varphi\in\Phi} \varphi(x) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{N}_\Phi\colon K^\times &\to \mathbb{C}^\times \\ x &\mapsto \prod_{\varphi\in\Phi} \varphi(x) \end{aligned} \tag{7.3}$$

are the *trace* and *norm* associated to the type Φ. Moreover, the number field

$$K^* := \mathbb{Q}(\mathrm{tr}_\Phi(K)) = \mathbb{Q}(\mathrm{N}_\Phi(K^\times))$$

is a CM field, endowed with an embedding $\iota_{K^*}\colon K^* \hookrightarrow \mathbb{C}$ coming from the maps tr_Φ and N_Φ. Finally, we have that

$$\mathrm{Aut}(\mathbb{C}/\iota_{K^*}(K^*)) = \{\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}) \mid \sigma \circ \varphi \in \Phi,\ \forall \varphi \in \Phi\} \tag{7.4}$$

and that for any $\varphi \in \Phi$, the set

$$\Phi^* := \{\sigma^{-1} \circ \iota_{K^*} \mid \sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}),\ \sigma \circ \varphi_1 \in \Phi\} \subseteq \mathrm{Hom}(K^*, \mathbb{C})$$

is a CM type of K^*, which does not depend on φ.

Definition 7.1.11 – Reflex of a CM pair

Let $(E, \Phi) = (K_1 \times \cdots \times K_n, \Phi_1 \times \cdots \times \Phi_n)$ be a CM pair, in the sense of Definition 7.1.7. Then the *reflex pair* (E^*, Φ^*) is defined as

$$(E^*, \Phi^*) := (K_1^* \times \cdots \times K_n^*, \Phi_1^* \times \cdots \times \Phi_n^*)$$

where K_i^* and Φ_i^* are the fields and the types provided by Proposition 7.1.10.

Let us observe that the type norm $\mathrm{N}_\Phi\colon K^\times \to \mathbb{C}^\times$ induces a map $\mathrm{N}_\Phi\colon K^\times \to (K^*)^\times$ which is *algebraic*, in the sense of the following definition.

Definition 7.1.12 – Algebraic maps of multiplicative groups

Let $f\colon K^\times \to F^\times$ be a group homomorphism between the multiplicative groups of two number fields K and F. Then we say that f is *algebraic* if one of the following equivalent conditions holds:

- for every $\mathbb{Q}$-basis $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ of K, there exists a rational function
$$f_{\mathcal{B}}(x_1, \dots, x_n) \in F(x_1, \dots, x_n)$$
such that
$$f\left(\sum_{j=1}^{n} a_j \alpha_j\right) = f_{\mathcal{B}}(a_1, \dots, a_n)$$
for every $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Q}^n \setminus \{0\}$;
- there exists a function $m_f\colon \mathrm{Hom}(K, \overline{F}) \to \mathbb{Z}$ such that
$$f(x) = \prod_{\tau\colon K \hookrightarrow \overline{F}} \tau(x)^{m_f(\tau)} \tag{7.5}$$
for every $x \in K^\times$;
- there exists a morphism of $\mathbb{Q}$-schemes
$$f_0\colon \mathrm{N}_{K/\mathbb{Q}}(\mathbb{G}_{m,K}) \to \mathrm{N}_{F/\mathbb{Q}}(\mathbb{G}_{m,F}) \tag{7.6}$$
which induces f when evaluated at $\mathbb{Q}$-points. Here $\mathrm{N}_{K/\mathbb{Q}}$ and $\mathrm{N}_{F/\mathbb{Q}}$ denote the Weil restriction functors.

Example 7.1.13. Let K be a CM field of degree $2d = [F\colon \mathbb{Q}]$, and let $\Phi \subseteq \mathrm{Hom}(K, \mathbb{C})$ be a CM type for K. Then the type norm $\mathrm{N}_\Phi\colon K^\times \to \mathbb{C}^\times$ defined in (7.3) induces an algebraic map $\mathrm{N}_\Phi\colon K^\times \to (K^*)^\times$. To see this, fix an element $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$, and consider the polynomial
$$f_{\Phi,\alpha}(t) := \prod_{\varphi \in \Phi} (t - \varphi(\alpha)) \in \mathbb{C}[t]$$
which has actually coefficients in $K^* \hookrightarrow \mathbb{C}$, in virtue of (7.4). Thus we see that there exists a unique injective map of $\mathbb{Q}$-algebras $\Phi_\alpha\colon K \hookrightarrow \mathrm{Mat}_{d\times d}(K^*)$ such that $\Phi_\alpha(\alpha) := M_{\Phi,\alpha}$, where $M_{\Phi,\alpha}$ is the companion matrix of $f_{\Phi,\alpha}$. Clearly this map Φ_α does not depend on the choice of α, and one can show that $\mathrm{N}_\Phi(x) = \det(\Phi_\alpha(x))$ for every $x \in K^\times$. This shows in particular that N_Φ is algebraic, using the first of the equivalent conditions appearing in Definition 7.1.12.

Remark 7.1.14. Let us observe that every algebraic map $f\colon K^\times \to F^\times$ induces maps
$$f_0(R)\colon (R \otimes_{\mathbb{Q}} K)^\times \to (R \otimes_{\mathbb{Q}} F)^\times$$

for every $\mathbb{Q}$-algebra R, simply by evaluating the map (7.6) at R-points. In particular, if we take $R = \mathbb{A}_{\mathbb{Q}}$ and $f = \mathrm{N}_\Phi$ we get the *idelic type norm*

$$\mathrm{N}_\Phi \colon \mathrm{A}_K^\times \to \mathbb{A}_{K^*}^\times \tag{7.7}$$

which is a continuous group homomorphism. Moreover, the reflex type Φ^* induces a map

$$\mathrm{N}_{\Phi^*} \colon \mathbb{A}_{K^*}^\times \to \mathbb{A}_K^\times \tag{7.8}$$

using the fact that $(K^*)^* \subseteq K$ for every CM field K, which follows directly from Definition 7.1.7.

We can now see under which conditions an abelian variety A with potential complex multiplication, defined over a field κ of characteristic zero, acquires all its complex multiplications after base-change to a given finite extension $\kappa' \supseteq \kappa$ (see [Shi98, Chapter II, Proposition 30]).

Proposition 7.1.15 – Field extensions and complex multiplications

Let A be an abelian variety defined over a field $\kappa \subseteq \mathbb{C}$, and suppose that A has complex multiplication (over κ). Then the action of $\mathrm{End}^0(A)$ over the tangent space of A at the origin induces a CM type Φ on the CM algebra $E := K_1 \times \cdots \times K_n$ coming from the decomposition (7.2). Moreover, this action induces embeddings $\iota_{K_i} \colon K_i^* \hookrightarrow \kappa$ for every $i \in \{1, \dots, n\}$.

Conversely, fix an abelian variety A defined over a field $\kappa \subseteq \mathbb{C}$, and suppose that there exists a finite extension $\kappa' \supseteq \kappa$ such that

$$\mathrm{End}^0(A_{/\kappa'}) \cong \prod_{i=1}^{n} \mathrm{Mat}_{e_i \times e_i}(K_i)$$

for some CM fields $K_1, \dots, K_n$, where $A \simeq A_1^{e_1} \times \cdots \times A_n^{e_n}$ is the isogeny decomposition given by Theorem 7.1.1. Then for every sub-extension $\kappa \subseteq \widetilde{\kappa} \subseteq \kappa'$ we have that

$$\mathrm{End}^0(A_{/\widetilde{\kappa}}) = \mathrm{End}^0(A_{/\kappa'}) \iff \iota_{K_i}(K_i) \subseteq \widetilde{\kappa},\ \forall i \in \{1, \dots, n\}$$

where $\iota_{K_i} \colon K_i \hookrightarrow \kappa'$ is the embedding defined in the previous paragraph.

Example 7.1.16. If E is an elliptic curve defined over a number field F, which has potential complex multiplication by an imaginary quadratic field K, we see from Proposition 7.1.15 that $\mathrm{End}^0(E) \cong \mathrm{End}^0(E_{/\overline{F}})$ if and only if $K \subseteq F$.

7.1.1 The main theorem of complex multiplication

One of the reasons why the theory of CM abelian varieties is much richer than the general one, is due to the fact that the L-function $L(A_{/F}, s)$ associated to the motive $\underline{H}^1(A/F)$ can be expressed in terms of L-functions of *Hecke characters*, which are certain characters of the group of idèles associated to the number field over which A is defined. In particular, the validity of Conjecture 3.3.4 and Conjecture 3.3.6 is established for L-functions of CM abelian varieties, thanks to work of Hecke (see [Neu99, Chapter VII, § 8]). We devote this section to explaining these claims, and to the presentation of the so-called *main theorem of complex multiplication*, the technical backbone upon which these results rely.

First of all, let us introduce the notion of *Hecke character*.

Definition 7.1.17 – Hecke character

Let F be a number field and let Ω be a topological ring. A *Ω-valued Hecke character* is a continuous group homomorphism

$$\psi\colon \mathbb{A}_F^\times \to \Omega^\times$$

such that $\psi(F^\times) = 1$, where $F^\times \hookrightarrow \mathbb{A}_F^\times$ via the diagonal embedding. A *Hecke character* is a $\mathbb{C}$-valued Hecke character. For every sub-field $K \subseteq F$ and every place $v \in M_K$ we denote by $\psi_v\colon F_v^\times \to \Omega^\times$ the restriction of ψ to $F_v^\times := (F \otimes_K K_v)^\times \cong \prod_{w|v} F_w^\times$.

Remark 7.1.18. Let us recall the related notion of algebraic Hecke character, which is not used in this thesis but is fundamentally related to the theory of complex multiplication. Fix a pair of number fields K and E, and consider the latter as a discrete topological field. Then an *E-valued algebraic Hecke character* for K is an E-valued Hecke character

$$\psi\colon \mathbb{A}_K^\times \to E^\times$$

such that the restriction $\psi_{\mathrm{alg}} := \psi \circ \iota_K\colon K^\times \to E^\times$ is algebraic, in the sense of Definition 7.1.12.

Let us observe that any algebraic Hecke character $\psi\colon \mathbb{A}_K^\times \to E^\times$ induces a family of Hecke characters $\psi^{(\lambda)}\colon \mathbb{A}_K^\times \to E_\lambda^\times$ indexed over the set of places $\lambda \in M_E$, where E_λ is endowed with the λ-adic topology. Indeed, it is sufficient to take $\psi^{(\lambda)} := \psi \cdot (\pi_\lambda \circ \psi_0(\mathbb{A}_\mathbb{Q}))^{-1}$ where $\psi_0(\mathbb{A}_\mathbb{Q})\colon \mathbb{A}_K^\times \to \mathbb{A}_E^\times$ is the map induced by evaluating (7.6) at $\mathbb{A}_\mathbb{Q}$-points, and $\pi_\lambda\colon \mathbb{A}_E^\times \twoheadrightarrow E_\lambda^\times$ is the canonical projection.

Hecke characters can be thought of as one dimensional *automorphic representations*. Thus, as we mentioned in the introduction of Chapter 3, there is a way to associate certain L-functions (defined as a suitable Euler product) to these Hecke characters. Let us recall this definition, following [RV99, § 7.4].

Definition 7.1.19 – Local L-factors

Let K be a local field of characteristic zero, and let $\chi\colon K^\times \to \mathbb{C}^\times$ be a continuous group homomorphism. Suppose that K is non-Archimedean, and let $\pi \in K^\times$ be a uniformiser. Then we define the *local L-factor* $L(\chi) \in \mathbb{C}$ as:

$$L(\chi) := \begin{cases} (1 - \chi(\pi)), & \text{if } \chi(O_K^\times) = 1 \\ 1, & \text{otherwise} \end{cases}$$

which does not depend on the choice of π, because for any two uniformisers $\pi, \pi' \in K^\times$ we have that $\pi/\pi' \in O_K^\times$. Suppose now that $K \cong \mathbb{C}$. In this case we define the local L-factor as

$$L(\chi) := \Gamma_\mathbb{C}\left(a(\chi) + \frac{|b(\chi)|}{2}\right) \in \mathbb{C}$$

where $a(\chi) \in \mathbb{C}$ and $b(\chi) \in \mathbb{N}$ are the unique numbers such that $\chi(z) = |z|^{a(\chi)}(z/|z|)^{b(\chi)}$ for every $z \in \mathbb{C}^\times$. Finally, if $K \cong \mathbb{R}$ we define

$$L(\chi) := \Gamma_{\mathbb{R}}(a(\chi) + b(\chi))$$

where $a(\chi) \in \mathbb{C}$ and $b(\chi) \in \{0, 1\}$ are such that $\chi(t) = |t|^{a(\chi)}(t/|t|)^{b(\chi)}$ for every $t \in \mathbb{R}^\times$.

Definition 7.1.20 – Hecke L-functions

Let F be a number field and $\psi\colon \mathbb{A}_F^\times \to \mathbb{C}^\times$ be a Hecke character. Then, for every finite set $S \subseteq M_F$ we define the L-function of ψ as the Euler product

$$L_S(\psi, s) = \prod_{v \in M_F \setminus S} L(\psi_v |\cdot|_v^s) \tag{7.9}$$

where $\psi_v\colon F_v^\times \to \mathbb{C}^\times$ denotes the character induced by the embedding $F_v \hookrightarrow \mathbb{A}_F$, and $|\cdot|_v\colon F_v^\times \to \mathbb{R}_{>0}$ denotes the v-adic absolute value, normalised in such a way that the product formula

$$\prod_{v \in M_F} |x|_v = 1$$

holds for every $x \in F^\times$. We finally write $L(\psi, s) := L_{M_F^\infty}(\psi, s)$ and $\widehat{L}(\psi, s) := L_\emptyset(\psi, s)$.

One of the reasons why Hecke L-functions are so interesting to study is because they are amongst the few which are known to admit a meromorphic continuation to $\mathbb{C}$, and to satisfy a functional equation, as explained by the following result (see [Neu99, Chapter VII, Theorem 8.5] and [RV99, Theorem 7-19]).

Theorem 7.1.21 – Meromorphic continuation and functional equations for Hecke L-functions

Let $\psi\colon \mathbb{A}_F^\times \to \mathbb{C}^\times$ be a Hecke character. Then the Euler product (7.9) converges for every $s \in \mathbb{C}$ such that $\Re(s) > 1$, and the L-function $L_S(\psi, s)\colon \Re_1 \to \mathbb{C}$ admits a meromorphic continuation to the whole complex plane. Furthermore, $\widehat{L}(\psi, s)$ satisfies a functional equation of the form

$$\widehat{L}(\psi, s) = \varepsilon(\psi, s) \cdot \widehat{L}(1 - s, \psi\|\cdot\|^{-1})$$

where $\varepsilon(\psi, s) := a(\psi) \cdot e^{b(\psi)s}$ is the ε-factor defined in [RV99, Theorem 7.2]. Here

$$\begin{aligned} \|\cdot\|\colon \mathbb{A}_F^\times &\to \mathbb{R}_{>0} \\ \alpha &\mapsto \prod_{v \in M_F} |\alpha|_v \end{aligned}$$

denotes the idelic norm map, and the complex numbers $a(\psi), b(\psi) \in \mathbb{C}$ are defined by setting

$$a(\psi) := W(\psi)\sqrt{|\Delta_F| \, \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{f}_\psi)}$$
$$b(\psi) := -\log||\Delta_F| \, \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{f}_\psi)|$$

where $\Delta_F \in \mathbb{Z}$ is the absolute discriminant of F and $W(\psi) \in \mathbb{T}^1$ is the *root number*, defined in [RV99, Page 259]. Moreover, $\mathfrak{f}_\psi \subseteq O_F$ denotes the *conductor* of ψ, which is defined as the product of prime ideals $\mathfrak{f}_\psi := \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f}_\psi)}$, where the integers $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f}_\psi)$ are defined by setting

$$\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f}_\psi) := \min\{n \in \mathbb{N} \mid \psi_{\mathfrak{p}}(O_F + \mathfrak{p}^n) = 1\} \tag{7.10}$$

where $\psi_{\mathfrak{p}} \colon F_{\mathfrak{p}}^\times \to \mathbb{C}^\times$ denotes the restriction of ψ to the $\mathfrak{p}$-adic completion of F. Finally, $L(\psi, s)$ is entire unless $\psi = \|\cdot\|^{-it}$ for some $t \in \mathbb{R}$. In the latter case the L-function $L(\psi, s)$ has two poles at $s = it$ and $s = 1 + it$.

Remark 7.1.22. We note that Theorem 7.1.21 uses the analytic normalisation for Hecke L-functions, which is the same one used in [RV99].

Remark 7.1.23. The definition of the conductor $\mathfrak{f}_\psi \subseteq O_F$ of a Hecke character $\psi \colon \mathbb{A}_F^\times \to \mathbb{C}^\times$ can be extended to any Hecke character $\psi \colon \mathbb{A}_F^\times \to \Omega^\times$ valued in a topological ring Ω such that the topological group of units $\Omega^\times$ has no small subgroup. This means that there exists an open subset $U \subseteq \Omega^\times$ such that $1 \in U$ and U does not contain any subgroup. In this case we know that the set

$$\{n \in \mathbb{N} \mid \psi_{\mathfrak{p}}(O_F + \mathfrak{p}^n) = 1\}$$

is non-empty for every prime ideal $\mathfrak{p} \subseteq O_F$, which shows that the definition (7.10) makes sense. The typical example of a topological ring Ω such that $\Omega^\times$ has no small subgroup is given by the product of finitely many copies of $\mathbb{C}$.

We have already mentioned in Example 3.3.8 that Theorem 7.1.21 can be seen as one of the few examples in which Conjecture 3.3.4 holds. In order to do this, we need to relate the Hecke L-function $L(\psi, s)$ to some motivic L-function. This can be achieved for every Hecke character $\psi \colon \mathbb{A}_F^\times \to \mathbb{C}^\times$ which arises as the Archimedean component of an algebraic Hecke character, as the following result shows (see [Sch88, § 4 and § 5])

Theorem 7.1.24 – Motives for algebraic Hecke characters

Let F and E be two number fields, and let $\psi \colon \mathbb{A}_F^\times \to E^\times$ be an algebraic Hecke character, as defined in Remark 7.1.18. Then there exists a motive $M \in \mathcal{MM}(F; E)$, which is unique up to isomorphism, such that the motivic L-function $L(M(\psi), s) \colon \mathbb{C} \to E \otimes_{\mathbb{Q}} \mathbb{C}$ decomposes as

$$L(M(\psi), s)_\lambda = \begin{cases} L\left(\psi^{(\lambda)}, s - \frac{w(\psi)}{2}\right), & \text{if } F_\lambda \cong \mathbb{R} \\ L\left(\psi^{(\lambda)}, s - \frac{w(\psi)}{2}\right) L\left(\overline{\psi^{(\lambda)}}, s - \frac{w(\psi)}{2}\right), & \text{if } F_\lambda \cong \mathbb{C} \end{cases}$$

where $L(M(\psi),s)_\lambda$ denotes the λ-adic component of $L(M(\psi),s)$, coming from the decomposition

$$E\otimes_{\mathbb{Q}}\mathbb{C}\cong\prod_{E\hookrightarrow\mathbb{C}}\mathbb{C}$$

and $\overline{\psi^{(\lambda)}}\colon \mathbb{A}_F^\times\to E_\lambda^\times\subseteq\mathbb{C}^\times$ denotes the complex conjugate of the Hecke character $\psi^{(\lambda)}\colon\mathbb{A}_F^\times\to E_\lambda^\times\subseteq\mathbb{C}^\times$ induced by the Archimedean place λ (see Remark 7.1.18). Moreover, $w(\psi)\in\mathbb{Z}$ denotes the *weight* of the algebraic Hecke character ψ, defined to be the unique integer such that for every embedding $\iota_{\overline{E}}\colon\overline{E}\hookrightarrow\mathbb{C}$ we have that

$$m_\psi(\tau)+m_\psi(\overline{\tau})=w(\psi)$$

for every $\tau\colon F\hookrightarrow\overline{E}$, where $\tau\mapsto\overline{\tau}$ denotes the action of complex conjugation on $\mathrm{Hom}(F,\overline{E})$ induced by the embedding $\iota_{\overline{E}}$, and $m_\psi\colon\mathrm{Hom}(F,\overline{E})\to\mathbb{Z}$ denotes the function appearing in (7.5).

The proof of Theorem 7.1.24 uses greatly the geometry of abelian varieties with complex multiplication, from which the motives $M(\psi)$ are constructed. In particular, it uses the fact that one can associate to every abelian variety A defined over a number field F, which has complex multiplication by the CM algebra $E\cong K_1^\times\times\cdots\times K_n^\times$, an algebraic Hecke character $\psi_A\colon\mathbb{A}_F\to E^\times$, *i.e.* a family of algebraic Hecke characters $\{\psi_A^{(i)}\colon\mathbb{A}_F\to K_i^\times\}_{i=1}^n$. These Hecke characters arise by looking at the action of the absolute Galois group $\mathcal{G}_F:=\mathrm{Gal}(\overline{F}/F)$ on the group of torsion points $A_{\mathrm{tors}}:=A(\overline{F})_{\mathrm{tors}}$. This action, which exists for every abelian variety A, gives rise to a Galois representation

$$\rho_A\colon\mathcal{G}_F\to\mathrm{Aut}_{\mathbb{Z}}(A_{\mathrm{tors}})$$

which induces an injection $\rho_A\colon\mathrm{Gal}(F(A_{\mathrm{tors}})/F)\hookrightarrow\mathrm{Aut}_{\mathbb{Z}}(A_{\mathrm{tors}})$, where $F(A_{\mathrm{tors}})$ denotes the *division field* associated to the abelian variety A, which is the compositum (in a fixed algebraic closure $\overline{F}$) of all the residue fields $F(P)$ associated to torsion points $P\in A_{\mathrm{tors}}$. Now, if we fix a prime $\ell\in\mathbb{N}$ we can look at the action of $\mathcal{G}_F$ on the group $A[\ell^\infty]:=\varinjlim_n A[\ell^n]$ of torsion points of ℓ-power order, which gives rise to the ℓ^∞-division field $F(A[\ell^\infty])$ and to an embedding $\rho_{A,\ell^\infty}\colon\mathrm{Gal}(F(A[\ell^\infty])/F)\hookrightarrow\mathrm{Aut}_{\mathbb{Z}}(A[\ell^\infty])$. These Galois representations are known, for a general abelian variety A, to be related to the L-function $L(\underline{H}^1(A),s)$. More precisely, the group $A[\ell^\infty]$ is a $\mathbb{Z}_\ell$-module, and one has the following identifications (see [Ser71, § 4.1] and [Mil86, Theorem 15.1]):

$$\begin{aligned}\mathrm{Aut}_{\mathbb{Z}}(A[\ell^\infty])&\cong\mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell(A))\\(T_\ell(A)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell)^\vee&\cong H_\ell^{1,0}(A)=R_\ell(\underline{H}^1(A))\end{aligned}\tag{7.11}$$

where $T_\ell(A):=\varprojlim_{n\in\mathbb{N}}A[\ell^n]$ is the ℓ-adic *Tate module* associated to the abelian variety A. Note that the transition maps appearing in the direct limit defining $A[\ell^\infty]$ are simply the inclusions $A[\ell^n]\hookrightarrow A[\ell^{n+1}]$, whereas the transition maps appearing in the inverse limit defining the Tate module $T_\ell(A)$ are the multiplication-by-ℓ maps $A[\ell^{n+1}]\to A[\ell^n]$. Thus using (7.11) we find that the relation

$$L(\underline{H}^1(A),s)=L(\rho_{A,\ell})\tag{7.12}$$

holds for every prime $\ell\in\mathbb{N}$ and every abelian variety A defined over a number field.

Let us go back to abelian varieties with complex multiplication, and to the problem of relating the L-function $L(\underline{H}^1(A),s)$ to the L-function of some Hecke character. Using the relation (7.12)

we see that this problem can be reduced to the problem of relating the Galois representation ρ_A to a Hecke character. This is precisely the content of the main theorem of complex multiplication, that we now recall.

Theorem 7.1.25 – The main theorem of complex multiplication

Let F be a number field, endowed with an embedding $\iota_F\colon F \hookrightarrow \mathbb{C}$, and let A be an abelian variety with complex multiplication (over F). Then:

- there exists a CM algebra $K = K_1 \times \cdots \times K_n$ of degree

$$[K\colon \mathbb{Q}] = \sum_{i=1}^{n} [K_i\colon \mathbb{Q}] = 2\dim(A)$$

which is endowed with an embedding $\iota_K\colon K \hookrightarrow \mathrm{End}_F^0(A)$. The emebddings ι_F and ι_K induce a CM type $\Phi \subseteq \mathrm{Hom}(K, \mathbb{C})$, and if we denote by

$$(K^* = K_1^* \times \cdots \times K_n^*, \Phi^*)$$

the reflex of the CM pair (E, Φ), then for every $i \in \{1, \dots, n\}$ we have an embedding $\iota_{K_i^*}\colon K_i^* \hookrightarrow F$.

Fix $\mathcal{O} := \mathrm{End}(A) \cap K$, so that $\mathcal{O}^\times \subseteq \mathrm{Aut}_F(A) \subseteq \mathrm{Aut}_{\mathbb{Z}}(A_{\mathrm{tors}})$. Then:

- the image of the Galois representation

$$\rho_A\colon \mathrm{Gal}(F(A_{\mathrm{tors}})/F) \hookrightarrow \mathrm{Aut}_{\mathbb{Z}}(A_{\mathrm{tors}})$$

is contained in the centraliser of $\mathrm{Aut}_F(A)$ inside $\mathrm{Aut}_{\mathbb{Z}}(A_{\mathrm{tors}})$. Moreover, the centraliser of $\mathcal{O}^\times$ inside $\mathrm{Aut}_{\mathbb{Z}}(A_{\mathrm{tors}})$ is isomorphic to $\widehat{\mathcal{O}}^\times$, where

$$\widehat{\mathcal{O}} := \varprojlim_{N \in \mathbb{N}} \mathcal{O}/N\mathcal{O}$$

denotes the profinite completion of the ring $\mathcal{O}$. Hence ρ_A induces a representation

$$\rho_A\colon \mathrm{Gal}(F(A_{\mathrm{tors}})/F) \hookrightarrow \widehat{\mathcal{O}}^\times$$

and in particular the extension $F \subseteq F(A_{\mathrm{tors}})$ is abelian.

Now, we associate to $\mathcal{O}$ the subgroup

$$U_{\mathcal{O}} := \left(\prod_p \mathcal{O}_p^\times\right) \cdot K_\infty^\times \subseteq \mathbb{A}_K^\times := (\mathbb{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} K)^\times \cong \mathbb{A}_{K_1}^\times \times \cdots \times \mathbb{A}_{K_n}^\times$$

where the product runs over all the rational primes $p \in \mathbb{N}$. Moreover, $\mathcal{O}_p := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $K_\infty := K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}^\Phi$ (compare with Definition 6.2.11). Then:

- for every sub-field $M \subseteq F$ such that $F(A_{\mathrm{tors}}) \subseteq M^{\mathrm{ab}} \cdot F$ and $\iota_{K_i^*}(K_i^*) \subseteq M$ for each $i \in \{1, \dots, n\}$, there exists a unique group homomorphism

$$\alpha \colon M^\times \cdot \mathrm{N}_{F/M}(\mathbb{A}_F^\times) \to K^\times$$

having the following properties:

(a) the image of the homomorphism

$$\begin{aligned} \xi_\alpha \colon M^\times \cdot \mathrm{N}_{F/M}(\mathbb{A}_F^\times) &\to \mathbb{A}_K^\times \\ s &\mapsto \alpha(s) \cdot \mathrm{N}_{\Phi^*}(\mathrm{N}_{M/K^*}(s^{-1})) \end{aligned}$$

is contained in U_O. Here $\mathrm{N}_{M/K^*} \colon \mathbb{A}_M^\times \to \mathbb{A}_K^\times$ is the idelic norm map (see Equation (6.7)) induced by the inclusions $\iota_{K_1}, \dots, \iota_{K_n}$, and $\mathrm{N}_{\Phi^*} \colon \mathbb{A}_{K^*}^\times \to \mathbb{A}_K^\times$ is the idelic type norm associated to the reflex type Φ^*. This is again induced by all the idelic reflex type norms $\mathrm{N}_{\Phi_i^*} \colon \mathbb{A}_{(K_i)^*}^\times \to \mathbb{A}_{K_i}^\times$ associated to each CM field K_i appearing as a factor of K (see Equation (7.8));

(b) the continuous group homomorphism

$$\begin{aligned} \psi_\alpha \colon M^\times \cdot \mathrm{N}_{F/M}(\mathbb{A}_F^\times) &\to (\mathbb{C}^\Phi)^\times \\ s &\mapsto \Phi(\alpha(s)) \cdot \mathrm{N}_{\Phi^*}(\mathrm{N}_{M/K^*}(s^{-1}))_\infty \end{aligned} \tag{7.13}$$

admits exactly $[M^{\mathrm{ab}} \cap F \colon M]$ continuous extensions $\widetilde{\psi} \colon \mathbb{A}_M^\times \to (\mathbb{C}^\Phi)^\times$ to the whole idèle group $\mathbb{A}_M^\times$. Moreover, each $\widetilde{\psi}$ is a Hecke character, *i.e.* a continuous group homomorphism which is trivial on $K^\times := K_1^\times \times \cdots \times K_n^\times$. Here the embedding $\Phi \colon K \hookrightarrow \mathbb{C}^\Phi$ is induced by the CM type $\Phi \subseteq \mathrm{Hom}(K, \mathbb{C})$, and for every idèle $t \in \mathbb{A}_K^\times$ we denote by $t_\infty \in K_\infty^\times \cong \mathbb{C}^\Phi$ its Archimedean component, coming from the decomposition $\mathbb{A}_K \cong \mathbb{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} K$;

(c) the following diagram

$$\begin{array}{ccc} M^\times \cdot \mathrm{N}_{F/M}(\mathbb{A}_F^\times) & \xrightarrow{\pi_O \circ \xi_\alpha} & U_O / K_\infty^\times \\ {\scriptstyle r_{M^{\mathrm{ab}} \cdot F/F(A_{\mathrm{tors}})} \circ \mathcal{A}_{F/M}} \big\downarrow & & \big\downarrow {\scriptstyle \wr\, \mathfrak{c}_O} \\ \mathrm{Gal}(F(A_{\mathrm{tors}})/F) & \xhookrightarrow[\rho_A]{} & \widehat{O}^\times \end{array} \tag{7.14}$$

commutes, where $\pi_O \colon U_O \twoheadrightarrow U_O/K_\infty^\times$ is the natural quotient map, and

$$\mathfrak{c}_O \colon \frac{U_O}{K_\infty^\times} \xrightarrow{\sim} \prod_{p \in \mathbb{N}} O_p^\times \xrightarrow{\sim} \prod_{p \in \mathbb{N}} \varprojlim_{n \in \mathbb{N}} \left(\frac{O}{p^n O} \right)^\times \xrightarrow{\sim} \varprojlim_{N \in \mathbb{Z}_{\geq 1}} \left(\frac{O}{NO} \right)^\times \xrightarrow{\sim} \widehat{O}^\times$$

is the natural isomorphism (compare with Equation (6.16)). Finally,

$$r_{M^{\mathrm{ab}} \cdot F/F(A_{\mathrm{tors}})} \colon \mathrm{Gal}(M^{\mathrm{ab}} \cdot F/F) \twoheadrightarrow \mathrm{Gal}(F(A_{\mathrm{tors}})/F)$$

denotes the restriction map, and the homomorphism $\mathcal{A}_{F/M}$ is defined by

$$\mathcal{A}_{F/M}\colon M^\times \cdot \mathrm{N}_{F/M}(\mathbb{A}_F^\times) \xrightarrow{[\cdot,M]} \mathrm{Gal}(M^{\mathrm{ab}}/M^{\mathrm{ab}}\cap F) \twoheadrightarrow \mathrm{Gal}(M^{\mathrm{ab}}\cdot F/F)$$

where $[\cdot, M]$ denotes the global Artin map (see Definition 6.1.10).

Proof. Theorem 7.1.25 is one of the main results of the theory developed by Shimura and Taniyama, which is exposed in the books [ST61; Shi94; Shi98]. Let us point at specific references for the general case, and then provide some more details for elliptic curves. The existence of a CM algebra K having the desired properties follows from Theorem 7.1.1 and Theorem 7.1.6. Indeed, suppose that

$$\mathrm{End}^0(A) \cong \prod_{i=1}^{n} \mathrm{Mat}_{e_i\times e_i}(\widetilde{K_i})$$

for some CM fields $\widetilde{K_1},\dots,\widetilde{K_n}$ and some $e_1,\ \dots, e_n \in \mathbb{N}$. Then we can choose, for every index $i \in \{1,\dots,n\}$, a totally real number field L_i of degree $[L_i\colon \mathbb{Q}] = e_i$ which is disjoint from $\widetilde{K_i}$, as follows easily from ramification theory. Then the compositum $K_i := L_i\cdot\widetilde{K_i}$ is again a CM field, of degree $2e_i\dim(A_i)$. Moreover, any choice of a $\widetilde{K_i}$-basis for K_i induces an embedding $K_i \hookrightarrow \mathrm{Mat}_{e_i\times e_i}(\widetilde{K_i})$. Thus, the CM algebra $K := K_1\times\cdots\times K_n$ has degree $2\dim(A)$ and admits an embedding $K\hookrightarrow \mathrm{End}^0(A)$. The other properties which are required from this CM algebra follow from Proposition 7.1.15.

Now, it is immediate to see that $\mathrm{Im}(\rho_A)\subseteq \mathrm{Aut}_{\mathbb{Z}}(A_{\mathrm{tors}})$ commutes with every $\sigma\in\mathrm{Aut}_F(A)$, precisely because the automorphism $\sigma\colon A\to A$ is defined over F, and is thus insensitive to the action of $\mathrm{Gal}(F(A_{\mathrm{tors}})/F)$. Thus in particular $\mathrm{Im}(\rho_A)$ centralises $\mathcal{O}^\times\subseteq\mathrm{Aut}_F(A)$. Now, it is only slightly more difficult to see that the centraliser of $\mathcal{O}^\times$ inside $\mathrm{Aut}_{\mathbb{Z}}(A_{\mathrm{tors}})$ is indeed $\widehat{\mathcal{O}}^\times$, and we refer the reader to [ST68, § 4, Corollary 1] for a proof. In the case of elliptic curves, this follows easily from the fact that $E[N]$ is a free module over $\mathcal{O}/N\mathcal{O}$ for every $N\in\mathbb{Z}_{\geq 1}$ (see Lemma 7.2.4).

The really challenging part of Theorem 7.1.25 is the last one, *i.e.* the existence and uniqueness of the group homomorphism $\alpha\colon M^\times\cdot\mathrm{N}_{F/M}(\mathbb{A}_F^\times)\to K^\times$. The reader can obtain a complete proof of all the properties stated in Theorem 7.1.25 by combining [Shi94, Proposition 7.40] and [Shi94, Proposition 7.41] when $M=F$, and using [Shi94, Theorem 7.44] for the general case.

Let us dive a little more in the details of the proofs for elliptic curves E. In this case K is an imaginary quadratic field and $\mathcal{O}\subseteq K$ is an order, in the sense of Definition 6.2.1. Moreover, the type $\Phi\subseteq\mathrm{Hom}(K,\mathbb{C})$ consists of a single embedding $\Phi\colon K\hookrightarrow\mathbb{C}$, which is exactly the composition of $\iota_K\colon K\hookrightarrow F$ and of the fixed embedding $F\hookrightarrow\mathbb{C}$. Hence $(K^*,\Phi^*)=(K,\Phi)$ and the idelic reflex type norm $\mathrm{N}_{\Phi^*}\colon\mathbb{A}_K^\times\to\mathbb{A}_K^\times$ is simply the identity map.

Fix a field $M\subseteq F$ as in the statement of the theorem, *i.e.* such that $K\subseteq M$ and $F(E_{\mathrm{tors}})\subseteq M^{\mathrm{ab}}\cdot F$. The first step in the construction of the group homomorphism $\alpha\colon M^\times\cdot\mathrm{N}_{F/M}(\mathbb{A}_F^\times)\to K^\times$ consists in applying the first main theorem of complex multiplication (see [Lan87, Chapter 10, Theorem 3]). This says that for every field automorphism $\sigma\colon\mathbb{C}\to\mathbb{C}$ such that $\sigma\big|_{K^{\mathrm{ab}}} = [s,K]$ for some idèle $s\in\mathbb{A}_K^\times$, and every complex analytic uniformisation $\xi\colon\mathbb{C}\twoheadrightarrow E(\mathbb{C})$ such that $\ker(\xi)\subseteq K$, there exists a unique complex analytic uniformisation $\xi'\colon\mathbb{C}\twoheadrightarrow E^\sigma(\mathbb{C})$ such that $\ker(\xi)=s\cdot\ker(\xi')$ and the following diagram

$$\begin{array}{ccc} \frac{K}{\ker(\xi)} & \xrightarrow{\cdot s^{-1}} & \frac{K}{\ker(\xi')} \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi'} \\ E(\mathbb{C}) & \xrightarrow{\sigma} & E^\sigma(\mathbb{C}) \end{array}$$

commutes. Here the notation $s \cdot \ker(\xi')$ refers to the action of $\mathbb{A}_K^\times$ on the set of lattices $\mathcal{L}(K)$ which we defined in Proposition 6.1.8, and analogously the map

$$\frac{K}{\ker(\xi)} \xrightarrow{\cdot s^{-1}} \frac{K}{\ker(\xi')}$$

is the one defined in (6.6).

Let us now see how to use the first main theorem of complex multiplication, that we just recalled, to define the continuous group homomorphism $\alpha\colon M^\times \cdot \mathrm{N}_{F/M}(\mathbb{A}_F^\times) \to K^\times$. This follows the same strategy of the proof of [Sil94, Chapter II, Theorem 9.1] First of all, fix $s \in M^\times \cdot \mathrm{N}_{F/M}(\mathbb{A}_F^\times)$, and take $\sigma\colon \mathbb{C} \to \mathbb{C}$ to be a field automorphism lifting $\mathcal{A}_{F/M}(s) \in \mathrm{Gal}(M^{\mathrm{ab}} \cdot F/F)$. Then we see that $\sigma\big|_{K^{\mathrm{ab}}} = [\mathrm{N}_{M/K}(s), K]$, thanks to the commutative diagram (6.8). Hence, fixing a complex uniformisation $\xi\colon \mathbb{C} \twoheadrightarrow E(\mathbb{C})$ we get, from the first main theorem of complex multiplication, another complex uniformisation $\xi'\colon \mathbb{C} \twoheadrightarrow E^\sigma(\mathbb{C})$ such that the following diagram

$$\begin{array}{ccc} \frac{K}{\Lambda} & \xrightarrow{(\mathrm{N}_{M/K}(s)^{-1})\cdot} & \frac{K}{\mathrm{N}_{M/K}(s)\Lambda} \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi'} \\ E(\mathbb{C}) & \xrightarrow{\sigma} & E^\sigma(\mathbb{C}) \end{array}$$

commutes, where $\Lambda := \ker(\xi)$. Now, observe that $E^\sigma = E$ because E is defined over F, and σ fixes F because $\mathcal{A}_{F/M}(s)$ does. Hence the two lattices Λ and $\mathrm{N}_{M/K}(s) \cdot \Lambda$ are homothetic, and there exists a unique $\beta(s) \in K^\times$ and a new, unique complex uniformisation $\xi''\colon \mathbb{C} \twoheadrightarrow E(\mathbb{C})$ such that $\ker(\xi'') = \Lambda$ and the following square

$$\begin{array}{ccc} \frac{K}{\Lambda} & \xrightarrow{(\beta(s)\,\mathrm{N}_{M/K}(s)^{-1})\cdot} & \frac{K}{\Lambda} \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi''} \\ E(\mathbb{C}) & \xrightarrow{\sigma} & E(\mathbb{C}) \end{array}$$

commutes. This shows that $\xi'' \circ \xi^{-1}\colon E(\mathbb{C}) \to E(\mathbb{C})$ is an automorphism, which implies that there exists a unique $\varepsilon(s) \in \mathcal{O}^\times$ such that the following square

$$\begin{array}{ccc} \frac{K}{\Lambda} & \xrightarrow{(\alpha(s)\,\mathrm{N}_{M/K}(s)^{-1})\cdot} & \frac{K}{\Lambda} \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\ E(\mathbb{C}) & \xrightarrow{\sigma} & E(\mathbb{C}) \end{array} \tag{7.15}$$

commutes, where $\alpha(s) := \beta(s) \cdot \varepsilon(s)$. It is now immediate to see that the map

$$\alpha\colon M^\times \cdot \mathrm{N}_{M/K}(\mathbb{A}_F^\times) \to K^\times$$

which we just defined is a group homomorphism, using the compatibility between the multiplication of lattices and the action of $\mathbb{A}_K^\times$ on the set of lattices $\mathcal{L}(K)$. Moreover, the fact that $\mathrm{Im}(\xi_\alpha) \subseteq U_{\mathcal{O}}$ follows from the fact that $\xi(\alpha) \cdot \Lambda = \Lambda$, as we see from (7.15), and from the fact that the complex uniformisation $\xi\colon \mathbb{C} \twoheadrightarrow E(\mathbb{C})$ can be chosen in such a way that $\Lambda := \ker(\xi)$ is an invertible ideal of $\mathcal{O}$ (see Proposition 7.1.33). Moreover, the commutativity of (7.15) is

clearly equivalent to the commutativity of (7.14), and α the unique homomorphism with these properties by construction. It remains now to be shown that α is continuous, and that the map

$$\begin{aligned} \psi_\alpha \colon M^\times \cdot \mathrm{N}_{F/M}(\mathbb{A}_F^\times) &\to \mathbb{C}^\times \\ s &\mapsto \alpha(s)\,\mathrm{N}_{M/K}(s^{-1})_\infty \end{aligned}$$

defined in (7.13) can be extended to exactly $[M^{\mathrm{ab}} \cap F \colon M]$ Hecke characters $\widetilde{\varphi} \colon \mathbb{A}_M^\times \to \mathbb{C}^\times$.

First of all, the fact that α is continuous is equivalent to say that $\ker(\alpha)$ is open inside the topological group $M^\times \cdot \mathrm{N}_{F/M}(\mathbb{A}_F^\times)$. To show this, one can use the fact that every division field $F(E[I]) \subseteq M^{\mathrm{ab}} \cdot F$ associated to an invertible ideal $I \subseteq \mathcal{O}$ contains the ray class field $H_{I,\mathcal{O}}$, as we show in Theorem 7.2.5. This can be combined with (7.15) to show that $\mathrm{N}_{M/K}(U_{I,\mathcal{O}})^{-1} \cap M^\times \cdot \mathrm{N}_{F/M}(\mathbb{A}_F^\times) \subseteq \ker(\alpha)$ for every invertible ideal $I \subseteq \mathcal{O}$, and this is enough to prove that $\ker(\alpha)$ is open. We refer the reader to the proof of [Sil94, Theorem 9.2] for more details.

Let us now observe that $\psi_\alpha(M^\times) = 1$ because for every $x \in M^\times$ we have $\mathcal{R}_{F/M}(x) = \mathrm{Id}_{M^{\mathrm{ab}} \cdot F}$, which implies that $\alpha(s) = \mathrm{N}_{M/K}(x) = \mathrm{N}_{M/K}(x^{-1})_\infty^{-1}$, where the last equality uses the compatibility between the idelic norm and the norm on number fields provided by (6.7).

To conclude, one needs to observe that ψ_α admits exactly $[M^{\mathrm{ab}} \cap F \colon M]$ extensions to $\mathbb{A}_M^\times$. To do so one may apply a general result about topological groups, which says that for every abelian topological group G endowed with a subgroup of finite index $H \subseteq G$, there are exactly $[G \colon H]$ ways to extend any given continuous group homomorphism $\psi \colon H \to \mathbb{C}^\times$ to the whole G. The proof is straightforward (see [Shi94, Lemma 7.45]): fix a decomposition of the finite abelian group G/H as a product of cyclic groups $G/H = C_1 \times \cdots \times C_r$, and choose $a_1, \dots, a_r \in G$ to be r-elements which reduce to generators of the cyclic groups $C_1, \dots, C_r$ modulo H. Then we see that any element $g \in G$ can be uniquely written as $g = h\, a_1^{n_1} \cdots a_r^{n_r}$ where $n_i \in \mathbb{Z}$ and $h \in H$, hence any continuous group homomorphism $\widetilde{\psi} \colon G \to \mathbb{C}^\times$ which extends ψ is of the form

$$\widetilde{\psi}(g) = \widetilde{\psi}(h\, a_1^{n_1} \cdots a_r^{n_r}) = \psi(h)\zeta_1^{n_1} \cdots \zeta_r^{n_r}$$

where $\zeta_1, \dots, \zeta_r \in \mathbb{C}^\times$ are $|P_i|$-th roots of $\psi(a_i^{|P_i|})$. Thus we see that ψ can be easily extended to the whole G (just take $\zeta_i = \psi(a_i)$ for every $i \in \{1, \dots, r\}$), and this extension can be achieved in $[G \colon H] = |P_1| \cdots |P_r|$ possible ways. Now, applying this general fact to $\psi = \psi_\alpha$ we see that the number of homomorphisms $\widetilde{\psi} \colon \mathbb{A}_M^\times \to \mathbb{C}^\times$ which extend ψ_α is exactly

$$[\mathbb{A}_M^\times \colon M^\times \cdot \mathrm{N}_{F/M}(\mathbb{A}_F^\times)] = [M^{\mathrm{ab}} \cap F \colon M] \tag{7.16}$$

where the equality (7.16) follows immediately from (6.9). This concludes the proof. □

Remark 7.1.26. The CM algebra K, whose existence is guaranteed by Theorem 7.1.25, need not be unique. More precisely, we see that this algebra is unique if and only if the abelian variety A is isogenous to a product of distinct simple abelian varieties, *i.e.* if $e_1 = \cdots = e_n = 1$ in Theorem 7.1.1.

Remark 7.1.27. If $(K_1^* \cdots K_n^*) \subseteq M \subseteq M' \subseteq F$ and $F(A_{\mathrm{tors}}) \subseteq M^{\mathrm{ab}}$ then $M \subseteq F$ is abelian and Theorem 7.1.25 gives us $[M^{\mathrm{ab}} \cap F \colon M] = [F \colon M]$ Hecke characters $\widetilde{\psi} \colon \mathbb{A}_M^\times \to (\mathbb{C}^\Phi)^\times$ and $[(M')^{\mathrm{ab}} \cap F \colon M'] = [F \colon M']$ Hecke characters $\widetilde{\psi}' \colon \mathbb{A}_{M'}^\times \to (\mathbb{C}^\Phi)^\times$. We can observe that

$$\frac{[M^{\mathrm{ab}} \cap F \colon M]}{[(M')^{\mathrm{ab}} \cap F \colon M']} = \frac{[F \colon M]}{[F \colon M']} = [M' \colon M] \in \mathbb{N}$$

and that for every Hecke character $\widetilde{\psi}' \colon \mathbb{A}_{M'}^{\times} \to (\mathbb{C}^{\Phi})^{\times}$ given by Theorem 7.1.25 there are exactly $[M' \colon M]$ Hecke characters $\widetilde{\psi} \colon \mathbb{A}_{M}^{\times} \to (\mathbb{C}^{\Phi})^{\times}$ such that $\widetilde{\psi}' = \widetilde{\psi} \circ \mathrm{N}_{M'/M}$.

Example 7.1.28. If we take $M = F$ in Theorem 7.1.25 then we have a unique Hecke character $\widetilde{\psi} \colon \mathbb{A}_{F}^{\times} \to (\mathbb{C}^{\Phi})^{\times}$ which coincides with the usual Hecke character

$$\psi_A \colon \mathbb{A}_{F}^{\times} \to (\mathbb{C}^{\Phi})^{\times}$$

associated to abelian varieties with complex multiplication (see [Shi94, Proposition 7.41]).

Moreover, in this case we have that $M^{\times} \cdot \mathrm{N}_{F/M}(\mathbb{A}_{F}^{\times}) = \mathbb{A}_{F}^{\times}$. Thus the homomorphism α constructed in Theorem 7.1.25 is actually an algebraic Hecke character $\alpha_A \colon \mathbb{A}_{F}^{\times} \to K^{\times}$, and $\psi_A = \alpha_A^{(\infty)}$ is the character induced by α_A on the Archimedean components (see Remark 7.1.18).

Remark 7.1.29. Theorem 7.1.25 can be used to show that the L-function $L(A, s) := L(\underline{H}^1(A), s)$ associated to a CM abelian variety A can be expressed in terms of L-functions for Hecke characters. For a number field $F \subseteq \mathbb{C}$ and a CM elliptic curve $E_{/F}$ which has potential complex multiplication by the order $\mathcal{O}$ inside the imaginary quadratic field K, this relation reads (see [Sil94, Chapter II, Theorem 10.5]):

$$L(E, s) = \begin{cases} L(\psi_E, s - 1/2) \cdot L(\overline{\psi_E}, s - 1/2), & \text{if } K \subseteq F \\ L(\psi_{E_{F \cdot K}}, s - 1/2), & \text{if } K \not\subseteq F \end{cases}$$

where $K \cdot F \subseteq \mathbb{C}$ denotes the compositum of K and F inside $\mathbb{C}$. We refer the reader to [Mil72, Pages 187-189] for a discussion concerning general abelian varieties.

We conclude this section by introducing Shimura's condition, which enables one to take M to be the as small as possible in Theorem 7.1.25 (see [Shi94, Pages 216-218]).

Definition 7.1.30 – Shimura's condiiton

Let A be an abelian variety with complex multiplication defined over a number field F, and let $K \hookrightarrow \mathrm{End}_{F}^{0}(A)$ be any CM algebra of degree $[K \colon \mathbb{Q}] = 2\dim(A)$, which exists by Theorem 7.1.25. Then we say that A satisfies Shimura's condition if the extension $(K_1^* \cdots K_n^*) \subseteq F(A_{\mathrm{tors}})$ is abelian, where $K_1^* \cdots K_n^* \subseteq F$ denotes the compositum of the reflex fields $K_1^*, \ldots, K_n^*$ inside F.

7.1.2 Conductors of elliptic curves with complex multiplication

Let A be an abelian variety with complex multiplication defined over a number field $F \subseteq \mathbb{C}$. We have seen in the previous section that the Galois representation

$$\rho_A \colon \mathrm{Gal}(F(A_{\mathrm{tors}})/F) \hookrightarrow \mathrm{Aut}_{\mathbb{Z}}(A_{\mathrm{tors}})$$

admits an explicit description, provided by the diagram (7.14), in terms of Hecke characters. For any abelian variety A we know, thanks to Grothendieck's monodromy theorem, that the conductor of the ℓ-adic Galois representation

$$\rho_{A,\ell^\infty} \colon \mathrm{Gal}(F(A[\ell^\infty])/F) \hookrightarrow \mathrm{Aut}_{\mathbb{Z}}(A[\ell^\infty])$$

does not depend on the prime ℓ, at least as long as A has good reduction at all the primes of O_F lying above ℓ. This conductor is thus denoted by $\mathfrak{f}_A \subseteq O_F$, and one knows that a prime ideal $\mathfrak{p} \subseteq O_F$ divides the conductor if and only if A has bad reduction at $\mathfrak{p}$ (see [Gro72, § 4]). The aim of this section is to recall how the ideal $\mathfrak{f}_A$ is related to the conductor $\mathfrak{f}_{\psi_A} \subseteq O_F$ of the Hecke character $\psi_A \colon \mathbb{A}_F^\times \to (\mathbb{C}^\Phi)^\times$ defined by Theorem 7.1.25 (see also Remark 7.1.27). The main result in this direction has been proved by Milne in [Mil72].

Theorem 7.1.31 – Conductors of CM abelian varieties and Hecke characters

Let A be an abelian variety defined over a number field F, let $L \supseteq F$ be a finite Galois extension and suppose that A_L has complex multiplication. Fix an embedding $L \hookrightarrow \mathbb{C}$, and let $\Phi \subseteq \mathrm{Hom}(L, \mathbb{C})$ be the CM type associated to A_L and to this embedding. Fix moreover a CM algebra $K \hookrightarrow \mathrm{End}^0(A_L)$ of degree $[K \colon \mathbb{Q}] = 2\dim(A)$, whose existence is guaranteed by Theorem 7.1.25. Assume finally that $K \cap \mathrm{End}^0(A)$ is a field and that $[K \colon K \cap \mathrm{End}^0(A)] = [L \colon F]$.

Then we have that $[L \colon F] \mid 2\dim(A)$ and the following formula

$$\mathfrak{f}_A = \left(\mathrm{N}_{L/F}(\mathfrak{f}_{\psi_{A_L}})\ \mathrm{disc}(L/F)\right)^{2\dim(A)/[L \colon F]}$$

holds, where $\mathfrak{f}_{\psi_{A_L}} \subseteq O_L$ is the conductor (see Equation (7.10)) of the Hecke character $\psi_{A_L} \colon \mathbb{A}_L^\times \to (\mathbb{C}^\Phi)^\times$ defined by Theorem 7.1.25.

Proof. Let $d := \dim(A)$ and $m := [L \colon F]$. Since $[K \colon \mathbb{Q}] = 2d$ it is immediate to see that $m \mid 2d$. Now, observe that for every $\varphi \in \Phi$ we have that $\mathfrak{f}_{\psi_{A_L}} = \mathfrak{f}_{\psi_{A_L,\varphi}}$, where $\psi_{A_L,\varphi} \colon \mathbb{A}_L^\times \to \mathbb{C}^\times$ denotes the φ-th component of $\mathfrak{f}$. This is indeed easy to see, and follows from the fact that the reflex norm N_{Φ^*} appears in the definition of ψ_A. Then the theorem follows from the fact that the Weil restriction $\mathrm{N}_{L/F}(A_L)$ is isogenous to A^m, as proved by Milne in [Mil72, Theorem 3], and from the two formulas:

$$\mathfrak{f}_{A_L} = \mathfrak{f}_{\psi_{A_L,\varphi}}^{2d}$$

$$\mathfrak{f}_{\mathrm{N}_{L/F}(A_L)} = \mathrm{N}_{L/F}(\mathfrak{f}_{A_L})\ \mathrm{disc}(L/F)^{2d}$$

proved by Serre and Tate in [ST68, Theorem 12] and by Milne in [Mil72, Theorem 1]. □

There are two extreme cases in which Theorem 7.1.31 holds. First of all, we can clearly take $F = L$ if A has already complex multiplication over F, in which case we get the formula $\mathfrak{f}_A = \mathfrak{f}_{\psi_A}^{2\dim(A)}$ proved by Serre and Tate in [ST68, Theorem 12]. On the other hand, we can take F to be the so-called field of moduli of our abelian variety A, *i.e.* the smallest number field over which there exists an abelian variety which is isomorphic to A over $\overline{F}$. In the case of elliptic curves E, this boils down to taking $F = \mathbb{Q}(j(E))$. Then, in virtue of Proposition 7.1.15, the smallest L that can be taken is given by the compositum of the field of moduli F with all the reflex fields $K_1^*, \dots, K_n^*$. Doing so in the case of elliptic curves, for which the hypotheses of Theorem 7.1.31 are clearly satisfied, we get the following formula, which is originally due to Deuring (see [Deu56]).

Proposition 7.1.32 – Conductors of CM elliptic curves

Let $\mathcal{O} \subseteq K$ be an order inside an imaginary quadratic field K. Let E be an elliptic curve defined over $\mathbb{Q}(j(E))$ with complex multiplication by $\mathcal{O}$. Denote by $\varphi\colon \mathbb{A}_{H_{\mathcal{O}}}^{\times} \to \mathbb{C}^{\times}$ the unique Hecke character associated by Theorem 7.1.25 to the base change of E over $K(j(E))$. Then, letting $j = j(E)$, one can write the conductor $\mathfrak{f}_E \subseteq \mathcal{O}_{\mathbb{Q}(j)}$ of E as

$$\mathfrak{f}_E = \mathrm{N}_{K(j)/\mathbb{Q}(j)}(\mathfrak{f}_\varphi) \cdot \mathrm{disc}(K(j)/\mathbb{Q}(j))$$

where $\mathrm{N}_{K(j)/\mathbb{Q}(j)}(\mathfrak{f}_\varphi) \subseteq \mathcal{O}_{\mathbb{Q}(j)}$ denotes the relative norm of the conductor $\mathfrak{f}_\varphi \subseteq \mathcal{O}_{K(j)}$ of the Hecke character φ and $\mathrm{disc}(K(j)/\mathbb{Q}(j))$ denotes the relative discriminant ideal associated to the quadratic extension $\mathbb{Q}(j) \subseteq K(j)$.

Finally, let us recall that the smallest field over which an elliptic curve E with complex multiplication by the order $\mathcal{O}$ inside the imaginary quadratic field K can be defined together with all its complex multiplications, *i.e.* the field $K(j(E))$, coincides with the ring class field $H_{\mathcal{O}}$, as stated in the following result.

Proposition 7.1.33 – Fields of moduli for CM elliptic curves

Let E be an elliptic curve defined over $\mathbb{C}$, such that $\mathrm{End}(E) \cong \mathcal{O}$ for some imaginary quadratic order $\mathcal{O}$. Then the j-invariant $j(E) \in \mathbb{C}$ is an algebraic integer, and the number field $\mathbb{Q}(j(E))$ is isomorphic to the ring class field $H_{\mathcal{O}}$ (see Definition 6.2.11). Furthermore, any isomorphism $\mathrm{End}(E) \cong \mathcal{O}$ induces an embedding $\mathcal{O} \hookrightarrow \mathbb{C}$, and there exists an invertible ideal $I \subseteq \mathcal{O}$ such that $E(\mathbb{C}) \cong \mathbb{C}/I$. This ideal is uniquely determined up to the multiplicative action of $\mathcal{O}^{\times}$, and in particular there are as many j-invariants of elliptic curves with complex multiplication by $\mathcal{O}$ as the class number $|\mathrm{Pic}(\mathcal{O})|$.

Proof. The fact that $j(E)$ is an algebraic integer can be proved analytically (see [Shi94, Theorem 4.14]) or algebraically, using the properties of good reduction of CM abelian varieties. We refer the interested reader to [Sil94, Chapter II, § 6] for a survey of these different approaches, in the case when $\mathcal{O} = \mathcal{O}_K$ is the maximal order of an imaginary quadratic field K. To see that the field $\mathbb{Q}(j(E))$ is isomorphic to $H_{\mathcal{O}}$ we refer the reader to [Cox13, Theorem 11.1]. Finally, the last properties concerning the complex analytic uniformisation of the elliptic curve E are proved in [Shi94, § 4.4]. □

7.2 Division fields of CM elliptic curves and ray class fields for imaginary quadratic orders

From now on, the rest of this chapter focuses on CM elliptic curves rather than higher dimensional abelian varieties. First of all, let us recall that in the case of elliptic curves one does not need to talk about CM types, because they are already hidden in the choice (usually tacitly assumed) of a complex embedding of the number field over which the elliptic curve is defined. More precisely, let F be a number field, and let $E_{/F}$ be an elliptic curve with complex multiplication. This means, in the language introduced in Definition 7.1.5, that we have an isomorphism $K \cong \mathrm{End}_F^0(E)$ between the $\mathbb{Q}$-algebra of endomorphisms $\mathrm{End}_F^0(E) := \mathrm{End}(E_{/F}) \otimes_{\mathbb{Z}} \mathbb{Q}$

defined over F, and an imaginary quadratic field K. This implies that there exists an order $\mathcal{O} \subseteq K$ such that $\mathcal{O} \cong \mathrm{End}_F(E)$. Moreover, the following result shows that this isomorphism can be fixed once we fix an embedding $F \hookrightarrow \mathbb{C}$ (see [Sil94, Chapter II, Proposition 1.1]).

Proposition 7.2.1 – Normalised isomorphism for CM elliptic curves

Let E be an elliptic curve with complex multiplication, defined over a number field F. Then there exists a map

$$\begin{aligned} \mathrm{Hom}(F, \mathbb{C}) &\to \mathrm{Isom}(\mathcal{O}, \mathrm{End}_F(E)) \\ \iota &\mapsto [\,]_{E,\iota} \end{aligned}$$

such that for every $\iota \in \mathrm{Hom}(F, \mathbb{C})$ and every $\alpha \in \mathcal{O}$ we have that $[\alpha]_\iota^*(\omega) = \iota(\alpha) \cdot \omega$ for every invariant differential $\omega \in \Omega^1_{E/\mathbb{C}}$. Here the base-change of E to the complex numbers is achieved by means of the embedding $\iota \colon F \hookrightarrow \mathbb{C}$.

Remark 7.2.2. Usually, we fix the embedding $F \hookrightarrow \mathbb{C}$, and denote the corresponding normalised isomorphism simply by $[\cdot]_E \colon \mathcal{O} \xrightarrow{\sim} \mathrm{End}_F(E)$.

Let us now introduce the notion of division fields of CM elliptic curves, which was used but not properly defined in the previous section.

Definition 7.2.3 – Division fields

Let $F \subseteq \mathbb{C}$ be a number field, and $E_{/F}$ be an elliptic curve with complex multiplication by the order $\mathcal{O}$ in the imaginary quadratic field K.. Then for every non-zero ideal $I \subseteq \mathcal{O}$ we denote by $E[I] \hookrightarrow E$ the sub-scheme whose points are given by

$$E[I](R) := E(R)[I] := \{P \in E(R) \ : \ [\alpha]_E(P) = 0 \text{ for all } \alpha \in I\}$$

for every F-algebra R.

When $I = \alpha \cdot \mathcal{O}$ for some $\alpha \in \mathcal{O}$ we write $E(L)[\alpha] := E(L)[I]$ and $E[\alpha] := E(\overline{\mathbb{Q}})[\alpha]$. For any non-zero ideal $I \subseteq \mathcal{O}$ we denote $E(\overline{\mathbb{Q}})[I]$ by $E[I]$, with a slight abuse of notation. These groups $E[I]$ are always finite and they give rise to finite extensions $F \subseteq F(E[I])$ obtained by adjoining to F the coordinates of every I-torsion point. We refer to the number field $F(E[I])$ as the *I-division field* of $E_{/F}$.

The next result summarises the main properties of the extension $F \subseteq F(E[I])$ when I is an invertible $\mathcal{O}$-ideal, as we defined in Lemma 6.2.7.

Lemma 7.2.4 – Division fields and invertible ideals

Let F be a number field and $E_{/F}$ an elliptic curve with complex multiplication by an order $\mathcal{O}$ in an imaginary quadratic field $K \subseteq F$. Then for every ideal $I \subseteq \mathcal{O}$ the extension $F \subseteq F(E[I])$ is Galois and there is a canonical inclusion $\mathrm{Gal}(F(E[I])/F) \hookrightarrow \mathrm{Aut}_{\mathcal{O}}(E[I])$.

Moreover, if I is invertible, the group $E[I]$ has a natural structure of free $\mathcal{O}/I$-module of rank one and, after choosing a generator, one gets an injective group homomorphism

$$\rho_{E,I}\colon \mathrm{Gal}(F(E[I])/F) \hookrightarrow (\mathcal{O}/I)^\times$$

which is denoted by $\rho_{E,N}$ when $I = N \cdot \mathcal{O}$ for some $N \in \mathbb{Z}$. Under the further assumption that I is coprime to the ideal $\mathfrak{f}_{\mathcal{O}} \cdot \mathcal{O}$ generated by the conductor $\mathfrak{f}_{\mathcal{O}} := |\mathcal{O}_K : \mathcal{O}|$ of the order $\mathcal{O}$, one has that $\mathcal{O}/I \cong \mathcal{O}_K/I\mathcal{O}_K$.

Proof. Since F contains the CM field K, the endomorphisms of E are all defined over F and this implies that $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ acts on $E[I]$ by $\mathcal{O}$-module automorphisms. In particular $F \subseteq F(E[I])$ is Galois and there is a canonical inclusion

$$\mathrm{Gal}(F(E[I])/F) \hookrightarrow \mathrm{Aut}_{\mathcal{O}}(E[I]).$$

If I is invertible, $E[I]$ has the structure of free $\mathcal{O}/I$-module of rank one by [BC20, Lemma 2.4], and the choice of a generator induces an isomorphism $\mathrm{Aut}_{\mathcal{O}}(E[I]) \cong (\mathcal{O}/I)^\times$ which gives the map $\rho_{E,I}$ appearing in the statement. The last assertion follows from [Cox13, Proposition 7.20]. □

We conclude this short section by showing that the division fields $F(E[I])$ cannot be too small. More precisely, we know already that $K \subseteq F$ for every elliptic curve E which has complex multiplication (over F) by an order $\mathcal{O}$ in an imaginary quadratic field K. Moreover, Proposition 7.1.33 shows that $H_{\mathcal{O}} \subseteq F$, where $H_{\mathcal{O}}$ denotes the ring class field of $\mathcal{O}$ (see Definition 6.2.11). Finally, the next result shows that $H_{I,\mathcal{O}} \subseteq F(E[I])$, where $H_{I,\mathcal{O}}$ denotes the ray class field defined again in Definition 6.2.11. In particular, we prove that the ray class field $H_{I,\mathcal{O}}$ is always generated over the imaginary quadratic field K by the values of the *Weber function*

$$\mathfrak{h}_E\colon E \twoheadrightarrow E/\mathrm{Aut}(E) \cong \mathbb{P}^1$$

associated to any elliptic curve $E_{/\mathbb{C}}$ which has complex multiplication by $\mathcal{O}$.

Theorem 7.2.5 – Ray class fields and Weber's function

Let $\mathcal{O}$ be an order inside an imaginary quadratic field $K \subseteq \mathbb{C}$, and let $I \subseteq \mathcal{O}$ be an invertible ideal. Then we have that

$$H_{I,\mathcal{O}} = H_{\mathcal{O}}(\mathfrak{h}_E(E[I])) = K(j(E), \mathfrak{h}_E(E[I]))$$

for any elliptic curve $E_{/\mathbb{C}}$ such that $\mathrm{End}(E) \cong \mathcal{O}$. In particular, if E is an elliptic curve defined over a number field F such that $\mathrm{End}_F(E) \cong \mathcal{O}$ then $H_{I,\mathcal{O}} \subseteq F(E[I])$.

Proof. We can assume that $j(E) \notin \{0, 1728\}$, because in this case we have that $\mathcal{O} = \mathcal{O}_K$, and an idelic proof of Theorem 7.2.5 is given by [Sil94, Chapter II, Theorem 5.6]. Fix a generator P of $E[I]$ as a module over $\mathcal{O}/I$, which exists by Lemma 7.2.4 because $I \subseteq \mathcal{O}$ is invertible. Then $H_{\mathcal{O}}(\mathfrak{h}_E(E[I])) = H_{\mathcal{O}}(\mathfrak{h}_E(P))$, as one can see by writing every endomorphism of E in the standard form described in [Was08, § 2.9] and applying [Lan87, Chapter I, Theorem 7].

Let now $\xi\colon \mathbb{C}/\mathfrak{a} \xrightarrow{\sim} E(\mathbb{C})$ be a complex parametrisation, where $\mathfrak{a} \subseteq \mathcal{O}$ is an invertible ideal (see Proposition 7.1.33). Fix moreover $z \in (\mathfrak{a}\colon I) \subseteq K \subseteq \mathbb{C}$ such that $\xi(\overline{z}) = P$, where $\overline{z} := \mathfrak{a}z/\mathfrak{a}$ denotes the image of z in the quotient $K/\mathfrak{a} \subseteq \mathbb{C}/\mathfrak{a}$. Then [Shi94, Theorem 5.5] shows that

$$H_{\mathcal{O}}(\mathfrak{h}_E(P)) = (K^{\mathrm{ab}})^{[W_P,K]}$$

where $W_P \subseteq \mathbb{A}_K^\times$ is the subgroup defined by $W_P := \left\{ s \in \mathbb{A}_K^\times \,\middle|\, s \cdot \mathfrak{a} = \mathfrak{a},\ s \cdot \overline{z} = \overline{z} \right\}$. In particular, we recall that for any $s \in \mathbb{A}_K^\times$ such that $s \cdot \mathfrak{a} = \mathfrak{a}$, the notation $s \cdot \overline{z}$ stands for the image of $\overline{z} \in K/\mathfrak{a}$ under the map $K/\mathfrak{a} \xrightarrow{s\cdot} K/\mathfrak{a}$. This map is defined by the commutative diagram

$$\begin{array}{ccccc}
\dfrac{K}{\mathfrak{a}} & \xrightarrow{\ s\cdot\ } & \dfrac{K}{s\cdot\mathfrak{a}} & = & \dfrac{K}{\mathfrak{a}} \\
\wr\downarrow & & \wr\downarrow & & \wr\downarrow \\
\displaystyle\bigoplus_{p\in M_{\mathbb{Q}}^0} \dfrac{K_p}{\mathfrak{a}_p} & \xrightarrow{(s_p\cdot)_p} & \displaystyle\bigoplus_{p\in M_{\mathbb{Q}}^0} \dfrac{K_p}{s_p\mathfrak{a}_p} & = & \displaystyle\bigoplus_{p\in M_{\mathbb{Q}}^0} \dfrac{K_p}{\mathfrak{a}_p}
\end{array}$$

where $\mathfrak{a}_p := \mathfrak{a} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathfrak{a}\,\mathcal{O}_p$ for any rational prime $p \in \mathbb{N}$. Since $H_{\mathcal{O}} = K(j(E))$, the theorem follows from the claim

$$W_P = U_{I,\mathcal{O}} \tag{7.17}$$

where $U_{I,\mathcal{O}} \subseteq \mathbb{A}_K^\times$ is the subgroup defined in (6.12).

Let us prove the claim (7.17). To show the inclusion $U_{I,\mathcal{O}} \subseteq W_P$ take any $s \in U_{I,\mathcal{O}}$. Then $s \cdot \mathfrak{a} = \mathfrak{a}$ because $s_p\mathfrak{a}_p = \mathfrak{a}_p$ for every rational prime $p \in \mathbb{N}$, since by definition $s_p \in \mathcal{O}_p^\times$. Moreover, $s \cdot \overline{z} = \overline{z}$ because $z \in (\mathfrak{a}\colon I)$ and $s_p \in 1 + I\mathcal{O}_p$ for every rational prime $p \in \mathbb{N}$, which implies that $(s_p - 1)z \in \mathfrak{a}_p$. This shows that $U_{I,\mathcal{O}} \subseteq W_z$

To prove the opposite inclusion $W_P \subseteq U_{I,\mathcal{O}}$, fix any rational prime $p \in \mathbb{N}$ and take $s \in W_P$, so that $s \cdot \mathfrak{a} = \mathfrak{a}$ and $s \cdot \overline{z} = \overline{z}$. Since $\mathfrak{a} \subseteq \mathcal{O}$ is invertible we have that $\mathfrak{a} \cdot (\mathcal{O}\colon \mathfrak{a}) = \mathcal{O}$ and

$$s \cdot \mathcal{O} = s \cdot (\mathfrak{a} \cdot (\mathcal{O}\colon \mathfrak{a})) = (s \cdot \mathfrak{a}) \cdot (\mathcal{O}\colon \mathfrak{a}) = \mathfrak{a} \cdot (\mathcal{O}\colon \mathfrak{a}) = \mathcal{O}$$

which shows that $s_p \in \mathcal{O}_p^\times$. Let us now prove that $s_p \in 1 + I \cdot \mathcal{O}_p$. Since $I \subseteq \mathcal{O}$ and $\mathfrak{a} \subseteq \mathcal{O}$ are both invertible we have that $I \cdot (\mathcal{O}\colon \mathfrak{a}) \cdot (\mathfrak{a}\colon I) = \mathcal{O}$, so that we can write $1 = \sum_{j=1}^{J} \alpha_j\beta_j\tau_j$ with $\alpha_j \in I$, $\beta_j \in (\mathcal{O}\colon \mathfrak{a})$ and $\tau_j \in (\mathfrak{a}\colon I)$. Notice that $s \cdot \overline{\tau_j} = \overline{\tau_j}$ for every $j \in \{1, \dots, J\}$ because $s \cdot \overline{z} = \overline{z}$ and $P = \xi(\overline{z})$ generates $E[I]$ as a module over $\mathcal{O}/I$. Hence $s_p - 1 \in I \cdot \mathcal{O}_p$, because we can write

$$s_p - 1 = \sum_{j=1}^{J} \alpha_j\,\beta_j\,(s_p\,\tau_j - \tau_j)$$

where $s_p\,\tau_j - \tau_j \in \mathfrak{a}_p = \mathfrak{a}\,\mathcal{O}_p$ and $\beta_j(s_p\,\tau_j - \tau_j) \in \mathcal{O}_p$ since $\beta_j \in (\mathcal{O}\colon \mathfrak{a})$ for every $j \in \{1, \dots, J\}$. Thus we have shown that $s_p \in \mathcal{O}_p^\times$ and $s_p \in 1 + I \cdot \mathcal{O}_p$ for every prime $p \in \mathbb{N}$, which gives $W_P \subseteq U_{I,\mathcal{O}}$. This shows the claimed equality (7.17), and allows us to conclude. □

Remark 7.2.6. Theorem 7.2.5 was already proved by Söhngen [Söh35], using the classical languange of class field theory (see also [Sch10, Theorem 6.2.3]) for a modern account. To claim indeed that Theorem 7.2.5 gives an idelic proof of Söhngen's result we need to appeal to Theorem 6.2.17, which shows that our idelic description of the ray class fields $H_{I,\mathcal{O}}$ coincides with the classical definition given by Söhngen. We also refer the interested reader to [Ste01, § 4] for a

presentation whose language is closer to ours, which focuses on the case $I = N \cdot \mathcal{O}$ for some $N \in \mathbb{Z}$.

7.3 Bounding the index of the image of Galois for elliptic curves with complex multiplication

We have seen that, for every number field F, the Galois representation

$$\rho_E \colon \mathrm{Gal}(F(E_{\mathrm{tors}})/F) \hookrightarrow \mathrm{Aut}_{\mathbb{Z}}(E_{\mathrm{tors}}) \tag{7.18}$$

associated to an elliptic curve $E_{/F}$ which has complex multiplication by an imaginary quadratic order $\mathcal{O}$, is related to Hecke characters by Theorem 7.1.25, and this allows to describe explicitly the ray class fields for the order $\mathcal{O}$ in terms of the values of the Weber function associated to the elliptic curve E.

The Galois representation (7.18) can in fact be associated to any elliptic curve E defined over the number field F. If E does not have complex multiplication, Serre's Open Image Theorem [Ser71, Théorème 3] shows that the subgroup $\mathrm{Im}(\rho_E)$ has finite index in $\mathrm{Aut}_{\mathbb{Z}}(E_{\mathrm{tors}})$. Giving an explicit bound on the index $|\mathrm{Aut}_{\mathbb{Z}}(E_{\mathrm{tors}}) \colon \mathrm{Im}(\rho_E)|$ is still an area of active research, related to Serre's "uniformity conjecture" (see [Lom15, Theorem 9.1]).

Let us now return to elliptic curves with complex multiplication. In this case, if F is a number field and $E_{/F}$ is an elliptic curve with complex multiplication by an order $\mathcal{O}$, the image of the Galois representation ρ_E is contained in $\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}}) \cong \widehat{\mathcal{O}}^\times$, which is the centraliser of $\mathcal{O}^\times = \mathrm{Aut}_F(E)$ inside $\mathrm{Aut}_{\mathbb{Z}}(E_{\mathrm{tors}})$. This shows in particular that $\mathrm{Im}(\rho_E)$ cannot have finite index inside $\mathrm{Aut}_{\mathbb{Z}}(E_{\mathrm{tors}})$. However, a theorem of Deuring shows that the index $|\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}}) \colon \mathrm{Im}(\rho_E)|$ is finite. The main result of this section, which is based on joint work in progress with Francesco Campagna, is the following theorem, which gives an explicit upper bound for this index.

Theorem 7.3.1 – An optimal bound for the index of the image of Galois, for CM elliptic curves

Let $F \subseteq \mathbb{C}$ be a number field and let E be an elliptic curve with CM by an order $\mathcal{O}$ inside an imaginary quadratic field $K \subseteq F$. Denote by ρ_E the associated Galois representation. Then the index $|\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}}) \colon \mathrm{Im}(\rho_E)|$ divides $B_E := |\mathrm{Aut}(E)| \cdot [F \cap K^{\mathrm{ab}} \colon H_{\mathcal{O}}]$, where $H_{\mathcal{O}}$ denotes the ring class field of $\mathcal{O}$.

Observe that, for every CM elliptic curve E defined over a number field F which contains the CM field K, we have that $B_E \leq B''_E \leq B'_E$, where

$$B'_E := \begin{cases} [F \colon \mathbb{Q}], \text{ if } j(E) \notin \{0, 1728\} \\ 3 \cdot [F \colon \mathbb{Q}], \text{ otherwise} \end{cases} \qquad \text{and} \qquad B''_E := |\mathrm{Aut}(E)| \cdot [F \colon H_{\mathcal{O}}]$$

are the quantities provided respectively by Lombardo [Lom17, Theorem 6.6] and Bourdon and Clark [BC20, Corollary 1.5] as upper bounds for the index $|\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}}) \colon \mathrm{Im}(\rho_E)|$.

We give two proofs of Theorem 7.3.1. The first one uses the fact that every elliptic curve $E_{/F}$ with complex multiplication by $\mathcal{O} \subseteq F$ is a twist of the base-change to F of an elliptic curve defined over the ring class field $H_{\mathcal{O}}$.

First proof of Theorem 7.3.1. Recall that $K^{\mathrm{ab}} \subseteq F(E_{\mathrm{tors}})$, thanks to Theorem 7.2.5, and observe that

$$\rho_E(\mathrm{Gal}(F(E_{\mathrm{tors}})/F \cdot K^{\mathrm{ab}})) = \mathrm{Aut}(E) \cap \mathrm{Im}(\rho_E)$$

because the values of the Weber function $\mathfrak{h}_E \colon E \twoheadrightarrow E/\mathrm{Aut}(E) \cong \mathbb{P}^1$ are all contained in K^{ab}, and generate the extension $K \subseteq K^{\mathrm{ab}}$. This shows that ρ_E induces an injective map

$$\sigma_E \colon \mathrm{Gal}(F \cdot K^{\mathrm{ab}}/F) \hookrightarrow \mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})/\mathrm{Aut}(E)$$

such that $|\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}}) \colon \mathrm{Im}(\rho_E)|$ divides $|\mathrm{Aut}(E)| \cdot |\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})/\mathrm{Aut}(E) \colon \mathrm{Im}(\sigma_E)|$. Hence to conclude it is sufficient to prove that $|\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})/\mathrm{Aut}(E) \colon \mathrm{Im}(\sigma_E)| = [F \cap K^{\mathrm{ab}} \colon H_{\mathcal{O}}]$.

To this end, let E_0 be an elliptic curve defined over the ring class field $H_{\mathcal{O}}$ such that $j(E) = j(E_0)$, which exists thanks to Proposition 7.1.33. In particular, there exists $\alpha \in F^{\times}$ such that $E_0 = E^{(\alpha)}$ is the twist of E by α (see [Sil09, Chapter X, Proposition 5.4]). Setting $n := |\mathrm{Aut}(E)|$ and writing E and E_0 in short Weierstraß forms, one has an isomorphism

$$\begin{aligned} \varphi \colon E &\xrightarrow{\sim} E_0 \\ (x, y) &\mapsto \left(\alpha^{2/n} \cdot x, \alpha^{3/n} \cdot y\right) \end{aligned} \tag{7.19}$$

defined over the finite extension $F(\sqrt[n]{\alpha})$. Then φ induces an isomorphism

$$\begin{aligned} \varphi_* \colon \mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}}) &\xrightarrow{\sim} \mathrm{Aut}_{\mathcal{O}}((E_0)_{\mathrm{tors}}) \\ f &\mapsto \varphi \circ f \circ \varphi^{-1} \end{aligned}$$

such that $\varphi_*(\mathrm{Aut}(E)) = \mathrm{Aut}(E_0)$. Moreover, for every $\widetilde{\tau} \in \mathrm{Gal}(\overline{F}/F)$ the two automorphisms

$$\varphi_*(\rho_E(\tau)), \rho_{E_0}(\tau_0) \in \mathrm{Aut}_{\mathcal{O}}((E_0)_{\mathrm{tors}})$$

differ by an element of $\mathrm{Aut}(E_0)$, where $\tau := \widetilde{\tau}|_{F(E_{\mathrm{tors}})}$ and $\tau_0 := \widetilde{\tau}|_{H_{\mathcal{O}}((E_0)_{\mathrm{tors}})}$. Indeed, for every $P_0 \in (E_0)_{\mathrm{tors}}$, if we write $P_0 = (x, y)$ in the short Weierstraß model for E_0 chosen above, we have:

$$\begin{aligned} \varphi_*(\rho_E(\tau))(P_0) &= (\chi_{\alpha,n}(\widetilde{\tau})^{-2}\tau(x), \chi_{\alpha,n}(\widetilde{\tau})^{-3}\tau(y)) = (\chi_{\alpha,n}(\widetilde{\tau})^{-2}\tau_0(x), \chi_{\alpha,n}(\widetilde{\tau})^{-3}\tau_0(y)) \\ \rho_{E_0}(\tau_0)(P_0) &= (\tau_0(x), \tau_0(y)) \end{aligned}$$

where $\chi_{\alpha,n} \colon \mathrm{Gal}(\overline{F}/F) \to \mu_n \subseteq K \subseteq H_{\mathcal{O}}((E_0)_{\mathrm{tors}})$ is the Kummer character, defined as

$$\chi_{\alpha,n}(\widetilde{\tau}) := \frac{\widetilde{\tau}(\sqrt[n]{\alpha})}{\sqrt[n]{\alpha}}$$

for every $\widetilde{\tau} \in \mathrm{Gal}(\overline{F}/F)$.

This shows that the following diagram

$$\begin{array}{ccccc} & & \mathrm{Gal}(F \cdot K^{\mathrm{ab}}/F) & \xhookrightarrow{\sigma_E} & \mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})/\mathrm{Aut}(E) \\ & \swarrow^{\sim} & \downarrow & & \downarrow \wr \, \varphi_* \\ \mathrm{Gal}(K^{\mathrm{ab}}/F \cap K^{\mathrm{ab}}) & \hookrightarrow & \mathrm{Gal}(K^{\mathrm{ab}}/H_{\mathcal{O}}) & \xhookrightarrow{\sigma_{E_0}} & \mathrm{Aut}_{\mathcal{O}}((E_0)_{\mathrm{tors}})/\mathrm{Aut}(E_0) \end{array} \tag{7.20}$$

commutes. Now, using [BC20, Theorem 1.4], which was already proved in [Ste01], one sees that σ_{E_0} is an isomorphism. In particular $|\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})/\mathrm{Aut}(E)\colon \mathrm{Im}(\sigma_E)| = [F \cap K^{\mathrm{ab}}\colon H_{\mathcal{O}}]$ as we wanted to prove. □

Our second proof uses the main theorem of complex multiplication, rather than the twist-structure of CM elliptic curves.

Second proof of Theorem 7.3.1. Recall that $K^{\mathrm{ab}} \subseteq F(E_{\mathrm{tors}})$, thanks to Theorem 7.2.5, and that

$$\rho_E(\mathrm{Gal}(F(E_{\mathrm{tors}})/F \cdot K^{\mathrm{ab}})) = \mathrm{Aut}(E) \cap \mathrm{Im}(\rho_E)$$

because the values of the Weber function $\mathfrak{h}_E\colon E \twoheadrightarrow E/\mathrm{Aut}(E) \cong \mathbb{P}^1$ are all contained in K^{ab}, and generate the extension $K \subseteq K^{\mathrm{ab}}$. This shows that ρ_E induces an injective map

$$\sigma_E\colon \mathrm{Gal}(F \cdot K^{\mathrm{ab}}/F) \hookrightarrow \mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})/\mathrm{Aut}(E)$$

such that $|\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})\colon \mathrm{Im}(\rho_E)|$ divides $|\mathrm{Aut}(E)| \cdot |\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})/\mathrm{Aut}(E)\colon \mathrm{Im}(\sigma_E)|$. To conclude it suffices to prove that $|\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})/\mathrm{Aut}(E)\colon \mathrm{Im}(\sigma_E)| = [F \cap K^{\mathrm{ab}}\colon H_{\mathcal{O}}]$, and we prove this by showing that there exists a group surjection $\psi_E\colon \mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}}) \twoheadrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/H_{\mathcal{O}})$ such that $\ker(\psi_E) = \mathrm{Aut}(E)$ and the following diagram

$$\begin{array}{ccc} \mathrm{Gal}(F(E_{\mathrm{tors}})/F) & \xhookrightarrow{\rho_E} & \mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}}) \\ \downarrow & & \downarrow{\scriptstyle \psi_E} \\ \mathrm{Gal}(K^{\mathrm{ab}}/F \cap K^{\mathrm{ab}}) & \hookrightarrow & \mathrm{Gal}(K^{\mathrm{ab}}/H_{\mathcal{O}}) \end{array} \tag{7.21}$$

commutes, where the map $\mathrm{Gal}(F(E_{\mathrm{tors}})/F) \twoheadrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/F \cap K^{\mathrm{ab}})$ is the canonical restriction.

Let us see how to define the map ψ_E. First of all, let

$$\mathfrak{t}_E\colon \mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}}) \xrightarrow{\sim} \varprojlim_N \mathrm{Aut}_{\mathcal{O}}(E[N]) \xrightarrow{\sim} \varprojlim_N \mathrm{Aut}_{\mathcal{O}}(\mathcal{O}/N\mathcal{O}) \xrightarrow{\sim} \widehat{\mathcal{O}}^\times$$

be the natural isomorphism induced by the various projection maps, and by the decomposition $E_{\mathrm{tors}} = \varinjlim_N E[N]$ of the torsion subgroup $E_{\mathrm{tors}} \subseteq E(\mathbb{C})$. Moreover, let

$$\mathfrak{c}_{\mathcal{O}}\colon \frac{U_{\mathcal{O}}}{K_\infty^\times} \xrightarrow{\sim} \widehat{\mathcal{O}}^\times$$

be the isomorphism defined by (6.16). Then we define ψ_E to be the map

$$\psi_E\ :\ \mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}}) \xrightarrow[\sim]{\mathfrak{t}_E} \widehat{\mathcal{O}}^\times \xrightarrow[\sim]{\mathfrak{c}_{\mathcal{O}}^{-1}} U_{\mathcal{O}}/K_\infty^\times \xrightarrow{\mathfrak{a}_{\mathcal{O}}} \mathrm{Gal}(K^{\mathrm{ab}}/H_{\mathcal{O}})$$

where $\mathfrak{a}_{\mathcal{O}}(s) := [s^{-1}, K]$ is the reciprocal of the Artin map. In particular $\ker(\mathfrak{a}_{\mathcal{O}}) = K_\infty^\times \cdot \mathcal{O}^\times/K_\infty^\times$, which shows that $\ker(\psi_E) = \mathrm{Aut}(E)$. Hence to conclude we only have to prove that the diagram (7.21) commutes.

To show this, let $\mathfrak{u}_E := \mathfrak{c}_{\mathcal{O}}^{-1} \circ \mathfrak{t}_E$, so that $\psi_E = \mathfrak{a}_{\mathcal{O}} \circ \mathfrak{u}_E$. Let moreover $\varphi\colon \mathbb{C} \twoheadrightarrow E(\mathbb{C})$ be any complex uniformisation such that $\Lambda := \ker(\varphi) \subseteq K$, and let $\widetilde{\varphi}\colon K/\Lambda \xrightarrow{\sim} E_{\mathrm{tors}}$ be the induced isomorphism of $\mathcal{O}$-modules. Then we have that $\mathfrak{u}_E^{-1} = \varphi_* \circ \varphi_\Lambda$, where $\varphi_*\colon \mathrm{Aut}_{\mathcal{O}}(K/\Lambda) \xrightarrow{\sim} \mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})$ is

defined as $\varphi_*(f) = \widetilde{\varphi} \circ f \circ \widetilde{\varphi}^{-1}$, and $\varphi_\Lambda \colon U_{\mathcal{O}}/K_\infty^\times \xrightarrow{\sim} \mathrm{Aut}_{\mathcal{O}}(K/\Lambda)$ is induced by the map which sends $s \in U_{\mathcal{O}}$ to the automorphism $K/\Lambda \xrightarrow{s\cdot} K/\Lambda$.

Now, applying Theorem 7.1.25 with $M = F$ we see that there exists a unique group homomorphism $\alpha_E \colon \mathbb{A}_F^\times \to K^\times$ with open kernel, such that the following diagram

$$\begin{array}{ccc} \mathbb{A}_F^\times & \xrightarrow{\pi_{\mathcal{O}} \circ \xi_E} & U_{\mathcal{O}}/K_\infty^\times \\ {\scriptstyle [\cdot,F]|_{F(E_{\mathrm{tors}})}}\downarrow & & \wr \downarrow {\scriptstyle \mathfrak{u}_E^{-1}} \\ \mathrm{Gal}(F(E_{\mathrm{tors}})/F) & \xhookrightarrow{\rho_E} & \mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}}) \end{array} \tag{7.22}$$

commutes, where $\pi_{\mathcal{O}} \colon U_{\mathcal{O}} \twoheadrightarrow U_{\mathcal{O}}/K_\infty^\times$ is the canonical projection and $\xi_E \colon \mathbb{A}_F^\times \to U_{\mathcal{O}}$ is the group homomorphism given by $\xi_E(s) := \alpha_E(s) \cdot \mathrm{N}_{F/K}(s^{-1})$. Since the vertical maps appearing in (7.22) are surjective, to show that the diagram (7.21) commutes it is sufficient to show the commutativity of the following square

$$\begin{array}{ccc} \mathbb{A}_F^\times & \xrightarrow{\beta_E} & U_{\mathcal{O}}/K_\infty^\times \\ {\scriptstyle [\cdot,F]|_{K^{\mathrm{ab}}}}\downarrow & & \downarrow {\scriptstyle \mathfrak{a}_{\mathcal{O}}} \\ \mathrm{Gal}(K^{\mathrm{ab}}/F \cap K^{\mathrm{ab}}) & \hookrightarrow & \mathrm{Gal}(K^{\mathrm{ab}}/H_{\mathcal{O}}) \end{array} \tag{7.23}$$

obtained by gluing (7.22) above (7.21). Now, since $K^\times \subseteq \ker([\cdot, K])$ we have that

$$\mathfrak{a}_{\mathcal{O}} \circ \beta_E = [\cdot, K] \circ \mathrm{N}_{F/K}$$

and the functoriality of class field theory (see Theorem 6.1.9) allows us to conclude that (7.23) commutes. □

Remark 7.3.2. The previous proof, when applied to an elliptic curve E defined over a number field F such that $F \cap K^{\mathrm{ab}} = H_{\mathcal{O}}$ shows that σ_E is an isomorphism. In particular, this shows that the map σ_{E_0} in the diagram (7.20) is an isomorphism.

We conclude this section by pointing the interested reader to Corollary 8.3.4, where we prove that that the bound

$$|\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}}) \colon \mathrm{Im}(\rho_E)| \leq B_E$$

provided by Theorem 7.3.1 is optimal. More precisely, for every imaginary quadratic order $\mathcal{O}$ we can combine Corollary 8.3.4 and Theorem 8.3.6 to see that

$$|\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}}) \colon \mathrm{Im}(\rho_E)| = B_E := |\mathrm{Aut}(E)| \cdot [F \colon K^{\mathrm{ab}} \colon H_{\mathcal{O}}]$$

for infinitely many elliptic curves E which are defined over the ring class field $H_{\mathcal{O}}$ and have complex multiplication by the order $\mathcal{O}$. These elliptic curves are precisely the ones satisfying Shimura's condition, which was defined in Definition 7.1.30. On the other hand, we show in Theorem 8.3.7 that there exist also infinitely many elliptic curves E defined over the ring class field $H_{\mathcal{O}}$ which have complex multiplication by $\mathcal{O}$ and do not satisfy Shimura's condition. For these elliptic curves, Corollary 8.3.4 shows that

$$|\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}}) \colon \mathrm{Im}(\rho_E)| = 1$$

which may be seen as a sign that there is still room for improving Theorem 7.3.1.

7.4 Beilinson's conjecture for elliptic curves with complex multiplication

The aim of this section is to present the sketch of a proof of the main result of [Roh87], which proves in an explicit way that the weak form of Beilinson's conjecture (see Conjecture 3.3.28) holds for the special value $L^*(E,0)$ associated to a CM elliptic curve E defined over $\mathbb{Q}$. This is a very special case of a general theorem due to Deninger, proved in [Den89; Den90] and surveyed in [Den88; DW88; Den94], which shows Beilinson's conjecture for elliptic curves with complex multiplication that satisfy Shimura's condition (see Definition 7.1.30). The main difference between the results of Deninger and Rohrlich is that the first one constructs, for every $k \geq 0$, a specific element $\xi_k \in H^{2,k+2}_{\mathcal{M}}(E)$ whose regulator, paired with a suitable homology class, is an explicit rational multiple of the special value $L^*(E,-k)$, whereas the second one shows that in fact $L^*(E,0)$ is a rational multiple of a multitude of regulators of elements in $H^{2,2}_{\mathcal{M}}(E)$, paired with a suitable homology class. We note that one can combine the Galois descent property of motivic cohomology with the fact that a CM elliptic curve E has potential good reduction everywhere to see that $H^{i,j}_{\mathcal{M}}(E) = H^{i,j}_{f}(E)$ for every $i,j \in \mathbb{Z}$. Hence, if Beilinson's predictions on the ranks of motivic cohomology groups are true (see Conjecture 3.3.25) then an implicit form of Rohrlich's result would of course follow from Deninger's theorem.

Let us now state Rohrlich's result, collecting first a certain amount of notation. First of all, we need to introduce the *diamond operator*, which operates on divisors defined over a curve.

Definition 7.4.1 – Diamond operator

Let X be a smooth, proper curve over a number field κ, whose absolute Galois group is denoted by $\mathcal{G}_\kappa := \mathrm{Gal}(\overline{\kappa}/\kappa)$. Then, for every ring Λ, the diamond operator $\diamond$ is a map

$$\diamond\colon \Lambda[X(\overline{\kappa})]^{\mathcal{G}_\kappa} \otimes \Lambda[X(\overline{\kappa})]^{\mathcal{G}_\kappa} \longrightarrow \Lambda[\mathcal{G}_\kappa \backslash J(\overline{\kappa})]$$

where $J := \mathrm{Jac}(X)$ denotes the Jacobian of X. This map is defined by

$$\left(\sum_{x \in X(\overline{\kappa})} a_x\,(x)\right) \diamond \left(\sum_{y \in X(\overline{\kappa})} b_y\,(y)\right) := \sum_{[x,y]_\kappa \in G_\kappa \backslash X(\overline{\kappa})^2} a_{[x]_\kappa}\, b_{[y]_\kappa}\, \frac{|[x,y]_\kappa|}{|[x-y]_\kappa|}\,([x-y]_\kappa)$$

where $[x]_\kappa \in \mathcal{G}_\kappa \backslash X(\overline{\kappa})$ denotes the Galois orbit of a point $X(\overline{\kappa})$, and analogously the notation $[x,y]_\kappa \in \mathcal{G}_\kappa \backslash X(\overline{\kappa})^2$ stands for the orbit of a pair $(x,y) \in X(\overline{\kappa})^2$ under the diagonal action of the Galois group $\mathcal{G}_\kappa$. Moreover, $x - y \in J(\overline{\kappa})$ denotes the difference of the images of x and y under any embedding $X \hookrightarrow J$, and it does not depend on this embedding.

Fix now an elliptic curve E defined over $\mathbb{Q}$ such that $\mathrm{End}(E) \cong \mathcal{O}_K$ for some imaginary quadratic field K. In particular, E is a twist of one of the elliptic curves appearing in Table A.11

which have complex multiplication by a maximal order. Besides the diamond operator defined in Definition 7.4.1, Rohrlich's theorem involves also a function

$$\mathcal{R}\colon E_{\mathrm{tors}} \to \mathbb{Q}$$

whose definition requires in turn the existence of a complex uniformisation $\theta_E\colon \mathbb{C} \twoheadrightarrow E(\mathbb{C})$ having certain peculiar properties (see [Roh87, Page 377]).

Proposition 7.4.2 – Uniformising CM elliptic curves over $\mathbb{Q}$

Let E be an elliptic curve defined over $\mathbb{Q}$ which has complex multiplication by the ring of integers O_K of an imaginary quadratic field K. Then, for every embedding $\sigma\colon K \hookrightarrow \mathbb{C}$ and every orientation of $E(\mathbb{R})^0$, there exists a unique complex analytic uniformisation $\theta_E\colon \mathbb{C} \twoheadrightarrow E(\mathbb{C})$ such that:

- $\theta_E(\mathbb{R}) \subseteq E(\mathbb{R})$;
- the induced isomorphism $\mathbb{R}/\mathbb{Z} \xrightarrow{\sim} E(\mathbb{R})^0$ preserves the orientations;
- $\ker(\theta_E) = \sigma(\mathfrak{b}_E)$, where $\mathfrak{b}_E \subseteq K$ is a fractional ideal defined as

$$\mathfrak{b}_E := \begin{cases} O_K, & \text{if } (\Delta_K = -3 \wedge a > 0) \vee (\Delta_K = -4 \wedge \Delta_E > 0) \vee (\Delta_K < -4 \wedge b > 0) \\ \mathfrak{D}_{K/\mathbb{Q}}^{-1}, & \text{otherwise} \end{cases}$$

for any short Weierstraß equation $y^2 = 4x^3 - a\,x - b$ defining E, with $a, b \in \mathbb{R}$ and $\Delta_E := a^3 - 27b^2$. Here $\mathfrak{D}_{K/\mathbb{Q}} \subseteq O_K$ denotes the different of the extension $\mathbb{Q} \subseteq K$, as defined in [Neu99, Chapter III, § 2].

Proof. Consider an invariant differential $\omega \in \Omega^1_{E_{\mathbb{R}}}$, which is defined over $\mathbb{R}$, and consider the Abel-Jacobi isomorphism

$$\begin{aligned} \alpha\colon E(\mathbb{C}) &\xrightarrow{\sim} \mathbb{C}/\Lambda \\ P &\mapsto \left[\int_O^P \omega\right] \end{aligned} \tag{7.24}$$

where $\Lambda \subseteq \mathbb{C}$ denotes the period lattice of ω, obtained by pairing it with singular homology classes $H_1^{\mathrm{sing}}(E(\mathbb{C}); \mathbb{Z})$. Observe that the lattice Λ is invariant by complex conjugation, hence $\Lambda \cap \mathbb{R} = \lambda\mathbb{Z}$ for some $\lambda \in \mathbb{R}^\times$. Thus, after scaling ω and Λ by λ^{-1} we may assume that $\Lambda \cap \mathbb{R} = \mathbb{Z}$. Using this in combination with the fact that ω is defined over $\mathbb{R}$, we see that α induces an isomorphism $E(\mathbb{R}) \cong \mathbb{R}/\mathbb{Z}$. We can assume that this isomorphism preserves the orientations, after composing it, if needed, with the inversion $z \mapsto -z$ on $\mathbb{C}/\Lambda$. We now define θ_E as:

$$\theta_E\colon \mathbb{C} \twoheadrightarrow \mathbb{C}/\Lambda \xrightarrow{\alpha^{-1}} E(\mathbb{C})$$

and observe that θ_E satisfies the first two conditions appearing in the statement.

To prove that $\ker(\theta_E) = \sigma(\mathfrak{b}_E)$, we claim first of all that $\Lambda = \ker(\theta_E)$ is of the form $\Lambda = b \cdot \sigma(O_K)$ or $\Lambda = b \cdot \sigma(\mathfrak{D}_{K/\mathbb{Q}}^{-1})$ for some $b \in \mathbb{Q}$. For ease of notation, we suppose fixed the inclusion $K \subseteq \mathbb{C}$ coming from σ. Now, observe that $\Lambda = \lambda \cdot O_K$ for some $\lambda \in K^\times$, because $\mathrm{Pic}(O_K) = 1$. Then, $\lambda/\overline{\lambda} \in O_K^\times$, since Λ is invariant under complex conjugation. If $\lambda = \overline{\lambda}$ then $\lambda \in K \cap \mathbb{R} = \mathbb{Q}$, hence

Λ is of the form $b \cdot \mathcal{O}_K$ for some $b \in \mathbb{Q}$. On the other hand, if $\Delta_K < -4$ then the only other option is $\lambda = -\overline{\lambda}$, which implies that $\lambda \in K \cap i\mathbb{R} = \sqrt{\Delta_K} \cdot \mathbb{Q}$. This allows us to conclude that $\Lambda = b \cdot \mathfrak{D}_{K/\mathbb{Q}}^{-1}$ for some $b \in \mathbb{Q}$, because $\mathfrak{D}_{K/\mathbb{Q}}^{-1} = \Delta_K^{-1/2} \cdot \mathcal{O} = \Delta_K \cdot (\sqrt{\Delta_K} \cdot \mathcal{O})$. Moreover, if $\Delta_K = -4$ then $K = \mathbb{Q}(i)$, and we have two more cases to consider, namely $\lambda = i \cdot \overline{\lambda}$ and $\lambda = -i \cdot \overline{\lambda}$. In these cases we have respectively that $\lambda \in (1+i) \cdot \mathbb{Q}$ and $\lambda \in (1-i) \cdot \mathbb{Q}$, which implies that $\Lambda = b \cdot \mathfrak{D}_{K/\mathbb{Q}}^{-1}$ for some $b \in \mathbb{Q}$ since $\mathfrak{D}_{K/\mathbb{Q}} = (1+i) \cdot \mathbb{Z}[i]$. Finally, we observe that $\lambda/\overline{\lambda} \notin \mu_6 \setminus \{\pm 1\}$ for every $\lambda \in \mathbb{C}^\times$, where $\mu_6 \subseteq \mathbb{C}^\times$ denotes the group of sixth roots of unity. This shows our claim for $K = \mathbb{Q}(\sqrt{-3})$, *i.e.* for $\Delta_K = -3$. Now, we point out that $\mathcal{O} \cap \mathbb{Q} = \mathfrak{D}_{K/\mathbb{Q}}^{-1} \cap \mathbb{Q} = \mathbb{Z}$, which implies that $\Lambda \in \{\mathcal{O}, \mathfrak{D}_{K/\mathbb{Q}}^{-1}\}$ using our previous claim.

To conclude that $\Lambda = \mathfrak{b}_E$ we proceed as follows. First of all, since θ_E is a complex uniformisation, we see that E admits a short Weierstraß model of the form $y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$, where

$$g_2(\Lambda) := 60 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-4} \quad \text{and} \quad g_3(\Lambda) := 140 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-6}$$

are the usual Eisenstein series computed on the lattice Λ. Now, if $\Delta_K < -4$ we see that

$$g_m(\mathcal{O}_K) = -g_m(\mathfrak{D}_{K/\mathbb{Q}}^{-1}) > 0 \tag{7.25}$$

for every $m \in \{2, 3\}$, which shows that $\Lambda = \mathfrak{b}_E$ as we wanted. The first equality in (7.25) follows from the fact that $\mathfrak{D}_{K/\mathbb{Q}} = \sqrt{\Delta_K} \cdot \mathcal{O}_K$ if $\Delta_K \neq -4$. Moreover, the fact that $g_m(\mathcal{O}_K) > 0$ can be proved by the following steps:

- $g_m(\mathbb{Z} + \tau\mathbb{Z}) \in \mathbb{R}^\times$ for every $\tau \in \mathbb{C}$ of the form $\tau = it$ with $t > 1$ or $\tau = \frac{1}{2} + it$ with $t > \sqrt{3}/2$. Indeed, the fact that $g_m(\mathbb{Z} + \tau\mathbb{Z}) \in \mathbb{R}$ follows easily from the definition, whereas the fact that $g_m(\mathbb{Z} + \tau\mathbb{Z}) \neq 0$ is a special case of a result of Rankin and Swinnerton-Dyer (see [RS70]);
- observe that

$$\lim_{t \to +\infty} g_m(\mathbb{Z} + it\mathbb{Z}) = \lim_{t \to +\infty} g_m\left(\mathbb{Z} + \left(\frac{1}{2} + it\right)\mathbb{Z}\right) = \begin{cases} 120 \cdot \zeta(4), & \text{if } m = 2 \\ 280 \cdot \zeta(6), & \text{if } m = 3 \end{cases}$$

  as follows again from the definition. This allows us to conclude that

$$\begin{gathered} g_m(\mathbb{Z} + it\mathbb{Z}) > 0, \ \forall t > 1, \\ g_m\left(\mathbb{Z} + \left(\frac{1}{2} + it\right)\mathbb{Z}\right) > 0, \ \forall t > \frac{\sqrt{3}}{2}. \end{gathered} \tag{7.26}$$

- use the fact that $\mathcal{O}_K = \mathbb{Z} + i\sqrt{2}\mathbb{Z}$ if $\Delta_K = -8$ and $\mathcal{O}_K = \mathbb{Z} + \left(\frac{1}{2} + i\frac{\sqrt{|\Delta_K|}}{2}\right)\mathbb{Z}$ if $2 \nmid \Delta_K < -4$.

Finally, if $\Delta_K = -3$ one can prove similarly that $g_2(\mathcal{O}_K) = -g_2(\mathfrak{D}_{K/\mathbb{Q}}) > 0$, whereas if $\Delta_K = -4$ we know that $g_3(\mathbb{Z}[i]) = 0$. This implies that $g_2(\mathbb{Z}[i]) \neq 0$, which can be combined with (7.26) to see that $\Delta_E = g_2(\mathbb{Z}[i])^3 > 0$.

To conclude the proof, we need to show that θ_E is unique. Indeed, any two complex uniformisations $\theta, \theta' \colon \mathbb{C} \twoheadrightarrow E(\mathbb{C})$ satisfy $\theta(z) = \theta'(cz)$ for some $c \in \mathbb{C}$. If both θ and θ' are defined over $\mathbb{R}$ then $c \in \mathbb{R}$. Moreover, if $\ker(\theta), \ker(\theta') \subseteq K$ then $c \in K \cap \mathbb{R} = \mathbb{Q}$, and if $\ker(\theta) \cap \mathbb{R} = \ker(\theta') \cap \mathbb{R} = \mathbb{Z}$ then necessarily $c \in \{\pm 1\}$. Finally, if θ and θ' induce isomorphisms

$\mathbb{R}/\mathbb{Z} \cong E(\mathbb{R})^0$ which are compatible with the orientations, then $c = 1$. Since θ_E satisfies all the aforementioned properties, the previous discussion shows that it is unique. □

Remark 7.4.3. To avoid unnecessary sign issues, whenever we have an elliptic curve E defined over $\mathbb{Q}$ which has potential complex multiplication we fix implicitly an embedding $\sigma\colon K \hookrightarrow \mathbb{C}$ and an orientation of $E(\mathbb{R})^0$.

Let us now introduce the function $\mathcal{R}\colon E_{\mathrm{tors}} \to \mathbb{Q}$ which appears in Rohrlich's theorem.

Definition 7.4.4 – The function $\mathcal{R}$

Let E be an elliptic curve defined over $\mathbb{Q}$, which has potential complex multiplication by the ring of integers O_K of an imaginary quadratic field $K \subseteq \mathbb{C}$. Then we define a function

$$\mathcal{R}\colon E_{\mathrm{tors}} \to \mathbb{Q}$$
$$x \mapsto \begin{cases} 0, \text{ if } \mathfrak{f}_{\psi_E} \nmid \mathrm{Ann}_{O_K}(x) \\ \mathfrak{s}(x), \text{ if } \mathfrak{f}_{\psi_E} \mid \mathrm{Ann}_{O_K}(x) \text{ and } [x]_K = [\overline{x}]_K \\ \mathfrak{s}(x) + \overline{\mathfrak{s}(x)}, \text{ if } \mathfrak{f}_{\psi_E} \mid \mathrm{Ann}_{O_K}(x) \text{ and } [x]_K \cap [\overline{x}]_K = \emptyset \end{cases}$$

where $\mathrm{Ann}_{O_K}(x) := \{\alpha \in O_K \mid [\alpha]_E(x) = 0\}$ denotes the annihilator of a point $x \in E_{\mathrm{tors}}$, and $\mathfrak{f}_{\psi_E}$ denotes the conductor of the Hecke character $\psi_E\colon \mathbb{A}_K^\times \to \mathbb{C}^\times$ associated to the base-change E_K (see Remark 7.1.27). Moreover, the function $\mathfrak{s}\colon E_{\mathrm{tors}} \to K$ is defined as

$$\mathfrak{s}(x) := \frac{1}{\xi_x} \prod_{\substack{\mathfrak{p} \mid \mathrm{Ann}_{O_K}(x) \\ \mathfrak{p} \nmid \mathfrak{f}_{\psi_E}}} (1 - \psi_E(\mathfrak{p}))$$

where $\xi_x \in O_K$ is the unique generator of the ideal $\mathrm{Ann}_{O_K}(x)\, \mathfrak{b}_E^{-1} \subseteq O_K$ such that $\theta_E(\xi_x^{-1}) \in \mathcal{G}_K\, x$. Finally, $z \mapsto \overline{z}$ denotes the complex conjugation map $\mathbb{C} \to \mathbb{C}$, and for every prime $\mathfrak{p} \subseteq O_K$ such that $\mathfrak{p} \nmid \mathfrak{f}_{\psi_E}$ we let $\psi_E(\mathfrak{p}) \in K^\times$ denote the image of any $\mathfrak{p}$-adic uniformiser $\pi_\mathfrak{p} \in K_\mathfrak{p}^\times \subseteq \mathbb{A}_K^\times$. This image does not depend on the choice of a uniformiser because $\psi_E(O_{K_\mathfrak{p}}^\times) = 1$, since $\mathfrak{p} \nmid \mathfrak{f}_{\psi_E}$.

We also denote by $\mathcal{R}\colon \mathbb{Q}[E_{\mathrm{tors}}] \to \mathbb{Q}$ the $\mathbb{Q}$-linear extension of the map $\mathcal{R}\colon E_{\mathrm{tors}} \to \mathbb{Q}$.

Finally, we are ready to state Rohrlich's theorem, and to provide a sketch of its proof.

Theorem 7.4.5 – Beilinson's conjecture at $s = 2$ for CM elliptic curves over $\mathbb{Q}$

Let E be an elliptic curve defined over $\mathbb{Q}$ having potential complex multiplication by the ring of integers O_K of an imaginary quadratic field $K \subseteq \mathbb{C}$. Let moreover $f, g\colon E \to \mathbb{P}^1$ be two functions defined over $\mathbb{Q}$ whose zeros and poles are torsion points. Then we have that

$$\langle r_E^\infty(\eta_{f,g}), \gamma_E \rangle = n_{f,g} \cdot \mathcal{R}(\mathrm{div}(f) \diamond \mathrm{div}(g))\, L'(E, 0)$$

where $r_E^\infty(\eta_{f,g}) \in H^{2,2}_{\mathcal{D}}(E) \cong H^1_{\mathrm{dR}}(X; \mathbb{R}(1))$ denotes the image, under Beilinson's regulator map r_E^∞ (see Example 2.4.6 and Remark 2.4.10) of the class $\eta_{f,g} \in H^{2,2}_{\mathcal{M}}(E)$ defined in (2.37).

Moreover, we have that $n_{f,g}\,\mathcal{R}(\mathrm{div}(f)\diamond \mathrm{div}(g)) \in \mathbb{Z}$, where $n_{f,g} \in \mathbb{N}$ denotes the smallest common multiple of the orders of the zeros and poles of f and g, which exists because the latter are assumed to be torsion points. Finally, $\gamma_E \in H_1^{\mathrm{sing}}(E;\mathbb{Q})^-$ denotes the homology class defined in Notation 2.5.6.

Proof (sketch). We provide a sketch of the proof, pointing at specific places of [Roh87] for details. The key idea, which is incidentally the same idea which supports the work of Deninger [Den89; Den90], is to express a regulator integral in terms of Eisenstein-Kronecker series, and then relate them to the L-function using complex multiplication.

Fix $\theta_E\colon \mathbb{C} \twoheadrightarrow E(\mathbb{C})$ to be the complex uniformisation given by Proposition 7.4.2, which allows us to view f and g as meromorphic on $\mathbb{C}$, which are invariant with respect to translation by the lattice $\mathfrak{b}_E$. Using this point of view, we see that

$$\langle r_E^{\infty}(\eta_{f,g}), \gamma_E\rangle = \frac{n_{f,g}}{\pi i}\int_{\mathbb{C}/\mathfrak{b}_E} \log|f(z)|\frac{\overline{g'(z)}}{g(z)}dz\wedge d\bar{z} \tag{7.27}$$

combining Remark 2.5.3 and Proposition 2.5.5. Now, Rohrlich shows in [Roh87, Equation (13)] that

$$\frac{1}{\pi i}\int_{\mathbb{C}/\mathfrak{b}_E} \log|f(z)|\frac{\overline{g'(z)}}{g(z)}dz\wedge d\bar{z} = \left.\frac{dM_{f,g}(s)}{ds}\right|_{s=0} \tag{7.28}$$

where $M_{f,g}$ is the function defined by

$$M_{f,g}(s) := \sum_{z,w\in\mathbb{C}} \mathrm{ord}_z(f)\,\mathrm{ord}_w(g)E_1(z-w,s;\mathfrak{b}_E) \tag{7.29}$$

which is defined in terms of the Eisenstein-Kronecker series $E_1(z,s;\mathfrak{b}_E)\colon \mathbb{C}^2\to\mathbb{C}$. This function, which is holomorphic on $\mathbb{C}^2$, is defined as the analytic continuation of the series expression

$$E_1(z,s;\mathfrak{b}_E) := \sum_{\beta\in\mathfrak{b}_E\setminus\{-z\}} \frac{z+\beta}{|z+\beta|^{2s}}$$

which converges for $\mathrm{Re}(s) > 3/2$. We refer the interested reader to [Wei99, Chapter VII, § 12-13] for the basic properties of the Eisenstein-Kronecker series, and we point out that Weil denotes $E_1(z,s;\mathfrak{b}_E)$ as $K_1(z,0,s)$. The proof of (7.28), which appears in [Roh87, Page 375], uses the following two results:

- for every lattice $\Lambda\subseteq\mathbb{C}$ and every elliptic function $\varphi\colon \mathbb{C}/\Lambda\to\mathbb{C}$ one has that

$$-\frac{1}{\pi i}\int_{\mathbb{C}/\Lambda}\log|\varphi(z)|E(z,s;\Lambda)dz\wedge d\bar{z} = \frac{1}{(s-1)^2}\sum_{z\in\mathbb{C}/\Lambda}\mathrm{ord}_z(\varphi)E(z,s-1;\Lambda) \tag{7.30}$$

  for every $s\in\mathbb{C}$ such that $\Re(s) < 1$. Here $E(z,s;\Lambda)$ denotes the double Eisenstein series, which is the analytic continuation of the double series

$$E(z,s;\Lambda) = \sum_{\lambda\in\Lambda\setminus\{-z\}} |z+\lambda|^{-2s}$$

which converges for every $s \in \mathbb{C}$ such that $\Re(s) > 1$. The formula (7.30) is proved in [Roh87, Page 372];

- every elliptic function $\varphi : \mathbb{C}/\Lambda \to \mathbb{C}$ can be factored as

$$\varphi(z) = c \prod_{z_0 \in \mathbb{C}} \mathfrak{g}(z - z_0; \Lambda)^{\mathrm{ord}_{z_0}(\varphi)}$$

where $c \in \mathbb{C}$ and $\mathfrak{g}(z;\Lambda) := \Delta(\Lambda)^{1/12} e^{-z\eta(z;\Lambda)/2} \sigma(z;\Lambda)$ is the Siegel function associated to Λ. This can be proved using the factorisation of elliptic functions in terms of the σ-function (see [Lan87, Chapter 18, § 1]). We observe moreover that $\mathfrak{g}(z)$ is related to the Eisenstein series $E(z,s)$ by the second Kronecker limit formula, which says that

$$-2\log|\mathfrak{g}(z;\Lambda)| = \left.\frac{dE(z,s;\Lambda)}{ds}\right|_{s=0}$$

for every $z \notin \Lambda$ (see [Lan87, Chapter 20, § 5]).

Now, using the diagonal action of $\mathcal{G}_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $E(\overline{\mathbb{Q}})^2 := E(\overline{\mathbb{Q}}) \times E(\overline{\mathbb{Q}})$, one can re-write the definition (7.29) as

$$M_{f,g}(s) = \sum_{[z,w]_{\mathbb{Q}}} \mathrm{ord}_z(f)\,\mathrm{ord}_w(g)\,\frac{|[z,w]_{\mathbb{Q}}|}{|[z-w]_{\mathbb{Q}}|} M_{z-w}(s;E) \tag{7.31}$$

where the sum runs over all the diagonal orbits $[z,w]_{\mathbb{Q}} \in E(\mathbb{Q})^2/\mathcal{G}_{\mathbb{Q}}$, and for every point $z \in K/\Lambda \cong E_{\mathrm{tors}}$, the functions $M_z(s)$ are defined as

$$M_z(s;E) := \sum_{w \in [z]_{\mathbb{Q}}} E_1(w,s;\mathfrak{b}_E)$$

where the sum runs over all the elements in the Galois orbit of $z \in E(\overline{\mathbb{Q}})$.

To conclude, Rohrlich proves in [Roh87, Pages 381-384] that

$$M_z(s;E) = A_z(s;E) \cdot L(E,s) \quad \text{and} \quad A_z(0,E) = \mathcal{R}(\mathrm{div}(f) \diamond \mathrm{div}(g)) \tag{7.32}$$

for every torsion point $z \in K/\mathfrak{b}_E$, where $A_z(s;E)$ is an explicitly defined holomorphic function. More precisely, $A_z(s;E)$ is defined as follows

$$A_z(s;E) := \begin{cases} 0, & \text{if } \mathfrak{f}_{\psi_E} \nmid \mathrm{Ann}_z(O_K) \\ B_z(s;E), & \text{if } \mathfrak{f}_{\psi_E} \mid \mathrm{Ann}_z(O_K) \text{ and } [z]_K = [\overline{z}]_K \\ B_z(s;E) + \overline{B_z(\overline{s};E)}, & \text{if } \mathfrak{f}_{\psi_E} \mid \mathrm{Ann}_z(O_K) \text{ and } [z]_K \cap [\overline{z}]_K = \emptyset \end{cases}$$

where $B_z(s;E)$ denotes the finite Euler product

$$B_z(s;E) := \frac{|\xi_z|^{2s}}{\xi_z} \prod_{\substack{\mathfrak{p} \mid \mathrm{Ann}_z(O_K) \\ \mathfrak{p} \nmid \mathfrak{f}_{\psi_E}}} \left(1 - \frac{\psi_E(\mathfrak{p})}{|\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{p})|^s}\right)$$

running over all the primes $\mathfrak{p} \subseteq O_K$ which divide the annihilator ideal $\mathrm{Ann}_z(O_K) \subseteq O_K$ and do not divide the conductor $\mathfrak{f}_{\psi_E} \subseteq O_K$. This indeed allows one to conclude, combining the fact that $L(E,0) = 0$ with (7.27), (7.28), (7.31) and (7.32). □

Now, to prove Beilinson's conjectures for the special value $L^*(E,0) = L'(E,0)$ one has to show that for every CM elliptic curve E defined over $\mathbb{Q}$ we can find a pair of functions $f, g\colon E \to \mathbb{P}^1$ such that $\mathcal{R}(\mathrm{div}(f) \diamond \mathrm{div}(g)) \neq 0$. This happens in fact for many pairs of functions. Since these pairs of functions are crucially used in Chapter 9 to construct suitable polynomials $P \in \mathbb{Z}[x,y]$ whose Mahler measure $m(P)$ is related to $L^*(E,0)$, we defer this last bit of the proof of Beilinson's conjectures for $L^*(E,0)$ to Section 9.1. More precisely, we use Section 9.1.1 to recall the construction of the pairs of functions $f, g\colon E \to \mathbb{P}^1$ defined by Deninger and Wingberg in [DW88, Theorem 4.10], and we devote Section 9.1.2 to recall the constructions of the pairs of functions $f, g\colon E \to \mathbb{P}^1$ defined by Rohrlich in [Roh87, Page 384].

8 Entanglement in the family of division fields of CM elliptic curves

> If your confusion leads you in the right direction,
> the results can be uncommonly rewarding.
>
> Haruki Murakami,
> *Hard-Boiled Wonderland and the End of the World*

The aim of this chapter, which is based on the preprint [CP20] written jointly with Francesco Campagna, is to study the ramification and entanglement of division fields associated to CM elliptic curves. These division fields were introduced in Definition 7.2.3, and we saw in Theorem 7.2.5 that they contain the ray class fields associated to the order by which the elliptic curve in question has complex multiplication.

Let us step back for a moment, and consider any elliptic curve E (not necessarily with complex multiplication) defined over a number field F. Fix also an algebraic closure $\overline{F} \supseteq F$. Then the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ acts on the group $E_{\mathrm{tors}} := E(\overline{F})_{\mathrm{tors}}$ of all torsion points of E. This action gives rise to a Galois representation

$$\rho_E : \mathrm{Gal}(F(E_{\mathrm{tors}})/F) \hookrightarrow \mathrm{Aut}_{\mathbb{Z}}(E_{\mathrm{tors}}) \cong \mathrm{GL}_2(\widehat{\mathbb{Z}})$$

where $F(E_{\mathrm{tors}})$ is the compositum of the family of fields $\{F(E[p^\infty])\}_p$ for $p \in \mathbb{N}$ prime. Each extension $F \subseteq F(E[p^\infty])$ is in turn defined as the compositum of the family $\{F(E[p^n])\}_{n\in\mathbb{N}}$, where, for every $N \in \mathbb{N}$, we denote by $F(E[N])$ the *division field* obtained by adjoining to F the coordinates of all the points belonging to the N-torsion subgroup $E[N] := E[N](\overline{F})$.

For an elliptic curve E without complex multiplication (CM), Serre's Open Image Theorem [Ser71, Théorème 3] asserts that the image of ρ_E has finite index in $\mathrm{GL}_2(\widehat{\mathbb{Z}})$. However, explicitly describing this image is a non-trivial problem in general, which is connected to the celebrated Uniformity Conjecture [Ser71, § 4.3]. A first step in this direction is to study the *entanglement* of the family $\{F(E[p^\infty])\}_p$ for p prime, *i.e.* to describe the image of the natural inclusion

$$\mathrm{Gal}(F(E_{\mathrm{tors}})/F) \hookrightarrow \prod_p \mathrm{Gal}(F(E[p^\infty])/F) \tag{8.1}$$

where the product runs over all primes $p \in \mathbb{N}$. For each non-CM elliptic curve $E_{/F}$ this has been done in [CS19] by Campagna and Stevenhagen. They identify a finite set S of "bad primes" (depending on E and F) such that the map (8.1) induces an isomorphism

$$\mathrm{Gal}(F(E_{\mathrm{tors}})/F) \xrightarrow{\sim} \mathrm{Gal}(F(E[S^\infty])/F) \times \prod_{p\notin S} \mathrm{Gal}(F(E[p^\infty])/F)$$

where $F(E[S^\infty])$ denotes the compositum of the family of fields $\{F(E[p^\infty])\}_{p\in S}$. In this case one says that the family $\{F(E[S^\infty])\}\cup\{F(E[p^\infty])\}_p$ is *linearly disjoint* over F. The first goal of this paper is to prove Theorem B (see also Theorem 8.2.6), which provides an analogous statement for CM elliptic curves.

A key ingredient in the proof of Theorem B is Proposition 8.2.2, which can be seen as an explicit version of Deuring's analogue, for CM elliptic curves, of Serre's Open Image Theorem (see [Ser71, § 4.5]). More precisely, if $E_{/F}$ is an elliptic curve with complex multiplication by an order $\mathcal{O}$ in an imaginary quadratic field K, the extension $F\subseteq F(E_{\mathrm{tors}})$ is abelian. This shows that the image of ρ_E has infinite index in $\mathrm{Aut}_{\mathbb{Z}}(E_{\mathrm{tors}})\cong \mathrm{GL}_2(\widehat{\mathbb{Z}})$, and in particular the conclusion of Serre's theorem does not hold in this setting. Nevertheless, the elements of $\mathrm{Gal}(\overline{F}/F)$ act on E_{tors} as $\mathcal{O}$-module automorphisms, so that the image of ρ_E is contained in the subgroup $\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})\subseteq \mathrm{Aut}_{\mathbb{Z}}(E_{\mathrm{tors}})$. Then Proposition 8.2.2 says that $\rho_E(\mathrm{Gal}(F(E[p^n])/F))=\mathrm{Aut}_{\mathcal{O}}(E[p^n])$ for every prime $p\notin S$ and every $n\in\mathbb{N}$. Hence one has the inclusion

$$\prod_{p\notin S}\mathrm{Aut}_{\mathcal{O}}(E[p^\infty])\subseteq \mathrm{Im}(\rho_E):=\rho_E(\mathrm{Gal}\,(F(E_{\mathrm{tors}})/F))$$

which can be used to show, as Deuring did, that $\mathrm{Im}(\rho_E)\subseteq \mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})$ has finite index. Proposition 8.2.2 is proved using some results concerning formal groups attached to CM elliptic curves, which are recalled in Section 8.1. We point out that another proof of Proposition 8.2.2 can also be deduced from previous work of Lozano-Robledo, as explained in Remark 8.2.3.

While Proposition 8.2.2 (combined with Lemma 7.2.4) gives the identification

$$\mathrm{Gal}(F(E[N])/F)\cong(\mathcal{O}/N\mathcal{O})^\times \tag{8.2}$$

for every $N\in\mathbb{N}$ coprime with $\mathfrak{b}_E$, we prove in Theorem 8.3.1 that, if the extension $K\subseteq F$ is abelian and $F(E_{\mathrm{tors}})\subseteq K^{\mathrm{ab}}$, the isomorphism (8.2) does not hold for infinitely many $N\in\mathbb{N}$ not coprime with $\mathfrak{b}_E$. Theorem 8.3.1 extends results of Coates and Wiles (see [CW77, Lemma 3]) and Kuhman (see [Kuh78, Chapter II, Lemma 3]) using the ray class fields for orders constructed in Section 6.2.

The condition $F(E_{\mathrm{tors}})\subseteq K^{\mathrm{ab}}$ was introduced by Shimura in [Shi94, Theorem 7.44], and we have provided a possible generalisation of this condition in Definition 7.1.30. Shimura also shows in [Shi94, Page 217] that, if K is an imaginary quadratic field with absolute discriminant $\Delta_K\neq -1\,(3)$, then there exists an elliptic curve E defined over the Hilbert class field H_K with complex multiplication by $\mathcal{O}_K$ such that $H_K(E_{\mathrm{tors}})\subseteq K^{\mathrm{ab}}$. We generalise Shimura's result in Theorem 8.3.6 by proving that, for every imaginary quadratic field K and any order $\mathcal{O}\subseteq K$, there exist infinitely many ellipic curves $E_{/H_{\mathcal{O}}}$ with complex multiplication by $\mathcal{O}$ which satisfy Shimura's condition, *i.e.* such that the extension $K\subseteq H_{\mathcal{O}}(E_{\mathrm{tors}})$ is abelian. Here $H_{\mathcal{O}}$ denotes the *ring class field of K relative to $\mathcal{O}$* (see [Cox13, § 9]), which is an abelian extension of K coinciding with the Hilbert class field H_K when $\mathcal{O}=\mathcal{O}_K$. We also show in Theorem 8.3.7 that there exist infinitely many elliptic curves $E_{/H_{\mathcal{O}}}$ which have complex multiplication by $\mathcal{O}$ and do not satisfy Shimura's condition. For these elliptic curves, we show in Corollary 8.3.4 that the whole family of division fields $\{H_{\mathcal{O}}(E[p^\infty])\}_p$ is linearly disjoint over $H_{\mathcal{O}}$.

In the final section, we use Theorem B and Theorem 8.3.1 to prove Theorem 8.4.4, which provides a complete description of the image of (8.1) when $F=K$ is an imaginary quadratic field and $E_{/K}$ is the base-change of an elliptic curve defined over $\mathbb{Q}$. In particular, as we note in Remark 8.4.5, Theorem 8.4.4 shows that the finite set of primes S appearing in Theorem B

cannot be made smaller in general. However, see Remark 8.2.8 for a broader discussion around this topic.

We finally remark that the work presented in this chapter, despite having different objectives, bears a connection with Lozano-Robledo's recent work [Loz19], which provides an explicit list of subgroups of $\mathrm{GL}_2(\mathbb{Z}_p)$ that can occur as the image of the p-adic Galois representations associated to a CM elliptic curve. We comment more punctually on this relation in Remark 8.2.5, Remark 8.3.5 and Remark 8.4.2.

8.1 Formal groups and elliptic curves

8.1.1 Formal groups

The aim of this subsection is to recall, following [Sil09, Chapter IV], some of the main points of the theory of one dimensional, commutative formal group laws defined over a ring R, which we call *formal groups* for short. Roughly speaking, these are power series $\mathcal{F} \in R[\![z_1, z_2]\!]$ for which the association $x +_{\mathcal{F}} y := \mathcal{F}(x, y)$ behaves like an abelian group law. More precisely, they are defined as follows.

Definition 8.1.1 – Formal groups

Let R be a commutative ring with unity. A power series $\mathcal{F} \in R[\![z_1, z_2]\!]$ is a one dimensional, commutative formal group law, which we call *formal group* for short, if:

- $\mathcal{F}(z_1, z_2) - z_1 - z_2 \in \langle \{z_1^2, z_1 z_2, z_2^2\} \rangle_R$. In particular, $\mathcal{F}(0, 0) = 0$;
- $\mathcal{F}(z_1, \mathcal{F}(z_2, z_3)) = \mathcal{F}(\mathcal{F}(z_1, z_2), z_3)$, which represents the associativity of the formal group law $z_1 +_{\mathcal{F}} z_2 := \mathcal{F}(z_1, z_2)$;
- $\mathcal{F}(z_1, z_2) = \mathcal{F}(z_2, z_1)$, which represents the commutativity of $+_{\mathcal{F}}$;
- there exists a unique power series in one variable $\iota_{\mathcal{F}}(t) \in t \cdot R[\![t]\!]$ such that $\mathcal{F}(t, \iota_{\mathcal{F}}(t)) = 0$.

Given a formal group $\mathcal{F} \in R[\![z_1, z_2]\!]$ we denote the set of endomorphisms of $\mathcal{F}$ by

$$\mathrm{End}_R(\mathcal{F}) := \{f \in tR[\![t]\!] \mid f(x +_{\mathcal{F}} y) = f(x) +_{\mathcal{F}} f(y)\}$$

which is a ring under the operations $(f +_{\mathcal{F}} g)(t) := \mathcal{F}(f(t), g(t))$ and $(g \circ f)(t) := g(f(t))$. We write $\mathrm{Aut}_R(\mathcal{F})$ for the unit group $\mathrm{End}_R(\mathcal{F})^\times$ and we denote by $[\cdot]_{\mathcal{F}}$ the unique ring homomorphism $\mathbb{Z} \to \mathrm{End}_R(\mathcal{F})$. For every $\phi \in \mathrm{End}_R(\mathcal{F})$ one has that $\phi \in \mathrm{Aut}_R(\mathcal{F})$ if and only if $\phi'(0) \in R^\times$, where $\phi'(t) := \frac{d}{dt}\phi \in R[\![t]\!]$ (see [Sil09, Chapter IV, Lemma 2.4]). Moreover, every $\phi \in \mathrm{End}_R(\mathcal{F})$ is uniquely determined by $\phi'(0)$ whenever R is torsion-free. More precisely, there exist two power series $\exp_{\mathcal{F}}, \log_{\mathcal{F}} \in (R \otimes_{\mathbb{Z}} \mathbb{Q})[\![t]\!]$ such that

$$\phi(t) = \exp_{\mathcal{F}}(\phi'(0) \cdot \log_{\mathcal{F}}(t)) \tag{8.3}$$

as explained in [Sil09, Chapter IV, § 5].

Let us now recall that if $(R, \mathfrak{m})$ is a complete local ring there is a well defined map

$$\begin{aligned} \mathfrak{m} \times \mathfrak{m} &\xrightarrow{+_{\mathcal{F}}} \mathfrak{m} \\ (x, y) &\mapsto \mathcal{F}(x, y) \end{aligned}$$

endowing the set $\mathfrak{m}$ with the structure of an abelian group, which is denoted by $\mathcal{F}(\mathfrak{m})$. We sometimes refer to $\mathcal{F}(\mathfrak{m})$ as the *group of* $\mathfrak{m}$*-points of* $\mathcal{F}$. Every $\phi \in \mathrm{End}_R(\mathcal{F})$ induces an endomorphism $\phi_{\mathfrak{m}} \colon \mathcal{F}(\mathfrak{m}) \to \mathcal{F}(\mathfrak{m})$, and for every ideal $\Phi \subseteq \mathrm{End}_R(\mathcal{F})$ we define the Φ-torsion subgroup $\mathcal{F}(\mathfrak{m})[\Phi] \subseteq \mathcal{F}(\mathfrak{m})$ as

$$\mathcal{F}(\mathfrak{m})[\Phi] := \bigcap_{\phi \in \Phi} \ker(\phi_{\mathfrak{m}}).$$

These Φ-torsion subgroups generalise the usual N-torsion subgroups $\mathcal{F}(\mathfrak{m})[N] \subseteq \mathcal{F}(\mathfrak{m})$ defined for every $N \in \mathbb{Z}$. The following lemma provides some information about the behaviour of $\mathcal{F}(\mathfrak{m})[p^n]$ under finite extensions of local rings with residue characteristic p (see [Sil09, Chapter IV, Exercise 4.6] and [Sil15, Page 15]).

Lemma 8.1.2 – Valuations of p-adic torsion points of formal groups

Let $R \subseteq S$ be a finite extension of complete discrete valuation rings of characteristic zero with maximal ideals $\mathfrak{m}_R \subseteq \mathfrak{m}_S$ and residue fields $\kappa_R \subseteq \kappa_S$. Let $p := \mathrm{char}(\kappa_R) > 0$ be the residue characteristic of R and S, and suppose that $\mathfrak{m}_R = pR$. Then for every formal group $\mathcal{F} \in R[\![z_1, z_2]\!]$ and every $x \in \mathcal{F}(\mathfrak{m}_S)[p^n] \setminus \mathcal{F}(\mathfrak{m}_S)[p^{n-1}]$ with $n \in \mathbb{Z}_{\geq 1}$ we have that

$$v_S(x) \leq \frac{v_S(p)}{p^{h(n-1)} \cdot (p^h - 1)}$$

where v_S denotes the normalised valuation on S, and

$$h = \mathrm{ht}(\overline{\mathcal{F}}) := \max \left\{ n \in \mathbb{N} \;\middle|\; [p]_{\overline{\mathcal{F}}} \in \kappa_R[\![t^{p^n}]\!] \right\}$$

is the height of the reduced formal group $\overline{\mathcal{F}} \in \kappa_R[\![z_1, z_2]\!]$.

Proof. Using that $h = \mathrm{ht}(\overline{\mathcal{F}})$ and that $\mathfrak{m}_R = p \cdot R$ we see that there exist $f, g \in R[\![t]\!]$ such that $[p]_{\mathcal{F}} = f(t^{p^h}) + p\, g(t)$. We can assume that $f, g \in t\, R[\![t]\!]$ and $g'(0) = 1$ because $[p]_{\mathcal{F}} \in t\, R[\![t]\!]$ and $[p]'_{\mathcal{F}}(0) = p$. Now, fix $x \in \mathcal{F}(\mathfrak{m}_S)[p^n] \setminus \mathcal{F}(\mathfrak{m}_S)[p^{n-1}]$ and proceed by induction on $n \in \mathbb{Z}_{\geq 1}$.

If $n = 1$ then $f(x^{p^h}) + p\, g(x) = [p]_{\mathcal{F}}(x) = 0$, hence $v_S(p) + v_S(g(x)) = v_S(f(x^{p^h}))$. Now $v_S(g(x)) = v_S(x)$ because $g(0) = 0$ and $g'(0) = 1$, and $v_S(f(x^{p^h})) \geq v_S(x^{p^h}) = p^h\, v_S(x)$ because $f(0) = 0$. Hence $v_S(p) \geq (p^h - 1) \cdot v_S(x)$, which is what we wanted to prove.

If $n \geq 2$ we know by induction that

$$\frac{v_S(p)}{p^{h(n-2)} \cdot (p^h - 1)} \geq v_S([p]_{\mathcal{F}}(x)) = v_S(f(x^{p^h}) + p\, g(x)) \geq \min(v_S(x^{p^h}), v_S(px))$$

because $[p]_{\mathcal{F}}(x) \in \mathcal{F}(\mathfrak{m}_S)[p^{n-1}] \setminus \mathcal{F}(\mathfrak{m}_S)[p^{n-2}]$ by assumption. This implies that

$$\min(v_S(x^{p^h}), v_S(px)) = v_S(x^{p^h})$$

as otherwise we would get the contradiction

$$v_S(p) \geq p^{h(n-2)} \cdot (p^h - 1) \cdot v_S(px) > v_S(p)$$

because $n \geq 2$, $v_S(x) > 0$ and $h \geq 1$. Hence we have that

$$v_S(x) = \frac{v_S(x^{p^h})}{p^h} \leq \frac{v_S(p)}{p^h \cdot (p^{h(n-2)} \cdot (p^h - 1))} = \frac{v_S(p)}{p^{h(n-1)} \cdot (p^h - 1)}$$

which is what we wanted to prove. □

8.1.2 Formal groups and elliptic curves

Given an elliptic curve E defined over a number field F by an integral Weierstraß equation, one can construct, following for example [Sil09, Chapter IV], a formal group $\widehat{E} \in O_F[\![z_1, z_2]\!]$, which can be thought of as the formal counterpart of the addition law on E. The association $E \mapsto \widehat{E}$ is functorial and in particular induces a map

$$\begin{aligned} \mathrm{End}_F(E) &\longrightarrow \mathrm{End}_F(\widehat{E}) \\ \phi &\mapsto \widehat{\phi} \end{aligned} \tag{8.4}$$

between the endomorphism rings of E and $\widehat{E}$. The power series lying in the image of (8.4) have integral coefficients, as proved in the following theorem, which is due to Streng (see [Str08, Theorem 2.9]).

Theorem 8.1.3 – Integrality of formal endomorphisms

Let E be an elliptic curve defined over a number field $F \subseteq \mathbb{C}$ and let $\widehat{E} \in O_F[\![z_1, z_2]\!]$ be the formal group law associated to a Weierstraß model of E with coefficients $a_1, \ldots, a_6 \in O_F$. Then for every $\phi \in \mathrm{End}_F(E)$ we have that $\widehat{\phi} \in O_F[\![t]\!]$.

Proof (sketch). One can show by induction that $\widehat{[n]}_E = [n]_{\widehat{E}} \in \mathbb{Z}[a_1, \ldots, a_6][\![t]\!] \subseteq O_F[\![t]\!]$ for every $n \in \mathbb{Z}$, where $[n]_E \in \mathrm{End}_F(E)$ denotes the multiplication-by-n map. This proves the theorem when $\mathrm{End}_F(E) \cong \mathbb{Z}$. Otherwise E has complex multiplication. Hence one can combine Example 7.1.9 and Proposition 7.2.1 with [Sil09, Chapter IV, Corollary 4.3] to see that there exists a unique isomorphism $[\cdot]_E \colon O \xrightarrow{\sim} \mathrm{End}_F(E)$ such that $\widehat{[\alpha]}_E'(0) = \alpha$ for every $\alpha \in O$, where O is an order in an imaginary quadratic field $K \subseteq F$.

Let $\{\psi_j\}_{j \in \mathbb{N}} \subseteq F[s]$ be the polynomials determined by the equality

$$\sum_{j=0}^{+\infty} \psi_j(s) \cdot t^j = \exp_{\widehat{E}}(s \cdot \log_{\widehat{E}}(t)) \in F[\![t, s]\!]$$

and observe that $\psi_j(\mathbb{Z}) \subseteq O_F$ for every $j \in \mathbb{N}$, because (8.3) shows that

$$\sum_{j=0}^{+\infty} \psi_j(n) \cdot t^j = [n]_{\widehat{E}}(t) \in O_F[\![t]\!]$$

for every $n \in \mathbb{Z}$.

To conclude, it is sufficient to show that $\psi_j(O) \subseteq O_{F_\mathfrak{P}}$ for every $j \in \mathbb{N}$ and every prime $\mathfrak{P} \subseteq O_F$, where $F_\mathfrak{P}$ denotes the completion of F at $\mathfrak{P}$. Indeed, in this case $\psi_j(O) \subseteq O_F$ for every $j \in \mathbb{N}$, and again (8.3) gives

$$\widehat{[\alpha]}_E(t) = \exp_{\widehat{E}}(\widehat{[\alpha]}_E'(0) \cdot \log_{\widehat{E}}(t)) = \exp_{\widehat{E}}(\alpha \cdot \log_{\widehat{E}}(t)) = \sum_{j=0}^{+\infty} \psi_j(\alpha) \cdot t^j \in O_F[\![t]\!]$$

for every $\alpha \in O$. The inclusion $\psi_j(O) \subseteq O_{F_\mathfrak{P}}$ is easily seen if $\mathfrak{P}$ lies above a rational prime $p \in \mathbb{N}$ which splits in K, because under this assumption $O \subseteq \mathbb{Z}_p$ and $\psi_j(\mathbb{Z}_p) \subseteq O_{F_\mathfrak{P}}$, since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$ and $\psi_j \colon F_\mathfrak{P} \to F_\mathfrak{P}$ is continuous with respect to the $\mathfrak{P}$-adic topology.

For the remaining cases, we refer the reader to the original proof contained in [Str08]. In fact, Streng provides two proofs: one uses the extension of a morphism $\phi \in \mathrm{End}(E)$ to the connected component of the identity of a Néron model, whereas the other is more elementary, and uses Vélu's isogeny formulas (see [Was08, § 12.3]). □

Let now $\mathfrak{P} \subseteq O_F$ be a prime of F with residue field $\kappa_\mathfrak{P}$ and corresponding maximal ideal $\mathfrak{m}_\mathfrak{P} \subseteq O_{F_\mathfrak{P}}$, where $F_\mathfrak{P}$ denotes the completion of F at $\mathfrak{P}$. Then [Str08, § 2] shows that there is a unique injective group homomorphism $\iota_\mathfrak{P} \colon \widehat{E}(\mathfrak{m}_\mathfrak{P}) \to E(F_\mathfrak{P})$ making the following diagram

$$\begin{array}{ccc} \widehat{E}(\mathfrak{m}_\mathfrak{P}) & \xrightarrow{\iota_\mathfrak{P}} & E(F_\mathfrak{P}) \\ \downarrow{\scriptstyle \widehat{\phi}_\mathfrak{P}} & & \downarrow{\scriptstyle \phi} \\ \widehat{E}(\mathfrak{m}_\mathfrak{P}) & \xrightarrow{\iota_\mathfrak{P}} & E(F_\mathfrak{P}) \end{array} \tag{8.5}$$

commute for every $\phi \in \mathrm{End}_{F_\mathfrak{P}}(E)$, where $\widehat{\phi}_\mathfrak{P} := (\widehat{\phi})_{\mathfrak{m}_\mathfrak{P}}$ (see Section 8.1.1). Moreover [Sil09, Chapter VII, Proposition 2.1 and Proposition 2.2] imply that $\iota_\mathfrak{P}$ fits in the following exact sequence

$$0 \to \widehat{E}(\mathfrak{m}_\mathfrak{P}) \xrightarrow{\iota_\mathfrak{P}} E(F_\mathfrak{P}) \xrightarrow{\pi_\mathfrak{P}} \widetilde{E}(\kappa_\mathfrak{P}) \to 0$$

in which $\widetilde{E}$ denotes the reduction of E modulo $\mathfrak{P}$ and $\pi_\mathfrak{P} \colon E(F_\mathfrak{P}) \twoheadrightarrow \widetilde{E}(\kappa_\mathfrak{P})$ is the canonical projection. Taking torsion and using (8.5) we get a left-exact sequence (extensively used in the next section):

$$0 \to \widehat{E}(\mathfrak{m}_\mathfrak{P})[\widehat{\Phi}] \xrightarrow{\iota_\mathfrak{P}} E(F_\mathfrak{P})[\Phi] \xrightarrow{\pi_\mathfrak{P}} \widetilde{E}(\kappa_\mathfrak{P})[\Phi] \tag{8.6}$$

for every ideal $\Phi \subseteq \mathrm{End}_{F_\mathfrak{P}}(E)$. Here $E(F_\mathfrak{P})[\Phi] \subseteq E(F_\mathfrak{P})$ is the Φ-torsion subgroup

$$E(F_\mathfrak{P})[\Phi] := \bigcap_{\phi \in \Phi} \ker(\phi)$$

and $\widetilde{E}(\kappa_\mathfrak{P})[\Phi]$ is defined analogously, noting that the map $\mathrm{End}_{F_\mathfrak{P}}(E) \to \mathrm{End}_{\kappa_\mathfrak{P}}(\widetilde{E})$ is injective (see [Sil94, Chapter II, Proposition 4.4]). We remark that $\widehat{E}(\mathfrak{m}_\mathfrak{P})[\widehat{\Phi}]$ is well defined since $\widehat{\Phi} \subseteq O_F[\![t]\!]$ by Theorem 8.1.3.

8.2 Division fields of CM elliptic curves: ramification and entanglement

The goal of this section is to prove Theorem B by studying the ramification properties of primes in division field extensions associated to CM elliptic curves, which are described in Proposition 8.2.1 and Proposition 8.2.2. The proof of these results is an application to the CM case of the theory of formal groups outlined in Section 8.1. We often tacitly assume that all our number fields are embedded into $\mathbb{C}$. This fixes in particular a unique, normalised isomorphism

$$[\cdot]_E \colon \mathcal{O} \xrightarrow{\sim} \mathrm{End}_F(E)$$

associated to every elliptic curve E which has complex multiplication by an order $\mathcal{O}$ inside an imaginary quadratic field K, and is defined over a number field $F \subseteq \mathbb{C}$ such that $K \subseteq F$ (see Proposition 7.2.1).

With the next proposition, we start our study concerning the ramification properties of the extensions $F \subseteq F(E[I])$ (see Definition 7.2.3), by finding an explicit finite set of primes outside which these are unramified.

Proposition 8.2.1 – Unramifiedness of division fields

Let $F \subseteq \mathbb{C}$ be a number field and $E_{/F}$ an elliptic curve with complex multiplication by an order $\mathcal{O}$ in an imaginary quadratic field $K \subseteq F$. Denote by $\mathfrak{f}_{\mathcal{O}} := |\mathcal{O}_K : \mathcal{O}|$ the conductor of the order $\mathcal{O}$, and by $\mathfrak{f}_E \subseteq \mathcal{O}_F$ the conductor ideal of the elliptic curve E. Then for every ideal $I \subseteq \mathcal{O}$ coprime with $\mathfrak{f}_{\mathcal{O}}$ the extension $F \subseteq F(E[I])$ is unramified at all primes not dividing $(I \cdot \mathcal{O}_F) \cdot \mathfrak{f}_E$.

Proof. Since I is coprime with the conductor of the order $\mathcal{O}$, it can be uniquely factored into a product of invertible prime ideals of $\mathcal{O}$ (see Lemma 6.2.7). The field $F(E[I])$ is then the compositum of all the division fields $F(E[\mathfrak{p}^n])$ with $\mathfrak{p}^n$ the prime power factors of I in $\mathcal{O}$. Hence it suffices to prove that for every invertible prime ideal $\mathfrak{p} \subseteq \mathcal{O}$ and every $n \in \mathbb{N}$, the field extension $F \subseteq F(E[\mathfrak{p}^n])$ is unramified at every prime of F not dividing $(\mathfrak{p}\mathcal{O}_F) \cdot \mathfrak{f}_E$.

Fix an invertible prime $\mathfrak{p} \subseteq \mathcal{O}$ and write $L := F(E[\mathfrak{p}^n])$. Let $\mathfrak{q} \nmid (\mathfrak{p}\mathcal{O}_F) \cdot \mathfrak{f}_E$ be a prime of F and fix a prime $\mathfrak{Q} \subseteq \mathcal{O}_L$ lying above $\mathfrak{q}$, with residue field κ. Since $\mathfrak{q}$ does not divide the conductor $\mathfrak{f}_E$ of the elliptic curve, E has good reduction $\widetilde{E}$ modulo $\mathfrak{q}$ and we then denote by $\pi \colon E(L) \to \widetilde{E}(\kappa)$ the reduction map. Take $\sigma \in I(\mathfrak{Q}/\mathfrak{q})$, where $I(\mathfrak{Q}/\mathfrak{q}) \subseteq \mathrm{Gal}(L/F)$ denotes the inertia subgroup of $\mathfrak{q} \subseteq \mathfrak{Q}$, and fix a torsion point $Q \in E[\mathfrak{p}^n] = E(L)[\mathfrak{p}^n]$. By definition of inertia, σ acts trivially on the residue field κ, hence

$$\pi(Q^\sigma - Q) = \pi(Q^\sigma) - \pi(Q) = \pi(Q) - \pi(Q) = 0 \tag{8.7}$$

i.e. the point $Q^\sigma - Q$ is in the kernel of the reduction map π. We are going to use the exact sequence (8.6) to show that the only $\mathfrak{p}^n$-torsion point contained in this kernel is 0. To this aim, we embed L in its $\mathfrak{Q}$-adic completion $L_{\mathfrak{Q}}$, with ring of integers $\mathcal{O}_{L_{\mathfrak{Q}}}$ and maximal ideal $\mathfrak{m}_{\mathfrak{Q}}$. Notice that the set $(\mathfrak{p}^n \cap \mathcal{O}) \setminus (\mathfrak{Q} \cap \mathcal{O})$ is non-empty because $\mathfrak{p} \nmid \mathfrak{f}_{\mathcal{O}}$ and $\mathfrak{q} \nmid (\mathfrak{p}\mathcal{O}_F)$. Consider then the formal group $\widehat{E} \in \mathcal{O}_F[\![z_1, z_2]\!]$ associated to an integral Weierstraß model of E, and let $\alpha \in (\mathfrak{p}^n \cap \mathcal{O}) \setminus (\mathfrak{Q} \cap \mathcal{O})$. The endomorphism $\widehat{[\alpha]}_E \in \mathrm{End}_F(\widehat{E})$ corresponding to $[\alpha]_E \in \mathrm{End}_F(E)$ via (8.4) becomes an automorphism over $L_{\mathfrak{Q}}$, because $\widehat{[\alpha]}_E'(0) = \alpha \in \mathcal{O}_{L_{\mathfrak{Q}}}^\times$.

Hence taking $\Phi = [\mathfrak{p}^n]_E$ in (8.6) shows that $E[\mathfrak{p}^n] \cap \ker(\pi) \subseteq E[\alpha] \cap \ker(\pi) = \{0\}$, where the last equality holds because $\widehat{E}(\mathfrak{m}_{\mathfrak{Q}})[\widehat{\alpha}]_E = 0$. Combining this with (8.7), we see that $Q^\sigma = Q$ for every $Q \in E[\mathfrak{p}^n]$ and $\sigma \in I(\mathfrak{Q}/\mathfrak{q})$. Since L is generated over F by the elements of $E[\mathfrak{p}^n]$, we deduce that the inertia group $I(\mathfrak{Q}/\mathfrak{q})$ is trivial. In particular, $F \subseteq L$ is unramified at every prime not dividing $(\mathfrak{p} \cdot \mathcal{O}_F)\,\mathfrak{f}_E$, as wanted. □

We now turn to the study of the primes which ramify in $F \subseteq F(E[I])$. To do this, it suffices to restrict our attention to the case $I = \mathfrak{p}^n$ for some prime $\mathfrak{p} \subseteq \mathcal{O}$ and some $n \in \mathbb{N}$, as we do in the following proposition.

Proposition 8.2.2 – Total ramification in division fields

Let $F \subseteq \mathbb{C}$ be a number field and $E_{/F}$ be an elliptic curve with complex multiplication by an order $\mathcal{O}$ in an imaginary quadratic field $K \subseteq F$. Denote by $\mathfrak{b}_E := \mathfrak{f}_{\mathcal{O}}\,\Delta_F\,N_{F/\mathbb{Q}}(\mathfrak{f}_E)$ the product of the conductor $\mathfrak{f}_{\mathcal{O}} := |\mathcal{O}_K : \mathcal{O}|$ of the order $\mathcal{O}$, the absolute discriminant $\Delta_F \in \mathbb{Z}$ of the number field F and the norm $N_{F/\mathbb{Q}}(\mathfrak{f}_E) := |\mathcal{O}_F/\mathfrak{f}_E|$ of the conductor ideal $\mathfrak{f}_E \subseteq \mathcal{O}_F$. Then, for any $n \in \mathbb{N}$ and any prime ideal $\mathfrak{p} \subseteq \mathcal{O}$ coprime with $\mathfrak{b}_E\,\mathcal{O}$ the extension $F \subseteq F(E[\mathfrak{p}^n])$ is totally ramified at each prime dividing $\mathfrak{p}\,\mathcal{O}_F$. Moreover, the Galois representation

$$\rho_{E,\mathfrak{p}^n} \colon \mathrm{Gal}(F(E[\mathfrak{p}^n])/F) \hookrightarrow (\mathcal{O}/\mathfrak{p}^n)^\times \cong (\mathcal{O}_K/\mathfrak{p}^n\mathcal{O}_K)^\times$$

defined in Lemma 7.2.4 is an isomorphism.

Proof. The statement is trivially true if $n = 0$, hence we assume that $n \geq 1$. Fix $\widehat{E} \in \mathcal{O}_F[\![z_1, z_2]\!]$ to be the formal group associated to an integral Weierstraß model of E, and let $\mathfrak{p} \subseteq \mathcal{O}$ be as in the statement. The hypothesis of coprimality with $\mathfrak{b}_E\,\mathcal{O}$ implies that $\mathfrak{p}$ is invertible in $\mathcal{O}$ and that it lies above a rational prime $p \in \mathbb{N}$ which is unramified in K. We divide the proof according to the splitting behaviour of p in $\mathcal{O}$, which is the same as the splitting behaviour in K, since $p \nmid \mathfrak{f}_{\mathcal{O}}$.

First, assume that p is inert in K, so that $\mathfrak{p} = p\mathcal{O}$. In this case, $L := F(E[\mathfrak{p}^n])$ coincides with the p^n-division field $F(E[p^n])$. The injectivity of the Galois representation

$$\rho_{E,p^n} \colon \mathrm{Gal}(L/F) \hookrightarrow (\mathcal{O}/p^n\mathcal{O})^\times \cong (\mathcal{O}_K/p^n\mathcal{O}_K)^\times$$

shows that the degree of the extension $F \subseteq L$ is bounded as

$$[L : F] \leq |(\mathcal{O}_K/p^n\mathcal{O}_K)^\times| = p^{2(n-1)}(p^2 - 1).$$

Let $\mathfrak{P} \subseteq \mathcal{O}_L$ be a prime of L lying above p and denote by $L_{\mathfrak{P}}$ the $\mathfrak{P}$-adic completion of L, with ring of integers $\mathcal{O}_{L_{\mathfrak{P}}}$, maximal ideal $\mathfrak{m}_{\mathfrak{P}}$ and residue field $\kappa_{\mathfrak{P}}$. We want to determine the ramification index $e(\mathfrak{P}/(\mathfrak{P} \cap \mathcal{O}_F))$.

Since p is inert in K, the reduced elliptic curve $\widetilde{E}$ is supersingular by [Lan87, § 14, Theorem 12], hence $\widetilde{E}(\kappa_{\mathfrak{P}})[p^n] = 0$. Taking $\Phi = [p^n]_E$ in (8.6), we see that the group $\widehat{E}(\mathfrak{m}_{\mathfrak{P}})$ contains a non-zero point of exact order p^n. We can now use Lemma 8.1.2 and the hypothesis $p \nmid \Delta_F$ to get

$$p^{h(n-1)}(p^h - 1) \leq v_{L_{\mathfrak{P}}}(p) = e(\mathfrak{P}/p) = e(\mathfrak{P}/(\mathfrak{P} \cap \mathcal{O}_F)) \leq [L : F] \leq p^{2(n-1)}(p^2 - 1) \tag{8.8}$$

where $h \in \mathbb{N}$ denotes the height of the reduction modulo $\mathfrak{P}$ of the formal group $\widehat{E}$. Since the latter is precisely the formal group associated to $\widetilde{E}$, we have that $h = 2$ by [Sil09, Chapter V,

Theorem 3.1]. Thus all the inequalities appearing in (8.8) are actually equalities, and we see at once that $e(\mathfrak{P}/(\mathfrak{P}\cap O_F)) = [L\colon F] = p^{2(n-1)}(p^2-1)$, which implies that $\rho_{E,\mathfrak{p}^n}$ is an isomorphism, and that $\mathfrak{P}\cap O_F$ is totally ramified in L. This concludes the proof of the inert case.

Suppose now that p splits in K, so that $pO = \mathfrak{p}\overline{\mathfrak{p}}$, where $\overline{\mathfrak{p}}$ is the image of $\mathfrak{p}$ under the unique non-trivial automorphism of K. If we put again $L := F(E[\mathfrak{p}^n])$, the injectivity of $\rho_{E,\mathfrak{p}^n}$ gives

$$[L : F] \leq |(O_K/p^nO_K)^\times| = p^{n-1}(p-1).$$

It is convenient in this case to work inside the bigger division field $\widetilde{F} := F(E[p^n])$, which contains both L and $L' := F(E[\,\overline{\mathfrak{p}}^n\,])$. We then fix $\mathfrak{P}, \overline{\mathfrak{P}} \subseteq O_{\widetilde{F}}$ to be two primes of $\widetilde{F}$, lying respectively above $\mathfrak{p}O_K$ and $\overline{\mathfrak{p}}O_K$, and we denote by $\mathcal{P} := \mathfrak{P}\cap O_L$ and $\overline{\mathcal{P}} := \overline{\mathfrak{P}}\cap O_L$ the corresponding primes in L. For every prime ideal $\mathfrak{q} \in \{\mathfrak{P}, \overline{\mathfrak{P}}\}$ we denote by $\widetilde{F}_\mathfrak{q}$ the $\mathfrak{q}$-adic completion of $\widetilde{F}$, with ring of integers $O_{\widetilde{F}_\mathfrak{q}}$ and residue field $\kappa_\mathfrak{q}$, and by $\widetilde{E}_\mathfrak{q}$ the reduction of $E_{/\widetilde{F}}$ modulo $\mathfrak{q}$. We use analogous notation for $\mathcal{P}$ and $\overline{\mathcal{P}}$. The goal is to compute the ramification index $e(\mathcal{P}/\mathcal{P}\cap O_F)$, and we divide our argument in three steps.

Step 1 First of all, we prove that the reduction map $E[\mathfrak{p}^n] \to \widetilde{E}_{\overline{\mathfrak{P}}}(\kappa_{\overline{\mathfrak{P}}})$ is injective. This is equivalent to say that $\ker(\pi_{\overline{\mathcal{P}}}) \cap E(L_{\overline{\mathcal{P}}})[\mathfrak{p}^n] = 0$, where

$$\pi_{\overline{\mathcal{P}}}\colon E(L_{\overline{\mathcal{P}}}) \twoheadrightarrow \widetilde{E}_{\overline{\mathcal{P}}}(\kappa_{\overline{\mathcal{P}}}) \subseteq \widetilde{E}_{\overline{\mathfrak{P}}}(\kappa_{\overline{\mathfrak{P}}})$$

denotes the reduction modulo $\overline{\mathcal{P}}$. Since p is coprime with the conductor of the order O by assumption, it is possible to find $\alpha \in \mathfrak{p}^n$ such that $\alpha \notin \overline{\mathfrak{p}}$. The endomorphism $\widehat{[\alpha]}_E \in \mathrm{End}_F(\widehat{E})$ corresponding to $[\alpha]_E \in \mathrm{End}_F(E)$ via (8.4) becomes an automorphism over $L_{\overline{\mathcal{P}}}$, because $\widehat{[\alpha]}'_E(0) = \alpha \in O^\times_{L_{\overline{\mathcal{P}}}}$. Hence taking $\Phi = [\mathfrak{p}^n]_E$ in (8.6) shows that

$$\ker(\pi_{\overline{\mathcal{P}}}) \cap E(L_{\overline{\mathcal{P}}})[\mathfrak{p}^n] \subseteq \ker(\pi_{\overline{\mathcal{P}}}) \cap E(L_{\overline{\mathcal{P}}})[\alpha] = 0$$

where the last equality holds because $\widehat{E}(\mathfrak{m}_{\overline{\mathcal{P}}})\widehat{[\alpha]}_E = 0$. In exactly the same way, using L' in place of L, one shows that the reduction map $E[\overline{\mathfrak{p}}^n] \to \widetilde{E}_{\overline{\mathfrak{p}}}(\kappa_{\overline{\mathfrak{p}}})$ is injective.

Step 2 We now claim that $\ker(\pi_\mathfrak{P}) \cap E[p^n] = E[\mathfrak{p}^n]$ where $\pi_\mathfrak{P} : E(\widetilde{F}) \to \widetilde{E}_\mathfrak{P}(\kappa_\mathfrak{P})$ denotes the reduction modulo $\mathfrak{P}$. Since $pO = \mathfrak{p}\overline{\mathfrak{p}}$, there is a decomposition of the group $E[p^n]$ into the direct sum of $E[\,\mathfrak{p}^n\,]$ and $E[\,\overline{\mathfrak{p}}^n\,]$, which are cyclic groups of order p^n by Lemma 7.2.4. In particular, there exist $A \in E[\mathfrak{p}^n]$ and $B \in E[\overline{\mathfrak{p}}^n]$ such that every p^n-torsion point $Q \in E[p^n]$ can be written as

$$Q = [a](A) + [b](B)$$

for unique $a, b \in \{0, \dots, p^n - 1\}$. If $\pi_\mathfrak{P}(Q) = 0$ then

$$\pi_\mathfrak{P}([b](B)) = \pi_\mathfrak{P}([-a](A)) \in \widetilde{E}_\mathfrak{P}[\,\mathfrak{p}^n\,] \cap \widetilde{E}_\mathfrak{P}[\,\overline{\mathfrak{p}}^n\,] = \{0\}$$

where the last equality follows from the fact that $\mathfrak{p}^n$ and $\overline{\mathfrak{p}}^n$ are coprime in O. In particular, $[b](B)$ is in the kernel of the reduction map $E[\overline{\mathfrak{p}}^n] \to \widetilde{E}_\mathfrak{P}(\kappa_\mathfrak{P})[p^n]$, which is the restriction of $\pi_\mathfrak{P}$ to $E[\overline{\mathfrak{p}}^n]$, and is injective by **Step 1**. Hence we have $Q = [a](A) \in E[\mathfrak{p}^n]$, and this shows the inclusion $\ker(\pi_\mathfrak{P}) \cap E[p^n] \subseteq E[\mathfrak{p}^n]$. To prove the other inclusion, first notice that the restriction

of $\pi_{\mathfrak{P}}$ to $E[p^n]$ gives rise to a surjection $E[p^n] \twoheadrightarrow \widetilde{E}_{\mathfrak{P}}(\kappa_{\mathfrak{P}})[p^n]$, because $E[\overline{\mathfrak{p}}^n] \to \widetilde{E}_{\mathfrak{P}}(\kappa_{\mathfrak{P}})[p^n]$ is injective and the elliptic curve $\widetilde{E}_{\mathfrak{P}}$ is ordinary by [Lan87, § 14, Theorem 12]. This gives

$$\frac{E[p^n]}{\ker(\pi_{\mathfrak{P}}) \cap E[p^n]} \cong \widetilde{E}_{\mathfrak{P}}(\kappa_{\mathfrak{P}})[p^n]$$

which in turn shows that

$$|\ker(\pi_{\mathfrak{P}}) \cap E[p^n]| = \frac{|E[p^n]|}{|\widetilde{E}_{\mathfrak{P}}(\kappa_{\mathfrak{P}})[p^n]|} = \frac{p^{2n}}{p^n} = p^n = |E[\mathfrak{p}^n]|.$$

We conclude that $\ker(\pi_{\mathfrak{P}}) \cap E[p^n] = E[\mathfrak{p}^n]$, as we wanted to prove.

Step 3 Using (8.6) with $\Phi = [p^n]_E$ and **Step 2**, after recalling that $\mathfrak{P}$ lies over $\mathcal{P}$, one can see that the group $\widehat{E}(\mathfrak{m}_{\mathcal{P}})$ contains a point of exact order p^n. We now apply Lemma 8.1.2, and the hypothesis $p \nmid \Delta_F$, to get

$$p^{h(n-1)}(p^h - 1) \leq v_{L_{\mathcal{P}}}(p) = e(\mathcal{P}/p) = e(\mathcal{P}/(\mathcal{P} \cap \mathcal{O}_F)) \leq [L\colon F] \leq p^{n-1}(p-1). \tag{8.9}$$

where $h \in \mathbb{N}$ denotes the height of the reduction modulo $\mathcal{P}$ of the formal group $\widehat{E}$. Since the latter is precisely the formal group associated to the ordinary elliptic curve $\widetilde{E}_{\mathcal{P}}$, we have that $h = 1$ by [Sil09, Chapter V, Theorem 3.1]. Thus all the inequalities appearing in (8.9) are actually equalities, and we see at once that $e(\mathcal{P}/(\mathcal{P} \cap \mathcal{O}_F)) = [L\colon F] = p^{n-1}(p-1)$, which implies that $\rho_{E,\mathfrak{p}^n}$ is an isomorphism, and that $\mathcal{P} \cap \mathcal{O}_F$ is totally ramified in L. This concludes the proof. □

Remark 8.2.3. As we already stated in the introduction, Proposition 8.2.2 can be obtained by combining various results of Lozano-Robledo. More precisely, see [Loz16, Proposition 5.6] for the inert case, and the proof of [Loz18, Theorem 6.10] for the split case. The arguments used by Lozano-Robledo for the inert case involve a formula for the valuation of the coefficient of t^p in the power series $[p]_{\widehat{E}}(t) \in \mathcal{O}_F[\![t]\!]$ (see [Loz13, Theorem 3.9]), and the study of the split case goes through a detailed investigation of Borel subgroups of $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ (see [Loz18, Section 4]). Our proof of Proposition 8.2.2, which concerns only CM elliptic curves and prime ideals not dividing $\mathfrak{b}_E \mathcal{O}$, appears to be shorter because it uses the same techniques to deal with the split and inert case. Notice as well that our discussion is explicitly written for general imaginary quadratic orders, whereas [Loz18, Theorem 6.10] is stated and proved only for maximal orders. We observe however that Lozano-Robledo uses [Loz18, Remark 6.12] to point out that the proof of [Loz18, Theorem 6.10] carries over to the general case.

We also remark that, if $\mathcal{O} = \mathcal{O}_K$ is a maximal order of class number 1 and $F = K$, Proposition 8.2.2 is proved by Coates and Wiles in [CW77, Lemma 5] (see also [Art78, Lemma 3] and [Coa13, Proposition 47]). The main tool used in their proof is Lubin-Tate theory.

Remark 8.2.4. Let $E_{/F}$ be any elliptic curve (not necessarily with complex multiplication) which has good supersingular reduction at a prime $\mathfrak{p} \subseteq \mathcal{O}_F$ lying above a prime $p \in \mathbb{N}$ which does not ramify in the extension $\mathbb{Q} \subseteq F$. Then one can use the same argument provided in the first part of the proof of Proposition 8.2.2 to show that the ramification index $e(\mathfrak{P}/\mathfrak{p})$ is bounded from below by $p^{2(n-1)}(p^2-1)$, where $\mathfrak{P} \subseteq F(E[p^n])$ is any prime lying above $\mathfrak{p}$. This result has already been proved by Lozano-Robledo in [Loz16, Proposition 5.6] and by Smith in [Smi18, Theorem 2.1].

Remark 8.2.5. Let E be an elliptic curve having complex multiplication by an imaginary quadratic order $\mathcal{O}$, and suppose that E is defined over the ring class field $H_{\mathcal{O}}$. Then, using the recent work

[Loz19] of Lozano-Robledo, and in particular [Loz19, Theorem 1.2.(4)] and [Loz19, Theorem 7.11], one can show that the Galois representation ρ_{E,p^n} is an isomorphism for every $n \in \mathbb{N}$ and every rational prime $p \in \mathbb{N}$ such that $p \nmid 2\mathfrak{f}_O\Delta_K$. This strengthens, for elliptic curves defined over H_O, the final assertion of Proposition 8.2.2.

We are now ready to prove Theorem B, whose statement we recall for convenience.

Theorem 8.2.6 – Entanglement and division fields of CM elliptic curves

Let F be a number field and $E_{/F}$ be an elliptic curve with complex multiplication by an order O in an imaginary quadratic field $K \subseteq F$. Denote by $\mathfrak{b}_E := \mathfrak{f}_O \Delta_F N_{F/\mathbb{Q}}(\mathfrak{f}_E)$ the product of the conductor $\mathfrak{f}_O := |O_K : O|$ of the order O, the absolute discriminant $\Delta_F \in \mathbb{Z}$ of the number field F and the norm $N_{F/\mathbb{Q}}(\mathfrak{f}_E) := |O_F/\mathfrak{f}_E|$ of the conductor ideal $\mathfrak{f}_E \subseteq O_F$.

Then the map (8.1) induces an isomorphism

$$\mathrm{Gal}(F(E_{\mathrm{tors}})/F) \xrightarrow{\sim} \mathrm{Gal}(F(E[S^\infty])/F) \times \prod_{p \notin S} \mathrm{Gal}(F(E[p^\infty])/F)$$

where $S \subseteq \mathbb{N}$ denotes the finite set of primes dividing $\mathfrak{b}_E$.

Remark 8.2.7. Recall that a family $\mathcal{F} = \{F_s\}_{s \in \mathcal{S}}$ of Galois extensions of a number field F, indexed over any set $\mathcal{S}$, is called *linearly disjoint* over F if the natural inclusion map

$$\mathrm{Gal}(L/F) \hookrightarrow \prod_{s \in \mathcal{S}} \mathrm{Gal}(F_s/F)$$

is an isomorphism, where L denotes the compositum of the fields F_s. Otherwise the family is called *entangled* over F.

Proof of Theorem 8.2.6. The family $\{F(E[p^\infty])\}_{q \notin S} \cup \{F(E[S^\infty])\}$ appearing in the statement of Theorem 8.2.6 is linearly disjoint over F if and only if $F(E[p^n]) \cap F(E[m]) = F$ for every prime $p \notin S$, every $n \in \mathbb{N}$ and every $m \in \mathbb{Z}$ coprime with p. To prove this latter statement, we first show that every non-trivial subextension of $\widetilde{F} := F(E[p^n])$ is ramified at some prime dividing p.

When p is inert in K, this follows immediately from Proposition 8.2.2. Suppose then that p is split in K, with $pO_K = \mathfrak{p}\overline{\mathfrak{p}}$. The division field $\widetilde{F}$ is the compositum over F of the extensions $F_{\mathfrak{p}} := F(E[\mathfrak{p}^n])$ and $F_{\overline{\mathfrak{p}}} := F(E[\overline{\mathfrak{p}}^n])$. By Proposition 8.2.2 the extension $F \subseteq F_{\mathfrak{p}}$ (respectively $F \subseteq F_{\overline{\mathfrak{p}}}$) is totally ramified at every prime of F lying over $\mathfrak{p}$ (resp. $\overline{\mathfrak{p}}$). Let $\mathfrak{P}$ be a prime of F lying above $\mathfrak{p}$, and denote by $I(\mathfrak{P}) \subseteq \mathrm{Gal}(\widetilde{F}/F)$ its inertia group and by $e(\mathfrak{P})$ its ramification index in the extension $F \subseteq \widetilde{F}$. If $F \subsetneq L$ is a subextension of $F \subseteq \widetilde{F}$ in which $\mathfrak{P}$ does not ramify, then L must be contained in the inertia field $T = (\widetilde{F})^{I(\mathfrak{P})}$ relative to $\mathfrak{P}$. Notice that the latter also contains $F_{\overline{\mathfrak{p}}}$, since by Proposition 8.2.1 the extension $F \subseteq F_{\overline{\mathfrak{p}}}$ is unramified at $\mathfrak{P}$. On the other hand, the fact that $F \subseteq F_{\mathfrak{p}}$ is totally ramified at $\mathfrak{P}$ gives the chain of inequalities

$$[F_{\overline{\mathfrak{p}}} : F] \leq [T : F] = \frac{[\widetilde{F} : F]}{|I(\mathfrak{P})|} = \frac{[\widetilde{F} : F]}{e(\mathfrak{P})} \leq \frac{[F_{\mathfrak{p}} : F] \cdot [F_{\overline{\mathfrak{p}}} : F]}{e(\mathfrak{P})} \leq [F_{\overline{\mathfrak{p}}} : F]$$

which shows that $T = F_{\overline{\mathfrak{p}}}$. Hence Proposition 8.2.2 implies that any extension $F \subseteq L$ which is unramified at every prime lying above $\mathfrak{p}$ is totally ramified at every prime lying above $\overline{\mathfrak{p}}$.

Now, it is easy to conclude that $\widetilde{F} \cap F(E[m]) = F$, since otherwise $F \subseteq F(E[m])$ would ramify at some prime of F dividing p, contradicting Proposition 8.2.1. □

Remark 8.2.8. Let F be a number field and $E_{/F}$ be an elliptic curve with complex multiplication by an order $\mathcal{O}$ in an imaginary quadratic field $K \subseteq F$. Denote by $S \subseteq \mathbb{N}$ the set of primes dividing $\mathfrak{b}_E$, as in Theorem 8.2.6. In this general setting, it is an interesting question to study the entanglement in the finite family of "bad" division fields $\{F(E[p^\infty])\}_{p \in S}$, as we do in Section 8.4 where we specify $F = K$ and E to be the base-change of an elliptic curve defined over $\mathbb{Q}$.

A first step towards a complete answer to the previous question in the general setting is to find the minimal set $S' \subseteq S$ such that the family of division fields

$$\{F(E[p^\infty])\}_{p \notin S'} \cup \{F(E[(S')^\infty])\}$$

is linearly disjoint over F. We partially answer the latter question in Corollary 8.3.4, where we prove that one can take $S' = \emptyset$ for every elliptic curve E defined over the ring class field $H_{\mathcal{O}}$ satisfying the condition $H_{\mathcal{O}}(E_{\text{tors}}) \not\subseteq K^{\text{ab}}$. There are infinitely many such elliptic curves when $\text{Pic}(\mathcal{O}) \neq \{1\}$, as we show in Theorem 8.3.7. On the other hand, if $\text{Pic}(\mathcal{O}) = \{1\}$ there are infinitely many examples of elliptic curves E having complex multiplication by $\mathcal{O}$ for which $S' = S$ can be arbitrarily large (see Remark 8.4.5).

Remark 8.2.9. Let F be a number field and E be a CM elliptic curve defined over F. Then, even when $K \not\subseteq F$, we have that $K \subseteq F(E[N])$ for every $N > 2$. This has been showed in [Mur83, Lemma 6] for $F = \mathbb{Q}$, and in [BCS17, Lemma 3.15] for arbitrary F. In particular, the statement of Theorem 8.2.6 does not hold when $K \not\subseteq F$.

The description of the set of primes S in Theorem 8.2.6 is actually redundant, since all the primes p dividing the conductor $\mathfrak{f}_{\mathcal{O}}$, with the possible exception of $p = 2$, also divide the absolute discriminant Δ_F of the field of definition of E. This can be seen using the fact that F contains the ring class field $K(j(E)) = H_{\mathcal{O}}$ (see Proposition 7.1.33). Indeed, the following proposition, which is a weaker form of [Cox13, Exercise 9.20], shows that the extension $\mathbb{Q} \subseteq H_{\mathcal{O}}$ is ramified at all the odd primes dividing the conductor $\mathfrak{f}_{\mathcal{O}}$, and thus allows us to conclude that for every prime $p \in \mathbb{N}$ such that $p \geq 3$ we have that $p \mid \mathfrak{f}_{\mathcal{O}} \Rightarrow p \mid \Delta_F$.

Proposition 8.2.10 – Ramification in the ring class fields associated to imaginary quadratic orders

Let $\mathcal{O}$ be an order of conductor $\mathfrak{f}_{\mathcal{O}} := |\mathcal{O}_K : \mathcal{O}|$ in an imaginary quadratic field K. Then the extension $\mathbb{Q} \subseteq H_{\mathcal{O}}$ is ramified at all the odd primes dividing $\mathfrak{f}_{\mathcal{O}}$. Moreover, if $4 \mid \mathfrak{f}_{\mathcal{O}}$ the same extension is also ramified at 2.

Proof. If $\mathfrak{f}_{\mathcal{O}} = 1$ there is nothing to prove. Otherwise, let $\mathfrak{f}_{\mathcal{O}} = p_1^{a_1} \cdots p_n^{a_n}$ be the prime factorisation of $\mathfrak{f}_{\mathcal{O}}$, and observe that, for every $i \in \{1, \ldots, n\}$, one has the chain of inclusions

$$K \subseteq H_{\mathcal{O}_K} \subseteq H_{\mathcal{O}_i} \subseteq H_{\mathcal{O}}$$

given by the *Anordnungsatz* for ring class fields (see Remark 6.2.15), where O_i denotes the unique order of conductor $p_i^{a_i}$ (see Example 6.2.8). Now, the class number formula [Cox13, Theorem 7.24] yields

$$[H_{O_i} : H_{O_K}] = \frac{[H_{O_i} : K]}{[H_{O_K} : K]} = \frac{h_{O_i}}{h_K} = \frac{p_i^{a_i}}{|O_K^\times : O_i^\times|} \left(1 - \left(\frac{\Delta_K}{p_i}\right) \frac{1}{p_i}\right). \tag{8.10}$$

where $h_{O_i} := [H_{O_i} : K] = |\mathrm{Pic}(O_i)|$ and analogously $h_K := [H_{O_K} : K] = |\mathrm{Pic}(O_K)|$. Since either $p_i \geq 3$ or $p_i = 2$ and $a_i \geq 2$, we see from (8.10) that $H_{O_i} \neq H_{O_K}$ except when $p_i = 3$, $a_i = 1$ and $K = \mathbb{Q}(\sqrt{-3})$. In this last case the extension $\mathbb{Q} \subseteq K$ is ramified at $p_i = 3$. Otherwise the extension $H_{O_K} \subsetneq H_{O_i}$ is ramified at some prime dividing p_i. Indeed, $H_{O_K} \subsetneq H_{O_i}$ is ramified at some prime because $K \subseteq H_{O_i}$ is abelian and H_{O_K} is the Hilbert class field of K, and this suffices to conclude because $K \subseteq H_{O_i}$ can ramify only at primes lying above p_i. □

Remark 8.2.11. If $2 \mid \mathfrak{f}_O$ but $4 \nmid \mathfrak{f}_O$, the extension $\mathbb{Q} \subseteq H_O$ could still be unramified at 2. This happens, for instance, if $\mathfrak{f}_O = 2$ and 2 splits in K, because in this case the ring class field H_O is equal to the Hilbert class field H_{O_K}.

Proposition 8.2.10 shows that the set S in Theorem 8.2.6 could be replaced by the set S' of primes dividing $2 \cdot \Delta_F \cdot \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{f}_E)$, even if this results in a slightly weaker statement. However, choosing the set S' instead of the set S allows to draw a comparison with a result of Lombardo on the image of p-adic Galois representations attached to CM elliptic curves, which is shown in [Lom17, Theorem 6.6]. In this paper, Lombardo proves the isomorphism

$$\mathrm{Gal}(F(E[p^\infty])/F) \cong (O \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$$

for every prime $p \nmid \Delta_F \cdot \mathrm{N}_{F/\mathbb{Q}}(\mathfrak{f}_E)$. If moreover $p \geq 3$, *i.e.* $p \notin S'$, this isomorphism follows also from Proposition 8.2.2 by taking inverse limits. The methods used in [Lom17] are different from ours, and generalise also to higher dimensional abelian varieties.

8.3 Minimality of division fields

We have seen in Proposition 8.2.2 that, for every CM elliptic curve E defined over a number field F with $\mathrm{End}_F(E) \cong O$ for some order O in an imaginary quadratic field $K \subseteq F$, the division fields $F(E[N])$ are *maximal* for all integers N coprime with a fixed integer $\mathfrak{b}_E \in \mathbb{N}$. This is to say that the associated Galois representation $\rho_{E,N}$ given by Lemma 7.2.4 is surjective. When E is defined over the ring class field H_O of K relative to O, the division fields $H_O(E[N])$ always contain the ray class field $H_{N,O}$ (see Definition 6.2.11), as we proved in Theorem 7.2.5. If the division field $H_O(E[N])$ is maximal and $N > 2$, then the containment $H_{N,O} \subseteq H_O(E[N])$ is strict. In this section, we want to study for which integers N the division fields are *minimal*, in the sense that $H_O(E[N]) = H_{N,O}$. Theorem 8.3.1, which is the main result of this section, provides an explicit set of integers $N \in \mathbb{N}$ for which such an equality occurs. In fact, Theorem 8.3.1 is formulated in a wider setting, with the integer N replaced by a general invertible ideal $I \subseteq O$. This minimality result is used in Section 8.4 to detect entanglement in families of division fields.

Before stating Theorem 8.3.1, we point out that the its proof uses crucially the main theorem of complex multiplication, which we stated as Theorem 7.1.25. Hence the entire Section 8.3 makes wide use of the concepts of lattices, idèles and Hecke characters that we introduced in Section 6.1. Finally, we recall that, for every order O contained in an imaginary quadratic field K and every ideal $I \subseteq O$, we denote by $H_{I,O}$ the ray class field of K modulo I relative to the order O, which was defined in Definition 6.2.11.

Theorem 8.3.1 – Minimality of division fields of CM elliptic curves

Let $F \subseteq \mathbb{C}$ be a number field and let $E_{/F}$ be an elliptic curve such that $\mathrm{End}_F(E) \cong \mathcal{O}$ for some order $\mathcal{O}$ inside an imaginary quadratic field $K \subseteq F$. Suppose that $F(E_{\mathrm{tors}}) \subseteq K^{\mathrm{ab}}$, *i.e.* that E satisfies Shimura's condition (see Definition 7.1.30). Let $H := H_{\mathcal{O}}$ the ring class field of $\mathcal{O}$, and fix $\alpha\colon K^\times \cdot \mathrm{N}_{F/K}(\mathbb{A}_F^\times) \to K^\times$ as in Theorem 7.1.25, with $M = K$. Then we have that $F(E[I]) = F \cdot H_{I,\mathcal{O}}$ for every invertible ideal $I \subseteq \mathcal{O}$ such that $I \subseteq \mathfrak{f}_\varphi \cap \mathcal{O}$, where $\mathfrak{f}_\varphi \subseteq \mathcal{O}_K$ is the conductor of any of the Hecke characters $\varphi\colon \mathbb{A}_K^\times \to \mathbb{C}^\times$ extending the group homomorphism $\psi_\alpha\colon K^\times \cdot \mathrm{N}_{F/K}(\mathbb{A}_F^\times) \to \mathbb{C}^\times$ defined in (7.13).

Proof. The containment $H_{I,\mathcal{O}} \subseteq F(E[I])$ is given by Theorem 7.2.5. Observe moreover that $K \subseteq F$ is an abelian extension, since $F \subseteq F(E_{\mathrm{tors}}) \subseteq K^{\mathrm{ab}}$ by assumption. Hence, to prove that $F(E[I]) \subseteq F \cdot H_{I,\mathcal{O}}$ it is sufficient to show that every I-torsion point of E is fixed by $[s, K]$, for any $s \in \mathbb{A}_K^\times$ such that $[s,K]\big|_{H_{I,\mathcal{O}}} = \mathrm{Id}$. Moreover, it suffices to consider only those $s \in \mathbb{A}_K^\times$ such that $s_\infty = 1$ and $s \in U_{I,\mathcal{O}}$, where $U_{I,\mathcal{O}} \leq \mathbb{A}_K^\times$ is the subgroup defined in (6.12). This follows from the fact that $[U_{I,\mathcal{O}}, K] = \mathrm{Gal}(K^{\mathrm{ab}}/H_{I,\mathcal{O}})$ and $K_\infty^\times \subseteq \ker([\cdot, K]) \cap U_{I,\mathcal{O}}$ by Definition 6.2.11 and Lemma 6.2.16.

Fix then $s \in U_{I,\mathcal{O}}$ with $s_\infty = 1$. To study the action of $[s, K]$ on $E[I]$, we fix an invertible ideal $\mathfrak{a} \subseteq \mathcal{O} \subseteq \mathbb{C}$ and a complex uniformisation $\xi\colon \mathbb{C}/\mathfrak{a} \xrightarrow{\sim} E(\mathbb{C})$, which exists by Proposition 7.1.33. Take a torsion point $P \in E[I]$, and let $z \in (\mathfrak{a}\colon I)$ be any element such that $\xi(\overline{z}) = P$, where $\overline{z} \in (\mathfrak{a}\colon I)/\mathfrak{a}$ denotes the image of z in the quotient. Since $s \in K^\times \cdot \mathrm{N}_{H/K}(\mathbb{A}_H^\times)$, we have that

$$P^{[s,K]} = \xi(\overline{z})^{[s,K]} = \xi\left((\alpha(s)\, s^{-1}) \cdot \overline{z}\right)$$

which follows from applying Theorem 7.1.25 with $M = K$. This result can be applied because

$$s \in U_{I,\mathcal{O}} \subseteq U_{\mathcal{O}} \subseteq K^\times \cdot U_{\mathcal{O}} = K^\times \cdot \mathrm{N}_{H/K}(\mathbb{A}_H^\times)$$

where the last equality is given by Lemma 6.2.16.

To conclude, it suffices to show that $s^{-1} \cdot \overline{z} = \overline{z}$ and $\alpha(s) = 1$. Notice that $s^{-1} \cdot \mathfrak{a} = \mathfrak{a}$, because $\mathfrak{a} \subseteq \mathcal{O}$ is invertible and $s_p \in \mathcal{O}_p^\times$ for every rational prime $p \in \mathbb{N}$. The equality $s^{-1} \cdot \overline{z} = \overline{z}$ then follows from the fact that, for every prime $p \in \mathbb{N}$, we have $s_p^{-1} z - z \in \mathfrak{a}_p$, because $z \in (\mathfrak{a}\colon I)$ and $s_p^{-1} \in 1 + I\mathcal{O}_p$. To prove the equality $\alpha(s) = 1$, notice that for every prime $p \in \mathbb{N}$ we have

$$1 + I\mathcal{O}_p \subseteq \prod_{\substack{w \mid p \\ w \in M_K^0}} (1 + \mathfrak{f}_\varphi \mathcal{O}_{K_w})$$

since $I \subseteq \mathfrak{f}_\varphi \cap \mathcal{O}$ by assumption. This implies that $\varphi_p(s_p) = 1$ for every prime $p \in \mathbb{N}$. Indeed $s_p \in 1 + I\mathcal{O}_p$ by the definition of $U_{I,\mathcal{O}}$ and for every $w \in M_K^0$ we have that $\varphi_w(1 + \mathfrak{f}_\varphi \mathcal{O}_{K_w}) = 1$, because $\mathfrak{f}_\varphi$ is the conductor of φ. Since $s_\infty = 1$ we get that $\alpha(s) = \varphi(s) = 1$, as was to be shown. □

Remark 8.3.2. Theorem 8.3.1 has been proved by Coates and Wiles (see [CW77, Lemma 3]) if $\mathcal{O} = \mathcal{O}_K$ is a maximal order of class number one. Their result has been generalised in the PhD thesis of Kuhman (see [Kuh78, Chapter II, Lemma 3]) to maximal orders $\mathcal{O} = \mathcal{O}_K$, under the hypothesis that $F \subseteq H_{I,\mathcal{O}_K}$.

Theorem 8.3.1 has a partial converse, as we show in the following proposition.

Proposition 8.3.3 – A partial converse to Theorem 8.3.1

Let $\mathcal{O}$ be an order in an imaginary quadratic field K and $F \supseteq K$ be an abelian extension. Let $E_{/F}$ be an elliptic curve with complex multiplication by the order $\mathcal{O}$. Suppose that there exists an invertible ideal $I \subseteq \mathcal{O}$ such that $F(E[I]) = F \cdot H_{I,\mathcal{O}}$, and that $I \cap \mathbb{Z} = N\mathbb{Z}$, with $N > 2$ if $j(E) \neq 0$ and $N > 3$ if $j(E) = 0$. Then $F(E_{\text{tors}}) = K^{\text{ab}}$.

Proof. It is sufficient to prove that $F(E_{\text{tors}}) \subseteq K^{\text{ab}}$, since the other inclusion follows from Theorem 7.2.5 and the fact that $K \subseteq F$ is abelian.

Fix an embedding $K \hookrightarrow \mathbb{C}$ and let $\xi : \mathbb{C}/\Lambda \xrightarrow{\sim} E(\mathbb{C})$ be a complex parametrization for E, where $\Lambda \subseteq K$ is a lattice. Take $\sigma \in \text{Aut}(\mathbb{C}/K^{\text{ab}})$. By [Shi94, Theorem 5.4] with $s = 1$, there exists a complex parametrization $\xi' : \mathbb{C}/\Lambda \xrightarrow{\sim} E(\mathbb{C})$ such that the following diagram

$$\begin{array}{ccc} E(\mathbb{C}) & \xrightarrow{\quad\sigma\quad} & E(\mathbb{C}) \\ & {\scriptstyle\xi}\nwarrow \quad \nearrow{\scriptstyle\xi'} & \\ & K/\Lambda & \end{array}$$

commutes. This means that σ acts on E_{tors} as an automorphism $\gamma = \xi' \circ \xi^{-1} \in \text{Aut}(E) \cong \mathcal{O}^\times$. In particular, for any point $P \in E[I]$ we have

$$\gamma(P) = \sigma(P) = P \tag{8.11}$$

since by assumption $F(E[I]) = F \cdot H_{I,\mathcal{O}} \subseteq K^{\text{ab}}$. Notice now that, if $j(E) \notin \{0, 1728\}$ we have $\text{Aut}(E) = \{\pm 1\}$, and equality (8.11) can occur for $\gamma = -1$ only when $I \cap \mathbb{Z} = 2\mathbb{Z}$. Similarly, if $j(E) = 1728$ or $j(E) = 0$, one sees that a non-trivial element of $\text{Aut}(E)$ can possibly fix only points of $E[2]$ or points of $E[2] \cup E[3]$, respectively. Our assumptions on I allow then to conclude that γ must be the identity on E.

We have shown that every complex automorphism which fixes the maximal abelian extension of K fixes also the torsion points of E. We conclude that $F(E_{\text{tors}}) \subseteq K^{\text{ab}}$, which finishes the proof. □

As a consequence of Proposition 8.3.3 we deduce that, for any order $\mathcal{O}$ in an imaginary quadratic field K, and any elliptic curve E with complex multiplication by $\mathcal{O}$ which is defined over the ring class field $H_{\mathcal{O}}$, the whole family of division fields $\{H_{\mathcal{O}}(E[p^\infty])\}_p$ is linearly disjoint over $H_{\mathcal{O}}$ as soon as the extension $K \subseteq H_{\mathcal{O}}(E_{\text{tors}})$ is not abelian.

Corollary 8.3.4 – The index of the image of the Galois representation attached to a CM elliptic curve

Let $\mathcal{O}$ be an order inside an imaginary quadratic field K, and let $E_{/H_{\mathcal{O}}}$ be an elliptic curve with complex multiplication by $\mathcal{O}$. Then we have that

$$|\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})\colon \mathrm{Im}(\rho_E)| = \begin{cases} |\mathcal{O}^\times|, & \text{if } K \subseteq H_{\mathcal{O}}(E_{\mathrm{tors}}) \text{ is abelian,} \\ 1, & \text{otherwise.} \end{cases}$$

In particular, if $H_{\mathcal{O}}(E_{\mathrm{tors}}) \not\subseteq K^{\mathrm{ab}}$ then all the Galois representations ρ_{E,p^n} defined in Lemma 7.2.4 are isomorphisms, and the family of division fields $\{H_{\mathcal{O}}(E[p^\infty])\}_p$ is linearly disjoint over $H_{\mathcal{O}}$.

Proof. Suppose that $K \subseteq H_{\mathcal{O}}(E_{\mathrm{tors}})$ is not abelian. Since $H_{\mathcal{O}}(E_{\mathrm{tors}}) \subseteq H_{\mathcal{O}}^{\mathrm{ab}}$, this shows in particular that $K \neq H_{\mathcal{O}}$ and hence that $j(E) \notin \{0, 1728\}$. Then Proposition 8.3.3 shows that

$$H_{\mathcal{O}}(E[N]) \neq H_{N,\mathcal{O}}$$

for every $N \in \mathbb{N}$ with $N \geq 2$. Since $j(E) \notin \{0, 1728\}$, this implies that the Galois representation

$$\rho_{E,N}\colon \mathrm{Gal}(H_{\mathcal{O}}(E[N])/H_{\mathcal{O}}) \to (\mathcal{O}/N\mathcal{O})^\times$$

introduced in Lemma 7.2.4 is an isomorphism for every $N \in \mathbb{Z}_{\geq 1}$. Hence the family of division fields $\{H_{\mathcal{O}}(E[p^\infty])\}_p$ is linearly disjoint over $H_{\mathcal{O}}$, and $\mathrm{Im}(\rho_E) = \mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})$.

Suppose now that $K \subseteq H_{\mathcal{O}}(E_{\mathrm{tors}})$ is abelian. Then Theorem 8.3.1 shows that there exists $N \in \mathbb{N}$ such that for every $M \in \mathbb{N}$ with $N \mid M$ we have that $H_{\mathcal{O}}(E[M]) = H_{M,\mathcal{O}}$. Combining this with Theorem 6.2.20 we get that $[\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})\colon \mathrm{Im}(\rho_E)] \geq |\mathcal{O}^\times|$. However, Theorem 6.2.20 and Theorem 7.2.5 imply that $[\mathrm{Aut}_{\mathcal{O}}(E_{\mathrm{tors}})\colon \mathrm{Im}(\rho_E)] \leq |\mathcal{O}^\times|$, which allows us to conclude. □

Remark 8.3.5. The previous Corollary 8.3.4 generalises [Loz19, Theorem 1.3], whose proof will appear in the forthcoming work [Loz]. Indeed, if $E_{/\mathbb{Q}}$ is an elliptic curve with complex multiplication by an order $\mathcal{O}$ in an imaginary quadratic field K then we clearly have that $K(E_{\mathrm{tors}}) \subseteq K^{\mathrm{ab}}$, hence Corollary 8.3.4 shows that the Galois representation $\rho_E\colon \mathrm{Gal}(K(E_{\mathrm{tors}})/K) \hookrightarrow \widehat{\mathcal{O}}^\times$ is not surjective. Let now $\widetilde{\rho}_E\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathcal{N}_{\delta,\phi}$ be the Galois representation associated to the elliptic curve E over $\mathbb{Q}$, where $\mathcal{N}_{\delta,\phi} \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is the subgroup defined by Lozano-Robledo in [Loz19, Theorem 1.1]. Then [Loz19, Theorem 1.1.(2)] and Corollary 8.3.4 show that

$$[\mathcal{N}_{\delta,\phi}\colon \mathrm{Im}(\widetilde{\rho}_E)] = [\widehat{\mathcal{O}}^\times\colon \mathrm{Im}(\rho_E)] = |\mathcal{O}^\times|$$

hence we get that $\widetilde{\rho}_E$ is not surjective. In particular, if $j(E) = 1728$ as in [Loz19, Theorem 1.3] we get that $[\mathcal{N}_{\delta,\phi}\colon \mathrm{Im}(\widetilde{\rho}_E)] = 4$.

We have seen that, for a CM elliptic curve E defined over an abelian extension F of the CM field K, having a minimal division field is essentially equivalent to Shimura's condition (see Definition 7.1.30), *i.e.* to the property that torsion points of E generate abelian extensions of K (and not only of F). It seems then natural to ask whether, for a fixed order $\mathcal{O}$ in an imaginary quadratic field K, there exists any elliptic curve E with complex multiplication by $\mathcal{O}$ and defined over the ring class field $H_{\mathcal{O}}$ (the smallest possible field of definition for E) with the property that

$H_O(E_{\text{tors}}) = K^{\text{ab}}$. This question is discussed by Shimura in [Shi94, Page 217]. Here the author proves that, if $O = O_K$ is a maximal order whose discriminant is a square modulo 3, then there exists an elliptic curve $E_{/H_O}$ such that $H_O(E_{\text{tors}}) = K^{\text{ab}}$. The next theorem generalises this result to arbitrary imaginary quadratic orders.

Theorem 8.3.6 – Infinitely many curves satisfy Shimura's condition

Let O be an order in an imaginary quadratic field K and let $j \in H_O$ be the j-invariant of any elliptic curve with complex multiplication by O. Then there exist infinitely many elliptic curves $E_{/H_O}$ with $j(E) = j$ but non-isomorphic over H_O, such that $H_O(E_{\text{tors}}) = K^{\text{ab}}$.

Proof. When O has class number 1 the statement is trivially true. We may then assume that $\text{Pic}(O) \neq \{1\}$, and in particular that $j \notin \{0, 1728\}$.

Let E_{0/H_O} be any elliptic curve with $j(E) = j$, and let $p \in \mathbb{N}$ be a prime satisfying

1. $p \equiv 3 \bmod 4$;
2. p does not divide $\mathfrak{f}_O \cdot N_{H_O/\mathbb{Q}}(\mathfrak{f}_{E_0})$, where $\mathfrak{f}_O := |O_K : O|$ denotes the conductor of the order O and $\mathfrak{f}_{E_0} \subseteq O_{H_O}$ is the conductor ideal of the elliptic curve E_0;
3. p splits completely in K.

There are infinitely many such primes. Indeed, it clearly suffices to show that there are infinitely many primes satisfying conditions 1 and 3, which are equivalent to

$$\left(\frac{-4}{p}\right) = -1 \quad \text{and} \quad \left(\frac{\Delta_K}{p}\right) = 1 \tag{8.12}$$

respectively. Here $\Delta_K \in \mathbb{Z}$ denotes the absolute discriminant of the imaginary quadratic field K, and $\left(\frac{\cdot}{p}\right)$ denotes Legendre's symbol (see [Neu99, Page 50]). The existence of an infinitude of primes such that (8.12) then follows from Dirichlet's theorem on primes in arithmetic progression (see [Neu99, Chapter VII, Theorem 5.14]), noticing that $\Delta_K \neq -4, -8$ by the assumption $\text{Pic}(O) \neq \{1\}$.

Let $\mathfrak{p} \subseteq O$ be a prime ideal lying over p and note that $\mathfrak{p}$ is invertible by condition 2. We define a new elliptic curve $E_{\mathfrak{p}}$ over H_O as follows: consider the division field $H_O(E_0[\mathfrak{p}])$. By Proposition 8.2.2, there is an isomorphism

$$\text{Gal}(H_O(E_0[\mathfrak{p}])/H_O) \cong (O/\mathfrak{p}O)^\times \cong \mathbb{F}_p^\times$$

where the last isomorphism follows from the fact that p splits in K. In particular, the group $\text{Gal}(H_O(E_0[\mathfrak{p}])/H_O)$ is cyclic of order $p-1$, so $H_O \subseteq H_O(E_0[\mathfrak{p}])$ contains unique sub-extensions of degree $(p-1)/2$ and of degree 2 over H_O. The first one is necessarily the ray class field $H_{\mathfrak{p},O}$ (see Theorem 7.2.5), the second one is of the form $H_O(\sqrt{\alpha})$ for some element $\alpha = \alpha_{\mathfrak{p}} \in H_O^\times$. By condition 1, the integer $p-1$ is not divisible by 4, hence these two extensions must be linearly disjoint over H_O. We deduce that $H_O(E_0[\mathfrak{p}]) = H_{\mathfrak{p},O}(\sqrt{\alpha})$. We set $E_{\mathfrak{p}} := E_0^{(\alpha)}$, where $E_0^{(\alpha)}$ denotes the twist of E_0 by $\alpha \in H_O^\times$.

By Proposition 8.4.1, which is proved in the next section, the Galois representation

$$\rho_{E_{\mathfrak{p}},\mathfrak{p}} : \mathrm{Gal}(H_{\mathcal{O}}(E_{\mathfrak{p}}[\mathfrak{p}])/H_{\mathcal{O}}) \hookrightarrow (\mathcal{O}/\mathfrak{p}\mathcal{O})^{\times}$$

is not surjective. This in particular implies that $H_{\mathcal{O}}(E_{\mathfrak{p}}[\mathfrak{p}]) = H_{\mathfrak{p},\mathcal{O}}$. It follows from Proposition 8.3.3 that $H_{\mathcal{O}}((E_{\mathfrak{p}})_{\mathrm{tors}}) = K^{\mathrm{ab}}$.

To conclude the proof, we want to show that the infinitely many elliptic curves $E_{\mathfrak{p}}$ with $\mathfrak{p} \subseteq \mathcal{O}$ chosen as above, are pairwise non-isomorphic over $H_{\mathcal{O}}$. To do so, it suffices to prove that the fields $H_{\mathcal{O}}(\sqrt{\alpha_{\mathfrak{p}}})$ associated to the quadratic twists are pairwise distinct. But this follows from Proposition 8.2.1 and Proposition 8.2.2, which show that the extension $H_{\mathcal{O}} \subseteq H_{\mathcal{O}}(\sqrt{\alpha_{\mathfrak{p}}})$ is ramified at all primes of $H_{\mathcal{O}}$ lying above $\mathfrak{p}$ and unramified at all primes of $H_{\mathcal{O}}$ which do not divide $\mathfrak{p} \cdot \mathfrak{f}_{E_{\mathfrak{p}}} \cdot \mathcal{O}_{H_{\mathcal{O}}}$, because $H_{\mathcal{O}}(\sqrt{\alpha_{\mathfrak{p}}}) \subseteq H_{\mathcal{O}}(E_0[\mathfrak{p}])$. This finishes the proof. □

We conclude this section by remarking that, under the assumption that $\mathrm{Pic}(\mathcal{O}) \neq \{1\}$, not all CM elliptic curves $E_{/H_{\mathcal{O}}}$ with $j(E) = j$ as in Theorem 8.3.6 satisfy Shimura's condition, *i.e.* have the property that $H_{\mathcal{O}}(E_{\mathrm{tors}}) = K^{\mathrm{ab}}$. We prove this by generalising and providing more detail to a remark of Shimura (see [Shi94, Pages 217-218]).

Theorem 8.3.7 – Infinitely many curves do not satisfy Shimura's condition

Let $\mathcal{O}$ be an order in an imaginary quadratic field K such that $\mathrm{Pic}(\mathcal{O}) \neq \{1\}$, and fix $j \in H_{\mathcal{O}}$ to be the j-invariant of any elliptic curve with complex multiplication by $\mathcal{O}$. Then there exist infinitely many elliptic curves $E_{/H_{\mathcal{O}}}$ with $j(E) = j$ but non-isomorphic over $H_{\mathcal{O}}$, and such that $H_{\mathcal{O}}(E_{\mathrm{tors}}) \neq K^{\mathrm{ab}}$.

Proof. Fix an elliptic curve E_0 defined over $H_{\mathcal{O}}$ such that $j(E_0) = j$ and $H_{\mathcal{O}}((E_0)_{\mathrm{tors}}) = K^{\mathrm{ab}}$. We know that infinitely many such elliptic curves E_0 exist by Theorem 8.3.6. We observe now that, for every $\alpha \in H_{\mathcal{O}}^{\times}$ such that the extension $K \subseteq H_{\mathcal{O}}(\sqrt{\alpha})$ is not abelian, we have that

$$H_{\mathcal{O}}((E_0^{(\alpha)})_{\mathrm{tors}}) \neq K^{\mathrm{ab}}$$

where $E_0^{(\alpha)}$ denotes the quadratic twist of E_0 by $\alpha \in H_{\mathcal{O}}^{\times}$. Indeed, Theorem 8.3.1 shows that $H_{\mathcal{O}}(E_0[N]) = H_{N,\mathcal{O}}$ for some $N \in \mathbb{N}$, and this, combined with Proposition 8.4.1 (which is proved in the next section), implies that $H_{\mathcal{O}}(E_0^{(\alpha)}[N]) = H_{N,\mathcal{O}}(\sqrt{\alpha}) \not\subseteq K^{\mathrm{ab}}$.

In order to conclude the proof it is thus sufficient to show that there exist infinitely many $\alpha \in H_{\mathcal{O}}^{\times}$ such that $\sqrt{\alpha} \notin K^{\mathrm{ab}}$ and the elliptic curves $E_0^{(\alpha)}$ are pairwise not isomorphic over $H_{\mathcal{O}}$. This is equivalent to say that there exist infinitely many distinct quadratic extensions of $H_{\mathcal{O}}$ which are not abelian over K. This can be shown, for instance, as follows.

Since $\mathrm{Pic}(\mathcal{O}) \neq \{1\}$ we have that $K \neq H_{\mathcal{O}}$. Hence Chebotarëv's density theorem (see [Neu99, Chapter VII, Theorem 13.4]) shows that there exists $r \in \mathbb{Z}_{\geq 2}$ and an infinite set of prime ideals $\Lambda_0 = \{\mathfrak{p}_j \subseteq \mathcal{O}_K\}_{j \in \mathbb{N}}$ such that for every index $j \in \mathbb{N}$ we have that $2 \notin \mathfrak{p}_j$ and

$$\mathfrak{p}_j \cdot \mathcal{O}_{H_{\mathcal{O}}} = \mathfrak{P}_{1,j} \cdots \mathfrak{P}_{r,j}$$

where $\mathfrak{P}_{1,j}, \ldots, \mathfrak{P}_{r,j} \subseteq \mathcal{O}_{H_{\mathcal{O}}}$ are distinct prime ideals. Fix now an index $j_0 \in \mathbb{N}$ (*e.g.* $j_0 = 0$), and take any $\alpha_0 \in \mathcal{O}_{H_{\mathcal{O}}}$ such that $\alpha_0 \in \mathfrak{P}_{1,j_0}$ and $\alpha_0 \notin \mathfrak{P}_{1,j_0}^2 \cup \mathfrak{P}_{2,j_0}$. Now, elementary ramification theory of quadratic extensions (see for instance [Gra03, Chapter I, Theorem 6.3]) shows that the extension $H_{\mathcal{O}} \subseteq H_{\mathcal{O}}(\sqrt{\alpha_0})$ ramifies at $\mathfrak{P}_{1,j_0}$ but not at $\mathfrak{P}_{2,j_0}$. This implies that the extension

$K \subseteq H_O(\sqrt{\alpha_0})$ is not Galois, hence in particular not abelian. Now, let Γ_0 be the finite set of prime ideals of O_K dividing $\mathrm{N}_{H_O/K}(\alpha_0)$, and put $\Lambda_1 := \Lambda_0 \setminus \Gamma_0$, which is still an infinite set. Fix an index $j_1 \in \mathbb{N}$ such that $\mathfrak{p}_{j_1} \in \Lambda_1$, and take any element $\alpha_1 \in \mathfrak{P}_{1,j_1} \setminus (\mathfrak{P}_{1,j_1}^2 \cup \mathfrak{P}_{2,j_1})$. Again $K \subseteq H_O(\sqrt{\alpha_1})$ is a non-abelian extension. Moreover, we have that $H_O(\sqrt{\alpha_0}) \neq H_O(\sqrt{\alpha_1})$, since the prime $\mathfrak{P}_{1,j_1}$ ramifies in the extension $H_O \subseteq H_O(\sqrt{\alpha_1})$, but the same prime does not ramify in $H_O \subseteq H_O(\sqrt{\alpha_0})$. Repeating this process, we construct an infinite set of pairwise distinct quadratic extensions $\{H_O \subseteq H_O(\sqrt{\alpha_j}) : j \in \mathbb{N}\}$ that are non-abelian over K. This concludes the proof. □

8.4 Entanglement in the family of division fields of CM elliptic curves over $\mathbb{Q}$

Let $E_{/\mathbb{Q}}$ be an elliptic curve with potential complex multiplication by some order in an imaginary quadratic field K. The aim of this section is to explicitly determine the image of the natural map

$$\mathrm{Gal}(K(E_{\mathrm{tors}})/K) \hookrightarrow \prod_q \mathrm{Gal}(K(E[q^\infty])/K) \tag{8.13}$$

where the product runs over all rational primes $q \in \mathbb{N}$, and $K(E[q^\infty])$ denotes the compositum of the q-power division fields of $E_{/K}$. In other words, we want to analyse the entanglement in the family of Galois extensions $\{K(E[q^\infty])\}_q$ over K. The conclusion of this study is Theorem 8.4.4, which provides a complete description of the image of (8.13) for all CM elliptic curves $E_{/\mathbb{Q}}$ such that $j(E) \notin \{0, 1728\}$.

Observe that there is essentially no difference in considering the division fields of the elliptic curve $E_{/\mathbb{Q}}$ and of its base change $E_{/K}$, because $\mathbb{Q}(E[n]) = K(E[n])$ for every $n > 2$, as explained in Remark 8.2.9. In particular, the family of division fields $\{\mathbb{Q}(E[q^\infty])\}_q$ is always entangled over $\mathbb{Q}$, but there are elliptic curves for which it is linearly disjoint over K, as we show in Theorem 8.4.4.

We briefly outline the strategy of our proof. Since E is defined over $\mathbb{Q}$, we have that

$$|\mathrm{Pic}(O)| = [K(j(E)) : \mathbb{Q}] = 1$$

as follows from Proposition 7.1.33. This implies that $j(E) \in \mathbb{Q}$ because $\mathbb{Q}(j(E)) \cap K = \mathbb{Q}$ (see [Cox13, Proposition 13.2]). Hence the elliptic curve E has complex multiplication by one of the thirteen imaginary quadratic orders O of class number 1, listed in [Cox13, Theorem 7.30]. For each of these orders O, we first find an elliptic curve $E_{0/\mathbb{Q}}$ with complex multiplication by O such that $N_{E_0} \in \mathbb{N}$ is minimal among all the conductors of elliptic curves defined over $\mathbb{Q}$ which have complex multiplication by O. Let us point out that, for every elliptic curve $A_{/\mathbb{Q}}$, the natural number $N_A \in \mathbb{N}$ is defined as the unique positive generator of the conductor ideal $\mathfrak{f}_A \subseteq \mathbb{Z}$. Having fixed E_0, we proceed to compute the full entanglement in the family of division fields of $E_{0/K}$, using Theorem 8.2.6, Theorem 8.3.1, and Proposition 7.1.32. Since O is an order of class number 1 and $j(E) \notin \{0, 1728\}$, we have that E is a quadratic twist of E_0. We then use Proposition 8.4.1, which describes how Galois representations attached to CM elliptic curves behave under quadratic twisting, to determine the complete entanglement in the family of division fields of $E_{/K}$.

In order to state Proposition 8.4.1, we introduce the following notation: given an elliptic curve E defined over a number field F and an element $\alpha \in F^\times$, we denote by $E^{(\alpha)}$ the *twist* of E by α, as described in [Sil09, Chapter X, § 5]. We recall that two twists $E^{(\alpha)}$ and $E^{(\alpha')}$ are isomorphic over

F if and only if α and α' represent the same class in $F^\times/(F^\times)^2$, *i.e.* if and only if $F(\sqrt{\alpha}) = F(\sqrt{\alpha'})$.

Proposition 8.4.1 – Twisting and surjectivity of Galois representations

Let $\mathcal{O}$ be an order of discriminant $\Delta_{\mathcal{O}} < -4$ in an imaginary quadratic field K, and let $H_{\mathcal{O}}$ be the ring class field of K relative to the order $\mathcal{O}$. Consider an elliptic curve $E_{/H_{\mathcal{O}}}$ with complex multiplication by $\mathcal{O}$, and fix $\alpha \in H_{\mathcal{O}}^\times$. Then, for every invertible ideal $I \subseteq \mathcal{O}$, the surjectivity of the Galois representation $\rho_{E,I}$ defined in Lemma 7.2.4 determines the surjectivity of $\rho_{E^{(\alpha)},I}$ as follows:

1. if $\rho_{E,I}$ is surjective, then $\rho_{E^{(\alpha)},I}$ is surjective if and only if
$$H_{\mathcal{O}}(E[I]) \neq H_{I,\mathcal{O}}(\sqrt{\alpha})$$
where $H_{I,\mathcal{O}}$ is the ray class field of K modulo I relative to $\mathcal{O}$ (see Definition 6.2.11);
2. if $\rho_{E,I}$ is not surjective, then $\rho_{E^{(\alpha)},I}$ is surjective if and only if
$$H_{\mathcal{O}}(E[I]) \cap H_{\mathcal{O}}(\sqrt{\alpha}) = H_{\mathcal{O}}.$$

Proof. First of all, observe that $\rho_{E,I}$ (respectively $\rho_{E^{(\alpha)},I}$) has maximal image if and only if there exists $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/H_{\mathcal{O}})$ such that $\rho_{E,I}(\sigma) = -1 \in (\mathcal{O}/I)^\times$ (respectively $\rho_{E^{(\alpha)},I}(\sigma) = -1$). Indeed, $H_{\mathcal{O}}(E[I])$ contains the ray class field $H_{I,\mathcal{O}}$, which is generated over $H_{\mathcal{O}}$ by the values of the Weber function $\mathfrak{h}_E\colon E \twoheadrightarrow E/\mathrm{Aut}(E) \cong \mathbb{P}^1$ at I-torsion points (see Theorem 7.2.5). Since $\mathfrak{h}_E([\varepsilon](P)) = \mathfrak{h}_E(P)$ for every $P \in E[I]$ and $\varepsilon \in \{\pm 1\} = \mathcal{O}^\times \cong \mathrm{Aut}(E)$, we see that $\rho_{E,I}$ induces the identification

$$\mathrm{Gal}(H_{\mathcal{O}}(E[I])/H_{I,\mathcal{O}}) \cong \mathrm{Im}(\pi_I^\times) \cap \mathrm{Im}(\rho_{E,I}) = \{\pm 1\} \cap \mathrm{Im}(\rho_{E,I}) \subseteq (\mathcal{O}/I)^\times \tag{8.14}$$

where $\pi_I^\times\colon \mathcal{O}^\times \to (\mathcal{O}/I)^\times$ denotes the map induced by the quotient $\pi_I\colon \mathcal{O} \twoheadrightarrow \mathcal{O}/I$. Hence, $\rho_{E,I}$ is surjective if and only if $-1 \in \mathrm{Im}(\rho_{E,I})$, and the same holds for $\rho_{E^{(\alpha)},I}$. Moreover, $\rho_{E^{(\alpha)},I}$ is linked to $\rho_{E,I}$, after choosing compatible generators of $E[I]$ and $E^{(\alpha)}[I]$ as $\mathcal{O}/I$-modules, by the formula

$$\rho_{E^{(\alpha)},I} = \rho_{E,I} \cdot \chi_\alpha \tag{8.15}$$

where $\chi_\alpha\colon \mathrm{Gal}(\overline{\mathbb{Q}}/H_{\mathcal{O}}) \to \{\pm 1\} \subseteq (\mathcal{O}/I)^\times$ is the quadratic character associated to $H_{\mathcal{O}}(\sqrt{\alpha})$.

To prove 1, suppose that $\rho_{E,I}$ has maximal image. First, assume that $H_{\mathcal{O}}(E[I]) \neq H_{I,\mathcal{O}}(\sqrt{\alpha})$. Then, either $H_{\mathcal{O}}(\sqrt{\alpha}) \cap H_{\mathcal{O}}(E[I]) = H_{\mathcal{O}}$ or we have $H_{\mathcal{O}}(\sqrt{\alpha}) \subseteq H_{I,\mathcal{O}}$. In the first case, we can certainly find $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/H_{\mathcal{O}})$ acting trivially on $H_{\mathcal{O}}(\sqrt{\alpha})$ and such that $\rho_{E,I}(\sigma) = -1$. Hence we can use (8.15) to see that $\rho_{E^{(\alpha)},I}(\sigma) = \rho_{E,I}(\sigma) \cdot \chi_\alpha(\sigma) = -1$. This implies, by the initial discussion, that $\rho_{E^{(\alpha)},I}$ has maximal image. In the second case, any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/H_{\mathcal{O}})$ with $\rho_{E,I}(\sigma) = -1$ acts trivially on $H_{I,\mathcal{O}} \supseteq H_{\mathcal{O}}(\sqrt{\alpha})$ by (8.14). As before, we can use (8.15) to conclude that $\rho_{E^{(\alpha)},I}$ has maximal image.

Assume now that $H_{\mathcal{O}}(E[I]) = H_{I,\mathcal{O}}(\sqrt{\alpha})$. This implies that the extensions $H_{\mathcal{O}} \subseteq H_{\mathcal{O}}(\sqrt{\alpha})$ and $H_{\mathcal{O}} \subseteq H_{I,\mathcal{O}}$ are linearly disjoint over $H_{\mathcal{O}}$, because $\rho_{E,I}$ has maximal image. In particular

$$\mathrm{Gal}(H_{\mathcal{O}}(E[I])/H_{\mathcal{O}}) \cong \mathrm{Gal}(H_{I,\mathcal{O}}/H_{\mathcal{O}}) \times \mathrm{Gal}(H_{\mathcal{O}}(\sqrt{\alpha})/H_{\mathcal{O}}).$$

We deduce that any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/H_{\mathcal{O}})$ with $\rho_{E,I}(\sigma) = -1$, being the identity on $H_{I,\mathcal{O}}$ by (8.14), must act non-trivially on $H_{\mathcal{O}}(\sqrt{\alpha})$. Then (8.15) gives

$$\rho_{E^{(\alpha)},I}(\sigma) = \rho_{E,I}(\sigma) \cdot \chi_{\alpha}(\sigma) = 1$$

and this suffices to see that $\rho_{E^{(\alpha)},I}$ is non-maximal. This concludes the proof of 1.

The proof of 2 can be carried out in a similar fashion. First of all, notice that the non-maximality of $\rho_{E,I}$ and (8.14) imply that $H_{I,\mathcal{O}} = H_{\mathcal{O}}(E[I])$. Now, by (8.15) the only possibility for $\rho_{E^{(\alpha)},I}$ to be surjective in this case is to find an automorphism $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/H_{\mathcal{O}})$ with $\rho_{E,I}(\sigma) = 1$ and $\chi_{\alpha}(\sigma) = -1$, which is clearly impossible if $H_{\mathcal{O}}(\sqrt{\alpha}) \subseteq H_{\mathcal{O}}(E[I]) = H_{I,\mathcal{O}}$. On the other hand, if $H_{\mathcal{O}}(E[I]) \cap H_{\mathcal{O}}(\sqrt{\alpha}) = H_{\mathcal{O}}$ one can certainly find $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/H_{\mathcal{O}})$ such that $\chi_{\alpha}(\sigma) = -1$ and $\rho_{E,I}(\sigma) = 1$, which shows by (8.15) that $\rho_{E^{(\alpha)},I}$ has maximal image. □

Remark 8.4.2. Let E be an elliptic curve with complex multiplication by an imaginary quadratic order $\mathcal{O}$ of discriminant $\Delta_{\mathcal{O}}$, and suppose that E is defined over the ring class field $H_{\mathcal{O}}$. Fix a rational prime $p \in \mathbb{N}$ such that $p \nmid 2\Delta_{\mathcal{O}}$ and $p \equiv \pm 1 \bmod 9$ if $\Delta_{\mathcal{O}} = -3$. Then, the recent results [Loz19, Theorem 4.4.(5)] and [Loz19, Theorem 7.11] of Lozano-Robledo show that, for every $\alpha \in H_{\mathcal{O}}^{\times}$ and every $n \in \mathbb{N}$, the Galois representation ρ_{E,p^n} is surjective if and only if $\rho_{E^{(\alpha)},p^n}$ is surjective. If moreover $\Delta_{\mathcal{O}} < -4$, then one can combine 1 of Proposition 8.4.1 with Remark 8.2.5 to show that $H_{\mathcal{O}}(E[p^n]) \neq H_{p^n,\mathcal{O}}(\sqrt{\alpha})$ for every $\alpha \in H_{\mathcal{O}}$ and every $n \in \mathbb{Z}_{\geq 1}$.

We want now to derive some consequences of Proposition 8.4.1 when $\alpha \in \mathbb{Q}^{\times}$, the class group $\mathrm{Pic}(\mathcal{O})$ is trivial, and the elliptic curve $E_{/K}$ is the base change to the imaginary quadratic field $K = H_{\mathcal{O}}$ of an elliptic curve defined over $\mathbb{Q}$. To do this, we make an essential use of Proposition 7.1.32. This result, originally due to Deuring, provides the formula

$$\mathfrak{f}_E = \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{f}_{\psi_E}) \cdot \mathrm{disc}(K/\mathbb{Q}) \tag{8.16}$$

which relates the conductor of a CM elliptic curve defined over $\mathbb{Q}$ to the conductor of the unique Hecke character $\psi_E \colon \mathbb{A}_K^{\times} \to \mathbb{C}^{\times}$ associated to its base change over K by Theorem 7.1.25 (see also Remark 7.1.27).

Now, let $E_{/K}$ be the base change to an imaginary quadratic field $K = H_{\mathcal{O}}$ of an elliptic curve $E_{/\mathbb{Q}}$ of conductor $\mathfrak{f}_E \subseteq \mathbb{Z}$, and suppose that E has complex multiplication by an order $\mathcal{O}$ of class number one and discriminant $\Delta_{\mathcal{O}} < -4$. Fix also some rational number $\alpha \in \mathbb{Q}^{\times}$. Under these hypotheses, we may assume that $\alpha = \Delta$, where $\Delta = \Delta_F \in \mathbb{Z}$ is the fundamental discriminant associated to some quadratic extension $\mathbb{Q} \subseteq F$. Since $E^{(\alpha\beta)} = (E^{(\alpha)})^{(\beta)}$ for any $\alpha, \beta \in \mathbb{Q}^{\times}$, we reduce the study of the Galois representation $\rho_{E^{(\Delta)},p^n}$, for any prime $p \in \mathbb{Z}_{\geq 1}$ and any $n \in \mathbb{N}$, to the following cases:

T.1 $\Delta = (-1)^{(q-1)/2} q$ for some prime $q \in \mathbb{Z}_{\geq 3}$ with $q \nmid p\,\mathfrak{f}_E$. In this case $K(\sqrt{\Delta}) \cap K(E[p^n]) = K$. Indeed, any prime $\mathfrak{q} \subseteq \mathcal{O}_K$ such that $\mathfrak{q} \mid q\mathcal{O}_K$ does not ramify in $K \subseteq K(E[p^n])$, as follows from Proposition 8.2.1 because $q \nmid p\,\mathfrak{f}_E$. On the other hand, any prime $\mathfrak{q} \mid q\mathcal{O}_K$ ramifies in $K \subseteq K(\sqrt{\Delta})$ since (8.16) shows that $q \nmid \Delta_K$, where $\Delta_K \in \mathbb{Z}_{<0}$ denotes the absolute discriminant of the imaginary quadratic field K. Thus, Proposition 8.4.1 implies that $\rho_{E^{(\Delta)},p^n}$ has maximal image independently from the behaviour of ρ_{E,p^n};

T.2 $p \geq 3$ and $\Delta = (-1)^{(p-1)/2} p$. In this case, class field theory shows that

$$\mathbb{Q}(\sqrt{\Delta}) \subseteq \mathbb{Q}(\mu_p) \subseteq H_{p^n,\mathcal{O}}$$

where for every $m \in \mathbb{N}$ we let $\mu_m \subseteq \overline{\mathbb{Q}}$ denote the group of m-th roots of unity. Hence, Proposition 8.4.1 implies that $\rho_{E^{(\Delta)},p^n}$ has maximal image if and only if ρ_{E,p^n} does;

T.3 $\Delta \in \{-4, -8, 8\}$ and $2 \nmid p\,\mathfrak{f}_E$. In this case $K(\sqrt{\Delta}) \cap K(E[p^n]) = K$, as in T.1, hence Proposition 8.4.1 shows that $\rho_{E^{(\Delta)},p^n}$ has maximal image independently from the behaviour of ρ_{E,p^n};

T.4 $\Delta \in \{-4, -8, 8\}$ and $p = 2$. In this case, $\mathbb{Q}(\sqrt{\Delta}) \subseteq \mathbb{Q}(\mu_{|\Delta|}) \subseteq H_{|\Delta|,\mathcal{O}}$ by class field theory. Hence, Proposition 8.4.1 implies that, for every $n \in \mathbb{N}$ such that $2^n \geq |\Delta|$, the representation $\rho_{E^{(\Delta)},2^n}$ has maximal image if and only if $\rho_{E,2^n}$ does, similarly to what we proved in T.2.

Remark 8.4.3. The previous discussion shows in particular that, under suitable hypotheses on Δ, if the Galois representation ρ_{E,p^n} is surjective then $\rho_{E^{(\Delta)},p^n}$ is surjective. This might not be the case if these assumptions on Δ are not satisfied, as it follows from Theorem 8.4.4.

We are now ready to study the entanglement of division fields of CM elliptic curves E defined over $\mathbb{Q}$ such that $j(E) \notin \{0, 1728\}$.

First of all, assume that E has complex multiplication by an order $\mathcal{O}$ with $\gcd(\Delta_{\mathcal{O}}, 6) = 1$. Here $\Delta_{\mathcal{O}} := \mathfrak{f}_{\mathcal{O}}^2 \Delta_K$ denotes the discriminant of $\mathcal{O}$, where $\Delta_K \in \mathbb{Z}$ denotes the absolute discriminant of K and $\mathfrak{f}_{\mathcal{O}} := [\mathcal{O}_K : \mathcal{O}]$ denotes the conductor of $\mathcal{O}$. Since $\mathrm{Pic}(\mathcal{O}) = \{1\}$ we have that $\mathcal{O} = \mathcal{O}_K$ and $\Delta_{\mathcal{O}} = \Delta_K = -p$, where $p \in \mathbb{N}$ is a prime number such that $p \geq 7$ and $p \equiv 3 \bmod 4$ (see [Cox13, Theorem 7.30]). Moreover, $E = E_0^{(\Delta)}$ for some fundamental discriminant $\Delta \in \mathbb{Z}$, where E_0 is one of the two elliptic curves with $j(E_0) = j(E)$ appearing in Table A.11, which lists the CM elliptic curves defined over $\mathbb{Q}$ whose conductor $N_{E_0} \in \mathbb{N}$ is minimal among its twists.

Let us study the division fields of E_0, as a first step towards the analysis of the division fields of E. Theorem 8.2.6 provides a decomposition

$$\mathrm{Gal}(K((E_0)_{\mathrm{tors}})/K) \cong \prod_q \mathrm{Gal}(K(E_0[q^\infty])/K) \tag{8.17}$$

where the product runs over all the rational primes $q \in \mathbb{N}$. Indeed in this case the set S_{E_0} appearing in Theorem 8.2.6 consists of the single prime p, because an inspection of Table A.11 shows that $N_{E_0} = p^2$. The isomorphism (8.17) implies that the family of division fields $\{K(E_0[q^\infty])\}_q$ is linearly disjoint over K, where $q \in \mathbb{N}$ runs over all the rational primes. Proposition 8.2.2 gives also that $\mathrm{Gal}(K(E_0[q^m])/K) \cong (\mathcal{O}/q^m\mathcal{O})^\times$ for every prime $q \neq p$ and every $m \in \mathbb{N}$. On the other hand we have that $\mathrm{Gal}(K(E_0[p^m])/K) \cong (\mathcal{O}/p^m\mathcal{O})^\times/\{\pm 1\}$ for every $m \in \mathbb{N}$. Indeed, it follows from (8.16) that $\mathfrak{f}_{\varphi_0} = \mathfrak{p}$, where $\mathfrak{p} \subseteq \mathcal{O}$ is the unique prime lying above p and $\varphi_0 \colon \mathbb{A}_K^\times \to \mathbb{C}^\times$ is the unique Hecke character associated by Theorem 7.1.25 to the base-change of E_0 over K. Hence, Theorem 8.3.1 shows that $K(E_0[p^m]) = H_{p^m,\mathcal{O}}$ for every $m \in \mathbb{N}$, where $H_{p^m,\mathcal{O}}$ is the ray class field of K modulo p^m because $\mathcal{O} = \mathcal{O}_K$. We can therefore conclude that $\mathrm{Gal}(K(E_0[p^m])/K) \cong (\mathcal{O}/p^m\mathcal{O})^\times/\{\pm 1\}$, using Theorem 6.2.20.

Let us now go back to the division fields of $E = E_0^{(\Delta)}$. We can assume that $p \nmid \Delta$, because otherwise $\Delta = -p\,\Delta'$ for some fundamental discriminant $\Delta' \in \mathbb{Z}$, hence $E \cong_K E_0^{(\Delta')}$, since $\sqrt{-p} \in K$. Here the symbol $\cong_K$ means that the two elliptic curves E and $E_0^{(\Delta')}$, which are defined

over $\mathbb{Q}$, become isomorphic when base-changed to K. Observe that $N_E = (p\,\Delta)^2$, which follows from (8.15) and [Ulm16, § 10, Proposition 1], because N_{E_0} is coprime with Δ. We see that

$$\mathrm{Gal}(K(E_{\mathrm{tors}})/K) \cong \left(\prod_{q\notin S} \mathrm{Gal}(K(E[q^\infty])/K)\right) \times \mathrm{Gal}(K(E[S^\infty])/K) \tag{8.18}$$

as a consequence of Theorem 8.2.6. The product appearing in (8.18) runs over the rational primes $q \in \mathbb{N}$ such that $q \notin S$, because in this case the finite set $S = S_E \subseteq \mathbb{N}$ appearing in Theorem 8.2.6 consists uniquely of the primes dividing $N_E = (p\,\Delta)^2$. Moreover, $\mathrm{Gal}(K(E[\ell^m])/K) \cong (\mathcal{O}/\ell^m\mathcal{O})^\times$ for every prime $\ell \in \mathbb{N}$ and every $m \in \mathbb{N}$, since T.1 and T.3 show that, for every $m \in \mathbb{N}$, the Galois representation ρ_{E,ℓ^m} has maximal image. On the other hand, Proposition 8.4.1 shows that $K(E[p^m]) = H_{p^m,\mathcal{O}}(\sqrt{\Delta})$, and that

$$K(E[p^m]) \cap K(E[\Delta]) = K(\sqrt{\Delta})$$

for every $m \in \mathbb{Z}_{\geq 1}$. Hence the family of division fields $\{K(E[q^\infty])\}_{q\in S}$ is entangled over K, and for every collection of integers $\{a_q\}_{q\in S} \subseteq \mathbb{Z}_{\geq 1}$ we get

$$\mathrm{Gal}(L/K) \cong \frac{\prod_{q\in S}(\mathcal{O}/q^{a_q}\mathcal{O})^\times}{\{\pm 1\}}$$

where L is the compositum of all the division fields $K(E[q^{a_q}])$ for $q \in S$.

Let us now consider orders $\mathcal{O}$ such that $\gcd(\Delta_{\mathcal{O}}, 6) \neq 1$. The analysis of the division fields of an elliptic curve $E_{/\mathbb{Q}}$ having complex multiplication by such an order $\mathcal{O}$ proceeds similarly to what happened before, with the only exception of the order $\mathcal{O} = \mathbb{Z}[\sqrt{-3}]$. Indeed, if

$$\mathcal{O} \in \{\mathbb{Z}[3\zeta_3], \mathbb{Z}[2i], \mathbb{Z}[\sqrt{-2}], \mathbb{Z}[\sqrt{-7}]\} \tag{8.19}$$

where $\zeta_3 := (-1+\sqrt{-3})/2$ and $i := \sqrt{-1}$, then all the elliptic curves E_0 appearing in Table A.11 which have complex multiplication by $\mathcal{O}$ share the property that N_{E_0} is a power of the unique rational prime $p \in \mathbb{N}$ which ramifies in the quadratic extension $\mathbb{Q} \subseteq K$. Hence Theorem 8.2.6 shows that

$$\mathrm{Gal}(K((E_0)_{\mathrm{tors}})/K) \cong \prod_q \mathrm{Gal}(K(E_0[q^\infty])/K)$$

where the product runs over all rational primes $q \in \mathbb{N}$, because in this case the finite set $S_{E_0} \subseteq \mathbb{N}$ appearing in Theorem 8.2.6 consists of the single prime p. This implies that the division fields of E_0 are linearly disjoint over K. Moreover, Proposition 8.2.2 gives that

$$\mathrm{Gal}(K(E_0[q^m])/K) \cong (\mathcal{O}/q^m\mathcal{O})^\times$$

for every rational prime $q \neq p$ and every $m \in \mathbb{N}$. On the other hand, (8.16) entails that $\mathfrak{f}_{\varphi_0} = \mathfrak{p}^k$ is a power of the unique prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K$ lying over p, with $k \leq 2$ if $\mathcal{O} \notin \{\mathbb{Z}[2i], \mathbb{Z}[\sqrt{-2}]\}$, and $k \leq 6$ otherwise. Therefore, Theorem 8.3.1 and Theorem 6.2.20 give that

$$\mathrm{Gal}(K(E_0[p^m])/K) \cong (\mathcal{O}/p^m)^\times/\{\pm 1\}$$

for every $m \in \mathbb{N}$ such that $m \geq 1$ if $\mathcal{O} \notin \{\mathbb{Z}[2i], \mathbb{Z}[\sqrt{-2}]\}$, and every $m \geq 3$ otherwise.

Let now $E_{/\mathbb{Q}}$ be any elliptic curve with complex multiplication by an order $\mathcal{O}$ belonging to the list (8.19). Since $j(E) = j(E_0) \notin \{0, 1728\}$ we know that $E = E_0^{(\Delta)}$ for some fundamental discriminant $\Delta \in \mathbb{Z}$. If $\mathcal{O} = \mathbb{Z}[3\zeta_3]$ or $\mathcal{O} = \mathbb{Z}[\sqrt{-7}]$, we can assume that $p \nmid \Delta$ because $\sqrt{-p} \in K$. Hence, Theorem 8.2.6 shows that

$$\mathrm{Gal}(K(E_{\mathrm{tors}})/K) \cong \left(\prod_{q \notin S} \mathrm{Gal}(K(E[q^\infty])/K) \right) \times \mathrm{Gal}(K(E[S^\infty])/K)$$

with the product running over the rational primes $q \in \mathbb{N}$ such that $q \notin S$, where in this case the finite set $S = S_E \subseteq \mathbb{N}$ appearing in Theorem 8.2.6 consists of the primes dividing $N_E = (p\,\Delta)^2$. Exactly as before, T.1 and T.3 show that

$$\mathrm{Gal}(K(E[\ell^m])/K) \cong (\mathcal{O}/\ell^m\mathcal{O})^\times$$

for every prime $\ell \in \mathbb{N}$ and every $m \in \mathbb{N}$. Moreover, Proposition 8.4.1 shows that

$$K(E[p^m]) = H_{p^m,\mathcal{O}}(\sqrt{\Delta}) \quad \text{and} \quad K(E[p^m]) \cap K(E[\Delta]) = K(\sqrt{\Delta})$$

for every $m \in \mathbb{Z}_{\geq 1}$. Hence, the family of division fields $\{K(E[q^\infty])\}_{q \in S}$ is entangled over K, and for every collection of integers $\{a_q\}_{q \in S} \subseteq \mathbb{Z}_{\geq 1}$ we get

$$\mathrm{Gal}(L/K) \cong \frac{\prod_{q \in S} (\mathcal{O}/q^{a_q}\mathcal{O})^\times}{\{\pm 1\}}$$

where L is the compositum of all the division fields $K(E[q^{a_q}])$ for $q \in S$.

Studying the entanglement in the family of division fields of E becomes slightly more complicated if $\mathcal{O} \in \{\mathbb{Z}[2i], \mathbb{Z}[\sqrt{-2}]\}$. First of all, note that there exists a unique $\Delta_2 \in \{1, -4, -8, 8\}$ such that $\Delta = \Delta_2\,\Delta'$, where $\Delta' \in \mathbb{Z}$ is an odd fundamental discriminant. We can now write $E = E_1^{(\Delta')}$, where $E_1 := E_0^{(\Delta_2)}$. One can check that, if $\mathcal{O} = \mathbb{Z}[\sqrt{-2}]$ then E_1 is isomorphic to one of the four elliptic curves with complex multiplication by $\mathbb{Z}[\sqrt{-2}]$ appearing in Table A.11. On the other hand, if $\mathcal{O} = \mathbb{Z}[2i]$ then E_1 can be either one of the two elliptic curves

$$y^2 = x^3 - 44x - 112$$
$$y^2 = x^3 - 44x + 112$$

or one of the two elliptic curves with complex multiplication by $\mathbb{Z}[2i]$ appearing in Table A.11. In each case, it is not difficult to see that $N_{E_1} \in \mathbb{N}$ is a power of 2, which shows that the division fields of E_1 behave similarly to the division fields of E_0. More precisely, Theorem 8.2.6 gives

$$\mathrm{Gal}(K((E_1)_{\mathrm{tors}})/K) \cong \prod_q \mathrm{Gal}(K(E_1[q^\infty])/K)$$

where the product runs over all the rational primes $q \in \mathbb{N}$. This shows that the division fields of E_1 are linearly disjoint over K. Moreover, Proposition 8.2.2 shows that

$$\mathrm{Gal}(K(E_1[q^m])/K) \cong (\mathcal{O}/q^m\mathcal{O})^\times$$

for every rational prime $q \geq 3$ and every $m \in \mathbb{N}$, and a combination of (8.16) and Theorem 8.3.1 gives $\mathrm{Gal}(K(E_1[2^m])/K) \cong (\mathcal{O}/2^m\mathcal{O})^\times/\{\pm 1\}$ for every $m \in \mathbb{N}$ such that $m \geq 3$. This concludes the analysis of the division fields of $E = E_1$ if $\Delta' = 1$. On the other hand, if $\Delta' \neq 1$ then $N_E = N_{E_1}\,(\Delta')^2$, where N_{E_1} is a power of 2. Therefore, Theorem 8.2.6 shows that

$$\mathrm{Gal}(K(E_{\mathrm{tors}})/K) \cong \left(\prod_{q \notin S} \mathrm{Gal}(K(E[q^\infty])/K)\right) \times \mathrm{Gal}(K(E[S^\infty])/K)$$

with the product running over the rational primes $q \in \mathbb{N}$ such that $q \notin S$, where $S = S_E$ denotes the finite set appearing in Theorem 8.2.6, which in this case consists of the primes dividing $2 \cdot \Delta'$. Similarly to what happened before, T.1 and T.4 show that $\mathrm{Gal}(K(E[\ell^m])/K) \cong (\mathcal{O}/\ell^m\mathcal{O})^\times$ for every prime $\ell \in \mathbb{N}$ and every $m \in \mathbb{N}$. Moreover, Proposition 8.4.1 gives $K(E[2^m]) = H_{2^m,\mathcal{O}}(\sqrt{\Delta'})$ and $K(E[2^m]) \cap K(E[\Delta']) = K(\sqrt{\Delta'})$ for every $m \geq 3$. Therefore, the family of division fields $\{K(E[q^\infty])\}_{q \in S}$ is entangled over K, and for all $\{a_q\}_{q \in S} \subseteq \mathbb{Z}_{\geq 1}$ with $a_2 \geq 3$ we get

$$\mathrm{Gal}(L/K) \cong \frac{\prod_{q \in S} (\mathcal{O}/q^{a_q}\mathcal{O})^\times}{\{\pm 1\}}$$

where L is the compositum of all the division fields $K(E[q^{a_q}])$ for $q \in S$.

We are left with the analysis of the entanglement between the division fields of an elliptic curve E defined over $\mathbb{Q}$ which has complex multiplication by $\mathcal{O} = \mathbb{Z}[\sqrt{-3}]$. As usual $E = E_0^{(\Delta)}$ for some fundamental discriminant $\Delta \in \mathbb{Z}$, where E_0 is one of the two elliptic curves with complex multiplication by $\mathbb{Z}[\sqrt{-3}]$ appearing in Table A.11. In contrast to what we have seen before, here $N_{E_0} = 2^2\,3^2$ is not a prime power. This forces us to study separately the division fields $K(E_0[2^\infty])$ and $K(E_0[3^\infty])$. First of all, one can compute that, for any of the two possibilities for E_0, given by the Weierstraß equations $y^2 = x^3 - 15x + 22$ and $y^2 = x^3 - 135x - 594$, the representation $\rho_{E_0,3}$ is not surjective, *i.e.* $K(E_0[3]) = H_{3,\mathcal{O}} = K(\sqrt[3]{2})$. This clearly shows that $\rho_{E_0,3^n}$ is not surjective for every $n \in \mathbb{Z}_{\geq 1}$. Moreover, $\rho_{E_0,2^n}$ is surjective for every $n \in \mathbb{Z}_{\geq 1}$. Indeed, Theorem 6.2.20 and Theorem 7.2.5 imply that

$$\left|\left(\frac{\mathcal{O}}{2^n\mathcal{O}}\right)^\times\right| = \frac{[H_{2^n\,3,\mathcal{O}} : K]}{[H_{3,\mathcal{O}} : K]} = \frac{[H_{2^n\,3,\mathcal{O}} : K]}{[K(E_0[3]) : K]} \leq \frac{[K(E_0[2^n\,3]) : K]}{[K(E_0[3]) : K]} \leq [K(E_0[2^n]) : K] \tag{8.20}$$

hence Lemma 7.2.4 shows that every inequality appearing in (8.20) is actually an equality, and $\rho_{E_0,2^n}$ is surjective. This gives that $K(E_0[2^n]) \cap K(E_0[3^m]) = K$ for every $n, m \in \mathbb{Z}_{\geq 1}$. These considerations, together with Theorem 8.2.6 and Proposition 8.2.2, give a decomposition

$$\mathrm{Gal}(K((E_0)_{\mathrm{tors}})/K) \cong \prod_q \mathrm{Gal}(K(E_0[q^\infty])/K)$$

where the product runs over all rational primes $q \in \mathbb{N}$. Moreover, for every $m \in \mathbb{N}$ we get

$$\mathrm{Gal}(K(E_0[q^m])/K) \cong \begin{cases} (\mathcal{O}/q^m\mathcal{O})^\times, & \text{if } q \neq 3 \\ (\mathcal{O}/3^m\mathcal{O})^\times/\{\pm 1\}, & \text{if } q = 3 \end{cases}$$

and the family of division fields $\{K(E[q^\infty])\}_q$ is linearly disjoint over K.

Let us go back to the division fields of $E = E_0^{(\Delta)}$, where we can assume that $3 \nmid \Delta$ because $\sqrt{-3} \in K$. Write now $\Delta = \Delta_2 \Delta'$ as above, where $\Delta_2 \in \{1, -4, -8, 8\}$ and $\Delta' \in \mathbb{Z}$ an odd fundamental discriminant, and let $E_1 := E_0^{(\Delta_2)}$. Then T.4 shows that $\rho_{E_1,2^n}$ is surjective for every $n \geq 3$. Moreover, $\rho_{E_1,3^n}$ is surjective for every $n \geq 1$, which follows from Proposition 8.4.1 after observing that $K(E_0[3]) \cap K(\sqrt{\Delta_2}) = K$ because $[K(E_0[3]) : K] = 3$. These considerations, together with Theorem 8.2.6, show that

$$\mathrm{Gal}(K((E_1)_{\mathrm{tors}})/K) \cong \left(\prod_{q \notin S} \mathrm{Gal}(K(E_1[q^\infty])/K) \right) \times \mathrm{Gal}(K(E_1[S^\infty])/K)$$

with the product running over the rational primes $q \in \mathbb{N}$ such that $q \notin S$, where $S = \{2, 3\}$ and $K(E_1[S^\infty])$ denotes the compositum of the division fields $K(E_1[2^\infty])$ and $K(E_1[3^\infty])$. Moreover, T.1, T.2, and the previous considerations show that $\mathrm{Gal}(K(E_1[\ell^m])/K) \cong (\mathcal{O}/\ell^m\mathcal{O})^\times$ for every prime $\ell \in \mathbb{N}$ and every $m \in \mathbb{N}$. Now, Proposition 8.4.1 shows that

$$K(E_1[3^m]) \cap K(E_1[\Delta_2]) = K(\sqrt{\Delta_2}) \quad \text{and} \quad K(E_1[3^m]) = H_{3^m,\mathcal{O}}(\sqrt{\Delta_2})$$

for every $m \in \mathbb{Z}_{\geq 1}$. Hence $K(E_1[2^\infty])$ and $K(E_1[3^\infty])$ are entangled over K, and for every pair of integers $a, b \in \mathbb{Z}_{\geq 1}$ we have that

$$\mathrm{Gal}(L/K) \cong \frac{(\mathcal{O}/2^a\mathcal{O})^\times \times (\mathcal{O}/3^b\mathcal{O})^\times}{\{\pm 1\}}$$

where L denotes the compositum of $K(E_1[2^a])$ and $K(E_1[3^b])$.

To conclude our analysis of the division fields of $E = E_0^{(\Delta)}$, we can observe that $E = E_1^{(\Delta')}$ and that $\gcd(\Delta', \mathfrak{f}_{E_1}) = \gcd(\Delta', 6) = 1$. Hence, Theorem 8.2.6 gives the decomposition

$$\mathrm{Gal}(K(E_{\mathrm{tors}})/K) \cong \left(\prod_{q \notin S} \mathrm{Gal}(K(E[q^\infty])/K) \right) \times \mathrm{Gal}(K(E[S^\infty])/K)$$

with the product running over the rational primes $q \in \mathbb{N}$ such that $q \notin S$, where $S \subseteq \mathbb{N}$ denotes the finite set of primes dividing $6\Delta'$. Now, T.1 and T.2 show that

$$\mathrm{Gal}(K(E[\ell^m])/K) \cong (\mathcal{O}/\ell^m)^\times$$

for all rational primes $\ell \in \mathbb{Z}$ and all $m \in \mathbb{N}$. Moreover, $K(E[3^m]) \cap K(E[\Delta]) = K(\sqrt{\Delta})$ and $K(E[3^m]) = H_{3^m,\mathcal{O}}(\sqrt{\Delta})$ for every $m \in \mathbb{Z}_{\geq 1}$, thanks to Proposition 8.4.1. Hence, the family $\{K(E[q^\infty])\}_{q \in S}$ is entangled over K, and for all $\{a_q\}_{q \in S} \subseteq \mathbb{Z}_{\geq 1}$ we get

$$\mathrm{Gal}(L/K) \cong \frac{\prod_{q \in S} (\mathcal{O}/q^{a_q}\mathcal{O})^\times}{\{\pm 1\}}$$

where L is the compositum of all the division fields $K(E[q^{a_q}])$ for $q \in S$.

The following theorem summarises the previous discussion. Recall that, for every rational prime $q \in \mathbb{N}$, we denote by $K(E[q^\infty])$ the compositum of all the division fields $\{K(E[q^n])\}_{n \in \mathbb{N}}$ associated to an elliptic curve E, and for every finite set of primes $S \subseteq \mathbb{N}$ we denote by $K(E[S^\infty])$ the compositum of all the fields $\{K(E[q^\infty])\}_{q \in S}$.

Theorem 8.4.4 – Entanglement of division fields of CM elliptic curves over $\mathbb{Q}$

Let $\mathcal{O}$ be an order inside an imaginary quadratic field K, such that $\mathrm{Pic}(\mathcal{O}) = 1$ and $\Delta_{\mathcal{O}} < -4$. We introduce the following notation:

- $n = n(\mathcal{O}) \in \mathbb{N}$ denotes the number $n := 4$ if $\mathcal{O} \in \{\mathbb{Z}[2i], \mathbb{Z}[\sqrt{-2}]\}$, and $n := 2$ otherwise;
- $p \in \mathbb{N}$ is the unique prime ramifying in $\mathbb{Q} \subseteq K$, which is well defined because $\mathrm{Pic}(\mathcal{O}_K) = \{1\}$;
- $\mathfrak{p} \subseteq \mathcal{O}_K$ is the unique prime lying above the rational prime $p \in \mathbb{N}$.

Label all the elliptic curves defined over $\mathbb{Q}$ which have complex multiplication by $\mathcal{O}$ as $\{A_r\}_{r \in \mathbb{Z}_{\geq 1}}$, in such a way that $N_{A_r} \leq N_{A_{r+1}}$ for every $r \in \mathbb{Z}_{\geq 1}$. Then $N_{A_n} < N_{A_{n+1}}$, and the properties of the division fields associated to the elliptic curve A_r depend on r as follows:

$\boxed{r \leq n}$

- the family $\{K(A_r[q^\infty])\}_q$, where $q \in \mathbb{N}$ runs over all the rational primes, is linearly disjoint over K;
- $\mathrm{Gal}(K(A_r[q^m])/K) \cong (\mathcal{O}/q^m\mathcal{O})^\times$, for every prime $q \neq p$ and every $m \in \mathbb{N}$;
- $\mathrm{Gal}(K(A_r[p^m])/K) \cong (\mathcal{O}/p^m\mathcal{O})^\times/\{\pm 1\}$, for every $m \geq n-1$;

$\boxed{r > n}$

- there exist a unique $r_0 \leq n$ and a unique fundamental discriminant $\Delta_r \in \mathbb{Z}$ coprime with p, such that $A_r = A_{r_0}^{(\Delta_r)}$;
- there is a decomposition

$$\mathrm{Gal}(K((A_r)_{\mathrm{tors}})/K) \cong \left(\prod_{q \notin S_r} \mathrm{Gal}(K(A_r[q^\infty])/K)\right) \times \mathrm{Gal}(K(A_r[S^\infty])/K)$$

  where $S_r \subseteq \mathbb{N}$ denotes the finite set of primes dividing $p \cdot \Delta_r$, and the product runs over the rational primes $q \in \mathbb{N}$ such that $q \notin S_r$. Hence the family

$$\{\, K(A_r[S_r^\infty]) \,\} \cup \{\, K(A_r[q^\infty]) \,\}_{q \notin S_r}$$

  is linearly disjoint over K;
- for every $m \in \mathbb{N}$ such that $m \geq n-1$ we have that

$$K(A_r[p^m]) = H_{p^m,\mathcal{O}}(\sqrt{\Delta_r}) \quad \text{and} \quad K(A_r[p^m]) \cap K(A_r[\Delta_r]) = K(\sqrt{\Delta_r})$$

  which shows that the family $\{K(A_r[q^\infty])\}_{q \in S_r}$ is entangled over K;
- $\mathrm{Gal}(K(A_r[q^m])/K) \cong (\mathcal{O}/q^m\mathcal{O})^\times$, for every prime $q \in \mathbb{N}$ and every $m \in \mathbb{N}$;
- for every collection of integers $\{a_q\}_{q \in S_r} \subseteq \mathbb{Z}_{\geq 1}$ with $a_p \geq n-1$, we have:

$$\mathrm{Gal}(L/K) \cong \frac{\prod_{q \in S_r} (\mathcal{O}/q^{a_q}\mathcal{O})^\times}{\{\pm 1\}}$$

  where L is the compositum of all the division fields $K(A_r[q^{a_q}])$ for $q \in S_r$.

Remark 8.4.5. Fix an imaginary quadratic order $\mathcal{O}$ having trivial class group $\mathrm{Pic}(\mathcal{O}) = \{1\}$, conductor $\mathfrak{f}_{\mathcal{O}} \neq 2$ and discriminant $\Delta_{\mathcal{O}} < -4$. Let $n = n(\mathcal{O}) \in \{2, 4\}$ be as in Theorem 8.4.4. We observe that, for every $r > n$, the set S_r appearing in Theorem 8.4.4 coincides with the set of primes $S = \{p : p \mid \mathfrak{b}_{A_r}\}$ appearing in Theorem 8.2.6. This shows that, even fixing the field of definition, the number of entangled division fields of an elliptic curve with complex multiplication can be arbitrarily large, as we already pointed out in Remark 8.2.8.

Remark 8.4.6. We exclude the two orders $\mathbb{Z}[i]$ and $\mathbb{Z}[\zeta_3]$ in the statement of Theorem 8.4.4, because elliptic curves having complex multiplication by these orders admit quartic (respectively sextic) twists (as explained in [Sil09, Chapter X, Proposition 5.4]). To study these we would need a generalisation of Proposition 8.4.1, which will be subject of future investigations.

9 Mahler measures and elliptic curves with complex multiplication

> Faith and mathematical proof
> are two irreconcilable things.
>
> Fyodor Dostoevsky,
> *A Writer's Diary*

This chapter, based on the preprint [Pen20], studies the special values $L^*(E,0) = L'(E,0)$ associated to CM elliptic curves E defined over $\mathbb{Q}$, and relates them to the Mahler measure of some two-variable polynomial $P \in \mathbb{Z}[x,y]$ associated to E. More specifically, the aim of this chapter is to prove Theorem A, which asserts that every CM elliptic curve $E_{/\mathbb{Q}}$ has a planar model $P \in \mathbb{Z}[x,y]$ such that

$$m(P) = rL'(E,0) + \log|s| \tag{9.1}$$

for two explicit numbers $r \in \mathbb{Q}^\times$ and $s \in \overline{\mathbb{Q}}^\times$. This implies in particular that the zero locus $V_P \hookrightarrow \mathbb{G}_m^2$ is birationally equivalent to the elliptic curve E. The formulas defining the two numbers r and s are made precise in Theorem 9.2.4, which provides also the explicit definition of the polynomial P. This polynomial has the following remarkable characteristics:

- in general, P is not *tempered* (see Definition 4.2.7), which implies in particular that the motivic cohomology class $\{x,y\} \in H^{2,2}_{\mathcal{M}}(V_P)$ does not generally extend to the smooth compactification E. This is the reason for the appearance of the logarithmic term in the identity (9.1). Nevertheless, Mahler measures of non-tempered polynomials have attracted much attention in recent years (see [LSZ16; LM18; MS19; Gia20; Sam20]). Most of them have been related to special values of L-functions via formulas comprising a logarithmic term, similarly to what happens in (9.1);
- in general, P has a very high degree, and thus the curve $V_P \hookrightarrow \mathbb{G}_m^2$ is generally highly singular. This is in contrast with the majority of previously known cases of Boyd's conjectures, where the polynomials appearing have small degree (see for instance Section 4.2 or Appendix A.1).

Theorem A fits into the vast landscape of conjectures and results which relate the Mahler measure of a polynomial with special values of certain L-functions. We have given an historical introduction to these questions in Section 4.2, and we devote the upcoming Appendix A.1 to list many known examples of such kinds of identities. In particular, we have seen in Question 4.2.9 that these relations often predict that the Mahler measure of a polynomial $P \in \mathbb{Z}[x_1,\dots,x_n]$ is a non-zero rational multiple of the special value at $s=0$ of the L-function associated to the motive $\underline{H}^{n-1}(\widetilde{V_P})$, where $\widetilde{V_P}$ is some desingularisation of some compactification of V_P. This type of question, inspired by Boyd's foundational work [Boy98], investigates the relations which

elapse between Mahler measures and special values of L-functions starting from the former, and constructing the latter accordingly. It is also interesting to do the opposite, as we expressed in Question 4.2.10. More precisely, it is an intriguing problem to start with the special value of some motivic L-function $L(M,s)$ and then find a polynomial whose Mahler measure is related to this special value. We view Theorem A as a step towards a positive answer to Question 4.2.10 for the motives $M := \underline{H}^1(E)$ associated to CM elliptic curves defined over $\mathbb{Q}$. Since there are only finitely many $\overline{\mathbb{Q}}$-isomorphism classes of CM elliptic curves defined over $\mathbb{Q}$, Theorem A can also be seen as a step towards a positive answer to Question 4.2.11, which refers to the problem of "twisting" identities between Mahler measures and special values of L-functions. We remark that most of the research conducted on the subject of Mahler measures and special values of L-functions revolves around Question 4.2.9, and not so much around the "inverse problem" posed in Question 4.2.10. Indeed, the only major line of research revolving around Question 4.2.10 is given by Chinburg's conjecture (see Remark 4.2.5) concerning the special values of Dirichlet L-functions, which was one of the main inspirations for the work contained in this chapter.

Let us explain what is the strategy behind the proof of Theorem A. We know, thanks to the work of Deninger and Wingberg (see [DW88]) and Rohrlich (see [Roh87]) which is recalled in Section 9.1, that for every CM elliptic curve E defined over $\mathbb{Q}$ there exist many pairs of functions $f,g \in \mathbb{Q}(E)$ such that the regulator of the *Milnor symbol* $\{f,g\}$ is related to the special value $L'(E,0)$. We prove in Section 9.1 that $\mathbb{Q}(E) = \mathbb{Q}(f,g)$, generalising a result of Brunault (see [Bru16a]). This allows us to construct the polynomial $P \in \mathbb{Z}[x,y]$ as the minimal polynomial of f and g. Finally, we can prove Theorem A by relating the regulator of $\{f,g\}$ to the Mahler measure of P. This is done in Section 9.2, using some generalisations of the seminal work of Deninger (see [Den97a]) that we recalled in Section 4.3.

This chapter makes wide use of the background that was developed in the previous parts of this thesis. First of all, we refer the reader to Chapter 2 for the required background on motives, motivic cohomology and regulators. In particular, Section 2.3.4 is essential to understand the computations presented in this chapter. More specifically, the results present in this chapter make use of *Bloch's trick* to construct a motivic cohomology class $\eta_{f,g} \in H^{2,2}_{\mathcal{M}}(E)$ starting from two functions $f,g\colon E \to \mathbb{P}^1$ whose set of zeros and poles $S_{f,g}$ consists of torsion points. This cohomology class can be expressed as

$$\eta_{f,g} := n_{f,g}\,\{f,g\} - \sum_{x \in S_{f,g}\setminus\{0\}} \{\partial_x(\{f,g\}), \varphi^{(x)}_{f,g}\}$$

as we have seen in Example 2.3.14. Here $n_{f,g} \in \mathbb{N}$ is the least common multiple of the orders of the points of $S_{f,g}$, which is a natural number because $S_{f,g} \subseteq E(\overline{\mathbb{Q}})_{\mathrm{tors}}$. Moreover, ∂_x denotes the map defined in Proposition 2.3.7, and for every $x \in S_{f,g}$ we denote by $\varphi^{(x)}_{f,g}\colon E \to \mathbb{P}^1$ any function defined over $\mathbb{Q}$ such that $\mathrm{div}(\varphi^{(x)}_{f,g}) = n_{f,g}\cdot((x)-(0))$.

This chapter uses also extensively the Deligne-Beilinson regulator maps

$$r^{\infty}_X\colon H^{\bullet,\bullet}_{\mathcal{M}}(X) \to H^{\bullet,\bullet}_{\mathcal{D}}(X)$$

associated to a given variety X (see Example 2.4.6). Here $H^{\bullet,\bullet}_{\mathcal{D}}$ denotes Deligne-Beilinson cohomology (see Example 2.1.22), which was studied more specifically for curves defined over the real numbers in Section 2.5. We recall in particular that $\gamma_E \in H_1(E(\mathbb{C}),\mathbb{Q})^-$ denotes the homology

class defined in Notation 2.5.6, which is the Poincaré dual of a differential form $\omega_E \in \mathcal{F}^1(E)$ such that $\int_{E(\mathbb{R})^0} \omega_E = 1$.

Secondly, we refer the reader to Chapter 3 for background on the construction of the L-function $L(E,s) := L(\underline{H}^1(E),s)$, and to Chapter 7 for background on the theory of complex multiplication. In particular, we recall that $L(E,s)$ coincides with the L-function $L(\psi_E, s-1/2)$ of the Hecke character $\psi_E \colon \mathbb{A}_K^\times \to \mathbb{C}^\times$ associated to the elliptic curve $E_{/K}$ obtained by base-changing E to the imaginary quadratic field K by which E has potential complex multiplication (see Theorem 7.1.25 and Remark 7.1.27). This entails that $L(E,s)$ has the analytic continuation predicted by Conjecture 3.3.4, and satisfies the functional equation expressed by Conjecture 3.3.6. In particular, we have that $4\pi^2\, L'(E,0) = N_E\, L(E,2)$, where $N_E \in \mathbb{N}$ denotes the unique generator of the conductor ideal $\mathfrak{f}_E \subseteq \mathbb{Z}$. Moreover, we recalled in Section 7.4 that the weak version of Beilinson's conjectures (see Conjecture 3.3.28) is known for the special value $L^*(E,0)$ associated to a CM elliptic curve E defined over $\mathbb{Q}$. More precisely, in this chapter we use the result of Rohrlich recalled in Theorem 7.4.5, which asserts that

$$\langle r_E^\infty(\eta_{f,g}), \gamma_E \rangle = n_{f,g} \cdot \mathcal{R}(\mathrm{div}(f) \diamond \mathrm{div}(g))\, L'(E,0) \tag{9.2}$$

for any pair of functions $f, g \colon E \to \mathbb{P}^1$ whose set of zeros and poles $S_{f,g}$ consists of torsion points. Here the pairing

$$\langle \cdot, \cdot \rangle \colon H^{2,2}_{\mathcal{D}}(E) \otimes H_1^{\mathrm{sing}}(E(\mathbb{C}), \mathbb{Q}) \to \mathbb{C}$$

is the one induced by (2.42). Moreover, $\mathcal{R} \colon \mathbb{Q}[E(\overline{\mathbb{Q}})_{\mathrm{tors}}] \to \mathbb{Q}$ denotes the function defined in Definition 7.4.4, and the diamond operator

$$\diamond \colon \mathbb{Q}[E(\overline{\mathbb{Q}})]^{\mathcal{G}_\mathbb{Q}} \otimes \mathbb{Q}[E(\overline{\mathbb{Q}})]^{\mathcal{G}_\mathbb{Q}} \to \mathbb{Q}[\mathcal{G}_\mathbb{Q} \backslash E(\overline{\mathbb{Q}})]$$

is defined in Definition 7.4.1.

Finally, we refer the reader to Chapter 4 for an outline of the theory of Mahler measures. In particular, we recall that the Mahler measure of a polynomial $P \in \mathbb{Z}[x,y] \setminus \{0\}$ is the real number

$$m(P) := \int_0^1 \int_0^1 \log|P(e^{2\pi i \theta_1}, e^{2\pi i \theta_2})|\, d\theta_1\, d\theta_2 \in \mathbb{R}_{\geq 0}$$

which was defined in Definition 4.1.1. This number is known to be related, under suitable conditions, to some specific regulator integral, by the foundational work of Deninger [Den97b] which we summarised in Section 4.3 (see in particular Theorem 4.3.4). We recall the relevant parts of this work in Section 9.2, and we use it to complete the proof of Theorem A.

9.1 Constructing the polynomials

The aim of this section is to associate to every elliptic curve E defined over $\mathbb{Q}$ which has potential complex multiplication by the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K, the polynomial $P \in \mathbb{Z}[x,y]$ appearing in Theorem A. To do so, we study the pairs of functions $f, g \colon E \to \mathbb{P}^1$ defined in [DW88, Theorem 4.10] and [Roh87, Page 384], for which we have that $\mathcal{R}(\mathrm{div}(f) \diamond \mathrm{div}(g)) \neq 0$, and we prove that $\mathbb{Q}(E) = \mathbb{Q}(f,g)$. Hence, if we take $P \in \mathbb{Z}[x,y]$ to be the minimal polynomial of f and g, we see immediately that V_P is birational to E, which was one of the conditions outlined in the statement of Theorem A. The Mahler measure $m(P)$ of P is related to $L'(E,0)$ in Section 9.2. Hence, combining the current section with the following one, we obtain a complete proof of Theorem A (see also Theorem 9.2.4).

9.1.1 Models of CM elliptic curves (according to Deninger and Wingberg)

The aim of this section is to construct the first pair of functions $f, g \in \mathbb{Q}(E)$ of the kind described in the introduction of Section 9.1. This construction is due to Deninger and Wingberg (see [DW88, Theorem 4.10]), and is expressed in the following result.

Lemma 9.1.1 – The Deninger-Wingberg pair of functions

Let E be an elliptic curve defined over $\mathbb{Q}$ having potential complex multiplication by the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field $K \subseteq \mathbb{C}$. Let moreover:

- $\nu_E \in \mathcal{O}_K$ be the number defined as

$$\nu_E := \mathrm{N}_{K/\mathbb{Q}(\mathfrak{f}_{\psi_E})} \cdot \frac{\min\{\mathfrak{b}_E \cap \mathbb{R}_{>0}\}}{b_E}$$

  where $b_E \in K$ is a fixed generator of the fractional ideal $\mathfrak{b}_E$ defined in Proposition 7.4.2, and $\mathfrak{f}_{\psi_E} \subseteq \mathcal{O}_K$ denotes the conductor of the Hecke character $\psi_E \colon \mathbb{A}_K^\times \to \mathbb{C}^\times$ associated to the base-change of E to the imaginary quadratic field K;

- $\mu_K := \mathcal{O}_K^\times$ denote the group of roots of unity contained in K, which coincides with the group of units of $\mathcal{O}_K$ since K is imaginary quadratic;

- $\widetilde{\chi_E}$ denote the map

$$\begin{aligned} \widetilde{\chi_E} \colon E[\nu_E](\mathbb{C}) &\to \mu_K \cup \{0\} \\ x &\mapsto \chi_E\left(\theta_E^{-1}(x)\,\frac{\overline{\nu_E}}{b_E}\right) \end{aligned} \tag{9.3}$$

  where $\chi_E \colon \mathcal{O}_K \to \mu_K \cup \{0\}$ is the multiplicative map defined by $\chi_E(x) := \psi_E(x)/x$ for every $x \in \mathcal{O}_K$ which is coprime to $\mathfrak{f}_{\psi_E}$, and by $\chi_E(x) = 0$ otherwise. Moreover, the map $\theta_E \colon \mathbb{C} \twoheadrightarrow E(\mathbb{C})$ appearing in (9.3) is the complex uniformisation constructed in Proposition 7.4.2.

Then there exists a pair of functions $f, g \colon E \to \mathbb{P}^1$ such that

$$\begin{aligned} \mathrm{div}(f) &= \sum_{x \in E[\nu_E](\overline{\mathbb{Q}}) \setminus \{0\}} ((x) - (0)) \\ \mathrm{div}(g) &= c_g \sum_{y \in E[\nu_E](\overline{\mathbb{Q}})/\mu_K} (([\widetilde{\chi_E}(\overline{x})](x)) - (0)) \end{aligned}$$

where $c_g \in \{1, 2\}$ denotes the order of the point $\sum_{y \in E[\nu_E](\overline{\mathbb{Q}})/\mu_K} [\widetilde{\chi_E}(\overline{x})](x) \in E[2](\mathbb{Q})$. Moreover we have that

$$\mathcal{R}(\mathrm{div}(f) \diamond \mathrm{div}(g)) = \frac{c_g\, \mathfrak{f}_E}{|\mathrm{disc}(K/\mathbb{Q})|} = c_g\, \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{f}_{\psi_E}) \in \mathbb{Z} \setminus \{0\} \tag{9.4}$$

where E_K is the base change of E over K.

Proof. The two divisors

$$\sum_{x\in E[\nu_E](\overline{\mathbb{Q}})\setminus\{0\}} ((x)-(0)) \qquad \text{and} \qquad 2\sum_{y\in E[\nu_E](\overline{\mathbb{Q}})/\mu_K} (([\widetilde{\chi_E}(\overline{x})](x))-(0))$$

are elements of $\mathbb{Q}[E(\overline{\mathbb{Q}})_{\mathrm{tors}}]^{0,\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$, as it is clear from the explicit description of the Galois action on torsion points (see [DW88, Section 4] and Section 7.4). Moreover, we have that

$$\sum_{x\in E[\nu_E](\overline{\mathbb{Q}})\setminus\{0\}} x = \begin{cases} 0, \text{ if } 2\nmid \mathrm{N}_{K/\mathbb{Q}}(\nu_E) \\ \sum_{x\in E[2](\overline{\mathbb{Q}})\setminus\{0\}} x = 0, \text{ otherwise} \end{cases}$$

because $E[\nu_E](\overline{\mathbb{Q}})$ is a group and $E[2](\overline{\mathbb{Q}}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. For similar reasons we have that

$$\sum_{y\in E[\nu_E](\overline{\mathbb{Q}})/\mu_K} [\widetilde{\chi_E}(\overline{x})](x) \in E[2](\mathbb{Q})$$

which implies that we can find two functions $f, g\colon E\to\mathbb{P}^1$ as in the statement of the theorem. Now the identity (9.4) follows from the computations carried out in [DW88, Section 4], after having observed that the regulator used by Rohrlich is twice the regulator used by Deninger and Wingberg (see [Roh87, Page 371] and [DW88, Equation 1.8] for a comparison) and that $\mathrm{div}(g)$ is twice the divisor β which appears in [DW88, Theorem 4.10]. □

Remark 9.1.2. It would in principle be possible to prove the identity (9.4) using directly the definition of $\mathcal{R}$ given in Definition 7.4.4. However this seems difficult, given the complexity of the divisors involved in Lemma 9.1.1.

We use now an idea due to Brunault (see [Bru16a, Lemma 3.3]) to prove that the function field $\mathbb{Q}(E)$ is generated, as a transcendental extension of $\mathbb{Q}$, by the functions f and g.

Lemma 9.1.3 – Generators for elliptic function fields

Let E be an elliptic curve defined over a field κ. For every $P\in E(\overline{\kappa})_{\mathrm{tors}}$, let $O_P := \mathrm{Gal}(\overline{\kappa}/\kappa)\cdot P$ be its Galois orbit, and let $f_P\in\kappa(E)$ be any function such that

$$\mathrm{div}(f_P) = c_P\sum_{x\in O_P}((x)-(0))$$

where $c_P\in\mathbb{Z}_{\geq 1}$ is the order of the point $\sum_{x\in O_P} x\in E(\kappa)_{\mathrm{tors}}$. Then we have that:

1. the extension $\kappa(f_P)\subset\kappa(E)$ contains no proper sub-extensions;
2. if $\kappa(f_P)=\kappa(f_Q)$ for some points $P,Q\in E(\overline{\kappa})_{\mathrm{tors}}$, and $\mathrm{char}(\kappa)=0$, then $|O_P|=|O_Q|$.

Proof. Consider a sub-extension $\kappa(f_P)\subseteq F\subseteq\kappa(E)$. Two possibilities can occur:

- $F=\kappa(g)$ for some function $g\in\kappa(E)$, which implies that $f_P = h\circ g$ for some $h\colon\mathbb{P}^1_\kappa\to\mathbb{P}^1_\kappa$. We can assume, up to applying two homographies $\mathbb{P}^1_\kappa\to\mathbb{P}^1_\kappa$, that $g(0)=\infty$ and that $h(0)=0$. These homographies can be taken to be defined over κ, because $0\in\mathbb{P}^1(\kappa)$ and $g(0)\in\mathbb{P}^1(\kappa)$. Then, every zero of g is a zero of f_P, and the converse also applies because

g is not constant (hence it has some zero $x \in O_P$) and defined over κ (hence all the points $y \in O_x = O_P$ are zeros of g). Moreover, $h(\infty) = h(g(0)) = f_P(0) = \infty$, which implies that 0 is the unique pole of g (since 0 is the unique pole of f_P). This implies that

$$\operatorname{div}(g) = d \sum_{x \in \operatorname{Gal}(\overline{\kappa}/\kappa)\, P} ((x) - (0))$$

for some $d \in \mathbb{Z}_{\geq 1}$. But then $c_P \mid d$ (since c_P is the order of $\sum_{x \in O_P} x \in E(\kappa)_{\text{tors}}$) and thus $c_P = d$ (because $f_P = h \circ g$). Hence $g = \alpha\, f_P$ for some $\alpha \in \kappa^\times$, which implies that $F = \kappa(f_P)$.

- there is an isogeny $\varphi\colon E \twoheadrightarrow E'$, which induces an embedding $\varphi^*\colon \kappa(E') \hookrightarrow \kappa(E)$, and we have that $F = \varphi^*(\kappa(E'))$. This implies that $f_P = g \circ \varphi$ for some function $g \in \kappa(E')$, which in turn implies that $f_P(x) = \infty$ for every $x \in \ker(\varphi)$. Hence φ is an isomorphism (because 0 is the unique pole of f_P), and thus $F = \kappa(E)$.

This shows that $\kappa(f_P) \subset \kappa(E)$ contains no proper sub-extensions.

Now, suppose that $\kappa(f_P) = \kappa(f_Q)$ for some points $P, Q \in E(\overline{\kappa})_{\text{tors}}$. Then we have that

$$c_P\, |O_P| = [\kappa(E)\colon \kappa(f_P)] = [\kappa(E)\colon \kappa(f_Q)] = c_Q\, |O_Q|$$

(see [Ful89, Proposition 8.4]). Moreover, we see that

$$f_P = \frac{a f_Q + b}{c f_Q + d} \qquad \text{for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\kappa)$$

and since both f_P and f_Q have 0 as their unique pole we must have that $c = 0$. Hence we get

$$|O_P|\, c_P + 1 \geq \sum_{x \in O_P} (c_P - 1)\, [\kappa(x)\colon \kappa] + \sum_{x \in O_Q} (c_Q - 1)\, [\kappa(x)\colon \kappa] \tag{9.5}$$

applying the Riemann-Hurwitz formula (see [SP, Section 0C1B]) for the covering $f_P\colon E \to \mathbb{P}^1$. This implies that $c_P = c_Q = 1$, and thus that $|O_P| = |O_Q|$. □

Now, in order to show that $\mathbb{Q}(E) = \mathbb{Q}(f, g)$ for any pair of functions $f, g\colon E \to \mathbb{P}^1$ as in Lemma 9.1.1, we need to use an explicit description of the action of the Galois group $\operatorname{Gal}(K(E[\nu_E])/K)$ over the set $E[\nu_E]/\mu_K$ (see [DW88, Page 264]).

Lemma 9.1.4 – Galois action on torsion points modulo units

We can describe the action of $\operatorname{Gal}(K(E[\nu_E])/K)$ on $E[\nu_E]/\mu_K$ as

$$\begin{aligned} \operatorname{Gal}(K(E[\nu_E])/K) \times E[\nu_E]/\mu_K &\to E[\nu_E]/\mu_K \\ (\sigma, x) &\mapsto [\varphi(\sigma|_{H_{\nu_E, O_K}})^{-1}\, \widetilde{\chi_E}(x)]_E(x) \end{aligned}$$

where $H_{\nu_E, O_K} \subseteq K(E[\nu_E])$ denotes the ray class field of K relative to $\nu_E\, O_K$ (see Definition 6.2.11) and

$$\varphi\colon \operatorname{Gal}(H_{\nu_E, O_K}/K) \xrightarrow{\sim} (O_K/\nu_E)^\times/\mu_K$$

denotes the isomorphism given by Theorem 6.2.20.

Proof. This follows immediately from Theorem 8.3.1, which was proved by Coates and Wiles in this setting (see [CW77, Lemma 3]). □

We are finally ready to prove that the pair of functions f, g defined in Lemma 9.1.1 generates the function field $\mathbb{Q}(E)$.

Theorem 9.1.5 – The Deninger-Wingberg polynomial

Let E be an elliptic curve defined over $\mathbb{Q}$ having potential complex multiplication by the ring of integers O_K of an imaginary quadratic field K. Let moreover $f, g \in \mathbb{Q}(E)$ be a pair of functions as in Lemma 9.1.1.

Then we have that $\mathbb{Q}(E) = \mathbb{Q}(f, g)$ and $\deg_x(P) = \mathrm{N}_{K/\mathbb{Q}}(\nu_E) - 1$, where $P \in \mathbb{Z}[x, y]$ denotes any minimal polynomial for f and g.

Proof. We know that $[\mathbb{Q}(E) \colon \mathbb{Q}(f)] = |E[\nu_E](\overline{\mathbb{Q}}) \setminus \{0\}| = \mathrm{N}_{K/\mathbb{Q}}(\nu_E) - 1$ (see [Ful89, Proposition 8.4]), which implies that $\deg_x(P) = \mathrm{N}_{K/\mathbb{Q}}(\nu_E) - 1$. Moreover, $[\mathbb{Q}(E) \colon \mathbb{Q}(g)] < [\mathbb{Q}(E) \colon \mathbb{Q}(f)]$ because $|E[\nu_E](\overline{\mathbb{Q}})/\mu_K| < |E[\nu_E](\overline{\mathbb{Q}})|$.

We also have that $\mathbb{Q}(g) = \mathbb{Q}(f_P)$, where $P = [\widetilde{\chi_E}(\overline{x_0})](x_0)$ for any $x_0 \in E[\nu_E](\overline{\mathbb{Q}})$. Indeed, we know that for every $x \in E[\nu_E](\overline{\mathbb{Q}})$ there exists $a \in (O_K/\nu_E)^\times$ such that $x = [a^{-1}](x)$, because $E[\nu_E](\overline{\mathbb{Q}})$ is a free (O_K/ν_E)-module of dimension one (see Lemma 7.2.4). We can now use Lemma 9.1.4 to see that

$$\begin{aligned}
\mathrm{div}(g) &= c_g \sum_{y \in E[\nu_E](\overline{\mathbb{Q}})/\mu_K} (([\widetilde{\chi_E}(\overline{x})](x)) - (0)) = \\
&= c_g \sum_{a \in (O_K/\nu_E)^\times/\mu_K} \Big(([\widetilde{\chi_E}(\overline{[a^{-1}](x_0)})]([a^{-1}](x_0))) - (0)\Big) = \\
&= c_g \sum_{a \in (O_K/\nu_E)^\times/\mu_K} \big(([a^{-1}\, \chi_E(a)\, \widetilde{\chi_E}(\overline{x})](x)) - (0)\big) = \\
&= c_g \sum_{\sigma \in \mathrm{Gal}(K(E[\nu_E](\overline{\mathbb{Q}}))/K)} ((\sigma([\widetilde{\chi_E}(\overline{x})](x))) - (0)) = c_P \sum_{y \in O_P} ((y) - (0)) = \mathrm{div}(f_P)
\end{aligned}$$

which implies that $g = \alpha\, f_P$ for some $\alpha \in \mathbb{Q}^\times$.

Now, to conclude that $\mathbb{Q}(E) = \mathbb{Q}(f, g)$ we apply Lemma 9.1.3, using the fact that $\mathbb{Q}(f) \neq \mathbb{Q}(f_P)$ since $[\mathbb{Q}(E) \colon \mathbb{Q}(f_P)] = [\mathbb{Q}(E) \colon \mathbb{Q}(g)] < [\mathbb{Q}(E) \colon \mathbb{Q}(f)]$. □

Remark 9.1.6. We know that $\deg_x(P) = \mathrm{N}_{K/\mathbb{Q}}(\nu_E) - 1$. Computing $\deg_y(P)$ is harder, but it can be done if we know $|E[\nu_E](\overline{\mathbb{Q}})/\mu_K|$ (which depends on $\gcd(\mathrm{N}_{K/\mathbb{Q}}(\nu_E), |\mu_K|)$) and $|S|$, where

$$S := \{x \in E[\nu_E](\overline{\mathbb{Q}}) \mid \widetilde{\chi_E}(x) = 0\} = \bigcup_{\substack{\alpha \mid \mathfrak{f}_{\psi_E} \\ \alpha \neq 1}} E\left[\frac{\nu_E}{\alpha}\right](\overline{\mathbb{Q}}) = \bigcup_{\substack{\alpha \mid \mathfrak{f}_{\psi_E} \\ \alpha\, O_K \in \mathrm{Spec}(O_K)}} E\left[\frac{\nu_E}{\alpha}\right](\overline{\mathbb{Q}}). \tag{9.6}$$

In particular, (9.6) shows that $|S|$ can be computed using an inclusion-exclusion principle.

9.1.2 Models of CM elliptic curves (according to Rohrlich)

Let us turn our attention to the pairs of functions f, g constructed by Rohrlich.

Lemma 9.1.7 – The Rohrlich pairs (see [Roh87, Pages 384-386])

Let E be an elliptic curve defined over $\mathbb{Q}$ having potential complex multiplication by the ring of integers O_K of an imaginary quadratic field K. Let $p \in \mathbb{N}$ be a prime such that $p \nmid N_E$ and $p\,O_K$ is also prime, where $N_E \in \mathbb{N}$ denotes the positive generator of the conductor ideal $\mathfrak{f}_E \subseteq \mathbb{Z}$. Let moreover $c \in \mathbb{N}$ be an integer such that

$$\mathfrak{f}_{\psi_E}\,\mathfrak{b}_E^{-1} \mid c\,O_K \mid \mathfrak{f}_{\psi_E}^{m} \qquad \text{for some} \qquad m \in \mathbb{N}$$

where $\mathfrak{b}_E \subseteq K$ is the fractional ideal defined in Proposition 7.4.2, and $\mathfrak{f}_{\psi_E} \subseteq O_K$ denotes the conductor of the Hecke character $\psi_E \colon \mathbb{A}_K^{\times} \to \mathbb{C}^{\times}$ associated to the base-change $E_{/K}$.

Then there exist two functions $f, g \colon E \to \mathbb{P}^1$ such that

$$\operatorname{div}(f) = k_p \sum_{x \in O_p} ((x) - (0))$$
$$\operatorname{div}(g) = k_c \sum_{y \in O_c} ((y) - (0))$$

where for every $m \in \mathbb{Z}_{\geq 1}$ we define $O_m := \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot \theta_E(1/m) \subseteq E[m](\overline{\mathbb{Q}})$, and we denote by $k_m \in \mathbb{Z}_{\geq 1}$ the order of the torsion point $\sum_{x \in O_m} x \in E[m](\mathbb{Q})$. Finally, we have that

$$\mathcal{R}\left(\operatorname{div}(f) \diamond \operatorname{div}(g)\right) = -\frac{k_p\,k_c\,(1+p^3)}{c\,p} \in \frac{1}{n_{f,g}}\,\mathbb{Z} \setminus \{0\}. \tag{9.7}$$

Proof. First of all, observe that such a number $c \in \mathbb{N}$ exists because $\mathfrak{b}_E^{-1} \mid \mathfrak{f}_{\psi_E}$, which follows from Deuring's formula (see Proposition 7.1.32) and the fact that $\operatorname{ord}_p(\mathfrak{f}_E) \neq 1$ for every prime $p \in \mathbb{N}$. Now, observe that $\sum_{x \in O_m} ((x) - (0)) \in \mathbb{Q}[E(\overline{\mathbb{Q}})_{\mathrm{tors}}]^{0,\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$ for every $m \in \mathbb{Z}_{\geq 1}$, which implies the existence of the pair $f, g \in \mathbb{Q}(E)$.

Let us now turn to the proof of (9.7). First of all, it is evident from the definition that $p \nmid c$, which implies that $|\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\,(x,y)| = |\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\,(x-y)|$ for every $x \in E[p](\overline{\mathbb{Q}})$ and $y \in E[c](\overline{\mathbb{Q}})$. Moreover, for every $\sigma_1, \sigma_2 \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ there exists $\tau \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that

$$\sigma_1\left(\theta_E\left(\frac{1}{p}\right)\right) = \tau\left(\theta_E\left(\frac{1}{p}\right)\right) \qquad \text{and} \qquad \sigma_2\left(\theta_E\left(\frac{1}{c}\right)\right) = \tau\left(\theta_E\left(\frac{1}{c}\right)\right)$$

because $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\,\theta_E(\alpha) = \operatorname{Gal}(\overline{\mathbb{Q}}/K)\,\theta_E(\alpha)$ for every $\alpha \in \mathbb{R}$ and $K(E[p](\overline{\mathbb{Q}})) \cap K(E[c](\overline{\mathbb{Q}})) = K$, as follows from Theorem B. This implies that

$$\begin{aligned}\mathcal{R}\left(\operatorname{div}(f) \diamond \operatorname{div}(g)\right) = k_p\,k_c\,\big(&\mathcal{R}(\theta_E(1/p) - \theta_E(1/c)) - |O_p|\,\mathcal{R}(\theta_E(1/c)) \\ &- |O_c|\,\mathcal{R}(\theta_E(-1/p)) + |O_p|\,|O_c|\,\mathcal{R}(0)\big)\end{aligned} \tag{9.8}$$

because $\{(0,0),(\theta_E(1/p),0),(0,\theta_E(1/c)),(\theta_E(1/p),\theta_E(1/c))\}$ is a full set of representatives for the diagonal action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $S_f \times S_g$. We have moreover that $\mathcal{R}(\theta_E(-1/p)) = \mathcal{R}(0) = 0$ and that $|O_p| = p^2 - 1$ (see [BC20, Theorem 7.8(c)]). Observe now that

$$\mathcal{R}(\theta_E(1/c)) = \frac{1}{c} \prod_{\mathfrak{p} \mid c\, \mathfrak{b}_E} (1 - \psi_E(\mathfrak{p})) = \frac{1}{c} \tag{9.9}$$

because $\theta_E(1/c) \in E(\mathbb{R})$, no prime ideal $\mathfrak{p} \mid c\, \mathfrak{b}_E$ is coprime to $\mathfrak{f}_{\psi_E}$ and $\mathrm{Ann}_{O_K}(\theta_E(1/c)) = c\, \mathfrak{b}_E$. Finally, we have that

$$\begin{aligned} \mathcal{R}(\theta_E(1/p) - \theta_E(1/c)) = \mathcal{R}\left(-\theta_E\left(\frac{\psi_E((c-p)\,O_K)}{c\,p}\right)\right) = \frac{-1}{c\,p} \prod_{\mathfrak{p} \mid (c\,p)\, \mathfrak{b}_E} (1 - \psi_E(\mathfrak{p})) = \\ = \frac{-(1 - \psi_E(p\,O_K))}{c\,p} = -\frac{1+p}{c\,p} \end{aligned} \tag{9.10}$$

because $\mathrm{Ann}_{O_K}(\theta_E(1/(c\,p))) = c\,p\,\mathfrak{b}_E$, and the only prime which divides $c\,p\,\mathfrak{b}_E$ and is coprime with $\mathfrak{f}_{\psi_E}$ is $p\,O_K$, for which we have that $\psi_E(p\,O_K) = \left(\frac{\mathrm{disc}(K/\mathbb{Q})}{p}\right) p = -p$. Putting together (9.8), (9.9) and (9.10) we obtain (9.7). □

Remark 9.1.8. Observe that $k_m \in \{1,2,3,4,6\}$ for every $m \in \mathbb{Z}_{\geq 1}$, which follows from the complete characterisation of the possible rational torsion subgroups $E(\mathbb{Q})_{\mathrm{tors}}$ associated to an elliptic curve E defined over $\mathbb{Q}$ which has potential complex multiplication (see [Ols74]).

We can now prove the analogue of Theorem 9.1.5 for Rohrlich's functions.

Theorem 9.1.9 – The Rohrlich polynomials

Let E be an elliptic curve defined over $\mathbb{Q}$, having potential complex multiplication by the ring of integers O_K of an imaginary quadratic field K. Let $p, c \in \mathbb{N}$ and $f, g \in \mathbb{Q}(E)$ be as in Lemma 9.1.7 and assume that $\phi(c) > p^2 - 1$, where ϕ denotes Euler's totient function. Then $\mathbb{Q}(f,g) = \mathbb{Q}(E)$, and if $P \in \mathbb{Z}[x,y]$ denotes a minimal polynomial of f and g we have that $\deg_x(P) = |O_p|$ and $\deg_y(P) = |O_c|$.

Proof. We see from Lemma 9.1.3 that either $\mathbb{Q}(E) = \mathbb{Q}(f,g)$ or $\mathbb{Q}(f) = \mathbb{Q}(g)$, and in this case we would have that $|O_p| = |O_c|$, but this is absurd. Indeed, $|O_p| = p^2 - 1$ and $\phi(c) < |O_c|$ (see [BC20, Section 6.5]). Then our hypothesis shows that $|O_c| > |O_p|$. The final part of the theorem follows simply from the fact that $\deg_x(P) = [\mathbb{Q}(E) : \mathbb{Q}(f)]$ and $\deg_y(P) = [\mathbb{Q}(E) : \mathbb{Q}(g)]$. □

9.2 Computing the Mahler measure

The aim of this section is to complete the proof of Theorem A, taking as $P \in \mathbb{Z}[x,y]$ a slightly modified version of the polynomials that we defined in Section 9.1. To do so, we use Lemma 4.3.2 and Lemma 4.3.3, both of which concern the action of the group

$$\Gamma := (\mathbb{Q}^\times)^3 \times \left(\mathbb{Z}^2 \rtimes_\varphi \mathrm{GL}_2(\mathbb{Z})\right) \tag{9.11}$$

on the ring of $\mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$. We recall that the actions of the elements

$$\begin{aligned}\mathbf{v} &= (v_0, v_1, v_2) \in (\mathbb{Q}^\times)^3\\ \mathbf{w} &= (w_1, w_2) \in \mathbb{Z}^2\\ M &= \begin{pmatrix} m_{1,1} & m_{1,2}\\ m_{2,1} & m_{2,2}\end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})\end{aligned}$$

on a Laurent polynomial $P \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$ are given by

$$\begin{aligned}\mathbf{v} * P &:= v_0\, P(v_1\, x, \ldots, v_2\, y)\\ \mathbf{w} * P &:= x^{w_1} y^{w_2}\, P(x, y)\\ M * P &:= P(x^{m_{1,1}} y^{m_{1,2}}, x^{m_{2,1}} y^{m_{2,2}}).\end{aligned}$$

For every $\alpha \in \Gamma$, we write $P_\alpha := \alpha * P$, and we denote by

$$\begin{aligned}\mathbf{v}_\alpha &\in (\mathbb{Z}^\times)^{n+1}\\ \mathbf{w}_\alpha &\in \mathbb{Z}^n\\ M_\alpha &\in \mathrm{GL}_n(\mathbb{Z})\end{aligned}$$

the components of α.

Remark 9.2.1. Let $P \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$ and let $\alpha \in \Gamma$. Then we have an isomorphism $V_P \xrightarrow{\sim} V_{P_\alpha}$ between the zero loci of P and P_α inside $\mathbb{G}_m^2$. This induces an isomorphism $\mathbb{Q}(\widetilde{V_{P_\alpha}}) \xrightarrow{\sim} \mathbb{Q}(\widetilde{V_P})$ between the function fields of the desingularisations of their compactifications, which identifies the functions $x, y \in \mathbb{Q}(\widetilde{V_P})$ with $x_\alpha := x^a\, y^b$ and $y_\alpha := x^c\, y^d$, where $a, b, c, d \in \mathbb{Z}$ are such that $M_\alpha = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$.

Let now $J_P := J_{\widetilde{V_P}}$ denote the Jacobian of $\widetilde{V_P}$, let $G \leq J_P(\overline{\kappa})$ denote any subgroup such that $S_{x,y} \subseteq G$ and let $\psi\colon \mathbb{Q}[G] \to \mathbb{Q}$ be any $\mathbb{Q}$-linear map which is odd, *i.e.* such that the equality $\psi((-x)) = -\psi((x))$ holds for every $x \in G$. Then we have that $S_{x_\alpha, y_\alpha} = S_{x,y}$, and

$$\begin{pmatrix}\mathrm{div}(x_\alpha)\\ \mathrm{div}(y_\alpha)\end{pmatrix} = M_\alpha \begin{pmatrix}\mathrm{div}(x)\\ \mathrm{div}(y)\end{pmatrix} \tag{9.12}$$

$$\psi(\mathrm{div}(x_\alpha) \diamond \mathrm{div}(y_\alpha)) = \det(M_\alpha)\, \psi(\mathrm{div}(x) \diamond \mathrm{div}(y)) \tag{9.13}$$

which follows simply from the fact that $\diamond$ is bilinear and that ψ is odd.

Using the action of the group Γ defined in (9.11), we can transform any Laurent polynomial to make the Deninger path (see Equation (4.24)) avoid the unit torus and the set of singular points. This can be done combining the work of Besser and Deninger (see Lemma 4.3.2) and Bornhorn (see Lemma 4.3.3).

Lemma 9.2.2 – Modifying polynomials in two variables

Let $Q \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}] \setminus \{0\}$ be any non-zero Laurent polynomial. Then there exists $\alpha \in \Gamma$ such that

$$Q_\alpha \in 1 + y\,\mathbb{Z}[x, y]$$
$$V_{Q_\alpha}(\mathbb{C}) \cap \mathbb{T}^2 = \emptyset$$
$$\gamma_{Q_\alpha} \cap V^{\mathrm{sing}}_{Q_\alpha}(\mathbb{C}) = \emptyset$$

where $\mathbb{T}^2 \subseteq (\mathbb{C}^\times)^2$ denotes the real unit torus, and $\gamma_Q, \gamma_{Q_\alpha} \subseteq (\mathbb{C}^\times)^2$ denote the Deninger paths defined in (4.24).

Proof. Combining Lemma 4.3.2 with Lemma 4.3.3 we see that there exists $\alpha' \in \mathbb{Z}^2 \rtimes_\varphi \mathrm{GL}_2(\mathbb{Z})$ such that $V_{Q_{\alpha'}}(\mathbb{C}) \cap \mathbb{T}^2 = \emptyset$ and $\gamma_{Q_{\alpha'}} \cap V^{\mathrm{sing}}_{Q_{\alpha'}}(\mathbb{C}) = \emptyset$.

To conclude it is sufficient to observe that there exist $p, q \in \mathbb{Q}^\times$ such that $V_{\widetilde{Q}}(\mathbb{C}) \cap \mathbb{T}^2 = \emptyset$, where $\widetilde{Q} := Q_{\alpha'}(px, qy)$. To show this, we can use the amoeba map

$$\begin{aligned} \mu\colon (\mathbb{C}^\times)^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (\log|x|, \log|y|) \end{aligned} \tag{9.14}$$

which deserves this name because for every Laurent polynomial $T \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ the set $\mu(V_T(\mathbb{C})) \subseteq \mathbb{R}^2$ is given by a bounded region to which are attached some "tentacles" going towards infinity (see [GKZ94, Page 194] for a picture). In particular, the complement $\mathbb{R}^2 \setminus \mu(V_T(\mathbb{C}))$ has at least one unbounded connected component (see [GKZ94, Corollary 6.1.8]). Now, the fact that $V_{\widetilde{Q}}(\mathbb{C}) \cap \mathbb{T}^2 = \emptyset$ is equivalent to say that $0 \notin \mu(V_{\widetilde{Q}}(\mathbb{C}))$. Moreover, we know that $\mu(V_{\widetilde{Q}}(\mathbb{C})) = \tau_{p,q}(\mu(V_Q(\mathbb{C})))$, where $\tau_{p,q}\colon \mathbb{R}^2 \to \mathbb{R}^2$ denotes the translation by the vector $-(\log|p|, \log|q|)$. Hence, we can use the fact that the set $\mathbb{R}^2 \setminus \mu(V_Q(\mathbb{C}))$ has at least one unbounded connected component to see that there exist $p, q \in \mathbb{Q}^\times$ sufficiently large such that $V_{\widetilde{Q}}(\mathbb{C}) \cap \mathbb{T}^2 = \emptyset$. Thus we can take $\alpha := (p, q) \times \alpha'$, so that $Q_\alpha = \widetilde{Q}$, and this concludes the proof. □

Remark 9.2.3. If we start from a tempered polynomial $P \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$, the resulting polynomial P_α is generally not tempered anymore, because we are scaling its variables and therefore its coefficients. Nevertheless, the functions x_α, y_α are still supported on torsion points, thanks to (9.12), and $r_{\overline{V_P}}(\eta_{x_\alpha, y_\alpha}) \neq 0$, thanks to (9.13). Hence, we are still able to apply Theorem 7.4.5, and we can find a relation between the Mahler measure of P_α and the L-value $L'(E, 0)$, despite the fact that P is not tempered.

We are now ready to prove Theorem A, which we state more precisely as follows.

Theorem 9.2.4 – Mahler measures and CM elliptic curves (see Theorem A)

Let E be an elliptic curve defined over $\mathbb{Q}$, having potential complex multiplication by the ring of integers O_K of an imaginary quadratic field K. Let $f, g \in \mathbb{Q}(E)$ be any pair of functions which generates the function field $\mathbb{Q}(E)$, such that $S_{f,g} \subseteq E(\overline{\mathbb{Q}})_{\mathrm{tors}}$ and $\langle r^\infty_E(\eta_{f,g}), \gamma_E \rangle \neq 0$. Let $Q \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$ be a minimal polynomial for f, g and let $P := Q_\alpha$

for any $\alpha \in \Gamma$ satisfying the conditions of Lemma 9.2.2. Let $c_P \in \mathbb{Q}^\times$ be defined by the identity $[\widetilde{j}_*(j_*(\gamma_P))] = (-c_P)\,\gamma_E$, where j denotes the open embedding $j\colon V_P^{\mathrm{reg}} \hookrightarrow E \setminus S_{f,g}$ and $\widetilde{j}$ denotes the open embedding $\widetilde{j}\colon E \setminus S_{f,g} \hookrightarrow E$. Let moreover:

- $\{[\gamma_z]\}_{z\in S_{f,g}} \subseteq H_1(E(\mathbb{C}) \setminus S_{f,g}; \mathbb{Z})$ be the homology classes associated to small loops around each point $z \in S_{f,g}$;
- $[\beta_1], [\beta_2] \in H_1(E(\mathbb{C}) \setminus S_{f,g}; \mathbb{Z})$ be such that the set $\{[\beta_i]\}_{i=1}^2 \cup \{[\gamma_z]\}_{z\in S_{f,g}}$ generates $H_1(E(\mathbb{C}) \setminus S_{f,g}; \mathbb{Z})$;
- $\{a_z\}_{z\in S_{f,g}\setminus\{0\}} \subseteq \mathbb{Z}$ and $\{b_i\}_{i=1}^2 \subseteq \mathbb{Z}$ be the numbers defined by the decomposition

$$[j_*(\gamma_P)] = \sum_{z\in S_{f,g}\setminus\{0\}} a_z\,[\gamma_z] + \sum_{i=1}^{2} b_i\,[\beta_i]$$

  which exists and is unique thanks to the exact sequence (2.43);
- $\tilde{x}, \tilde{y} \in \mathbb{Q}(E)$ be the functions given by $\tilde{x} := f_\alpha$ and $\tilde{y} := g_\alpha$ (see Remark 9.2.1)

Then V_P is birational to E and

$$m(P) = r\,L'(E,0) + \log|s|$$

where $r \in \mathbb{Q}$ and $s \in \overline{\mathbb{Q}}$ are defined by

$$\begin{aligned} r &:= \frac{c_P\,\mathcal{R}(\mathrm{div}(\tilde{x}) \diamond \mathrm{div}(\tilde{y}))}{n_{\tilde{x},\tilde{y}}} \in \mathbb{Q} \\ s &:= \prod_{z\in S_{\tilde{x},\tilde{y}}\setminus\{0\}} \partial_z(\{\tilde{x},\tilde{y}\})^{a_z} \in \overline{\mathbb{Q}}^\times \end{aligned} \tag{9.15}$$

and $r \neq 0$ for a suitable choice of α.

Proof. Recall first of all that $S_{\tilde{x},\tilde{y}} = S_{f,g}$. Observe moreover that

$$[\widetilde{j}_*(j_*(\gamma_P))], \gamma_E \in H_1(E(\mathbb{C}); \mathbb{Q})^- \cong \mathbb{Q}$$

which implies that $c_P \in \mathbb{Q}$ exists. We have now the following chain of identities:

$$m(P) = -\left\langle r^\infty_{V_P}(\{x,y\}), [\gamma_P]\right\rangle = \tag{9.16}$$

$$= -\left\langle r^\infty_{V_P^{\mathrm{reg}}}(\iota^*(\{x,y\})), [\gamma_P]\right\rangle = \tag{9.17}$$

$$= -\left\langle r^\infty_{E\setminus S_{\tilde{x},\tilde{y}}}(\{\tilde{x},\tilde{y}\}), [j_*(\gamma_P)]\right\rangle = \tag{9.18}$$

$$= -\frac{1}{n_{\tilde{x},\tilde{y}}}\left(\left\langle r^\infty_E(\eta_{\tilde{x},\tilde{y}}), [\widetilde{j}_*(j_*(\gamma_P))]\right\rangle - \sum_{z\in S_{\tilde{x},\tilde{y}}\setminus\{0\}} \left\langle r^\infty_{E\setminus S_{\tilde{x},\tilde{y}}}(\{\partial_z(\{\tilde{x},\tilde{y}\}), \varphi^{(z)}_{\tilde{x},\tilde{y}}\}), [j_*(\gamma_P)]\right\rangle\right) = \tag{9.19}$$

$$= \frac{c_P}{n_{\tilde{x},\tilde{y}}}\left\langle r^\infty_E(\eta_{\tilde{x},\tilde{y}}), \gamma_E\right\rangle + \sum_{z\in S_{\tilde{x},\tilde{y}}\setminus\{0\}} a_z\,\log|\partial_z(\{\tilde{x},\tilde{y}\})| = \tag{9.20}$$

$$= \left(\frac{c_P \, \mathcal{R}(\operatorname{div}(\tilde{x}) \diamond \operatorname{div}(\tilde{y}))}{n_{\tilde{x},\tilde{y}}} \right) L'(E,0) + \log \left| \prod_{z \in S_{\tilde{x},\tilde{y}} \setminus \{0\}} \partial_z(\{\tilde{x},\tilde{y}\})^{a_z} \right| \tag{9.21}$$

where ι denotes the open embedding $\iota \colon V_P^{\mathrm{reg}} \hookrightarrow V_P$.

To explain these identities we observe that (9.16) is an application of Theorem 4.3.4, using the fact that $a_0 = 1$ in this case, and (9.17) is a consequence of the fact that $\gamma_P \subseteq V_P^{\mathrm{reg}}(\mathbb{C})$. Moreover, (9.18) follows from the fact that $\iota^*(\{x,y\}) = j^*(\{\tilde{x},\tilde{y}\})$ and (9.19) follows from the definition of $\eta_{\tilde{x},\tilde{y}}$. Finally, (9.20) follows from Proposition 2.5.5 and (9.21) follows from Theorem 7.4.5.

Now, observe that $c_P = 0$ if $V_P(\mathbb{C}) \cap \{(x,y) \in \mathbb{C}^2 \colon |x| = 1\} \subseteq \{(x,y) \in \mathbb{C}^2 \colon |y| < 1\}$ (see [Boy98, Page 48]). Clearly, the same holds if we take $|y| > 1$ in the set on the right and if we change x with y. In other words, if the amoeba $\mu(V_P(\mathbb{C}))$ does not intersect all the four semi-axes we have that $c_P = 0$. Nevertheless, it is clear that we can translate the amoeba sufficiently enough so that, with a convenient rotation, it intersects all the four semi-axes. When this happens, we have that $c_P \neq 0$. □

Remark 9.2.5. Pairs of functions like the ones described in the statement of Theorem 9.2.4 are given by the constructions of Deninger and Wingberg (see Lemma 9.1.1) and Rohrlich (see Lemma 9.1.7).

9.3 Some open questions

We conclude this chapter with some questions which may serve as a guide for future research.

First of all, it is interesting to ask whether one can remove the logarithmic term appearing in Theorem 9.2.4, henceforth giving a positive answer to Question 4.2.10 for each motive $M = \underline{H}^1(E)$ which arises from a CM elliptic curve $E_{/\mathbb{Q}}$. This would be equivalent to finding a planar model for E which is *weakly tempered*, in the sense of the following definition.

Definition 9.3.1 – Weakly tempered polynomials

Let $P \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial, with zero locus $V_P \hookrightarrow \mathbb{G}_m^2$. Fix also some compactification $V_P \hookrightarrow \overline{V_P}$, and some desingularisation $\widetilde{V_P} \twoheadrightarrow \overline{V_P}$. Let $g := \dim(J_{\widetilde{V_P}}) \in \mathbb{N}$ be the genus of $\widetilde{V_P}$ and let $\{[\gamma_z]\}_{z \in S_{\tilde{x},\tilde{y}}} \subseteq H_1(\widetilde{V_P}(\mathbb{C}) \setminus S_{\tilde{x},\tilde{y}}; \mathbb{Z})$ be the homology classes associated to small loops around each point $z \in S_{\tilde{x},\tilde{y}}$. Let γ_P denote Deninger's path (see Equation (4.24)) and let $j \colon V_P^{\mathrm{reg}} \hookrightarrow \widetilde{V_P} \setminus S_{\tilde{x},\tilde{y}}$ denote the obvious inclusion.

Then P is said to be *weakly tempered* if $\partial\gamma_P = \emptyset$ and if there exist some homology classes

$$\{[\beta_i]\}_{i=1}^{2g} \subseteq H_1(\widetilde{V_P}(\mathbb{C}) \setminus S_{\tilde{x},\tilde{y}}; \mathbb{Z})$$

and a point $z_0 \in S_{\tilde{x},\tilde{y}}$ such that:

- the set $\{[\gamma_z]\}_{z \in S_{\tilde{x},\tilde{y}}} \cup \{[\beta_i]\}_{i=1}^{2g}$ generates $H_1(\widetilde{V_P}(\mathbb{C}) \setminus S_{\tilde{x},\tilde{y}}; \mathbb{Z})$;
- we have that

$$\left| \prod_{z \in S_{\tilde{x},\tilde{y}} \setminus \{0\}} \partial_z(\{\tilde{x},\tilde{y}\})^{a_z} \right| = 1$$

where $\{a_z\}_{z\in S_{f,g}\setminus\{0\}} \subseteq \mathbb{Z}$ (and $\{b_i\}_{i=1}^{2g} \subseteq \mathbb{Z}$) are defined by the decomposition

$$[j_*(\gamma_P)] = \sum_{z\in S_{\tilde{x},\tilde{y}}\setminus\{z_0\}} a_z\,[\gamma_z] + \sum_{i=1}^{2g} b_i\,[\beta_i]$$

which exists and is unique (see Proposition 2.5.5).

Remark 9.3.2. If a polynomial $P \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ is tempered, in the sense of Definition 4.2.7 and Remark 4.2.8, then P is weakly tempered in the sense of Definition 9.3.1.

Since Rohrlich's construction produces infinitely many pairs of functions $f, g\colon E \to \mathbb{P}^1$ (see Lemma 9.1.7), it could be possible that at least for one of these pairs the resulting polynomial $P \in \mathbb{Z}[x, y]$ defined in Theorem 9.1.9 is weakly tempered.

Secondly, it is interesting to ask whether there exists an algorithm for computing the polynomial $P \in \mathbb{Z}[x, y]$ associated to a CM elliptic curve E. This can easily be done if $S_{f,g} \subseteq E(\mathbb{Q})_{\text{tors}}$, where $S_{f,g}$ denotes the set of zeros and poles of the functions $f, g\colon E \to \mathbb{P}^1$ used to construct the polynomial P. This is unfortunately not the case in general, and actually the degree of the number field where the set $S_{f,g}$ is defined can be quite large. Nevertheless, it should be possible to device an algorithm which computes P without having to compute all the points in $S_{f,g}$, simply using for example the expressions which are known for the divisors of the functions represented by division polynomials (see [Sil09, Chapter III, Exercise 3.7]).

Thirdly, it is interesting to ask whether the techniques explained in this chapter can help to relate some other special value $L^*(E, -n)$, associated to a CM elliptic curve $E_{/\mathbb{Q}}$, to the Mahler measure of some polynomial $P \in \mathbb{Z}[x_1, \ldots, x_{n+2}]$. In fact, the work of Deninger (see [Den89] and [Den90]) shows that the weak form of Beilinson's conjectures (see Conjecture 3.3.28) is true for the special values $L^*(E, -n)$, for every $n \in \mathbb{N}$, by constructing explicitly a motivic cohomology class $\mathfrak{D}_n \in H^{2,n+2}_{\mathcal{M}}(E)$ whose regulator (paired with the homology class γ_E defined in Notation 2.5.6) is a non-zero rational multiple of $L^*(E, -n)$. The cohomology class $\mathfrak{D}_n$ is even defined starting from an $(n+2)$-tuple of functions $f_1, \ldots, f_{n+2} \in \mathbb{Q}(E)$, using Beilinson's Eisenstein symbol (see [Bei86a]). In particular, we have that $\mathfrak{D}_0 = \eta_{f,g}$, where $f, g\colon E \to \mathbb{P}^1$ are the two functions defined in Lemma 9.1.1. Now, it remains the problem to show how to define a polynomial P from this tuple of $(n+2)$-functions. The natural guess is of course to take a polynomial such that $P(f_1, \ldots, f_{n+2}) = 0$. Moreover, such a polynomial would probably need to be n-exact, in the sense of Section 5.2, in order to have a relation between $m(P)$ and $L^*(E, -n)$. How to precisely construct the polynomial, as well as a precise relation between the Mahler measure and the special value, remain open questions which will be the subject of future research.

Some tables

A

> Yea, from the table of my memory
> I'll wipe away all trivial fond records,
> All saws of books, all forms, all pressures past,
> That youth and observation copied there.
>
> William Shakespeare, *Hamlet*

The aim of this appendix is to collect various tables which have been mentioned in the main body of this thesis. First of all, Appendix A.1 exhibits various lists containing many of the known identities which relate the Mahler measure of a polynomial to some special values of L-functions. Then, Appendix A.2 contains a complete list of all the families of two-variable polynomials $P \in \mathbb{Z}[k][x, y]$ which are tempered (see Definition 4.2.7) and reflexive, *i.e.* such that the Newton polygon $\Delta_P \subseteq \mathbb{R}^2$ contains only one interior point whose coordinates are both integers. Finally, Table A.11 contains the minimal Weierstraß equations of those CM elliptic curves defined over $\mathbb{Q}$ which have minimal conductor amongst their twists.

A.1 Known Mahler measure identities

As we just stated in the introduction of this appendix, the aim of this section is to gather an almost complete list (to the author's knowledge) of known identities between Mahler measures and special values of L-functions. We do this with the hope of giving to the reader a sense of the abundance and variety of works dedicated to this topic. We refer the reader to Chapter 4 for the necessary background on the Mahler measure, and in particular to Section 4.2 for an historical overview of the relations between special values of L-functions and Mahler measures.

First of all, let us start with some identities which fit in the framework of Question 4.2.9. In other words, we take $P_k(t_1, t_2) \in \mathbb{Z}[k][t_1^{\pm 1}, t_2^{\pm 1}]$ to be a family of polynomials, and we present a list of values $k \in \mathbb{C}$ such that the ratio

$$r_k := \frac{L'(\underline{H}^1(\widetilde{V_{P_k}}), 0)}{m(P_k)}$$

is known to be a rational number. Here $\widetilde{V_{P_k}}$ denotes any smooth compactification of the curve $V_{P_k} \hookrightarrow \mathbb{G}_m^2$, defined as the zero locus of P_k inside $\mathbb{G}_m^2$. The most studied polynomial family by far is given by

$$P_k^{(1)}(t_1, t_2) := t_1 + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + k \tag{A.1}$$

which is sometimes called *Boyd-Deninger family*, because it was thoroughly investigated in the work of Deninger [Den97b] and Boyd [Boy98]. It is easy to see that, for every $k \in \mathbb{C} \setminus \{-4, 0, 4\}$ such that $k^2 \in \mathbb{Z}$, the curve $\widetilde{V_{P_k^{(1)}}}$ is isomorphic (over $\mathbb{Q}$) to the elliptic curve

$$E_k^{(1)} \; : \; y^2 = x^3 + \frac{k^2}{8}\left(\frac{k^2}{8} - 1\right)x^2 + \frac{k^4}{256}x. \tag{A.2}$$

Moreover, for $k^2 \in \{-4, 0, 4\}$ we know that

$$m(P_0^{(1)}) = m((t_1^2 + t_2)(t_2 + 1)) = 0 \tag{A.3}$$

$$m(P_{-4}^{(1)}) = m(P_4^{(1)}) = 2m(1 + i \cdot t_1 + i \cdot t_2 + t_1 t_2) = 2L'(\chi_{-4}, -1) \tag{A.4}$$

where $i := \sqrt{-1}$ denotes the imaginary unit. Indeed, (A.3) follows immediately from Theorem 4.1.15, and (A.4) follows from the identity

$$m(1 + t_1 + t_2 - t_1 t_2) = L'(\chi_{-4}, -1)$$

which is due to Ray (see Table A.4). Finally, we present in Table A.1 a complete list of values of k for which the quotient

$$r_k^{(1)} := \frac{L^*(E_k^{(1)}, 0)}{m(P_k^{(1)})}$$

is known to be a rational number. This table is taken from the recent article [Sam20] by Samart. Note that $m(P_k^{(1)}) = m(P_{-k}^{(1)})$, hence Table A.1 records only one of the two values $\pm k$. As a matter of notation, we remark that the column *LMFDB* present in Table A.1 refers to the labels of the elliptic curves $E_k^{(1)}$ in the *L-functions and modular forms database* [LMFDB]. Finally, Table A.1 shows that different Mahler measures in the family $m(P_k^{(1)})$ are rationally related to the same L-value, even if the ratios $r_k^{(1)}$ differ as k varies. More specifically, the elliptic curves $E_k^{(1)}$ corresponding to $k \in \{\sqrt{2}, 3\sqrt{2}, 8, \sqrt{2}i\}$ are all in the isogeny class 24.a, as well as all the elliptic curves corresponding to $k \in \{1, 5, 3i\}$ belong to the isogeny class 15.a and all the elliptic curves corresponding to $k \in \{2\sqrt{2}, 4i\}$ belong to the isogeny class 32.a.

The second family of polynomials whose Mahler measure has been widely studied is given by

$$P_k^{(2)}(t_1, t_2) := t_1 + \frac{1}{t_1} + t_2 + \frac{1}{t_2} + \frac{t_1}{t_2} + \frac{t_2}{t_1} + k. \tag{A.5}$$

This family was introduced by Boyd in [Boy98, Equation (1-31)], and was the main protagonist of Theorem 4.4.3. Boyd also showed that for every $k \in \mathbb{Z} \setminus \{-6, 2, 3\}$, the polynomial $P_k^{(2)}$ gives a planar model for the elliptic curve $E_k^{(2)}$ given by the Weierstraß equation:

$$E_k^{(2)} : y^2 + kxy - 2y = (x - 1)^3 \tag{A.6}$$

which is identified with $V_{P_k^{(2)}}$ via the mutually inverse birational maps (4.42) and (4.43). As we already stated in Section 4.4.1 (see in particular Equation (4.41)), we have that

$$m(P_{-6}^{(2)}) = 5\,L'(\chi_{-3}, -1)$$

$$m(P_2^{(2)}) = m(x+y) + m(y+1) + m(x+1) = 0$$
$$m(P_3^{(2)}) = m(x+y+1) + m(x+y+xy) = 2\,L'(\chi_{-3}, -1)$$

which takes care of the polynomials $P_k^{(2)}$ whose zero locus is singular. For the other ones, we report in Table A.2 the values of k for which the ratio

$$r_k^{(2)} := \frac{L^*(E_k^{(2)}, 0)}{m(P_k^{(2)})}$$

is known to be a rational number.

Now, another family of polynomials which give interesting Mahler measures is given by the Weierstraß forms

$$P_{\mathbf{a}}^{(3)}(t_1, t_2) := t_2^2 + a_1 t_1 t_2 + a_3 t_2 - t_1^3 - a_2 t_1^2 - a_4 t_1 - a_6 \tag{A.7}$$

which depend on a vector of parameters $\mathbf{a} = [a_1, a_2, a_3, a_4, a_6]$. We note that most of these Weierstraß forms are not *tempered*, in the sense of Definition 4.2.7. Nevertheless, some of them are, like the ones for which $a_3 = 1$ and $a_2 = a_4 = a_6 = 0$, which are known as *Deuring normal forms* (see [Sil09, Appendix A, Proposition 1.3]). Some non-tempered Weierstraß forms are nevertheless known to satisfy relations of the kind

$$m(P_{\mathbf{a}}^{(3)}) = \frac{1}{r_{\mathbf{a}}^{(3)}} L'(E_{\mathbf{a}}^{(3)}, 0) + \log|s_{\mathbf{a}}^{(3)}| \tag{A.8}$$

where $E_{\mathbf{a}}^{(3)}$ denotes the elliptic curve defined by $P_{\mathbf{a}}^{(3)}$. Moreover, $r_{\mathbf{a}}^{(3)} \in \mathbb{Q}^\times$ is a non-zero rational number, and $s_{\mathbf{a}}^{(3)} \in \overline{\mathbb{Q}}^\times$ is a non-zero algebraic number. We gather all the identities of this kind which are known to the author in Table A.3. We also mention that Lalín and Mittal managed to prove the infinite family of identities (see [LM18, Corollary 3]):

$$m(t_2^2 + 2t_1t_2 - at_1^3 + a^{-1}t_1) = \begin{cases} 2L'(E,0), & \frac{\sqrt{5}-1}{2} \le a \le \frac{1+\sqrt{5}}{2} \\ \log(a), & a \ge \frac{3+\sqrt{13}}{2} \\ -\log(a), & 0 < a \le \frac{-3+\sqrt{13}}{2} \end{cases}$$

where a is allowed to be any real number, and $E\colon y^2 + 2xy = x^3 + x$ is the elliptic curve identified by the LMFDB label 20.a3. Finally, we refer the reader to [Bru05, § 3.9] for a proof of the two "sporadic identities"

$$m\left((t_1+1)t_2^2 + (t_1^2 + 5t_1 + 2)t_2 + (t_1^2 + 2t_1 + 1)\right) = 7L'(X_1(11), 0)$$
$$m\left(t_2^2 + (t_1^2 + 2t_1 - 1)t_2 + t_1^3\right) = 5\,L'(X_1(11), 0)$$

associated to the elliptic modular curve $X_1(11)\colon y^2 + y = x^3 - x^2$.

To conclude this section, let us mention something about the other kinds of identities which relate the Mahler measure of some polynomial to some special value. First of all, we have identities which fit into the framework of Question 4.2.9, for polynomials $P \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with $n \ge 3$. We refer the interested reader to the works [Ber08; Ber10; Ber+13; PRS14; BN18; ZGQ20] for examples of these kinds of identities. Moreover, we have identities that answer a

question analogous to Question 4.2.9 where $\underline{H}^{n-1}(\widetilde{V_P})$ is replaced by a convenient sub-motive. Typical examples of these identities arise when P is a polynomial defining a curve of genus $g \geq 2$, whose Jacobian has a one-dimensional factor in its Poincaré decomposition (see Theorem 7.1.1). Such identities have been investigated in the works [Boy98; Bor99; Bor15; BZ16; BZ17; LW18; LQ19; LW20]. Finally, there are many known examples of identities between Mahler measures and special values of L-functions which involve polynomials that are conjectured or proved to be *successively exact* (see Chapter 5). All rigorously proved identities of this kind involve Laurent polynomials $P \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ which are conjectured to be $(n-1)$-exact, although we refer the reader to [Lal15] and to Section 5.3 for a conjectural identity involving the three-variable polynomial $P(x_1, x_2, x_3) := x_3 - (1 - x_1)(1 - x_2)$, which is only 1-exact. On the other hand, we refer the interested reader to [Smy81; Ray87; BR02; BRD03; Lal03; Lal06; DL07; Tou08b; Lal16; LQ19] for a plethora of results which relate Mahler measures to special values L-functions arising from zero-dimensional objects. A particular example of these is given by the special values $L^*(\chi_K, -1) = L'(\chi_K, -1)$ arising from the Dirichlet character χ_K associated to an imaginary quadratic field K. This is the subject of Chinburg's conjecture (see Remark 4.2.5), which can be seen as a specialisation of Question 4.2.10 to the motives $M = \underline{H}^0(\chi_K)$ arising from these Dirichlet characters. We have collected in Table A.4 all the imaginary quadratic fields $K = \mathbb{Q}(\sqrt{\Delta_K})$ for which, to our knowledge, Chinburg's conjecture is known to hold. As a last remark, we recall that there exist examples of identities between Mahler measures and special values of L-functions which involve a linear combination of the previous types. We refer the interested reader to [Bor99; Sam13; Bor15; Sam15; ZGQ20] for examples of these kinds of identities.

A.2 Tempered reflexive polynomials

The aim of this section is to describe explicitly the set of tempered polynomials $P \in \mathbb{Z}[x, y]$ whose Newton polygon $\Delta_P \subseteq \mathbb{R}^2$ is *reflexive*. First of all, we recall that $\Delta_P \subseteq \mathbb{R}^2$ is defined as the convex hull of the set $\{\mathbf{j} \in \mathbb{R}^2 \mid a_{\mathbf{j}}(P) \neq 0\}$, where $a_{\mathbf{j}}(P) \in \mathbb{Z}$ denotes the $\mathbf{j}$-th coefficient of P, when P is written in multi-index notation $P(x, y) = \sum_{\mathbf{j}} a_{\mathbf{j}}(P) x^{\mathbf{j}}$. Then Δ_P is a convex polygon, whose vertices lie on the lattice $\mathbb{Z}^2 \subseteq \mathbb{R}^2$. As such, we can talk about the faces $\sigma < \Delta_P$ of Δ_P. Any such face σ consists of a list of points $\sigma = \{\mathbf{j}_1^\sigma, \dots, \mathbf{j}_{r(\sigma)}^\sigma\} \subseteq \Delta_P \cap \mathbb{Z}^2$, which are ordered by reading them counter-clockwise on the polygon Δ_P. To each face $\sigma < \Delta_P$, one can therefore associate a *face polynomial*

$$P_\sigma(t) := \sum_{k=1}^{r(\sigma)} a_{\mathbf{j}_k^\sigma}(P)\, t^k$$

and we know from Remark 4.2.8 that $P \in \mathbb{Z}[x, y]$ is *tempered* (in the sense of Definition 4.2.7) if and only if $m(P_\sigma) = 0$ for every face $\sigma < \Delta_P$, where $m(\cdot)$ denotes the Mahler measure (see Definition 4.1.1).

Now, we recall that a convex lattice polygon $\Delta \subseteq \mathbb{R}^2$ is called *reflexive* if $|\Delta_P^\circ \cap \mathbb{Z}^2| = 1$, *i.e.* if there is only one point with integral coordinates lying inside the polygon Δ_P. It turns out that there are exactly 16 orbits of convex, reflexive lattice polygons, with respect to the natural action of $\mathbb{Z}^2 \rtimes \mathbb{GL}_2(\mathbb{Z}^2)$ (see [Rab89]). This action is obtained as the combination of the translation action of $\mathbb{Z}^2$, and the multiplication action of $\mathrm{GL}_2(\mathbb{Z})$ on the lattice $\mathbb{Z}^2$. We refer the reader to Table A.5 for a list of representatives of the sixteen orbits of convex, reflexive lattice polygons, and to [BZ20, Page 47] for a picture. In particular, all our representatives have the origin $(0, 0)$ as their unique interior lattice point. Let us finally remark that the action of $\mathbb{Z}^2 \rtimes \mathbb{GL}_2(\mathbb{Z}^2)$ on the set of convex, lattice polygons corresponds to the action of the same group on the ring of Laurent polynomials $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$, given by (4.30).

A little thinking shows that, for every fixed convex lattice polygon $\Delta \subseteq \mathbb{R}^2$, there are only finitely many families of tempered Laurent polynomials $P_{\mathbf{k}} \in \mathbb{Z}[\mathbf{k}][x^{\pm 1}, y^{\pm 1}]$ such that $\Delta_{P_{\mathbf{k}}} = \Delta$. Here $\mathbf{k}$ denotes a set of free parameters, which are as many as $|\Delta^\circ \cap \mathbb{Z}^2|$. Then, to find all the families $P_{\mathbf{k}}$, one has simply to find all the possible face polynomials, which amounts to find all the polynomials $f \in \mathbb{Z}[t]$ such that $m(f) = 0$, up to a given degree. This can be done in particular for the sixteen reflexive, convex lattice polygons collected in Table A.5. In this case we see that, if $\Delta_{P_k} \notin \{\Delta_{(13)}, \Delta_{(16)}\}$, we can write

$$P_k(x, y) = \frac{1}{xy} \cdot \big(A(x)y^2 + B(x)y + C(x)\big) - k$$

for some $A, B, C \in \mathbb{Z}[x]$. On the other hand, if $\Delta_{P_k} = \Delta_{(13)}$ then

$$P_k(x, y) = \frac{1}{x^2 y} \cdot \big(A(x)y^2 + B(x)y + C(x)\big) - k$$

and if $\Delta_{P_k} = \Delta_{(16)}$ then

$$P_k(x, y) = \frac{1}{xy} \cdot \big(y^3 + A(x)\, y^2 + B(x)\, y + C(x)\big) - k$$

for some $A, B, C \in \mathbb{Z}[x]$. As we said, imposing the temperedness of $P_k(x, y)$ gives automatically a finite list of possible polynomials A, B, C associated to every polygon $\Delta \in \{\Delta_{(1)}, \ldots, \Delta_{(16)}\}$. We write down explicitly this list in Table A.6 and Table A.7. In total, we have 668 families of polynomials $P_k(x, y)$, although we want to remark that some of them parametrise exactly the same family of curves. Finally, we know that $P_k(x, y) = 0$ gives rise to an elliptic surface, because the polygons $\Delta_{(1)}, \ldots, \Delta_{(16)}$ are reflexive. This is an instance of a celebrated theorem of Baker, which computes the genus of a curve in terms of the interior lattice point of the Newton polygon of its planar model (see for instance [BP00, Theorem 4.2] or [Dok18, § 2]). We devote Table A.8 and Table A.9 to the collection of Weierstraß models for all of these elliptic surfaces, except from the ones having Newton polygon $\Delta_{(13)}$. Indeed, for all the other ones, we were able to use the algorithm described in [ART05], which is implemented in PARI/GP, in order to find closed expressions for the Weierstraß forms. On the other hand, for any of the 220 families $P_k(x, y)$ having Newton polygon $\Delta_{(13)}$, one can certainly use the more sophisticated algorithm implemented in SageMath (see [SAGE, Construct elliptic curves as Jacobians]), to get a Weierstraß equation for the family P_k. We have skipped this step in order to save space and spare the patience of our readers.

A.3 The tables

This section contains the tables that were mentioned in Appendix A.1 and Appendix A.2.

Paper	$\pm k$	LMFDB	$r_k^{(1)}$	Paper	$\pm k$	LMFDB	$r_k^{(1)}$
[RZ14]	1	15.a7	1	[RZ12]	8	24.a2	1/4
[Zud14]	$\sqrt{2}$	56.a4	4	[Bru16b]	12	64.a2	1/2
[LR07]	2	24.a5	1	[Lal10; RZ14]	16	48.a1	1/11
[Rod99]	$2\sqrt{2}$	32.a4	1	[Zud14]	i	17.a4	1/2
[Bru16b]	3	21.a6	1/2	[Rod99; LR07]	$\sqrt{2}i$	24.a4	2/3
[Rod99]	$3\sqrt{2}$	24.a4	2/5	[Zud14]	$2i$	40.a3	1
[Lal10; RZ14]	5	15.a3	1/6	[Lal10; RZ14]	$3i$	15.a6	1/5
[Rod99]	$4\sqrt{2}$	64.a2	1	[Rod99]	$4i$	32.a2	1/2

Table A.1.: Values of $k \in \mathbb{C} \setminus \{-4, 0, 4\}$ such that $k^2 \in \mathbb{Z}$ and the quotient $r_k^{(1)} := L'(E_k^{(1)}, 0)/m(P_k^{(1)})$ is known to be rational, where $P_k^{(1)}$ is defined as in (A.1) and $E_k^{(2)}$ is the elliptic curve defined in (A.2)

Paper	k	LMFDB	$r_k^{(2)}$	Paper	k	LMFDB	$r_k^{(2)}$
[Mel19]	-5	14.a6	1/6	[RZ12]	4	20.a3	1/3
[RZ12]	-2	20.a4	1/2	[Rod99]	6	36.a4	1/2
[Rod99]	0	24.a5	1/2	[Mel19]	10	14.a4	1/10
[Mel19]	1	14.a5	1				

Table A.2.: Values of $k \in \mathbb{Z} \setminus \{-6, 2, 3\}$ such that the quotient $r_k^{(2)} := L'(E_k^{(2)}, 0)/m(P_k^{(2)})$ is known to be rational, where $P_k^{(2)}$ is defined as in (A.5) and $E_k^{(2)}$ is the elliptic curve defined in (A.6).

Paper	$\mathbf{a} = [a_1, a_2, a_3, a_4, a_6]$	LMFDB	$r_{\mathbf{a}}^{(3)}$	$s_{\mathbf{a}}^{(3)}$
[Tou08a; Ber15]	$[2, 0, 0, 1, 0]$	20.a3	1/2	1
[Bru16a]	$[-3, 0, 1, 0, 0]$	54.a2	1	1
[Bru16a]	$[-2, 0, 1, 0, 0]$	35.a2	1	1
[Bru16a]	$[-1, 0, 1, 0, 0]$	14.a5	1/2	1
[LR17]	$[3, 0, 0, -1, 0]$	17.a4	2/7	1
[Gia20]	$[4, 0, 2, 0, 0]$	20.a3	3/8	$\sqrt[3]{2}$

Table A.3.: List of vectors $\mathbf{a}$ for which the corresponding Weierstraß form, defined in (A.7), satisfies an identity of the kind (A.8).

Paper	$-\Delta_K$	Paper	$-\Delta_K$
[Smy81]	3	[BRD03]	19, 40, 120
[Ray87]	4, 7, 8, 20	[LQ19]	23, 303, 755
[BR02]	11, 15, 24, 35, 39, 55, 84		

Table A.4.: List of discriminants $-\Delta_K$ of imaginary quadratic fields K for which Chinburg's conjecture is known (see Remark 4.2.5)

Δ	Vertices	Δ	Vertices
$\Delta_{(1)}$	{ (-1,0), (0,1), (1,-1) }	$\Delta_{(9)}$	{ (-1,-1), (-1,0), (0,1), (1,0), (1,-1) }
$\Delta_{(2)}$	{ (-1,-1), (0,1), (1,-1) }	$\Delta_{(10)}$	{ (-1,0), (-1,1), (0,1), (1,0), (1,-1), (0, -1) }
$\Delta_{(3)}$	{ (-1,0), (0,1), (1,-1), (0, -1) }	$\Delta_{(11)}$	{ (-1,-1), (-1,0), (1,0), (2,-1) }
$\Delta_{(4)}$	{ (-1,0), (0,1), (1,0), (0,-1) }	$\Delta_{(12)}$	{ (-1,-1), (-1,0), (1,0), (1,1), (1,-1) }
$\Delta_{(5)}$	{ (-1,-1), (0,1), (1,0), (1,-1) }	$\Delta_{(13)}$	{ (-2,-1), (0,1), (2,-1) }
$\Delta_{(6)}$	{ (-1,0), (0,1), (1,0), (1,-1), (0,-1) }	$\Delta_{(14)}$	{ (-1,-1), (-1,1), (0,1), (2,-1) }
$\Delta_{(7)}$	{ (-1,-1), (0,1), (2,-1) }	$\Delta_{(15)}$	{ (-1,-1), (-1,1), (1,1), (1,-1) }
$\Delta_{(8)}$	{ (-1,-1), (0,1), (1,1), (1,-1) }	$\Delta_{(16)}$	{ (-1,-1), (-1,2), (2,-1) }

Table A.5.: Convex, reflexive lattice polygons $\Delta \subseteq \mathbb{R}^2$. See [BZ20, Page 47] for a picture.

Δ_P	Number	$A(x)$	$B(x)$	$C(x)$
$\Delta_{(1)}$	1	x	-1	x^2
$\Delta_{(2)}$	4	x	0	$x^2 + \alpha\, x + 1$
		x	0	$-x^2 + 1$
$\Delta_{(3)}$	2	x	1	$x^2 + x$
		x	1	$x^2 - x$
$\Delta_{(4)}$	4	x	$x^2 + 1$	x
		$-x$	$x^2 + 1$	x
		x	$-x^2 + 1$	x
		$-x$	$-x^2 + 1$	x
$\Delta_{(5)}$	4	x	x^2	$x^2 + \alpha\, x + 1$
		x	x^2	$-x^2 + 1$
$\Delta_{(6)}$	4	x	$x^2 + 1$	$x^2 + x$
		x	$x^2 - 1$	$x^2 + x$
		x	$x^2 + 1$	$x^2 - x$
		$-x$	$x^2 - 1$	$x^2 + x$
$\Delta_{(7)}$	20	x	$\alpha\, x^2$	$x^3 + \beta\,(x^2 + x) + 1$
		x	0	$-x^3 + \beta\,(x^2 - x) + 1$
$\Delta_{(8)}$	16	$x^2 + x$	$\alpha_1\, x^2$	$x^2 + \alpha_2\, x + 1$
		$x^2 + x$	0	$-x^2 + 1$
		$-x^2 + x$	0	$x^2 + \alpha\, x + 1$
		$x^2 + x$	$\alpha\, x^2$	$x^2 - 1$

Table A.6.: Tempered reflexive polynomials with Newton polygon $\Delta_{(1)} - \Delta_{(8)}$ (see Table A.5). Here $\alpha_j \in \{0, 1, 2\}$ and $\beta_j \in \{-1, 0, 1, 2, 3\}$.

Δ_P	Number	$A(x)$	$B(x)$	$C(x)$
$\Delta_{(9)}$	8	x	x^2+1	$x^2+\alpha\,x+1$
		x	$-x^2+1$	$x^2+\alpha\,x+1$
		x	x^2+1	x^2-1
		x	$-x^2+1$	x^2-1
$\Delta_{(10)}$	8	$x+\varepsilon_1$	$x^2+\varepsilon_2$	$x^2+\varepsilon_3\,x$
$\Delta_{(11)}$	20	x	$\alpha\,x^2+1$	$x^3+\beta\,(x^2+x)+1$
		$-x$	1	$x^3+\beta\,(x^2+x)+1$
$\Delta_{(12)}$	16	x^2+x	$\alpha_1\,x^2+1$	$x^2+\alpha_2\,x+1$
		$-x^2+x$	1	$x^2+\alpha\,x+1$
		x^2+x	1	$-x^2+1$
		x^2+x	$\alpha\,x^2+1$	x^2-1
$\Delta_{(13)}$	220	x^2	$\alpha_1\,x^3+\alpha_2\,x$	$x^4+\varepsilon v_1\,(x^3+x)+v_2\,x^2+1$
		x^2	$\alpha\,x$	$-x^4+(\beta-1)\,(x^3-x)+1$
		x^2	$\alpha\,x^3$	$x^4+(\beta-1)\,(x^3-x)-1$
		$-x^2$	0	$x^4+\varepsilon v_1\,(x^3+x)+v_2\,x^2+1$
$\Delta_{(14)}$	80	$x+1$	$\alpha_1\,x^2+\alpha_2$	$x^3+\beta\,(x^2+x)+1$
		$x+1$	α	$-x^3+\beta\,(x^2-x)+1$
		$x+1$	$\alpha\,x^2$	$x^3+\beta\,(x-x^2)-1$
		$x+1$	0	$-x^3+\beta\,(x-x^2)-1$
$\Delta_{(15)}$	136	$x^2+\alpha_1\,x+1$	$\alpha_2\,x^2+\alpha_3$	$x^2+\alpha_4\,x+1$
		x^2-1	$\alpha_1\,x^2$	$x^2+\alpha_2\,x+1$
		$x^2+\alpha_1\,x+1$	α_2	$-x^2+1$
		$-x^2+1$	α_1	$x^2+\alpha_2\,x+1$
		$-x^2-\alpha_1\,x-1$	0	$x^2+\alpha_2\,x+1$
		$-x^2+1$	$-\alpha_1\,x^2+\alpha_2$	$-x^2+1$
		$x^2+\alpha_1\,x+1$	$\alpha_2\,x^2$	x^2-1
		x^2-1	0	$-x^2+1$
$\Delta_{(16)}$	125	$\beta_1\,x+\beta_2$	$\beta_1\,x^2+\beta_2$	$x^3+\beta_3(x^2+x)+1$

Table A.7.: Tempered reflexive polynomials with Newton polygon $\Delta_{(1)}-\Delta_{(8)}$ (see Table A.5). Here

$$\left(\begin{smallmatrix}v_1\\v_2\end{smallmatrix}\right)\in\{\left(\begin{smallmatrix}0\\-2\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\2\end{smallmatrix}\right),\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right),\left(\begin{smallmatrix}2\\2\end{smallmatrix}\right),\left(\begin{smallmatrix}2\\3\end{smallmatrix}\right),\left(\begin{smallmatrix}3\\4\end{smallmatrix}\right),\left(\begin{smallmatrix}4\\6\end{smallmatrix}\right)\}.$$

and $\alpha_j\in\{0,1,2\}$. Moreover, we have that $\beta_j\in\{-1,0,1,2,3\}$, and $\varepsilon_j\in\{\pm1\}$.

Δ_{P_k}	a_1	a_2	a_3	a_4	a_6
$\Delta_{(1)}$	k	0	1	0	0
$\Delta_{(2)}$	k	$-\alpha$	0	1	0
	k	0	0	-1	0
$\Delta_{(3)}$	k	-1	-1	0	0
	k	1	-1	0	0
$\Delta_{(4)}$	k	-2	0	1	0
	k	0	0	-1	0
	k	0	0	-1	0
	k	2	0	1	0
$\Delta_{(5)}$	k	$-\alpha$	-1	1	0
	k	0	-1	-1	0
$\Delta_{(6)}$	k	-2	-1	1	0
	k	0	1	-1	0
	k	0	-1	-1	0
	k	2	-1	1	0
$\Delta_{(7)}$	k	$-\beta$	$-\alpha$	β	-1
	k	β	0	β	1
$\Delta_{(8)}$	k	$-\alpha_2 - 1$	$-\alpha_1$	$\alpha_2 + 1$	-1
	k	-1	0	-1	1
	k	$-\alpha + 1$	0	$-\alpha + 1$	1
	k	1	α	-1	-1
$\Delta_{(9)}$	k	$-\alpha - 1$	-2	$\alpha + 1$	-1
	k	$-\alpha + 1$	0	$-\alpha + 1$	1
	k	-1	0	-1	1
	k	1	-2	-1	-1
$\Delta_{(10)}$	k	$-(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$	$-(\varepsilon_3\varepsilon_1 + \varepsilon_2)$	$\varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_2\varepsilon_3$	$-\varepsilon_1\varepsilon_2\varepsilon_3$

Table A.8.: Weierstraß equations for the elliptic surfaces defined in Table A.6 and Table A.7.

Δ_{P_k}	a_1	a_2	a_3	a_4	a_6
$\Delta_{(11)}$	k	$-(\alpha+\beta)$	$-(\alpha+\beta)$	$\beta\alpha+\beta+1$	$k-(\beta\alpha+\beta+1)$
	k	β	β	$\beta-1$	$k-\beta+1$
$\Delta_{(12)}$	k	$-(\alpha_1+\alpha_2+1)$	$-(\alpha_1+\alpha_2+1)$	$(\alpha_1+1)(\alpha_2+1)+1$	$k-(\alpha_1+1)(\alpha_2+1)-1$
	k	$1-\alpha$	$\alpha-1$	$-\alpha$	$\alpha-k$
	k	-1	1	-2	$2-k$
	k	$1-\alpha$	$\alpha-1$	$-\alpha$	$\alpha-k$
$\Delta_{(14)}$	k	$-(\alpha_1\alpha_2+2\beta)$	$-(\alpha_1+\alpha_2)(\beta+1)$	$(\alpha_1+\alpha_2)^2+(\beta+1)^2-4$	$P_{\alpha_1,\alpha_2,\beta}(k)$
	k	0	$\alpha(1-\beta)$	$3+2\beta-\alpha^2-\beta^2$	$k^2+(\beta-1)ak$
	k	0	$\alpha(1-\beta)$	$3+2\beta-\alpha^2-\beta^2$	$k^2+(\beta-1)ak$
	k	0	0	$-(\beta^2+3)$	$2(\beta^2+1)-k^2$
$\Delta_{(15)}$	k	$-\alpha_1\alpha_4-\alpha_2a_3-2$	$-(\alpha_1+\alpha_4)(\alpha_2+\alpha_3)$	$Q(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$	$R_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(k)$
	k	0	$\alpha_1\alpha_2$	$4-\alpha_1^2-\alpha_2^2$	$k^2-\alpha_1\alpha_2k$
	k	0	$\alpha_1\alpha_2$	$4-\alpha_1^2-\alpha_2^2$	$k^2-\alpha_1\alpha_2k$
	k	0	$\alpha_1\alpha_2$	$4-\alpha_1^2-\alpha_2^2$	$k^2-\alpha_1\alpha_2k$
	k	$\alpha_1\alpha_2+2$	0	$(\alpha_1+\alpha_2)^2-4$	$-k^2+2(\alpha_1^2+\alpha_2^2-4)$
	k	$\alpha_1\alpha_2+2$	0	$(\alpha_1+\alpha_2)^2-4$	$-k^2+2(\alpha_1^2+\alpha_2^2-4)$
	k	0	$\alpha_1\alpha_2$	$4-\alpha_1^2-\alpha_2^2$	$k^2-\alpha_1\alpha_2k$
	k	-2	0	-4	$8-k^2$
$\Delta_{(16)}$	k	$-(\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3)$	$S(\beta_1,\beta_2,\beta_3)$	$T(\beta_1,\beta_2,\beta_3)$	$V_{\beta_1,\beta_2,\beta_3}(k)$

Table A.9.: Weierstraß equations for the elliptic surfaces defined in Table A.6 and Table A.7. The polynomials P, Q, R, V are defined in Table A.10.

$$
\begin{aligned}
P_{\alpha_1,\alpha_2,\beta}(k) &= -k^2+(\alpha_1+\alpha_2)(\beta+1)k+2\beta(2-\alpha_1\alpha_2-(\alpha_1+\alpha_2)^2)-(\alpha_1\alpha_2+2)(\beta^2)+1\\
Q(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= (\alpha_1+\alpha_4)^2+(\alpha_2+\alpha_3)^2+\alpha_1\alpha_2\alpha_3\alpha_4-4\\
S(\beta_1,\beta_2,\beta_3) &= 9-\beta_1^2-\beta_2^2-\beta_3^2-2\beta_1\beta_2\beta_3\\
T(\beta_1,\beta_2,\beta_3) &= \beta_3^2\beta_2\beta_1+\beta_3^2\beta_2+\beta_3^2\beta_1+\beta_3\beta_2^2\beta_1+\beta_3\beta_2^2\\
&\quad+\beta_3\beta_2\beta_1^2-3\beta_3\beta_2+\beta_3\beta_1^2-3\beta_3\beta_1+\beta_2^2\beta_1+\beta_2\beta_1^2-3\beta_2\beta_1\\
R_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(k) &= -k^2+(\alpha_1+\alpha_4)(\alpha_2+\alpha_3)k+\alpha_3(\alpha_2(\alpha_4(-2\alpha_1-\alpha_4)-\alpha_1^2)\\
&\quad+\alpha_3(-\alpha_1\alpha_4-2))-2\alpha_1^2+\alpha_2^2(-\alpha_1\alpha_4-2)-2\alpha_4^2+8\\
V_{\beta_1,\beta_2,\beta_3}(k) &= k^3+((-\beta_2-\beta_3)\beta_1-\beta_3\beta_2)k^2+T(\beta_1,\beta_2,\beta_3)k-2\beta_3^3\beta_2\beta_1\\
&\quad-2\beta_3^3-2\beta_3^2\beta_2^2\beta_1-\beta_3^2\beta_2^2\beta_1^2-\beta_3^2\beta_2^2-2\beta_3^2\beta_2\beta_1^2\\
&\quad+4\beta_3^2\beta_2\beta_1+9\beta_3^2-\beta_3^2\beta_1^2-2\beta_3\beta_2^3\beta_1+\\
&\quad-2\beta_3\beta_2^2\beta_1^2+4\beta_3\beta_2^2\beta_1+-2\beta_3\beta_2\beta_1^3+4\beta_3\beta_2\beta_1^2\\
&\quad+6\beta_3\beta_2\beta_1+-2\beta_2^3+9\beta_2^2-\beta_2^2\beta_1^2+-2\beta_1^3+9\beta_1^2-27
\end{aligned}
$$

Table A.10.: Coefficient polynomials featured in Table A.9.

Δ_K	$\mathfrak{f}_O$	$j(E)$	N_E	Equations
-3	1	0	3^3	$y_2^2 + y = x_3^3 - 7$ $y^2 + y = x^3$
	2	$2^4\,3^3\,5^3$	$2^2 3^2$	$y_2^2 = x_3^3 - 15x + 22$ $y^2 = x^3 - 135x - 594$
	3	$-2^{15}\,3\,5^3$	3^3	$y_2^2 + y = x_3^3 - 30x + 63$ $y^2 + y = x^3 - 270x - 1708$
-4	1	$2^6\,3^3$	2^5	$y_2^2 = x_3^3 - x$ $y^2 = x^3 + 4x$
	2	$2^3\,3^3\,11^3$	2^5	$y_2^2 = x_3^3 - 11x - 14$ $y^2 = x^3 - 11x + 14$
-7	1	$-3^3\,5^3$	7^2	$y_2^2 + xy = x_3^3 - x_2^2 - 2x - 1$ $y^2 + xy = x^3 - x^2 - 107x + 552$
	2	$3^3\,5^3\,17^3$	7^2	$y_2^2 + xy = x_3^3 - x_2^2 - 37x - 78$ $y^2 + xy = x^3 - x^2 - 1822x + 30393$
-8	1	$2^6\,5^3$	2^8	$y_2^2 = x_3^3 - x_2^2 - 3x - 1$ $y_2^2 = x_3^3 + x_2^2 - 3x + 1$ $y_2^2 = x_3^3 - x_2^2 - 13x + 21$ $y^2 = x^3 + x^2 - 13x - 21$
-11	1	-2^{15}	11^2	$y_2^2 + y = x_3^3 - x_2^2 - 7x + 10$ $y^2 + y = x^3 - x^2 - 887x - 10143$
-19	1	$-2^{15}\,3^3$	19^2	$y_2^2 + y = x_3^3 - 38x + 90$ $y^2 + y = x^3 - 13718x - 619025$
-43	1	$-2^{18}\,3^3\,5^3$	43^2	$y_2^2 + y = x_3^3 - 860x + 9707$ $y^2 + y = x^3 - 1590140x - 771794326$
-67	1	$-2^{15}\,3^3\,5^3\,11^3$	67^2	$y_2^2 + y = x_3^3 - 7370x + 243528$ $y^2 + y = x^3 - 33083930x - 73244287055$
-163	1	$-2^{18}\,3^3\,5^3\,23^3\,29^3$	163^2	$y_2^2 + y = x_3^3 - 2174420x + 1234136692$ $y^2 + y = x^3 - 57772164980x - 5344733777551611$

Table A.11.: Minimal Weierstraß equations of CM elliptic curves defined over $\mathbb{Q}$ having the smallest conductor N_E amongst all their twists. Here $N_E \in \mathbb{N}$ denotes the unique positive generator of the conductor ideal $\mathfrak{f}_E \subseteq \mathbb{Z}$, and $\mathfrak{f}_O := |O_K : O|$ denotes the conductor of the imaginary quadratic order O (see Example 6.2.8).

Postface: conclusions and future research

> Every phrase and every sentence
> is an end and a beginning.
>
> T.S.Eliot, *Little gidding*

We hope that this thesis has given the reader a taste of the richness of the various theories involved, from the theory of heights to that of L-functions, from Mahler measures to abelian varieties with complex multiplication. We have contributed our grain of sand to these mountains so difficult to climb, and yet so overwhelmingly beautiful when one stares at them from the valley down below, by exploring the relations between special values of L-functions associated to CM elliptic curves and Mahler measures of polynomials in Chapter 9, and by studying various properties of the division fields attached to CM elliptic curves in Chapter 8. We have also dived into the depths of exact polynomials in Chapter 5, and we have explored the various definitions of ray class fields for orders in Chapter 6.

This thesis in particular aims to prove, once again, how objects with extra symmetries, such as elliptic curves with complex multiplication, can be used as mathematical guinea pigs, on which testing broader conjectures is both easier and sometimes more enlightening than trying to attack immediately the general case. On the other hand, we have striven to present in every context the most general picture that our technical abilities managed to portray, as we do indeed believe that often the effort of doing so is paid back by the insight one obtains after gaining an aerial view of the mathematical surroundings, allowed by the generality pursued.

This being said, the rest of this final section is devoted to give an overview of the future perspectives opened by this thesis, and to provide a list of possible future research themes. The first examples of these are clearly given by the ongoing projects joint with Fabien Pazuki (see Section 3.4), François Brunault (see Chapter 5) and Francesco Campagna (see Chapter 6 and Section 7.3). We won't spend more words on these, and we refer the reader to the ends of the aforementioned sections and chapters for an outline of the future, previewed steps of those projects. Other than this, here are some further questions which might be interesting to explore in future work:

Mahler determinants

As we have already pointed out in Remark 4.2.3, it would be interesting to prove at least one relation expressing the special value of an L-function as the determinant of a matrix whose entries would be Mahler measures. For this to be a truly new result of course this matrix would need to have at least dimension two, and this determinant should not result in a Mahler measure itself, at least not for "obvious reasons", *e.g.* the determinant should not

be equal to the sum of two entries. Vice-versa, it would be interesting to find a Mahler measure which can be related to the determinant of a matrix whose entries are special values of L-functions. Obvious candidates for the special values of L-functions amenable to these kinds of computations would be the numbers $\zeta_K^*(1)$, associated to some number field K such that $\mathrm{rk}(O_K^\times) \geq 2$, or the values $L^*(C, 0)$ for some curve C defined over $\mathbb{Q}$ whose Jacobian is a simple CM abelian variety of dimension $g \geq 2$.

Exact polynomials and Hecke characters

We have seen already in Chapter 5 how we aim to use modular techniques in order to prove Lalín's conjecture. On the other hand, one could try to use the constructions of motivic cohomology elements coming from complex multiplication, in order to prove new types of identities going beyond Question 4.2.9. For example, one could use Beilinson/Neukirch's elements in the motivic cohomology of abelian number fields, and in particular their geometric counterparts constructed by Huber and Kings in [HK99], to find new 1-exact polynomials $P \in \mathbb{Z}[x, y]$ adapted to attack Chinburg's conjecture (see Remark 4.2.5). On the other hand, as we outlined in Section 9.3, one could use Deninger's results [Den89] to construct new k-exact polynomials $P \in \mathbb{Z}[x_1, \ldots, x_{k+2}]$ whose Mahler measure is related to the L-value $L^*(E, -k)$ associated to an elliptic curve E with complex multiplication.

Motives for Mahler measures

Deninger's construction of mixed motives which have the Mahler measure of a polynomial as one of their periods (see Remark 4.3.6) should be generalised to encompass any kind of polynomial, and not only those satisfying the hypotheses of Theorem 4.3.4. To do so, one needs to pursue until the end Bornhorn's computations, which could be generalised to higher dimensional cases, as well as some of the relative cohomology techniques that were outlined in Chapter 5.

The Mahler measure and Deninger's dynamical system

It is already known that the Mahler measure $m(P)$ of a polynomial can be computed as the entropy of a dynamical system. Recently, Deninger has introduced in [Den18] a way to attach a dynamical system to many arithmetic schemes, *i.e.* schemes of finite type over $\mathrm{Spec}(\mathbb{Z})$. Does the Mahler measure of a polynomial $P \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with integer coefficients appear as an entropy of one of these dynamical systems, for example the one associated to the zero locus $V_P \hookrightarrow \mathbb{G}_{m,\mathbb{Z}}^n$?

Division fields of CM abelian varieties

It would be interesting of course to extend the results of Chapter 8 to higher dimensional abelian varieties. While much of the theory carries over to the general context, like the main theorem of complex multiplication (see Theorem 7.1.25), many arguments featured in Chapter 8 must be changed, if they are to be adapted to the higher dimensional case. Most notably, formal groups need to be replaced by p-divisible groups, or by formal group schemes.

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